Remarks on the homology of a regular 2-cover

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REMARKS ON THE HOMOLOGY OF A REGULAR 2-COVER

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Abstract

Let $p : X \to Y$ be a regular 2-cover. We assume that the groups $H_*(X;\mathbb{Z})$ are torsion free. It is easy to show that $H_*(Y;\mathbb{Z})$ are odd-torsion free. It is also well-known that the torsion subgroup of $H_*(Y;\mathbb{Z})$ has exponent 2. Several applications are also known. Since the results are scattered and somehow hidden in the literature, a survey of these researches may be useful. This is the purpose of this note. Our applications include the study of the homology of planar polygon spaces.

1 Introduction

Let $p : X \to Y$ be a regular 2-cover. We assume that the groups $H_*(X;\mathbb{Z})$ are torsion free. It is well-known that $H_*(Y;\mathbb{Z})$ are odd-torsion free. In fact, if $\mathbb{F}$ is $\mathbb{Q}$ or $\mathbb{Z}_p$ (where p is an odd prime), then we have

$$H_*(Y;\mathbb{F}) \cong H_*(X;\mathbb{F})^{\Delta},$$

where Δ denotes the group os the deck transformations. Since $\dim_{\mathbb{F}} H_*(X;\mathbb{F})^{\Delta}$ is constant for all $\mathbb{F}$, it follows that $H_*(Y;\mathbb{Z})$ is odd-torsion free.

Then we naturally encounter the following question: Does the torsion subgroup of $H_*(Y;\mathbb{Z})$ have exponent 2? The following theorem is well-known.

Theorem A . (i) *Let $p : X \to Y$ be a regular 2 cover. We assume that $H_*(X;\mathbb{Z})$ are torsion free. If $x \in H_*(Y;\mathbb{Z})$ is a torsion element, then we have $2x = 0$.*

(ii) *In particular, for every $q \in \mathbb{N} \cup \{0\}$, there exist a_q and $b_q \in \mathbb{N} \cup \{0\}$ such that*

$$H_q(Y;\mathbb{Z}) \cong \bigoplus_{a_q} \mathbb{Z} \oplus \bigoplus_{b_q} \mathbb{Z}_2.$$

Corollary B . (i) *We keep the assumption of Theorem A. Let $PS_{\mathbb{F}}(Y)$ be the Poincaré polynomial of Y in variable t with coefficients in a field $\mathbb{F}$. If we know $PS_{\mathbb{Q}}(Y)$ and $PS_{\mathbb{Z}_2}(Y)$, then we can determine $H_*(Y;\mathbb{Z})$.*

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(ii) *More precisely, we set*

$$\Gamma(Y) = \sum_{q=0}^{\infty} b_q t^q.$$

Then we have

$$\Gamma(Y) = \frac{PS_{\mathbb{Z}_2}(Y) - PS_{\mathbb{Q}}(Y)}{1+t}.$$

Since the above results and their applications are scattered and somehow hidden in the literature, a survey of these researches may be useful. This is the purpose of this note. Our applications include the study of the homology of planar polygon spaces.

2 Proofs of the main results

Proof of Theorem A. (i) The proof is given implicitly in [5, 9.3.2]. Let

$$\mathrm{tr}^* : H_*(Y;\mathbb{Z}) \to H_*(X;\mathbb{Z})$$

be the transfer homomorphism. (See, for example, [5, 1.3.2].) Then we have

$$\pi_* \circ \mathrm{tr}^* = 2.$$

Since $H_*(X;\mathbb{Z})$ are torsion free, we have $\mathrm{tr}^*(x) = 0$. Hence $2x = 0$.

(ii) Note that (i) also implies that $H_*(Y;\mathbb{Z})$ are odd-torsion free. Hence the result follows from the fundamental theorem of finitely generated abelian groups. □

Proof of Theorem B. (ii) According to the universal coefficient theorem, there exists a split short exact sequence

$$0 \to H_q(Y;\mathbb{Z}) \otimes \mathbb{Z}_2 \to H_q(Y;\mathbb{Z}_2) \to \mathrm{Tor}(H_{q-1}(Y;\mathbb{Z});\mathbb{Z}_2) \to 0.$$

Recall that

$$\mathbb{Z} \otimes \mathbb{Z}_2 = \mathbb{Z}_2 \otimes \mathbb{Z}_2 = \mathbb{Z}_2,\ \mathrm{Tor}(\mathbb{Z},\mathbb{Z}_2) = 0 \ \text{ and } \ \mathrm{Tor}(\mathbb{Z}_2,\mathbb{Z}_2) = \mathbb{Z}_2.$$

This implies that if $H_{q-1}(Y;\mathbb{Z})$ contains $\mathbb{Z}_2$ as a direct summand, then this causes that

$$\dim_{\mathbb{Z}_2} H_i(Y;\mathbb{Z}_2) = \dim_{\mathbb{Q}} H_i(Y;\mathbb{Q}) + 1$$

for $i = q-1, q$. Hence we have

$$PS_{\mathbb{Z}_2}(Y) = PS_{\mathbb{Q}}(Y) + (1+t)\,\Gamma(Y)$$

and (ii) follows.

(i) is an immediate consequence of (ii). □

3 Applications

3.1 Closed surfaces

We set

$$T(n) = \underset{n}{\#} T^2 \quad \text{and} \quad P(n) = \underset{n}{\#} \mathbb{R}P^2.$$

For every $P(n)$, there is a regular 2-cover $p : T(n-1) \to P(n)$. In order to apply Corollary B, recall that

$$PS_{\mathbf{Q}}(P(n)) = 1 + (n-1)t$$

and

$$PS_{\mathbf{Z}_2}(P(n)) = 1 + nt + t^2.$$

Hence we obtain the well-known result that $\Gamma(P(n)) = t$.

3.2 X is a Stiefel manifold

We set $H_k = \{I_k, -I_k\}$. We define

$$X_{n,k} = SO(n+k)/SO(n) \quad \text{and} \quad Y_{n,k} = SO(n+k)/(SO(n) \times H_k).$$

The projection

$$p : X_{n,k} \to Y_{n,k} \tag{1}$$

is a regular 2-cover.

In order that $H_*(X_{n,k}; \mathbb{Z})$ are torsion free, we assume that $k = 1$ and $n \geq 1$, or $k = 2$ and n is even.

3.2.1 The case $k = 1$

Note that $p : X_{n,1} \to Y_{n,1}$ is the well-known regular 2-cover

$$p : S^n \to \mathbb{R}P^n. \tag{2}$$

Recall that

$$PS_{\mathbf{Q}}(\mathbb{R}P^n) = \begin{cases} 1 + t, & n \text{ is odd} \\ 1, & n \text{ is even.} \end{cases}$$

and

$$PS_{\mathbf{Z}_2}(\mathbb{R}P^n) = \sum_{q=0}^{n} t^q.$$

In fact, using the Gysin sequence for (2), we have

$$H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^{n+1})$$

and this gives $PS_{\mathbb{Z}_2}(\mathbb{R}P^n)$.

Using Corollary B, we obtain the well-known result:

$$\Gamma(\mathbb{R}P^n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} t^{2i+1}.$$

Note that the case for $n = 2$ is treated in 3.1.

3.2.2 The case $k = 2$

Note that $X_{n,2}$ is the unit tangent bundle of S^{n+1}. Since we are assuming n to be even, we have

$$H^*(X_{n,2}; \mathbb{Z}) \cong H^*(S^n \times S^{n+1}; \mathbb{Z}).$$

Using the Gysin sequence for (1), it is possible to prove that

$$PS_{\mathbf{Q}}(Y_{n,2}) = 1 + t^n + t^{n+1} + t^{2n+1}$$

and

$$PS_{\mathbb{Z}_2}(Y_{n,2}) = \left(\sum_{q=0}^{2n+1} t^q \right) + t^n + t^{n+1}.$$

Using Corollary B, we have

$$\Gamma(Y_{n,2}) = \sum_{i=1}^{n} t^{2i-1}.$$

But we can compute $H_*(Y_{n,2}; \mathbb{Z})$ more directly. Consider the oriented Grassmann manifold

$$\tilde{G}_2(\mathbb{R}^{n+2}) = SO(n+2)/\left(SO(n) \times SO(2)\right).$$

The cohomology ring $H^*(\tilde{G}_2(\mathbb{R}^{n+2}); \mathbb{Z})$ is well known. (See, for example, [10, p.129, Remark 4.8].)

$$H^*(\tilde{G}_2(\mathbb{R}^{n+2}); \mathbb{Z}) = \mathbb{Z}[t, s]/(t^{\frac{n}{2}+1} - 2ts, s^2),$$

where $\deg t = 2$ and $\deg s = n$.

Consider the Gysin sequence for the fiber bundle

$$S^1 \to Y_{n,2} \to \tilde{G}_2(\mathbb{R}^{n+2}).$$

Since $\pi_1(Y_{n,2}) = \mathbb{Z}_2$, the Euler class of the bundle is $2t$. Then, as modules, we have

$$H^*(Y_{n,2}; \mathbb{Z}) \cong H^*(S^n \times \mathbb{R}P^{n+1}; \mathbb{Z}).$$

3.3 X is a Lie group

3.3.1 $X = SP(n)$

Recall that $Z(Sp(n))$, the center of $Sp(n)$, is given by $Z(Sp(n)) = \{I_n, -I_n\}$. We apply Theorem A to the regular 2-cover

$$p : Sp(n) \to Sp(n)/Z(Sp(n)).$$

(The base space is sometimes written as $PSp(n)$.) Then the torsion subgroup of $H_*(Sp(n)/Z(Sp(n)); \mathbb{Z})$ has exponent 2. Let us apply Corollary B and study $\Gamma(Sp(n)/Z(Sp(n)))$.

First, [3, Proposition 9.2] tells us that

$$PS_{\mathbb{Q}}(Sp(n)/Z(Sp(n))) = PS_{\mathbb{Q}}(Sp(n)). \tag{3}$$

Second, [3, Théorme 11.3] tells us the following result: Let $s = 2^k$ be the maximal power of 2 which divides n. Then we have

$$PS_{\mathbb{Z}_2}(Sp(n)/Z(Sp(n))) = \left(\sum_{q=0}^{4s-1} t^q\right) \prod_{\substack{i=1 \\ i \neq s}}^{n} t^{4i-1}. \tag{4}$$

We can compute $\Gamma(Sp(n)/Z(Sp(n)))$ from (3) and (4).

Example 1.

$$\Gamma(Sp(1)/Z(Sp(1))) = t$$
$$\Gamma(Sp(2)/Z(Sp(2))) = t + t^3 + t^4 + t^5 + t^6 + t^8$$

and

$$\begin{aligned}\Gamma(Sp(3)/Z(Sp(3))) = t &+ t^3 + t^4 + t^5 + t^6 + t^7 + 2t^8 + t^9 + 2t^{10} \\ &+ t^{11} + 2t^{12} + t^{13} + t^{14} + t^{15} + t^{16} + t^{17} + t^{19}.\end{aligned}$$

Note that $Sp(1)/Z(Sp(1)) = SO(3)$.

3.3.2 $X = SU(n)$

Recall that

$$Z(SU(n)) = \{cI_n \mid c^n = 1\}.$$

In order that $-I_n \in Z(SU(n))$, we assume that n is even. As in 3.2, we set $H_n = \{I_n, -I_n\}$. Similarly to 3.3.1, we apply Theorem A to the covering map

$$p : SU(n) \to SU(n)/H_n.$$

Then the torsion subgroup of $H_*(SU(n)/H; \mathbb{Z})$ has exponent 2. Let us apply Corollary B and study $\Gamma(SU(n)/H)$.

First, [3, Proposition 9.2] tells us that

$$PS_{\mathbb{Q}}(SU(n)/H_n) = PS_{\mathbb{Q}}(SU(n)). \tag{5}$$

Second, [3, Théorme 11.4] tells us the following result: Let $s = 2^k$ be the maximal power of 2 which divides n. Then we have

$$PS_{\mathbb{Z}_2}(SU(n)/H_n) = \left(\sum_{q=0}^{2s-1} t^q\right) \prod_{\substack{i=2 \\ i \neq s}}^{n} t^{2i-1}. \tag{6}$$

We can compute $\Gamma(SU(n)/H)$ from (5) and (6).

Example 2.

$$\begin{aligned}
\Gamma(SU(2)/H_2) &= t \\
\Gamma(SU(4)/H_4) &= t + t^3 + t^4 + t^5 + 2t^6 + 2t^8 \\
&\quad + t^9 + t^{10} + t^{11} + t^{13}
\end{aligned}$$

and

$$\begin{aligned}
\Gamma(SU(6)/H_6) &= t + t^6 + t^8 + t^{10} + t^{12} + t^{13} + t^{15} + 2t^{17} \\
&\quad + t^{19} + t^{21} + t^{22} + t^{24} + t^{26} + t^{28} + t^{33}.
\end{aligned}$$

Note that $SU(2)/H_2 = SO(3)$.

3.4 $X = \mathbb{C}P^{2m+1}$

We define an involution $\tau : \mathbb{C}P^{2m+1} \to \mathbb{C}P^{2m+1}$ by

$$\tau[z_0, \ldots, z_{2m+1}] = [w_0, \ldots, w_{2m+1}],$$

where

$$w_{2i} = -\bar{z}_{2i+1} \quad \text{and} \quad w_{2i+1} = \bar{z}_{2i}, \quad 0 \leq i \leq m.$$

We set

$$X_m = \mathbb{C}P^{2m+1} \quad \text{and} \quad Y_m = \mathbb{C}P^{2m+1}/\tau.$$

Since τ is fixed point free, we have a regular 2-cover

$$p : X_m \to Y_m.$$

Theorem A tells us that the torsion subgroup of $H_*(Y_m;\mathbb{Z})$ has exponent 2. But much stronger result is known: By [11, Proposition 3.2], there are isomorphisms

$$H_*(Y_m;\mathbb{Z}) \cong H_*(\mathbb{H}P^m;\mathbb{Z}) \otimes H_*(\mathbb{R}P^2;\mathbb{Z}).$$

Note that $Y_0 = \mathbb{R}P^2$.

3.5 Generalized Enriques surfaces

Let X be a connected closed manifold of dimension 4. We assume that

(i) X is orientable,

(ii) $H_1(X;\mathbb{Z}) = 0$, and

(iii) there is a fixed point free involution $\tau : X \to X$ which preserves orientation.

We set $Y = X/\tau$.

Example 3. We have the following example for X and Y. (See [5, 9.1].) A nonsingular compact complex surface X is called a *generalized* $K3$*-surface* if

$$H_1(X;\mathbb{Z}) = 0 \quad \text{and} \quad w_2(X) = 0.$$

A *generalized Enriques surface* is a complex surface Y which

(i) has $w_2(Y) \neq 0$, and

(ii) can be obtained as the orbit space X/τ of a generalized $K3$-surface by a fixed point free holomorphic involution $\tau : X \to X$.

As an application of Theorem A, we claim the following:

Proposition 4. *We have*

$$\Gamma(Y) = t + t^2.$$

Proof. From the Gysin sequence for the regular 2-cover $p : X \to Y$, we have

$$H_q(Y;\mathbb{Z}_2) = \mathbb{Z}_2 \quad \text{for } q = 0, 1, 3 \text{ and } 4.$$

Since $H_1(X;\mathbb{Z}) = 0$, we have $H_1(Y;\mathbb{Q}) = 0$. Using Theorem A, we have $H_1(Y;\mathbb{Z}) = \mathbb{Z}_2$.

Since Y is orientable, Poincaré duality tells us that $H^3(Y;\mathbb{Z}) = \mathbb{Z}_2$. Hence $H_2(Y;\mathbb{Z}) = \mathbb{Z}_2$.

Using the fact that $H^1(Y;\mathbb{Z}) = 0$, we have $H_3(Y;\mathbb{Z}) = 0$. Since Y is orientable, we have $H_4(Y;\mathbb{Z}) = \mathbb{Z}$. □

3.6 Polygon spaces

Given a string $\ell = (l_1, \dots, l_n)$ of n positive real numbers $l_i > 0$, one considers the moduli space X_ℓ of closed planar polygonal curves having side lengths l_i. Points of X_ℓ parametrize different shapes of such polygons. Formally X_ℓ is defined as the orbit space

$$X_\ell = \left\{ (z_1, \dots, z_n) \in S^1 \times \cdots S^1 \;\middle|\; \sum_{i=1}^{n} l_i z_i = 0 \in \mathbb{C} \right\} / SO(2). \tag{7}$$

Here $z_i \in S^1 \subset \mathbb{C}$ denote the unit vectors in the directions of the sides of a polygon; the group of rotations $SO(2)$ acts diagonally on $(z_1, \dots, z_n)$.

The polygon spaces (7) come with a natural involution

$$\tau : X_\ell \to X_\ell, \quad \tau(z_1, \dots, z_n) = (\bar{z}_1, \dots, \bar{z}_n) \tag{8}$$

induced by complex conjugation. Geometrically, this involution associates to a polygonal shape the shape of the reflected polygon. We set $Y_\ell = X_\ell/\tau$. Let

$$p : X_\ell \to Y_\ell \tag{9}$$

be the projection.

The length vector ℓ is called *generic* if $\sum_{i=1}^{n} l_i \epsilon_i \neq 0$ for any choice $\epsilon = \pm 1$. It is known that for a generic length vector ℓ, the space X_ℓ is a closed smooth manifold of dimension $n-3$. Moreover, the involution (8) is fixed point free. Hence (9) is a regular 2-cover.

The moduli space X_ℓ of planar polygonal linkages were studied extensively by many mathematicians. For example, the groups $H_*(X_\ell; \mathbb{Z})$ were determined in [6] for all ℓ. In particular, $H_*(X_\ell; \mathbb{Z})$ are torsion free for all ℓ.

We are interested in the groups $H_*(Y_\ell; \mathbb{Z})$. We recall the following results.

(i) The cohomology ring $H^*(Y_\ell; \mathbb{Z}_2)$ is determined in [7, Corollary 9.2] for generic ℓ.

(ii) The Poincaré polynomial $PS_{\mathbb{Q}}(Y_\ell)$ is determined in [9] for all ℓ.

Theorem 5. *When ℓ is generic, the torsion subgroup of $H_*(Y_\ell; \mathbb{Z})$ has exponent 2.*

Proof. By [6, Theorem 1], the groups $H_*(X_\ell; \mathbb{Z})$ are torsion free. If we apply Theorem A to the regular 2-cover (9), then the result follows. □

Moreover, if we apply Corollary B to the above (i) and (ii), we can determine $\Gamma(Y_\ell)$. As an example, we consider the case for $\ell = \underbrace{(1, \dots, 1)}_{n}$. In order that ℓ is generic, n must be odd. Hence we set $n = 2m+1$.

First, from [9], we have

$$PS_{\mathbb{Q}}(Y_\ell) = \sum_{\substack{0 \le q \le m-2 \\ q:\ \text{even}}} \binom{2m}{q} t^q + \binom{2m}{m-1} t^{m-1} + \sum_{\substack{m \le q \le 2m-3 \\ q:\ \text{odd}}} \binom{2m}{q+2} t^q. \tag{10}$$

(The result is also obtained in [8, Theorem C].) In particular, the manifold Y_ℓ is non-orientable.

Second, from [7, Corollary 9.2], we have

$$PS_{\mathbb{Z}_2}(Y_\ell) = \sum_{q=0}^{m-1} \left(\sum_{i=0}^{q} \binom{2m}{i} \right) t^q + \sum_{q=m}^{2m-2} \left(\sum_{i=0}^{2m-2-q} \binom{2m}{i} \right) t^q.$$

(The result is also obtained in [8, Proposition 5.1] by a different method.)

Using Corollary B, we have the following result.

Proposition 6. *For an odd n, we set $\ell = \underbrace{(1,\dots,1)}_{n}$. Then we have*

$$\Gamma(Y_\ell) = \sum_{\substack{0\le q\le m-2\\ q:\ odd}} \left(\sum_{i=0}^{q} \binom{2m}{i}\right) t^q + \sum_{\substack{m-1\le q\le 2m-3\\ q:\ odd}} \left(\sum_{i=q+3}^{2m} \binom{2m}{i}\right) t^q.$$

Remark 7. (i) Proposition 6 is already proved in [8, Theorem F]. For the proof of the assertion that the torsion subgroup of $H_*(Y_\ell;\mathbb{Z})$ has exponent 2, the author failed to notice Theorem A in our paper and proved by rather complicated method. Moreover, some torsion subgroup was left unknown. More precisely, consider $H_{m-1}(Y_\ell;\mathbb{Z})$ for the case that m is even. It is stated in [8, Theorem F] that

$$H_{m-1}(Y_\ell;\mathbb{Z}) = \bigoplus_{\binom{2m}{m-1}} \mathbb{Z} \oplus G$$

and G satisfies that

$$G \otimes \mathbb{Z}_2 = \bigoplus_{\sum_{i=0}^{m-2}\binom{2m}{i}} \mathbb{Z}_2.$$

But in fact, Theorem 5 tells us that

$$G = \bigoplus_{\sum_{i=0}^{m-2}\binom{2m}{i}} \mathbb{Z}_2.$$

(ii) In summary, the above steps for determining $H_*(Y_\ell;\mathbb{Z})$ is as follows.

$$\text{Theorem } 5 + PS_{\mathbb{Q}}(Y_\ell) + PS_{\mathbb{Z}_2}(Y_\ell) \Rightarrow H_*(Y_\ell;\mathbb{Z}). \tag{11}$$

Here we recall that Theorem 5 is a consequence of Theorem A in our paper. We remark that if we use the ring structure $H^*(Y_\ell;\mathbb{Z}_2)$ instead of $PS_{\mathbb{Z}_2}(Y_\ell)$, then the following assertions are true:

$$\text{Theorem } 5 + H^*(Y_\ell;\mathbb{Z}_2) \Rightarrow PS_{\mathbb{Q}}(Y_\ell) \Rightarrow H_*(Y_\ell;\mathbb{Z}) \tag{12}$$

and

$$PS_{\mathbb{Q}}(Y_\ell) + H^*(Y_\ell;\mathbb{Z}_2) \Rightarrow \text{Theorem } 5 \Rightarrow H_*(Y_\ell;\mathbb{Z}). \tag{13}$$

Proof of (12). [7, Corollary 9.2] tells us that the cohomology ring $H^*(Y_\ell;\mathbb{Z}_2)$ is of the form

$$H^*(Y_\ell;\mathbb{Z}_2) = \mathbb{Z}_2[R, V_1,\dots,V_{n-1}]/\mathcal{I}_\ell,$$

where R and V_i are of degree 1 and the ideal $\mathcal{I}_\ell$ is generated by three families. One of them is the monomials

$$V_i^2 + RV_i \quad \text{for } i = 1,\dots,n-1. \tag{14}$$

Consider the Bockstein spectral sequence $\{B_r, d^r\}$ with $B_1^q \cong H^q(Y_\ell;\mathbb{Z}_2)$ and $d^1 = Sq^1$. We compute B_2^q using (14). Since we can use Theorem 5, we have $B_2^q \cong B_\infty^q$. Since

$$B_\infty^q \cong (H^q(Y_\ell;\mathbb{Z})/\text{torsion}) \otimes \mathbb{Z}_2,$$

we can determine $PS_{\mathbb{Q}}(Y_\ell)$. Now we have the three information on the left-hand side of (11), we know $H_*(Y_\ell;\mathbb{Z})$. □

Proof of (13). Using the above Bockstein spectral sequence, we compute B_2^q. Comparing the result with $PS_{\mathbb{Q}}(Y_\ell)$, we see that $B_2^q \cong B_\infty^q$. This implies Theorem 5. Then by (11), know $H_*(Y_\ell;\mathbb{Z})$. □

3.7 Configuration spaces

For a manifold M, we define spaces $X(M)$ and Y_M by

$$X_M = M \times M - \Delta(M) \quad \text{and} \quad Y_M = X_M/\mathbb{Z}_2.$$

(The spaces X_M and Y_M are usually denoted by $F(M,2)$ and $B(M,2)$, respectively.) Then there is a regular 2-cover

$$p : X_M \to Y_M. \tag{15}$$

Example 8. There are homotopy equivalences

$$Y_{\mathbb{R}^d} \simeq \mathbb{R}P^{d-1} \quad \text{and} \quad Y_{S^d} \simeq \mathbb{R}P^d.$$

A situation for which Theorem A can be applied is the following:

Lemma 9. *If the manifold M is compact and orientable such that $H_*(M;\mathbb{Z})$ are torsion free, then $H_*(X_M;\mathbb{Z})$ are torsion free.*

Proof. We set $\dim M = d$. From the cohomology long exact sequence of the pair $(X_M, \Delta(M))$ and Poincaré duality, we have the following long exact sequence:

$$\cdots \xrightarrow{\Delta^*} H^{q-1}(M;\mathbb{Z}) \to H^{2d-q}(X_M;\mathbb{Z}) \to H^q(M^2;\mathbb{Z}) \xrightarrow{\Delta^*} H^q(M;\mathbb{Z}) \to \cdots. \tag{16}$$

Since Δ^* are surjective and $H^*(M^2;\mathbb{Z})$ are torsion free, $H^*(X_M;\mathbb{Z})$ are torsion free. Hence $H_*(X_M;\mathbb{Z})$ are torsion free. □

3.7.1 $X = T(n)$

As in 3.1, we set $T(n) = \underset{n}{\#} T^2$.

Proposition 10. *There are isomorphisms*

$$H_q(Y_{T(n)};\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & q = 0 \\ \bigoplus_{2n} \mathbb{Z} \oplus \mathbb{Z}_2, & q = 1 \\ \bigoplus_{2n^2-n} \mathbb{Z} \oplus \bigoplus_{2n} \mathbb{Z}_2, & q = 2 \\ 0, & q \geq 3. \end{cases}$$

Proof. By Lemma 9, we can apply Theorem A to (15). Then the torsion subgroup of $H_*(Y_{T(n)};\mathbb{Z})$ has exponent 2. By [1, Theorem C] (see also Remark 11), we have

$$PS_{\mathbb{Q}}(Y_{T(n)}) = 1 + (2n)t + (2n^2 - n)t^2. \tag{17}$$

By [2, p.120], we have

$$PS_{\mathbb{Z}_2}(Y_{T(n)}) = 1 + (2n+1)t + (2n^2 + n + 1)t^2 + (2n)t^3.$$

Hence Proposition 10 holds. □

Remark 11. (i) We can prove (17) more directly. In fact, we simply use a similar sequence to (16) for the pair $(\mathrm{SP}^2(T(n)), \Delta(T(n)))$.

(ii) Proposition 10 is a generalization of the right homotopy equivalence in Example 8.

3.7.2 The complement of a knot in $\mathbb{R}^3$

Let M be the complement of a knot in $\mathbb{R}^3$. Although we do not use our main theorems nor Lemma 9 in the following computations, we give some results for our reference.

Proposition 12. *There are isomorphisms*

$$H_q(Y_M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2, & q = 1,2 \\ \mathbb{Z}, & q = 3,4 \\ 0, & q \geq 5. \end{cases}$$

Proof. According to [2, 5.2], the generators of $H_*(Y_M;\mathbb{F})$ are given by the following table.

Table 1: The generators of $H_*(Y_M;\mathbb{F})$.

q	$H_q(Y_M;\mathbb{Z}_2)$	$H_q(Y_M;\mathbb{Z}_p)$ (where p is an odd prime)
0	z_{00}^2	z_{00}^2
1	$y_0 z_{00}$, z_{10}	$y_0 z_{00}$
2	$x z_{00}$, y_0^2, z_{01}	$x z_{00}$
3	$x y_0$, y_1	$x y_0$
4	x^2	x^2

More precisely, $H_*(Y_M;\mathbb{Z}_2)$ are given explicitly in 5.2. Moreover, [2, p.111] tells us that Theorem A in the paper holds for homology with coefficients in $\mathbb{Z}_p$. Hence the arguments in 5.2 remains valid for such coefficients and the above table holds.

Now since $\beta(y_1) = y_0^2$ and $\beta(z_{01}) = z_{10}$, the torsion subgroup of $H_*(Y_M;\mathbb{Z})$ has exponent 2. Hence Proposition 12 follows from Table 1 and the universal coefficient theorem. □

Remark 13. Concerning Lemma 9, if $H_*(M;\mathbb{Z})$ have 2-torsion then $H_*(Y_M;\mathbb{Z})$ may have higher 2-torsion. For example, [4] shows that $H_*(Y_{\mathbb{R}P^n};\mathbb{Z})$ have 2-torsion (for $n \geq 2$), 4-torsion (for $n \geq 3$), but no 8-torsion.

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