APPLICATION OF DISCRETE VARIATIONAL TECHNIQUES TO THE ANALYSIS OF LATTICED SHELLS

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by

Takeshi Oshiro*

ABSTRACT

The techniques of discrete field mechanics, a new concept in structural analysis, are used in conjunction with energy methods to obtain an exact mathematical model to represent a latticed shell subjected to flexure and corresponding solutions. The method developed is designated here as the discrete variational approach and its usefulness has proven especially effective for the analysis of latticed shells with general types of boundary supports, such as free or ribbed polygonal edges.

Essentially, the method is based on the application of the calculus of variations in discrete field mechanics developed in Appendix A to the concept of the Micro Approach used in field analysis. The immediate results are:

(a) The mathematical model which can be used for the linear or non-linear analysis of latticed structures

(b) A clear statement of the natural boundary conditions associated with each system

(c) Closed form solutions to the total model described by the steps (a) and (b)

A further development of the method, the modified discrete variational method analogous to the method of Lagrange multipliers, is presented in the same appendix and enables one to obtain with relative ease closed form solutions to structures which were not amenable by conventional methods because of the complexity of the boundary conditions. Such solutions are valid over the entire structure and are independent of the size of the system.

The buckling condition of latticed shells is also investigated by this method in the work presented in Chapter IV which clarifies on a rational basis the behavior of the compressed members as an integral part of the entire system.

Each solution presented in this paper has been investigated numerically and compared with results obtained by open form methods. The comparison shows significant accuracy and the great reliability of the technique proposed here.

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NOTATION

The letter symbols used in this dissertation have the following definitions:

A_1 A_2	: Cross sectional area of a member for the α_1-and α_2-polygons
A	: Projection of the unit vector along the polygon member on the tangent at the corresponding node
a_1, a_2	: Length of element cable of a net
E	: Boole's displacement operator defined in Appendix B
EI_2, EI_3 $E\bar{I}_2$, $E\bar{I}_3$	: Flexural rigidities of the polygon members
$\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$	: Principal coordinate system of a member

$\bar{F}^R$, $\bar{F}^L$	: Force vectors at the right or the left end of a member
	$\bar{F}^R = F_1^R\bar{e}_1 + F_2^R\bar{e}_2 + F_3^R\bar{e}_3$
	$\bar{F}^L = F_1^L\bar{e}_1 + F_2^L\bar{e}_2 + F_2^L\bar{e}_3$
$\bar{F}^e$	: External force vector acting at a node
	$\bar{F}^e = \bar{F}_1^e + \bar{F}_2^e + \bar{F}_3^e$
F_{mn}^1, F_{mn}^2, F_{mn}^2	: Euler coefficients for the external loads
F_A, F_B	: Axial force prior to and during buckling
GJ_1, $G\bar{J}_1$	: Torsional rigidity of a member in α_1- and α_2- polygons
K	: Curvature of α_1-polygon defined in difference geometry as $K = 2\sin\psi_\alpha$
L_1, L_2	: Length of a member in α_1- and α_2- polygons
$\ell\alpha$, ℓ, L	: Span lengths and total length of the continuous beam shown in Appendix A
M	: Number of nodes in the α_1- direction
$\bar{M}^R$, $\bar{M}^L$	: Moment vector at the right or the left end of a member
	$\bar{M}^R = M_1^R\bar{e}_1 + M_2^R\bar{e}_2 + M_2^R\bar{e}_3$
	$\bar{M}^L = M_1^L\bar{e}_1 + M_2^L\bar{e}_2 + M_2^L\bar{e}_3$
$\bar{M}^e$	: :External moment vector acting on a node
	$\bar{M}^e = M_1^e\bar{t}_1 + M_2^e\bar{t}_2 + M_3^e\bar{N}$
N	: Number of nodes in α_2- direction
R	: Tension forces of cables in the α_1 direction shown in Appendix A
S, $\bar{S}$	: Tension forces of cables in the α_2 direction shown in Appendix A
T_k^e	: With k=1, 2, ……… 6, it respectively represents all the external loads
	F_1^e, F_2^e, F_3^e, M_1^e, M_2^e, M_3^e, i. e. $T_1^e = F_1^e$
$[T_{ij}]$, $[T_{ij}']$	: Matrix coefficients defined by Eqs. (2-16) and (2-17)
U	: Total potential energy of a structural system
$\bar{u}^R$	: Displacement vector at the right end of a member,
	$\bar{u}^R = u_1^R\bar{e}_1 + u_2^R\bar{e}_2 + u_3^R\bar{e}_3$
$\bar{u}\alpha_1$	: Displacement vector of a polygon at node α_1.
	$\bar{u}_{\alpha_1} = \bar{u}_{t\alpha}\,\bar{t}_{\alpha_1} + \bar{u}_{n\alpha}\,\bar{n}_{\alpha_1} + \bar{u}_{b\alpha}\,\bar{b}_{\alpha_1}$
$\bar{u}$	: Displacement vector of the joint (α_1, α_2) of a latticed shell
	$\bar{u} = u_1\bar{t}_1 + u_2\bar{t}_2 + u_3\bar{N}$
U_{mn}^1, U_{mn}^2, U_{mn}^3	: Euler coefficients for the displacement functions
$V\alpha_1$, $V\alpha_2$	: :Strain energy in a member $\nabla_1\bar{X}$ and in a member $\nabla_2\bar{X}$
V	: Total strain energy of the parametric polygon
w_1, w_2	: Weighting functions defined by Eqs. (3-4), (3-5)
W_{kA}, W_{kB}	: Deformation prior to and during buckling; with k = 1, 2, 3, 4, they represent respectively u_1, u_3, θ_1, θ_2

$\bar{X}(\alpha_1, \alpha_2)$	: Position vector of a typical joint (α_1, α_2)
Y_k	: Represent deformations. $k = 1, \cdots\cdots 6$ gives respectively $u_1, u_2, u_3, \theta_1, \theta_2, \theta_3$
$\bar{t}\alpha_1, \bar{n}\alpha_1, \bar{b}\alpha_1$	: Local trihedron of the α_1- polygon
$\bar{t}_1, \bar{t}_2, \bar{N}$	: Local trihedron of a latticed shell at a joint (α_1, α_2)
$\bar{\theta}^R, \bar{\theta}^L$	: Rotation vector at the right and at the left end member $\bar{\theta}^R = \theta_1^R \bar{e}_1 + \theta_2^R \bar{e}_2 + \theta_3^R \bar{e}_3$ $\bar{\theta}^L = \theta_1^L \bar{e}_1 + \theta_2^L \bar{e}_2 + \theta_3^L \bar{e}_3$
$\bar{\theta}\alpha_1$	: Rotation vector of a polygon $\bar{\theta}\alpha_1 = \theta_{t\alpha}\bar{t}\alpha_1 + \theta_{n\alpha}\bar{n}\alpha_1 + \theta_{b\alpha}\bar{b}\alpha_1$
$\bar{\theta}$	: Rotation vector of a latticed shell at a joint (α_1, α_2) $\bar{\theta} = \theta_1\bar{t}_1 + \theta_2\bar{t}_2 + \theta_3\bar{N}$
$\theta_{mn}^1, \theta_{mn}^2, \theta_{mn}^3$	: Euler coefficients for the rotation functions
α_1, α_2	: Discrete variables; $\alpha_1, = 0, 1, 2, \cdots\cdots M$ $\alpha_2 = 0, 1, 2, \cdots\cdots N$
δU	: First energy variation
$b_1, \gamma_1, \bar{b}_1, \bar{\gamma}_1$	: Coefficients related to axial force in the α_1 - polygon defined by Eq. (2-10) with the subscrip 2, they denote similar quantities for the α_2 - polygon
δ_m^{m1}	: Kronecker delta defined by Eq. (3-23)
$\epsilon_k h_k$	: Variation of the deformations
$\bar{\lambda}_m$	: Euler coefficient for the modification function
$\Delta_1^{-1}, \Delta_2^{-1}$	: Inverse difference operator or summation operators defined in Appendix B
∇, M, N	: First difference operators defined in Appendix B
N B	: Second difference operators defined in Appendix B

CHAPTER I

INTRODUCTION

A latticed shell can be defined as a three dimensional assembly of one dimensional element that resists arbitrary loads. The capacity and efficiency of such structures to carry loads are obvious, since every member is a part of the three dimensional latticed shell path which is chosen to be the most effective. This has been demonstrated in many applications of latticed shells, such as in roofs, space vehicles, communication towers and reflectors.

With a repetitive framework pattern, a latticed shell provides the advantage of standardization of member length and size, although for some structures there is the difficulty of joining in space the members which meet at different angles. This chief barrier is now being overcome. Several excellent connectors have been produced mainly for prefabricated steel or aluminum latticed shells and are illustrated in a reference, "Space Structures" (8). Through mass production their cost can be kept low and their use even enables the erection of highly complex latticed shells by semi-skilled personnel.

Since the demand for these structures has increased more effective and efficient methods for their analysis must be developed. Presently there is no unified approach to the rational analysis of these structures.

The analytical methods appearing in the literature can be divided into the following two methods:

(1) Continuum Approach

This method approximates the actual discrete system by an equivalent shell membrane or anisotropic shell. The equivalent shell membrane method was presented by M. Pagano (31) in 1962, and by D. T. Wright (42) in 1965. Essentially the membrane forces and buckling loads for latticed shells are predicted by using modified shell formulas obtained by replacing the discrete structure by an equivalent continuum. The anistropic shell method was explained by W. Flügge (15). A work done by J. D. Renton (32) in 1967, and Heki, K and Y Fujitani (18) in 1967, relates the discrete variables to their equivalen s in the continuum by use of Taylor's series expansions. Using this technique the governing difference equations of plane and space grids are transformed to differential equations, thus yielding continuum models.

It is obvious that the continuum approximation may lead sometimes to erroneous resul s as it approximates discrete properties by continuum ones when no clear analogy between both exists. However, this approach may be useful for an approximate analysis in the preliminary design stage.

(2) Discrete Approach

Two categories are found in this approach. They are the open form methods of which the matrix methods are the most popular and the discrete field methods on which this work is based.

(a) Open Form Methods

Matrix methods are becoming very popular in the computer age. Typical works on this method have been presented by Eiseman, Lin Woo, and Namyet (14) in 1962, P. H. Cheng (6) in 1964, M. Berenyl (2) in 1967, and J. Michalos (28) in 1967. This method requires the solution of a set of simultaneous equations for the unknown forces or deformations of all the joints of the structure. It will give correct solutions for a latticed shell with a limited number of joints. However, as the number of these joints increases, the round-off error and the excessive computation time will make the application of this method impractical.

(b) Discrete Field Analysis

In this analysis the concepts of discrete field mechanics and of difference geometry are utilized to obtain the mathematical model, a two dimensional partial difference equation.

Pioneering work for latticed shells was introduced by L. A. Larkin (26), D. L. Dean (9, 10, 11) and C. P. Ugarte (10, 39). The first of these appeared in publication in 1960. Exact closed form field solutions were obtained for latticed shells with momentless connections. These analyses are feasible and satisfactory for certain types of loading conditions, but a more realistic approach requires the consideration of flexure in such structures since most joints provide at least partial restraint. Field solutions are valid over the entire latticed shell and are essentially independent of the number of joints and the size of the latticed shell.

W. Gutkowski (17) presented a circular cylindrical latticed shell with rigid joints in 1965, but his solution did not satisfy all the boundary conditions. A master's thesis by S. Ch. Shrivastava (34) in 1967 also takes the flexure of the members into consideration. However, his solu ion is limited to a special boundary condition. Mithaiwala's treatment (29) has also similar restrictions.

The difficulty in directly finding proper solutions of high order difference equations for arbitrary boundary conditions proves to be the major weakness in the previous works. This suggests that a new method be found to overcome the difficulty.

The primary objective of this dissertation is to provide a rational method to utilize the concepts of difference geometry and the calculus of finite difference to obtain in an efficient manner solutions to the latticed shells with general boundary supports. This is accomplished by the application of the calculus of variations to discrete field mechanics. To the knowledge of this author this is the first attempt to apply this new branch of discrete field mechanics to the analysis of latticed shells.

The calculus of variations in continuum mechanics was applied by Bernoulli, Euler, and Lagrange in such fields as geometry and physics. Today it is a highly advanced branch of modern mathematics closely related to the theory of differential equations by which various statics and dynamics problems have been effectively handled. Applications in engineering have been presented by Bleich (3), Sokolnikoff (35) and other authors. This theory deals with the calculation of the extreme values of functions defined by certain integrals whose integrands contain one or more functions of continuous variables. In con inuum mechanics this problem is concerned with finding equilibrium states and the condi ions necessary to achieve such states.

As in continuum mechanics, the equilibrium state of a latticed shell can be related to an extremum. However, since the variables are discrete and the functional describing the problem is a summation instead of an integral, the existing theory needs to be modified to establish the properties needed in discrete field mechanics problems. An introductory work done by Goudreau (16) in 1963 applied the technique to the problems of a lamella beam. This author extended further this work and developed a theory which provides a more general mathematical treatment of the calculus of variations in discrete field mechanics. A significant application of this theory enables the author to obtain closed form solutions of latticed shells with general boundary conditions for which no solution is available in the literature.

Buckling is a serious problem that should be considered in the design and construction of most latticed shells. An inadequate resistance to buckling contributed to the recent failure of a large span latticed dome in Bucharest, Rumania in January, 1963.

Considerable research on the stability of latticed shells has been performed in recent years, and it can be divided into the following categories:

(a) Modification of the known linear theory for shell membranes, using energy criteria, which yields a differential equation. This is represented by von Karman and Tsien (40), Pagano (31), Klöppel and Jungbluth (24), and Wright (42).

(b) Application of perturbation techniques, using digital computers were presented by Keller and Reiss (23) and Weinitschke (41).

Most of the works except that of Pagano have been performed on spherical domes. However, no rational analysis for the buckling of cylindrical latticed shells has yet been obtained.

A second objective of this dissertation is to develop an analysis for the elastic stability of cylindrical latticed shells. It is believed that this is the first attempt at a rigorous treatment of this problem. The concept of calculus of variations has been utilized in this analysis.

Although the principles of the calculus of variations was applied to circular cylindrical latticed shells, it can also be applied in the same manner to solve other types of latticed shells.

The effectiveness of this technique will be demonstrated through a comparison of the numerical results of the closed form solution with those of the open form method.

CHAPTER II

ENERGY FORMULATION OF CYLINDRICAL LATTICED SHELLS

A key step in the analysis of a structural system is an adequate and efficient mathematical model to represent the system under consideration. For latticed shells, such a model can be obtained by the application of the concepts of difference geometry and of the calculus of finite differences, or by the application of the calculus of variations in discrete field mechanics as it will be demonstrated in this chapter. The later technique proves to be a powerful tool for the analysis of latticed shells as it exploits certain broad minimum principles that characterize the equilibrium states of such structures. Knowledge of the principles involved in the energy methods is indispensable for a thorough understanding of the calculus of variations and the mathematical procedures to be applied.

The fundamental theorem of the calculus of variations and the corresponding derivations are shown in Appendix A. The results given there will be applied to obtain the governing equations for the general flexural analysis of cylindrical latticed shells, and the natural boundary conditions associated with the corresponding mathematical model.

II. 1. TOTAL POTENTIAL ENERGY OF CYLINDRICAL LATTICED SHELLS

A latticed shell may be described as a discrete surface generated by two independent sets of parametric polygons α_1 and α_2. The latticed shell surface can then be described by the position vector, $\bar{X}$ (α_1, α_2), of its typical node (α_1, α_2). The interval of definition of the independent variables (α_1, α_2) is given by the field of integer numbers, $0 \leqq \alpha_1 \leqq M$ and $0 \leqq \alpha_2 \leqq N$ as shown in Figure 1.

A local coordinate system defined by the unit tangents to the space polygons and by the normal to the latticed surface shown in Figure 2 has proven to be the most convenient reference system for the formulation of the total potential energy of latticed shells. The unit vector $\bar{t}_1$ and $\bar{t}_2$ denote the unit tangent vectors to the α_1-polygon and the α_2-polygon, respectively. The unit vector $\bar{N}$ is called the latticed shell normal defined by

$$\bar{N} = \frac{\bar{t}_1 \times \bar{t}_2}{\sin \Psi} \qquad (2-1)$$

where Ψ is the angle between $\bar{t}_1$ and $\bar{t}_2$, (Fig. 2b).

Since this study is of cylindrical latticed shells as shown in Fig. 3, the case of interest is the orthogonal one, i. e. $\Psi = \frac{\pi}{2}$.

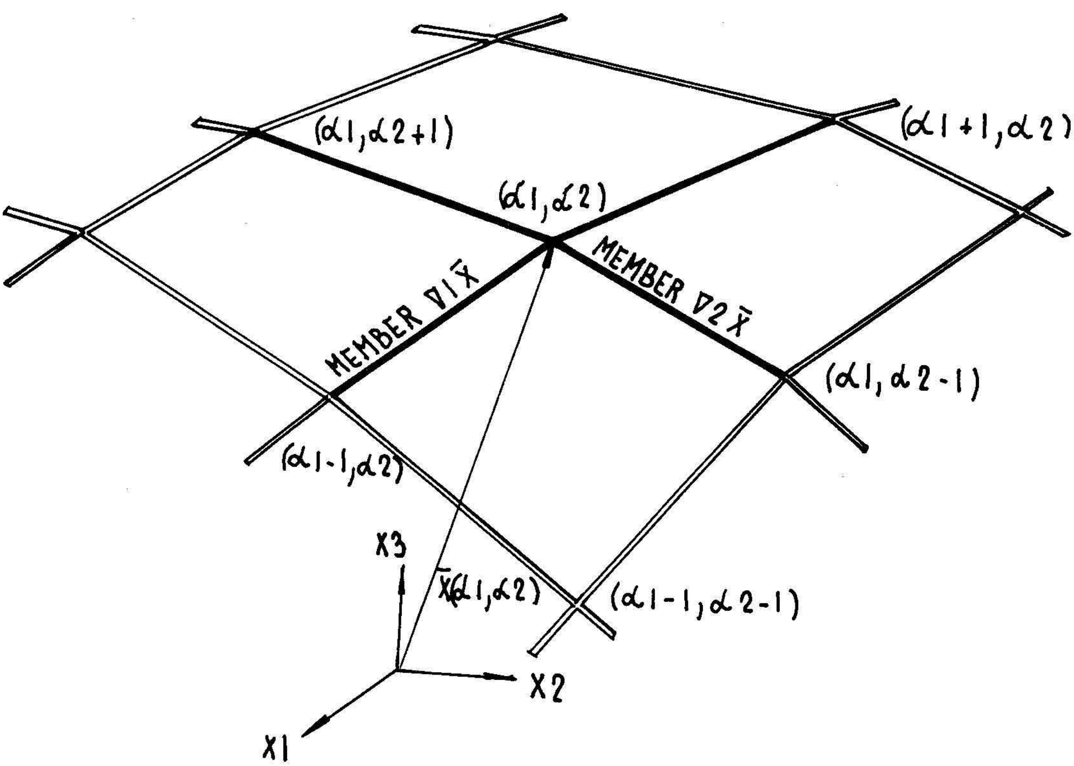

FIG. 1. LATTICED SHELL ELEMENT

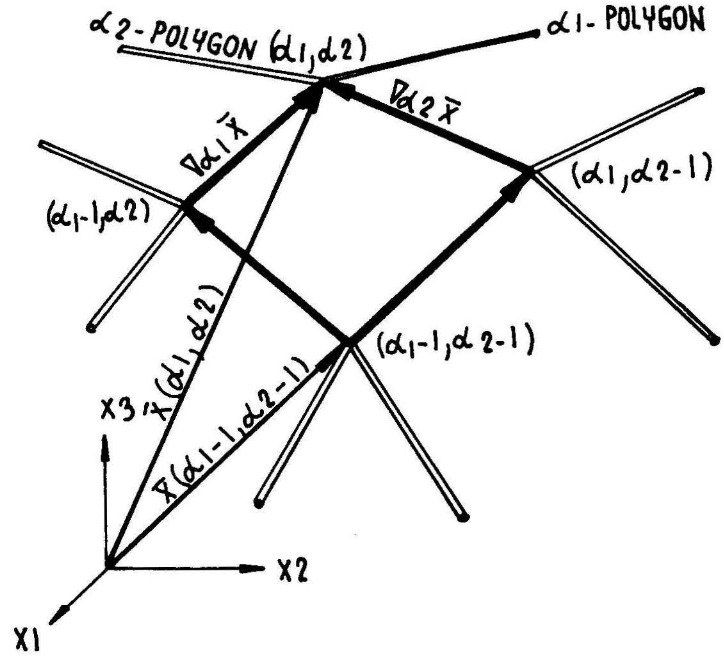

FIG. 2a SURFACE ELEMENT

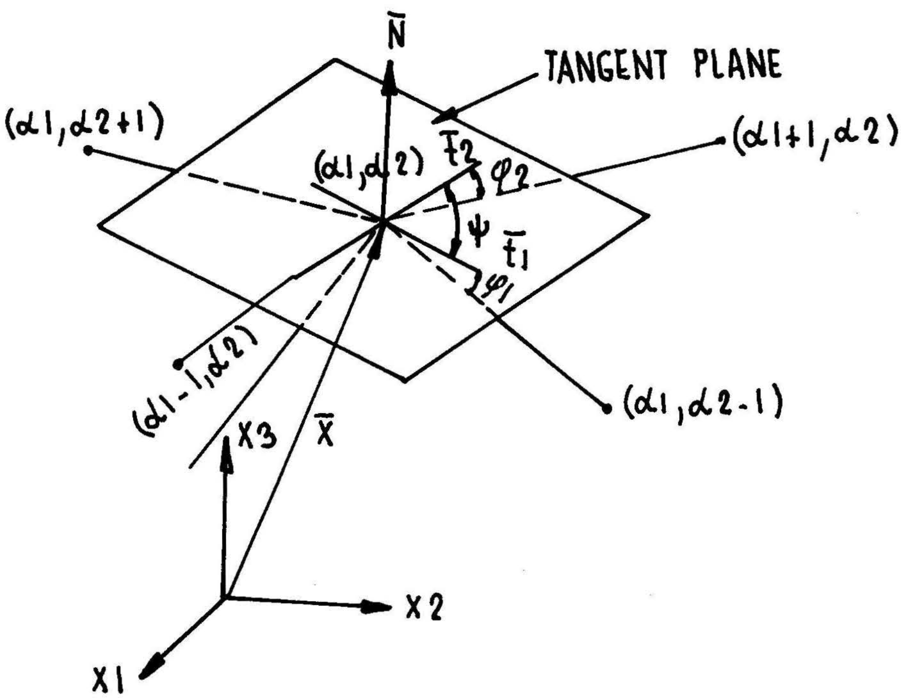

FIG. 2b TANGENT PLANE AND LATTICED SURFACE NORMAL

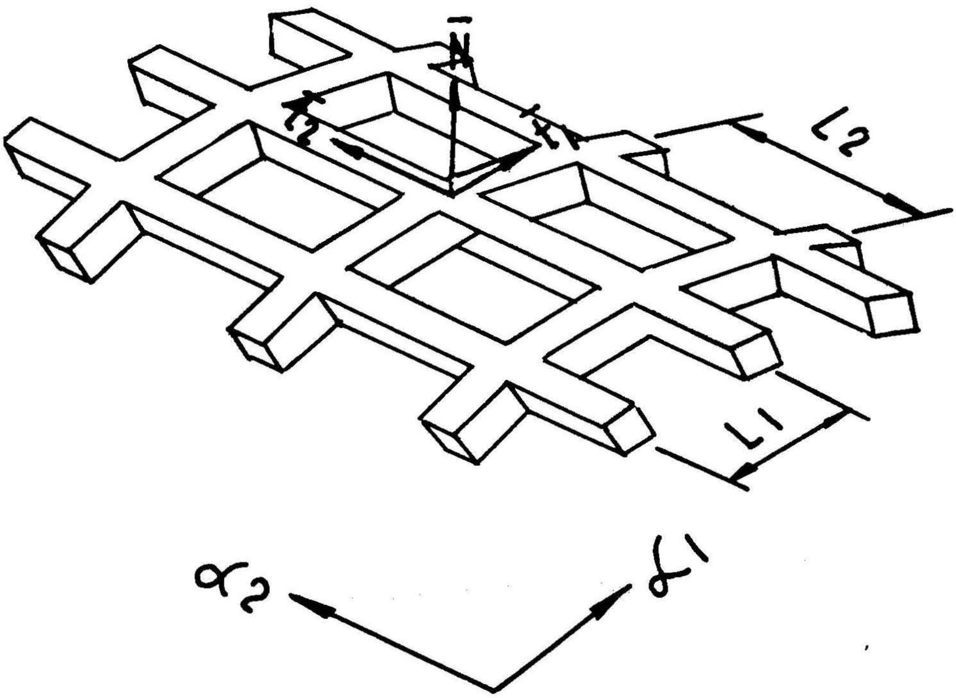

FIG. 3 ELEMENT OF A CIRCULAR CYLINDRICAL LATTICED SHELL

Before any attempt is made to apply the variational techniques of Appendix A to the flexural analysis of latticed shells, it should be understood that the connections of such structures can develop restraint against all types of flexural effects.

Using the basic knowledge of the surface the total potential energy of a cylindrical latticed shell is formulated using the equivalent moment and force vectors acting at the nodes.

II. 1a STRAIN ENERGY IN A TYPICAL MEMBER $\nabla_1 \bar{X}$ OF THE α_1-POLYGONS

The objective of this section is to formulate the strain energy of a member of a latticed shell using the local trihedron $(\bar{t}_1, \bar{t}_2, \bar{N})$. However, since this energy is more easily obtained when the principal coordinate system $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ of Fig. 6a, b is used this will be done first as an intermediate step. The desired energy can then be obtained by using a proper matrix transformation. Consider a typical member $\nabla_1 \bar{X}$ wtih its forces and deformations about the principal coordinate system $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ shown in Fig. 4 and Fig. 6a, b. The strain energy, $V\alpha_1$, in the member $\nabla_1 \bar{X}$ is equal to the work done by the forces (applied gradually) as they induce corresponding deformations (33),

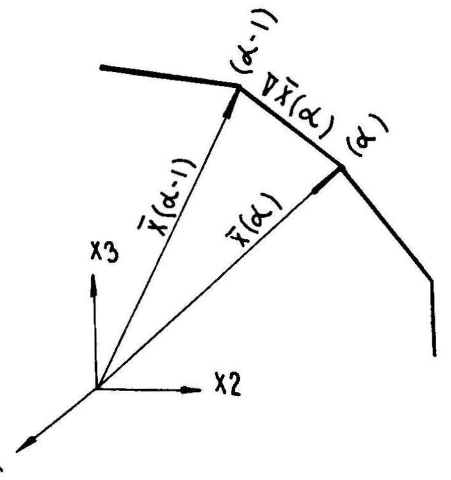

FIG. 4 PLANE POLYGON

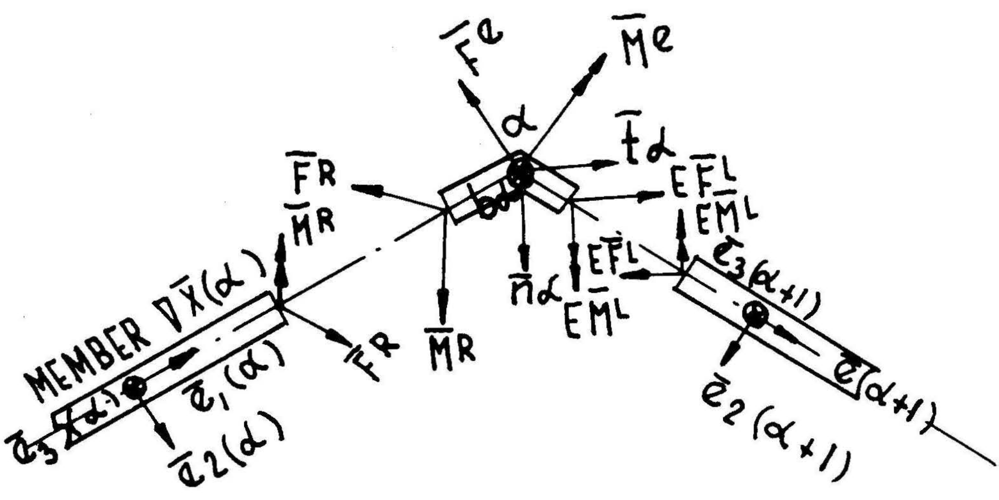

FIG. 5 FORCE AND MOMENT VECTORS AT A NODE (α)

that is

$$V_{\alpha 1} = M_{\alpha} (\bar{F}^R \cdot \bar{u}^R + \bar{M}^R \cdot \bar{\theta}^R) \tag{2-2}$$

in which $M_{\alpha}=(1+E_{\alpha}^{-1})/2$ is the backward mean operator and $\bar{F}^R$, $\bar{M}^R$, $\bar{u}^R$, $\bar{\theta}^R$ denote the force, moment, displacement and rotation vectors, respectively. The superscripts R and L serve to indicate the quantities acting respectively at the right or left of the joint

α_1, α_2. It can easily be seen that compatibility of deformations requires

$$E_{\alpha}^{-1}\ (\bar{F}^R \cdot \bar{u}^R) = \bar{F}^L \cdot \bar{u}^L \qquad (2-3)$$

The force-deformation relations for a straight prismatic beam subjected to an axial force are available (13). In difference notation, these relations are

$$M_1^R = -\ M_1^L = \frac{GJ_1}{L_1}\ \nabla_\alpha\ \ \theta_1^R \qquad (2-4a,\ b)$$

$$\begin{pmatrix} M_2^R \\ \\ M_2^L \end{pmatrix} = \bar{b}_1\ \frac{EI_2}{L_1} \begin{pmatrix} \bar{\gamma}_1 - \nabla_\alpha & \frac{\bar{\gamma}_1}{L_1}\ \nabla_\alpha \\ \\ (1-\bar{\gamma}_1)\ \nabla_\alpha + \bar{\gamma}_1 & \frac{\bar{\gamma}_1}{L_1}\ \nabla_\alpha \end{pmatrix} \begin{pmatrix} \theta_2^R \\ \\ u_3^R \end{pmatrix} \qquad (2-5a,\ b)$$

$$\begin{pmatrix} M_3^R \\ \\ M_2^L \end{pmatrix} = b_1\ \frac{EI_3}{L_1} \begin{pmatrix} \gamma_1 - \nabla_\alpha & \frac{\gamma_1}{L_1}\ \nabla_\alpha \\ \\ (1-\gamma_1)\ \nabla_\alpha + \gamma_1 & \frac{\gamma_1}{L_1}\ \nabla_\alpha \end{pmatrix} \begin{pmatrix} \theta_2^R \\ \\ u_2 \end{pmatrix} \qquad (2-6a,\ b)$$

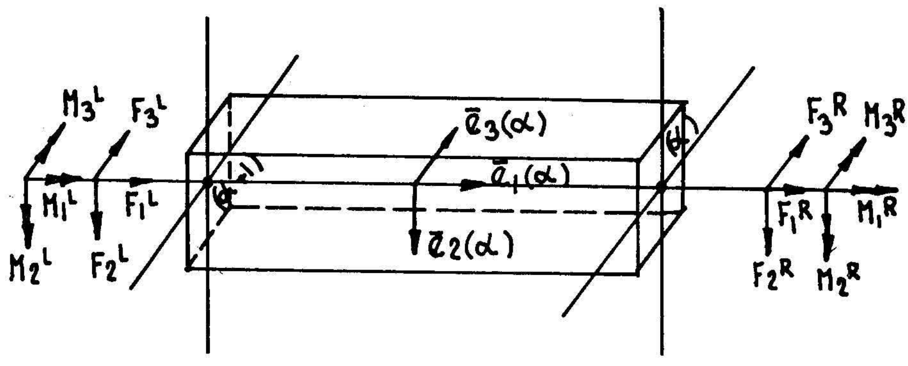

FIG. 6a FORCES ON MEMBER $\nabla\bar{X}\ (\alpha)$

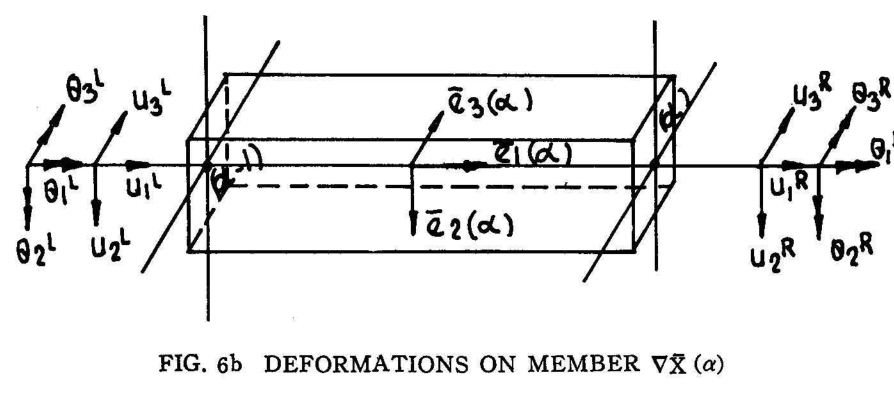

FIG. 6b DEFORMATIONS ON MEMBER $\nabla\bar{X}\ (\alpha)$

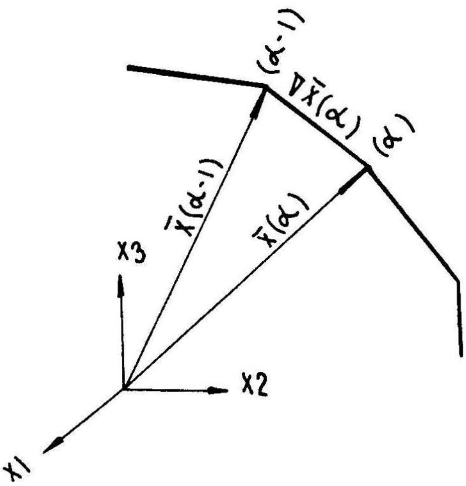

FIG. 4 PLANE POLYGON

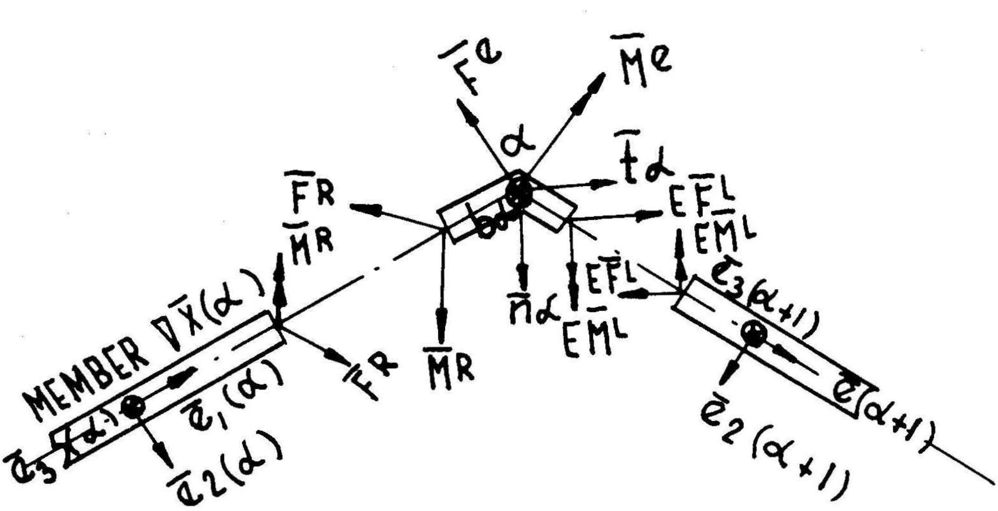

FIG. 5 FORCE AND MOMENT VECTORS AT A NODE (α)

that is

$$V_{\alpha 1} = M_{\alpha} \left(\bar{F}^R \cdot \bar{u}^R + \bar{M}^R \cdot \bar{\theta}^R \right) \tag{2-2}$$

in which $M_{\alpha}=(1+E_{\alpha}^{-1})/2$ is the backward mean operator and $\bar{F}^R$, $\bar{M}^R$, $\bar{u}^R$, $\bar{\theta}^R$ denote the force, moment, displacement and rotation vectors, respectively. The superscripts R and L serve to indicate the quantities acting respectively at the right or left of the joint

α_1, α_2. It can easily be seen that compatibility of deformations requires

$$E_{\alpha}^{-1}\,(\bar{F}^R \cdot \bar{u}^R) = \bar{F}^L \cdot \bar{u}^L \tag{2-3}$$

The force-deformation relations for a straight prismatic beam subjected to an axial force are available (13). In difference notation, these relations are

$$M_1^R = -M_1^L = \frac{GJ_1}{L_1}\nabla_\alpha\ \theta_1^R \tag{2-4a, b}$$

$$\begin{pmatrix} M_2^R \\ M_2^L \end{pmatrix} = \bar{b}_1 \frac{EI_2}{L_1} \begin{pmatrix} \bar{\gamma}_1 - \nabla_\alpha & \frac{\bar{\gamma}_1}{L_1}\nabla_\alpha \\ (1-\bar{\gamma}_1)\nabla_\alpha + \bar{\gamma}_1 & \frac{\bar{\gamma}_1}{L_1}\nabla_\alpha \end{pmatrix} \begin{pmatrix} \theta_2^R \\ u_3^R \end{pmatrix} \tag{2-5a, b}$$

$$\begin{pmatrix} M_3^R \\ M_2^L \end{pmatrix} = b_1 \frac{EI_3}{L_1} \begin{pmatrix} \gamma_1 - \nabla_\alpha & \frac{\gamma_1}{L_1}\nabla_\alpha \\ (1-\gamma_1)\nabla_\alpha + \gamma_1 & \frac{\gamma_1}{L_1}\nabla_\alpha \end{pmatrix} \begin{pmatrix} \theta_2^R \\ u_2 \end{pmatrix} \tag{2-6a, b}$$

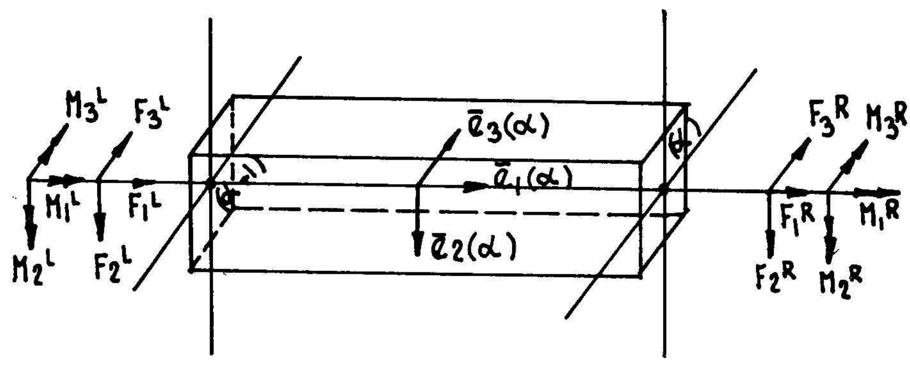

FIG. 6a FORCES ON MEMBER $\nabla\bar{X}\,(\alpha)$

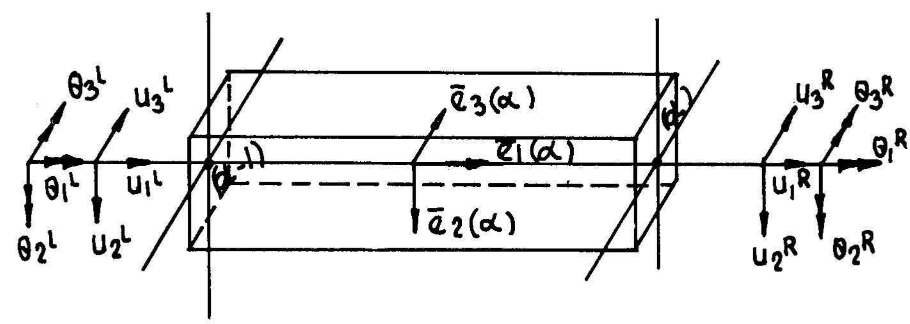

FIG. 6b DEFORMATIONS ON MEMBER $\nabla\bar{X}\,(\alpha)$

$$F_1^R = -F_1^L \frac{EA}{L_1} \nabla_\alpha u_1^R \tag{2-7a, b}$$

$$F_2^R = -F_2^L = b_1 \frac{EI_3}{L_1} \frac{\gamma_1}{L_1} \left\{ (\nabla_\alpha - 2)\, \theta_3^R + \frac{2}{L_1} \nabla_\alpha\, u_2^R \right\} \tag{2-8a, b}$$

$$F_3^R = -F_3^L = -\bar{b}_1 \frac{EI_2}{L_1} \frac{\bar{\gamma}_1}{L_1} \left\{ (\nabla_\alpha - 2)\, \theta_2^R - \frac{2}{L_1} \nabla_\alpha\, u_3^R \right\} \tag{2-9a, b}$$

where

$\nabla\alpha$ is the partial backward difference operator.

i. e., $\nabla\alpha\, u_1^R = u_1^R - u_1^L$

GJ_1, EI_2, EI_3 are the torsional and flexural rigidities in $\bar{e}_1 - \bar{e}_2$ plane and in $\bar{e}_1 - \bar{e}_3$ plane, respectively.

A_1, L_1 represent the area of the cross section and the length of the member $\nabla_1 \bar{X}$ respectively

b_1, γ_1 are the coefficients related to the axial force defined by

$$b_1 = \frac{\phi_1 \csc \phi_1 - 1}{\frac{2}{\phi_1} \tan \frac{\phi_1}{2} - 1} \qquad \gamma_1 = \frac{\phi_1 (1 - \cos \phi_1)}{\phi_1 - \sin \phi_1} \tag{2-10a, b}$$

in which

$$\phi_1^2 = -\frac{L_1^2}{EI_3} F_1 \tag{2-11}$$

$\bar{b}_1$, $\bar{\gamma}_1$ are the coefficients obtained by replacing $\bar{\phi}_1$ by $\bar{\phi}_1^2 = -\frac{L_1^2}{EI_2} F_1$ into Eqs. (2-10 a. b).

The substitution of Eq. (2-4) through Eq. (2-9) into Eq. (2-2) yields the expression for the strain energy of the member as follows:

$$\begin{aligned} V\alpha_1 = \tfrac{1}{2}\, b_1 \frac{EI_3}{L_1} \Bigg(& \left\{ (\gamma_1 - \nabla\alpha)\, \theta_3^R - \frac{\gamma_1}{L_1} \nabla\alpha\, u_2^R \right\} \theta_3^R \\ & + \left(\left\{ (1 - \gamma_1)\, \nabla\alpha + \gamma_1 \right\} \theta_3^R - \frac{\gamma_1}{L_1} \nabla\alpha\, u_2^R \right) \\ & \times (1 - \nabla\alpha)\, \theta_3^R + \frac{\gamma_1}{L_1} \left\{ (\nabla\alpha - 2)\, \theta_3^R + \frac{2}{L_1} \nabla\alpha\, u_2^R \right\} \nabla\alpha\, u_2^R \Bigg) \\ & + \tfrac{1}{2} \bar{b}_1 \frac{EI_2}{L_1} \Bigg(\left\{ (\bar{\gamma}_1 - \nabla\alpha)\, \theta_2^R + \frac{\bar{\gamma}_1}{L_1} \nabla\alpha\, u_3^R \right\} \theta_2^R + \Bigg(\left\{ (1 - \bar{\gamma}_1)\, \nabla\alpha + \bar{\gamma}_1 \right\} \theta_2^R \\ & + \frac{\bar{\gamma}_1}{L_1} \nabla\alpha\, u_3^R \Bigg\} \Bigg) (1 - \nabla\alpha)\, \theta_2^R - \frac{\bar{\gamma}_1}{L_1} \left\{ (\nabla\alpha - 2)\, \theta_2^R - \frac{2}{L_1} \nabla\alpha\, u_3^R \right\} \nabla\alpha\, u_3^R \Bigg) \\ & + \tfrac{1}{2} \frac{EA_1}{L_1} (\nabla\alpha\, u_1^R)^2 + \tfrac{1}{2} \frac{GJ_1}{L_1} (\nabla\alpha\, \theta_1^R)^2 \end{aligned} \tag{2-12}$$

To transform the above equation into a function of the deformations along the local trihedron of the polygon, consider the orthogonal system composed by the unit tangent, $\bar{t}_\alpha$, the unit normal, $\bar{u}_\alpha$, and the unit binormal vector, $\bar{b}_\alpha$, at its typical node shown in Fig. 5. The displacement components of a node along the principal coordinate system can be related to displacement components along the local trihedron by the matrix transformation

$$\begin{pmatrix} u_1^R \\ u_2^R \\ u_3^R \end{pmatrix} = \begin{pmatrix} T_{ij} \end{pmatrix} \begin{pmatrix} u_{t\alpha}(\alpha_1, \alpha_2) \\ u_{n\alpha}(\alpha_1, \alpha_2) \\ u_{b\alpha}(\alpha_1, \alpha_2) \end{pmatrix} \tag{2-13}$$

Similarly, the relation between the roation components becomes

$$\begin{pmatrix} \theta_1^R \\ \theta_2^R \\ \theta_3^R \end{pmatrix} = \begin{pmatrix} T_{ij} \end{pmatrix} \begin{pmatrix} \theta_{t\alpha}(\alpha_1, \alpha_2) \\ \theta_{n\alpha}(\alpha_1, \alpha_2) \\ \theta_{b\alpha}(\alpha_1, \alpha_2) \end{pmatrix} \tag{2-14}$$

where

$$[T_{ji}] = \begin{pmatrix} \bar{e}_1(\alpha)\cdot\bar{t}_{\alpha 1}(\alpha) & \bar{e}_1(\alpha)\cdot\bar{n}_{\alpha 1}(\alpha) & \bar{e}_1(\alpha)\cdot b_{\alpha 1}(\alpha) \\ \bar{e}_2(\alpha)\cdot\bar{t}_{\alpha 1}(\alpha) & \bar{e}_2(\alpha)\cdot\bar{n}_{\alpha 1}(\alpha) & \bar{e}_1(\alpha)\cdot b_{\alpha 1}(\alpha) \\ \bar{e}_3(\alpha)\cdot\bar{t}_{\alpha 1}(\alpha) & \bar{e}_3(\alpha)\cdot\bar{n}_{\alpha 1}(\alpha) & \bar{e}_1(\alpha)\cdot b_{\alpha 1}(\alpha) \end{pmatrix} \tag{2-15}$$

Transformation of the matrices $[u_1^L, u_2^L, u_3^L]^T$ and $[\theta_1^L, \theta_2^L, \theta_3^L]^T$ into the matrices $[u_{t\alpha}(\alpha_1-1, \alpha_2), u_{n\alpha}(\alpha_1-1, \alpha_2), u_{b\alpha}(\alpha_1-1, \alpha_2)]^T$ and $[\theta_{t\alpha}(\alpha_1-1, \alpha_2), \theta_{n\alpha}(\alpha_1-1, \alpha_2), \theta_{b\alpha}(\alpha_1-1, \alpha_2)]^T$ respectively, can be done by the transformation matrix coefficient $[T'_{ij}]$.

The transformation matrix coefficients $[T_{ij}]$ and $[T'_{ij}]$ are obviously functions of the intrinsic geometric properties of the space polygon. Since, in general, torsion and curvature determine these properties, the matrix coefficients $[T_{ij}]$ and $[T'_{ij}]$ will exhibit quantities which measure these properties. For plane polygons such as the ones encountered in cylindrical latticed shells, torsion vanishes.

The matrix transformation coefficients then become

$$[T_{ij}] = \begin{pmatrix} A(\alpha) & -K(\alpha)/2 & 0 \\ K(\alpha)/2 & A(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2-16}$$

$$[T'_{ij}] = \begin{pmatrix} A(\alpha-1) & K(\alpha-1)/2 & 0 \\ -K(\alpha-1)/2 & A(\alpha-1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2-17}$$

where $A(\alpha) = \cos \Psi\alpha$

$K(\alpha) = 2 \sin \Psi\alpha$

$\Psi\alpha$ is the angle between the forward or backward polygon member and the unit tangent $\bar{t}(\alpha_1)$ s own in Fig. 2b.

To formulate the total potential energy of a cylindrical latticed shell, one needs the components of deformations along the local trihedron composed of the units tangents, $\bar{t}_1$ and $\bar{t}_2$, and the normal to the latticed shell, $\bar{N}$. Therefore, proper transformation matrix coefficients for transforming vector components from $\bar{t}_{\alpha 1}$, $\bar{u}_{\alpha 1}$, $\bar{b}_{\alpha 1}$ to $\bar{t}_1$, $\bar{t}_2$, $\bar{N}$ are required. Using the theory of difference geometry these relations are found easily as follows:

$$\bar{t}_{\alpha 1} = \bar{t}_1,\ \bar{u}_{\alpha 1} = -\bar{N},\ \bar{b}_{\alpha 1} = \bar{t}_2 \tag{2-18}$$

Therefore, the deformations in the local coordinate system are transformed accordingly as follows:

$$u_{t\alpha} = u_1,\quad u_{n\alpha} = -u_3,\quad u_{b\alpha} = u_2 \tag{2-19}$$

$$\theta_{t\alpha} = \theta_1,\quad \theta_{n\alpha} = -\theta_3,\quad \theta_{b\alpha} = \theta_2 \tag{2-20}$$

Substituting the results mentioned above, Eq. (2−13) through Eq. (2−20) into Eq. (2−12) one obtains the strain energy of a typical member $\nabla_1\bar{X}$ as a function of the deformations in the latticed shell coordinate system. Thus

$$\begin{aligned}
V\alpha_1 = {} & \frac{b_1}{2}\frac{EI_3}{L_1}\Big(2\gamma_1 M_1 \theta_2 \cdot \theta_2 - \Big\{(1-\gamma_1)\nabla_1 + \gamma_1\Big\}\theta_2 \cdot \nabla_1 \theta_2 \\
& + \frac{\gamma_1}{L_1}\Big(-4M_1\theta_2 + \frac{2}{L_1}\Big\{KM_1u_1 - A\nabla_1 u_3\Big\}\Big)\Big\{KM_1u_1 - A\nabla_1 u_3\Big\}\Big) \\
& + \frac{1}{2}\bar{b}_1\frac{EI_2}{L_1}\Big(\Big\{K(\bar{\gamma}_1-2)M_1\theta_1 - A(\bar{\gamma}_1-2)\nabla_1\theta_3\Big\}\frac{K}{2}\theta_1 \\
& + \Big(K(M_1 + \frac{\bar{\gamma}_1}{2}\nabla_1 - \frac{\bar{\gamma}_1}{2})\theta_1 - A\Big\{(1-\bar{\gamma}_1)\nabla_1 + \bar{\gamma}_1\Big\}\theta_3 \\
& \times\Big(\frac{K}{2}\nabla_1\theta_1 + A\nabla_1\theta_3\Big) + \frac{\bar{\gamma}_1}{L_1}\Big\{K\nabla_1\theta_3 - 4AM_1\theta_3 + \frac{2}{L_1}\nabla_1 u_2\Big\}\nabla_1 u_2 \\
& + \frac{1}{2}\frac{A_1 E}{L_1}(A\nabla_1 u_1 + KM_1 u_3)^2 + \frac{1}{2}\frac{GJ_1}{L_1}\times(A\nabla_1\theta_1 + KM_1\theta_3)^2
\end{aligned} \tag{2-21}$$

Following a similar procedure to that mentioned above, one obtains the strain energy of a typical member $\nabla_2\bar{X}$ of the α_2-polygon, shown in Fig. 1. It is

$$\begin{aligned}
V\alpha_2 = {} & \frac{1}{2}b_2\frac{E\bar{I}_3}{L_2}\Big[2\gamma_2 M_2\theta_1 \cdot \theta_1 - \Big\{(1-\gamma_2)\nabla_2 + \gamma_2\Big\}\theta_1 \cdot \nabla_2\theta_1 \\
& + \frac{\gamma_2}{L_2}\Big\{-4M_2\theta_1 + \frac{2}{L_2}\nabla_2 u_3\Big\}\nabla_2 u_3\Big] + \frac{1}{2}\bar{b}_2\frac{E\bar{I}_2}{L_2} \\
& \times\Big[2\bar{\gamma}_2 M_2\theta_3 \cdot \theta_3 - \Big\{(1-\bar{\gamma}_2)\nabla_2 + \bar{\gamma}_2\Big\}\theta_3 \cdot \nabla_2\theta_3 \\
& + \frac{\bar{\gamma}_2}{L_2}(4M_2\theta_3 + \frac{2}{L_2}\nabla_2 u_1)\nabla_2 u_1\Big] + \frac{1}{2}\frac{A_2 E}{L_2}(\nabla_2 u_2)^2 + \frac{1}{2}\frac{G\bar{J}_1}{L_2}(\nabla_2\theta_2)^2
\end{aligned} \tag{2-22}$$

where

$G\bar{J}_1$, $E\bar{I}_2$, $E\bar{I}_3$ are the torsional and flexural rigidity in the $\bar{e}_1(\alpha_2)-\bar{e}_2(\alpha_2)$ plane and in the $\bar{e}_1(\alpha_2)-\bar{e}_3(\alpha_2)$ plane, respectively. Unit vectors $\bar{e}_1(\alpha_2)$, $\bar{e}_2(\alpha_2)$, $\bar{e}_3(\alpha_2)$ form the principal coordinate system of the member $\nabla_2\bar{X}$.

A_2, L_2 are the area of the cross section and length of the member, respectively.

b_2, $\bar{b}_2$, γ_2, $\bar{\gamma}_2$ are the cooefficients defined by similar relations to Equations (2−10) & (2−11).

Ⅱ. 1b TOTAL POTENTIAL ENERGY OF CYLINDRICAL LATTICED SHELLS

The total potential energy of the cylindrical latticed shell shown in Fig. 3 is obtained by adding the total strain energy of the parametric polygons, V, and the potential euergy due to the external load, W, as follows:

$$U = V + W \tag{2-23}$$

Since the strain energy stored in an individual member of a cylindrical latticed shell is obtained by Eqs. (2−21) and (2−22), the total strain energy is obtained by summing that of all members, that is,

$$\begin{aligned} V &= \sum_{\alpha 1=1}^{M}\sum_{\alpha 2=1}^{N-1} V_{\alpha 1} + \sum_{\alpha 1=1}^{M-1}\sum_{\alpha 2=1}^{N} V_{\alpha 2} \\ &= \Delta_1^{-1}\Delta_2^{-1}\left[V_{\alpha 1}+V_{\alpha 2}\right]\Big|_{\alpha_1=1}^{M}\Big|_{\alpha_2=1}^{N} + \Delta_2^{-1} V_{\alpha 1} \text{ at } \alpha_1=M\Big|_{\alpha_2=1}^{N} \\ &\quad + \Delta_1^{-1} V_{\alpha 2} \text{ at } \alpha_2=N\Big|_{\alpha_1=1}^{M} \end{aligned} \tag{2-24}$$

where the inverse delta operator is

$$\Delta_1^{-1} V_{\alpha 1}\Big|_{\alpha_1=1}^{M} = \sum_{\alpha 1=1}^{M-1} V_{\alpha 1} \tag{2-25}$$

The potential energy due to external loads is

$$\begin{aligned} W &= \sum_{\alpha 1=1}^{M-1}\sum_{\alpha 2=1}^{N-1} W(\alpha_1, \alpha_2) + \sum_{\alpha 2=1}^{N-1} W(\alpha_1, \alpha_2)\Big|_{\alpha_1=0,\ \alpha_1=M} \\ &\quad + \sum_{\alpha 1=1}^{M-1} W(\alpha_1, \alpha_2)\Big|_{\alpha_2=0,\ \alpha_2=N} \\ &= \Delta_1^{-1}\Delta_2^{-1} W(\alpha_1, \alpha_2)\Big|_{\alpha_1=1}^{M}\Big|_{\alpha_2=1}^{N} \end{aligned}$$

$$+ \Delta_2^{-1} W(\alpha_1, \alpha_2) \text{ at } \begin{matrix}\alpha_1 = 0 \\ \alpha_1 = M\end{matrix} \Bigg|_{\alpha_2=1}^{N}$$

$$+ \Delta_1^{-1} W(\alpha_1, \alpha_2) \text{ at } \begin{matrix}\alpha_2 = O \\ \alpha_2 = N\end{matrix} \Bigg|_{\alpha_1=1}^{M} \qquad (2\text{-}26)$$

where

$$W(\alpha_1, \alpha_2) = -\bar{F}^e \cdot \bar{u} - \bar{M}^e \cdot \bar{\theta} \qquad (2\text{-}27)$$

$\bar{F}^e$ and $\bar{M}^e$ are the force and moment vectors, respectively, acting at a joint (α_1, α_2). These vectors can be expressed by the following components:

$$\bar{F}^e = F_1^e \bar{t}_1 + F_2^e \bar{t}_2 + F_3^e \bar{N}$$

$$\bar{M}^e = M_1^e \bar{t}_1 + M_2^e \bar{t}_2 + M_3^e \bar{N} \qquad (2\text{-}28a, b)$$

Therefore, the total potential energy of a cylindrical latticed shell is expressed as

$$U = \Delta_1^{-1} \Delta_2^{-1} \left[V\alpha_1 + V\alpha_2 + W(\alpha_1, \alpha_2) \right] \Bigg|_{\alpha_1=1}^{M} \Bigg|_{\alpha_2=1}^{N}$$

$$+ \Delta_2^{-1} V\alpha_1 \text{ at } \alpha_1 = M \Bigg|_{\alpha_2=1}^{N} + \Delta_1^{-1} V\alpha_2 \text{ at } \alpha_2 = N \Bigg|_{\alpha_1=1}^{M}$$

$$+ \Delta_2^{-1} W(\alpha_1, \alpha_2) \text{ at } \begin{matrix}\alpha_1 = O \\ \alpha_1 = M\end{matrix} \Bigg|_{\alpha_2=1}^{N} + \Delta_1^{-1} W(\alpha_1, \alpha_2) \text{ at } \begin{matrix}\alpha_2 = O \\ \alpha_2 = N\end{matrix} \Bigg|_{\alpha_1=1}^{M} \qquad (2\text{-}29)$$

The resultant expression, Eq. (2-29), is a summation equation with respect to the two variables, α_1 and α_2, as opposed to the integral equation which would be obtained for anisotropic shells.

Substituting Eq. (2-21), (2-22) and (2-26) into Eq. (2-29) one obtains the expression for the total potential energy in terms of the deformation components along the local trihedron of the latticed shell.

II. 2 APPLICATION OF THE CALCULUS OF VARIATIONS TO THE ANALYSIS OF CYLINDRICAL LATTICED SHELLS

From the theorem of the minimum potential energy, it is known that a stable equilibrium configuration requires that the total potential energy in Eq. (2-29) must have a stationary value, that is

$$U = V + W = \text{Stationary} \qquad (2\text{-}30)$$

Therefore, applying the theorem of the discrete variational calculus developed in Appendix A one obtains the necessary condition for U to be stationary. This is

$$\delta U = \sum_{K=1}^{6} \delta U_K = O \tag{2-31}$$

where

$$\delta U_k = \Delta_1^{-1} \Delta_2^{-1} \left[\frac{\partial V\alpha_1}{\partial Y_k} + \frac{\partial V\alpha_2}{\partial Y_k} - T_k^e(\alpha_1, \alpha_2) - \Delta_1 \frac{\partial V\alpha_1}{\partial \nabla_1 Y_k} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_k} \right]$$

$$\times \epsilon_k h_k(\alpha_1, \alpha_2) \Bigg|_{\alpha_1=1}^{M} \Bigg|_{\alpha_2=1}^{N} + \Delta_2^{-1} \left[\left\{ \frac{\partial V\alpha_1}{\partial Y_k} + \frac{\partial V\alpha_1}{\partial \nabla_1 Y_k} - T_k^e \right\}_{\alpha_1=M} \epsilon_k h_k(M, \alpha_2) \right.$$

$$\left. - \left\{ E_1 \frac{\partial V\alpha_1}{\partial \nabla_1 Y_k} + T_k^e \right\}_{\alpha_1=0} \epsilon_k h_k(0, \alpha_2) \right]_{\alpha_2=1}^{N}$$

$$+ \Delta_1^{-1} \left[\left\{ \frac{\partial V\alpha_2}{\partial Y_k} + \frac{\partial V\alpha_2}{\partial \nabla_2 Y_k} - T_k^e \right\}_{\alpha_2=N} \epsilon_k h_k(\alpha_1, N) \right.$$

$$\left. - \left\{ E_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_k} + T_k^e \right\}_{\alpha_2=0} \epsilon_k h_k(\alpha_1, 0) \right]_{\alpha_1=1}^{N} \tag{2-32}$$

where Y_k, T_k^e, $k = 1, 2, \cdots\cdots, 6$ refer to all deformations ($u_1, u_2, u_3, \theta_1, \theta_2, \theta_3$) and external loads ($F_1, F_2, F_3, M_1, M_2, M_3$) at a typical joint ($\alpha_1, \alpha_2$). For example, the terms involved in the double summation of the above equation are written for $k = 1$, i. e. $Y_1 = u_1$ as

$$\Delta_1^{-1} \Delta_2^{-1} \left[\frac{\partial V\alpha_1}{\partial u_1} + \frac{\partial V\alpha_2}{\partial u_1} - F_1^e(\alpha_1, \alpha_2) - \Delta_1 \frac{\partial V\alpha_1}{\partial \nabla_1 u_1} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 u_1} \right]$$

$$\times \epsilon_1 h_1(\alpha_1, \alpha_2) \Bigg|_{\alpha_1=1}^{M} \Bigg|_{\alpha_2=1}^{N} \tag{2-33}$$

As stated in Appendix A, the variations $\epsilon_k h_k$, $k = 1, 2, \cdots\cdots, 6$ are completely arbitrary. Thus Eq. (2-32) can vanish only if the coefficients of the variations each vanish individually. Using this condition one obtains from the coefficients of the variations in the double summation, six equations which represent the equilibrium equations for the flexural analysis of a latticed shell and a set of conditions which are designated as the natural boundary conditions.

The equilibrium equations are compactly given by the expression

$$\frac{\partial V\alpha_1}{\partial Y_k} + \frac{\partial V\alpha_2}{\partial Y_k} - \Delta_1 \frac{\partial V\alpha_1}{\partial \nabla_1 Y_k} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_k} - T_k^e = 0 \tag{2-34}$$

where $k = 1, 2, \cdots\cdots 6$, i. e. $u_1, u_2, u_3, \theta_1, \theta_2, \theta_3$

For example, the above expression for $k = 1$, i. e. u_1, is shown in the bracket of

Eq. (2-33).

These are valid over all interior nodes, i. e. $1 \leqq \alpha_1 \leqq M-1,\ 1 \leqq \alpha_2 \leqq N-1$. Substitution of Eqs. (2-21), (2-22) and (2-26) into Eq. (2-32) yields the governing partial difference equations of latticed shells listed in Table I.

Natural Boundary Conditions

If one considers the summations in Eq. (2-32) one finds that the summation over the edges of constant α_1 and α_2 are respectively

$$\Delta_2^{-1}\left[\left\{\frac{\partial V\alpha_1}{\partial Y_k}+\frac{\partial V\alpha_1}{\partial \nabla_1 Y_k}-T_k^e\right\}_{\alpha_1=M}\epsilon_k h_k(M,\alpha_2)-\left\{E_1\frac{\partial V\alpha_1}{\partial \nabla_1 Y_k}+T_k^e\right\}_{\alpha_1=0}\epsilon_k h_k(0,\alpha_2)\right]_{\alpha_2=1}^{N}$$

$$k = 1,\ 2,\ \cdots\cdots,\ 6 \qquad (2\text{-}35)$$

$$\Delta_1^{-1}\left[\left\{\frac{\partial V\alpha_2}{\partial Y_k}+\frac{\partial V\alpha_2}{\partial \nabla_2 Y_k}-T_k^e\right\}_{\alpha_2=N}\epsilon_k h_k(\alpha_1,N)-\left\{E_2\frac{\partial V\alpha_2}{\partial \nabla_2 Y_k}+T_k^e\right\}_{\alpha_2=0}\epsilon_k h_k(\alpha_1,0)\right]_{\alpha_1=1}^{M}$$

$$k = 1,\ 2,\ \cdots\cdots,\ 6 \qquad (2\text{-}36)$$

As a consequence of Eq. (2-33) the terms involving the arbitrary deformation variations in the above summation must each vanish. Thus, it is required that

$$\text{at } \alpha_1 = 0 \quad \left\{E_1\frac{\partial V\alpha_1}{\partial \nabla_1 Y_k}+T_k^e\right\}\epsilon_k h_k = 0$$

$$k = 1,\ \cdots\cdots\ 6 \qquad (2\text{-}37a,\ b)$$

$$\text{at } \alpha_1 = M \quad \left\{\frac{\partial V\alpha_1}{\partial Y_k}+\frac{\partial V\alpha_1}{\partial \nabla_1 Y_k}-T_k^e\right\}\epsilon_k h_k = 0$$

$$k = 1,\ \cdots\cdots\ 6$$

The above expressions yield the following six "natural" boundary conditions at edges of constant α_1:

$$\text{at } \alpha_1 = 0,\ F_k = E_1\frac{\partial V\alpha_1}{\partial \nabla_1 Y_k}+T_k^e = 0 \qquad Y_k = 0 \qquad (2\text{-}38a,\ b)$$

$$\text{at } \alpha_1 = M,\ F_k = \frac{\partial V\alpha_1}{\partial Y_k}+\frac{\partial V\alpha_1}{\partial \nabla_1 Y_k}-T_k^e = 0 \qquad Y_k = 0 \qquad (2\text{-}39a,\ b)$$

where F_k, $k = 1, 2\cdots\cdots, 6$ represent the total resultant forces at the boundaries, i. e. F_1, F_2, F_3, M_1, M_2, M_3. For example, the above expression is written for $k = 1$ as

$$F_1 = E_1\frac{\partial V\alpha_1}{\partial \nabla_1 u_1}+F_1^e = 0, \qquad u_1 = 0 \qquad (2\text{-}40a,\ b)$$

Similarly, on the edge of constant α_2, one obtains

$$\text{at } \alpha_2 = 0, \; F_k = E_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_k} + T_k^e = 0, \qquad Y_k = 0 \tag{2,41a, b}$$

$$\alpha_2 = N, \; F_k = \frac{\partial V\alpha_2}{\partial Y_k} + \frac{\partial V\alpha_2}{\partial \nabla_2 Y_k} - T_k^e = 0, \qquad Y_k = 0 \tag{2,42a, b}$$

A study of the above conditions shows that Eqs. (2-38a), (2-39a), (2-41a) and (2-42a) represent the physical boundary conditions and Eqs. (2-38b), (2-39b), (2-41b), and (2-42b) the geometric boundary conditions.

Alternate expressions of the above natural boundary conditions in terms of deformations are shown in Table 2 and Table 3, and specific examples are given in Eq. (3-54) of Chapter Ⅲ.

Some combinations of the natural boundary conditions which often arise in the analysis of a latticed shell are the following:

a. Ribbed Support

$$F_1 = F_2 = F_3 = M_1 = M_2 = M_3 = O \tag{2-43}$$

b. Clamped Support

$$u_1 = u_2 = u_3 = \theta_1 = \theta_2 = \theta_3 = O \tag{2-44}$$

c. Diaphragm Support (or Simply Support)

$$F_2 = M_1 = M_3 = u_1 = u_3 = \theta_2 = O \tag{2-45}$$

d. Diaphragm Support with Rotational Constraint

$$F_2 = M_3 = u_1 = u_3 = \theta_1 = \theta_2 = O \tag{2-46}$$

Other combinations can of course be conceived but, in any event, one must be certain to select only one condition from each of the pairs given by Eqs. (2-38a, b), (2-39a, b), (2-41a, b), (2-42a, b) or else the first energy variation Eq. (2-35) will be violated. That is, one cannot specify both the force and the deformation in a given direction at the edge of a latticed shell.

CHAPTER Ⅲ

FLEXURAL ANALYSIS OF CIRCULAR CYLINDRICAL LATTICED SHELLS WITH VARIOUS BOUNDARY CONDITIONS

In recent years cylindrical latticed shells have been widely used. Their applications range from recreational stadiums to attractive world's fair pavilions. However, despite the obvious structural advantages, the theoretical analysis poses many fundamental problems.

The energy approach obtained in the previous chapter has been shown useful in deriving the governing equations and in particular in determining the number and nature of the feasible boundary conditions to be used for any latticed shell. However, the direct use of the first energy variation is a more powerful method to obtain the difficult closed form solutions and also enables a clear mathematical insight into the problem under consideration.

The following three cases are treated in this chapter and for each of these cases numerical comparisons between their closed form solutions and their open form solutions are presented.

Ⅲ. 1. CIRCULAR CYLINDRICAL LATTICED SHELLS WITH DIAPHRAGM BOUNDARY CONDITIONS

It is assumed that the two end circular polygons of the latticed shell as shown in Fig. 7 have stiffnesses equal to one-half that of the interior polygons at $\alpha_2 = O$ and $\alpha_2 = N$. It is further assumed that the two end spans of the circular polygons have stiffnesses equal to one-half that of the regular span. These assumptions are made to consider a practical problem which can be solved using the mathematical properties of finite trigonometric series.

Although the first energy variation will be used directly to obtain the closed form solutions, the governing equations and the natural boundary conditions, as derived from the energy expression, are shown in Tables 1, 2 and 3. Only the boundary conditions at $\alpha_1 = O$ and $\alpha_2 = O$ are listed since those conditions will be applicable at the edges $\alpha_1 = M$ and $\alpha_2 = N$, when the problem is unfolded into its symmetric and anti-symmetric solutions.

For half-stiffness members at the boundaries of the latticed shell, the expression for the first energy variation in Eq. (2-32) must be modified accordingly as follows:

$$\delta U = \sum_{k=1}^{6} \delta U_k = 0 \qquad (3\text{-}1)$$

TABLE 1 GOVERNING EQUATION

U_1	U_2	U_3	θ_1	θ_2	θ_3	Load
$\frac{K^2}{2L_1}(\Delta_1+4) - \frac{C_1A^2}{L_1}\Delta_1 - \frac{2C_2}{L_2}\Delta_2$	0	$-\frac{AK}{2L_1}(2+C_1)\Theta_1$	0	$-\frac{K}{2}(\Delta_1+4)$	$-C_2\Theta_2$	$\frac{F_1^e}{C_8}$
0	$-\frac{2C_3}{L_1}\Delta_1 - \frac{C_4}{L_2}\Delta_2$	0	$-\frac{C_3K}{2}\Delta_1$	0	$C_3A\Theta_1$	$\frac{F_2^e}{C_8}$
$\frac{AK}{2L_1}(2+C_1)\Theta_1$	0	$-\frac{2A^2}{L_1}\Delta_1 + \frac{C_1K^2}{4L_1} \times (\Delta_1+4) - \frac{2C_5}{L_2}\Delta_2$	$C_5\Theta_2$	$-A\Theta_1$	0	$\frac{F_3^e}{C_8}$
0	$-\frac{C_3K}{2}\Delta_1$	$-C_5\Theta_2$	$-\frac{C_3K^2L_1}{4\gamma_1}(\Delta_1+4-2\gamma_1) - \frac{C_6A^2L_1}{\gamma_1}\Delta_1 + \frac{C_5L_2}{\gamma_2}(\Delta_2+2\gamma_2)$	0	$\frac{KAL_1}{2\gamma_1}(C_3-C_6)\Theta_2$	$\frac{M_1^e}{C_8}$
$-\frac{K}{2}(\Delta_1+4)$	0	$A\Theta_1$	0	$\frac{L_1}{\gamma_1}(\Delta_1+2\gamma_1) - \frac{C_7L_1}{\gamma_1}\Delta_2$	0	$\frac{M_2^e}{C_8}$
$C_2\Theta_2$	$-C_3A\Theta_1$	0	$\frac{KAL_1}{2\gamma_1}(C_6-C_3)\Theta_1$	0	$\frac{C_3A^2L_1}{\gamma_1}(\Delta_1+2\gamma_1) + \frac{C_6K^2L_1}{4\gamma_1}(\Delta_1+4) + \frac{C_2L_2}{\gamma_2}(\Delta_2+2\gamma_2)$	$\frac{M_N^e}{C_8}$

$C_1=\frac{A_1L_1^2}{b_1\gamma_1I_3}$ $C_2=\frac{L_1^2\bar{\gamma}_2\bar{b}_2\bar{I}_2}{L_2^2\gamma_1b_1I_3}$ $C_3=\frac{\bar{b}_1I_2}{b_1I_3}$ $C_4=\frac{A_2L_1^2}{b_1\gamma_1I_3}$ $C_5=\frac{L_1^2\gamma_2b_2I_3}{L_2^2\gamma_1b_1I_3}$ $C_6=\frac{GJ_1}{b_1EI_3}$ $C_7=\frac{GJ_2L_1}{b_1EI_3L_2}$ $C_8=\frac{b_1\gamma_1EI_3}{L_1^2}$

TABLE 2 NATURAL BOUNDARY CONDITION AT $\alpha_1 = 0$

U_1	U_2	U_3	θ_1	θ_2	θ_3	Load	
$\frac{K^2}{L_1} N_1 - \frac{C_1 A^2}{L_1} \Delta_1 - \frac{C_2}{L_2} \nabla_2$	0	$-\frac{AK}{L_1}(\Delta_1 + C_1 N_1)$	0	$-K N_1$	$-\frac{C_2}{2} \boxminus_2$	$\frac{F_1^e}{C_8}$	$U_1(0, \alpha_2) = 0$
0	$-\frac{2C_3}{L_1} \Delta_1 - \frac{C_4}{2L_2} \nabla_2$	0	$-\frac{C_3 K}{2} \Delta_1$	0	$2 C_3 A N_1$	$\frac{F_2^e}{C_8}$	$U_2(0, \alpha_2) = 0$
$\frac{AK}{2L_1}(4N_1 + C_1 \Delta)$	0	$-\frac{2A^2}{L_1} \Delta_1 + \frac{C_1 K^2}{2L_1} N_1 - \frac{C_5}{L_2} \nabla_2$	$\frac{C_5}{2} \boxminus_2$	$-2AN_1$	0	$\frac{F_3^e}{C_8}$	$U_3(0, \alpha_2) = 0$
0	$-\frac{C_3 K}{2} \Delta_1$	$-\frac{C_5}{2} \boxminus_2$	$-\frac{C_3 K^2 L_1}{4\gamma_1}(\Delta_1 + 2 - \gamma_1) - \frac{C_6 A^2 L_1}{\gamma_1} \Delta_1 + \frac{C_5 L_2}{2\gamma_2}(\nabla_2 + 2\gamma_2)$	0	$\frac{KAL_1}{2\gamma_1}\{C_3(\Delta_1 + \gamma) - 2C_6 N_1\}$	$\frac{M_1^e}{C_8}$	$\theta_1(0, \alpha_2) = 0$
$-\frac{K}{2} N_1$	0	$A \Delta_1$	0	$\frac{L_1}{\gamma_1}(\Delta_1 + \gamma_1) - \frac{C_7 L_1}{2\gamma_1} \nabla_2$	0	$\frac{M_2^e}{C_8}$	$\theta_2(0, \alpha_2) = 0$
$\frac{C_2}{2} \boxminus_2$	$-C_1 A \Delta_1$	0	$\frac{KAL_1}{2\gamma_1}\{C_3(\Delta_1 + 2 - \gamma_1) + C_6 \Delta_1\}$	0	$\frac{C_3 A^2 L_1}{\gamma_1}(\Delta_1 + \gamma_1) + \frac{C_6 L_1 K^2}{2\gamma_1} N_1 + \frac{C_2 L_1}{2\gamma_2}(\nabla_2 + 2\gamma_2)$	$\frac{M_3^e}{C_8}$	$\theta_3(0, \alpha_2) = 0$

TABLE 3 NATURAL BOUNDARY CONDITION AT $\alpha_2=0$

U_1	U_2	U_3	θ_1	θ_2	θ_3	Load	
$\frac{K^2}{4L_1}(\nabla_1+4) - \frac{C_1A^2}{2L_1}\nabla_1 - \frac{2C_2}{L_2}\Delta_2$	0	$-\frac{AK}{2L_1}(2+C_1)\boxminus_1$	0	$-\frac{K}{4}(\nabla_1+4)$	$-2C_2N_2$	$\frac{F_1^e}{C_8}$	$U_1(\alpha_1,0)=0$
0	$-\frac{C_3}{L_1}\nabla_1-\frac{C_4}{L_2}\Delta_2$	0	$-\frac{C_3K}{2}\nabla_1$	0	$\frac{C_3A}{2}\boxminus_1$	$\frac{F_2^e}{C_8}$	$U_2(\alpha_1,0)=0$
$\frac{AK}{4L_1}(2+C_1)\boxminus_1$	0	$-\frac{A^2}{L_1}\nabla_1+\frac{C_1K^2}{8L_1}(\nabla_1+4)-\frac{C_5}{L_2}\nabla_2$	$2C_5N_2$	$-\frac{A}{2}\boxminus_1$	0	$\frac{F_3^e}{C_8}$	$U_3(\alpha_1,0)=0$
0	$-\frac{C_3K}{4}\nabla_1$	$-C_5\Delta_2$	$-\frac{C_3K^2L_1}{8\gamma_1}(\nabla_1+4-2\gamma_1)-\frac{C_6A^2L_1}{2\gamma_1}\nabla_1+\frac{C_5L_2}{\gamma_2}(\Delta_2+\gamma_2)$	0	$\frac{KAL_1}{4\gamma_1}(C_3-C_6)\boxminus_2$	$\frac{M_1^e}{C_8}$	$\theta_1(\alpha_1,0)=0$
$-\frac{K}{4}(\nabla_1+4)$	0	$\frac{A}{2}\boxminus_1$	0	$\frac{L_1}{2\gamma_1}(\nabla_1+2\gamma_1)-\frac{C_7L_1}{\gamma_1}\Delta_2$	0	$\frac{M_2^e}{C_8}$	$\theta_2(\alpha_1,0)=0$
$C_2\Delta_2$	$-\frac{C_3A}{2}\boxminus_1$	0	$\frac{KAL_1}{4\gamma_1}(C_6-C_3)\boxminus_1$	0	$\frac{C_3A^2L_1}{2\gamma_1}(\nabla_1+2\gamma_1)+\frac{C_6L_1K^2}{8\gamma_1}(\nabla_1+4)+\frac{C_2L_2}{\gamma_2}(\Delta_2+\gamma_2)$	$\frac{M_3^e}{C_8}$	$\theta_3(\alpha_1,0)=0$

where

$$\delta U_k = \Delta_1^{-1}\Delta_2^{-1}\left[\frac{\partial V\alpha_1}{\partial Y_k}+\frac{\partial V\alpha_2}{\partial Y_k}-T_k^e(\alpha_1,\alpha_2)-\Delta_1\frac{\partial V\alpha_1}{\partial\nabla_1 Y_k}-\Delta_2\frac{\partial V\alpha_2}{\partial\nabla_2 Y_k}\right]\epsilon_k h_k\Bigg|_{\alpha_1=1}^{M}\Bigg|_{\alpha_2=1}^{N}$$

$$+\Delta_2^{-1}\left[w_2\left\{\frac{\partial V\alpha_1}{\partial Y_k}+\frac{\partial V\alpha_1}{\partial\nabla_1 Y_k}-w_1T_k^e(\alpha_1,\alpha_2)\right\}_{\alpha_1=M}\epsilon_k h_k(M,\alpha_2)\Bigg|_0^{N+1}\right.$$

$$+\frac{1}{2}\left(\frac{\partial V\alpha_2}{\partial Y_k}-\Delta_2\frac{\partial V\alpha_2}{\partial\nabla_2 Y_k}\right)_{\alpha_1=M}\epsilon_k h_k(M,\alpha_2)\Bigg|_1^{N}+w_2\left\{-E_1\frac{\partial V\alpha_1}{\partial\nabla_1 Y_k}-w_1T_k^e(\alpha_1,\alpha_2)\right\}_{\alpha_1=0}$$

$$\left.\times\epsilon_k h_k(0,\alpha_2)\Bigg|_0^{N+1}+\frac{1}{2}\left(\frac{\partial V\alpha_2}{\partial Y_k}-\Delta_2\frac{\partial V\alpha_2}{\partial\nabla_2 Y_k}\right)_{\alpha_1=0}\epsilon_k h_k(0,\alpha_2)\Bigg|_1^{N}\right]$$

$$+\Delta_1^{-1}\left[w_1\left\{\frac{\partial V\alpha_2}{\partial Y_k}+\frac{\partial V\alpha_2}{\partial\nabla_2 Y_k}-w_2T_k^e(\alpha_1,\alpha_2)\right\}_{\alpha_2=N}\epsilon_k h_k(\alpha_1,N)\Bigg|_0^{M+1}\right.$$

$$+\frac{1}{2}\left(\frac{\partial V\alpha_1}{\partial Y_k}-\Delta_1\frac{\partial V\alpha_1}{\partial\nabla_1 Y_k}\right)_{\alpha_2=N}\epsilon_k h_k(\alpha_1,N)\Bigg|_1^{M}+w_1\left\{-E_2\frac{\partial V\alpha_2}{\partial\nabla_2 Y_k}\right.$$

$$\left.\left.-w_2T_k^e(\alpha_1,\alpha_2)\right\}_{\alpha_2=0}\epsilon_k h_k(\alpha_1,0)\Bigg|_0^{M+1}+\frac{1}{2}\left(\frac{\partial V\alpha_1}{\partial Y_k}-\Delta_1\frac{\partial V\alpha_1}{\partial\nabla_1 Y_k}\right)_{\alpha_2=0}\epsilon_k h_k(\alpha_1,0)\Bigg|_1^{M}\right]$$

(3-2)

Extending the range of the double summation over the boundaries one obtains the expression for the first energy variation as follows:

$$\delta U_k = w_1w_2\Delta_1^{-1}\Delta_2^{-1}\left\{\frac{\partial V\alpha_1}{\partial Y_k}+\frac{\partial V\alpha_2}{\partial Y_k}-T_k^e-\Delta_1\frac{\partial V\alpha_1}{\partial\nabla_1 Y_k}-\Delta_2\frac{\partial V\alpha_2}{\partial\nabla_2 Y_k}\right\}\epsilon_k h_k\Bigg|_0^{M+1}\Bigg|_0^{N+1}$$

$$+\frac{w_2}{2}\Delta_2^{-1}\left[\left\{\frac{\partial V\alpha_1}{\partial Y_k}+(\Delta_1+2)\frac{\partial V\alpha_1}{\partial\nabla_1 Y_k}\right\}_{\alpha_1=M}\epsilon_k h_k(M,\alpha_2)+\left\{\frac{\partial V\alpha_1}{\partial Y_k}+(\Delta_1+2)\frac{\partial V\alpha_1}{\partial\nabla_1 Y_k}\right\}_{\alpha_1=0}\right.$$

$$\left.\times\epsilon_k h_k(0,\alpha_2)\right]\Bigg|_0^{N+1}+\frac{w_1}{2}\Delta_1^{-1}\left[\left\{\frac{\partial V\alpha_2}{\partial Y_k}+(\Delta_2+2)\frac{\partial V\alpha_2}{\partial\nabla_2 Y_k}\right\}_{\alpha_2=N}\right.$$

$$\times\epsilon_k h_k(\alpha_1,N)+\left\{\frac{\partial V\alpha_2}{\partial Y_k}+(\Delta_2+2)\frac{\partial V\alpha_2}{\partial\nabla_2 Y_k}\right\}_{\alpha_2=0}\epsilon_k h_k(\alpha_1,0)\Bigg|_0^{M+1} \qquad (3\text{-}3)$$

where the loading components at the boundary nodes are expressed with the weighting functions defined by

$$w_1=\begin{cases}1 & \text{for } \alpha_1=1,\ (1),\ M-1\\ \frac{1}{2} & \text{for } \alpha_1=0,\ \alpha_1=M\end{cases} \qquad (3\text{-}4)$$

$$w_2=\begin{cases}1 & \text{for } \alpha_2=1,\ (1),\ N-1\\ \frac{1}{2} & \text{for } \alpha_2=0,\ \alpha_2=N\end{cases} \qquad (3\text{-}5)$$

These functions are related to the half stiffnesses of the edge members.

It should be noti ed that the range of summation in Eq. (3-3) has been extended over the boundary nodes which enables one to use the orthogonality properties with respect to the special weighting functions, w_1 and w_2, of he finite trigonometric series. As a result, the boundary conditions are modified accordingly. These modified conditions derived from Eq. (3-3) are:

$$\frac{\partial V\alpha_1}{\partial Y_k} + (\Delta_1+2)\frac{\partial V\alpha_1}{\partial \nabla_1 Y_k} = 0 \quad \text{or} \quad Y_k = 0 \tag{3-6a, b}$$

$$\text{at } \alpha_1 = 0,\ M, \text{ and } \alpha_2 = \text{arbitrary}$$

$$\frac{\partial V\alpha_2}{\partial Y_k} + (\Delta_2+2)\frac{\partial V\alpha_2}{\partial \nabla_2 Y_k} = 0 \quad \text{or} \quad Y_k = 0 \tag{3-7a, b}$$

$$\text{at } \alpha_2 = 0,\ N, \text{ and } \alpha_1 = \text{arbitrary}$$

The relation between the actual and the modified boundary condition must be examined carefully since this is one of the key points of this analysis. To illustrate this consider the boundary node (o, α_2). From Eq. (3-2) the boundary condition requires

$$\left[- E_1 \frac{\partial V\alpha_1}{\partial \nabla_1 Y_k} - w_1 T_k^e + \frac{1}{2}\left(\frac{\partial V\alpha_2}{\partial Y_k} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_k}\right)\right]_{\substack{\alpha_1=0 \\ 1\leqslant\alpha_2\leqslant N-1}} = 0 \tag{3-8}$$

which can be rewritten as

$$\left[\left\{ w_1\left(\frac{\partial V\alpha_1}{\partial Y_k} - \Delta_1 \frac{\partial V\alpha_1}{\partial \nabla_1 Y_k} - T_k^e\right) + \frac{1}{2}\left(\frac{\partial V\alpha_2}{\partial Y_k} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_k}\right)\right\}\right.$$
$$\left. - w_1 \left\{\frac{\partial V\alpha_1}{\partial Y_k} + (\Delta_1+2)\frac{\partial V\alpha_1}{\partial \nabla_1 Y_k}\right\}\right]_{\substack{\alpha_1=0 \\ 1\leqslant\alpha_2\leqslant N-1}} = 0 \tag{3-9}$$

The first bracket shown in the first line of Eq. (3-9) which appears in the double summation, yields the modified governing equation; the second bracket represents the modified boundary conditions. Therefore, if the solution assumed satisfies the modified boundary condition as well as the modified governing equation, this solution satisfies the true conditions, Eq. (3-9).

The procedure for finding the solutions is to assume a set of deformation functions for u_1, u_2, u_3, θ_1, θ_2 and θ_3 that satisfy the vanishing of the first energy variation Eq. (3-3).

Boundary Conditions

Feasible boundary conditions for the general case of cylindrical latticed shells which frequently occur were discussed in the preceding chapter. It is assumed that the cylindrical latticed shell is supported by a plane diaphragm. In practice, this condition can be met by a plane structure or a wall which is rigid in its own plane but offers no resistance against displacement perpendicular to its plane (15). Mathematically this results in the following boundary conditions:

At $\alpha_1 = 0$

$$u_2 = u_3 = \theta_1 = 0 \qquad \text{(3-10a, b, c)}$$

$$F_1 = M_2 = M_3 = 0 \qquad \text{(3-11a, b, c)}$$

The first three boundary conditions are self-explanatory. The fourth condition, Eq. (3-11, a), requires the vanishing of the total axial force in the $\bar{t}_1$ direction. The fifth and the sixth conditions, Eq. (3-11d, c), require the vanishing of the total moment in the $\bar{t}_2$ and $\bar{N}$ directions, respectively.

Similar conditions are assumed at $\alpha_2 = 0$. These are

$$u_1 = u_3 = \theta_2 = 0 \qquad \text{(3-12a, b, c)}$$

$$F_2 = M_1 = M_3 = 0 \qquad \text{(3-13a, b, c)}$$

It will be assumed that the boundary conditions are symmetric or anti-symmetric with respect to half of the span, i. e. $\frac{M}{2}$, $\frac{N}{2}$, and therefore similar statements hold for the boundaries $\alpha_1 = M$ and $\alpha_2 = N$.

To examine the foregoing boundary conditions, Eqs. (3-11a, b, c), these conditions are written in expanded form through Eq. (3-8). The resultant expressions are

$$F_1 = -E_1 \frac{\partial V\alpha_1}{\partial \nabla_1 u_1} + \frac{1}{2}\left(\frac{\partial V\alpha_2}{\partial u_1} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 u_1}\right) - w_1 F_1^e$$

$$= c_8\left(\frac{K^2}{L_1}N_1 - \frac{c_1 A^2}{L_1}\Delta_1 - \frac{c_2}{L_2}\not\nabla_2\right)u_1 - \frac{c_8 AK}{L_1}(\Delta_1 + c_1 N_1)\, u_3 - c_8\, KN_1\, \theta_2 - \frac{c_2\, c_8}{2}\not\Delta_2\, \theta_3$$

$$- w_1 F_1^e = 0$$

$$M_2 = -E_1 \frac{\partial V\alpha_1}{\partial \nabla_1 \theta_2} + \frac{1}{2}\left(\frac{\partial V\alpha_2}{\partial \theta_2} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 \theta_2}\right) - w_1 M_2^e$$

$$= -\frac{c_8 K}{2} N_1\, u_1 + c_8\, A\Delta_1\, u_3 + c_8 \left\{\frac{L_1}{\gamma_1}(\Delta_1 + \gamma_1) - \frac{L_1 c_7}{2\gamma_1}\not\nabla_2\right\}\theta_2 - w_1 M_2^e = 0$$

$$M_3 = -E_1 \frac{\partial V\alpha_2}{\partial \nabla_1 \theta_3} + \frac{1}{2}\left(\frac{\partial V\alpha_2}{\partial \theta_3} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 \theta_3}\right) - w_1 M_3^e$$

$$= \frac{c_2\, c_8}{2}\not\Delta_2\, u_1 - c_1\, c_8\, A\, u_2 + \frac{c_8\, KAL_1}{2\gamma_1}\left\{c_3\,(\Delta_1 + 2 - \gamma_1) + c_6\, \Delta_1\right\}\theta_1$$

$$+ c_8\left\{c_3\, A^2 \frac{L_1}{\gamma_1}(\Delta_1 + \gamma_1) + \frac{c_6\, K^2\, L_1}{2\gamma_1}N_1 + \frac{c_2\, L_1}{2\gamma_2}(\not\nabla_2 + 2\gamma_2)\right\}\theta_3 - w_1 M_3^e = 0$$

(3-14a, b, c)

It is seen that Eqs. (3-14, a, b, c) appear complicate and it will be almost impossible to seek solutions which satisfy these conditions. However, the modified boundary conditions obtained by the energy approach result in simple expressions which yield exactly the same physical meanings as shown in Eq. (3-9). The modified boundary conhitions are obtained from Eq. (3-6a). They are

$$F_1' = \frac{\partial V\alpha_1}{\partial u_1} + (\Delta_1+2)\frac{\partial V\alpha_1}{\partial \nabla_1 u_1} = 0$$

$$M_2' = \frac{\partial V\alpha_1}{\partial \theta_2} + (\Delta_1+2)\frac{\partial V\alpha_1}{\partial V_1 \theta_2} = 0$$

$$M_3' = \frac{\partial V\alpha_1}{\partial \theta_3} + (\Delta_1+2)\frac{\partial V\alpha_1}{\partial \nabla_1 \theta_3} = 0 \qquad \text{(3-15a, b, c)}$$

Using Eqs. (2-21), (2-22) one obtains the following equations:

$$F_1' = c_8\left(-\frac{K}{L_1} + 2c_1A^2\right)\bar{\nabla}_1 u_1 + c_8\left\{\frac{2A}{L_1}\nabla\!\!\!/_1 + c_1AK(\nabla\!\!\!/_1+4)\right\}u_3 + c_8\bar{\nabla}_1\theta_2 = 0$$

$$M_2' = \frac{2c_8\gamma_1K}{L_1}\bar{\nabla}_1 u_1 - \frac{2c_8\gamma_1A}{L_1}\nabla\!\!\!/_1 u_3 - 2c_8\bar{\nabla}_1\theta_2 = 0$$

$$M_3' = \frac{2c_2c_8\gamma_1}{L_1}\nabla\!\!\!/_1 u_2 + c_8\left\{c_2AK(\nabla\!\!\!/_1+4-2\gamma_1) - \frac{c_6AK}{2}\right\}\theta_1$$

$$+ c_8\left(-2c_2A^2 - \frac{c_6K^2}{2}\right)\bar{\nabla}_1\theta_3 = 0 \qquad \text{(3-16a, b, c)}$$

Similarly, by the use of Eq. (3-7a) the modified boundary condition at $\alpha_2 = 0$ is obtained as follows:

$$F_2' = \frac{4c_8}{L_2}\bar{\nabla}_2 u_1 + 2c_8(4-\nabla\!\!\!/_2)\theta_3$$

$$M_1' = \frac{2c_8\gamma_2}{L_2}\nabla\!\!\!/_2 u_3 - 2c_8\bar{\nabla}_2\theta_1$$

$$M_3' = \frac{-2c_8\gamma_2}{L_2}\nabla\!\!\!/_2 u_1 - 2c_8\bar{\nabla}_2\theta_3 \qquad \text{(3-17a, b, c)}$$

Thus, Eqs. (3-16a, b, c), (3-17a. b. c) are the required statements of the diaphragmed cylindrical latticed shell in terms of the corresponding deformations.

A procedure which can be applied to obtain solutions for latticed shells consists in arbitrarily assuming functions involving undetermined coefficients of the deformations u_1, u_2, u_3, θ_1 θ_2 and θ_3. As in the energy method in continuum mechanics the assumed functions must be able to describe the particular deformed shape of the latticed shell under consideratoin. The coefficients of the terms are the parameters to be found by the condition that the total potential energy is stationary. It is essential, however, in choosing the functions that they satisfy at least partially the boundary conditions of the problem.

For the analysis the solutions are assumed to de double finite Fourier series. Therefore, their Euler coefficeints become the parameters or undetermined coefficients.

Similarly, it is convenient to represent the external loads F_1^e, F_2^e, F_3^e, M_1^e, M_2^e, and M_3^e by double finite Fourier series in which the Euler coefficients of these fun tions are obtained by the orthogonality properties of the corresponding trigonometric series.

The following double finite Fourier series are assumed for the solutions of the cylindrical latticed shell with diaphragm boundary conditions:

$$u_1(\alpha_1, \alpha_2) = \sum_{m=0}^{M} \sum_{n=1}^{N-1} U_{mn}^1 \cos \lambda m \, \alpha_1 \sin \lambda n \, \alpha_2$$

$$u_2(\alpha_1, \alpha_2) = \sum_{m=1}^{M-1} \sum_{n=1}^{N} U_{mn}^2 \sin \lambda m \, \alpha_1 \cos \lambda n \, \alpha_2$$

$$u_3(\alpha_1, \alpha_2) = \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} U_{mn}^3 \sin \lambda m \, \alpha_1 \sin \lambda n \, \alpha_2$$

$$\theta_1(\alpha_1, \alpha_2) = \sum_{m=1}^{M-1} \sum_{n=0}^{N} \theta_{mn}^1 \sin \lambda m \, \alpha_1 \cos \lambda n \, \alpha_2$$

$$\theta_2(\alpha_1, \alpha_1) = \sum_{m=0}^{M} \sum_{n=1}^{N-1} \theta_{mn}^2 \cos \lambda m \, \alpha_1 \sin \lambda n' \alpha_2$$

$$\theta_3(\alpha_1, \alpha_2) = \sum_{m=0}^{M} \sum_{n=0}^{N} \theta_{mn}^3 \cos \lambda m \, \alpha_1 \cos \lambda n \, \alpha_2 \qquad (3\text{-}18a, \cdots\cdots f)$$

where

$$\lambda_m = \frac{m\pi}{M} \qquad \lambda_n = \frac{n\pi}{N}$$

The solutions described by Eqs. (3-18a,…f) satisfy the proposed boundary conditions. This is obvious with regard to the three conditions Eqs. (3. 10a, b, c). To show that Eqs. (3-16a, b, c) are satisfied by the assumed solutions, Eqs. (3-18a,…f) are substituted into Eqs. (3-16a, b, c). The results yield

$$F_1' = \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} c_8 \Bigg[\left(-\frac{K}{L_1} + 2c_1 A^2\right)(-2\sin \lambda m) U_{mn}^1 - \left\{ \frac{8A}{L_1} \sin^2 \frac{\lambda m}{2} + 4c_1 AK \cos^2 \frac{\lambda m}{2} \right\} U_{mn}^3 - 2\sin \lambda m \, \theta^2{}_{mn} \Bigg]$$

$$\times \sin \lambda m \, \alpha_1 \quad \sin \lambda n \, \alpha_2$$

$$M_2' = \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} c_8 \left[-\frac{4\gamma_1 K}{L_1} \sin \lambda m\, U^1_{mn} + \frac{8\gamma_1 A}{L_1} \sin^2 \frac{\lambda m}{2} U^3_{mn} \right.$$

$$\left. + 4 \sin \lambda m\, \theta^2{}_{mn} \right] \sin \lambda m\, \alpha_1 \quad \sin \lambda n\, \alpha_2$$

$$M_3' = \sum_{m=1}^{M-1} \sum_{n-1}^{N-1} c_8 \left[-\frac{8c_2\gamma_1}{L_1} \sin^2 \frac{\lambda m}{2} U^2_{mn} + \left\{ c_2\, AK \left(-4 \sin^2 \frac{\lambda m}{2} + 4 - 2\gamma_1 \right) \right. \right.$$

$$\left. \left. - \frac{c_6\, AK}{2} \right\} \theta^1{}_{mn} + 2 \left(2c_2\, A^2 + \frac{c_6\, K^2}{2} \right) \sin \lambda m\, \theta^3{}_{mn} \right]$$

$$\times \sin \lambda m\, \alpha_1 \quad \sin \lambda n\, \alpha_2 \qquad \text{(3-19a, b, c)}$$

From the above expressoins it is clear that the modified boundary conditions are satisfied at $\alpha_1 = 0$. A similar procedure can be followed for the boundary at $\alpha_2 = 0$. It shows that the condition Eqs. (3-12a, b, c) and (3-17a, b, c) are also satisfied.

As it has been demonstrated, all the boundary conditions are satisfied by the assumed solutions. The case of using these solutions for other supports in which some of the boundary conditions are violated will be discussed later.

To proceed with the solution, the external loads must be expanded into appropriate finite double series:

$$\frac{F_1^e}{w_1} = \sum_{m=0}^{M} \sum_{n=1}^{N-1} F^1_{mn} \cos \lambda m\, \alpha_1 \quad \sin \lambda n\, \alpha_2$$

$$\frac{F_2^e}{w_2} = \sum_{m=1}^{M-1} \sum_{n=0}^{N} F^2_{mn} \sin \lambda m\, \alpha_1 \quad \cos \lambda n\, \alpha_2$$

$$F_3^e = \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} F^3_{mn} \sin \lambda m\, \alpha_1 \quad \sin \lambda n\, \alpha_2 \qquad \text{(3-20a, b, c)}$$

The Euler coefficients can be derived from the following expressions:

$$F^1_{mn} = \frac{2}{M\Gamma_m} \Delta_1^{-1} \Delta_2^{-1} F_1^e (\alpha_1 . \alpha_2) \cos \lambda m\, \alpha_1 \quad \sin \lambda n\, \alpha_2 \Bigg|_{\alpha_1=0}^{M+1} \Bigg|_{\alpha_2=1}^{N}$$

$$F^2_{mn} = \frac{2}{N\Gamma_n} \Delta_1^{-1} \Delta_2^{-1} F_2^e (\alpha_1, \alpha_2) \sin \lambda m\, \alpha_1 \quad \cos \lambda n\, \alpha_2 \Bigg|_{\alpha_1=1}^{M} \Bigg|_{\alpha_2=0}^{N+1}$$

$$F^3_{mn} = \frac{4}{MN} \Delta_1^{-1} \Delta_2^{-1} F_3^e (\alpha_1, \alpha_2) \sin \lambda m\, \alpha_1 \quad \sin \lambda n\, \alpha_2 \Bigg|_{\alpha_1=1}^{M-1} \Bigg|_{\alpha_2=1}^{N-1}$$

(3-21a, b, c)

where the following general orthogonality properties of finite series have been utilized

$$\sum_{\alpha 1=1}^{M-1} \sin \frac{i\pi \alpha_1}{M} \sin \frac{k\pi \alpha_1}{M} = \pm \frac{M}{2} \delta_i^{2IM \pm K}$$

$$\sum_{\alpha 1=0}^{M} w_1 \cos\frac{i\pi\alpha_1}{M}\cos\frac{k\pi\alpha_1}{M} = \Gamma_i\ \delta_i^{2IM\pm K} \qquad (3\text{-}22a, b)$$

in which I = 0, (1), ∞, $\delta_i^{2IM\pm k}$ is the Kronecker delta defined dy

$$\delta_i^{2IM\pm K} = \begin{cases} 0 & i \neq 2IM \pm K \\ 1 & i = 2iM \pm K \end{cases} \qquad (3\text{-}23)$$

and Γ_m is a normalization factor defined by

$$\Gamma_m = \sum_{\alpha 1=0}^{M} w_1 \cos\frac{m\pi\alpha_1}{M} \qquad (3\text{-}24)$$

Although the inclusion of the external loads M_1^e, M_2^e, M_3^e does not require special techniques, they are disregarded here as they are of less importance in practical problems.

In order to give specific examples of expanding loads into finite Fourier series, consider an arbitrarily placed unit joint load which is represented by the Kronecker delta

$$F_1^e(\alpha_1, \alpha_2) = F_2^e(\alpha_1, \alpha_2) = F_3^e(\alpha_1, \alpha_2) = \delta_{\alpha 1}^{\xi}\,\delta_{\alpha 2}^{\eta} \qquad (3\text{-}25)$$

where, the Kronecker deltas, $\delta_{\alpha 1}^{\xi}$ and $\delta_{\alpha 2}^{\eta}$, are defined by Eq. (2-23). The Euler coefficients of the assumed series (3-21a, b, c) become

$$F_{mn}^1 = \frac{2}{M\Gamma_m}\cos\lambda m\,\xi\,\sin\lambda n\,\eta$$

$$F_{mn}^2 = \frac{2}{M\Gamma_n}\sin\lambda m\,\xi\,\cos\lambda n\,\eta$$

$$F_{mn}^3 = \frac{4}{MN}\sin\lambda m\,\xi\,\sin\lambda n\,\eta \qquad (3\text{-}26a, b, c)$$

To secure the desired solution the variations of the deformations in the first energy variation, $\epsilon_k h_k$ in Eq. (3-3) need to be considered. They are regarded as kinematically infinitesimal deformations. For the case when k = 1 and 4, these variations will take the forms

$$\epsilon_1 h_1 = \delta u_1 = \sum_{m=0}^{M}\sum_{n=1}^{N-1} \delta U_{mn}^1 \cos\lambda m\,\alpha_1 \sin\lambda n\,\alpha_2$$

$$\epsilon_4 h_4 = \delta\theta_1 = \sum_{m-1}^{M-1}\sum_{m=0}^{N} \delta\theta_{mn}^1 \sin\lambda m\,\alpha_1 \cos\lambda n\,\alpha_2 \qquad (3\text{-}27a, b)$$

Substituting the solutions assumed for the deformations Eqs. (3-18a,…f), the external loads Eqs. (3-20a, b, c) and the variations of the deformations Eqs. (3-27a, b) into the first energy variation, Eq. (3-3), and letting the coefficients of δU_{mn}^1, δU_{mn}^2, δU_{mn}^3,

$\delta\theta^1_{mn}$, $\delta\theta^2_{mn}$, $\delta\theta^3_{mn}$ vanish, one obtains six simultaneous algebraic equations for the unknown Euler coefficients of the double series Eq. (3-18, a…f). This procedure will be illustrated for the first of these six equations as follows:

$$w_1 w_2 \Delta_1^{-1} \Delta_2^{-1} \left\{ \frac{\partial V\alpha_1}{\partial u_1} + \frac{\partial V\alpha_2}{\partial u_1} - F^e_1 - \Delta_1 \frac{\partial V\alpha_1}{\partial \nabla_1 u_1} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 u_1} \right\} \epsilon_1 h_1 \Bigg|_0^{M+1} \Bigg|_0^{N+1}$$

$$= w_1 w_2 \Delta_1^{-1} \Delta_2^{-1} \Bigg[c_8 \left\{ \frac{K^2}{2L_1} (\nabla_1 + 4) - c_1 \frac{A^2}{L_1} c_2 \nabla_2 \right\} u_1 - \frac{c_8 AK}{2L_1} (2+C_1)\, \mathit{\Delta}_1 u_3$$

$$- \frac{c_8 K}{2} (\nabla_1 + 4)\, \theta_2 - c_2 c_8 \mathit{\Delta}_2 \theta_3 - F^e_1 \Bigg] \epsilon_1 h_1 \Bigg|_0^{M+1} \Bigg|_0^{N+1}$$

$$= \frac{M-N}{4} \Bigg[c_8 \left\{ \frac{K^2}{L_1} + \frac{2c_1 A^2}{L_1} + \frac{4K^2}{L_2} + \left(\frac{K^2}{L_1} - \frac{2c_1 A^2}{L_1} \right) \cos \lambda m - \frac{4c_2}{L_2} \cos \lambda n \right\} U^1_{mn}$$

$$- \frac{c_8 AK}{L_1} (2+c_1) \sin \lambda m\, U^3_{mn} - c_8 K (1 + \cos \lambda m)\, \theta^2_{mn} + 2c_8 c_8 \sin \lambda n\, \theta^3_{mn}$$

$$- F^1_{mn} \Bigg] \delta U^1_{mn} = 0$$

It should be pointed out that the terms in the brackets shown in the adove equations are zero by the equilibrium equation at any node including the boundaries in the $\bar{t}_1$ direction. The complete list of these equilibrium equations are shown in Table 1. To simplify the forthcoming expression, it will be assumed that the members of the latticed shell are prismatic. It follows that the coefficients of the force-deformation relations for the bending of the lattice member become

$$b_1 = \bar{b}_1 = 2$$

$$\gamma_1 = \bar{\gamma}_1 = 3$$

The resultant expressions of the six simultaneous equations can be represented.

$$\left(A_{mn} \right) \left(X_{mn} \right) = \left(F_{mn} \right) \qquad \text{(3-29)}$$

where, matrix $\left(A_{mn} \right)$ is shown in Table 4.

$$\text{matrix} \left(X_{mn} \right) = \left(U^1_{mn},\ U^2_{mn},\ U^3_{mn},\ \theta^1_{mn},\ \theta^2_{mn},\ \theta^3_{mn} \right)^T$$

$$\text{matrix} \left(F_{mn} \right) = \left(F^1_{mn}/c_8,\ F^2_{mn}/c_8\ F^3_{mn}/c_8\ 0,\ 0,\ 0, \right)^T$$

Therefore, $\{X_{mn}\}$ is given by

$$\{X_{mn}\} = [A_{mn}]^{-1} \{F_{mn}\} \tag{3-30}$$

Numerical Example

As an illustration of the preceding formulas, consider the circular cylindrical latticed shell shown in Fig. 7. The following two cases are considered:

Case 1 Vertical Load, $P_0 = 0.1$ kip at every node except at the boundary nodes, where $P_0/2$ is applied.

TABLE 4 MARIX $[A_{mn}]$

A_{11}	O	A_{13}	O	A_{15}	A_{16}
O	A_{22}	O	A_{24}	O	A_{26}
A_{13}	O	A_{33}	A_{34}	A_{35}	O
O	A_{24}	A_{34}	A_{44}	O	A_{46}
A_{15}	O	A_{35}	O	A_{55}	O
A_{16}	A_{26}	O	A_{46}	O	A_{66}

where

$$A_{11} = \frac{K^2}{L_1}(1+\cos\lambda m) + \frac{2c_1 A^2}{L_1}(1-\cos\lambda m) + \frac{4c_2}{L_2}(1-\cos\lambda n)$$

$$A_{13} = -\frac{AK}{L_1}(2+c_1)\sin\lambda m$$

$$A_{15} = -K(1+\cos\lambda m)$$

$$A_{16} = 2c_2 \sin\lambda n$$

$$A_{22} = \frac{4c_3}{L_1}(1-\cos\lambda m) + \frac{2c_4}{L_2}(1-\cos\lambda n)$$

$$A_{24} = c_3 K(1-\cos\lambda m)$$

$$A_{26} = -2c_3 A \sin\lambda m$$

$$A_{33} = \frac{4A^2}{L_1}(1-\cos\lambda m) + \frac{c_1K^2}{2L_1}(1+\cos\lambda m) + \frac{4c_5}{L_2}(1-\cos\lambda n)$$

$$A_{34} = -2c_5\sin\lambda m$$

$$A_{35} = 2A\sin\lambda m$$

$$A_{44} = \frac{L_1}{3}(c_3K^2+2c_6A^2) + \frac{4c_5L_2}{3} - \frac{2L_1}{3}\left(\frac{c_3K^2}{4}+c_6A^2\right)\cos\lambda m + \frac{2c_5L_2}{3}\cos\lambda n$$

$$A_{46} = -\frac{KAL_1}{3}(c_3-c_6)\sin\lambda m$$

$$A_{55} = \frac{L_1}{3}(4+2c_7) + 2\cos\lambda m - 2c_7\cos\lambda n)$$

$$A_{66} = \frac{L_1}{3}\left(4c_3A^2+\frac{c_6K^2}{2}\right) + \frac{4c_2L_2}{3} + \frac{L_1}{3}\left(2c_3A^2+\frac{c_6K^2}{2}\right)\cos\lambda m + \frac{2c_2L_2}{3}\cos\lambda n$$

Case 2 Vertical Load, P = 0.1 kip at nodes (1, 2), (3, 2), (1, 3) and (3, 3). Under these symmetric loading conditions only one quarter of the latticed shell needs to be considered.

The geometric and member properties used in the examples are:

$A = \cos\frac{\pi}{12}$	$K = 2\sin\frac{\pi}{12}$
$A_1 = A_2 = 2.21\text{in.}^2$	$I_3 = \bar{I}_3 = 6.00\text{in.}^4$
$I_2 = \bar{I}_2 = 0.77\text{in.}^4$	$J_1 = \bar{J}_1 = 0.054\text{in.}^4$
$L_1 = L_2 = 60.\text{ in.}$	

The Euler coefficients $[X_{mn}]$ in Eq. (3-30) are computed for the two cases above. They require the inversion of a 6×6 matrix for each combination of indices m and n. Since only symmetrical loads with respect to $\frac{M}{2}$ and $\frac{N}{2}$ are considered, the indices take only odd integer values. The Euler coefficients are substituted into Eqs. (3-18a, ⋯f) to obtain the displacements and rotations at each node. The calculations were carried out by a digital computer. The results which represent the closed form solutions for the latticed shell are listed in Table 5. The above cases were also computed by open form method. Thirty simultaneous equations were established using the governing difference equation in Table 1, and the boundary conditions in Tables 2 and 3. The results obtained after inverting a 30×30 matrix are also listed in Table 5. It should be noticed that if the number of joinst is increased, the closed form solutions will still require the inversion of a 6×6 matrix. However, the matrix needed for the solution by open form method will increase considerable as one needs to add six unknown deformations for each additional node.

The comparisons of these results show surprisingly good agreement and give confidence in tbe method proposed here. The displacements in the $\bar{t}_2$ direction, u_2 (α_1, α_2) are not listed, since the results show that the order of these values is about 1/1000 of those for u_1 (α_1, α_2) and u_3 (α_1, α_2). The largest values for the displacements u_1 (α_1, α_2) and u_3 (α_1, α_2) and the rotations θ_1 (α_1, α_2), θ_2 (α_1, α_2) and θ_3 (α_1, α_2) occur at nodes (0, 2), (2, 2), (2, 0), (0, 2) and (0, 0) respectively.

For the design of the member of the latticed shell, one may desire to have the values of the deformations about the principal coordinates, but this should not present a problem as they can be easily obtained by using the relations shown in Eqs. (2-13), (2-14), (2-19) and (2-20). The values thus obtained are substituted into the force-deformation relations Eqs. (2-2) through (2-9), which may be needed for the design of members of the latticed shell.

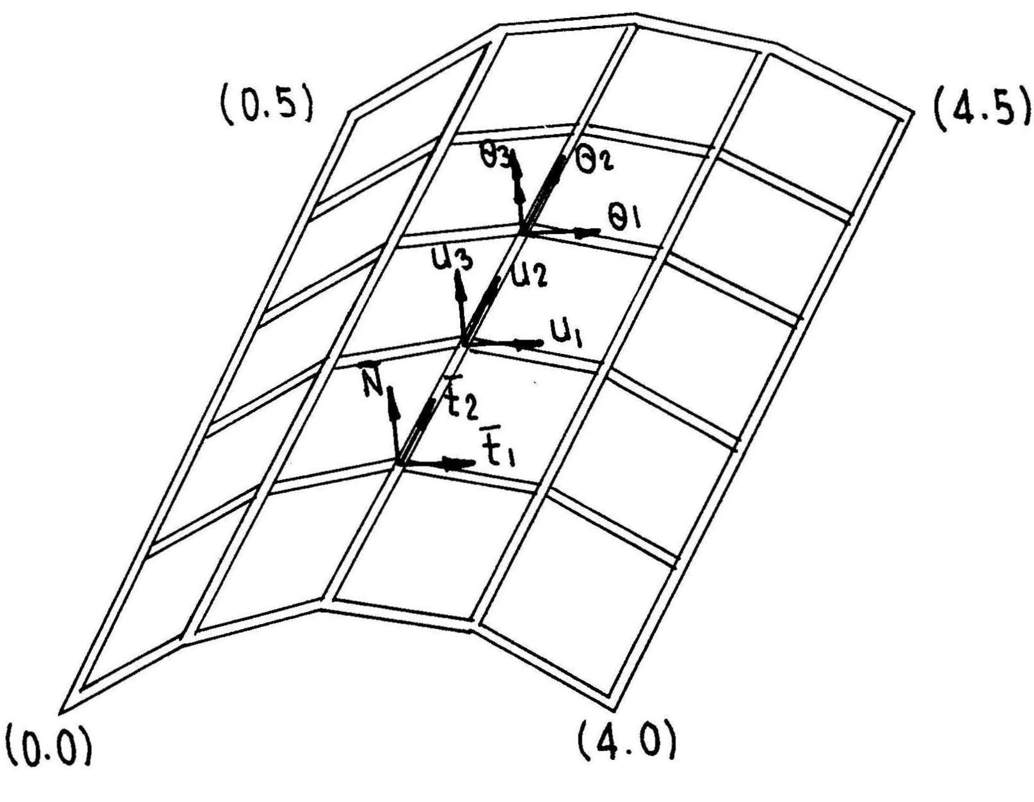

FIG. 7 CIRCULAR CYLINDRICAL LATTICED SHELL

this analysis. The other set of boundary conditions are

$$\alpha_1 = 0 \text{ and } \alpha_1 = M$$

$$u_2 = u_3 = \theta_1 = F_1 = M_2 = M_3 = 0 \qquad (3\text{-}32a,\cdots f)$$

For a more efficient treatment the general problem will be unfolded into the following two cases:

(a) Symmetric behavior of the latticed shell with respect to $\frac{N}{2}$, thus the deformation components u_1, u_3, θ_2 are symmetric with respect to $\frac{N}{2}$, while components u_2, θ_1, θ_3 are anti-symmetric.

(b) Anti-symmetric behavior of the latticed shell with respect to $\frac{N}{2}$, which requires that the deformation components u_1, u_3, θ_2 be anti-symmetric with respect to $\frac{N}{2}$, while the components u_2, θ_1, θ_3 would be symmetric.

Only the first case will be discussed in detail since the second case can be treated by following the same procedure.

Symmetric Case

Solutions are ssumed to be

$$u_1(\alpha_1, \alpha_2) = \sum_{m=0}^{M} \sum_{n=1}^{N-1} U^1_{mn} \cos \lambda m \alpha_1 \sin \lambda n \alpha_2$$

$$u_2(\alpha_1, \alpha_2) = \sum_{m=1}^{M-1} \sum_{n=1}^{N} U^2_{mn} \sin \lambda m \alpha_1 \cos \lambda n \alpha_2$$

$$u_3(\alpha_1, \alpha_2) = \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} U^3_{mn} \sin \lambda m \alpha_1 \sin \lambda n \alpha_2$$

$$\theta_1(\alpha_1, \alpha_2) = \sum_{m=1}^{M-1} \sum_{n=0}^{N} \theta^1_{mn} \sin \lambda m \alpha_1 \cos \lambda n \alpha_2$$

$$\theta_2(\alpha_1, \alpha_2) = \sum_{m=0}^{M} \sum_{n=1}^{N-1} \theta^2_{mn} \cos \lambda m \alpha_1 \sin \lambda n \alpha_2$$

$$\theta_3(\alpha_1, \alpha_2) = \sum_{m=0}^{M} \sum_{n=0}^{N} \theta^3_{mn} \cos \lambda m \alpha_1 \cos \lambda n \alpha_2 \qquad (3\text{-}33a,\cdots f)$$

where, $m = 0, 1, 2, \cdots, M$. $n = 1, 3, 5,$ N (or $N-1$) odd numbers only.

As it has been pointed out in the previous section the above solutions satisfy all the modified boundary conditions except one, i. e.

$$\theta_1 = 0 \text{ at } \alpha_2 = 0 \text{ and } \alpha_2 = N \qquad (3\text{-}34)$$

By using Eq. (3-33d) this condition becomes

TABLE 5 - Numerical Result (Diaphragm Supports)

	Case 1 Closed Form	Case 1 Open Form	Case 2 Closed Form	Case 2 Open Form	Note
u_1 (0,1)	− 0. 4213	− 0. 4219	− 0. 1292	− 0. 1292	in
u_1 (1,1)	− 0. 2977	− 0. 2977	− 0. 0913	− 0. 0926	
u_1 (0,2)	− 0. 6702	− 0. 6703	− 0. 2190	− 0. 2190	
u_1 (1,2)	− 0. 4737	− 0. 4738	− 0. 1541	− 0. 1541	
u_3 (1,1)	− 0. 4609	− 0. 4610	− 0. 1417	− 0. 1418	in
u_3 (2,1)	− 0. 6497	− 0. 6498	− 0. 1988	− 0. 1988	
u_3 (1,2)	− 0. 7332	− 0. 7332	− 0. 2423	− 0. 2423	
u_3 (2,2)	− 1. 0344	− 1. 0344	− 0. 3326	− 0. 3326	
θ_1 (1,0)	− 1. 2325	− 1. 2326	− 0. 2134	− 0. 2134	$\times \frac{1}{150}$
θ_1 (2,0)	− 1. 7366	− 1. 7367	− 0. 2993	− 0. 2993	
θ_1 (1,1)	− 0. 9693	− 0. 9694	− 0. 1677	− 0. 1677	
θ_1 (2,1)	− 1. 3678	− 1. 3679	− 0. 2360	− 0. 2360	
θ_1 (1,2)	− 0. 3531	− 0. 3531	− 0. 0610	− 0. 0610	
θ_1 (2,2)	− 0. 4995	− 0. 4995	− 0. 0863	− 0. 0863	
θ_2 (0,1)	0. 7200	0. 7201	0. 1252	0. 1252	$\times \frac{1}{150}$
θ_2 (1,1)	0. 5032	0. 5033	0. 0862	0. 0862	
θ_2 (0,2)	1. 1441	1. 1442	0. 1986	0. 1986	
θ_2 (1,2)	0. 8022	0. 8022	0. 1377	0. 1377	
θ_3 (0,0)	0. 5373	0. 5374	0. 0931	0. 0931	$\times \frac{1}{150}$
θ_3 (1,0)	0. 3798	0. 3798	0. 0657	0. 0657	
θ_3 (0,1)	0. 4262	0. 4262	0. 0737	0. 0737	
θ_3 (1,1)	0. 3013	0. 3013	0. 0521	0. 0521	
θ_3 (0,2)	0. 1575	0. 1575	0. 0272	0. 0272	
θ_3 (1,2)	0. 1114	0. 1114	0. 0192	0. 0192	

Ⅲ. 2 CIRCULAR CYLINDRICAL LATTICED SHELLS WITH CLAMPED CONDITIONS

The objective of the following section is to modify the double series solutions, Eqs. (3-18a, f) to satisfy the boundary conditions of the two other practical cases for which closed form solutions are desired.

If, instead of the boundary condition at $\alpha_2 = 0$ and $\alpha_2 = N$, Eqs. (3-12) and (3-13), the edges which are clamped but free to move axially are considered. The boundary conditions at these edges are

$$u_1 = u_3 = \theta_1 = \theta_2 = F_2 = M_3 = 0 \qquad \text{(3-31a,}\cdots\text{f)}$$

A study of the feasibility of using solutions Eqs. (3-18a,$\cdots$f) to solve this problem shows that they satisfy all the boundary conditions except condition Eq. (3-31c).

A search in the literature reveals no solution to these boundary supports but the following discrete modified variational method solves the problem.

As in the previous section, the edge members with half-stiffnesses are considered in

$$\sum_m \sum_n \theta^1_{mn} \sin \frac{m\pi\alpha_1}{M} = 0 \tag{3-35}$$

Since the above expression should be satisfied for all values of α_1, it follows that

$$\sum_n \theta^1_{mn} = 0 \tag{3-36}$$

This condition, which would be the one violated by solutions Eq. (3-33a, …f), is called the constraint condition.

The first energy variation Eq. (3-2) will be modified as follows:

$$\delta U_k = w_1 w_2 \Delta_1^{-1} \Delta_2^{-1} \left\{ \frac{\partial V\alpha_1}{\partial Y_k} + \frac{\partial V\alpha_2}{\partial Y_k} - T^e_k - \Delta_1 \frac{\partial V\alpha_1}{\partial \nabla_1 Y_k} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_k} - \left(\delta^0_{\alpha 2} - \delta^N_{\alpha 2}\right) \right.$$

$$\left. \times \delta^4_k \lambda(\alpha_1) \right\} \epsilon_k h_k \Bigg|_0^{M+1} \Bigg|_0^{N+1} + \frac{w_2}{2} \Delta_2^{-1} \left[\left\{ \frac{\partial V\alpha_1}{\partial Y_k} + (\Delta_1 + 2) \frac{\partial V\alpha_1}{\partial \nabla_1 Y_k} \right\} \right.$$

$$\left. \times \epsilon_k h_k (\alpha_1, \alpha_2) \right] \begin{matrix} \alpha_1 = 0 \\ \alpha_1 = M \end{matrix} \Bigg|_0^{N+1} + \frac{w_1}{2} \Delta_1^{-1} \left[\left\{ \frac{\partial V\alpha_2}{\partial Y_k} + (\Delta_2 + 2) \frac{\partial V\alpha_2}{\partial \nabla_2 Y_k} \right. \right.$$

$$\left. \left. + \delta^4_k \left(\delta^0_{\alpha 2} - \delta^N_{\alpha 2}\right) \lambda(\alpha_1) \right\} \epsilon_k h_k (\alpha_1, \alpha_2) \right] \begin{matrix} \alpha_2 = 0 \\ \alpha_2 = N \end{matrix} \Bigg|_0^{M+1} = 0 \tag{3-37}$$

where, δ^4_k, is the Kronecker delta defined by $\delta^4_k = \begin{cases} 1 \text{ for } K = 4 \\ 0 \text{ for } K \neq 4 \end{cases}$ $\lambda(\alpha_1)$ is a modification function defined at $\alpha_2 = 0$ and N

and w_1, w_2 represent weighting functions Eqs. (3-4), (3-5).

Expanding the modification function $\lambda(\alpha_1)$ in the following series one obtains

$$\lambda(\alpha_1) = \sum_{m=1}^{M-1} \bar{\lambda} m \sin \lambda m\, \alpha_1 \tag{3-38}$$

The modification function $\lambda(\alpha_1)$ is defined only at the boundaries and has characteristics similar to the Lagrange multiplier in a functional form. Physically the functions are related directly to forces which should be applied to the boundaries in order to obtain the solution which will satisfy the true boundary conditions.

It should be observed that the term $(\delta^0_{\alpha 2} - \delta^N_{\alpha 2}) \times \delta^4_K \lambda(\alpha_1)$ appearing in Eq. (3-37) is subtracted from the first bracket and exactly the same term is added in the third term. Therefore, there is no change in the first energy variation.

The terms in the double summation which are shown in the first line of Eq. (3-37) will be designated as the modified first energy variation. They take the form,

$$\delta \bar{U} = w_1 w_2 \Delta_1^{-1} \Delta_2^{-1} \left[\frac{\partial V\alpha_1}{\partial Y_K} + \frac{\partial V\alpha_2}{\partial Y_K} - T^e_K - \Delta_1 \frac{\partial V\alpha_1}{\partial \nabla_1 Y_K} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_K} \right.$$

$$\left. - \left(\delta^0_{\alpha 2} - \delta^N_{\alpha 2}\right) \delta^4_K \lambda(\alpha_1) \right] \epsilon_K h_K \Bigg|_{\alpha_1 = 0}^{M+1} \Bigg|_{\alpha_2 = 0}^{N+1} \tag{3-39}$$

From the above considerations it is seen that the problem reduces to one of solving the modified first energy variation Eq. (3-39) and the boundary constraint condition Eq. (3-36).

The external loads F_1^e, F_2^e, and F_3^e are expressed by the same type of finite double series considered in the previous section, Eqs. (3-20a, b, c), but the index n takes only odd integer values.

Substituting the external loads Eqs. (3-20a, b, c), the double finite series solution for the deformations, Eqs. (3-33a, … f), and the modification function Eq. (3-38), into the modified first energy variation, Eq. (3-39), one obtains a set of simultaneous algebraic equations represented by the following matrix

$$\left[Amn\right]\left[Xmn\right] = \left[Fmn\right] + \left[\lambda m\right] \tag{3-40}$$

where

$\left[Amn\right]$ is shown in Table 4

$$\left[Xmn\right] = \left[U^1_{mn}.\ U^2_{mn}.\ U^3_{mn}.\ \theta^1_{mn}.\ \theta^2_{mn}.\ \theta^3_{mn}\right]^T$$

$$\left[Fmn\right] = \left[F^1_{mn}/c8.\ F^2_{mn}/c8.\ F^3_{mn}/c8,\ 0,\ 0,\ 0\right]^T$$

$$\left[\lambda m\right] = \left[0,\ 0,\ 0,\ \frac{4wn}{N}\lambda m,\ 0,\ 0\right]^T$$

The resultant matrix, Eq. (3-40), establishes the relation between the Euler coefficients $[X_{mn}]$, the coefficients of the external loads $[F_{mn}]$ and the coefficients of the yet unknown modification function, $\bar{\lambda}m$.

A solution for the coefficients of the unknown deformations is indicated below

$$\left[Xmn\right] = \left[Amn\right]^{-1}\left[Fmn\right] + \left[Amn\right]^{-1}\left[\lambda m\right] \tag{3-41}$$

If the matrix $[D_{ij}\ (m, n)]$ represents the inverse of the 6×6 matrix $[A_{mn}]$, the term θ^1_{mn} can be written as

$$\theta^1_{mn} = \left[D_{4j}\ (m, n)\right]\left[Fmn\right] + \frac{4wn}{N} D_{44}\ (m, n)\ \bar{\lambda}m \tag{3-42}$$

where $[D_{4j}\ (m, n)]$ $j = 1.2,\ 6$, are the components of the inverse matrix $[D_{ij}\ (m, n)]$.

Substitution of the solution Eq. (3-42) into the constraint condition, Eq. (3-36) yields the solution for $\bar{\lambda}m$. These steps are shown in the equations which follows:

$$\sum_{n} \theta^{1}_{mn} = \sum_{n} \left(D_{4j}\ (m, n)\right) \left(F_{mn}\right) + \frac{4}{N} \sum_{n} w_n\, D_{44}\ (m, n)\ \bar{\lambda} m = 0 \tag{3-43}$$

$$\bar{\lambda} m = - \frac{N \sum_{n} \left(D_{4j}\ (m, n)\right) \left(F_{mn}\right)}{4 \sum_{n} wn\ D_{44}\ (m, n)} \tag{3-44}$$

where $n = 1, 3, 5, \cdots, N$ (or $N-1$) odd number only
N : number of nodes in the α_2 – direction

Introducing the above solution into Eq. (3-41) the Euler coefficients, U^1_{mn}, U^2_{mn}, U^3_{mn}, θ^1_{mn}, θ^2_{mn}, θ^3_{mn}, are obtained in terms of the konwn coefficients $[F_{mn}]$. These solutions satisfy the necessary condition that the total potential energy be stationary, Eq. (3-37), since they satisfy the modified firste energy variation, Eq. (3-39), as well as all of the modified boundary conditions.

Numerical Example 2

To illustrate the preceding formulas the circular cylindrical latticed shell shown in Fig. 7 is again considered. The two cases of loading of Example 1 are also investigated by closed form and open form methods. Therefore, there is no need to list the data shown in the previous example.

Computational Procedure

As shown in Eqs. (3-41) and (3-44), the closed form solutions require 6×6 matrix arithmetic. The open form formulations obtained by using Tables 1, 2, and 3 give 28 simultaneous equations since the boundary conditions $\theta_1\ (1, 0) = \theta_1\ (2, 0) = 0$, are prescribed. The values obtained by the two different methods are listed in Table 6 and the comparison of these values shows an excellent agreement. From the above presentation and the illustrated examples, it can be concluded that the discrete modified variational method is an effective procedure in obtaining exact closed form solutions to structural problems.

A study of the results shown indicates that the largest values of the deformations, $u_1\ (\alpha_1, \alpha_2)$, $u_3\ (\alpha_1, \alpha_2)$, $\theta_1\ (\alpha_1, \alpha_2)$, $\theta_2\ (\alpha_1, \alpha_2)$ and $\theta_3\ (\alpha_1, \alpha_2)$ occur at nodes (0, 2), (2, 2), (0, 2), (2, 1) and (0, 0), while the maximum value of $\theta_3 (\alpha_1, \alpha_2)$ occurs at (0, 1) for Case 2.

It is interesting to compare the results of Examples 1 and 2, since they differ only by the restraint condition imposed at $\alpha_2 = 0$, N in Example 2. All the deformations corresponding to the later example decrease significantly as expected. For example, the largest displacement in the $\tilde{N}$ direction for Case 1 is -1.034 in. in Example 1, while it has the value of -0.418 in. in Example 2.

The above results also imply that the proper design of supports can provide great saving in materials.

TABLE 6 - Numerical Result 2 (Diaphragm Supports With Rotation Constraints)

	Case 1 Closed Form	Case 1 Open Form	Case 2 Closed Form	Case 2 Oqen Form	Note
u_1 (0,1)	− 0. 1257	− 0. 1258	− 0. 0412	− 0. 0412	in
u_1 (1,1)	− 0. 0888	− 0. 0888	− 0. 0290	− 0. 0290	
u_1 (0,2)	− 0. 2717	− 0. 2717	− 0. 1003	− 0. 1003	
u^1 (1,2)	− 0. 1919	− 0. 1919	− 0. 0702	− 0. 0701	
u_3 (1,1)	− 0. 1376	− 0. 1376	− 0. 0455	− 0. 0455	in
u_3 (2,1)	− 0. 1927	− 0. 1928	− 0. 0626	− 0. 0626	
u_3 (1,2)	− 0. 2975	− 0. 2976	− 0. 1125	− 0. 1126	
u_3 (2,2)	− 0. 4183	− 0. 4183	− 0. 1491	− 0. 1491	
θ_1(1,1)	− 0. 4958	− 0. 4958	− 0. 1831	− 0. 1831	$\times \frac{1}{150}$
θ_1(2,1)	− 0. 6965	− 0. 6966	− 0. 2442	− 0. 2443	
θ_1(1,2)	− 0. 2318	− 0. 2318	− 0. 1053	− 0. 1053	
θ_1(2,2)	− 0. 3276	− 0. 3277	− 0. 1332	− 0. 1333	
θ_2(0,1)	0. 2162	0. 2162	0. 0732	0. 0732	$\times \frac{1}{150}$
θ_2(1,1)	0. 1479	0. 1479	0. 0470	0. 0470	
θ_2(0,2)	0. 4662	0. 4664	0. 1892	0. 1892	
θ_2(1,2)	0. 3228	0. 3229	0. 1059	0. 1060	
θ_3(0,0)	0. 1715	0. 1715	0. 0546	0. 0546	$\times \frac{1}{150}$
θ_3(1,0)	0. 1209	0. 1209	0. 0384	0. 0384	
θ_3(0,1)	0. 1682	0. 1682	0. 0627	0. 0627	
θ_3(1,1)	0. 1189	0. 1189	0. 0441	0. 0441	
θ_3(0,2)	0. 1010	0. 1009	0. 0409	0. 0409	
θ_3(1,2)	0. 0713	0. 0713	0. 0288	0. 0288	

III. 3 CIRCULAR CYLINDRICAL LATTICED SHELLS WITH POLYGONAL RIBBED BOUNDARY AT $\alpha_2 = 0$, N

In the previous two sections it has been assumed that diaphragm supports exist at $\alpha_1 = 0$, M and $\alpha_2 = 0$, N. The following section analyses a more general type of boundary condition which occurs frequently in the design of roofs.

It is assumed that the two straight edge members at $\alpha_1 = 0$ and M have stiffnesses which are those half of the interior ones, but the two polygonal edge members at $\alpha_2 = 0$ and N are the same as the interior ones.

With these boundary members, the expression for the first energy variation Eq. (2-32) must be modified accordingly. Similar to Eqs. (3-1) and (3-2) one can write as

$$\delta U = \sum_{K=1}^{6} \delta U_K \tag{3-45}$$

where

$$\delta U_K = \Delta_1^{-1} \Delta_2^{-1} \left[\frac{\partial V\alpha_1}{\partial Y_K} + \frac{\partial V\alpha_2}{\partial Y_K} - T_K^e - \Delta_1 \frac{\partial V\alpha_1}{\partial \nabla_1 Y_K} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_K} \right] \epsilon_K h_K \Bigg|_{\alpha_1=1}^{M} \Bigg|_{\alpha_2=1}^{N}$$

$$+\Delta_2^{-1}\left[\left\{\frac{\partial V\alpha_1}{\partial Y_K}+\frac{\partial V\alpha_2}{\partial Y_K}-w_1 T_K^e\right\}_{\alpha_1=M}\epsilon_K h_K(M,\alpha_2)\Bigg|_0^{N+1}+\frac{1}{2}\left(\frac{\partial V\alpha_2}{\partial Y_K}-\Delta_2\frac{\partial V\alpha_2}{\partial\nabla_2 Y_K}\right)_{\alpha_1=M}\right.$$

$$\times\epsilon_K h_K(M,\alpha_2)\Bigg|_1^N+\left(-E_1\frac{\partial V\alpha_1}{\partial\nabla_1 Y_K}-w_1 T_K^e\right)_{\alpha_1=0}\epsilon_K h_K(0,\alpha_2)\Bigg|_0^{N+1}$$

$$\left.+\frac{1}{2}\left(\frac{\partial V\alpha_2}{\partial Y_K}-\Delta_2\frac{\partial V\alpha_2}{\partial\nabla_2 Y_K}\right)_{\alpha_1=0}\epsilon_K h_K(0,\alpha_2)\Bigg|_1^N\right]$$

$$+\Delta_1^{-1}\left[w_1\left\{\frac{\partial V\alpha_2}{\partial Y_K}+\frac{\partial V\alpha_2}{\partial\nabla_2 Y_K}-T_K^e\right\}_{\alpha_2=N}\epsilon_K h_K\Bigg|_0^{M+1}\right.$$

$$+\left(\frac{\partial V\alpha_1}{\partial Y_K}-\Delta_1\frac{\partial V\alpha_1}{\partial\nabla_1 Y_K}\right)_{\alpha_2=0}\epsilon_K h_K(\alpha_1,N)\Bigg|_1^M$$

$$+w_1\left\{-E_2\frac{\partial V\alpha_2}{\partial\nabla_2 Y_K}-T_K^e\right\}_{\alpha_2=0}\epsilon_K h_K\Bigg|_0^{M+1}$$

$$\left.+\left(\frac{\partial V\alpha_1}{\partial Y_K}-\Delta_1\frac{\partial V\alpha_1}{\partial\nabla_1 Y_K}\right)_{\alpha_2=0}\epsilon_K h_K(\alpha_1,0)\Bigg|_1^M\right] \tag{3-46}$$

Extending the range of double summation in order to use the orthogonality properties of trigonometric series the following expression for the first energy variation is obtained:

$$\delta U_K=w_1\Delta_1^{-1}\Delta_2^{-1}\left\{\frac{\partial V\alpha_1}{\partial Y_K}+\frac{\partial V\alpha_2}{\partial Y_K}-T_K^e-\Delta_1\frac{\partial V\alpha_1}{\partial\nabla_1 Y_K}-\Delta_2\frac{\partial V\alpha_2}{\partial\nabla_2 Y_K}\right\}\epsilon_K h_K\Bigg|_0^{M+1}\Bigg|_0^{N+1}$$

$$+\frac{1}{2}\Delta_2^{-1}\left[\left\{\frac{\partial V\alpha_1}{\partial Y_K}+(\Delta_1+2)\frac{\partial V\alpha_1}{\partial\nabla_1 Y_K}\right\}\epsilon_K h_K\right]\begin{matrix}\alpha_1=0\\ \alpha_1=M\end{matrix}\Bigg|_0^{N+1}$$

$$+w_1\Delta_1^{-1}\left[\left(E_2\frac{\partial V\alpha_2}{\partial\nabla_2 Y_K}\right)_{\alpha_2=N}\epsilon_K h_K(\alpha_1,N)-\left(\frac{\partial V\alpha_2}{\partial Y_K}+\frac{\partial V\alpha_2}{\partial\nabla_2 Y_K}\right)_{\alpha_2=0}\epsilon_K h_K(\alpha_1,0)\right]\Bigg|_1^{M+1}=0 \tag{3-47}$$

The modified boundary conditions derived from Eq. (3-47) are written as

At $\alpha_1 = 0$ and M, and α_2 = arbitrary

$$\frac{\partial V\alpha_1}{\partial Y_K}+(\Delta_1+2)\frac{\partial V\alpha_1}{\partial\nabla_1 Y_K}=0 \quad\text{or}\quad Y_K(\alpha_1,\alpha_2)=0 \tag{3-48a, b}$$

At $\alpha_2 = 0$, and α_1 = arbitrary

$$\frac{\partial V\alpha_2}{\partial Y_K}+\frac{\partial V\alpha_2}{\partial\nabla_2 Y_K}=0 \quad\text{or}\quad Y_K(\alpha_1,\alpha_2)=0 \tag{3-49a, b}$$

At α_2 = N, and α_1 = arbitrary

$$E_2\frac{\partial V\alpha_2}{\partial\nabla_2 Y_K}=0 \quad\text{or}\quad Y_K(\alpha_1,\alpha_2)=0 \tag{3-50a, b}$$

The following boundary conditions are to be imposed:

(a) Diaphragm supports at $\alpha_1 = 0$, and $\alpha_1 = M$, that is

$$u_2 = u_3 = \theta_1 = F_1 = M_2 = M_3 = 0 \qquad \text{(3-51a, } \cdots \text{ f)}$$

(b) Polygonal ribbed supports at $\alpha_2 = 0$ and $\alpha_2 = N$, which imply that these edges are free supports. The mathematical statement of these boundary conditions requires

$$F_1 = F_2 = F_3 = M_1 = M_2 = M_3 = 0 \qquad \text{(3-52a, } \cdots \text{ f)}$$

Using Eq. (3-46) the boundary conditions in Eqs. (3-52a, ⋯ f) are represented by the following expression:

$$H_K = -E_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_K} - T^e_K + \frac{\partial V\alpha_1}{\partial Y_K} - \Delta_1 \frac{\partial V\alpha_1}{\partial \nabla_1 Y_K} = 0 \qquad \text{(3-53)}$$

where, $k = 1, 2, \cdots, 6$ gives $H_1 = F_1$, $H_2 = F_2$, $H_3 = F_3$, $H_4 = M_1$, $H_5 = M_2$, $H_6 = M_3$.

If the above equations are written in terms of the deformations and external loads, they take the form

$$F_1 = c_8 \left\{ \frac{K^2}{2L_1} (\boxslash_1 + 4) - \frac{c_1 A^2}{L_1} \boxslash_1 - \frac{2c_2}{L_2} \Delta_2 \right\} u_1 - \frac{c_8 AK}{L_1} (2 + c_1) \boxminus_1 u_3$$

$$- \frac{c_8 K}{2} (\boxslash_1 + 4)\ \theta_2 - 2c_2 c_8 N_2\ \theta_3 - F^e_1 = 0$$

$$F_2 = c_8 \left(- \frac{2c_3}{L_1} \boxslash_1 - \frac{c_4}{L_2} \Delta_2 \right) u_2 - Kc_3 c_8 \boxslash_1\ \theta_1 + Ac_3 c_8 \boxminus_1\ \theta_3 - F^e_2 = 0$$

$$F_3 = \frac{c_8 AK}{2L_1} (2 + c_1) \boxminus_1 u_1 + c_8 \left\{ - \frac{2A^2}{L_1} \boxslash_1 + \frac{K^2 c_1}{4c_1} (\boxslash_1 + 4) - \frac{c_5}{L_2} \boxslash_2 \right\} u_3$$

$$+ 2c_5 c_8 N_2\ \theta_1 - Ac_8 \boxminus_1\ \theta_2 - F^e_3 = 0 \qquad \text{(3-54a, } \cdots \text{ f)}$$

$$M_1 = - \frac{c_3 c_8 K}{2} \boxslash_1 u_2 - c_8 c_5 \Delta_2 u_3 + c_8 \left\{ \frac{c_3 K^2 L_1}{4\gamma_1} (\boxslash_1 + 4 - 2\gamma_1) - A^2 \frac{c_6 L_1}{\gamma_1} \boxslash_1 \right.$$

$$\left. + \frac{c_5 L_2}{\gamma_2} (\Delta_2 + \gamma_2) \right\} \theta_1 + \frac{c_8 KAL_1}{4\gamma_1} (b_3 - b_6) \boxminus_2\ \theta_3 - M^e_1 = 0$$

$$M_2 = - \frac{c_8 K}{2} (\boxslash_1 + 4)\ u_1 + c_8 A \boxminus_1 u_3 + c_8 \left\{ \frac{L_1}{\gamma_1} (\boxslash_1 + 2\gamma_1) - \frac{L_1}{\gamma_1} c_7 \Delta_2 \right\} \theta_2 - M^e_2 = 0$$

$$M_3 = c_2 c_8 \Delta_2 u_1 - Ac_3 c_8 \boxminus_1 u_2 + \frac{c_8 KAL_1}{2\gamma_1} (c_6 - c_3) \boxminus_1\ \theta_1 + c_8 \left\{ A^2 \frac{c_3 L_1}{\gamma_1} (\boxslash_1 + 2\gamma_1) \right.$$

$$\left. + \frac{c_6 L_1 K^2}{4\gamma_1} (\boxslash_1 + 4) + \frac{c_2 L_2}{\gamma_2} (\Delta_2 + \gamma_2) \right\} \theta_3 - M^e_3 = 0$$

A close examination of the above equations shows that it will be very difficult to find solutions which satisfy the above conditions by ordinary methods. However, the following approach reduces them to simple expressions called modified boundary conditions.

Eq. (3-53) can be rewritten as

$$H_K = H_K^1 + H_K^2 \tag{3-55}$$

where

$$H_K^1 = \frac{\partial V\alpha_1}{\partial Y_K} + \frac{\partial V\alpha_2}{\partial Y_K} - T_K^e - \Delta_1 \frac{\partial V\alpha_1}{\partial \nabla_1 Y_K} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_K}$$

$$H_K^2 = -\left(\frac{\partial V\alpha_2}{\partial Y_K} + \frac{\partial V\alpha_2}{\partial \nabla_2 Y_K}\right) \tag{3-56a, b}$$

It is observed that H_K^1, Eq. (3-56a), is the same expression which appears in the bracket of the double summation of Eq. (3-46). Since $H_K^1 = 0$ is an equilibrium condition at $\alpha_2 = 0$, $H_K^2 = 0$ can be considered the modified boundary condition stated in Eq. (3-49a). They are

$$H_1^2 = F_1^1 = c_8 \left\{ -\frac{2c_2}{L_2} \nabla_2 u_1 - c_2 (2 - \nabla_2)\, \theta_3 \right\} = 0$$

$$H_2^2 = F_2^1 = -\frac{c_4 c_8}{L_2} \nabla_2 u_2 = 0$$

$$H_3^2 = F_3^1 = c_8 \left\{ -\frac{2c_5}{L_2} \nabla_2 u_3 + c_5 (2 - \nabla_2)\, \theta_1 \right\} = 0$$

$$H_4^2 = M_1^1 = c_8 \left\{ c_5 \nabla_2 u_3 - \frac{c_5 L_2}{3} (3 - \nabla_2)\, \theta_1 \right\} = 0$$

$$H_5^2 = M_2^1 = -\frac{c_7 c_8 L_1}{3} \nabla_2 \theta_2 = 0$$

$$H_6^2 = M_3^1 = c_8 \left\{ - c_2 \nabla_2 u_1 - \frac{c_2 L_2}{3} (3 - \nabla_2)\, \theta_3 \right\} = 0 \tag{3-57a, ... f}$$

For simplicity the properties of the members of the latticed shell used in the above expressions correspond to those of prismatic members.

To accomplish the desired solution, one needs to find expressions which would satisfy the modified boundary conditions, Eq. (3-57,...f) and the vanishing of the first energy variation, Eq. (3-46).

Prior to this work no closed form solution which satisfies all these conditions has been obtained.

The objective of this section is to modify the available solutions to obtain solutions to this new problem. As in the previous section, the solution of the general problem is obtained by superimposing solutions for the cases of symmetry and anti-symmetry about N/2.

The following solutions are assumed for the displacments and rotations. The assumption takes account of the possibility that neither the displacements nor the rotations at the boundaries, $\alpha_2 = 0$ and N, are zero. Only the symmetric case will be given in detail since the anti-symmetric case follows exactly the same procedure. The solutions are

$$u_1(\alpha_1, \alpha_2) = \sum_m \sum_n U^1_{mn} \cos\lambda m\, \alpha_1 \sin \lambda n \left(\alpha_2 + \frac{1}{2}\right)$$

$$u_2(\alpha_1, \alpha_2) = \sum_m \sum_n U^2_{mn} \sin \lambda m \alpha_1 \cos \lambda n \left(\alpha_2 + \frac{1}{2}\right)$$

$$u_3(\alpha_1, \alpha_2) = \sum_m \sum_n U^3_{mn} \sin \lambda m \alpha_1 \sin \lambda n \left(\alpha_2 + \frac{1}{2}\right)$$

$$\theta_1(\alpha_1, \alpha_2) = \sum_m \sum_n \theta^1_{mn} \sin \lambda m \alpha_1 \cos \lambda n \left(\alpha_2 + \frac{1}{2}\right)$$

$$\theta_2(\alpha_1, \alpha_2) = \sum_m \sum_n \theta^2_{mn} \cos \lambda m \alpha_1 \sin \lambda n \left(\alpha_2 + \frac{1}{2}\right)$$

$$\theta_3(\alpha_1, \alpha_2) = \sum_m \sum_n \theta^3_{mn} \cos \lambda m \alpha_1 \cos \lambda n \left(\alpha_2 + \frac{1}{2}\right) \qquad (3\text{-}58,\cdots f)$$

where m = 1, (1), , M. n = 1, 3, 5, , N (or N +1) odd number only

$$\lambda m = \frac{m\pi}{M} \qquad \lambda n = \frac{n\pi}{N+1}$$

The solutions, Eqs. (3–58a,…f) were assumed to be composed of functions of the independent variables α_1 and α_2. For the problem under consideration, the boundary conditions at $\alpha_1 = 0$ and M are the same in the last section and for this reason that part of the solution will remain the same. Substituting Eqs. (3–58a,…f) into the modified boundary conditions one finds that only $F'_2 = 0$ is satisfied.

Therefore, the modification of the solutions used in the previous cases is required. This is better explained by considering the first energy variation,

$$\delta U = \sum_{K=1}^{6} \delta U_K = 0 \qquad (3\text{-}59)$$

where

$$\delta U = w_1 \Delta_1^{-1} \Delta_2^{-1} \left\{ \frac{\partial V\alpha_1}{\partial Y_K} + \frac{\partial V\alpha_2}{\partial Y_K} - T^e_K - \Delta_1 \frac{\partial V\alpha_1}{\partial \nabla_1 Y_K} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_K} - \lambda^1_K(\alpha_1)\, \delta^0_{\alpha 2} - \lambda^2_k(\alpha_1)\, \delta^N_{\alpha 2} \right\} \epsilon_k h_k(\alpha_1, \alpha_2) \Bigg|_0^{M+1} \Bigg|_0^{N+1}$$

$$+ \frac{1}{2} \Delta_2^{-1} \left[\left\{ \frac{\partial V\alpha_1}{\partial Y_K} + (\Delta_1 + 2) \frac{\partial V\alpha_1}{\partial \nabla_1 Y_K} \right\} (\delta^M_{\alpha 1} - \delta^0_{\alpha 1})\, \epsilon_K h_K(\alpha_1, \alpha_2) \right] \Bigg|_0^{N+1}$$

$$+ w_1 \Delta_1^{-1} \left[\left\{ (\Delta_2 + 1) \frac{\partial V \alpha_2}{\partial \nabla_2 Y_K} + \lambda_K^2 (\alpha_1) \right\}_{\alpha_2 = N} \epsilon_K h_K (\alpha_1, N) - \left\{ \frac{\partial V \alpha_2}{\partial Y_K} + \frac{\partial V \alpha_2}{\partial \nabla_2 Y_K} \right. \right.$$

$$\left. \left. - \lambda_K^1 (\alpha_1) \right\}_{\alpha_2 = 0} \epsilon_K h_K (\alpha_1, 0) \right] \Bigg|_{\alpha_1 = 0}^{M+1} \tag{3-60}$$

where $\lambda_K^1 (\alpha_1)$ and $\lambda_K^2 (\alpha_1)$, $k = 1, 2, \cdots 6$ are modification functions defined at $\alpha_2 = 0$ and $\alpha_2 = N$, respectively, and assumed as follows:

$$\lambda_1^1 (\alpha_1) = \bar{\lambda}_1^2 (\alpha_1) = \sum_{m=0}^{M} \bar{\lambda}_{1m} \cos \lambda m \, \alpha_1$$

$$\lambda_2^1 (\alpha_1) = - \bar{\lambda}_2^2 (\alpha_1) = \sum_{m=1}^{M-1} \bar{\lambda}_{2m} \sin \lambda m \, \alpha_1$$

$$\lambda_3^1 (\alpha_1) = \bar{\lambda}_3^2 (\alpha_1) = \sum_{m=1}^{M-1} \bar{\lambda}_{3m} \sin \lambda m \, \alpha_1$$

$$\lambda_4^1 (\alpha_1) = - \bar{\lambda}_4^2 (\alpha_1) = \sum_{m=1}^{M-1} \bar{\lambda}_{4m} \sin \lambda m \, \alpha_1$$

$$\lambda_5^1 (\alpha_1) = \bar{\lambda}_5^2 (\alpha_1) = \sum_{m=0}^{M} \bar{\lambda}_{5m} \cos \lambda m \, \alpha_1$$

$$\lambda_6^1 (\alpha_1) = - \bar{\lambda}_6^2 (\alpha_1) = \sum_{m=0}^{M} \bar{\lambda}_{6m} \cos \lambda m \, \alpha_1 \tag{3-61}$$

It should be noted that the terms λ_K^1, λ_K^2 are subtracted from the first bracket and exactly the same terms are added to the third bracket. Therefore, by comparing Eq. (3-60) with Eq. (3-46), one finds no changes in the first energy variation. It will be seen from these expressions that the bracket of the double summation yields equilibrium equations which contain the modification fuctions. The bracket of the single summation with respect to α_2 and α_1 yields respectively the modified boundary conditions at $\alpha_1 = 0$ and M, and the modified boundary condition with the modification function at $\alpha_2 = 0$ and N.

The modification functions λ (α_1) in the previous section were used to satisfy the geometric boundary condition which was not fulfilled by the solutions assumed.

A similar use will be required of the modification functions, $\lambda_K^1 (\alpha_1)$ and $\lambda_K^2 (\alpha_2)$, $k = 1, 2, \cdots 6$. Physically, these functions relate the forces which should be applied along the boundary edges $\alpha_2 = 0$ and $\alpha_2 = N$, in order to satisfy the prescribed boundary conditions.

A study of the boundary conditions which includes the modification functions can be made by considering Eqs. (3-46) and (3-60). The results yield the condition required at $\alpha_2 = 0$, which is

$$H_k = -E_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_K} - T^e_K + \frac{\partial V\alpha_1}{\partial Y_K} - \Delta_1 \frac{\partial V\alpha_1}{\partial \nabla_1 Y_K} \tag{3-62}$$

The above expression can be rewritten as

$$H_K = \bar{H}^1_K + \bar{H}^2_K \tag{3-63}$$

where

$$\bar{H}^1_K = \frac{\partial V\alpha_1}{\partial Y_K} + \frac{\partial V\alpha_2}{\partial Y_K} - T^e_K - \Delta_1 \frac{\partial V\alpha_1}{\partial \nabla_1 Y_K} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_K} - \lambda^1_K(\alpha_1)\,\delta^0_{\alpha 2}$$

$$\bar{H}^2_K = -\frac{\partial V\alpha_2}{\partial Y_K} - \frac{\partial V\alpha_2}{\partial \nabla_2 Y_K} + \lambda^1_K(\alpha_1)\,\delta^0_{\alpha 2} \tag{3-64a, b}$$

Similar equations will be obtained at $\alpha_2 = N$. It should be noticed that $\bar{H}^1_K = 0$, Eq. (3-64a), represents the governing equation of the system. Therefore, the problem becomes that of solving this equation extended over the boundary and the modified boundary conditions prescribed as $\bar{H}^2_K$.

Since the symmetric case is being considered only the boundary conditions at $\alpha_2 = 0$ need be examined.

The modified boundary conditions with modification function, $\bar{H}^2_K$, are represented by the following equations:

$$\bar{H}^2_1 = \bar{F}^1_1 = -\frac{2c_2 c_8}{L_2}\nabla_2 u_1 - c_2 c_8 (2-\nabla_2)\,\theta_3 + \lambda^1_1(\alpha_1) = 0$$

$$\bar{H}^2_2 = \bar{F}^1_2 = -\frac{c_4 c_8}{L_2}\nabla_2 u_2 + \lambda^1_2(\alpha_1) = 0$$

$$\bar{H}^2_3 = \bar{F}^1_3 = -\frac{2c_5 c_8}{L_2}\nabla_2 u_3 + c_5 c_8 (2-\nabla_2)\,\theta_1 + \lambda^1_3(\alpha_1) = 0$$

$$\bar{H}^2_4 = M^1_1 = +c_5 c_8 \nabla_2 u_3 - \frac{c_5 c_8 L_2}{3}(3-\nabla_2)\,\theta_1 + \lambda^1_4(\alpha_1) = 0$$

$$\bar{H}^2_5 = M^1_2 = -\frac{c_7 c_8 L_1}{3}\nabla_2 \theta_2 + \lambda^1_5(\alpha_1) = 0$$

$$\bar{H}^2_6 = M^1_3 = -c_2 c_8 \nabla_2 u_1 - \frac{c_2 c_8 L_2}{3}(3-\nabla_2)\,\theta_3 + \lambda^1_6(\alpha_1) = 0 \tag{3-65a,…f}$$

Substitution of Eqs. (3-58a, … f) and (3-61a, … f) into Eqs. (3-65a, … f) yields the following expressions at α_2 = constant.

$$\sum_n \left[\frac{4c_2}{L_2}\sin\frac{\lambda n}{2} U^1_{mn} + 2c_2 \cos\frac{\lambda n}{2}\theta^3_{mn}\right] = \frac{\bar{\lambda}_{1m}}{c_8}$$

$$\bar{\lambda}_{2m} = 0$$

$$\sum_n \left[\frac{4c_5}{L_2}\sin\frac{\lambda n}{2} U^3_{mn} - 2c_5 \cos\frac{\lambda n}{2}\theta^1_{mn}\right] = \frac{\bar{\lambda}_{3m}}{c_8}$$

$$\sum_n \left[-2c_5 \sin\frac{\lambda n}{2} U^3_{mn} + c_5 L_2 \cos\frac{\lambda n}{2} \theta^1_{mn} \right] = \frac{\bar{\lambda}_{4m}}{c_8}$$

$$\sum_n \left[\frac{2c_7 L_1}{3} \sin\frac{\lambda n}{2} \theta^2_{mn} \right] = \frac{\bar{\lambda}_{5m}}{c_8}$$

$$\sum_n \left[2c_2 \sin\frac{\lambda n}{2} U^1_{mn} + c_2 L_2 \cos\frac{\lambda n}{2} \theta^3_{mn} \right] = \frac{\bar{\lambda}_{6m}}{c_8} \qquad (3\text{-}66a,\cdots f)$$

where n = 1, 3, 5, N (or N + 1) odd integer numbers only

It can be easily concluded from Eqs. (3-66a, c, d, f) that

$$\bar{\lambda}_{4m} = -\frac{L_2}{2}\bar{\lambda}_{3m}, \qquad \bar{\lambda}_{6m} = \frac{L_2}{2}\bar{\lambda}_{1mn} \qquad (3\text{-}67a, b)$$

Therefore, the foregoing conditions which will be called the constraint conditions, are reduced to the three algebraic equations summed over n. For convenience, they are written in the following matrix form:

$$\sum_n \left[C_{2n} \right] \cdot \left[Xmn \right] = \left[I \right] \left[\bar{\bar{\lambda}}m \right] \qquad (3\text{-}68)$$

where

$$\left[C_{2n} \right] = \begin{pmatrix} \frac{4}{L_2} c_2 \sin\frac{\lambda n}{2} & 0 & 0 & 0 & 0 & 2c_2\cos\frac{\lambda n}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4c_5}{L_2}\sin\frac{\lambda n}{2} & -2c_5\cos\frac{\lambda n}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2c_7L_1}{3}\sin\frac{\lambda n}{2} & 0 \end{pmatrix} \qquad (3\text{-}69)$$

$$\left[Xmn \right] = \left[U^1_{mn},\ U^2_{mn},\ U^3_{mn},\ \theta^1_{mn},\ \theta^2_{mn},\ \theta^3_{mn} \right]^T \qquad (3\text{-}70)$$

$$\left[\bar{\bar{\lambda}}m \right] = \left[\bar{\lambda}_1/c_8,\ \bar{\lambda}_2/c_8,\ \bar{\lambda}_3/c_8,\ \bar{\lambda}_5/c_8 \right]^T \qquad (3\text{-}71)$$

and $[I]$ is the unit matrix with a 4×4 dimension.

To satisfy $\delta U = 0$, Eq. (3-60), the external loads will be expanded into appropriate finite double series:

$$F^e_1(\alpha_1, \alpha_2) = \sum_m \sum_n F^1_{mn} \cos\lambda m\alpha_1 \sin\lambda n\left(\alpha_2 + \frac{1}{2}\right)$$

$$F^e_2(\alpha_1, \alpha_2) = \sum_m \sum_n F^2_{mn} \sin\lambda m\alpha_1 \cos\lambda n\left(\alpha_2 + \frac{1}{2}\right)$$

$$F_3^e(\alpha_1, \alpha_2) = \sum_m \sum_n F^3_{mn} \sin \lambda m \alpha_1 \sin \lambda n \left(\alpha_2 + \frac{1}{2}\right) \quad \text{(3-72a, b, c)}$$

The Euler coefficients, F^1_{mn}, F^2_{mn}, F^3_{mn} are obtained for arbitrarily unit joint loads and the solutions corresponding to these loads are usually known as Green's function solutions.

For the loads, one can write

$$F_1^e(\alpha_1, \alpha_2) = F_2^e(\alpha_1, \alpha_2) = F_3^e(\alpha_1, \alpha_2) = \delta^{\xi}_{\alpha 1} \delta^{\eta}_{\alpha 2} \quad \text{(3-73a, b, c)}$$

After use is made of the orthogonality properties of the trigonometric functions one obtains the coefficients as follows:

$$F^1_{mn} = \frac{2}{\Gamma_m (N+1)} \cos \lambda m \, \xi \sin \lambda n \left(\eta + \frac{1}{2}\right)$$

$$F^2_{mn} = \frac{4\psi n}{M(N+1)} \sin \lambda m \, \xi \cos \lambda n \left(\eta + \frac{1}{2}\right)$$

$$F^3_{mn} = \frac{4}{M(N+1)} \sin \lambda m \, \xi \sin \lambda n \left(\eta + \frac{1}{2}\right) \quad \text{(3-74a, b, c)}$$

where m = 1, (1), M, n = 1, 3, 5 ⋯ N (or N + 1) odd numbers only.

Γ_m is a normalization factor defined by Eq. (3-24).

ψn is a weighting function defined by

$$\psi n = 1 - \frac{1}{2} \delta^0_n \quad \text{(3-75)}$$

General orthogonality properties which were used in the above derivations are

$$\sum_{\alpha 2=0}^{N} \cos \frac{i\pi}{N+1}\left(\alpha_2 + \frac{1}{2}\right) \cos \frac{k\pi}{N+1}\left(\alpha_2 + \frac{1}{2}\right) = \frac{N+1}{2\psi_i} \delta_i^{2I(N+1) \pm K} \quad \text{(3-76)}$$

in which ψ_i is defined by Eq. (3-75) replacing n by i.

$$\sum_{\alpha 2=0}^{N} \sin \frac{i\pi}{N+1}\left(\alpha_2 + \frac{1}{2}\right) \sin \frac{k\pi}{N+1}\left(\alpha_2 + \frac{1}{2}\right) = \pm \frac{N+1}{2} \delta_i^{2I(N+1) \pm K} \quad \text{(3-77)}$$

Following the same procedure shown in the previous section, the solutions assumed for the defomations, Eqs. (3-58a, ⋯ f), the modification fuctions, Eqs. (3-61a, ⋯ f), and the external loads, Eqs. (3-72a, b, c), are substituted into Eqs. (3-59) and (3-60).

By establishing that the coefficients of δU^1_{mn}, δU^2_{mn}, δU^3_{mn}, $\delta\theta^1_{mn}$, $\delta\theta^2_{mn}$, $\delta\theta^3_{mn}$ must vanish, six simultaneous algebraic equations are obtained. They are written in the following matrix form:

$$\left[A_{mn}\right]\left[X_{mn}\right] = \psi n \left[F_{mn}\right] + \frac{4\psi n}{N+1}\left[C_{1n}\right]\left[\bar{\lambda}_m\right] \quad \text{(3-78)}$$

where $[A_{mn}]$ is listed in Table 4 in which

$$\lambda m = \frac{m\pi}{M} \cdot \lambda n = \frac{n\pi}{N+1}$$

$\left[Xmn \right]$ as defined by Eq. (3-70)

$\left[\bar{\bar{\lambda}}m \right]$ as defined by Eq. (3-71)

ψn is the weighting function defined by Eq. (3-75)

and

$$\left[C_{1n} \right] = \begin{pmatrix} \sin\frac{\lambda n}{2} & 0 & 0 & 0 \\ 0 & \cos\frac{\lambda n}{2} & 0 & 0 \\ 0 & 0 & \sin\frac{\lambda n}{2} & 0 \\ 0 & 0 & -\frac{L_2}{2}\cos\frac{\lambda n}{2} & 0 \\ 0 & 0 & 0 & \sin\frac{\lambda n}{2} \\ \frac{L_2}{2}\cos\frac{\lambda n}{2} & 0 & 0 & 0 \end{pmatrix} \tag{3-79}$$

The resultant matrix form, Eq. (3-78), relates the Euler coefficients of the deformations, $[X_{mn}]$, to the coefficients of external load, $[F_{mn}]$, and to the coefficients of the modification functions, $[\bar{\bar{\lambda}}_{mn}]$. Substituting the matrix $[X_{mn}]$ into the constraint equation, Eq. (3-68), one obtains a matrix equation from which the solution for the matrix $[\bar{\bar{\lambda}}_m]$ can be obtaineid. The solution is then substituted into Eq. (3-78), and the resultant matrix equation is solved for a new set of modified coefficients with known coefficients.

The steps to be followed are indicated below. Solving Eq. (3-78) for $[X_{mn}]$ one finds

$$\left[Xmn \right] = \psi n \left[Amn \right]^{-1} \left[Fmn \right] + \frac{4\psi n}{N+1} \left[Amn \right]^{-1} \left[C_{1n} \right] \left[\bar{\bar{\lambda}}m \right] \tag{3-80}$$

Substitution of Eq. (3-80) into Eq. (3-68) yields

$$\sum_n \left[\psi n \left[C_{2n} \right] \left[Amn \right]^{-1} \left[Fmn \right] + \frac{4\psi n}{N+1} \left[C_{2n} \right] \left[Amn \right]^{-1} \left[C_{1n} \right] \left[\bar{\bar{\lambda}}m \right] \right]$$

$$= \left[I \right] \left[\bar{\bar{\lambda}}m \right] \tag{3-81}$$

From the above result one obtains

$$\left[\bar{\bar{\lambda}}m\right] = \left[\left[I\right] - \frac{4}{N+1}\sum_n \psi n\left[C_{2n}\right]\left[Amn\right]^{-1}\left[C_{1n}\right]\right]^{-1} \times \sum_n\left[\psi n\left[C_{2n}\right]\left[Amn\right]^{-1}\left[Fmn\right]\right] \tag{3-82}$$

and

$$\left[Xmn\right] = \psi n\left[Amn\right]^{-1}\left[Fmn\right] + \frac{4\psi n}{N+1}\left[Amn\right]^{-1}\left[C_{1n}\right] \times \left[\left[I\right] - \frac{4}{N+1}\sum_n \psi n\left[C_{2n}\right]\left[Amn\right]^{-1}\left[C_{1n}\right]\right]^{-1} \times \sum_n\left[\psi n\left[C_{2n}\right]\left[Amn\right]^{-1}\left[Fmn\right]\right] \tag{3-83}$$

Numerical Example 3

As an illustration of the preceding formulas, the following two models are considered. The first model is identical to the one considered in the previous sections and is shown in Fig. 7. The second one is a smaller model in which M = 2 and N = 3 as shown in Fig. 8. It will be designated as the 2 × 3 model.

The following loading cases are illustrated: For the 4 × 5 model,

Case 1: A vertical load Po = 0.1 kip at every node except at the boundary nodes $\alpha_1 = 0$ and $\alpha_1 = M$, where Po/2 is applied.

Case 2. A vertical load Po = 0.1 kip at nodes (1, 2), (1, 3), (3, 2) and (3, 3)

For the next 2 × 3 model

A vertical load Po = 0.1 kip at nodes (1, 1) and (1, 2)

Under these symmetric loading conditions, one needs to consider only one quarter of the latticed shell. The data used here are the same as that used in the previous examples.

Computational Procedure

Eqs. (3-58a, ··· f), (3-82) and (3-83) were programmed for a digital computer and the results are listed in Table 7. The displacements, u_2 (α_1, α_2) were neglected in the calculations of the closed form solutions since they were small in comparison with the other displacements. Therefore, the procedure required a 5 × 5 matrix arithmetic. The formulation needed for the open form solutions were obtained by using Tables 1 and 2 and Eqs. (3-54a,···f). It yielded 36 simultaneous equations in the unknown deformations.

The result obtained for Case 1 indicates that each circular polygon of the latticed shell behaves similar to the others; that is, the interconnecting members do not carry any load. However, under the loading condition considered in Case 2 they distribute the external loads more efficiently.

Numerical results for both cases are listed in Table 7 and their comparisons show a good agreement between the values obtained by the two different methods. The largest displacements of $u_1(\alpha_1, \alpha_2)$, $u_3(\alpha_1, \alpha_2)$ and the rotation $\theta_2(\alpha_1, \alpha_2)$ for Case 1 are respectively $u_1(0, \alpha_2)$, $u_3(2, \alpha_2)$, $\theta_2(0, \alpha_2)$ with $\alpha_2 = 0, 1, 2$. For Case 2 the largest deformations, $u_1(\alpha_1, \alpha_2)$, $u_3(\alpha_1, \alpha_2)$, $\theta_1(\alpha_1, \alpha_2)$, $\theta_2(\alpha_1, \alpha_2)$ and $\theta_3(\alpha_1, \alpha_2)$ are respectively $u_1(0, 2)$, $u_3(2, 2)$, $\theta_1(2, 0)$, $\theta_2(0, 2)$ and $\theta_3(0, 1)$.

Similar computations were performed for the 2 × 3 model and the results are listed in Table 8. For this case, the open form methods require the solution of a 12 × 12 matrix inversion.

From the potential value of the method represented and the numerical comparisons shown, the effectiveness of the closed form solutions proposed here appears obvious.

TABLE 7 - Numerical Result (Diaphragm and Ribbed Supports)

	Case 1 Closed Form	Case 1 Open Form	Case 2 Closed Form	Case 2 Open Form	Note
u_1 (0,0)	− 2. 1053	− 2. 0993	− 0. 3032	− 0. 3027	in
u_1 (1,0)	− 1. 4800	− 1. 4842	− 0. 2138	− 0. 2141	
u_1 (0,1)	− 2. 1053	− 2. 0993	− 0. 3684	− 0. 3673	
u_1 (1,1)	− 1. 4800	− 1. 4842	− 0. 2588	− 0. 2596	
u_1 (0,2)	− 2. 1053	− 2. 0993	− 0. 4220	− 0. 4197	
u^1 (1,2)	− 1. 4800	− 1. 4842	− 0. 2943	− 0. 2960	
u_3 (1,0)	− 2. 2970	− 2. 2944	− 0. 3310	− 0. 3305	in
u_3 (2,0)	− 3. 2389	− 3. 2425	− 0. 4674	− 0. 4679	
u_3 (1,1)	− 2. 2970	− 2. 2944	− 0. 4019	− 0. 4018	
u_3 (2,1)	− 3. 2389	− 3. 2425	− 0. 5665	− 0. 5667	
u_3 (1,2)	− 2. 2970	− 2. 2944	− 0. 4595	− 0. 4615	
u_3 (2,2)	− 3. 2389	− 3. 2425	− 0. 6453	− 0. 6426	
θ_1(1,0)	0. 0	0. 0	− 0. 1759	− 0. 1758	$\times \frac{1}{150}$
θ_1(2,0)	0. 0	0. 0	− 0. 2481	− 0. 2486	
θ_1(1,1)	0. 0	0. 0	− 0. 1730	− 0. 1778	
θ_1(2,1)	0. 0	0. 0	− 0. 2422	− 0. 2357	
θ_1(1,2)	0. 0	0. 0	− 0. 0839	− 0. 0889	
θ^1(2,2)	0. 0	0. 0	− 0. 1169	− 0. 1100	
θ_2(0,0)	3. 5560	3. 5730	0. 5136	0. 5135	$\times \frac{1}{150}$
θ_2(1,0)	2. 5320	2. 5199	0. 3645	0. 3645	
θ_2(0,1)	3. 5560	3. 5730	0. 6220	0. 6275	
θ_2(1,1)	2. 5320	2. 5199	0. 4432	0. 4393	
θ_2(0,2)	3. 5560	3. 5730	0. 7066	0. 7322	
θ_3(1,2)	2. 5320	2. 5199	0. 5078	0. 4898	
θ_3(0,0)	0. 0	0. 0	0. 0456	0. 0455	$\times \frac{1}{150}$
θ_3(1,0)	0. 0	0. 0	0. 0315	0. 0319	
θ_3(0,1)	0. 0	0. 0	0. 0801	0. 0786	
θ_3(1,1)	0. 0	0. 0	0. 0541	0. 0553	
θ_3(0,2)	0. 0	0. 0	0. 0349	0. 0337	
θ_3(1,2)	0. 0	0. 0	0. 0227	0. 0237	

TABLE 8 - Numerical Result 4 (Ribbed Support)

	Closed Form	Open Form	Note
u_1 (0,0)	− 0. 0375	− 0. 0375	
u_1 (0,1)	− 0. 1629	− 0. 1629	$\times \frac{1}{15}$ (in)
8_3 (1,0)	− 0. 1395	− 0. 1395	
u_3 (1,1)	− 0. 6070	− 0. 6071	
θ_1(1,0)	− 0. 9178	− 0. 9180	
θ_1(1,1)	− 0. 4663	− 0. 4663	
θ_2(0,0)	0. 3130	0. 3130	$\times \frac{1}{1500}$
θ_2(0,1)	1. 3600	1. 3601	
θ_3(0,0)	0. 0206	0. 0211	
θ_3(0,1)	0. 0559	0. 0560	

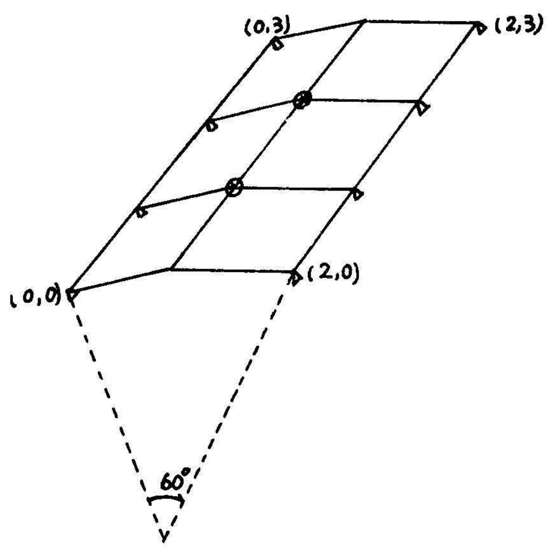

FIG. 8 2 × 3 MODEL

CHAPTER Ⅳ

STABILITY OF CIRCULAR CYLINDRICAL LATTICED SHELLS

The advantages of latticed shells have been pointed out in the previous discussions; however, very little has been said about one of the most important problems, stability. One must consider this when the members of the latticed shell are subjected to compressive forces. Because of the rigid connections between the members of the latticed shell, the deflection of one member in the buckling state causes distortion of the other members. Each member is elastically restrained by the others and the degree of restraint of any particular element depends upon the flexural rigidity of all members. Thus, the study of the stability of the latticed shell is necessary in order to obtain the actual buckling condition of the entire system or to clarify on a rational basis the role of compressed members as a part of the latticed shell rather than as isolated members.

The problems of framework stability have been treated by Bleich (3) and Timoshenko and Gere (37). The equivalent continuum method has been studied for domes by several current researches (31, 42). However, no rational analysis is available for the problem under consideration.

The following study may be the first attempt to treat the problem by discrete field mechanics, which deals with the exact mathematical model.

A similar procedure to that presented in Chapter Ⅱ will be followed to derive the governing equations. To avoid duplication only the additional terms needed to obtain the corresponding mathematical model will be presented.

Two aspects of the buckling problem are usually distinguished.

(a) Local Buckling: This case may be illustrated in a cylindrical latticed shell, by observing that a member of a circular or other type of plane polygon might buckle locally under the given loading. This would have the effect of reducing the stiffness of the latticed shell which might then buckle overall.

(b) General Buckling: Under this type of buckling the latticed shell might itself buckle before the members buckle locally.

This work considers only the general buckling of a latticed shell under a constant joint load applied in the normal direction.

Ⅳ. 1 DERIVATION OF THE EQUILIBRIUM EQUATIONS

Due to the influence of the axial forces in the members, the force-deformation relations, Eqs. (2-8a, b) must be modified. The new expression takes the form

$$F_2^R = -F_2^L = b_1 \frac{EI_3}{L_1} \frac{\gamma_1}{L_1} \left\{ (\nabla_\alpha - 2)\, \theta_3^R + \frac{2}{L_1} \nabla_\alpha\, u_2^R \right\} + \frac{EA_1}{L_1^2} (\nabla_\alpha\, u_1^R)^2 \qquad (4\text{-}1)$$

The other force-deformation relations are the same as those given by Eq. (2-4) through Eq. (2-9). Following the same procedure shown in Chapter Ⅱ. 1 the additional strain energy due to the axial force is obtained. It is

$$V\alpha_1,\ \text{axial} = \frac{AE}{2L_1^2}\left\{ (KM_1 u_1 - A\nabla_1 u_3)^2 (A\nabla_1 u_1 + KM_1 u_3 \right\} \tag{4-2}$$

Thus, the strain energy stored in a typical member of the α_1-polygon is

$$\bar{V}\alpha_1 = V\alpha_1 + V\alpha_1,\ \text{axial} \tag{4-3}$$

where $V\alpha_1$ is defined by Eq. (2-21).

The total potential energy of a cylindrical latticed shell shown in Fig. 3, is obtained by adding the total strain energy of the parametric polygons, V, and the potential enery due to external load, W, as follows:

$$U = V + W \tag{4-4}$$

where

$$V = \sum_{\alpha 1=1}^{M} \sum_{\alpha 2=1}^{N-1} \bar{V}\alpha_1 + \sum_{\alpha 1=1}^{M-1} \sum_{\alpha 2=1}^{N} V\alpha_2 \tag{4-5}$$

$$W = \sum_{\alpha 1=1}^{M-1} \sum_{\alpha 2=1}^{N-1} W(\alpha_1, \alpha_2) + \sum_{\alpha 2=1}^{N-1} W(\alpha_1, \alpha_2) \Bigg|_{\substack{\alpha_1=0 \\ \alpha_1=M}}$$

$$+ \sum_{\alpha 1=1}^{M-1} W(\alpha_1, \alpha_2) \Bigg|_{\substack{\alpha_2=0 \\ \alpha_2=N}} \tag{4-6}$$

$$W(\alpha_1, \alpha_2) = -F_3^e \cdot \bar{N} \tag{4-7}$$

Applying the theorem of the calculus of variations in discrete field mechanics given in Appendix A, one obtains the necessary condition for U, given by Eq. (4-4), to be stationary. This is

$$\delta U = \Delta_1^{-1} \Delta_2^{-1} \left[\frac{\partial \bar{V}\alpha_1}{\partial Y_K} + \frac{\partial V\alpha_2}{\partial Y_K} - \delta_K^3 T_K^e - \Delta_1 \frac{\partial V\alpha_1}{\partial \nabla_1 Y_K} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_K} \right] \epsilon_K h_K \Bigg|_{\alpha_1=1}^{M} \Bigg|_{\alpha_2=1}^{N}$$

$$+ \Delta_2^{-1} \left[\left\{ \frac{\partial \bar{V}\alpha_1}{\partial Y_K} - \frac{\partial \bar{V}\alpha_1}{\partial \nabla_1 Y_K} - \delta_K^3 T_K^e \right\}_{\alpha_1=M} \epsilon_K h_K (M, \alpha_2) \right.$$

$$\left. - \left\{ E_1 \frac{\partial \bar{V}\alpha_1}{\partial \nabla_1 Y_K} + \delta_K^3 T_K^e \right\}_{\alpha_1=0} \epsilon_K h_K (0, \alpha_2) \right]_{\alpha=1}^{M}$$

$$+ \Delta_1^{-1} \left[\left\{ \frac{\partial V\alpha_2}{\partial Y_K} + \frac{\partial V\alpha_2}{\partial \nabla_2 Y_K} - \delta_K^3 T_K^e \right\}_{\alpha_2=N} \epsilon_K h_K (\alpha_1, N) \right.$$

$$\left. - \left\{ E_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_K} + \delta_K^3 T_K^e \right\}_{\alpha_2=0} \epsilon_K h_K (\alpha_1, 0) \right]_{\alpha_1=0}^{M} = 0 \tag{4-8}$$

It will be assumed that members of the end polygons at $\alpha_2 = 0$, $\alpha_2 = N$ and those of end spans at $\alpha_1 = 0$ and $\alpha_1 = M$ have a flexural stiffness which is half those of the corresponding interior members.

For diaphragm type boundary supports, the first energy variation, Eq. (4-8), is written as

$$
\begin{aligned}
\delta U = {} & w_1\, w_2\, \Delta_1^{-1}\, \Delta_2^{-1} \left\{ \frac{\partial \bar{V}\alpha_1}{\partial Y_k} + \frac{\partial V\alpha_2}{\partial Y_k} - \delta_k^3\, T_k^e - \Delta_1 \frac{\partial \bar{V}\alpha_1}{\partial \nabla_1 Y_k} - \Delta_2 \frac{\partial V\alpha_2}{\partial \nabla_2 Y_k} \right\} \\
& \times \epsilon_k\, h_k \Big|_0^{M+1} \Big|_0^{N+1} \\
& + \frac{w_2}{2} \Delta_2^{-1} \left[(\delta_{\alpha 1}^0 + \delta_{\alpha 1}^M) \left\{ \frac{\partial \bar{V}\alpha_1}{\partial Y_k} + (\Delta_1 + 2) \frac{\partial \bar{V}\alpha_1}{\partial \nabla_1 Y_k} \right\} \epsilon_k\, h_k \Big|_0^{N+1} \right. \\
& + \frac{w_1}{2} \Delta_1^{-1} \left[(\delta_{\alpha 2}^0 + \delta_{\alpha 2}^N) \left\{ \frac{\partial V\alpha_2}{\partial Y_k} + (\Delta_2 + 2) \frac{\partial V\alpha_2}{\partial \nabla_2 Y_k} \right\} \epsilon_k\, h_k \Big|_0^{M+1} = 0 \right.
\end{aligned}
\tag{4-9}
$$

It should be observed that Eq. (4-9) includes the equilibrium conditions valid at all nodes as well as the modified boundary conditions.

Substitution of Eqs. (2-4), (2-22), (4-2) and (4-7) into Eq. (4-9) yields the expression for the term δU, which appears in the double summation of Eq. (4-9). It takes the following form:

$$
\begin{aligned}
\delta U_1 = {} & w_1\, w_2\, \Delta_1^{-1}\, \Delta_2^{-1} \left[\left[c_8 \left\{ \frac{K^2}{2L_1} (\varDelta_1 + 4) - c_1 \frac{A_2}{L_1} \varDelta_1 - c_2 \frac{2}{L_2} \varDelta_2 \right\} u_1 \right.\right. \\
& - \frac{c_8 AK}{2L_1} (2 + c_1)\, \bar{\varDelta}_1\, u_3 - \frac{c_8 K}{2} (\varDelta_1 + 4)\, \theta_2 \\
& \left. + \frac{A_1 E}{L_1{}^2} (A\, \nabla_1\, u_1 + KM_1\, u_3) \left\{ \frac{K^2}{4} (\varDelta_1 + 4)\, u_1 - \frac{AK}{2} \bar{\varDelta}_1\, u_3 \right\} \right] \epsilon_1\, h_1 \\
& + \left[\frac{c_8 AK}{2L_1} (2 + c_1)\, \bar{\varDelta}_1\, u_1 + c_8 \left\{ \frac{2A^2}{L_1} \varDelta_1 + \frac{c_1 K^2}{4L_1} (\varDelta_1 + 4) - \frac{2c_5}{L_2} \varDelta_2 \right\} u_3 \right. \\
& + c_5\, c_8\, \bar{\varDelta}_2\, \theta_1 - c_8\, A\, \bar{\varDelta}_1\, \theta_2 - F_3^e + \frac{A_1 E}{L_1^2} (A\, \nabla_1\, u_1 + KM\, u_3) \\
& \left. \times \left(\frac{AK}{2} \bar{\varDelta}_1\, u_1 - A^2\, \varDelta_1\, u_3 \right) \right] \epsilon_3\, h_3 \\
& + \left[- c_5\, c_8\, \bar{\varDelta}_2\, u_3 + c_8 \left\{ - \frac{c_3 K^2 L_1}{4\gamma_1} (\varDelta_1 + 4 - 2\bar{\gamma}_1) \right.\right. \\
& \left.\left. - \frac{c_6 A^2 L_1}{\gamma_1} \varDelta_1 \frac{c_5 L_2}{\gamma_2} (\varDelta_2 + 2\gamma_2) \right\} \theta_1 \right] \epsilon_4\, h_4 + \left[- \frac{c_8 K}{2} (\varDelta_1 + 4)\, u_1 + c_8\, A \bar{\varDelta}_1\, u_3 \right. \\
& \left. \left. + \frac{c_8 L_1}{\gamma_1} \left\{ (\varDelta_1 + 2\gamma_1) - c_7\, \varDelta_2 \right\} \theta_2 \right] \epsilon_5\, h_5 \right] \Big|_{\alpha_1 = 0}^{M+1} \Big|_{\alpha_2 = 0}^{N+1} = 0
\end{aligned}
\tag{4-10}
$$

where the deformations u_2 and θ_3 have been neglected since they are very small for the type of loading and supports considered. It should be pointed out that the above equation involves non-linear terms which arise from the interaction of the axial forces. This author will not solve this non-linear equation but will use this to obtain the buckling condition.

Buckling Condition

The prebuckling (equilibrium prior to buckling) and the buckling conditions are obtained using Eq. (4-10). The total deformations during buckling, $u_1, u_3, \theta_1, \theta_2$ are assumed to be separable as

$$W_K = W_{KA} + W_{KB} \qquad K = 1, 2, 3, 4 \tag{4-11}$$

where W_K represents the deformations, K takes respectively the values 1, 2, 3, 4 W_{KA} designates the deformations which appear prior to buckling, and W_{KB} defines the infinitesimal deformations which appear during buckling.

Since the latticed shell must be in equilibrium during buckling, the total potential energy must be stationary. Using Eq. (4-11) the deformations appearing in Eq. (4-10) are replaced by terms with the subscripts A and B, However, since the latticed shell must also be in equilibrium prior to the buckling state, the condition, Eq. (4-10), obtained as a result of replacing the terms W_K by W_{KA} must also be satisfied. Subtracting the second condition, $\delta U\ (W_{KA}) = 0$, from the first condition, $\delta U\ (W_{KA} + W_{KB}) = 0$, one obtains the buckling equations which must be satisfied when a state of buckling is reached.

Following the same notations used for the deformations, the axial force in a member is written as

$$F(\alpha_1, \alpha_2) = F_A(\alpha_1, \alpha_2) + F_B(\alpha_1, \alpha_2) \tag{4-12}$$

where $F_A(\alpha_1, \alpha_2)$ is the axial force prior to buckling and $F_B(\alpha_1, \alpha_2)$ denotes the axial force during buckling.

The following assumptions are made for this problem; the members considered are prismatic and the effect of axial force upon the bending moments are ignored. Therefore, the coefficients used in the force-deformation relations, Eq. (2-4) through Eq. (2-9) are:

For the prebuckling condition

$$b_1 = \bar{b}_1 = b_2 = \bar{b}_2 = 2, \qquad \gamma_1 = \bar{\gamma}_1 = \gamma_2 = \bar{\gamma}_2 = 3$$

For the buckling condition

$$\bar{b}_1 = b_2 = \bar{b}_2 = 2 \qquad \bar{\gamma}_1 = \gamma_2 = \bar{\gamma}_2 = 3$$

The coefficients b_1 and γ_1 are defined by Eq. (2-10a, b) and (2-11).

It will be noticed that the prebuckling condition is exactly the same as the equation obtained in Chapter Ⅲ. 1, if we retain only linear terms. This equation will not be duplicated here.

The governing equation for buckling condition is obtained by satisfying the following requirement:

$$\delta U_1\,(W_{KB}) = \delta U_1\,(W_{KA} + W_{KB}) - \delta U_1\,(W_{KA}) \tag{4-13}$$

Retaining only the linear terms in the bubkling deformation, W_{KB}, the following equation is obtained:

$$\begin{aligned}
\delta U_1\,(W_{KB}) = {} & w_1\, w_2\, \Delta_1^{-1}\, \Delta_2^{-1} \Bigg[\Bigg[c_8 \left\{ \frac{K^2}{2L_1} (\not\nabla_1 + 4) - \frac{c_1 A^2}{L_1} \not\nabla_1 \right. \\
& \left. - \frac{2c_2}{L_2} \not\nabla_2 \right\} u_{1B} - \frac{c_8 AK}{2L_1} (2 + c_1)\, \not\Delta_1\, u_{3B} - \frac{c_8 K}{2} \\
& \times (\not\nabla_1 + 4)\, \theta_{2B} + F_A \left\{ \frac{K^2}{4L_1} (\not\nabla_1 + 4)\, u_{1B} - \frac{AK}{2L_1} \not\nabla_1\, u_{3B} \right\} \Bigg] \epsilon_1\, h_1 \\
& + \Bigg[\frac{c_8 AK}{2L_1} (2 + c_1)\, \not\Delta_1\, u_{1B} + c_8 \left\{ \frac{2 A^2}{L_1} \not\nabla_1 + \frac{c_1 K^2}{4L_1} (\not\nabla_1 + 4) \right. \\
& \left. - \frac{2c_5}{L_2} \not\nabla_2 \right\} u_{3B} - c_5\, c_8\, \not\Delta_2\, \theta_{1B} - c_8\, A\, \not\Delta_1\, \theta_{2B} \\
& + F_A \left(\frac{AK}{2} \not\Delta_1\, u_{1B} - A^2 \not\nabla_1\, u_{3B} \right] \epsilon_3\, h_3 \\
& + \Bigg[- c_5\, c_8\, \not\Delta_2\, u_{3B} + c_8 \left\{ - \frac{c_3 K^2 L_1}{4\gamma_1} (\not\nabla_1 + 4 - 2\bar{\gamma}_1) \right. \\
& \left. - \frac{c_6 A^2 L_1}{\gamma_1} \not\nabla_1\, \frac{c_5 L_2}{\gamma_2} (\not\nabla_2 + 2\gamma_2) \right\} \theta_{2B} \Bigg] \epsilon_4\, h_4 \\
& + \Bigg[- \frac{c_8 K}{2} (\not\nabla_1 + 4)\, u_{1B} + c_8\, A\, \not\Delta_1\, u_{3B} \\
& + \frac{c_8 L_1}{\gamma_1} \Big\{ (\not\nabla_1 + 2\gamma_1) - c_7 \not\nabla_2 \Big\}\, \theta_{2B} \Bigg] \epsilon_5\, h_5 \Bigg|_{\alpha_1 = 0}^{M+1} \Bigg|_{\alpha_2 = 0}^{N+1} = 0
\end{aligned} \tag{4-14}$$

The term, $\frac{A_1 E}{L_1}$ $(A\, \nabla_1\, u_1\, A + KM_1\, u_3\, A)$, which appeard in obtaining Eq. (4-14) and defines the axial force prior to buckling, is assumed to be constant and written as F_A in Eq. (4-14).

Following the procedure explained, the modified boundary condition for the diaphragm supports similar to Eqs. (3-15) and (3-17) will be obtained.

Ⅳ. 2 DOUBLE FINITE SERIES SOLUTIONS

To obtain solutions for the buckling state which satisfy the diaphragm boundary conditions, the buckling deformations are taken as

$$u_{1B}(\alpha_1, \alpha_2) = \sum_{m=0}^{M} \sum_{n=1}^{N-1} U^1_{mn} \cos \lambda m \alpha_1 \sin \lambda n \alpha_2$$

$$u_{3B}(\alpha_1, \alpha_2) = \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} U^3_{mn} \sin \lambda m \alpha_1 \sin \lambda n \alpha_2$$

$$\theta_{1B}(\alpha_1, \alpha_2) = \sum_{m=1}^{M-1} \sum_{n=0}^{N} \theta^1_{mn} \sin \lambda m \alpha_1 \cos \lambda n \alpha_2$$

$$\theta_{2B}(\alpha_1, \alpha_2) = \sum_{m=0}^{M} \sum_{n=1}^{N-1} \theta^2_{mn} \cos \lambda m \alpha_1 \sin \lambda n \alpha_2$$

where $\lambda m = \frac{m\pi}{M}$ and $\lambda n = \frac{n\pi}{N}$ (4-15a, ··· d)

Substitution of Eqs. (4-15a,···d) into Eq. (4-14) yields four simultaneous algebraic equations which give the criteria for buckling. Since these equations are homogeneous, the feasibility of a solution is expressed by the well-known condition.

$$\begin{vmatrix} A_1 & A_2 & 0 & A_3 \\ A_2 & A_4 & A_5 & A_6 \\ 0 & A_5 & A_7 & 0 \\ A_3 & A_6 & 0 & A_8 \end{vmatrix} = 0 \qquad (4\text{-}16)$$

where

$$A_1 = -\frac{K^2}{L_1}\gamma(\phi) D_1 \qquad A_2 = \frac{AK}{L_1}\gamma(\phi) D_2 \qquad A_3 = -KD_1$$

$$A_4 = \frac{4A^2}{L_1}\gamma(\phi) D_3 - \frac{4c_5}{L_2} D_4, \quad A_5 = -c_5 D_5, \quad A_6 = AD_2$$

$$A_7 = \frac{2c_5 L_2}{3} D_6, \qquad A_8 = \frac{2L_1}{\gamma_1}(D_3 + \gamma_1)$$

$$D_1 = 1 + \cos \lambda m, \quad D_2 = \sin \lambda m, \quad D_3 = \cos \lambda m - 1 \qquad (4\text{-}17a, \cdots t)$$

$$D_4 = \cos \lambda n - 1, \quad D_5 = \sin \lambda n, \quad D_6 = 2 + \cos \lambda n$$

$$\ell = \frac{L_2}{L_1}, \qquad k = \frac{\bar{I}_3}{I_3}, \qquad \gamma(\phi) = \frac{\phi^2}{2\gamma_1 b_1} - 1$$

$$\phi^2 = \frac{N^0 L_1^2}{EI_3}, \qquad c_5 = \ell . k, \qquad N^0 = -F_A$$

The expansion of the determinant, Eq. (4-16), yields the following transcendental equation as follows:

$$B_{.}\ \gamma^2(\phi) + B_2\,\gamma^2(\phi)\,\gamma_1 + B_3\,\gamma(\phi) + B_4\,\gamma(\phi)\,\gamma_1 + B_5\,\gamma_1 = 0 \qquad (4\text{-}18)$$

where

$$\gamma(\phi) = \frac{\phi^2}{2\gamma_1 b_1} - 1, \qquad \gamma_1 = \frac{\phi(1-\cos\phi)}{\phi - \sin\phi} \qquad b_1 = \frac{\phi\csc\phi - 1}{\frac{2}{\phi}\tan\frac{\phi}{2} - 1} \qquad (4\text{-}19a, b, c)$$

$$B_1 = -4D_3\,D_6\,A^2\left(4D_3\,\ell + \frac{D_2^2}{D_1}\right)$$

$$B_2 = -4A^2\,D_6\left(4D_3\,\ell + \frac{D_2^2}{D_1}\right)$$

$$B_3 = 2c_5\,D_3\left(8D_4\,D_6 + 3D_5^2\right)$$

$$B_4 = 2A^2\,D_6\,\ell\left(\frac{8c_5\,D_4}{A^2\,\ell} - 2D_2^2 + \frac{3c_5\,D_5^2}{A^2\,D_6\,\ell} + D_2^2 - 4D_1\,D_3\right)$$

$$B_5 = c_5\,D_1\left(8\,D_4\,D_6 + 3D_5^2\right) \qquad (4\text{-}20a, \cdots, e)$$

The functions, $\gamma(\phi)$, γ_1, b_1 appearing in the transcendental equation depend on ϕ which is a function of the parameters m, n, 1, and k. The parameter 1 defines the ratio between the length of a member of a generator, L_2, to the length of a member of a circular polygon, L_1. The parameter K defines the ratio between the moment of inertia of a member of a circular polygon, I_3, to that of a generator, $\bar{I}_3$.

The transcendental equation Eq. (4-18) can be solved directly by a digital computer but some modifications were made in order to use the subprogram provided by the computer center at the University of Delaware.

The series expansion for the transcendental functions b_1, γ_1, used are as follows (13)

$$b_1 = 2\left(1 + \frac{\phi^2}{60} + \frac{13\phi^4}{25,200} + \cdots\cdots\right)$$

$$\gamma_1 = 3\left(1 - \frac{\phi^2}{30} - \frac{\phi^4}{12,600} - \cdots\cdots\right) \qquad (4\text{-}21a, b)$$

Substitution of the first two terms of the above series into Eq. (4-18) yields a polynomial of the third degree in the parameter $\bar{\phi}$

$$\frac{1}{250}\left(-36\,B_2 + 6\,B_4 - B_5\right)\bar{\phi}^3$$

$$+\frac{1}{25}\left(36\,B_1 + 180\,B_2 - 6\,B_3 - 63\,B_4 + 15\,B_5\right)\bar{\phi}^2$$

$$+\frac{1}{5}\left(144\,B_1 - 504\,B_2 + 84\,B_3 + 324\,B_4 - 124\,B_5\right)\bar{\phi}$$

$$+ 144 \, (B_1 + 3\,B_2 - B_3 - 3\,B_4 + 3\,B_5\,) = 0 \tag{4-22}$$

where $$\bar{\phi} = \phi^2 = \frac{N^0 \, L_1^2}{E \, I_3} \tag{4-23}$$

Solving the above equation, Eq. (4-22), for $\bar{\phi}$, the buckling load of a member can be calculated as follows:

$$N^0 = \bar{\phi} \, \frac{E \, I_3}{L_1^2} \tag{4-24}$$

The nondimensional parameter $\bar{\phi}$ is calculated for all values of m, n, k, 1. They are shown in the numerical example which follows.

Once the buckling load of a member is known, the applied load in the direction opposite to the latticed shell normal, P, is obtained from the relation

$$P = 2N^0/K \tag{4-25}$$

where K represents the curvature of the circular polygon defined by $K = 2\sin \psi_1$.

When $\bar{\phi}$ is π^2, N^0 represents the Euler load for a pinned-end column. As seen in the numerical example, the values obtained for $\bar{\phi}$ are always smaller than π^2, depending upon the geometry, K, and the member properties, 1, k.

Numerical Example 4

The following numerical calculations have been performed to illustrate the buckling problem of a circular cylindrical latticed shell.

Consider a 3 × 4 model, that is, M = 3 and N = 4. As discussed previously, the geometric properties of the latticed shell, the ratio of the members length, $1 = \frac{L_2}{L_1}$, and the ratio of the moment of inertia of the two types of members, $k = \frac{3}{I_3}$, are important factors to be considered. The data used in the numerical example are:

$A = \cos \psi \alpha_1 = \cos \frac{\pi}{12} = 0.9659$

$1 = 1.0,\ 2.0,\ 3.0,\ 4.0,\ 5.0$

$k = 0.25,\ 0.50,\ 0.75,\ 1.00,\ 2.00,\ 3.00,\ 4.00,\ 5.00,\ 10.00,\ 100.00$

m = number of half waves in α_1 – direction = 1, 2, 3

n = number of half waves in α_2 – direction = 1, 2, 3, 4

Computational Procedure

Eq. (4-23) is solved for all combinations of m, n and the parameters described above. The minimum value of $\bar{\phi}$ obtained is listed in Table 9. It is interesting to note that the buckling load occurs when m = 3 and n = 1 for most of the values of k and 1, with the buckling mode depending upon the values of k and 1. It is observed that k is decreased, that is, when the flexural rigidity of the members of the α_2 - polygon (straight generator) becomes weak in bending, the latticed shell tends to buckle with m = 1 and n = 1. On the other hand, by increasing the values up to k = 100.00 (computed only for the theoretical interest) the buckling mode becomes m = 1 and n = 4. However, this is not the only factor determines the buckling mode, as it can be shown that value of 1 has also considerable influence. For example, for 1 = 2.0 and k = 10.00, the buckling load occurred when m = 3 and n = 1.

The results of the calculations can be presented in a graph in which the abscissas represent the k values and the ordinates the buckling parameter $\bar{\phi}$. Then, for each value of 1, a line is obtained. Several lines of this type are shown in Fig. 9. It is seen that for smaller values of 1, the buckling parameter $\bar{\phi}$ increases rapidly when the value of k increases. For large values of 1, the buckling parameter $\bar{\phi}$ increases slightly.

As an illustration of the results shown in Table 9, consider the case of k = 1.00 and 1=1. 0. One obtains $\bar{\phi}$ = 1. 2281 from Table 9. If this value is substituted into Eqs. (4-24) and (4-25), one obtains

$$P = 4.7450 \frac{EI_3}{L_1{}^2}$$

which representsthe buckling load of the latticed shell under the conditions considered.

TABLE 9 - Buckling Parameter For 3 × 4 Model

k \ 1	1.0	2.0	3.0	4.0	5.0	Note
0.25	*0.9769	0.9435	0.9355	0.9335	0.9328	
0.50	1.1106	0.9550	0.9389	0.9349	0.9335	
0.75	1.1968	0.9663	0.9423	0.9364	0.9343	M= 3
1.00	1.2281	0.9777	0.9457	0.9378	0.9350	N = 1
2.00	1.6007	1.0228	0.9592	0.9435	0.9379	except
3.00	1.8944	1.0675	0.9727	0.9493	0.9409	*M=1, N=1
4.00	2.1657	1.1116	0.9862	0.9550	0.9438	**M=1, N=4
5.00	2.4171	1.1551	0.9996	0.9607	0.9468	
10.00	**2.5893	1.3659	1.0661	0.9892	0.9614	
100.00	**2.6361	**2.6061	2.0941	1.4684	1.2164	

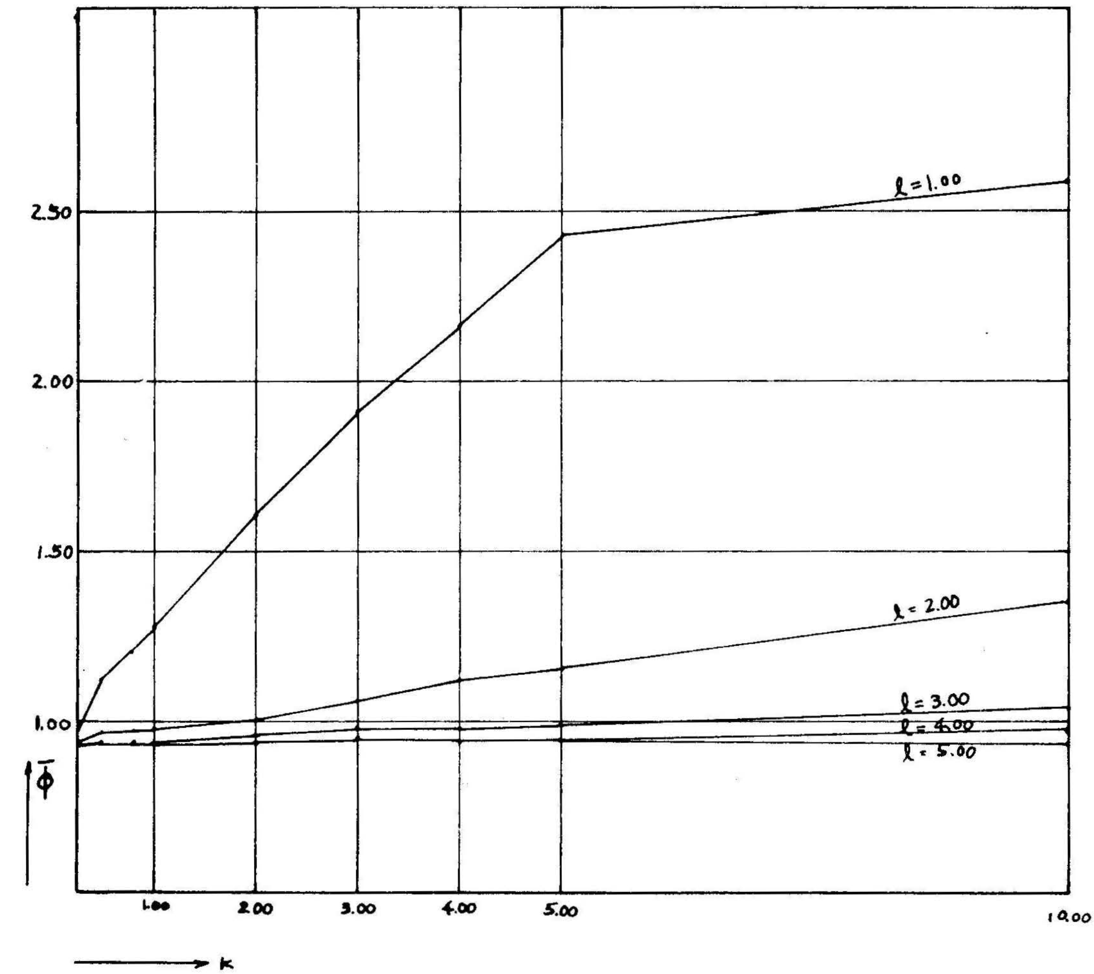

FIG. 9 BUCKLING PARAMETER $\bar{\Phi}$

CHAPTER V

CONCLUSIONS

In the preceding chapters the application of the concepts of difference geometry, calculus of finite difference, and discrete variational calculus has proved useful in formulating an adequate and efficient mathematical model for latticed shells. Indeed, the application of the calculus of variations enables one to find closed form solutions for cases in which the mathematical model and its corresponding solution appears intractable. The technique has also proven very effective in the stability analysis of latticed structures.

In Chapter II the energy formulation for the flexural analysis of cylindrical latticed shells has been presented in detail. The total potential energy has been formulated as a function of the discrete variables and reflects the geometrical properties of the latticed shell. The concepts of the discrete variational calculus developed in Appendix A were then utilized to obtain the first energy variation which is the condition to be satisfied for a state of stable equilibrium. Even though the energy formulations were used directly to obtain the solutions which appear in the next chapter, the equilibrium equations and the natural boundary conditions derived from the variational technique were shown and discussed in detail. These equations were utilized in expanded matrix form to solve the latticed shells by open form methods.

Applications of the first energy variation to perform a flexural analysis of circular cylindrical latticed shells with various boundary conditions were presented in Chapter Ⅲ. The technique proposed here proved a powerful tool in obtaining closed form solutions to difficult problems and in providing a clear insight into the behavior of the mathematical model.

Closed form solutions were obtained for the following three cases:

(1) Diaphragm supports at all four edges.

(2) Diaphragm supports at the edges, α_1 = constant, and diaphragm supports with rotational constraints at the edges, α_2 = constant.

(3) Diaphragm supports at two edges and ribbed polygonal supports at the other two edges.

The comparison of numerical results calculated by closed and open form methods shows that the two results are identical, even though the size of the matrix used for these methods is quite different. The results may not have the close agreement for latticed shells with large numbers of nodes as the error from the open form methods may increase considerably.

In Chapter Ⅳ the stability of circular cylindrical latticed shells was presented. Because of the rigid connectivity, each member is elastically restrained by the type of connector provided and by the flexural rigidity of the other members. The energy techniques derived in Appendix A were directly utilized to obtain the buckling load of the system. A numerical example is illustrated which shows the influence of the various parameters on the buckling loads and mode shape of failure. The interaction of these factors is reflected on the factor, $\bar{\phi}$, which must be considered on design of latticed shells.

In Appendix A the calculus of variations in continuum mechanics has been modified, yielding the formulation of the fundamental theorem of calculus of variations in discrete field mechanics. The usefulness of this theorem has been described in the preceding chapters.

Two illustrative examples, the analysis of a continuous beam with spring boundaries and that of a cable net with boundary deflections were presented to show specific applications of the theorem in obtaining closed form solutions for one-dimensional and two-dimensional structures.

Two distinctive types of field analysis (12), the Micro and the Macro Approach, have been defined in discrete field mechanics. The method proposed here may be considered as the application of energy methods to the Micro Approach. The author believes that similar energy formulations can be obtained for the Macro Approach.

A modified discrete variational method, similar to the method of Lagrange multipliers in continuum mechanics, has also been presented in the appendix. This method has proven especially useful in obtaining closed form solutions to problems for which this type of solution appears very cumbersome or intractable, and has simplified the procedure of obtaining the solutions of a structure for which the analysis for other types of boundary conditions is known.

Through the numerical computation of the problems considered in this dissertation, the author feels that the method proposed here is practical, more accurate and less time consuming than the methods in use.

It is hoped that the present work can be effectively extended to solve other types of discrete systems.

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APPENDIX A

CALCULUS OF VARIATION IN DISCRETE FIELD MECHANICS

The calculus of variations has been until recently a branch of modern mathematics closely related to the theory of differential equations, which has been successfully applied in continuum mechanics to solve various problems in statics and dynamics.

The objective of this appendix is to transform the theory used in continuum mechanics to one applicable to discrete field mechanics and to establish a mathematical model, two dimensional difference equations, for latticed shells.

(A) One-dimensional Case

It has been shown in mechanics that a stable equilibrium configuration is reached when the total potential energy of the system is stationary. When the corresponding necessary conditions to reach such a state are applied one may be able to obtain the governing equations and the associated natural boundary conditions. Therefore, for the equilibrium state of one-dimensional system with an unknown deformation, $Y(\alpha)$, as function of a discrete variable, α, the problem reduced to that of finding this function from the stationary potential energy.

Let the total potential energy take the form

$$U(Y) = \sum_{\alpha=1}^{N} F(\alpha, Y, \nabla Y, \nabla^2 Y) = \Delta^{-1} F(\alpha, Y, \nabla Y, \nabla^2 Y) \Big|_{1}^{N+1} \tag{A-1}$$

where Y, ∇Y, $\nabla^2 Y$ are functions of the discrete variable α, and ∇ and Δ^{-1} are respectively the standard backward and inverse difference operators defined as follows:

$$\nabla Y(\alpha) = Y(\alpha) - Y(\alpha - 1)$$

$$\nabla^r Y(\alpha) = \nabla (\nabla^{r-1} Y(\alpha) \qquad \text{(A-2a, b)}$$

$$\sum_{\alpha=1}^{\alpha=N} Y(\alpha) = \Delta^{-1} Y(\alpha) \Big|_{\alpha=1}^{\alpha=N+1} \tag{A-3}$$

It is assumed that $Y = Y(\alpha)$ is the discrete function which will make Eq. (A-1) stationary.

Let $\epsilon h(\alpha)$ be an arbitrary but discrete function defined in the interval $o \leqslant \alpha \leqslant N$, where α takes only integer values and ϵ be a smaller number.

A new function can then be defined as follows:

$$\bar{Y}(\alpha) = Y(\alpha) + \epsilon h(\alpha) \tag{A-4}$$

The second term in Eq. (A-4), $\epsilon h(\alpha)$, will be designated as the variation of $Y(\alpha)$. If ϵ is taken sufficiently small such that $\epsilon h(\alpha)$ remains below a small quantity for all integer values of α, the new function, $\bar{Y}$, will lie in the close neighborhood of Y. The definite sum,

$$U(\bar{Y}) = U[Y + \epsilon h(\alpha)] \tag{A-5}$$

becomes a continuous function, $U(\epsilon)$, of the parameter ϵ, and this function will coincide with this stationary value sought when $\epsilon = 0$. With reference to Eq. (A-1) one can write

$$U(\bar{Y}) = U(\epsilon) = \Delta^{-1} F(\alpha, Y + \epsilon h, \nabla Y + \epsilon \nabla Y, \nabla^2 Y + \epsilon \nabla^2 Y) \Big|_{\alpha=1}^{\alpha=N+1} \tag{A-6}$$

Since Eq. (A-6) is a continuous function of the parameter ϵ, the necessary condition that $U(\epsilon)$ be stationary is obtained as

$$\frac{\partial U(\epsilon)}{\partial \epsilon} \Big|_{\epsilon=0} = 0 \tag{A-7}$$

Performing the differentiation indicated by Eq. (A-7) on Eq. (A-6) according to the familiar rules of the differential calculus, one obtains

$$\frac{\partial U(\epsilon)}{\partial \epsilon} \Big|_{\epsilon=0} = \Delta^{-1} \left(\frac{\partial F}{\partial \bar{Y}} \frac{\partial \bar{Y}}{\partial \epsilon} + \frac{\partial F}{\partial \nabla \bar{Y}} \frac{\partial \nabla \bar{Y}}{\partial \epsilon} + \frac{\partial F}{\partial \nabla^2 \bar{Y}} \frac{\partial \nabla^2 \bar{Y}}{\partial \epsilon} \right) \Big|_{\alpha=1}^{N+1} \Big|_{\epsilon=0}$$

$$= \Delta^{-1} \left(\frac{\partial F}{\partial Y} h + \frac{\partial F}{\partial \nabla Y} \nabla h + \frac{\partial F}{\partial \nabla^2 Y} \nabla^2 Y \right) \Big|_{\alpha=1}^{N+1} \Big|_{\epsilon=0} = 0 \tag{A-8}$$

Applying the techniques of the summation by parts of the calculus of finite differences (22), Eq. (A-8) is written as

$$\Delta^{-1} \left[\left\{ \frac{\partial F}{\partial Y} - \Delta \left(\frac{\partial F}{\partial \nabla Y} + \Delta^2 \left(\frac{\partial F}{\partial \nabla^2 Y} \right) \right\} h + \left\{ \frac{\partial F}{\partial \nabla Y} - \Delta \left(\frac{\partial F}{\partial \nabla^2 Y} \right) \right\} E^{-1} h \right.$$

$$\left. + \frac{\partial F}{\partial \nabla^2 Y} E^{-1} \nabla h \right]_{\alpha=1}^{N+1} = 0 \tag{A-9}$$

In which E^{-1} and Δ represent respectively the Boole's displacement operator and the first forward difference operator, defined as follows:

$$E^{-1} f(\alpha) = f(\alpha - 1)$$
$$\Delta f(\alpha) = f(\alpha + 1) - f(\alpha) \qquad \text{(A-10)}$$

Multiplying Eq. (A-9) by ϵ and rearranging the limits of the summation, Eq. (A-9) can be rewritten in the form

$$\Delta^{-1}\left[\left\{\frac{\partial F}{\partial Y} - \Delta\left(\frac{\partial F}{\partial \nabla Y}\right) - \Delta^2\left(\frac{\partial F}{\partial \nabla^2 Y}\right)\right\}\epsilon h(\alpha)\right]_{\alpha=1}^{\alpha=N}$$
$$+\left[\left\{\frac{\partial F}{\partial Y} + \frac{\partial F}{\partial \nabla Y} - \Delta\left(\frac{\partial F}{\partial \nabla^2 Y}\right)\right\}\epsilon h(\alpha) + \left(E\frac{\partial F}{\partial \nabla^2 Y}\right)\nabla\epsilon h(\alpha)\right]_{\alpha=N}$$
$$-\left[\left\{E\frac{\partial F}{\partial \nabla Y} - \Delta\frac{\partial F}{\partial \nabla^2 Y}\right)\right\}\epsilon h(\alpha) + \left(E\frac{\partial F}{\partial \nabla^2 Y}\right)\nabla\epsilon h(\alpha)\right]_{\alpha=0} = 0 \qquad \text{(A-11)}$$

The left side of Eq. (A-11) is called the first variation of the definite sum, U, defined by Eq. (A-1). If one designates this variation by δU, the necessary condition for U being stationary becomes

$$\delta U = 0 \qquad \text{(A-12)}$$

It can be shown that the first factor defined at all interior nodes $1 \leqslant \alpha \leqslant N-1$, represents the governing difference equation of the system, while the other factors defined at $\alpha = 0$, N represent the feasible boundary conditions. The method will be illustrated on a multispan beam with spring constraints at the end supports.

(B) Two-dimentional Case

The problem of determining stationary values of a double summation leads to a partial difference equation which defines the unknown function $Y(\alpha_1, \alpha_2)$.

Consider the following example which is used in analysis of cylindrical latticed shell:

$$U(Y) = \sum_{\alpha 1=1}^{M}\sum_{\alpha 2=1}^{N-1} F(\alpha_1, \alpha_2, Y, \nabla_1 Y, \nabla_2 Y)$$
$$+ \sum_{\alpha 1=1}^{M-1}\sum_{\alpha 2=1}^{N} F(\alpha_1, \alpha_2, Y, \nabla_1 Y, \nabla_2 Y) \qquad \text{(A-13)}$$

where Y is a function of two discrete independent variables, α_1, α_2.

The partial difference operators, ∇_1, ∇_2 and the inverse difference operators Δ_1^{-1}, Δ_2^{-1}, are defined as

$$\nabla_1 Y(\alpha_1, \alpha_2) = Y(\alpha_1, \alpha_2) - Y(\alpha_1 - 1, \alpha_2)$$
$$\nabla_2 Y(\alpha_1, \alpha_2) = Y(\alpha_1, \alpha_2) - Y(\alpha_1, \alpha_2 - 1) \qquad \text{(A-14a, b)}$$

$$\sum_{\alpha 1=1}^{M} Y(\alpha_1, \alpha_2) = \Delta_1^{-1} Y(\alpha_1, \alpha_2)\Big|_{\alpha_1=1}^{M+1}$$

$$\sum_{\alpha 2=1}^{M} Y(\alpha_1, \alpha_2) = \Delta_2^{-1} Y(\alpha_1, \alpha_2)\Big|_{\alpha_2=1}^{N+1} \qquad \text{(A-15a, b)}$$

Considering an arbitrary function h (α_1, α_2) of the two discrete variables α_1 and α_2, defined in the interval $0 \leqq \alpha_1 \leqq M$, $0 \leqq \alpha_2 \leqq N$, and following the procedure obtained for the one-dimensional case one finds that the condition for U being stationary is as follows:

$$\delta U = \Delta_1^{-1}\Delta_2^{-1}\left\{\left(\frac{\partial F}{\partial Y} - \Delta_1\frac{\partial F}{\partial \nabla_1 Y} - \Delta_2\frac{\partial F}{\partial \nabla_2 Y}\right)\epsilon h(\alpha_1, \alpha_2)\right\}\Big|_{\alpha_1=1}^{M}\Big|_{\alpha_2=1}^{N}$$

$$+ \Delta_2^{-1}\left[\left\{\left(\frac{\partial F}{\partial Y} + \frac{\partial F}{\partial \nabla_1 Y}\right)\epsilon h(\alpha_1, \alpha_2)\right\}_{\alpha_1=M} - \left\{(\Delta_1 + 1)\frac{\partial F}{\partial \nabla_1 Y}\epsilon h(\alpha_1, \alpha_2)\right\}_{\alpha_1=0}\right]_{\alpha_2=1}^{N}$$

$$+ \Delta_1^{-1}\left[\left\{\left(\frac{\partial F}{\partial Y} + \frac{\partial F}{\partial \nabla_2 Y}\right)\epsilon h(\alpha_1, \alpha_2)\right\}_{\alpha_2=N} - \left\{(\Delta_2 + 1)\frac{\partial F}{\partial \nabla_2 Y}\epsilon h(\alpha_1, \alpha_2)\right\}_{\alpha_2=0}\right]_{\alpha_1=1}^{M}$$

$$= 0 \qquad \text{(A-16)}$$

The significance of the above expression can only be obtained by its careful examination. It has been previously stated that the variation, $\epsilon h(\alpha_1, \alpha_2)$ is completely arbitrary. Thus, the first factor in Eq. (A-16) can vanish as required only if the coefficient of the variation vanishes. Using this reasoning one obtains, from the vanishing of the coefficient of the variation in the double summation, the following difference equation:

$$\frac{\partial F}{\partial Y} - \Delta_1\frac{\partial F}{\partial \nabla_1 Y} - \Delta_2\frac{\varrho F}{\partial \nabla_2 Y} = 0 \qquad \text{(A-17)}$$

The above equation represents the conditions which must be fulfilled to secure at stable equilibrium at all interior nodes and, therefore, constitutes the governing difference equation defined at $1 \leqq \alpha_1 \leqq M-1$, and $1 \leqq \alpha_2 \leqq N-1$.

By considering the second factor, a single summation with respect to the variable α_2, one finds, as a consequence of Eq. (A-16), that the terms involving the arbitrary variations must each vanish. Thus, it is required that

$$\left(\Delta_1 + 1\right)\frac{\partial F}{\partial \Delta_1 Y}\epsilon h(\alpha_1, \alpha_2)\Big|_{\alpha_1=0} = 0$$

$$\left(\frac{\partial F}{\partial Y} + \frac{\partial F}{\partial \nabla_1 Y}\right)\epsilon h(\alpha_1, \alpha_2)\Big|_{\alpha_1=M} = 0 \qquad \text{(A-18a, b)}$$

Each of the above equations will be satisfied if the appropriate conditions are prescribed. Therefore, the following natural boundary conditions are provided at α_2 = constant,

at $\alpha_1 = 0$

$$\left(\Delta_1 + 1\right)\frac{\partial F}{\partial \nabla_1 Y} = 0 \;, \quad \text{or} \quad \epsilon h(0, \alpha_2) = 0 \qquad \text{(A-19a, b)}$$

at $\alpha_1 = \mathrm{M}$

$$\frac{\partial F}{\partial Y} + \frac{\partial F}{\partial \nabla_1 Y} = 0 \ , \qquad \text{or} \qquad \epsilon h\ (M, \alpha_2) = 0 \qquad \text{(A-20a, b)}$$

Similar conditions can beobtained at α_1 = constant; they are:

at $\alpha_2 = 0$

$$(\Delta_2 + 1) \frac{\partial F}{\partial \nabla_2 Y} = 0 \qquad \text{or} \qquad \epsilon h\ (\alpha_1, 0) = 0 \qquad \text{(A-21a, b)}$$

at $\alpha_2 = \mathrm{N}$

$$\frac{\partial F}{\partial Y} + \frac{\partial F}{\partial \nabla_2 Y} = 0 \qquad \text{or} \qquad \epsilon h\ (\alpha_1, N) = 0 \qquad \text{(A-22a, b)}$$

In order to gain more insight into the applicability and effectiveness of the method just described, the analysis of a single layer cable net with boundary conditions different to those treated in the literature will be presented.

(C) Modified Discrete Variational Method

In the previous sections, a discrete variational technique has been demonstrated in obtaining the equilibrium equations and the natural boundary conditions of a standard system. However, a more valuable application of this technique is to provide closed form solutions for general boundary conditions for which no such solutions are available.

Similar to the method used in continuum mechanics, the procedure in finding a solution is to choose an algebraic or trigonometric series to be capable of describing the particular deformed shape. Since trigonometric series are functions whose behavior is well known, they will be the type of functions to be used in connection with the discrete variational methods. The proper orthogonality properties of trigonometric series are shown in Eqs. (3-22a, b), (3-76) and (3-77). A study of these properties requires the extension of the range of summation over the boundaries and, accordingly, the terms in the first energy variation have to be rearranged.

Considering the one-dimentional case one can rewrite the first energy variation Eq. (A-11) as

$$\begin{aligned} \delta U = \Delta^{-1} & \left[\left\{ \frac{\partial F}{\partial Y} - \Delta \left(\frac{\partial F}{\partial \nabla Y} \right) \right\} \epsilon h\ (\alpha) \right]_{\alpha=1}^{\alpha=N} \\ & + \left[\left(\frac{\partial F}{\partial Y} + \frac{\partial F}{\partial \nabla Y} \right) \epsilon h\ (\alpha) \right]_{\alpha=N} \\ & - \left[\left\{ \left(\Delta + 1 \right) \frac{\partial F}{\partial \nabla Y} \right\} \epsilon h\ (\alpha) \right]_{\alpha=0} = 0 \end{aligned} \qquad \text{(A-23)}$$

where the terms involving $\nabla^2 Y$, $\nabla^2 \epsilon h\ (\alpha)$ were disregarded in order to have a more simple presentation.

The above equation can be rearranged by extending the range of summation over the boundaries as

$$\delta U = \Delta^{-1}\left[\left\{\frac{\partial F}{\partial Y} - \Delta\left(\frac{\partial F}{\partial \nabla Y}\right)\right\}\epsilon h\ (\alpha)\right]_{\alpha=0}^{\alpha=N+1}$$
$$+\left[\left\{\left(\Delta+1\right)\frac{\partial F}{\partial \nabla Y}\right\}\epsilon h\ (\alpha)\right]_{\alpha=N}$$
$$-\left[\left(\frac{\partial F}{\partial Y}+\frac{\partial F}{\partial \nabla Y}\right)\epsilon h\ (\alpha)\right]_{\alpha=0} = 0 \qquad \text{(A-24)}$$

A close examination of Eq. (A-24) shows that this equation contains the equilibrium equation valid also at the boundaries and a set of modified natural boundary conditions.

If the solution assumed is composed by a sum of orthogonal functions which satisfies the modified boundary conditions at $\alpha = 0$ and $\alpha = N$, the solution can be obtained by standard procedures. But, such a solution cannot be found easily unless special restrictions are made.

Structural systems with general boundary conditions will be analyzed by the use of modification parameters λ^1, λ^2 which are defined only at boundaries. These parameters will modify a solution which does not satisfy boundary conditions and thus behave similar to the Lagrange multipliers. Physically, the parameters λ^1 and λ^2 are related to the forces which must be applied at the boundaries in order to satisfy that part of the boundary condition not fulfilled by assumed solution. By use of the modified parameters Eq. (A-24) can be rewritten as

$$\delta U = \Delta^{-1}\left[\left\{\frac{\partial F}{\partial Y} - \Delta\left(\frac{\partial F}{\partial \nabla Y}\right) - \lambda^1\,\delta^0_\alpha - \lambda^2\,\delta^N_\alpha\right\}\epsilon h\ (\alpha)\right]_{\alpha=0}^{\alpha=N+1}$$
$$+\left[\left\{\left(\Delta+1\right)\frac{\partial F}{\partial \nabla Y} + \lambda^2\,\delta^N_\alpha\right\}\epsilon h\ (\alpha)\right]_{\alpha=N}$$
$$-\left[\left\{\frac{\partial F}{\partial Y}+\frac{\partial F}{\partial \nabla Y} - \lambda^1\,\delta^0_\alpha\right\}\epsilon h\ (\alpha)\right]_{\alpha=0} = 0 \qquad \text{(A-25)}$$

It should be observed that the unknown quantities λ^1 at $\alpha=0$ and λ^2 at $\alpha=N$ have been subtracted from the first bracket, which results the equilibrium equation at all nodes and exactly the same quantities have been added to the second and third bracket, respectively. Therefore, the value of the first energy variation has not been changed. The terms involved in the summation operator is designated as the modified first energy variation, δU.

A study of Eq. (A-25) shows that the problem, $\delta U = 0$ has been reduced to

$$\delta J = \Delta^{-1}\left[\frac{\partial F}{\partial Y} - \Delta\left(\frac{\partial F}{\partial \nabla Y}\right) - \lambda^1\,\delta^0_\alpha - \lambda^2\,\delta^N_\alpha\right]\epsilon h\ (\alpha)\Bigg|_{\alpha=0}^{\alpha=N+1} = 0 \qquad \text{(A-26)}$$

$$\frac{\partial F}{\partial Y} + \frac{\partial F}{\partial \nabla Y} - \lambda^1 = 0\,, \qquad \text{or} \quad Y\,(O) = 0 \quad \text{at } \alpha = 0 \tag{A-27a, b}$$

and

$$\left(\Delta + 1\right)\frac{\partial F}{\partial \nabla Y} + \lambda^2 = 0\,, \qquad \text{or} \quad Y\,(N) = 0 \quad \text{at } \alpha = N \tag{A-28a, b}$$

The expression $\delta \bar{U}$ represents the modified first energy variation and Eqs. (A-27) and (A-28) are the corresponding modified natural boundary conditions at $\alpha = O$ and $\alpha = N$, respectively.

When the orthogonality property with respect to a specific weighting function is defined, as shown in Eq. (3-22b), the above equations have to be changed as follows:

$$\delta \bar{U} = \Delta^{-1}\, w\alpha \left[\frac{\partial F}{\partial Y} - \Delta\left(\frac{\partial F}{\partial \nabla_1 Y}\right) - \lambda^1\, \delta^0_\alpha - \lambda^2\, \delta^N_\alpha\right] \epsilon h \,\Bigg|_{\alpha=0}^{\alpha=N+1} \tag{A-29}$$

$$\frac{\partial F}{\partial Y} + \left(\Delta + 2\right)\frac{\partial F}{\partial \nabla Y} + \lambda^1 = 0\,, \qquad \text{or} \quad Y\,(0) = 0 \quad \text{at } \alpha = 0 \tag{A-30a, b}$$

and

$$\frac{\partial F}{\partial Y} + \left(\Delta + 2\right)\frac{\partial F}{\partial \nabla Y} - \lambda^2 = 0\,, \qquad \text{or} \quad Y(N) = 0 \quad \text{at } \alpha = N \tag{A-31a, b}$$

Similar technique is also applicable for the two dimensional case. The first energy variation, Eq. (A-16), is written by extending the range of double summation over the boundaries, that is

$$\begin{aligned}
\delta U = {} & \Delta_1^{-1}\, \Delta_2^{-1} \left\{\left(\frac{\partial F}{\partial Y} - \Delta_1 \frac{\partial F}{\partial \nabla_1 Y} - \Delta_2 \frac{\partial F}{\partial \nabla_2 Y}\right) \epsilon h\,(\alpha_1, \alpha_2)\right\} \Bigg|_{\alpha_1=0}^{M+1} \Bigg|_{\alpha_2=0}^{N+1} \\
& + \Delta_2^{-1} \Bigg[\left\{\left(\Delta_1 + 1\right)\frac{\partial F}{\partial \nabla_1 Y}\, \epsilon h\,(\alpha_1, \alpha_2)\right\}_{\alpha_1=M} \\
& - \left\{\left(\frac{\partial F}{\partial Y} + \frac{\partial F}{\partial \nabla_1 Y}\right) \epsilon h\,(\alpha_1, \alpha_2)\right\}_{\alpha_1=0}\Bigg]_{\alpha_2=1}^{N} \\
& + \Delta_1^{-1} \Bigg[\left\{\left(\Delta_2 + 1\right)\frac{\partial F}{\partial \nabla_2 Y}\, \epsilon h\,(\alpha_1, \alpha_2)\right\}_{\alpha_2=N} \\
& - \left\{\left(\frac{\partial F}{\partial Y} + \frac{\partial F}{\partial \nabla_2 Y}\right) \epsilon h\,(\alpha_1, \alpha_2)\right\}_{\alpha_2=0}\Bigg]_{\alpha_1=1}^{M} = 0
\end{aligned} \tag{A-32}$$

Consider the case where the solutions assumed satisfy the modified boundary conditions at α_1 = constant as shown in the second line of Eq. (A-32). Then, one needs to modify the solution at the boundaries $\alpha_2 = 0$ and $\alpha_2 = N$. Following a similar technique to that used in the one-dimensional case, modification parameters can be defined at each unsatisfied node, $\alpha_1 = 0, 1, \cdots, M$. However, this approach will yield a set of simultaneous equations, which for the case of large number of nodes may be cumbersome to solve. This author proposes to express the parameters in a functional form called a modification function. In order to make effective use of the orthogonality properties of the assumed solution the modification function is assumed to have the same form as the solution does along the α_1-direction. The modification functions defined at $\alpha_2 = 0$ and $\alpha_2 = N$ are designated by $\lambda^1(\alpha_1)$ and $\lambda^2(\alpha_1)$, respectively.

By use of these functions, Eq. (A-32) is rewritten

$$\delta U = \Delta_1^{-1}\Delta_2^{-1}\left[\left\{\frac{\partial F}{\partial Y} - \Delta_1\frac{\partial F}{\partial \nabla_1 Y} - \nabla_2\frac{\partial F}{\partial \nabla_2 Y} - \lambda^1(\alpha_1)\,\delta^0_{\alpha 2} - \lambda^2(\alpha_1)\,\delta^N_{\alpha 2}\right\}\epsilon h\right]\Bigg|_{\alpha_1=0}^{M+1}\Bigg|_{\alpha_2=0}^{N+1}$$
$$+\Delta_2^{-1}\left[\left\{\left(\Delta_1+1\right)\frac{\partial F}{\partial \Delta_1 Y}\epsilon h\,(\alpha_1,\alpha_2)\right\}_{\alpha_1=M} - \left\{\left(\frac{\partial F}{\partial Y}+\frac{\partial F}{\partial \nabla_1 Y}\right)\epsilon h\,(\alpha_1,\alpha_2)\right\}_{\alpha_1=0}\right]\Bigg|_{\alpha_2=1}^{N}$$
$$+\Delta_1^{-1}\left(\left[\left\{\left(\Delta_2+1\right)\frac{\partial F}{\partial \nabla_2 Y} + \lambda^2(\alpha_1)\right\}\epsilon h\,(\alpha_1,\alpha_2)\right]_{\alpha_2=N}\right.$$
$$\left.-\left[\left\{\frac{\partial F}{\partial Y} + \frac{\partial F}{\partial \nabla_2 Y} - \lambda^1(\alpha_1)\right\}\epsilon h\,(\alpha_1,\alpha_2)\right]_{\alpha_2=0}\right)\Bigg|_{\alpha_1=1}^{M} = 0 \qquad \text{(A-33)}$$

Therefore, the problem being considered in Eq. (A-33) is reduced to thefollowing:

$$\delta U = \Delta_1^{-1}\Delta_2^{-1}\left[\left\{\frac{\partial F}{\partial Y} - \Delta_1\frac{\partial F}{\partial \nabla_1 Y} - \Delta_2\frac{\partial F}{\partial \nabla_2 Y} - \lambda^1(\alpha_1)\,\delta^0_{\alpha 2}\right.\right.$$
$$\left.\left.-\lambda^2(\alpha_1)\,\delta^N_{\alpha 2}\right\}\epsilon h\,(\alpha_1,\alpha_2)\right]\Bigg|_{\alpha_1=0}^{M+1}\Bigg|_{\alpha_2=0}^{N+1} = 0 \qquad \text{(A-34)}$$

$$\frac{\partial F}{\partial Y} + \frac{\partial F}{\partial \nabla_1 Y} = 0 \quad \text{or } Y(\alpha_1,\alpha_2) = 0 \quad \text{at } \alpha_1 = 0,\ \alpha_2 = \text{constant} \qquad \text{(A-35a, b)}$$

$$\left(\Delta_1 + 1\right)\frac{\partial F}{\partial \nabla_1 Y} = 0 \quad \text{or } Y(\alpha_1,\alpha_2) = 0 \quad \text{at } \alpha_1 = M,\ \alpha_2 = \text{constant} \qquad \text{(A-36a,b)}$$

$$\frac{\partial F}{\partial Y} + \frac{\partial F}{\partial \nabla_2 Y} - \lambda^1(\alpha_1) = 0, \quad \text{or } Y(\alpha_1,\alpha_2) = 0 \quad \text{at } \alpha_2 = 0,\ \alpha_1 = \text{constant} \qquad \text{(A-37a,b)}$$

$$\left(\Delta_2 + 1\right)\frac{\partial F}{\partial \nabla_2 Y} + \lambda^2(\alpha_1) = 0, \quad \text{or } Y(\alpha_1,\alpha_2) = 0 \quad \text{at } \alpha_2 = N,\ \alpha_1 = \text{constant} \qquad \text{(A-38a, b)}$$

When several functions Y_k, $k = 1, 2\cdots$, Which do not satisfy the boundary conditions are considered in the problem, one can assume modification functions, $\lambda^1{}_k(\alpha_1)$ and $\lambda^2{}_k(\alpha_1)$, $k = 1, 2, \cdots$, corresponding to Y_k. This problem is illustrated in the analysis of a cylindrical latticed shell with polygonal ribbed supports.

EXAMPLE 1 - ONE-DIMENSIONAL CASE: CONTINUOUS BEAM WITH SPRING CONSTRAINTS

To illustrate the one-dimensional discrete variational techniques, consider the regular continuous beam with spring constraints at the ends shown in Fig. A-1.

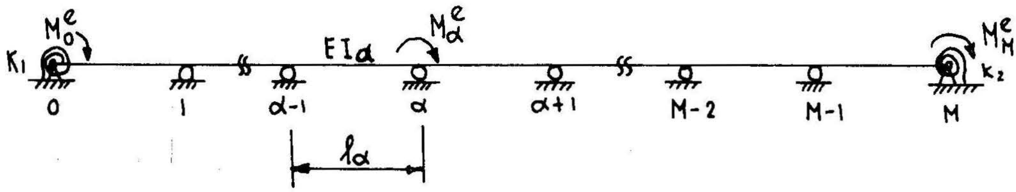

FIG. A-1 CONTINUOUS BEAM WITH SPRING CONSTRAINTS

FIG. A-2 ELEMENT MEMBER $(\alpha-1, \alpha)$

The symbols used in the figures have the following definitions:

K_1, K_2	represent the spring constants at $\alpha=0$ and $\alpha=M$, respectively
EI_α	is flexural rigidity of a member $(\alpha-1, \alpha)$
M_α^e, M_0^e, M_M^e	are external loads applied at a typical point, $\alpha=0$ and $\alpha=M$, respectively

Fig. A-2 shows a typical member at the ends α and $\alpha-1$ with moments M_α, $M_{\alpha-1}$, and rotations θ_α, $\theta_{\alpha-1}$. The strain energy V_α, in this element which is represented by its deformed configuration, is equal to the work done by the moments as they induce the rotations (33). That is

$$V_\alpha = \frac{1}{2}(M_\alpha \cdot \theta_\alpha + M_{\alpha-1} \cdot \theta_{\alpha-1}) \tag{A-39}$$

Using a difference notation, the moments $M\alpha$, and $M_{\alpha-1}$ are written as functions of the rotations in the following form

$$M_\alpha = b_\alpha\ k_\alpha\ (\gamma_\alpha - \nabla_\alpha)\ \theta_\alpha$$

$$M_{\alpha-1} = b_\alpha\ k_\alpha \left\{(1-\gamma_\alpha)\ \nabla_\alpha + \gamma_\alpha\right\} \theta_\alpha \tag{A-40a, b}$$

where for prismatic beam, $b_\alpha = 2$, $\gamma_\alpha = 3$.

Substituting Eqs. (A-40a, b) into Eq. (A-39), one obtains the strain energy of the typical element member

$$V_\alpha = \frac{1}{2} b_\alpha k_\alpha \left[(\gamma_\alpha - \nabla_\alpha) \theta_\alpha \cdot \theta_\alpha + \left\{ (1-\gamma_\alpha) \nabla_\alpha + \gamma_\alpha \right\} \theta_\alpha (1-\nabla_\alpha) \theta_\alpha \right]$$

For the continuous beam of Fig. A-1, the total strain energy, V, is obtained as the sum of the strain energy in the individual elements and that due to end springs. The result is

$$V = \sum_{\alpha=1}^{M} V_\alpha + \frac{K_1}{2} \theta_0^2 + \frac{K_2}{2} \theta_M^2 \tag{A-42}$$

where $V\alpha$ is the strain energy of a typical member, Eq. (A-41), and M denotes the total number of elements.

It can easily be recognized that the last two terms of Eq. (A-42) represent the strain energy of the elastic constraints where k_1 and k_2 are the spring constants indicating the degree of resistance against rotations at $\alpha = 0$ and M.

The potential energy due to external loads is

$$W = - \sum_{\alpha=0}^{M} M_\alpha^e \theta_\alpha \tag{A-43}$$

Therefore, the total potential energy of the continuous beam is

$$\begin{aligned} U &= V + W \\ &= \frac{b\alpha\, k\alpha}{2} \sum_{\alpha=1}^{M} \left[(\gamma\alpha - \nabla) \theta\alpha \cdot \theta\alpha + \left\{ (1-\gamma\alpha) \nabla + \gamma\alpha \right\} \theta\alpha (1-\nabla) \theta\alpha \right] \\ &+ \frac{K_1}{2} \theta_0^2 + \frac{K_2}{2} \theta_M^2 - \sum_{\alpha=0}^{M} M_\alpha^e \cdot \theta\alpha \end{aligned} \tag{A-44}$$

Substitution of Eq. (A-44) into Eq. (A-11) yields the following expression for the first energy variation:

$$\begin{aligned} \delta U &= \Delta^{-1} \left[bK (\Delta\!\!\!/ + 2\gamma) \theta\alpha - M_\alpha^e \right] \epsilon\theta\alpha \Big|_1^M \\ &+ \left[\left\{ bK (\Delta + \gamma) \theta\alpha + K_1 \theta\alpha - M_\alpha^e \right\} \epsilon\theta\alpha \right]_{\alpha=0} \\ &+ \left[\left\{ bK (\gamma - \nabla) \theta\alpha + K_1 \theta\alpha - M_\alpha^e \right\} \epsilon\theta\alpha \right]_{\alpha=M} \\ &= 0 \end{aligned} \tag{A-45}$$

where $b\alpha = b$ and $\gamma\alpha = \gamma$ are assumed to be constant.

The following condition established in the first bracket of the above equation represents the governing difference equation of the structure

$$\mathrm{bk}\ (\nabla\!\!\!\!/ + 2\gamma)\ \theta\alpha - \mathrm{M}^{e}_{\alpha} = 0 \qquad \text{(A-46)}$$

The second and the third expressions yield the natural boundary conditions at $\alpha = 0$ and $\alpha = \mathrm{M}$

$$\mathrm{bk}\ (\Delta + \gamma)\ \theta_0 + \mathrm{K}_1\ \theta_0 - \mathrm{M}^{e}_{0}\ 0, \quad \text{or} \quad \theta_0 = 0$$

$$\text{at } \alpha = 0 \qquad \text{(A-47a, b)}$$

$$\mathrm{bk}\ (\gamma - \nabla)\ \theta_\mathrm{M} + \mathrm{K}_2\ \theta_\mathrm{M} - \mathrm{M}^{e}_{\mathrm{M}} = 0, \quad \text{or} \quad \theta_\mathrm{M} = 0$$

$$\text{at } \alpha = \mathrm{M} \qquad \text{(A-48a, b)}$$

To solve the above equations by a finite Fourier series a modification is necessary in order to apply the proper orthogonality relation of the finite series. The range of summation $\alpha = 1$ to $\alpha = \mathrm{M} - 1$ is extended over $\alpha = 0$ and $\alpha = \mathrm{M}$. Therefore, Eq. (A-45) is rewritten as

$$\delta\mathrm{U} = \Delta^{-1}\ \mathrm{w}\alpha \left[\mathrm{bk}\ (\nabla\!\!\!\!/ + 2\gamma)\ \theta\alpha - \frac{\mathrm{M}^{e}_{\alpha}}{\mathrm{w}\alpha} \right] \epsilon\theta\alpha \Bigg|_{0}^{\mathrm{M}+1}$$
$$+ \left[\left\{ \frac{\mathrm{bk}}{2}\ \Delta\!\!\!\!/\ \theta\alpha + \mathrm{K}_1\ \theta\alpha \right\} \epsilon\theta\alpha \right]_{\alpha=0}$$
$$+ \left[\left\{ -\frac{\mathrm{bk}}{2}\ \Delta\!\!\!\!/\ \theta\alpha + \mathrm{K}_2\ \theta\alpha \right\} \epsilon\theta\alpha \right]_{\alpha=\mathrm{M}} = 0 \qquad \text{(A-49)}$$

Since the standard finite Fourier series will not satisfy the arbitrary types of boundary conditions given in the above equations, the method similar to that of the Lagrange multipliers will be used here.

Defining two unknown parameters λ_1 and λ_2, to be used to satisfy the required boundary conditions, Eq. (A-49) can be written as

$$\delta\mathrm{U} = \Delta^{-1}\ \mathrm{w}\alpha \left[\mathrm{bk}\ (\nabla\!\!\!\!/ + 2\gamma)\ \theta\alpha - \frac{\mathrm{M}^{e}_{\alpha}}{\mathrm{w}\alpha} - \lambda_1\ \delta^{0}_{\alpha} - \lambda_2\ \delta^{\mathrm{M}}_{\alpha} \right] \epsilon\theta\alpha \Bigg|_{0}^{\mathrm{M}+1}$$
$$+ \left[\left\{ \frac{\mathrm{bk}}{2}\ \Delta\!\!\!\!/\theta\alpha + \mathrm{K}_1\ \theta\alpha + \frac{\lambda_1}{2} \right\} \epsilon\theta\alpha \right]_{\alpha=0}$$
$$+ \left[\left\{ \frac{\mathrm{bk}}{2}\ \Delta\!\!\!\!/\theta\alpha + \mathrm{K}_2\ \theta\alpha + \frac{\lambda_2}{2} \right\} \epsilon\theta\alpha \right]_{\alpha=\mathrm{M}} = 0 \qquad \text{(A-50)}$$

It should be observed that the unknown quantities λ_1 at $\alpha = 0$ and λ_2 at $\alpha = \mathrm{M}$ have been subtracted from the first bracket and exactly the same quantities have been added to the second and third bracket, respectively. Therefore, the value of the first energy variation has not been changed. The terms involved in the summation operator will be designated as the modified first energy variation, $\delta \bar{U}$. It has the form

$$\delta \bar{U} = \Delta^{-1} w\alpha \left[bk\ (\nabla + 2\gamma)\ \theta\alpha - \frac{M^e_\alpha}{w\alpha} - \lambda_1 \delta^0_\alpha - \lambda_2 \delta^M_\alpha \right] \epsilon\theta\alpha \Bigg|_1^{M+1} \tag{A-51}$$

The problem has been reduced from Eq. (A-50) to that of solving the following equations

$$\delta \bar{U} = 0$$

$$\frac{bk}{2} \nabla\theta\alpha + K_1\ \theta\alpha + \frac{\lambda_1}{2} = 0 \qquad \text{at } \alpha = 0$$

$$\frac{bk}{2} \nabla\theta\alpha + K_2\ \theta\alpha \frac{\lambda_2}{2} = 0 \qquad \text{at } \alpha = M \tag{A-52a, b, c}$$

where

$\delta \bar{U}$ represents the modified first energy variation and the second, Eq. (A-52b), and the third, Eq. (A-52c) are the modified boundary conditions.

To solve the above equation the following solution is assumed

$$\theta\alpha = \sum_{m=0}^{M} \theta_m \cos \lambda m\alpha$$

where $\lambda m = \frac{m\pi}{M}$ (A-53)

The external load is expanded in a similar series with the weighting function

$$\frac{M^e_\alpha}{w\alpha} = \sum_{m=0}^{M} Mm \cos \lambda m\alpha \tag{A-54}$$

The Euler coefficient Mm is obtained by using the orthogonality property of the trigonometric series, Eq. (3-22b).

$$Mm = \frac{1}{\Gamma m} \Delta^{-1} M^e_\alpha \cos \lambda m\alpha \Bigg|_0^{M+1} \tag{A-55}$$

where

$$\Gamma m = \sum_{\alpha=0}^{M} w\alpha \cos^2 \lambda m\alpha \tag{A-56}$$

The variation of the rotation is assumed as

$$\epsilon\theta\alpha = \delta\theta\alpha = \sum_{m=0}^{M} \delta\theta m \cos \lambda m\alpha \tag{A-57}$$

Substitution of Eqs. (A-53), (A-54), (A-57) into the modified first energy variation, $\delta \bar{U}$, Eq. (A-51), one obtains

$$bk \left\{ 2 (\cos \lambda m - 1) + 2\gamma \right\} \Gamma_m \theta m - \Gamma_m Mm - \lambda_1 - \lambda_2 \cos m\pi = 0 \qquad \text{(A-58)}$$

from which

$$\theta m = \left\{ Mm + \frac{1}{\Gamma_m} (\lambda_1 + \lambda_2 \cos m\pi) \right\} / Cm \qquad \text{(A-59)}$$

where

$$Cm = 2bk (\cos \lambda m - 1 + \gamma) \qquad \text{(A-60)}$$

Substitution of Eq. (A-53) into Eqs. (A-52b, c) gives

$$K_1 \sum_{m=0}^{M} \theta m + \frac{\lambda_1}{2} = 0$$

$$K_2 \sum_{m=0}^{M} \theta m \cos m\pi + \frac{\lambda_2}{2} = 0 \qquad \text{(A-61a, b)}$$

Since θm was obtained as a function of λ_1 and λ_2, one substitutes Eq. (A-59) into Eqs. (A-61a, b) yielding

$$K_1 \sum_{m=0}^{M} \left\{ Mm + \frac{1}{\Gamma_m} (\lambda_1 + \lambda_2 \cos m\pi) \right\} / Cm + \frac{\lambda_1}{2} = 0$$

$$K_2 \sum_{m=0}^{M} \left\{ Mm + \frac{1}{\Gamma_m} (\lambda_1 + \lambda_2 \cos m\pi) \right\} (\cos m\pi) / Cm + \frac{\lambda_2}{2} = 0 \qquad \text{(A-62a,b)}$$

The above equations can be written in a matrix form as

$$\left[A \right] \begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} -K_1 \sum_{m=0}^{M} \dfrac{Mm}{Cm} \\ -K \sum_{m=0}^{M} \dfrac{Mm \cos m\pi}{Cm} \end{Bmatrix} \qquad \text{(A-63)}$$

where the matrix [A] is defined as

$$\left[A \right] = \begin{bmatrix} K_1 \sum_{m=0}^{M} \dfrac{1}{\Gamma_m Cm} + \dfrac{1}{2}, & K_1 \sum_{m=0}^{M} \dfrac{\cos m\pi}{\Gamma_m Cm} \\ K_2 \sum_{m=0}^{M} \dfrac{\cos m\pi}{\Gamma_m Cm}, & K_2 \sum_{m=0}^{M} \dfrac{(\cos m\pi)^2}{\Gamma_m Cm} + \dfrac{1}{2} \end{bmatrix} \qquad \text{(A-64)}$$

Therefore, using Eq. (A-63) it is obtained

$$\begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} = \left[A \right]^{-1} \begin{Bmatrix} -K_1 \sum_{m=0}^{M} \dfrac{Mm}{Cm} \\ -K_2 \sum_{m=0}^{M} \dfrac{Mm \cos m\pi}{Cm} \end{Bmatrix} \qquad \text{(A-65)}$$

Once λ_1 and λ_2 are obtained they are substituted into Eq. (A-59) to obtain the complete Euler coefficient which satisfies the spring boundary conditions.

If one assumes equal spring constant at $\alpha = 0$ and $\alpha = M$, that is, $K_1 = K_2 = K$, the problem is simplified. It is always convenient to treat the problem as the superposition of the symmetric and the anti-symmetric cases with respect to $\frac{1}{2}M$. Only the symmetric case is discussed in detail, since the anti-symmetric case follows the same procedure.

For the symmetric case (i. e. $\theta\alpha = \theta_{M-\alpha}$), the solution is assumed in the form

$$\theta\alpha = \sum_{m=0,2,4,}^{M} \theta m \cos \lambda m\, \alpha \tag{A-66}$$

where the index m takes only even integer values.

Similarly the external loads are expressed as

$$\frac{M^e_\alpha}{w\alpha} = \sum_{m=0,2,4}^{M} Mm \cos \lambda m\, \alpha \tag{A-67}$$

The substitution of $\theta\alpha$, Eq. (A-66), into Eq. (A-52, b. c.) yields the result

$$\lambda_1 = \lambda_2 = \lambda^s \tag{A-68}$$

Accordingly, the modified first energy variation is written

$$\delta \bar{U} = \Delta^{-1}\, w\alpha \left[bk\,(\nabla + 2\gamma)\,\theta\alpha - \frac{M^e_\alpha}{w\alpha} - 2\lambda^s\, \delta^0_\alpha \right] \epsilon\theta\alpha \Bigg|_0^{M+1} \tag{A-69}$$

Introducing Eqs. (A-46) and (A-67) into Eq. (A-69) one obtains

$$\theta m = \left(Mm + \frac{2\lambda^s}{\Gamma_m} \right) / Cm \tag{A-70}$$

where $\qquad Cm = 2bk\,(\cos \lambda m - 1 + \gamma)$

With the results of Eqs. (A-70) and (A-61a), it is found

$$K \sum_{m=0,2,4}^{M} \left(Mm + \frac{2\lambda^s}{\Gamma_m} \right) / Cm + \frac{\lambda^s}{2} = 0 \tag{A-71}$$

from which

$$\lambda^s = \frac{-K \displaystyle\sum_{m=0,2,4}^{M} \frac{Mm}{Cm}}{2K \displaystyle\sum_{m=0,2}^{M} \left(\frac{1}{Cm\,\Gamma_m} + \frac{1}{2} \right)} \tag{A-72}$$

Introducing the above results, Eq. (A-72), into Eq. (A-30) one obtains the Euler coefficient for the symmetric case.

The value of the parameter for the anti-symmetric case can be obtained by letting the index m to take only integer values in the preceding equations. One finds

$$\lambda^{a/s} = \frac{-K \sum_{m=1,3}^{M} \frac{Mm}{Cm}}{2K \sum_{m=1,3}^{M} \left(\frac{1}{Cm\,\Gamma_m} + \frac{1}{2}\right)} \qquad \text{(A-73)}$$

EXAMPLE 2 - TWO-DIMENSIONAL CASE: ANALYSIS OF A CABLE NET

To illustrate the two-dimensional case the analysis of the cable net shown in Fig. A−3 is presented.

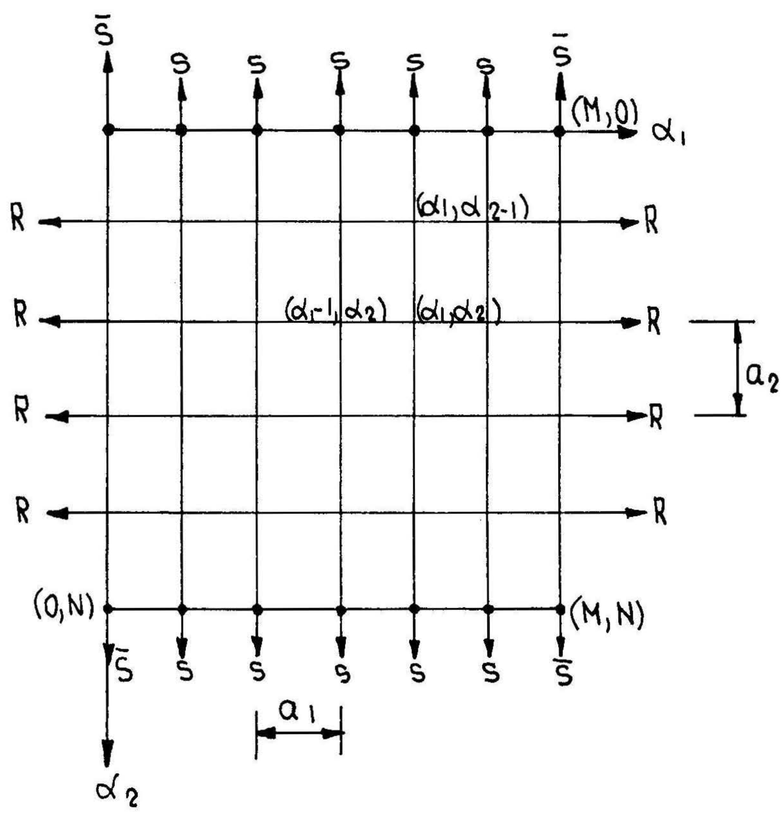

FIG. A-3 NET WITH FREE BOUNDARY AT α_1=O, M

The net of Fig. A − 3 is assumed to have simple supports at $\alpha_2 = 0$, N and free supports at $\alpha_1 = 0$, M. Therefore, it allows boundary deflection at the later edges.

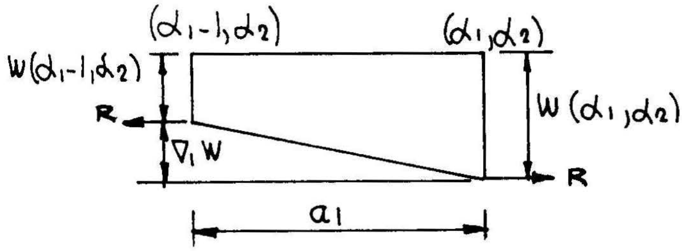

FIG. A-4a CABLE ELEMENT AT α_2 = CONSTANT

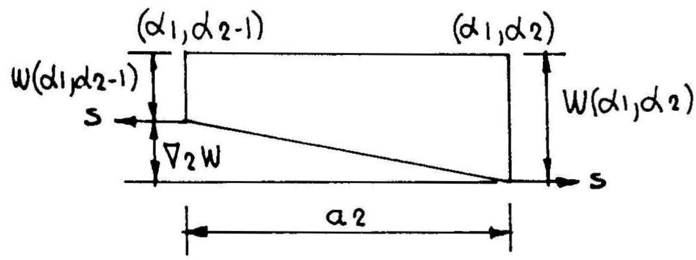

FIG. A-4b CABLE ELEMENT AT α_1 = CONSTANT

The cables parallel to the α_1 axis are assumed to have the horizontal components of the cable tension, i. e. R = constant. Similar assumption applies to the horizontal components of the tension in the α_2 – cables. However, a different component $\bar{S}$ is considered for the boundary cables.

The strain energy of an element ($\alpha_1 - 1$, α_2) (α_1, α_2) shown in Fig. A – 4a is obtained as

$$V(\alpha_1, \alpha_2) = \frac{R}{2a} \nabla_1 W . W - \frac{R}{2a} \nabla_1 W . E^{-1} W = \frac{R}{2a} (\nabla_1 W)^2 \qquad \text{(A-74)}$$

It follows that the total strain energy of cables parallel to α_1 axis is

$$V_1 = \frac{R}{2a_1} \sum_{\alpha_1=1}^{M} \sum_{\alpha_2=1}^{N-1} (\nabla_1 W)^2 \qquad \text{(A-75)}$$

Similarly, the total strain energy of the cables parallel to α_2 – axis, Fig. A – 4b, is found to be

$$V_2 = \frac{S}{2a_2} \sum_{\alpha_1=1}^{M-1} \sum_{\alpha_2=1}^{N} (\nabla_2 W)^2 + \frac{\bar{S}}{2a_2} \Sigma (\nabla_2 W)^2 \left(\delta^0_{\alpha_1} + \delta^M_{\alpha_1}\right) \qquad \text{(A-76)}$$

The potential energy due to external loads is

$$W = - \sum_{\alpha_1=0}^{M} \sum_{\alpha_2=1}^{N-1} P(\alpha_1, \alpha_2)\, W(\alpha_1, \alpha_2) \qquad \text{(A-77)}$$

Therefore, the total potential energy of the cable net is

$$U = \frac{R}{2a_1} \sum_{\alpha_1=1}^{M} \sum_{\alpha_2=1}^{N-1} (\nabla_1 W)^2 + \frac{S}{2a_2} \sum_{\alpha_1=1}^{M-1} \sum_{\alpha_2=1}^{N} (\nabla_2 W)^2$$

$$+ \frac{\bar{S}}{2a_2} \sum_{\alpha_2=1}^{N} (\nabla_1 W)^2 \left(\delta^0_{\alpha_1} + \delta^M_{\alpha_1}\right) - \sum_{\alpha_1=0}^{M} \sum_{\alpha_2=1}^{N-1} P(\alpha_1, \alpha_2). W \qquad \text{(A-78)}$$

where the boundary conditions $W(\alpha_1, 0) = W(\alpha_1, N) = 0$ are used.

To satisfy the nonhomogeneous boundary conditions at the edges α_1 = constant, the modified discrete variational method in Appendix A is applied. Thus, two modification functions λ^1 and λ^2 are defined accordingly

$$\delta U = \Delta_1^{-1} \Delta_2^{-1} \left[-\frac{R}{a} \varDelta_1 W - \frac{S}{b} \varDelta_2 W - P - \lambda^1(\alpha_2)\, \delta^0_{\alpha_1} - \lambda^2(\alpha_2)\, \delta^M_{\alpha_1} \right] \epsilon W \Bigg|_0^{M+1} \Bigg|_1^{N}$$

$$- \Delta_2^{-1} \left[\left\{ -\frac{R}{a} \Delta_1 W + \frac{\bar{S}-S}{b} \varDelta_2 W - \lambda^2(\alpha_2)\, \delta^M_{\alpha_1} \right\} \epsilon W_{\alpha_1 = M} \right.$$

$$\left. + \left\{ \frac{R}{a} \nabla_1 W + \frac{\bar{S}-S}{b} \varDelta_2 W - \lambda^2(\alpha_2)\, \delta^0_{\alpha_1} \right\} \epsilon W_{\alpha_1} \right]_0^N = 0 \qquad \text{(A-79)}$$

The modification functions $\lambda^1(\alpha_2)$, $\lambda^2(\alpha_2)$ have been subtracted from the first bracket and the same functions added to the second and third bracket. Therefore, the value of the first energy variation has not changed.

The problem under consideration is further simplified if one considers separately the symmetric and the antisymmetric solutions.

For the symmetric case it is assumed that the applied joint loads are such that $W(\alpha_1, \alpha_2)$ are symmetric with respect to $\frac{M}{2}$, while for the anti-symmetric case the loads applied result in deformations, $W(\alpha_1, \alpha_2)$, which are antisymmetric with respect to $\frac{M}{2}$.

Only the symmetric case will be illustrated in detail and the anti-symmetric case can be obtained by following a similar procedure.

For the symmetric case the terms in the first bracket of Eq. (A-79) can be reduced to

$$\delta \bar{U} = \Delta_1^{-1} \Delta_2^{-1} \left[-\frac{R}{a_1} \varDelta_1 W - \frac{S}{a_2} \varDelta_2 W - P - 2\lambda^s(\alpha_2)\, \delta^0_{\alpha_1} \right] \epsilon W \Bigg|_0^{M+1} \Bigg|_1^{N} = 0 \qquad \text{(A-80)}$$

where $\delta \bar{U}$ represents the modified first energy variation and $\lambda^1(\alpha_2) = \lambda^2(\alpha_2) = \lambda^s(\alpha_2)$ is introduced.

Because of symmetry only one boundary condition needs to be considered in the α_1-direction. This condition relates the modification function $\lambda^s(\alpha_2)$ to the boundary statements and is designated as the boundary constraint.

At $\alpha_1 = 0$ it takes the form

$$\frac{R}{a_1} \nabla_1 W + \frac{\bar{S} - S}{a_2} \Delta_2 W - \lambda^s (\alpha_2) = 0 \tag{A-81}$$

The joint deformations W (α_1, α_2) is assumed in the series

$$W (\alpha_1, \alpha_2) = \sum_{m=0,2}^{M} \sum_{n=1,2}^{N-1} Wmn \cos \frac{m\pi}{M+1} \left(\alpha_1 + \tfrac{1}{2}\right) \sin \frac{n\pi \alpha_2}{N} \tag{A-82}$$

where the index m takes only even integer values.

Similarly the external loads are expressed as

$$P (\alpha_1, \alpha_2) = \sum_{m=0,2}^{M} \sum_{n=1,2}^{N-1} Pmn \cos \frac{m\pi}{M+1} \left(\alpha_1 + \tfrac{1}{2}\right) \sin \frac{n\pi \alpha_2}{N} \tag{A-83}$$

The modification function is assumed as

$$\lambda^s (\alpha_2) = \sum_{n=1,2}^{N-1} \bar{\lambda}^s n \sin \frac{n\pi \alpha_2}{N} \tag{A-84}$$

Substituting Eqs. (A-82), (A-83), (A-84) into Eq. (A-83) and using the orthogonality relations of the trigonometric series, one obtains

$$\left(\frac{2R}{a_1} \gamma_m + \frac{2S}{a_2} \gamma_n\right) Wmn \frac{(M+1)N}{4\phi m} = \frac{Pmn(M+1)N}{4\phi m} - \bar{\lambda}^s_n N \cos \frac{m\pi}{2(M+1)} \tag{A-85}$$

from which

$$Wmn = \frac{Pmn + \frac{4\phi m}{M+1} \bar{\lambda}^s_n \cos \frac{m\pi}{2(M+1)}}{2\frac{R\gamma m}{a_1} + \frac{2S}{a_2} \gamma_n} \tag{A-86}$$

In the above expression the following notations have been used:

$$\gamma_m = 1 - \cos \frac{m\pi}{M+1} \qquad \gamma_n = 1 - \cos \frac{n\pi}{N}$$

$$\phi m = 1 - \frac{1}{2} \delta^0_m$$

The substitution of Eqs. (A-84) and (A-86) into Eq. (A-81) yields the expression

$$\sum_{m=0,2}^{M} \frac{\bar{S} - S}{b} (-2\gamma_n) \cos \frac{m\pi}{2(M+1)} Wmn - \bar{\lambda}^s_n = 0 \tag{A-87}$$

If the results given by Eq. (A-87) are substituted into Eq. (A-81), the following solution is obtained:

$$\bar{\lambda}_n^s = \frac{-2\gamma_n \dfrac{\bar{S}-S}{a_2} \displaystyle\sum_m \frac{\cos\dfrac{m\pi}{2(M+1)} P_{mn}}{\dfrac{2R}{a_1} r_m + \dfrac{2S}{a_2} r_n}}{1 + \dfrac{8}{M+1} \dfrac{\bar{S}-S}{a_2} r_u \displaystyle\sum_m \frac{\cos^2 \dfrac{m\pi}{2(M+1)}}{\dfrac{2R\gamma_m}{a_1} + \dfrac{2S\gamma_n}{a_2}}} \tag{A-88}$$

where the index m = 0, 2, 4, M even integer values.

The displacement of the cable net can now be obtained by substituting the solution for $\bar{\lambda}_n^s$, Eq. (A-88) into Eq. (A-86).

A similar solution can be found for the anti-symmetric case by letting the index m take odd integer values only in Eq. (A-88)

APPENDIX B

DEFINITIONS AND FORMULAS FROM THE CALCULUS OF FINITE DIFFERENCES

A list of operators from the Calculus of Finite Differences used in this work follows:

Let f (α) be a discrete function defined only in the region of the integer numbers. Then, the following operators can be properly defined:

First Order Operators

Boole's Displacement Operator

$Ef(\alpha) = f(\alpha+1)$

$E^{\pm n} f(\alpha) = f(\alpha \pm n)$

First Forward Difference Operator: Delta

$\Delta f(\alpha) = f(\alpha+1) - f(\alpha) = (E-1) f(\alpha)$

First Backward Difference Operator: Nabla

$\nabla f(\alpha) = f(\alpha) - f(\alpha-1) = (1-E^{-1}) f(\alpha)$

Forward Mean Operator: Nu

$N f(\alpha) = \frac{1}{2}[f(\alpha)+f(\alpha+1)] = \frac{1}{2}(E+1) f(\alpha)$

Backward Mean Operator: Un

И$f(\alpha) = \frac{1}{2}[f(\alpha)+f(\alpha-1)] = \frac{1}{2}(1+E^{-1}) f(\alpha)$

Second Order Operators

Second Central Difference Operator: Debla

$\not\nabla f(\alpha) = f(\alpha+1) - 2f(\alpha) + f(\alpha-1) = (E-2+E^{-1}) f(\alpha)$

Mean Difference Operator: Multa

$\not\Delta f(\alpha) = f(\alpha+1) - f(\alpha-1) = (E-E^{-1}) f(\alpha)$

For functions of more than one independent variable partial operators E_i, Δ_i, ∇_i, N_i', M_i, $\mathscr{N}_i$, $\mathscr{B}_i$ are defined in an analoguous manner. For example:

$$E_1 f(\alpha_1, \alpha_2) = f(\alpha_1+1, \alpha_2)$$

$$\Delta_1 f(\alpha_1, \alpha_2) = f(\alpha_1+1, \alpha_2) - f(\alpha_1, \alpha_2)$$

Inverse Delta or Summation Operator

The inverse delta represented by Δ^{-1} behaves similar to the integration symbol in the continuum. It is defined as

$$\Delta^{-1} f(\alpha) \Big|_{\alpha=1}^{M+1} = \sum_{\alpha=1}^{M} f(\alpha)$$

Formulas for the above operation are listed in standard book of Calculus of Finite Differences. The inversion of a product of two functions is usually obtained by using a technique similar to the integration by parts. The summation by parts can be accomplished by the following formula:

$$\Delta^{-1} [f(\alpha) \Delta g(\alpha)] = f(\alpha) g(\alpha) - \Delta^{-1} [\Delta f(\alpha) Eg(\alpha)]$$

Difference Equations

The equation

$$[A_n \Delta^n + A_{n-1} \Delta^{n-1} \cdots\cdots + A_2 \Delta^2 + A_1 \Delta + A_0] f(\alpha) = U(\alpha)$$

which related the unknown function $f(\alpha)$ to its difference, is called a difference equation of the order n. The coefficients An may be functions of α. If they are constants the equation is a linear ordinary difference equation of the order n.

Partial Difference Equations

A difference equation which is a function of two or more independent variables is called partial difference equation. For example:

$$G(\alpha_1, \alpha_2, f, \Delta f, \Delta_2 f, \Delta_1^2 f \cdots\cdots \Delta_1^m f, \Delta_2^n f) = 0$$

involves difference of two variables and, therefore, is designated as a partial difference equation of the order m and n with respect to the variables α_1 and α_2.

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