THE RIEMANNIAN VOLUME OF THE MODULI SPACE OF PLANAR QUASI-EQUILATERAL POLYGONS WITH VERTEX NUMBER 4, 5 OR 6

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Abstract

Let $P(n,r)$ be the moduli space of n-gons in $\mathbb{R}^2$ whose edge lengths are $1,\dots,1$ and r. There is an inclusion $P(n,r) \hookrightarrow (\mathbb{R}^2)^n$. Restricting the usual Riemannian metric on $(\mathbb{R}^2)^n$, we regard $P(n,r)$ as a Riemannian manifold. In this paper, with the aid of a computer, we compute the volume of $P(n,r)$ for $n = 4, 5$ and 6.

1 Introduction

Given a string $(r_1,\dots,r_n)$ of n positive real numbers, one considers the moduli space of closed polygonal linkages in $\mathbb{R}^d$ having side lengths r_i. Two n-gons are identified if there exists an orientation-preserving isometry of $\mathbb{R}^d$ which sends vertices of one polygon to another one. For these 20 years, the space has been studied by many people. Particular interests were paid for the cases $d = 2$ and 3 because the moduli spaces are typical examples in certain fields in mathematics. The details are explained below.

First, we explain the case $d = 2$. Recall that Jordan-Steiner [6] and Kapovich-Millson [10] proved the universality theorem of planar mechanical linkages: Every compact real algebraic variety is homeomorphic to some components of the configuration space of a planar mechanical linkage. (As explained in [10], such a problem was already studied in the 19th century.)

Since the polygonal linkages are typical examples for planar mechanical linkages, it is natural to study this in detail. Motivated by this, the study of the moduli space of planar polygon spaces was started by Hausmann [2], Kapovich-Millson [8] and Walker [13]. The study has a long history and there are many references. For example, the homology groups of planar polygon spaces of arbitrary edge lengths have been determined in Farber and Schütz [1]. (See Theorem 1 for some other results.)

Next, we explain the case $d = 3$. The study originated in Kapovich and Millson [9]. They proved that the moduli space is a typical example for the symplectic

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quotient, and hence the space has a natural symplectic structure. Motivated by this, Tezuka and the author [7] computed the symplectic volume of the moduli space for the case $(r_1, \ldots, r_n) = (1, \ldots, 1)$. The result is given by a simple formula involving the binomial coefficient.

The work of [7] leads us to the following question: What is the volume of the planar polygon space? In contrast to the spatial case, the planar polygon does not have a symplectic structure. Instead of this, we can naturally regard the space as a Riemannian submanifold of $(\mathbb{R}^2)^n$. In this paper, we numerically compute the volume for $n = 4, 5, 6$ and $(r_1, \ldots, r_n) = (1, \ldots, 1, r)$. (Such a polygon is sometimes called a quasi-equilateral polygon.)

This paper is organized as follows. In §2, we summarize our results. In §3, we explain the method of computations. In §§4,5 and 6, we state our results for $n = 4, 5$ and 6, respectively. In §7, we compute the volume when r approaches 0 by another method.

2 Summary of the results

Let r be a positive real number. As in §1, let $P(n, r)$ be the moduli space of planar quasi-equilateral n-gons of edge lengths $1, \ldots, 1$ and r. Since two n-gons are identified if there exists an orientation-preserving isometry of $\mathbb{R}^2$ which sends vertices of one polygon to another one, we may assume that the end points of the last edge is O and $(r, 0)$. Thus we can define $P(n, r)$ as follows:

$$P(n,r) = \{(u_1, \ldots, u_n) \in (\mathbb{R}^2)^n ; u_1 = O, u_n = (r, 0) \quad \text{and } \|u_{i+1} - u_i\| = 1 \text{ for } 1 \leq i \leq n-1\}. \tag{1}$$

(See Fig.1.)

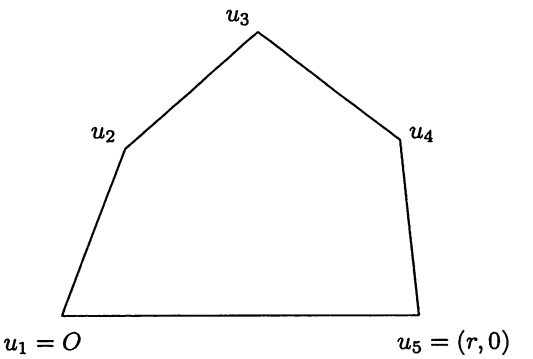

Figure 1: $P(5, r)$

Many topological properties of $P(n.r)$ are known. For example, it is clear that $P(n,r) = \varnothing$ for $r > n-1$ and $P(n, n-1) = \{\text{one point}\}$. For $0 < r < n-1$, $P(n,r)$ has singular points if and only if $n - r$ is odd. In general, $\dim P(n,r) = n - 3$.

The main interest in this paper is the cases for $n = 4, 5$ and 6. The following results are known.

Theorem 1. *For $0 < r < n-1$, the following results are known.*

(i) *The case of $n = 4$.*

  (a) *For $0 < r < 1$, there is a homeomorphism $P(4, r) \cong S^1 \coprod S^1$.*

  (b) *The topological type of $P(4, 1)$ is given by the following figure.*

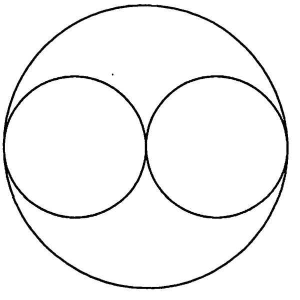

Figure 2: $P(4, 1)$

In particular, there is a homotopy equivalence $P(4,1) \simeq \bigvee_4 S^1$.

  (c) *For $1 < r < 3$, we have $P(4, r) \cong S^1$.*

(ii) *The case of $n = 5$.*

  (a) ([3, 13]). *Havel and Walker proved that for $0 < r < 2$, there is a homeomorphism $P(5, r) \cong \Sigma_4$, where Σ_4 denotes a connected closed orientable surface of genus 4.*

  (b) ([12]). *Toma determined the topological type of $P(5, 2)$. The result is given by Fig.3. More precisely, $P(5, 2)$ is obtained from $P(4, 1)$ by attaching a space as indicated by the first figure and the attaching map is given by the second figure.*

  Thus $P(5, 2)$ is obtained from Σ_4 by collapsing the middle circle contained in each of four handles to a point. Hence, there is a homotopy equivalence $P(5, 2) \simeq S^2 \vee \bigvee_4 S^1$.

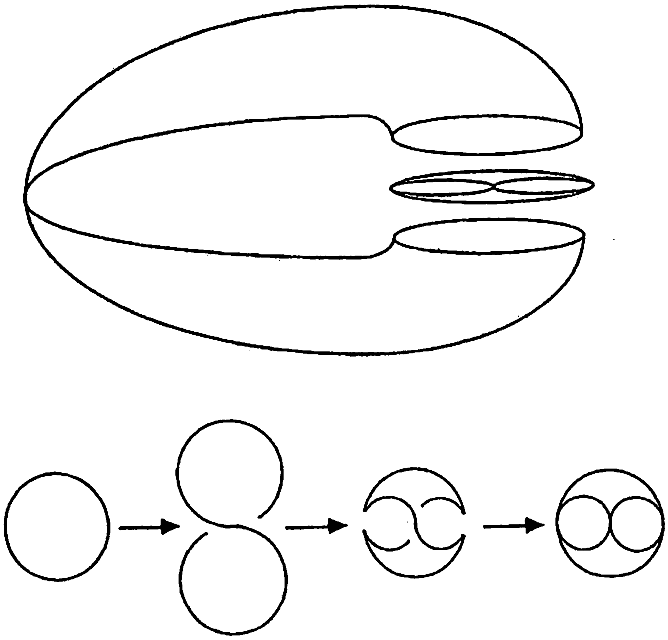

Figure 3: $P(5,2)$. (Taken from [12, Corrections, p.95].)

(c) *For* $2 < r < 4$, *we have* $P(5,r) \cong S^2$.

(iii) *The case of* $n = 6$.

(a) *For* $0 < r < 1$, *we have* $P(6,r) \cong S^1 \times P(5,1)$.

(b) ([11, 5]) *By* [2, 8], *it was known that* $P(6,1)$ *has* 10 *singular points such that the neighborhood of a singular point is* $C(S^1 \times S^1)$.

- *We define* $P(6,1)' = P(6,1) - \{\textit{singular points}\}$. *Kojima, Nishi and Yamashita proved that* $P(6,1)'$ *is homeomorphic to the moduli space of semi-stable marked* 6 *point configurations on the circle, which was known to be homeomorphic to a hyperbolic* 3*-manifold with ten cusps.*
- *Hirano advanced one step further: The space* $P(6,1)$ *is actually equal to the space obtained from* $P(6,1)'$ *by compactifying each end with a point. In order to prove this, he constructed a natural cellular decomposition of* $P(6,1)$ *by* 16 *copies of a polyhedron with* 10 *faces. (See §6 for related topics.)*

(c) ([4]). *Hinokuma and Shiga proved that for* $1 < r < 3$, *there is a homeomorphism* $P(6,r) \cong \#_5(S^1 \times S^2)$.

(d) ([4]). *There is a homotopy equivalence* $P(6,3) \simeq S^3 \vee \bigvee_5 S^1$.

(e) *For* $3 < r < 5$, *we have* $P(6,r) \cong S^3$.

Note that (1) gives an inclusion $P(n,r) \hookrightarrow (\mathbb{R}^2)^n$. Restricting the usual Riemannian metric on $(\mathbb{R}^2)^n$, we regard $P(n,r)$ as a Riemannian manifold. We define $V(n,r)$ be the volume of $P(n,r)$ with respect to the Riemannian metric.

Remark 2. In order to define a Riemannian metric on $P(n,r)$, we embedded $P(n,r)$ into $(\mathbb{R}^2)^n$ using the vertices. It is also possible to embed $P(n,r)$ into $(\mathbb{R}^2)^n$ using the edges. More precisely, we define an embedding $f : P(n,r) \to (\mathbb{R}^2)^n$ as follows: Setting $a_i = u_{i+1} - u_i$, we define

$$f(u_1, \dots, u_n) = (a_1, \dots, a_n).$$

In contrast to (1), we have

$$\begin{aligned} f(P(n,r)) = \{(a_1, \dots, a_n) \in (\mathbb{R}^2)^n;\ & a_1 + \cdots + a_n = 0, \\ & \|a_i\| = 1\ (1 \le i \le n-1) \text{ and } a_n = (-r, 0)\}. \end{aligned}$$

The Riemannian metric using vertices and that using edges are not isometric. But since the results are quite similar, we consider only $V(n,r)$ in this paper.

We give the results on $V(n,r)$ for $n = 4, 5$ and 6 in §§4,5 and 6, respectively. We study the value $\lim_{r \to +0} V(n,r)$ in §7. In this section, we summarize the results.

Theorem A . (i) *The graph of* $V(4,r)$ *is given by Fig.6. The graph increases when* $0 < r < 1$ *and decreases when* $1 < r < 3$.

(ii) *We also have the graph of* $\dfrac{V(4,r)}{dr}$ *in Fig.7. From this, we can read the concavity of Fig.6.*

(iii) *We have*

$$\lim_{r \to +0} V(4,r) = 4\sqrt{2}\pi.$$

Theorem B . (i) *The graph of* $V(5,r)$ *is given in Fig.9. The graph is monotone decreasing.*

(ii) *We also have the graph of* $\dfrac{V(5,r)}{dr}$ *in Fig.10. From this, we can read the concavity of Fig.9.*

Before we state the results on $V(6,r)$, we remark that it is not easy to find parametrizations of $P(6,r)$ from which the multiple integral for $V(6,r)$ is computable.

Theorem C . *There are parametrizations of* $V(6,r)$ *from which* $V(6,r)$ *is computable. The graph of* $V(6,r)$ *is given in Fig.12.*

Our final result is about $\lim_{r \to +0} V(n,r)$.

Theorem D . (i) *We set*

$$V(n,0) = \lim_{r \to +0} V(n,r). \tag{2}$$

Then, there is a method to compute $V(n,0)$ from the parametrizations of $P(n-1,1)$.

(ii) *In particular, $V(4,0)$ is as given in Theorem A (iii). The values of $V(5,0)$ and $V(6,0)$ are given in Theorem 5. The results are consistent with the data of $V(5,r)$ in Theorem B and $V(6,r)$ in Theorem C, respectively.*

3 Preliminaries

3.1 The definition of $V(n,r)$

For completeness, we recall the definition of $V(n,r)$. We fix r in $0 < r \leq n-1$. Let D be a closed domain in $\mathbb{R}^{n-3}$. We call a C^∞ function $p : D \to P(n,r)$ to be a parametrization if rank $df = n-3$ holds for all points on Int D.

Let $s \in \mathbb{N}$ and assume that a parametrization $p_i : D \to P(n,r)$ is given for each $i = 1, 2, \ldots, s$. (Actually, we will take s to be 2, 4 or 4 according as $n = 4, 5$ or 6.) We assume the following conditions:

(i) $\displaystyle\bigcup_{i=1}^{s} p_i(D) = P(n,r)$.

(ii) $p_i(\text{Int } D) \cap p_j(\text{Int } D) = \emptyset$ for all $1 \leq i < j \leq s$.

In this case, we have

$$V(n,r) = \sum_{i=1}^{s} c_i,$$

where c_i is defined as follows. Let $(x^1, \ldots, x^{n-3})$ be the coordinate of $\mathbb{R}^{n-3}$. Let M_i be the $(n-3) \times (n-3)$ matrix whose (j,k)-th element is given by

$$\frac{\partial p_i}{\partial x^j} \cdot \frac{\partial p_i}{\partial x^k}.$$

Moreover, we set $G_i = \sqrt{\det M_i}$. (Note that $G_i \, dx^1 \wedge \cdots \wedge x^{n-3}$ is the pull-back the volume form of $P(n,r)$ by $p_i : D \to P(n,r)$.) Then, we define c_i to be

$$c_i = \int \cdots \int_D G_i \, dx^1 \ldots dx^{n-3}.$$

3.2 About $P_{n,0}$

Theorem D is proved by considering a new space $P(n,0)$. Formally, $P_{n,0}$ is defined by setting $r = 0$ in (1). More precisely, we set

$$P(n,0) = \{(u_1, \ldots, u_{n-1}) \in (\mathbb{R}^2)^{n-1};\ u_1 = O \\ \text{and } \|u_{i+1} - u_i\| = 1 \text{ for } 1 \leq i \leq n-1\},$$

where we understand u_n to be u_1. (See Fig.4.)

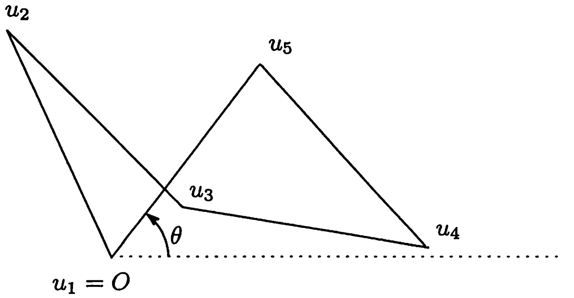

Figure 4: $P(6,0)$

Lemma 3. *Regard u_i to be a column vector. Then, the map*

$$\phi : SO(2) \times P(n-1,1) \to P(n,0)$$

defined by

$$\phi\left(g, (u_1, \dots, u_{n-1})\right) = (gu_1, \dots, gu_{n-1})$$

is a homeomorphism. Recall that $u_1 = O$ and $u_{n-1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Proof. Let $(w_1, \dots, w_{n-1}) \in P(n,0)$. If $w_{n-1} = \begin{pmatrix} w^1_{n-1} \\ w^2_{n-1} \end{pmatrix}$, then we set

$$g = \begin{pmatrix} w^1_{n-1} & -w^2_{n-1} \\ w^2_{n-1} & w^1_{n-1} \end{pmatrix}.$$

Then, it is easy to check that ϕ^{-1} is given by

$$\phi^{-1}(w_1, \dots, w_{n-1}) = \left(g, \left(g^{-1}w_1, \dots, g^{-1}w_{n-1}\right)\right).$$

$\square$

Now we have the following commutative diagram, where i is the inclusion.

$$\begin{tikzcd}
SO(2) \times P(n-1,1) \arrow[rr, "\phi", "\cong"'] \arrow[dr, "i\circ\phi"'] & & P(n,0) \arrow[dl, "i"] \\
& (\mathbb{R}^2)^{n-1} &
\end{tikzcd}$$

By the restriction using i, we define the Riemannian metric on $P(n,0)$. We denote by $\mathrm{vol}(P(n,0))$ the volume of $P(n,0)$ with respect to the Riemannian metric. Then, by the continuity of the volume form, we have

$$V(n,0) = \mathrm{vol}(P(n,0)),$$

where $V(n,0)$ is defined in (2).

Actually, $\mathrm{vol}(P(n,0))$ is computable by the following method: By the pull-back using $i \circ \phi$, we define the Riemannian metric on $SO(2) \times P(n-1,1)$. Let v_n be the volume of $SO(2) \times P(n-1,1)$ with respect to the metric. Since ϕ is isometric, we have

$$\mathrm{vol}(P(n,0)) = v_n.$$

Thus we have

$$V(n,0) = v_n. \tag{3}$$

The above is a content of Theorem D (i). In §7, we explain more details.

4 Results on $V(4,r)$

We give results about Theorem A. We define parametrizations of $P(4,r)$ as follows. We set $u_2 = (\cos\theta, \sin\theta)$. Note that u_3 is determined in two positions. Hence we have $s = 2$ in §3.1. (See Fig.5.)

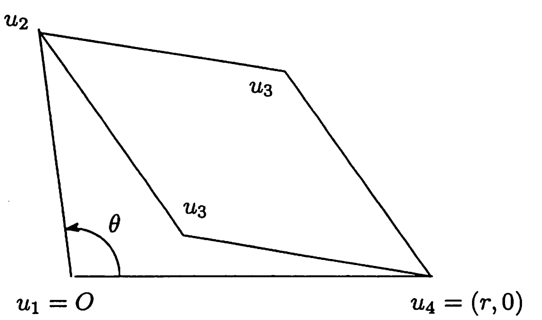

Figure 5: The parametrizations of $P(4,r)$

In the notation of §3.1, we have

$$c_i = \int_D G_i(\theta)\, d\theta,$$

where

$$D = \{\theta \in [0,2\pi]; \|(r,0) - (\cos\theta, \sin\theta)\| \le 2\}.$$

Hence

$$c_i = \int_{-h(\theta)}^{h(\theta)} G_i(\theta)\, d\theta,$$

where

$$h(\theta) = \begin{cases} \pi & 0 < r \le 1 \\ \arccos(\frac{r^2-3}{2r}) & 1 \le r \le 3. \end{cases}$$

Using this, we compute the values of $V(4,r)$ as follows.

r	$V(4,r)$	r	$V(4,r)$	r	$V(4,r)$
0.1	17.78	1.1	15.90	2.1	7.30
0.2	17.83	1.2	13.95	2.2	6.79
0.3	17.91	1.3	12.61	2.3	6.29
0.4	18.02	1.4	11.59	2.4	5.76
0.5	18.18	1.5	10.76	2.5	5.21
0.6	18.41	1.6	10.06	2.6	4.63
0.7	18.70	1.7	9.43	2.7	3.98
0.8	19.12	1.8	8.85	2.8	3.22
0.9	19.73	1.9	8.32	2.9	2.26
1.0	15.16	2.0	7.80	3.0	0

Table 1: $V(4,r)$

The graph of $V(4,r)$ is given as follows.

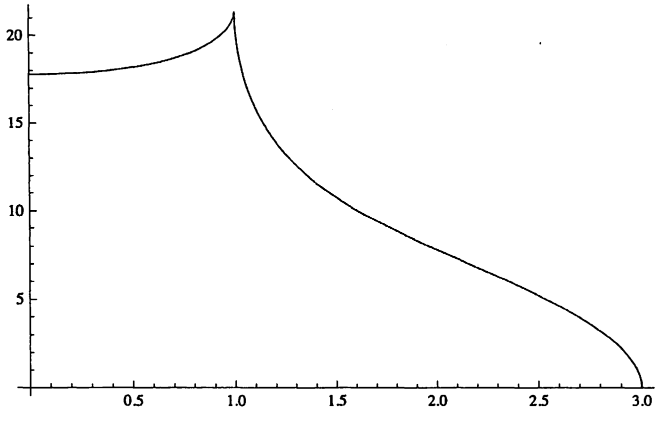

Figure 6: $V(4,r)$

Note that the graph increases when $0 < r < 1$ and decreases when $1 < r < 3$.

Next, the graph of $\dfrac{dV(4,r)}{dr}$ is given as follows.

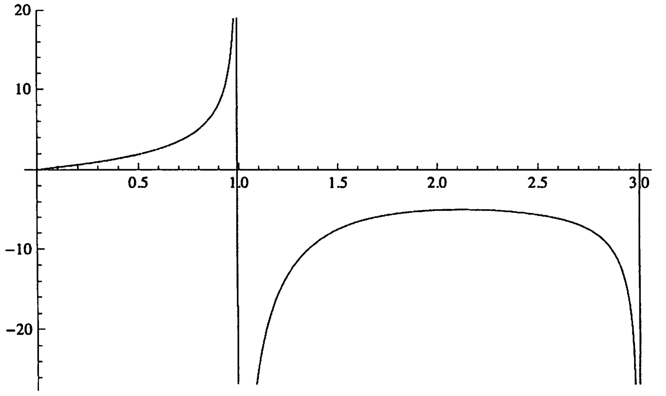

Figure 7: $\dfrac{dV(4,r)}{dr}$

From Fig.7, we can read the concavity of Fig.6.

5 Results on $V(5,r)$

We give results about Theorem B. We define parametrizations of $P(5,r)$ as follows. We set $u_3 = (u,v)$. Note that u_2 and u_4 are determined in two positions. Hence we have $s = 4$ in §3.1. (See Fig.8.)

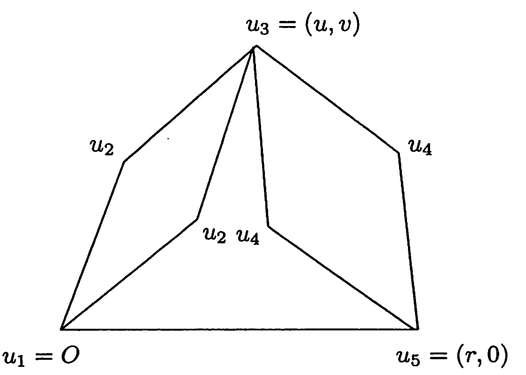

Figure 8: The parametrizations of $P(5,r)$

In the notation of §3.1, we have

$$D = \{(u,v) \in \mathbb{R}^2; \|(u,v)\| \leq 2 \quad \text{and} \quad \|(r,0)-(u,v)\| \leq 2\}.$$

By definition, we have

$$c_i = \iint_D G_i(u,v)\, dudv.$$

From this and the symmetry of D, it is easy to see that

$$c_i = 4\int_{r-2}^{r/2}\int_0^{\sqrt{4-(u-r)^2}} G_i(u,v)\, dudv. \tag{4}$$

But if we set

$$v = \sqrt{4-(u-r)^2} - \frac{\sqrt{4-(u-r)^2}}{w}$$

and transform (4) to an infinite integral

$$c_i = 4\int_{r-2}^{r/2}\int_1^{\infty} G_i\left(u, \sqrt{4-(u-r)^2} - \frac{\sqrt{4-(u-r)^2}}{w}\right)\frac{\sqrt{4-(u-r)^2}}{w^2}\, dudw,$$

then one may obtain the results more quickly. Now using this, we compute the values of $V(5,r)$ as follows.

r	$V(5,r)$	r	$V(5,r)$	r	$V(5,r)$	r	$V(5,r)$
0.1	154.37	1.1	105.67	2.1	54.91	3.1	18.80
0.2	145.75	1.2	102.10	2.2	49.02	3.2	16.37
0.3	139.21	1.3	98.49	2.3	44.19	3.3	14.05
0.4	133.74	1.4	94.79	2.4	40.00	3.4	11.81
0.5	128.91	1.5	90.95	2.5	36.26	3.5	9.65
0.6	124.53	1.6	86.90	2.6	32.86	3.6	7.57
0.7	120.45	1.7	82.53	2.7	29.70	3.7	5.57
0.8	116.59	1.8	77.67	2.8	26.75	3.8	3.63
0.9	112.88	1.9	71.96	2.9	23.97	3.9	1.78
1.0	109.25	2.0	63.70	3.0	21.32	4.0	0

Table 2: $V(5,r)$

The graph of $V(5,r)$ is given as follows.

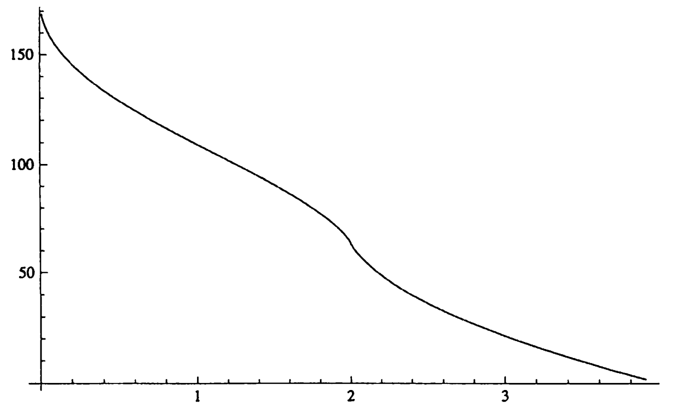

Figure 9: $V(5,r)$

Note that the graph is monotone decreasing.

Next, the graph of $\dfrac{dV(5,r)}{dr}$ is given as follows.

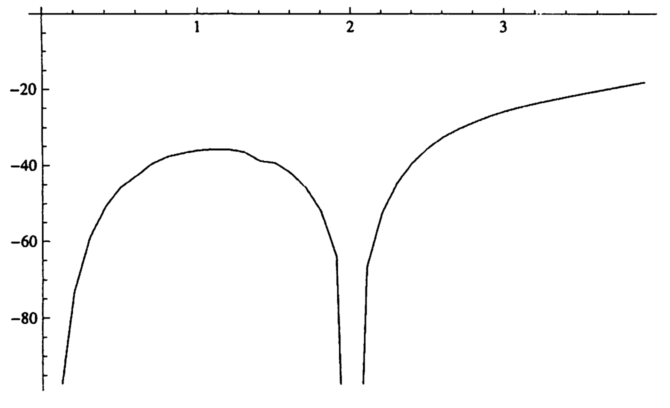

Figure 10: $\dfrac{dV(5,r)}{dr}$

From Fig.10, we can read the concavity of Fig.9.

6 Results on $V(6,r)$

We give results about Theorem C. It is not easy to find parametrizations of $P(6,r)$ from which we can compute the multiple integral. For most parametrizations, computers return data that $V(6,r) \approx \infty$. As far as the author tries, only the following parametrizations are successful for computer calculations: We set

$$u_3 = \left(u - \frac{\cos\theta}{2}, v - \frac{\sin\theta}{2}\right) \quad \text{and} \quad u_4 = \left(u + \frac{\cos\theta}{2}, v + \frac{\sin\theta}{2}\right).$$

Note that u_2 and u_4 are determined in two positions. Hence we have $s = 4$ in §3.1. (See Fig.11.)

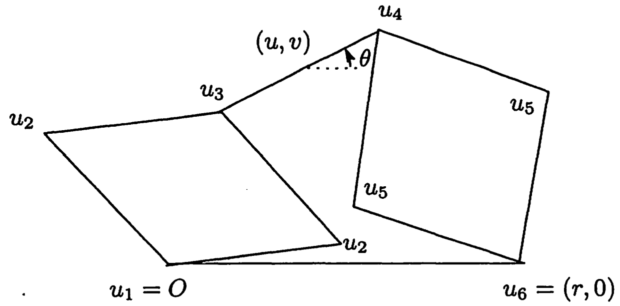

Figure 11: The parametrizations of $P(6,r)$

In the notation of §3.1, we have

$$D = \left\{ (u,v,\theta) \in \mathbb{R}^2 \times [0, 2\pi]; \| \left(u - \frac{\cos\theta}{2}, v - \frac{\sin\theta}{2}\right) \| \leq 2 \right. \tag{5}$$

$$\left. \text{and} \quad \|(r,0) - \left(u + \frac{\cos\theta}{2}, v + \frac{\sin\theta}{2}\right) \| \leq 2 \right\}.$$

By definition, we have

$$c_i = \iiint_D G_i(u,v,\theta)\, dudvd\theta.$$

Using this, we compute the values of $V(6,r)$ as follows.

r	$V(6,r)$	r	$V(6,r)$	r	$V(6,r)$	r	$V(6,r)$	r	$V(6,r)$
0.1	875.09	1.1	749.92	2.1	437.40	3.1	160.03	4.1	41.40
0.2	863.55	1.2	718.14	2.2	409.27	3.2	141.89	4.2	38.88
0.3	874.81	1.3	685.42	2.3	381.41	3.3	124.77	4.3	32.69
0.4	890.25	1.4	649.92	2.4	347.96	3.4	110.00	4.4	4.00
0.5	869.69	1.5	618.50	2.5	314.16	3.5	90.94	4.5	3.52
0.6	872.39	1.6	581.38	2.6	290.44	3.6	74.41	4.6	3.08
0.7	856.26	1.7	555.66	2.7	265.26	3.7	67.53	4.7	2.62
0.8	825.58	1.8	515.55	2.8	233.76	3.8	57.16	4.8	1.82
0.9	800.97	1.9	486.70	2.9	206.83	3.9	48.75	4.9	1.07
1.0	778.94	2.0	443.66	3.0	182.16	4.0	37.75	5.0	0

Table 3: $V(6,r)$

The graph of $V(6,r)$ is given as follows.

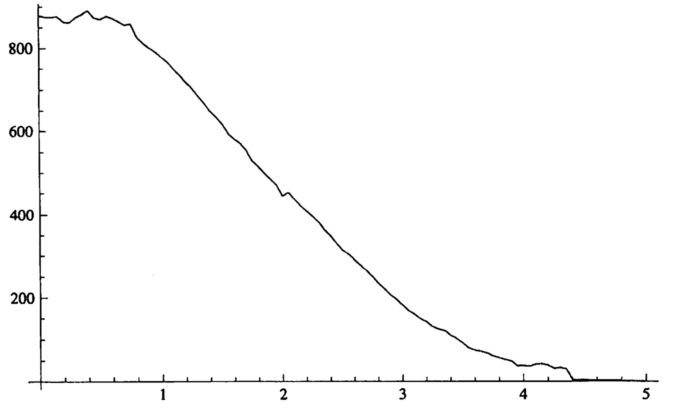

Figure 12: $V(6,r)$

Remark 4. For our reference, we give two other parametrizations of $P(6,r)$. But we cannot get satisfactory data from them.

(i) One method is to set

$$u_2 = (\cos\theta_1, \sin\theta_1), \quad u_3 = (\cos\theta_1 + \cos\theta_2, \sin\theta_1 + \sin\theta_2)$$
$$\text{and} \quad u_4 = (\cos\theta_1 + \cos\theta_2 + \cos\theta_3, \sin\theta_1 + \sin\theta_2 + \sin\theta_3).$$

In this case, u_5 is determined in two positions. (See Fig.13.)

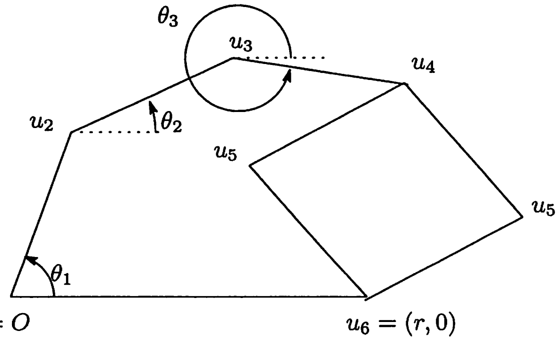

Figure 13: The parametrizations $P(6, r)$ by θ_1, θ_2 and θ_3

In the notation of §3.1, we have

$$D = \{(\theta_1, \theta_2, \theta_3) \in [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]; \\ \|(r, 0) - (\cos\theta_1 + \cos\theta_2 + \cos\theta_3, \sin\theta_1 + \sin\theta_2 + \sin\theta_3)\| \leq 2\}.$$

The author thinks that the reason why these parametrization return bad answers is that this D is much more complicated than (5).

(ii) As explained in Theorem 1 (iii)(b), Hirano constructed a natural cellular decomposition of $P(6, 1)$ by 16 copies of a polyhedron with 10 faces. We can use his idea to construct parametrizations of $P(6, r)$, in particular, we have $s = 16$ in the notation of §3.1. We explain this below. For $(u_1, \ldots, u_6) \in P(6, r)$, we set

$$\ell_1 = \|u_3\|, \quad \ell_2 = \|u_5 - u_3\| \quad \text{and} \quad \ell_3 = \|u_1 - u_5\|.$$

We use (ℓ_1, ℓ_2, ℓ_3) as parametrizations of $P(6, r)$. In each of the following figures, u_2 and u_4 are determined in two positions. Hence there are 16 parametrizations. (See Fig.14.)

Note that Fig.11 gives the inner triangle $\triangle Ou_3u_4$ by the coordinates of the vertices. On the other hand, Fig.14 gives $\triangle Ou_3u_5$ by its edge lengths.

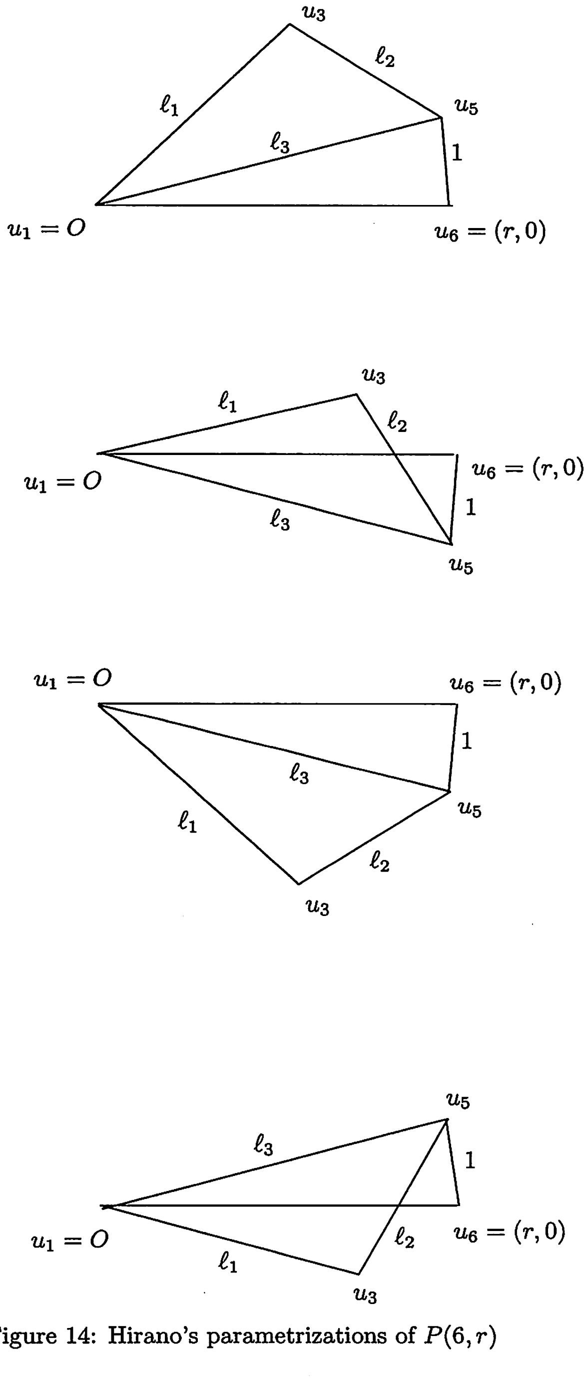

Figure 14: Hirano's parametrizations of $P(6, r)$

In the notation of §3.1, D is given by the triangle inequalities:

$$D = \{(\ell_1, \ell_2, \ell_3) \in \mathbb{R}^3; 0 \le \ell_1, \ell_2, \ell_3 \le 2 \quad \text{and} \quad |\ell_1 - \ell_2| \le l_3 \le \ell_1 + \ell_2\}.$$

7 Results on $V(n, 0)$

We prove Theorem D. (3) tells us that in order to compute $V(n, 0)$, it will suffice to compute v_n. As we see below, v_n is computable in the same way as in §3.1: Recall that the Riemannian metric on $SO(2) \times P(n-1, 1)$ is defined by identifying the space with $(i \circ \phi)(SO(2) \times P(n-1, 1))$. For parametrizations $p_i : D \to P(n-1, 1)$ (where $1 \le i \le s$), we define parametrizations

$$\tilde{p}_i : [0, 2\pi] \times D \to (i \circ \phi)(SO(2) \times P(n-1, 1))$$

by

$$\tilde{p}_i(\alpha, x^1, \ldots, x^{n-4}) = \left(\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} p_i(x^1, \ldots, x^{n-4}) \right).$$

Note that this is the right-hand side of Lemma 3.

To the parametrization $\tilde{p}_i$, we associate an $(n-3) \times (n-3)$ matrix N_i in the same way as in §3.1. More precisely, we define

$$\xi_{j,k} = \frac{\partial \tilde{p}_i}{\partial x^j} \cdot \frac{\partial \tilde{p}_i}{\partial x^k} \quad \text{for } 1 \le j, k \le n-4$$
$$\zeta_j = \frac{\partial \tilde{p}_i}{\partial x^j} \cdot \frac{\partial \tilde{p}_i}{\partial \alpha} \quad \text{for } 1 \le j \le n-4$$

and

$$\lambda = \frac{\partial \tilde{p}_i}{\partial \alpha} \cdot \frac{\partial \tilde{p}_i}{\partial \alpha}.$$

Then, we set

$$N_i = \begin{pmatrix} \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,n-4} & \zeta_1 \\ \xi_{2,1} & \xi_{2,2} & \cdots & \xi_{2,n-4} & \zeta_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_{n-4,1} & \xi_{n-4,2} & \cdots & \xi_{n-4,n-4} & \zeta_{n-4} \\ \zeta_1 & \zeta_2 & \cdots & \zeta_{n-4} & \lambda \end{pmatrix}.$$

We define $H_i = \sqrt{\det N_i}$ and

$$d_i = \int_0^{2\pi} \int \cdots \int_D H_i \, dx^1 \ldots dx^{n-4} d\alpha.$$

Then, by the definition of v_n, we have

$$v_n = \sum_{i=1}^{s} d_i. \tag{6}$$

The computations of $\xi_{j,k}$ and ζ_j are easy. In fact,

$$\xi_{j,k} = \frac{\partial p_i}{\partial x^j} \cdot \frac{\partial p_i}{\partial x^k} \quad \text{and} \quad \zeta_j = \left(g \frac{\partial p_i}{\partial x^j} \right) \cdot \left(\frac{\partial g}{\partial \alpha} p_j \right),$$

where we write $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ by g. But we need to be careful about λ. In fact, due to the last term of Lemma 3, we have

$$\lambda = \|p_i\|^2, \tag{7}$$

where if $p_i = (u_1, \ldots, u_{n-1})$, then we set $\|p_i\|^2 := \sum_{\nu=1}^{n-1} \|u_\nu\|^2$.

Theorem 5. *We have the following results.*

(i) $v_4 = 4\sqrt{2}\pi \approx 17.77$.

(ii) $v_5 = 169.58$.

(iii) $v_6 = 885.11$.

Proof. (i) We claim that $H_1(\theta) = H_2(\theta) = \sqrt{2}$ for all $\theta \in [0, 2\pi]$. In fact, recall that $P(3,1) = \{\text{two points}\}$. For $(g, (u_1, u_2, u_3)) \in SO(2) \times P(3,1)$, we have $\|u_2\| = \|u_3\| = 1$. Hence by (7), $\lambda = 2$ and we have $H_1(\theta) = H_2(\theta) = \sqrt{2}$. Now (i) follows from (6).

(ii) and (iii) also follow from (6). □

Remark 6. Comparing with Theorem 5, Table 1 is probably a result with high numerical precision and Tables 2 and 3 are reliable results.

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