An Approach to the Commutation Relations in Quantum Field Theory

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Abstract

A general rule for obtaining both the commutator and anticommutator for any two points in space−time is studied in the case of the absence of interaction. The anticommutator is determined except for its sign. Examples of the use of this method are given.

§1. Introduction

As is well known, the commutator for any two points in space−time plays an important role in quantum field theory. There are several ways of obtaining the commutator. The usual method is to introduce the canonical conjugates of field variables and postulate the equal−time commutators between them.

A general rule for obtaning the commutator was first given by Peierls.[1] He treated the commutator as the one being related directly to the Lagrangian of the system.

The purpose of the present paper is not to obtain any new results, but to simplify Peierls' method. When the total Lagrangian density of a system consists of the free and interaction Lagrangians, the propagation functions for the asymptotic fields have been determined on the basis of the Yang−Feldman formalism.[2]**

In §2 and §3 we present a simple rule for forming the commutation relations, and the last section (§4) is devoted to its application.

§2. Commutation relations based on Peierls' method

In this section, we consider general four−dimensional commutation relations according to Peierls' method.

The free Lagrangian density L of a system is in general given by

$$\mathrm{L}=\mathrm{L}(\phi_\alpha,\ \partial_\mu\phi_\alpha), \qquad (2.1)$$

where the field operators ϕ_α are chosen to represent Bose fields, the subscript α denotes the different types of field as well as the components of each field.

The field equations for ϕ_α are expressed by[2]

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**After this work was completed, the author became aware that Nishijima had already obtained a similar conclusion.[3]

$$\partial_\mu[\partial L/\partial(\partial_\mu\phi_\alpha)]-\partial L/\partial\phi_\alpha\equiv D_{\alpha\beta}(\partial)\phi_\beta=0, \tag{2.2}$$

where ∂ is any differential operator, $D(\partial)$ a matrix. Let us define the retarded (Δ^R) and advanced (Δ^A) Green's functions in such a way that

$$D_{\alpha\beta}(\partial)\Delta^R_{\beta\gamma}(x)=D_{\alpha\beta}(\partial)\Delta^A_{\beta\gamma}(x)=\delta_{\alpha\gamma}\delta^4(x). \tag{2.3}$$

A modified Lagrangian $\bar{L}'$ can be given

$$\bar{L}'=\bar{L}+\lambda A, \tag{2.4}$$

where $\bar{L}$ is the Lagrangian of the system, λ an infinitesimal parameter, and A any function of the field operators. For definiteness $A=\phi_\varepsilon(y)$ is employed in this section. Then, the modified Lagrangian density L' becomes

$$L'(x)=L(x)+\lambda\phi_\varepsilon(y)\delta^4(x-y). \tag{2.5}$$

Considering the modified field operators $\phi'_\eta(x)$ as expansions in powers of λ, we can write them to first order as

$$\phi'_\eta(x)=\phi_\eta(x)+\lambda D_R\phi_\eta(x), \tag{2.6}$$

with

$$D_R\phi_\eta(x)\rightarrow 0, \quad \text{as } t\rightarrow-\infty. \tag{2.7}$$

Similarly we can define other modified field operators

$$\phi'_\eta(x)=\phi_\eta(x)+\lambda D_A\phi_\eta(x), \tag{2.8}$$

with

$$D_A\phi_\eta(x)\rightarrow 0, \quad \text{as } t\rightarrow+\infty. \tag{2.9}$$

According to Peierls, four-dimensional commutation relation between two field operators is given by

$$[\phi_\varepsilon(y),\ \phi_\eta(x)]=i[D_R\phi_\eta(x)-D_A\phi_\eta(x)], \tag{2.10}$$

where $[A,B]=AB-BA$. From (2.5) and (2.2), we have

$$L'(\phi'_\eta,\ \partial_\mu\phi'_\eta)=L(\phi'_\eta,\ \partial_\mu\phi'_\eta)+\lambda\phi'_\varepsilon(y)\delta^4(x-y), \tag{2.11}$$

$$\partial_\mu[\partial L'/\partial(\partial_\mu\phi'_\eta)]-\partial L'/\partial\phi'_\eta=D_{\eta\zeta}(\partial)\phi'_\zeta-\lambda\delta_{\varepsilon\eta}\delta^4(x-y). \tag{2.12}$$

Hence, by using (2.6), we obtain

$$D_{\eta\zeta}(\partial)\cdot D_R\phi_\zeta(x)=\delta_{\varepsilon\eta}\delta^4(x-y). \tag{2.13}$$

Therefore, using (2.3), we rewrite (2.13) as

$$\begin{aligned} D_R\phi_\eta(x)&=[D^{-1}(\partial)]_{\eta\varepsilon}\delta^4(x-y)\\ &\equiv\Delta^R_{\eta\varepsilon}(x-y), \end{aligned} \tag{2.14}$$

where the notation $D^{-1}(\partial)$ denotes the inverse of $D(\partial)$. Similarly $D_A\phi_\eta(x)$ is also expressed by

$$D_A\phi_\eta(x)=[D^{-1}(\partial)]_{\eta\xi}\delta^4(x-y)\equiv\Delta^A_{\eta\xi}(x-y). \tag{2.15}$$

Hence, from (2.10), we find four-dimensional commutation relation with respect to $\phi_\xi(y)$ and $\phi_\eta(x)$

$$[\phi_\xi(y),\ \phi_\eta(x)]=i[\Delta^R_{\eta\xi}(x-y)-\Delta^A_{\eta\xi}(x-y)]$$

$$=-i\Delta_{\eta\xi}(x-y), \tag{2.16}$$

where

$$\Delta_{\eta\xi}(x)=\Delta^A_{\eta\xi}(x)-\Delta^R_{\eta\xi}(x), \tag{2.17}$$

which satisfies

$$D_{\alpha\beta}(\partial)\Delta_{\beta\gamma}(x)=0. \tag{2.18}$$

With the aid of (2.14), (2.15), (2,17), and (2.18), the matrix $\Delta(x)$ in (2.17) is directly determined by rewriting the elements $1/(\Box+m^2)$, $1/\Box$, and $1/\Box^2$ of the matrix $D^{-1}(\partial)$ into $\Delta(x;m^2)$, $D(x)$, and $E(x)$, respectively, where $\Box$ is the d'Alembert operator, $\Delta(x;m^2)$ is a singular function which satisfies $(\Box+m^2)\Delta(x;m^2)=0$, $D(x)=\Delta(x;0)$, and $E(x)=-[(\partial/\partial m^2)\Delta(x;m^2)]_{m=0}$.

§ 3. Commutation relations of the Dirac fields

On the basis of Peierls' method, we postulate the existence of some operator θ, which is itself of such a character as to change sign if the coordinate system is rotated by 2π, and which anticommutes with all components of the Dirac field at all points in space-time.

The free Lagrangian density L of a system is

$$L=L(\psi_\alpha,\ \partial_\mu\psi_\alpha). \tag{3.1}$$

Here, ψ_α is the component of the Dirac field. The field equations and the retarded and advanced Green's functions are exactly the same as those of the preceding section.

Consider first the modified Lagrangian density L' with the added term $A=\lambda\theta\psi_\xi(y)$:

$$L'(\psi'_\alpha,\ \partial_\mu\psi'_\alpha)=L(\psi'_\alpha,\ \partial_\mu\psi'_\alpha)+\lambda\theta\psi'_\xi(y)\delta^4(x-y), \tag{3.2}$$

where $\psi'_\alpha(x)$ is a solution of the modified field equations. From (3.2) and the field equations, we have

$$D_{\alpha\beta}(\partial)\psi'_\beta(x)-\lambda\theta\delta_{\xi\alpha}\delta^4(x-y)=0, \tag{3.3}$$

with

$$\psi'_\beta(x)=\psi_\beta(x)+\lambda D_R\psi_\beta(x), \tag{3.4}$$

$$\psi'_\beta(x)=\psi_\beta(x)+\lambda D_A\psi_\beta(x), \tag{3.5}$$

where

$$D_R\psi_\beta(x)\to 0, \quad \text{as } t\to-\infty, \tag{3.6}$$

$$D_A\psi_\beta(x)\to 0, \quad \text{as } t\to+\infty. \tag{3.7}$$

Hence we obtain

$$D_{\alpha\beta}(\partial)D_C\psi_\beta(x)=\theta\delta_{\xi\alpha}\delta^4(x-y), \tag{3.8}$$

$$D_C\psi_\beta(x)=\theta[D^{-1}(\partial)]_{\beta\xi}\delta^4(x-y)$$

$$\equiv\theta\bar{S}^C_{\beta\xi}(x-y), \tag{3.9}$$

where $C=R$ or A.

Therefore four-dimensional anticommutation relation is

$$[\theta\psi_\xi(y),\ \psi_\beta(x)]=\theta\{\psi_\xi(y),\ \psi_\beta(x)\}=i[D_R\psi_\beta(x)-D_A\psi_\beta(x)]$$

$$=i\theta[\bar{S}^R_{\beta\xi}(x-y)-\bar{S}^A_{\beta\xi}(x-y)]=-i\theta\bar{S}_{\beta\xi}(x-y), \tag{3.10}$$

where $\{A,B\}=AB+BA$ and $\bar{S}_{\beta\xi}(x)=\bar{S}^A_{\beta\xi}(x)-\bar{S}^R_{\beta\xi}(x)$.

As is obvious from (3.2) we can have another modified Lagrangian density

$$L'(\psi'_\alpha,\ \partial_\mu\psi'_\alpha)=L(\psi'_\alpha,\ \partial_\mu\psi'_\alpha)+\lambda\psi'_\xi(y)\theta\delta^4(x-y). \tag{3.11}$$

From above expression we find

$$\{\psi_\xi(y),\ \psi_\beta(x)\}=i\theta\bar{S}_{\beta\xi}(x-y). \tag{3.12}$$

Therefore, from (3.10) and (3.12), it can be concluded that the anticommutator is merely determined except for its sign :

$$\{\psi_\xi(y),\ \psi_\beta(x)\}=\pm i\bar{S}_{\beta\xi}(x-y). \tag{3.13}$$

§ 4. Examples

In this section a number of cases will be treated by the methods developed in the preceding sections.

The first case is that of the free charged field ϕ which has the Lagrangian density

$$L=\partial^\nu\phi^\dagger\cdot\partial_\nu\phi-\mu^2\phi^\dagger\phi, \tag{4.1}$$

where μ the bare mass of ϕ and the Minkowski metric employed is $(1,\ -1,\ -1,\ -1)$. The field equations for ϕ and $\phi^\dagger$ are

$$\begin{bmatrix} 0 & \Box+\mu^2 \\ \Box+\mu^2 & 0 \end{bmatrix}\begin{bmatrix}\phi \\ \phi^\dagger\end{bmatrix}=0, \tag{4.2}$$

where $\Box=\partial^{\nu}\partial_{\nu}$. Hence the matrix $D^{-1}(\partial)$ is

$$\begin{bmatrix} 0 & 1/(\Box+\mu^2) \\ 1/(\Box+\mu^2) & 0 \end{bmatrix}. \tag{4.3}$$

Therefore, from (2.16), we have

$$\begin{aligned}[\phi(x),\ \phi^{\dagger}(y)] &= -i\Delta_{\phi^{\dagger}\phi}(y-x) \\ &= i\Delta(x-y;\mu^2), \\ [\phi(x),\ \phi(y)] &= -i\Delta_{\phi\phi}(y-x) \\ &= 0, \\ [\phi^{\dagger}(x),\ \phi^{\dagger}(y)] &= -i\Delta_{\phi^{\dagger}\phi^{\dagger}}(y-x) \\ &= 0.\end{aligned} \tag{4.4}$$

The second example is a neutral vector field U_{μ}. The Lagrangian density of the system is given by [1]

$$L=-(1/4)F^{\mu\nu}F_{\mu\nu}+(1/2)m^2U^{\mu}U_{\mu}+B\partial^{\nu}U_{\nu}, \tag{4.5}$$

where $F_{\mu\nu}=\partial_{\mu}U_{\nu}-\partial_{\nu}U_{\mu}$, m the bare mass of U_{μ}, and B an auxiliary scalar field having negative norm.

The field equations for U_{μ} and B are

$$\begin{bmatrix} -(\Box+m^2)g^{\mu\nu}+\partial^{\mu}\partial^{\nu} & \partial^{\mu} \\ -\partial^{\nu} & 0 \end{bmatrix}\begin{bmatrix} U_{\nu} \\ B \end{bmatrix}=0, \tag{4.6}$$

where $g^{00}=-g^{kk}=1(k=1,\ 2,\ 3)$ and $g^{\mu\nu}=0\,(\mu\neq\nu)$. The matrix $D^{-1}(\partial)$ is

$$\begin{bmatrix} [-1/(\Box+m^2)](g_{\mu\nu}-\partial_{\mu}/\Box) & -\partial_{\mu}/\Box \\ \partial_{\nu}/\Box & -m^2/\Box \end{bmatrix}. \tag{4.7}$$

Four-dimensional commutation relations are

$$\begin{aligned}[U_{\mu}(x),\ U_{\nu}(y)] &= -i\Delta_{\nu\mu}(y-x) \\ &= -i(g_{\mu\nu}+m^{-2}\partial^{x}_{\mu}\partial^{x}_{\nu})\Delta(x-y;m^2) \\ &\quad +im^{-2}\partial^{x}_{\mu}\partial^{x}_{\nu}D(x-y), \\ [U_{\mu}(x),\ B(y)] &= -i\Delta_{B\mu}(y-x) \\ &= -i\partial^{x}_{\mu}D(x-y), \\ [B(x),\ B(y)] &= -i\Delta_{BB}(y-x) \\ &= -im^{-2}D(x-y),\end{aligned} \tag{4.8}$$

where $\partial^{x}_{\mu}=\partial/\partial x^{\mu}$.

The last example is the Dirac field, with the Lagrangian density

$$L=-\bar{\psi}(-i\gamma^{\mu}\partial_{\mu}+m)\psi, \tag{4.9}$$

where γ is the Dirac matrix and $\bar{\psi}=\psi^{\dagger}\gamma^{0}$. The field equations are

$$\begin{bmatrix} 0 & i\tilde{\gamma}^{\mu}\partial_{\mu}+m \\ -i\gamma^{\mu}\partial_{\mu}+m & 0 \end{bmatrix}\begin{bmatrix} \psi \\ \tilde{\bar{\psi}} \end{bmatrix}=0, \tag{4.10}$$

where $\tilde{\bar{\psi}}$ and $\tilde{\gamma}$ are the transpose of ψ and γ, respectively. The matrix $D^{-1}(\partial)$ is

$$\begin{bmatrix} 0 & (i\gamma^{\mu}\partial_{\mu}+m)/(\Box+m^2) \\ (-i\tilde{\gamma}^{\mu}\partial_{\mu}+m)/(\Box+m^2) & 0 \end{bmatrix}. \tag{4.11}$$

From (3.13), four-dimesional anticommutation relations are

$$\begin{aligned} \{\psi_{\alpha}(x),\ \tilde{\bar{\psi}}_{\beta}(y)\} &= \pm i\bar{S}_{\beta\alpha}(y-x) \\ &= \pm i(-i\tilde{\gamma}^{\mu}\partial_{\mu}^{y}+m)_{\beta\alpha}\Delta(y-x;m) \\ &= \mp iS_{\alpha\beta}(x-y;m)\ , \\ \{\tilde{\bar{\psi}}_{\alpha}(x),\ \psi_{\beta}(y)\} &= \pm\bar{S}_{\beta\alpha}(y-x) \\ &= \pm i(i\gamma^{\mu}\partial_{\mu}^{y}+m)_{\beta\alpha}\Delta(y-x;m) \\ &= \pm iS_{\beta\alpha}(y-x;m), \end{aligned} \tag{4.12}$$

where $S(x;m)=(i\gamma^{\mu}\partial_{\mu}+m)\Delta(x;m^2)$.

Acknowledgement

The author wishes to thank Professor S. Ishijima of the University of the Ryukyus for his helpful suggestions.

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