Locally Trivial Fiber Spaces and Stiefel-Whitney Classes

Locally Trivial Fiber Spaces and Stiefel-Whitney Classes

Ryuji MAEHARA*

E. Fadell [3] generalized the notion of a plane bundle, and gave a definition of generalized tangent bundle τ_M for a topological manifold M. In this paper, we prove

Theorem *Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a locally trivial fiber space such that F, B, E are topological manifolds Then there exist generalized plane bundles ξ, η over E with the properties:*

$$\tau_F = j^*(\eta), \quad \xi \overset{*}{\simeq} p^*(\tau_B) \quad \textit{and} \quad \tau_E \overset{*}{\simeq} \xi \oplus \eta$$

where $j^*(\eta)$, $p^*(\tau_B)$ denote the generalized plane bundles indvced from η, τ_B by j, p, respectively; $\overset{*}{\simeq}$ denotes fiber homotopy equivalence; and $\oplus$ denotes the Whitney sum.

Some consequences and applications of the theorem will be discussed in sections 4, 5.

1. Preliminaries

Consider the following commutative diagram of spaces and maps:

$$(\xi) \qquad \begin{array}{ccccc} F & \longrightarrow & E & \xrightarrow{p} & B \\ \uparrow & & \uparrow & & \| \\ F_0 & \longrightarrow & E_0 & \xrightarrow{p_0} & B \end{array}$$

where the unlabelled arrows are inclusion maps and $F = p^{-1}(b_\circ)$, $F_\circ = p_\circ^{-1}(b_\circ)$ $(b_\circ \in B)$. Such a diagram (denoted $\xi = (E, E_\circ, p, B)$) is called *a (locally trivial) fibered pair with fiber* $(F, F_\circ)$ if for each point b in B we can find an open set U containing b and a homeomorphism of pairs

$$\phi : (U \times F, U \times F_\circ) \longrightarrow (p^{-1}(U), p_\circ^{-1}(U))$$

with the property $p\,\phi\,(b', x) = b'$. When $E_\circ$ is the empty subset of E, the above fibered pair reduces to a (locally trivial) fiber space $F \xrightarrow{j} E \xrightarrow{p} B$.

In a fibered pair $\xi = (E, E_\circ, p, B)$, suppose the base space B is paracompact. Then it is known that $p: E \longrightarrow B$ and $p_\circ : E_\circ \longrightarrow B$ are Hurewicz fiber spaces. In fact, the map p admits a lifting function

$$\lambda : \{(z, \ell) \in E \times B^I \mid p(z) = \ell(o)\} \longrightarrow E^I$$

such that $p\lambda(z, \ell) = \ell$ and if $z \in E_\circ$ then $\lambda(z, \ell)$ is a path in $E_\circ$ (where X^I denotes

Received October 31, 1973

* Dept. of Math., Sci. & Eng. Div., Univ. of the Ryukyus

the space of paths in X with the compact-open topology). See [3] .

Let $\xi=(E, E_0, p, B)$ and $\xi'=(E', E'_0, p', B)$ be fibered pairs with the same base space B. A map of fibered pairs $\alpha: \xi \longrightarrow \xi'$ is a map $\alpha: (E, E_0) \longrightarrow (E', E'_0)$ such that $p'\alpha=p$. i.e. α is fiber preserving. If $\beta: \xi \longrightarrow \xi'$ is another map of fibered pairs, then $\alpha \overset{*}{\simeq} \beta$ (read fiberwise homotopic) provided there exists a homotopy $h: (E, E_0)\times I \rightarrow (E', E'_0)$ such that $h(z,o)=\alpha(z)$, $h(z,1)=\beta(z)$ and $p'h(z,t)=p(z)$ for all $t \in I$. ξ and ξ' are said to be fiber homotopy equivalent if there are maps of fibered pairs $\xi \underset{\alpha'}{\overset{\alpha}{\leftrightarrows}} \xi'$ such that $\alpha'\alpha \overset{*}{\simeq} 1$ and $\alpha\alpha' \overset{*}{\simeq} 1$. Both α and α' will be called fiber homotopy equivalences.

Let ξ and ξ' be as in the preceding paragraph. The Whitney sum $\xi \oplus \xi'$ of ξ and ξ' is defined by $\xi \oplus \xi' = (\overline{E}, \overline{E}_0, \overline{p}, B)$ where

$$\begin{cases} \overline{E} = \{(z, z') \in E\times E' \mid p(z)=p'(z'),\} \\ \overline{E}_0 = [(E\times E'_0) \cup (E_0 \times E')] \cap \overline{E}, \\ \overline{p}(z, z') = p(z) \ (=p'(z')). \end{cases}$$

It is not difficult to see that $\xi \oplus \xi'$ is a fibered pair.

A fibered pair $\xi=(E, E_0, p, B)$ with fiber (F, F_0) is called *a generalized p-plane bundle* (abbreviated *n-gpb*) if it satisfies the following properties:

i) $p: E \rightarrow B$ admits a cross-section $s: B \rightarrow E$ (i.e., $ps=1$) such that $E_0 = E - s(B)$,

ii) $(F, F_0) \sim (R^n, R^n - o)$ where R^n is a Euclidean n-space, o is the origin of R^n, and $\sim$ designates homotopy equivalence of pairs. If ξ is an m-gpb and if η is an n-gpb with the same base space as that of ξ, then $\xi \oplus \eta$ is an $(m+n)$-gpb (see [3]).

An *n-manifold* is a connected paracompact space which is locally homeomorphic to Euclidean n-space R^n $(n \geqq 1)$. Given an n-manifold M, let

$$T_0(M) = \{ \ell \in M^I \mid \ell(t) \neq \ell(o) \text{ for } o < t \leqq 1 \},$$

let $T(M)$ be the union of $T_0(M)$ and the constant paths on M, and give $T(M)$ the compact-open topology. Define $\pi: T(M) \rightarrow M$ by $\pi(\ell)=\ell(o)$. Then

$$\tau_M = (T(M), T_0(M), \pi, M)$$

is an n-gpb (see [3]). τ_M will be called the *tangent n-gpb* of M. If M possesses a differentiable structure and if we let (E, q, M) denote the tangent bundle of M and (E_0, q_0, M) the sub-bundle of non-zero vectors, then (E, E_0, q, M) is clearly an n-gpb. It is known that there exists a fiber homotopy equivalence $\tau_M \overset{*}{\simeq} (E, E_0, q, M)$. See [3] .

2. Two propositions.

Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a locally trivial fiber space. We define

$\xi = (H, H_o, \pi, E)$ and $\eta = (V, V_o, \pi, E)$ as follows:

$$(\xi)\quad \begin{cases} H_o = \{ \ell \in E^I \mid p\ell(o) \neq p\ell(t) \text{ for } o < t \leqq 1 \}, \\ H = H_o \cup \{\text{constant paths in } E\}, \\ \pi(\ell) = \ell(o); \end{cases}$$

$$(\eta)\quad \begin{cases} V_o = \{ \ell \in E^I \mid \ell(o) \neq \ell(t) \text{ and } p\ell(o) = p\ell(t) \text{ for } o < t \leqq 1 \}, \\ V = V_o \cup \{\text{constant paths in } E\}, \\ \pi(\ell) = \ell(o). \end{cases}$$

The theorem stated in the beginning devides into the following two propositions:

Proposition 1. *With the above notations, if F is an n-manifold, then η is an n-gpb and $\tau_F = j^*(\eta)$ (=the n-gpb induced from η by j).*

Proposition 2. *If F is an n-manifold and B is an m-manifold, then ξ is an m-gpb and*

$$\xi \overset{*}{\simeq} p^*(\tau_B), \quad \tau_E \overset{*}{\simeq} \xi \oplus \eta.$$

In the proof of these propositions, the following elementary lemma will be needed. Let D^n denote the n-ball in Euclidean n-space R^n, i.e. $D^n = \{x \in R^n \mid \|x\| \leqq 1\}$ and let V^n denote the interior of D^n. If $k<n$ we may regard $R^k = \{(x_1, \cdots, x_n) \in R^n \mid x_{k+1} = \cdots = x_n = 0\}$ and hence $D^k \subset D^n$, $V^k \subset V^n$.

Lemme 3. (See [3, p.492]). *Let M be an n-manifold. Suppose U is an open set in M such that its closure $\overline{U}$ is homeomorphic to the unit ball D^n with U corresponding to the interior V^n of D^n. For $k<n$, let $U^{(k)}$ be the subset of U which corresponds to the subset V^k ($\subset V^n$). Finally, let $G(M)$ be the space of homeomorphisms of M with the compact-open topology. Then there exists a map*

$$\gamma : U \times U \longrightarrow G(M)$$

satisfying the following properties:

i) $\gamma(a, b)(a) = b$,

ii) $\gamma(a, a) = 1$,

iii) $\gamma(b,c)\,\gamma(a,b) = \gamma(a,c)$,

iv) $\gamma(a,b)(z) = z$ *for* $z \in M - U$,

v) *if* $a, b \in U^{(k)}$, $\gamma(a,b)$ *maps* $U^{(k)}$ *onto* $U^{(k)}$.

3. Proof of the propositions.

Proof of proposition 1. Let $z_o \in E$. Choose an open neighborhood U_1 of $p(z_o)$ in B for which there exists a homeomorphism $\phi : U_1 \times F \to p^{-1}(U_1)$ with the property $p\,\phi(b,x) = b$. This is possible because (E, p, B) is a locally trivial fiber space. When $\phi(b, x) = z$, we will write $x = q(z)$. Let $p(z_o) = b_o$ and $q(z_o) = x_o$, i.e., $\phi(b_o, x_o) = z_o$. Since F is an n-manifold, there is a neighborhood U_2 of x_o with $(\overline{U}_2, U_2)$ homeomorphic to (D^n, V^n). Let $W = \phi(U_1 \times U_2)$. Obviously,

W is a neighborhood of z_0. We shall show the product structure of $\pi : (\pi^{-1}(W), \pi_0^{-1}(W)) \to W$.

Let (V', V_o') be the fiber at z_0, i. e., $(V', V_o') = (\pi^{-1}(z_0), \pi_0^{-1}(z_0))$. Define $\alpha : W \times (V', V_o') \to (\pi^{-1}(w), \pi_0^{-1}(w))$ by

$$\alpha(z, \ell)(t) = \phi(pz, \gamma(x_0, qz)\, q\, \ell(t))$$

where $\gamma : U_2 \times U_2 \to G(F)$ is the map stated in Lemma 3. Clearly, α maps $W \times V_o'$ into $\pi_0^{-1}(W)$. To see the continuity of α, it suffices to see that $\gamma(x_0, qz)(q\, \ell(t))$ is continuous with respect to (z, ℓ, t). But this follows from the continuity of γ and ,q and from the fact that the evaluation map $M^M \times M^I \times I \to M$ ($(q,w,t) \to qw(t)$) is continuous owing to the local compactness of M and I. That α is fiber preserving, i.e., $\pi\, \alpha(z, \ell) = z$ or equivalently $\alpha(z, \ell)(o) = z$, is easily verified. Now, define $\alpha' : \pi^{-1}(W) \to W \times V'$ by

$$\alpha'(\bar{\ell}) = (\bar{\ell}(o), (b_0, \gamma(q\bar{\ell}(o), x_0) q\bar{\ell}).$$

An easy calculation shows that α' is the inverse of α. Hence, α is a homeomorphism. Therefore, $\eta = (V, V_0, \pi, E)$ is a fibered pair. Note that η has the same fiber as the fiber of the tangent n-gpb of F. Thus η is an n-gpb. The last assertion $\tau_F = j^*(\eta)$ is clear.

Proof of Proposition 2. (1) First, we see the local product structure in $\xi = (H, H_0, \pi, E)$. For $z_0 \in E$, let $U_1 (\subset B)$, $U_2 (\subset F)$, $W (\subset E)$, ϕ, b_0, x_0, and q be as in the preceding proof. We may assume $(\bar{U}_1, U_1)$ is homeomorphic to (D^m, V^m) since B is an m-manifold. Let $(H', H_0') = (\pi^{-1}(z_0), \pi_0^{-1}(z_0)) (\subset (H, H_0))$. The object is to define a homeomorphism of pairs $\beta : W \times (H', H_0') \to (\pi^{-1}(W), \pi_0^{-1}(W))$ with $\pi\beta = p_1$ (=the projection onto W).

We can find a homeomorphism $(W, W) \cong (D^{m+n}, V^{m+n})$ so that $W^{(n)} = p^{-1}(b_0) \cap W$ corresponds to $V^n (\subset V^{m+n})$. So, Lemma 3 gives us the map $\gamma_2 : W^{(n)} \times W^{(n)} \subset W \times W \to G(E)$. Note that $\gamma_2(a, b)(z) = z$ for $z \in E - W$ and $\gamma_2(a,b)$ maps $p^{-1}(b_0)$ onto $p^{-1}(b_0)$. On the other hand, since $(\bar{U}_1, U_1) \cong (D^m, V^m)$, we have the map $\gamma_1 : U_1 \times U_1 \to G(B)$. Define $\Gamma : U_1 \times U_1 \to G(p^{-1}(U))$ by

$$\Gamma(b_1, b_2)(z) = \phi(\gamma_1(b_1, b_2)(pz), qz).$$

We can regard Γ as a map from $U_1 \times U_1$ into $G(E)$ by defining $\Gamma(b_1, b_2)(z) = z$ for $z \in E - p^{-1}(U)$.

Now define $\beta : W \times (H', H_0') \to (\pi^{-1}(W), \pi_0^{-1}(W))$ by

$$\beta(z, \ell)(t) = \Gamma(b_0, p(z))\, \gamma_2(z_0, \phi(b_0, q(z)))\, \ell(t).$$

The continuity of β is quaranteed by the local compactness of E and I. β is fiber preserving; in fact,

$$\pi\beta(z, \ell) = \beta(z, \ell)(o) = \Gamma(b_0, pz)\, \gamma_2(z_0, \phi(b_0, qz))z_0$$
$$= \Gamma(b_0, pz)\, \phi(b_0, qz)\, \phi(pz, qz) = z.$$

β is a homeomorphism since it has inverse β' given by $\beta'(\bar{\ell}) = (z, \ell)$ where

$$\begin{cases} z = \bar{\ell}(o) \\ \ell = \gamma_2(\phi(b_o, q\bar{\ell}(o)), z_o)\,\Gamma(p\bar{\ell}(o), b_o)\bar{\ell}. \end{cases}$$

Therefore, ξ has a local product structure.

(2) To see that $\xi = (H, H_o, \pi, E)$ is an m-gpb, define $S : E \to H$ by $S(z) = \tilde{z}$ (the constant path at z). S is a cross-section, and $H_o = H - S(E)$ holds. All that remains is to show $(H', H_o') \sim (R^m, R^m - O)$. But this follows from [3, Prop. 4.1] because (H', H_o') is identical to the fiber of the "normal fiber space" of the imbedding $F \subset E$.

(3) Next, we show that $\xi \overset{*}{\simeq} p^*(\tau_B)$. Let $\tau_B = (T(B), T_o(B), \pi', B)$ be the tangent m-gpb of B. Then, by definition, we have $p^*(\tau_B) = (\bar{H}, \bar{H}_o, p_1, E)$ where

$$\begin{cases} \bar{H} = \{ (z, \ell) \in E \times T(B) \mid p(z) = \ell(o) \}, \\ \bar{H}_o = \{ (z, \ell) \in E \times T_o(B) \mid p(z) = \ell(o) \} = \bar{H} \cap (E \times T_o(B)), \\ p_1(z, \ell) = z. \end{cases}$$

Let λ be a lifting function of the fiber space (E, p, B) such that if ℓ is the constant path at $p(z)$ then $\lambda(z, \ell)$ is the constant path at z. We define maps of pairs

$$(\bar{H}, \bar{H}_o) \underset{g}{\overset{f}{\rightleftarrows}} (H, H_o) \text{ by}$$

$$\begin{cases} f = \lambda, \text{ i.e., } f(z, \ell)(t) = \lambda(z, \ell)(t), \\ g(\bar{\ell}) = (\bar{\ell}(o), p\bar{\ell}). \end{cases}$$

It is easy to see that f and g are fiber preserving, i.e., $\pi f = p_1$ and $p_1 g = \pi$. Furthermore, since $gf(z, \ell) = g(\lambda(z, \ell)) = (\lambda(z, \ell)(o), p\lambda(z, \ell)) = (z, \ell)$, we have $gf = 1$. Now, let us show $fg \overset{*}{\simeq} 1$. For a path $w \in E^I$ and $s \in I$, let w_s denote the path given by

$$w_s(t) = \begin{cases} w(s+t), & o \leqq t \leqq 1-s \\ w(1), & 1-s \leqq t \leqq 1. \end{cases}$$

We define a homotopy $h : (H, H_o) \times I \to (H, H_o)$ by

$$h(w,s)(t) = \lambda(w(t(1-s)), pw_t(1-s))(st).$$

h is a fiberwise homotopy since

$$\pi h(w, s) = h(w, s)(o) = \lambda(w(o), pw_o)(o) = w(o) = \pi(w).$$

h is a homotopy between 1 and fg; in fact,

$h(w, o)(t) = \lambda(w(t), pw_t)(o) = w(t)$, i.e., $h(w, o) = w$, and

$h(w, 1)(t) = \lambda(w(o), pw_o)(t) = \lambda(w(o), pw)(t) = fg(w)(t)$, i.e.,

$h(w, 1) = fg(w)$. Hence, $fg \overset{*}{\simeq} 1$. This proves $\xi \overset{*}{\simeq} p^*(\tau_B)$.

(4) The proof of the final assertion $\tau_E \overset{*}{\simeq} \xi \oplus \eta$ is essentially same as the argument in the proof of [3, Theorem 4.11], and hence, is omitted.

4. Characteristic classes.

Let $\xi=(E, E_0, p, B)$ be a fibered pair with fiber (F, F_0). Then the fundamental group $\pi_1(B, b_0)$ acts on $H_*(F, F_0; G)$. If this action is trivial, ξ is called a G-orientable fibered pair. An n-manifold M is said to be G-orientable when its tangent n-gpb τ_M is G-orientable as a fibered pair.

Let Λ denote a commutative ring with unit.

Theorem 4 (See Fadell [3]). *If (E, E_0, p, B) is a Λ-orientable n-gpb with fiber (F, F_0), then there exists an element $U \in H^n(E, E_0; \Lambda)$ satisfying the following properties:*

i) The inclusion map $j:(F, F_0)\to(E, E_0)$ induces an isomorphism j^*: $H^n(E, E_0; \Lambda)\to H^n(F, F_0; \Lambda)$, and if we identify $H^n(F, F_0; \Lambda)$ with Λ, U corresponds to the unit of Λ under j^*

ii) The homomorphism defined by cup product,

$$\cup U : H^i(E; \Lambda)\to H^{i+n}(E, E_0; \Lambda)$$

is an isomorphism for every i.

U is called a Λ-orientation of the n-gpb, and from property (i), it is determined uniquely by the choice of identification $\Lambda=H^n(F, F_0; \Lambda)$. Now, note that $p^*: H^i(B; \Lambda)\to H^i(E; \Lambda)$ is an isomorphism since the total fiber F is contractible. Hence, from (ii), the composition

$$\phi : H^i(B; \Lambda)\xrightarrow{P^*}H^i(E; \Lambda)\xrightarrow{\cup U}H^{i+n}(E, E_0; \Lambda)$$

is an isomorphism. ϕ is the Thom isomorphism associated to U. The Euler class $X(\xi)$ of the n-gpb ξ is defined by

$$X(\xi)=\phi^{-1}(U\cup U)\in H^n(B; \Lambda).$$

When $\Lambda=Z_2$ (= the ring of integers mod 2), the Stiefel-Whitney classes $W_i(\xi)$ $(i=0, 1, 2\cdots)$ of ξ are defined by

$$W_i(\xi)=\phi^{-1}S_q^i(U)\in H^i(B; Z_2)$$

where $S_q^i: H^n(E, E_0; Z_2)\to H^{n+i}(E, E_0; Z_2)$ denotes the i-th Steenrod operation. Note that in defining $W_i(\xi)$ we do not need to worry about the orientability of ξ since every n-gpb is Z_2-orientable.

Characteristic classes $X(\xi)$, $W_i(\xi)$ satisfy the naturality property in the following sense : Let $f: B'\to B$ be a map and let $f^*\xi$ be the n-gpb induced from ξ by f;then

$$X(f^*\xi)=f^*X(\xi),\quad W_i(f^*\xi)=f^*W_i(\xi).$$

Furthermore, characteristic classes are invariances of fiber homotopy equivalence; that means, $\xi\overset{*}{\simeq}\xi'$ implies

$$X(\xi)=X(\xi') \text{ and } W_i(\xi)=W_i(\xi).$$

Therefore, Proposition 1 gives the following corollary. The fact stated there was recently proved by Gottlieb [4] in a slightly different method.

Corollary 5 (to Proposition 1). *Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a locally trivial fiber space such that F is an n-manifold. Then*

$$W_k(\tau_F) \in j^* H^k(E; Z_2),\ k = o, 1, 2, \cdots, n.$$

If F is orientable (ie, Z-orientable) and $\pi_1(E) = O$, then

$$X(\tau_F) \in j^* H^n(E; Z).$$

The author does not know whether or not the above corollary remains valid for a Hurewicz fiber space $F \longrightarrow E \longrightarrow B$ instead for a locally trival fiber space.

Example *Let $P^{2n} \xrightarrow{j} E \xrightarrow{p} B$ be a locally trivial fiber space where P^{2n} is the real projective space of dimension $2n$. Then $j^* : H^k(E, Z_2) \rightarrow H^k(P^{2n}, Z_2)$ is onto for every k.*

Proof The cohomology ring $H^*(P^{2n}, Z_2)$ is generated by the unique nonzero element u in $H^1(P^{2n}, Z_2)$. u is the first Stiefel-Whitney class of P^{2n}(see [5]) and hence, u is in the image of j^*.

Proposition 2 gives

Corollary 6. *Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a locally trivial fiber space such that both B and F are (topological) manifolds. Then*

$$W_k(\tau_F) = j^* W_k(\tau_E),\ k = 0, 1, \cdots, \dim F.$$

If $\pi_1(E) = O$, $X(\tau_E)$ is a decomposable element.

Proof Proposition 2 says:

$$\tau_E \overset{*}{\simeq} \xi \oplus \eta, \quad \tau_F = j^*(\eta), \quad \xi \overset{*}{\simeq} p^*(\tau_B).$$

Hence,

$$\begin{aligned} W_k(\tau_E) &= W_k(\xi \oplus \eta) \\ &= \sum_{i=o}^{k} W_i(\xi) \cup W_{k-i}(\eta) \quad \text{(see [3 , Theorem 6.11])} \\ &= \sum_{i=o}^{k} p^* W_i(\tau_B) \cup W_{k-i}(\eta). \end{aligned}$$

Therefore,

$$\begin{aligned} j^* W_k(\tau_E) &= \sum_{i=o}^{k} j^* p^* W_i(\tau_B) \cup j^* W_{k-i}(\eta) \\ &= W_o(\tau_B) \cup j^* W_k(\eta) \quad \text{(since } pj = 0\text{)} \\ &= j^* W_k(\eta) = W_k(\tau_F). \end{aligned}$$

The second assertion follows from

$$X(\tau_E) = X(\xi \oplus \eta) = X(\xi) \cup X(\eta).$$

The assumption $\pi_1(E) = O$ quarantees the orientability of τ_E, ξ, η.

Example *Let $F \xrightarrow{j} E \xrightarrow{p} B$ be as in Corallary 6. If E is orientable then so is F.*

Proof In general, a (topological) manifold M is orientable if and only if $W_1(\tau_M) = 0$ (see [6,p. 349]). Corollary 6 says $W_1(\tau_F)=j^* W_1(\tau_E)$. Hence, $W_1(\tau_E) = 0$ implies $W_1(\tau_F) = 0$.

5. Applications

Let K be a field. For a map $f: X \to Y$, we will use the following notations.

Notation

Imf^* = the image of $f^* : H^*(Y, K) \to H^*(X, K)$.

$(Imf^*)' = \{x \in H^*(X, K) \mid x \cup y = 0$ for some nonzero y in $Imf^*\}$.

Let PB be the space of paths in B starting at b_0. The path fibration $\pi: PB \to B$ is the map defined by $\pi(\ell) = \ell(1)$. If $f: Y \to B$ is a map, f induces a fibration(= Hurewicz fiber space) $q: X \to Y$;

$$\begin{cases} X = \{(y, \ell) \in Y \times PB \mid f(y) = \ell(1)\}, \\ q(y, \ell) = y. \end{cases}$$

The fiber of q at $y_0 (\in f^{-1}(b_0))$ is $y_0 \times \Omega B$, which we shall identify with ΩB (the loop space of B at b_0). A fibration, as above, which is induced from a path fibration will be called a *principal fibration.*

Lemma 7. *Let* $\Omega B \xrightarrow{i} X \xrightarrow{q} Y$ *be a principal fibration. Then* $x \in (Im\, q^*)'$ *implies* $i^*(x) = 0$.

Proof Define $m: \Omega B \times X \to X$ by

$$m(\lambda, (y, \ell)) = (y, \lambda * \ell)$$

where $\lambda * \ell$ denotes the product path. Observe that if λ_0 is the constent path at b_0 then the map $(y, \lambda_0) \to m(\lambda_0, (y, \ell))$ is homotopic to the identity map of X, and the map $\lambda \to m(\lambda, (y_0, \lambda_0)$ is homotopic to the inclusion map $i: \Omega B \to X$. Consider the diagram

$$\begin{array}{ccc} \Omega B \times X & \xrightarrow{1 \times q} & \Omega B \times Y \\ m \downarrow & & \downarrow p_r \\ X & \xrightarrow{q} & Y \end{array}$$

where p_r is the projection. Clearly the diagram is commutative. Now, from the above observation on m, we can write

$$m^*(x) = i^*(x) \otimes 1 + 1 \otimes x + \Sigma x' \otimes x'' \quad \in H^*(\Omega B, K) \otimes H^*(X, K)$$

where $o < \deg x' < \deg x$. On the other hand, since $x \in (Im\ q^*)'$, there is a nonzero element $y \in Im\ q^*$ with $x \cup y = o$. Let $y = q^*(z)$ for $z \in H^*(Y, K)$. Then $m^*(y) = m^* q^*(z) = (1 \times q)^* p_r{}^*(z) = (1 \times q)^*(1 \otimes z)$

$$= 1 \otimes q^*(z) = 1 \otimes y.$$

Hence,

$$\begin{aligned} o &= m^*(x \cup y) = m^*(x) \cup m^*(y) \\ &= (i^*(x) \otimes 1 + 1 \otimes x + \Sigma x' \otimes x'') \cup (1 \otimes y) \\ &= i^*(x) \otimes y + 1 \otimes xy + \Sigma x' \otimes x'' y \\ &= i^*(x) \otimes y + \Sigma x' \otimes x'' y. \end{aligned}$$

The assumption on deg x' implies $i^*(x) \otimes y = o$, and hence $i^*(x) = o$.

Theorem 8 *Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fiber space. Let $f: X \to F$ be a map with $jf \simeq O$. Then $x \in (Im\ j^*)'$ implies $f^*(x) = o$.*

Proof It is well-known that the fiber inclusion map j factors as $F \xrightarrow{h} \bar{F} \xrightarrow{q} E$, where h is a homotopy equivalence and q is a principal fibration with fiber ΩB ($\bar{F} \xrightarrow{q} E$ is the fibration induced by p from the path fibration over B; for example, see [2]). By assumbtion, $O \simeq jf = qhf$; hence, hf factors through ΩB in the homotopy sense; more precisely there is a map $k : X \to \Omega B$ such that the left triangle in the following diagram commutes in the homotopy sense.

$$\begin{tikzcd} \Omega B \arrow[r, "i"] & \bar{F} \arrow[r, "q"] & E \\ & F \arrow[u, "h"] \arrow[ur, "j"'] & \\ & X \arrow[u, "f"] \arrow[uul, "k"] & \end{tikzcd}$$

It is easy to see that $x \in (Im\ j^*)'$ comes from some $x' \in (Im\ q^*)'$ via h^*. Thus

$$f^*(x) = f^* h^*(x') = k^* i^*(x') = o$$

since $i^*(x') = o$ by the preceding lemma.

Let M be a manifold. Let G be the space of homeomorphisms of M onto itself and G_o the subspace consisting of such homeomorphisms that do not move the base point x_o of M. Then the evaluation map $w : G \to M (w(g) = g(x_o))$ is a locally trivial fiber space with fiber $w^{-1}(x_o) = G_o$. The local product structure in $w : G \to M$ can be easily shown by using Lemma 3.

Recall now the following fact ([1, p. 55]): *There is a locally trival fiber space $M \xrightarrow{j} B_{G_o} \xrightarrow{p} B_G$ with $jw \simeq O$.* We have information about the image of $j^* : H^*(B_{G_o}, Z_2) \to H^*(M, Z_2)$ (Corollary 5). Thus Theorem 8 gives some results on the evaluation map $w : G \to M$.

Example *If M is nonorientable, then*

$$w^* = O : H^n(M, Z_2) \to H^n(G, Z_2)$$

where $n = \dim M$.

Proof If M is nonorientable, there is the nonzero element $W_1(\tau_M)$ in the image of $j^* : H^1(B_{G_0}, Z_2) \to H^1(M, Z_2)$. Hence, $H^n(M, Z_2) \subset (Im\, j^*)' \subset ker\, w^*$.

Example (Gottlieb [4]). *If M is compact and its Euler-Poincare number is odd, then*

$$w^* = O : H^k(M, Z_2) \to H^k(G, Z_2)$$

for every $k > O$.

Proof The hypothesis implies $W_n(\tau_M) \neq O$ ($n = \dim M$). See [6, p. 348]. $W_n(\tau_M)$ is in Imj^*. Hence, $H^k(M, Z_2) \subset (Imj^*)' \subset ker\, w^*$.

Proposition 9. *Let M be a compact triangulated manifold with odd Euler-Poincare number. Let $M \xrightarrow{i} E \xrightarrow{p} B$ be a locally trivial fiber space. Then $H^k(M, Z_2) \neq O$ ($k \neq o$) implies $i \mid M^{k+1} \neq O$, where M^{k+1} denotes the $(k+1)$-skelton.*

Proof Similar to the preceding example. If $i \mid M^{k+1} \simeq O$, the homomorphism $H^k(M, Z_2) \to H^k(M^{k+1}, Z_2)$ would be trivial.

References

1 A. Borel, *Topics in the homotopy theory of fibre bundles*, Lecture notes in mathematics # 36, Springer, 1967.

2 B. Eckmann and P. J. Hilton, *Operators and cooperators in homotopy theory*, Math. Annalen 141 (1960), 1-12.

3 E. Fadell, *Generalized normal bundles for locally flat imbeddings*, Trans. Amer. Math. Soc 114 (1965), 488-513.

4 D. H. Gottlieb, *Homology tangent bundles and universal bundles*, Proc. of Amer. Math. Soc. 36 (1972), 246-252.

5 J. Milnor, *Lectures on characteristic classes*, Princeton University, 1957.

6 E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.