Analysis of Continuous Beams & Cable Nets by Finite Difference Calculus

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Takeshi OSHIRO* Naohiko TOKASHIKI**

1. Introduction

With the application of high-speed electronic computers, the matrix methods have become prominent means for analyzing complex engineering structures and the advantages of the matrix methods have been shown in many cases. While the techniques of the matrix methods are simple and straight forwards, a problem of large matrix orders becomes apparent in general caces, which results in the need for large storage facilities in a computer. Also the round-off error and the excessive computation time make the methods less applicable.

Generally, many engineering structures are arranged in uniform (i. e., beams on equidistant supports, cables with equal spaces.) For such structures, the discrete field analysis gives more advantages and it will be shown that this method is superior to the matrix methods. This is the direct application of finite difference calculus and the mathematical models are difference equations. Exact closed form field solution can be found for many regular cases and such solutions are valid over entire structures. Therefore, the solutions are essentially independent of the size of structures, that is, the same solution form holds for the structure with a very small or a very large number of nodes or elements. For more complex structures with irregular patterns, this method also can be applied. A closed form solutions are not to be found easily for these cases. However, the numerical technique called as "walk through" gives solutions, which is out of presentation of this paper. This paper presents the application of the discrete field analysis using simple examples such as continuous beams and cable nets. The closed form solutions are obtained, which are in the forms of single and double finite Fourier series.

While the same examples are analyzed by the first author using the variational technique,[5] this paper presents the direct application of the force equilibrium on node points. It will be noticed that same difference equations are obtained. The validity of this technique can be proven theoretically by the general concept of the variational technique. Comparing with the variational technique, this method yields simple forms. This paper emphasizes on the numerical calculations, which were not presented in the previous paper.[5]

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2. Analysis of Continuous Beams

A continuous beam on equidistant supports in Fig. 1 is considered. The governing equation and the boundary conditions are obtained which are a set of second order difference equations. Corresponding closed form solution is obtained in the form of single Fourier series.

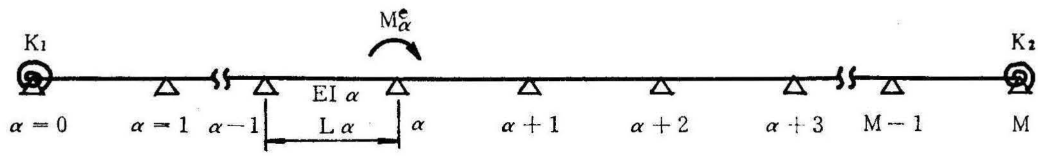


Fig. 1 Continuous Beam

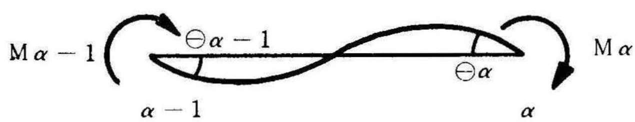


Fig. 2 Beam Element

2. 1 Governing Equation

The governing equation for the subject problem is found by substituting the general force-defomation relations(2) into the equilibrium equation which is obtained by summing joint moments at a typical support.

The equilibrium equation at a typical supports (α) in Fig. 3 is

$$M_\alpha^R + M_\alpha^L = M_\alpha^e \tag{1}$$

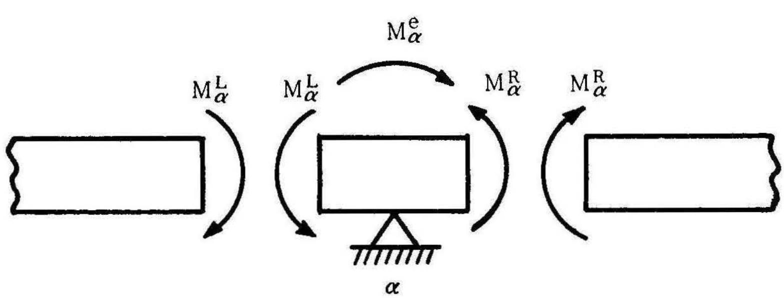


Fig. 3 Equilibrium of Moments at a Typical Node

in which M_e^α is the equivalent applied joint moment load, that is, the actual applied joint moment minus the result of fixed end moments. Fixed end moments are these dueto midspan loads when joint deformations are zero. The general force-deformation relations for this problem are written as(2)

$$\left.\begin{aligned} M_\alpha^R &= bk(\triangle + r)\theta_\alpha \\ M_\alpha^L &= bk(r - \nabla)\theta_\alpha \end{aligned}\right\} \qquad (2\,a.\,b)$$

where k, beam stiffness, equals flexural rigidity at a reference point divided by beam length; $EI\alpha / L\alpha$, and b and r are slope-deflection coefficients, which are 2 and 3 for prismatic members, respectively. The symbols, $\triangle$ and ∇, are the first forward and backward operators, respectively and are defined as $\triangle\theta(\alpha) = \theta(\alpha+1) - \theta(\alpha)$, and $\nabla\theta(\alpha) = \theta(\alpha) - \theta(\alpha-1)$, respectively.

Substitution of the force-deformation relations, Eqs. (2a, b) into the equilibrium equation, Eq. (1) yields the following equation as,

$$bk(\nabla\!\!\!/ + 2r)\theta_\alpha - M_\alpha^e = 0 \qquad (3)$$

where $\nabla\!\!\!/$ denotes the second central difference operator and is defined as $\nabla\!\!\!/\,\theta(\alpha) = \theta(\alpha+1) - 2\theta(\alpha) + \theta(\alpha-1)$.

Equation (3) represents the governing difference equation of the continuous beam.

2.2 Boundary Conditions

Considering the moment equilibrium at $\alpha = 0$ in Fig. 4 the boundary condition is obtained as

$$M_0^R + K_0\,\theta_0 - M_0^e = 0 \qquad (4)$$

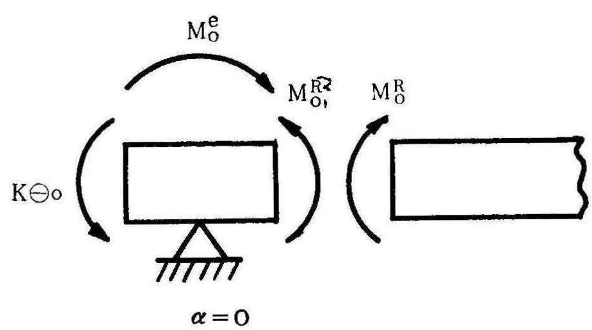


Fig. 4 Equilibrium of Moments at Boundary ($\alpha = 0$)

where K_0 is the spring constant at the boundary.

The substitution of the force-deformation relation, Eq. (2a), into the above equation, Eq. (4), yields

$$bk(\triangle + r)\theta_0 + K_0\,\theta_0 - M_0^e = 0 \qquad (5)$$

Similarly, at the other boundary at $\alpha = M$, the boundary condition is obtained as

$$bk(r - \nabla)\theta_M + K_M\theta_M - M_M^e = 0 \qquad (6)$$

where K_M is the spring constant at $\alpha = M$.

2. 3 Modification of Governing Equation and Boundary Conditions

Several attempts have been tried to solve the second order difference equations, Eqs. (3), (5) and (6).[(1) (2) (6)] However, none of these gave a satisfactory solution. If possible, Fourier series solution is known to be the simplest form. The difficulties is to find the series solution to satisfy both the governing equation and the boundary conditions. The significance of the following technique is to modified the series solution to satisfy all the necessary conditions.

The solution is going to be assumed for this problem as Eq. (18) and the following proper orthogonality relation in Eq. (7), is to be applied.

$$\sum_{\alpha=0}^{M} w_\alpha \cos \lambda_m \alpha \cdot \cos \lambda'_m \alpha = \begin{cases} 0 & \lambda_m \neq \lambda'_m \\ \sum_{\alpha=0}^{M} w_\alpha \cos^2 \lambda_m \alpha & \lambda_m = \lambda_m \end{cases} \qquad (7)$$

where $\lambda_m = m\pi/M$ and $\lambda'_m = m'\pi/M$.

Accordingly, the governing equation, Eq. (3), and the boundary conditions, Eqs. (5) and (6), have to be modified. The governing equation, Eq. (3), is to be written including both boundaries; $\alpha = 0$ and $\alpha = M$:

$$w_\alpha \left[bk(\nabla + 2r)\theta_\alpha - \frac{M^e_\alpha}{w_\alpha} \right] = 0 \qquad (8)$$

where w_α is the weighting function, and is defined as

$$w_\alpha = \begin{cases} \frac{1}{2} & \alpha = 0 \text{ and } M \\ 1 & \alpha \neq 0 \text{ and } M \end{cases}$$

Since the governing equation extended over the boundaries, the boundary conditions, Eqs. (5) and (6), have to be modified accordingly. The introduction of modification factors made possible to find closed form Fourier series solution[(5)] and the same factors are also introduced here. The physical meanings of the factors are explained as the forces applied at boundaries which connect the modified boundary conditions and the governing equations. Using the modification factors, the assumed finite Fourier series solution is modified to satisfy all the neccesary conditions.

The boundary condition at $\alpha = 0$, Eq. (5), can be rewritten as

$$bk(\Delta + r)\theta_0 + K_0\theta_0 - M^e_0$$
$$= \left(\frac{bk}{2}\nabla\theta_0 + K_0\theta_0 + \frac{\lambda^1}{2}\right) + \frac{1}{2}\left[bk(\nabla + 2r)\theta_0 - 2M^e_0 - \lambda^1\right] \qquad (9)$$

Also, the boundary condition at $\alpha = M$ can be rewritten as

$$b k(\alpha-\nabla)\theta_M+K_M\theta_M-M_M^e$$
$$=\left(\frac{-bk}{2}\varDelta\theta_M+K_M\theta_M+\frac{\lambda^2}{2}\right)+\frac{1}{2}\left[b k(\varDelta+2\gamma)\theta_M-2M_M^e-\lambda^2\right] \tag{10}$$

where the symbol $\varDelta$, multa, denotes the second mean difference operotor, and is defined as $\varDelta\theta(\alpha)=\theta(\alpha+1)-\theta(\alpha-1)$.

The terms in the first bracket on the right side of Eqs. (9) and(10) represent the modified boundary conditions at $\alpha=o$ and $\alpha=M$, respectively. Therefore, the problem becomes to find the solution to satisfy the following equations.

At $\alpha=0,1,\sim,M$

$$w_\alpha\left[b k(\varDelta+2\gamma)\theta_\alpha-\frac{M_\alpha^e}{w_\alpha}-\lambda^1\delta_\alpha^o-\lambda^2\delta_\alpha^M\right]=0$$

At $\alpha=0$,

$$\frac{bk}{2}\varDelta\theta_o+K_o\theta_o+\frac{\lambda^1}{2}=0 \qquad \text{(11 a, b, c,)}$$

At $\alpha=M$,

$$\frac{-bk}{2}\varDelta\theta_M+K_M\theta_M+\frac{\lambda^2}{2}=0$$

where kronecker delta function, $\delta_\alpha^{\alpha_1}=0$ for $\alpha\neq\alpha_1$ and $\delta_\alpha^{\alpha_1}=1$ for $\alpha=\alpha_1$, is used.

Exactly same equations are obtained by the first author[5] and, consequently, the procedures to find the solution are same. Therefore, further explanations are considered to be unnecessary. Instead, several examples are calculated to present the significance of this technique.

3. Numerical Procedures

While the following examples can be calculated even by a desk calculator, the computer FACOM 230−15, of computer-center, university of the Ryukyus, was used for convenience.

The examples are treated as the superposition of symmetric and anti-symmetric case, which makes the calculation simple. The following procedures are followed for this calculation.

For symmetric case, the modification factor, λ^s, is calculated first, which is written as

$$\lambda^s=\frac{-2K\sum_{m=0,2,4,}^{M}\frac{M_m}{C_m}}{K\sum_{m=0,2,4,}^{M}\frac{1}{C_m\,\Gamma_m}+\frac{1}{2}} \tag{12}$$

where $C_m = 2bk(\cos\lambda_m - 1 + r)$, K is spring constant and M_m is the Euler coefficient of finite Fourier series which is written as

$$M_m = \frac{1}{\Gamma_m}\sum_{\alpha=0}^{M} M_{\alpha}^{e}\cos\lambda_m\alpha \tag{13}$$

where

$$\Gamma_m = \sum_{\alpha=0}^{M} w_{\alpha}\cos^2\lambda_m\alpha = \begin{cases}\frac{M}{2} & m\neq 0, M\\ M & m=0, M\end{cases} \tag{14}$$

For an external load with the magnitude of one applied at $\alpha=\alpha_o$, Kronecker delta function is used as

$$M^{e}_{\alpha=\alpha_o} = \delta^{\alpha_o}_{\alpha} \tag{15}$$

and the coefficient, M_m, is obtained as

$$M_m = \frac{1}{\Gamma_m}\cos\lambda_m\alpha_o \tag{16}$$

The coefficientM_m and $\lambda^{\$}$ are substituted into the coefficient of rotations, θ_m, which takes the form as

$$\theta_m = \left[M_m + \frac{\lambda^{\$}}{\Gamma_m}\right]/C_m \tag{17}$$

Finally, the rotations at any nodes are calculated by the following solution

$$\theta(\alpha) = \sum_{m=0,2,4}^{M}\theta_m\cos\lambda_m\alpha \tag{18}$$

For anti-symmetric case, exactly same equations are applied except that the integer number m takes only odd.

4. Example : Continuous Beam on Equidistant Supports with various Spring Constants at End Boundaries

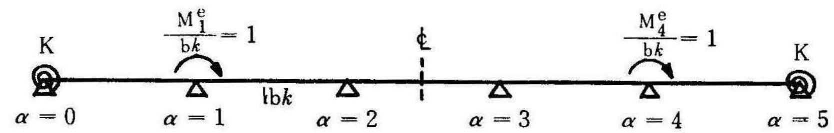
4.1 Symmetirc Case

The following data are used to calculate rotations at $\alpha = 0, 1, 2, 3, 4, 5$ and the resuts are listed in Table 1. Data: $M=5$, $M_1^e/bk = M_4^e/bk = 1$, $r=3$, K=variable.

While the spring constant takes several values, the equations are not necessary to be changed. They are calculated changing the value of K in the Euler coefficient, θ_m. External loads are applied at $\alpha=1$ and $\alpha=4$, symmetrically with respect to M/2, that is, $\alpha=1$ and $\alpha=4$ for this case. The values of M_1^e/bk and M_4^e/bk are assumed to be magnitude of one.

According to general structural analyses with Fourier series, Eq. (18), is never assumed to solve the continuous beam with fixed end (i.e., K=∞), because zero deformations at boundaries are not obtained by cosine series. However, the modification of cosine series makes the solution exact, which is one of the significance of this techniques.

Table 1 Numerical Results for Symmetric Case

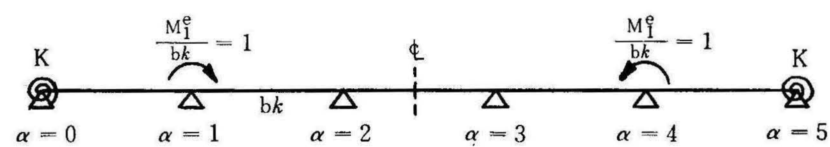


K \ θ α	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5
0	−0.1515	0.3030	−0.0606	−0.0606	0.3030	−0.1515
1	−0.0962	0.2885	−0.0577	−0.0577	0.2885	−0.0962
10	−0.0224	0.2691	−0.0538	−0.0538	0.2691	−0.0224
100	−0.0026	0.2638	−000528	−0.0528	0.2638	−0.0026
∞	0.0000	0.2632	−0.0526	−0.0526	0.2632	0.0000

4.2 Anti-symmetric Case.

Same problem as described above is calculated with external loads at $\alpha=1$ and 4, applied anti-symmetricaly with respect to M/2. For this case, the integer number m takes only odd numbers, that is, m=1, 3, 5,. Results are listed in Table 2.

Table 2 Numerical Results for Anti-Symmetric Case



K \ θ α	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5
0	−0.1579	0.3158	−0.1053	0.1053	−0.3158	0.1579
1	−0.1000	0.3000	−0.1000	0.1000	−0.3000	0.1000
10	−0.0233	0.2791	−0.0930	0.0930	−0.2791	0.0233
100	−0.0027	0.2735	−0.0912	0.0912	−0.2735	0.0027
∞	0.0000	0.2727	−0.0909	0.0909	−0.2727	0.0000

4.3 Continuous Beam with 20 Spans

The significance of this technique can be understood easily if the number of the span becomes large. According to general matrix methods, corresponding simultaneous equations have to be solved, where high matrix inversion is required. For instance, the example shown in Fig. 5 requires the inversion of 11×11 matrix. However the calculation proposed here can be done even by using a desk calculator since the summation with respect to integer number m in Eq. (18), is only required.

This example is calculated by two methods and the results are listed in Table 3, which shows the same results if allowable errors are considered.

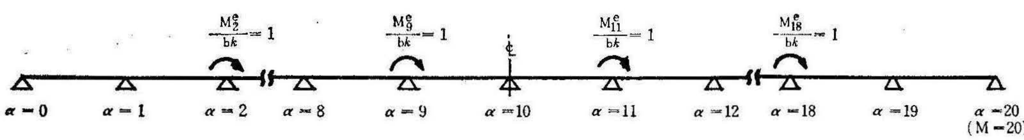


Fig. 5 A Continuous Beam with Simple Supports

Table 3 Numerical Comparisons for Continuous Beam in Fig. 5

θ \ α	θ_0	θ_1	θ_2	θ_3	θ_7	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}
Closed Form	0.0414	−0.0829	0.2901	−0.0776	0.0204	−0.0040	−0.0045	0.0218	−0.0828	0.3094	−0.1547
Matrix Form	0.0414	−0.0829	0.2901	−0.0776	0.0204	−0.0039	−0.0044	0.0218	−0.0827	0.3094	−0.1547

5. Analysis of Cable Net

The application of the technique to the two dimensional case is illustrated in the analysis of cable net in Fig. 6.

The net shown in Fig. 6 is assumed to have simple supports at $\alpha_2 = 0$, N and free supports at $\alpha_1 = 0$, M. Therefore, it allows boundary deflections at the later edges. R, S, and $\overline{S}$ are horizontal component of cable tensions in cables parallel to α_1 axis, α_2 axis, (except boundaries) and at boundaries, respectively.

5.1 Governing Equation

The general governing equation at a typical node (α_1, α_2) is obtained by summing the forces nomal to the reference line as shown in Fig. 7.

The nomal components of the forces are obtained by simple geometry shown in Fig. 7 and the result takes the form as

$$R\left(\frac{\Delta_1 w}{a_1}\right) - RE^{-1}\left(\frac{\Delta_1 w}{a_1}\right) + S\left(\frac{\Delta_2 w}{a_2}\right) - SE^{-1}\left(\frac{\Delta_2 w}{a_2}\right) = -P(\alpha_1, \alpha_2) \tag{19}$$

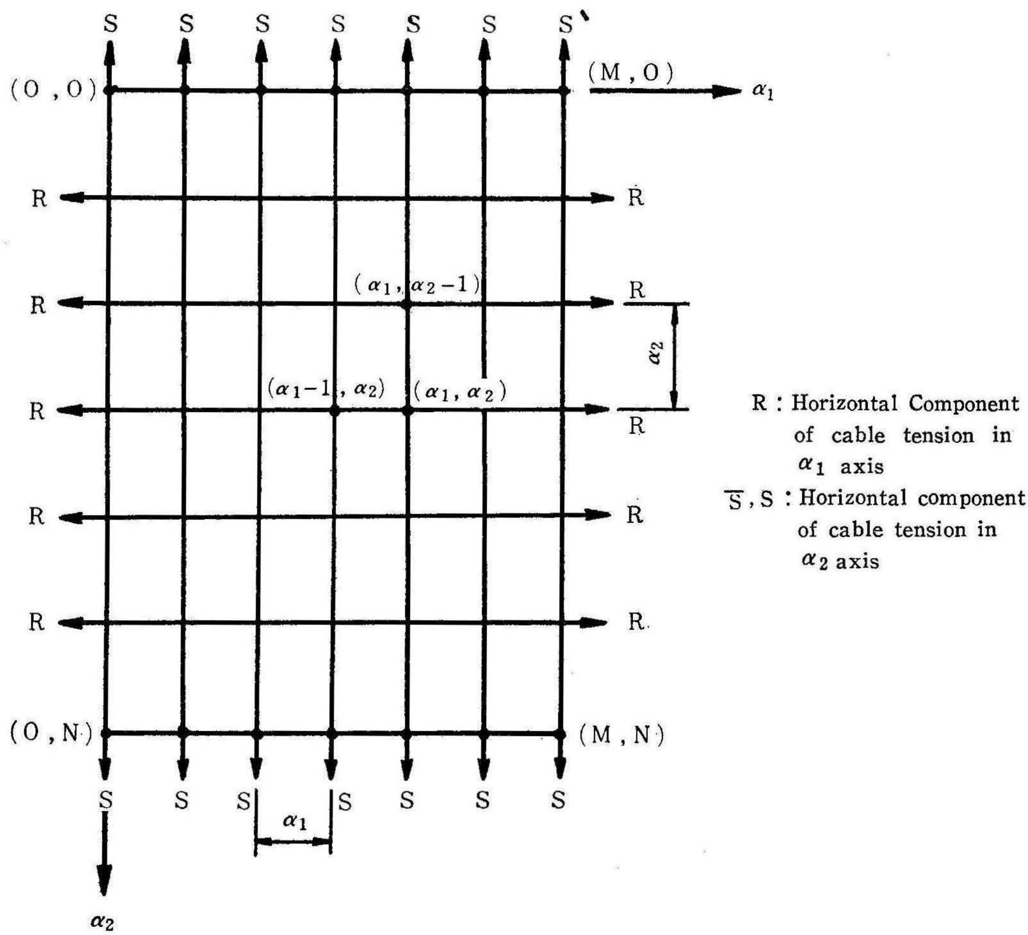


Fig. 6 Cable Net with Simple and Free Supports

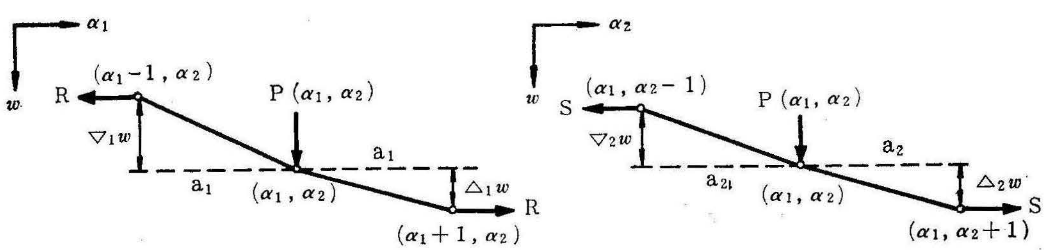


Fig. 7 Polygon Elements for α_1 and α_2 Direction

where E^{-1} is the Boole's backward displacement oprator defined as $E_1^{-1} w(\alpha_1, \alpha_2) = w(\alpha_1 - 1, \alpha_2)$ and $E_2^{-1} w(\alpha_1, \alpha_2) = w(\alpha_1, \alpha_2 - 1)$.

Eq. (19), is the governing equation of the cable net and is rewritten using difference operetors as

$$\frac{R}{a_1} \Delta_1 \omega + \frac{S}{a_2} \Delta_2 \omega = -P \tag{20}$$

where symbols ∇_1 and ∇_2, denote the second partial difference operators with respect to the variable α_1 and α_2, respectively. For instance, the definition of the operator for the variable α_1 is, $\nabla_1 w(\alpha_1, \alpha_2) = w(\alpha_1+1, \alpha_2) - 2w(\alpha_1, \alpha_2) + w(\alpha_1-1, \alpha_2)$.

It has to be noticed that Eq. (20) is the second order of the partial difference equation.

5.2 Bounbary Conditions

Similarly, the equilibrium of the nomal forces at the boundaries $\alpha_1 = 0$ and $\alpha_1 = M$ are obtained, which is written as at $\alpha = 0$,

$$R\left(\frac{\Delta_1 w}{a_1}\right) + \bar{S}\left(\frac{\Delta_2 w}{a_2}\right) - \bar{S}E^{-1}\left(\frac{\Delta_2 w}{a_2}\right) = -P \tag{21}$$

or, $$\frac{R}{a_1}\Delta_1 w + \frac{\bar{S}}{a_2}\nabla_2 w = -P \tag{21a}$$

and, at $\alpha = N$,

$$-R\left(\frac{\nabla_1 w}{a_1}\right) + \bar{S}\left(\frac{\Delta_2 w}{a_2}\right) - \bar{S}E^{-1}\left(\frac{\Delta_2 w}{a_2}\right) = -P \tag{22}$$

or, $$-\frac{R}{a_1}\nabla_1 w + \frac{\bar{S}}{a_2}\nabla_2 w = -P \tag{22a}$$

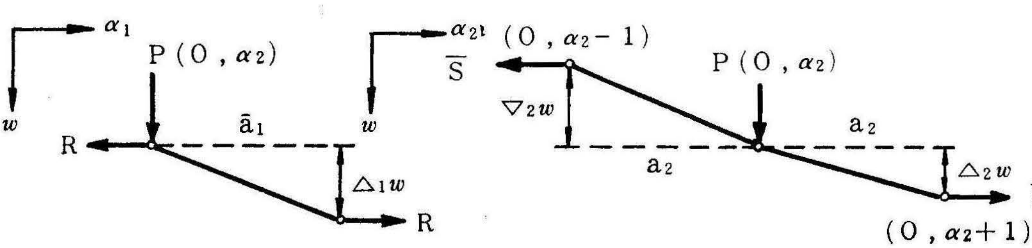


Fig. 8 Polygon Elements for α_1 and α_2 Direction at Boundary $\alpha_1 = 0$

The boundary conditions at $\alpha_2 = 0$ and N are simple supports, which take the equation as

$$w(\alpha_1, 0) = w(\alpha_1, N) = 0 \tag{23}$$

To provide closed form solutions for considered above, the modification factors, $\lambda^1(\alpha_2)$, $\lambda^2(\alpha_2)$ are introduced into Eqs. (20), (21*a*) and (22*a*) as similar to the previous case, where $\lambda^1(\alpha_2)$ and $\lambda^2(\alpha_2)$ are considered physically as forces applied at the boundaries, i. e., $\alpha_1 = 0$, $\alpha_1 = M$, and $0 < \alpha_2 < N$. The boundary conditions, Eqs. (21*a*) and (22*a*), are modified as follows;

At $\alpha_1 = 0$

$$\frac{R}{a_1}\Delta_1 w + \frac{\overline{S}}{a_2}\mathcal{D}_2 w + P = \left[\frac{R}{a_1}\nabla_1 w + \frac{\overline{S}-S}{a_2}\mathcal{D}_2 w - \lambda^1(\alpha_2)\,\delta^0_{\alpha_1}\right] + \left[\frac{R}{a_1}\mathcal{D}_1 w + \frac{S}{a_2}\mathcal{D}_2 w + P + \lambda^1(\alpha_2)\,\delta^0_{\alpha_1}\right] \tag{24}$$

and, at $\alpha_1 = M$

$$-\frac{R}{a_1}\nabla_1 w + \frac{\overline{S}}{a_2}\mathcal{D}_2 w + P = \left[-\frac{R}{a_1}\Delta_1 w + \frac{\overline{S}-S}{a_2}\mathcal{D}_2 w - \lambda^2(\alpha_2)\,\delta^M_{\alpha_1}\right] + \left[\frac{R}{a_1}\mathcal{D}_1 w + \frac{S}{a_2}\mathcal{D}_2 w + P + \lambda^2(\alpha_2)\,\delta^M_{\alpha_1}\right] \tag{25}$$

where $\delta^{\overline{\alpha}_1}_{\alpha_1}$ is Kronecker delta function defined as $\delta^{\overline{\alpha}_1}_{\alpha_1} = 1$ for $\alpha_1 = \overline{\alpha}_1$ and $\delta^{\overline{\delta}_1}_{\alpha_1} = 0$ for $\alpha_1 \neq \overline{\alpha}_1$.

The factors, $\lambda^1(\alpha_2)$ and $\lambda^2(\alpha_2)$, are the modification factors at $\alpha_1 = 0$, and $\alpha_1 = M$, respectively, where α_2 take the value $0 < \alpha_2 < N$.

The terms in the first bracket of the right side of Eqs. (24) and (25), are the modified boundary conditions at $\alpha_1 = 0$ and $\alpha_1 = M$, and the terms in the second one are exactly same form as the governing equation, Eq. (20). Therefor, the governing equation, Eq. (20), can be extended over the boundaries at $\alpha_1 = 0$ and $\alpha_1 = M$, and is written as

$$\frac{R}{a_1}\mathcal{D}_1 w + \frac{S}{a_2}\mathcal{D}_2 w + P + \lambda^1(\alpha_2)\,\delta^0_{\alpha 1} + \lambda^2(\alpha_2)\,\delta^M_{\alpha 1} = 0 \tag{26}$$

which is valid over all interior and also boundary nodes at $\alpha_1 = 0$, M. From Eqs. (24) and (25), the modified boundary conditions are written as

at $\alpha_1 = 0$

$$\frac{R}{a_1}\nabla_1 w + \frac{\overline{S}-S}{a_2}\mathcal{D}_2 w - \lambda^1(\alpha_2)\,\delta^0_{\alpha_1} = 0 \tag{27}$$

at $\alpha_1 = M$

$$-\frac{R}{a_1}\Delta_1 w + \frac{\overline{S}-S}{a_2}\mathcal{D}_2 w - \lambda^2(\alpha_2)\,\delta^M_{\alpha_1} = 0 \tag{28}$$

and, at $\alpha_2 = 0$ and N

$$w(\alpha_1, \alpha_2) = 0 \tag{29}$$

It has to be noticed that above equations are exactly same as ones obtained.[5] Therefore, further procedures follow same. Only the results are shown in the numerical example without any explanations.

6. Numerical Example of Cable Net

The following example is treated as symmetric and anti - symmetric case which describe the load conditions applied symmetrically and anti - symmetrically with respect to $\alpha_1 = M/2$.

The numerical calculation for symmetric case starts to find the Euler coefficient of the external load which is assumed as double Fourier series

$$P(\alpha_1, \alpha_2) = \sum_{m=0,2,4,}^{M} \sum_{n=1,2,}^{N-1} P_{mn} \cos\frac{m\pi}{M+1}\left(\alpha_1 + \frac{1}{2}\right) \cdot \sin\frac{n\pi}{N}\alpha_2 \tag{30}$$

$$P_{mn} = \frac{4\phi_m}{N(M+1)} \sum_{\alpha_1=0,2}^{M} \sum_{\alpha_2=1,2}^{N-1} P(\alpha_1, \alpha_2) \cos\frac{m\pi}{M+1}\left(\alpha_1 + \frac{1}{2}\right) \cdot \sin\frac{n\pi}{N}\alpha_2 \tag{31}$$

A unit load applied at a node $(\overline{\alpha}_1, \overline{\alpha}_2)$ is written as

$$P(\alpha_1, \alpha_2) = \delta_{\alpha_1}^{\overline{\alpha}_1} \delta_{\alpha_2}^{\overline{\alpha}_2} \tag{32}$$

for which the Euler coefficient is obtained as

$$P_{mn} = \frac{4\phi_m}{N(M+1)} \sum_{\alpha_2=1,2,}^{M} \sum_{\alpha_1=0,2}^{N-1} \cos\frac{m\pi}{M+1}\left(\overline{\alpha}_1 + \frac{1}{2}\right) \cdot \sin\frac{n\pi}{N}\overline{\alpha}_2 \tag{33}$$

The modification factors for symmetric case are assumed as the following series function in α_2 axis as

$$\lambda^{\$}(\alpha_2) = \sum_{n=1,2,}^{N-1} \lambda_n^{\$} \sin\frac{n\pi}{N}\alpha_2 \tag{34}$$

where $\lambda_n^{\$}$ denotes the Euler coefficient of the modification function. The Euler coefficient, $\lambda_n^{\$}$, is obtained as

$$\lambda_n^{\$} = \frac{-2r_n \dfrac{\overline{S}-S}{a_2} \displaystyle\sum_{m=0,2,4,}^{M} \dfrac{\cos\dfrac{m\pi}{2(M+1)} \cdot P_{mn}}{\dfrac{2Rr_m}{a_1} + \dfrac{2Sr_n}{a_2}}}{1 + \dfrac{\overline{S}-S}{a_2} r_n \displaystyle\sum_{m=0,2,4,}^{M} \dfrac{\dfrac{8\phi_m}{M+1}\cos^2\dfrac{m\pi}{2(M+1)}}{\dfrac{2Rr_m}{a_1} + \dfrac{2Sr_n}{a_2}}} \tag{35}$$

where

$$\left.\begin{aligned} \gamma_m &= 1-\cos\frac{m\pi}{M+1} \\ \gamma_n &= 1-\cos\frac{n\pi}{N} \\ \phi_m &= 1-\frac{1}{2}\delta_m^o \end{aligned}\right\} \qquad (36\ a,\ b,\ c)$$

The values of $\lambda_n^{\$}$ for n=1,~, N are substituted into the Euler coefficient of the displacement, Eq. (37) and the displacements at any nodes, Eq. (38), are obtained as follow;

$$w_{mn}=\frac{P_{mn}+\dfrac{4\phi_m}{M+1}\lambda_n^{\$}\cos\dfrac{m\pi}{2(M+1)}}{2\dfrac{R\gamma_m}{a_1}+2\dfrac{S\gamma_n}{a_2}} \qquad (37)$$

$$w(\alpha_1,\ \alpha_2)=\sum_{m=0,2,4,}^{M}\ \sum_{n=1,2,}^{N-1} w_{mn}\cos\frac{m\pi}{M+1}(\alpha_1+\frac{1}{2})\cdot\sin\frac{n\pi}{N}\alpha_2 \qquad (38)$$

For anti-symmetric case, same equations hold except the integer number, m, takes only odd numbers.

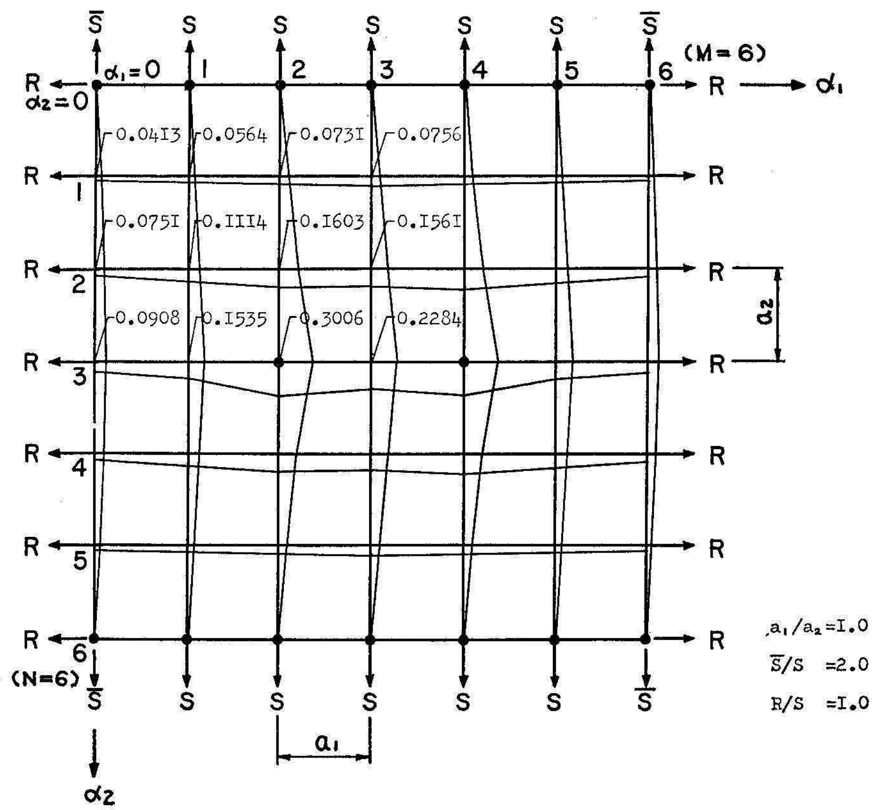


Fig. 9 Results of Calculation for Cable Net

The results of the numerical calculation for symmetric case are shown in Fig. 9, where the following data are used; $M=6$, $N=6$, $a_1/a_2=1$, $\overline{S}/S=2$, $R/S=1$, $P(2,3)=P(4,3)=1$.

Same example is calculated using a matrix method, which requires the inversion of 12×12 matrix. The numerical comparisons between two methods are shown in Table 4 which indicates validity of the technique proposed. Eqs. (35), (37) and (38), require only summations with respect to the integer numbers, which are calculated without any difficulties.

Table 4 Numerical Comparisons for Cable Net in Fig. 9

(α_1, α_2)	$w(\alpha_1, \alpha_2)$	(α_1, α_2)	$w(\alpha_1, \alpha_2)$	(α_1, α_2)	$w(\alpha_1, \alpha_2)$	(α_1, α_2)	$w(\alpha_1, \alpha_2)$
(0,1)	0.0413*	(1,1)	0.0564*	(2,1)	0.0731*	(3,1)	0.0756*
	0.0413		0.0564		0.0731		0.0755
(0,2)	0.0751*	(1,2)	0.1114*	(2,2)	0.1603*	(3,2)	0.1561*
	0.0751		0.1114		0.1603		0.1561
(0,3)	0.0908*	(1,3)	0.1535*	(2,3)	0.3006*	(3,3)	0.2284*
	0.0908		0.1535		0.3006		0.2284

Note: Displacements indicated by * are the ones calculated by closed form solution and the others are caluculated by a matrix method.

7. Conclusion

The governing equation and the boundary conditions are obtained for a continuous beam and a cable net directly from the force equilibrium, where the derivations are easier than the variational method.[5] With the general concept shown in the variational method, the modification factors are also introduced to derive closed form solutions. The techniqus of the discrete field analysis are discussed with emphasis on the significance of finite Fourier series.

Numerical examples are illustrated in the analyses of continuous beams and cable nets. The results show the validity of the techniques proposed. Also, the procedures in the numerical examples indicate the significance in the calculations for which any matrix inversions are not required.

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