SURVEY ON THE RANK AND STRUCTURE THEORY OF GROUP C*-ALGEBRAS OF LIE GROUPS

Ryukyu Math. J., 13(2000), 79-95

SURVEY ON THE RANK AND STRUCTURE THEORY OF GROUP C^*-ALGEBRAS OF LIE GROUPS

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ABSTRACT. This article is a survey on the rank and structure theory of group C^*-algebras of Lie groups, studied and developed mainly by the author recently or formerly. The main points are the stable rank and connected stable rank estimations of these C^*-algebras in terms of groups, and the decomposition of group C^*-algebras of some Lie groups into finite or infinite composition series. In addition, some improvements of former results are obtained.

§0. INTRODUCTION

This paper is organized as follows:

§1. Solvable Lie groups of type I

§2. Amenable Lie groups of type I

§3. Non-amenable Lie groups of type I

§4. Solvable Lie semi-direct products of non type I

§5. Solvable Lie groups of type R or non type R

§6. Tables of examples of Lie groups

In each section except §6, we will show some results and their methods for the matter explained in the abstract (cf.[Sd1-7] and [ST1, ST2]).

Received November 30, 2000.

2000 Mathematics Subject Classification: Primary 46L05. Secondary 22D25, 46L80, 19K56.

We shall start with reviewing some notations and facts about the stable rank and connected stable rank for C^*-algebras.

Notations: For a C^*-algebra $\mathfrak{A}$, we denote by $\hat{\mathfrak{A}} = \mathfrak{A}^\wedge$ its spectrum of all irreducible representations of $\mathfrak{A}$ up to unitary equivalence. We denote by $C^*(G)$ the (full) group C^*-algebra of a Lie group G (cf.[Dx]), and by $\hat{G}_1$ the space of all 1-dimensional representations of G. Note that the unitary dual $\hat{G}$ of G is identified with $C^*(G)^\wedge$. Denote by $\hat{G}_\infty$ the subspace of all infinite dimensional, unitary representations of G in $\hat{G}$. For a locally compact Hausdorff space X, we denote by $C_0(X)$ the C^*-algebra of all continuous complex-valued functions on X vanishing at infinity, and let $C(X) = C_0(X)$ when X is compact. We denote by $\mathbb{K} = \mathbb{K}(H)$ the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space H.

For a C^*-algebra $\mathfrak{A}$, we denote by $\mathrm{sr}(\mathfrak{A})$, $\mathrm{csr}(\mathfrak{A}) \in \{1, 2, \cdots, \infty\}$ its stable rank and connected stable rank respectively [Rf].

Theorem [Rf], [Ns1]. *Let X be a compact T_2-space. Then*

$$\mathrm{sr}(C(X)) = [\dim X/2] + 1 \equiv \dim_{\mathbb{C}} X,$$
$$\mathrm{csr}(C(X)) \leq [(\dim X + 1)/2] + 1$$

where $[x]$ means the least integer $\geq x$.

Remark. We have by [Sh] that for S^n the n-dimensional sphere,

$$\mathrm{csr}(C_0(\mathbb{R}^n)) = \mathrm{csr}(C(S^n)) = \begin{cases} 2 \text{ if } n = 1, \text{ and } 1 \text{ if } n = 2, \\ [(n+1)/2] + 1, \quad n \geq 3. \end{cases}$$

Theorem [Rf], [Sh]. *Let $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$ be an exact sequence of C^*-algebras. Then we have that*

$$\mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \leq \mathrm{sr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I})$$
$$\mathrm{csr}(\mathfrak{A}) \leq \mathrm{csr}(\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I})$$

where $\vee$ means the maximum.

Theorem [Rf], [Sh], [Ns1]. *For any C^*-algebra $\mathfrak{A}$, we have that*

$$\mathrm{sr}(\mathfrak{A} \otimes \mathbb{K}) = 2 \wedge \mathrm{sr}(\mathfrak{A}), \quad \mathrm{csr}(\mathfrak{A} \otimes \mathbb{K}) \leq 2 \wedge \mathrm{csr}(\mathfrak{A})$$

where $\wedge$ means the minimum.

§1. Solvable Lie groups of type I

Nilpotent Lie groups.

Theorem [ST1], [Sd4]. *Let G be a connected, nilpotent Lie group. Then*

$$\mathrm{sr}(C^*(G)) = \dim_{\mathbb{C}} \hat{G}_1.$$

Remarks. The simply connected case is due to [ST1], and its generalization to the connected case is due to [Sd4]. The above formula for G a nilpotent semi-direct product of the form $\mathbb{R}^m \rtimes \mathbb{R}$ was first obtained by [Sh].

For a connected solvable Lie group G, we have the following isomorphisms as a topological group:

$$\hat{G}_1 \cong (G/[G,G])^\wedge \cong \mathbb{R}^n \times \mathbb{Z}^k$$

for some $n, k \geq 0$, where $[G,G]$ is the commutator group of G. If G is simply connected, we have that

$$\hat{G}_1 \cong (G/[G,G])^\wedge \cong (\mathfrak{G}^*)^G \cong \mathbb{R}^n$$

for $n \geq 1$, where $\mathfrak{G}^*$ is the real dual space of the Lie algebra $\mathfrak{G}$ of G, and $(\mathfrak{G}^*)^G$ is the fixed point space under the coadjoint action of G.

Note that $C^*(G)$ for G a connected solvable Lie group has the following exact sequence:

$$0 \to \mathfrak{I}_G \to C^*(G) \to C_0(\hat{G}_1) \to 0$$

where $\hat{\mathfrak{I}}_G = \hat{G}_\infty$. When G is of type I, $\mathfrak{I}_G$ is also of type I. Then $\mathfrak{I}_G$ has a composition series $\{\mathfrak{I}_j\}_{j=1}^N$ ($N \leq \infty$) such that its subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ have continuous trace. In the case of G a simply connected, nilpotent Lie group, by [Sd1] we may take N finite and $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ are liminary with Hausdorff spectrums. In both situations, the subquotients are stable C^*-algebras of continuous trace by [Sd6, Proposition 3.6], i.e., of the form

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \Gamma_0(X_j, \{\mathbb{K}\}_{t\in X_j}) \cong \Gamma_0(X_j, \{\mathbb{K}\}_{t\in X_j}) \otimes \mathbb{K}$$

where the middle side means a C^*-algebra of continuous fields on a locally compact T_2-subspace X_j of $\hat{\mathfrak{I}}_G$.

To prove the rank theorem above, we applied the following theorem inductively to another composition series of $C^*(G)$ ([ST2]):

Theorem [Ns2]. *Let $\mathfrak{A}$ be a C^*-algebra having a closed ideal $\mathfrak{I}$ of continuous trace with any element of $\hat{\mathfrak{I}}$ ∞-dimensional. Then*

$$\mathrm{sr}(\mathfrak{A}) \leq 2 \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}).$$

Moreover, we used the following:

Theorem [ST1], [Sd4]. *Let G be a connected nilpotent Lie group. Then the following are equivalent:*

(1) $\mathrm{sr}(C^*(G)) = 1$.
(2) *G is isomorphic to either $\mathbb{R}$ or $\mathbb{T}^k$ or $\mathbb{R} \times \mathbb{T}^k$.*
(3) $\dim_{\mathbb{C}} \hat{G}_1 = 1$.

If G is simply connected, then the conditions (1) and (3) are equivalent to that $G \cong \mathbb{R}$.

Example 1.1. We denote by H_{2n+1} the real $(2n+1)$-dimensional Heisenberg Lie group, which consists of the following matrices:

$$(c, b, a) = \begin{pmatrix} 1 & a & c \\ & 1_n & b^t \\ 0 & & 1 \end{pmatrix} \in H_{2n+1} \subset GL_{n+2}(\mathbb{R})$$

with $(c, b, a) = (c, b_1, \cdots, b_n, a_1, \cdots, a_n)$ identified with elements of the semi-direct product $\mathbb{R}^{n+1} \rtimes \mathbb{R}^n$. Then we have that

$$\mathrm{sr}(C^*(H_{2n+1})) = [2n/2] + 1 = \dim_{\mathbb{C}}(H_{2n+1})_1^\wedge,$$

and we have the following exact sequence (cf.[Sd7]):

$$0 \to C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K} \to C^*(H_{2n+1}) \to C_0(\mathbb{R}^{2n}) \to 0.$$

Solvable Lie groups of type I.

Theorem [ST2]. *Let G be a simply connected, solvable Lie group of type I. Then*

$$\mathrm{sr}(C^*(G)) = (2 \vee \dim_{\mathbb{C}} \hat{G}_1) \wedge \dim G.$$

To prove this theorem, we applied Nistor's theorem above to a composition series of $C^*(G)$ with its suquotients having continuous trace, obtained from being of type I, and used the following theorem:

Theorem [ST2]. *Let G be a simply connected, solvable Lie group. Then the following are equivalent:*

(1) $\mathrm{sr}(C^*(G)) = 1$.

(2) *G is isomorphic to $\mathbb{R}$.*

Example 1.2. Let A_{n+1} $(n \geq 1)$ be the real $(n+1)$-dimensional, extended $ax + b$ group defined by the following matrices:

$$(a, t) = \begin{pmatrix} e^t & & & a_1 \\ & \ddots & & \vdots \\ & & e^t & a_n \\ 0 & & & 1 \end{pmatrix} \in A_{n+1} \subset GL_{n+1}(\mathbb{R})$$

with $(a, t) = (a_1, \cdots, a_n, t)$ identified with elements of the semi-direct product $\mathbb{R}^n \rtimes \mathbb{R}$. Then we have that

$$\mathrm{sr}(C^*(A_{n+1})) = 2 \vee ([1/2] + 1) = 2 \vee \dim_{\mathbb{C}}(A_{n+1})^{\wedge}_{1},$$

and $C^*(A_{n+1})$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^{n+1}$ such that

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \begin{cases} C_0(\mathbb{R}) & j = n+1, \\ \oplus^{\binom{n}{n+1-j}} C_0((\mathbb{R} \setminus \{0\})^{n+1-j}) \rtimes \mathbb{R} & 1 \leq j \leq n \end{cases}$$

where $\oplus^{\binom{n}{n+1-j}}$ means the combination $\binom{n}{n+1-j}$ times direct sum, and the direct factors are C^*-crossed products of $C_0((\mathbb{R} \setminus \{0\})^{n+1-j})$ by $\mathbb{R}$. Since each connected component of $(\mathbb{R}\setminus\{0\})^{n+1-j}$ are invariant under the action of $\mathbb{R}$, and the action on each is free and wandering (cf.[Gr]), we have that

$$\begin{aligned} C_0((\mathbb{R} \setminus \{0\})^{n+1-j}) \rtimes \mathbb{R} &\cong \oplus^{2^{n+1-j}} C_0(\mathbb{R}_+^{n+1-j}) \rtimes \mathbb{R} \\ &\cong \oplus^{2^{n+1-j}} C_0(\mathbb{R}_+^{n-j}) \otimes \mathbb{K}. \end{aligned}$$

§2. Amenable Lie groups of type I

Theorem [Sd3]. *Let G be a connected, amenable Lie group of type I. Then*

$$\dim_{\mathbb{C}} \hat{G}_1 \leq \mathrm{sr}(C^*(G)) \leq 2 \vee \dim_{\mathbb{C}} \hat{G}_1.$$

Remark. The solvable case is due to [ST2], which is included in this theorem.

We have the following exact sequence: for G a simply connected, amenable Lie group,

$$0 \to \mathfrak{I}_G \to C^*(G) \to C_0(\hat{R}_1^S) \otimes C^*(S) \to 0$$

where R is the radical of G and S is a compact group such that $G \cong R \rtimes S$, and $\hat{R}_1^S$ is the fixed point space of $\hat{R}_1$ under the adjoint action by S, and $\hat{\mathfrak{I}}_G = \hat{G}_\infty$. Note that $\hat{R}_1^S = \hat{G}_1$.

Without using Nistor's theorem, if we use the composition series as given above and in the nilpotent case, we obtain that

Theorem 2.1. *Let G be a connected, amenable Lie group of type I. Then*

$$\dim_{\mathbb{C}} \hat{G}_1 \leq \mathrm{sr}(C^*(G)) \leq 2 \vee \dim_{\mathbb{C}} \hat{G}_1 \vee \mathrm{csr}(C_0(\hat{G}_1)),$$
$$\mathrm{csr}(C^*(G)) \leq 2 \vee \mathrm{csr}(C_0(\hat{G}_1)).$$

Proof. We first show that

$$\mathrm{sr}(\mathfrak{I}_G) \leq 2, \quad \mathrm{csr}(\mathfrak{I}_G) \leq 2$$

where $\hat{\mathfrak{I}}_G = \hat{G}_\infty$, which is deduced from applying the rank formulas in the introduction to the structure of $\mathfrak{I}_G$ (a composition series with its subquotients stable). Next, applying the rank formula for exact sequences of C^*-algebras, we get the conclusion (cf.[Sd3]). $\square$

Remark. In the above theorem, we note that

$$2 \vee \dim_{\mathbb{C}} \hat{G}_1 \vee \mathrm{csr}(C_0(\hat{G}_1)) \leq \begin{cases} 2 \vee \dim_{\mathbb{C}} \hat{G}_1 & \text{if } \dim \hat{G}_1 \text{ even}, \\ \dim_{\mathbb{C}} \hat{G}_1 + 1 & \text{if } \dim \hat{G}_1 \text{ odd}. \end{cases}$$

As a note, there exist some connected, amenable Lie groups of type I such that their group C^*-algebras have stable rank one. For example, it is the semi-direct product of $\mathbb{R}^n$ by $SO(n)$ with the action by matrix multiplication. As a simply connected example, we may take the semi-direct product of $\mathbb{R}^n$ by $\mathrm{Spin}(n)$ the universal covering group of $SO(n)$ (cf.[Sd3]).

Theorem [Sd3]. *Let G be a simply connected, amenable Lie group with its radical commutative. Then G is of type I and*

$$\mathrm{sr}(C^*(G)) = (2 \wedge \dim_{\mathbb{C}}(\hat{R}_1/G)) \wedge \dim_{\mathbb{C}} \hat{G}_1$$

where $\hat{R}_1/G$ means the orbit space under the adjoint action of G.

Theorem [Sd3]. *Let G be a simply connected, amenable Lie group of type I with R its radical. Then*

$$\begin{aligned}(2 \wedge \dim_{\mathbb{C}}(\hat{R}_1/G)) \wedge \dim_{\mathbb{C}} \hat{G}_1 &\leq \mathrm{sr}(C^*(G)) \\ &\leq (2 \vee \dim_{\mathbb{C}} \hat{G}_1) \wedge (\dim R \vee 1).\end{aligned}$$

Example 2.2. Let $G = \mathbb{R}^n \rtimes_\alpha \mathrm{Spin}(n)$ $(n \geq 2)$, where $\mathrm{Spin}(n)$ is the universal covering group of $SO(n)$ and α is induced from the action of $SO(n)$ by matrix multiplication (cf.[Sd3]). Then we have that

$$\mathrm{sr}(C^*(G)) = 1 = [0/2] + 1 = \dim_{\mathbb{C}} \hat{G}_1$$

where $\hat{G}_1$ is the trivial representation of G, and $\mathbb{R}^n/G = \mathbb{R}$, and

$$0 \to C_0(\mathbb{R}_+) \otimes C^*(\mathrm{Spin}(n)_\chi) \otimes \mathbb{K} \to C^*(G) \to C^*(\mathrm{Spin}(n)) \to 0$$

where $\mathrm{Spin}(n)_\chi$ is the stabilizer of the trivial representation χ of $\mathbb{R}^n$, and we note that $C^*(\mathrm{Spin}(n))$, $C^*(\mathrm{Spin}(n)_\chi)$ are isomorphic to restricted direct sums of matrix algebras over $\mathbb{C}$ (See [Sd3]).

§3. Non-amenable Lie groups of type I

We denote by $C_r^*(G)$ the reduced group C^*-algebra of a nonamenable Lie group G. Note that $C_r^*(G)$ is a quotient of $C^*(G)$.

Theorem [Sd2]. *Let G be a non-compact, connected, real semisimple Lie group. Then*

$$\mathrm{sr}(C_r^*(G)) = 2 \wedge \mathrm{rr}(G)$$

where $\mathrm{rr}(G)$ *means the real rank of G.*

Theorem 3.1 ([Sd2]). *Let G be a non-amenable, connected, real reductive Lie group with Z its center. Then*

$$\mathrm{sr}(C_r^*(G)) = 2 \wedge (\mathrm{rr}([G,G]) \vee (\dim \hat{Z} + 1)).$$

Remark. The following structure of $C^*(G)$ for G a connected reductive Lie group was obtained by [Ws]:

$$C_r^*(G) \cong \oplus_{(P,w)} q_w [(C_0(\hat{A}) \otimes \mathbb{K}(H_w)) \rtimes_{\alpha \otimes 1} W_w] q_w$$

where $\oplus_{P,w}$ means a restricted direct sum over equivalence classes (P, w) with $P = MAN$ a cuspidal parabolic subgroup of G and w a representation of discrete series of M, and H_w is a representation space of a representation of G induced from w, and W_w is the stabilizer of w in the Weyl group of A, and q_w is a suitable projection of the multiplier of the crossed product $(C_0(\hat{A}) \otimes \mathbb{K}(H_w)) \rtimes_{\alpha\otimes 1} W_w$ associated with a W_w-cocycle. See [Ws] for details.

Remark. A mistake of the proof of Theorem 3.1 was pointed out by A. Vallett [Math. Review, 99a:46126]. However, the proof is right if G is assumed to be simply connected. So we will make a correction of the proof in the following.

Proof of Theorem 3.1. Note that if G is simply connected, by using the methods of [Sd2] we have the above rank formula. Now suppose G is not simply connected. Let $\tilde{G}$ be the universal covering group of G. Then $G \cong \tilde{G}/\Gamma$ for Γ a discrete, central subgroup of $\tilde{G}$. Then we have the quotient

$$C^*(\tilde{G}) \to C^*(G) \to 0$$

since Γ is an amenable, closed normal subgroup of $\tilde{G}$. Therefore we get that

$$\mathrm{sr}(C^*(\tilde{G})) \geq \mathrm{sr}(C^*(G)).$$

On the other hand, $\dim \tilde{Z}^\wedge$ for the center $\tilde{Z}$ of $\tilde{G}$ is preserved under the quotient by Γ. Hence, applying the proof of [Sd2] to $C^*(G)$ similarly, we get the conclusion. $\square$

Remark. On board to Tokashiki Island on the Ichi-San Kenshu 2000, Professor T. Maeda suggested me the map from $\tilde{G}$ to G in the case of $GL_2(\mathbb{C})$ such as

$$(-1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}) \mapsto (-1)\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

from $(\mathbb{C}\setminus\{0\})\times SL_2(\mathbb{C})$ to $GL_2(\mathbb{C})$, which inspired me to the above proof. I would like to thank him here.

In general, we obtain that

Theorem 3.2 ([Sd2]). *Let G be a non-amenable, connected Lie group of type I. Then*

$$\operatorname{sr}(C_r^*(G)) \leq 2, \quad \operatorname{csr}(C_r^*(G)) \leq 2.$$

Remark. The stable rank estimation is due to [Sd2]. If $\operatorname{rr}(G/R) \geq 2$ for R the radical of G, then $\operatorname{sr}(C_r^*(G)) = 2$ ([Sd2]). If S is a connected semi-simple Lie group with real rank one and finite center and its discrete series non empty, then the K-group $K_1(C_r^*(\mathbb{R}\times S))$ is non empty (cf.[Vl]). By [Eh], we get $\operatorname{csr}(C_r^*(\mathbb{R}\times S)) \geq 2$.

On the other hand, we have that

Theorem 3.3. *Let G be a minimally almost periodic, topological group of type I. Then*

$$\begin{cases} \operatorname{sr}(C^*(G)) \leq 2, & \operatorname{csr}(C^*(G)) \leq 2, \\ \operatorname{sr}(C_r^*(G)) \leq 2, & \operatorname{csr}(C_r^*(G)) \leq 2. \end{cases}$$

Proof. We recall that a topological group is minimally almost periodic if there exist no continuous almost periodic functions on G except constant functions, which is equivalent to that there exist no finite dimensional, unitary representations of G except the trivial representation (cf.[JMS, 33 almost periodic function], [Dx]). Hence we have that

$$0 \to \mathfrak{I}_G \to C^*(G) \to \mathbb{C} \to 0$$

where $\hat{\mathfrak{I}}_G = \hat{G}_\infty$, and $\mathfrak{I}_G$ is the kernel of the representation of $C^*(G)$ associated with the trivial representation of G. Next, we use the same methods as in Theorem 2.1. Also note that $C_r^*(G)^\wedge = C_r^*(G)^\wedge_\infty$. $\square$

Remark. Any non-compact, connected simple Lie group is a minimally almost periodic group (cf.[JMS, Section 33]). I would like to thank Professor K. Kikuchi for drawing my attention to the connection between almost periodic functions on groups and finite dimensional, unitary representations of groups.

In the above theorem, we note that $\operatorname{sr}(C^*(G)) = 1$ if and only if $\operatorname{sr}(\mathfrak{I}_G) = 1$, which is obtained by [Ns2] since the index map of K-groups from $K_1(\mathbb{C}) = 0$ to $K_0(\mathfrak{I}_G)$ is trivial.

Example 3.4. We note that there exists a minimally almost periodic, topological group of non type I. Let $M_5 = \mathbb{C}^2 \rtimes_\alpha \mathbb{R}$ be the real 5-dimensional Mautner group, where $\alpha_t(z,w) = (e^{2\pi i t}z, e^{2\pi i t\theta}w)$ for $t \in \mathbb{R}$, $z, w \in \mathbb{C}$ with θ an irrational number. Then we let $G = (M_5 \times M_5) \rtimes_\beta SL_2(\mathbb{R})$, where β is the matrix multiplication on $\mathbb{R} \times \mathbb{R}$ of $M_5 \times M_5$ and trivial elsewhere. Then we have that

$$\begin{aligned}
&0 \to \mathfrak{I} \otimes C^*(SL_2(\mathbb{R})) \to C^*(G) \to C_0(\mathbb{R}^2) \rtimes_{\hat{\beta}} SL_2(\mathbb{R}) \to 0, \\
&0 \to C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes_{\hat{\beta}} SL_2(\mathbb{R}) \to C_0(\mathbb{R}^2) \rtimes_{\hat{\beta}} SL_2(\mathbb{R}) \\
&\quad \to C^*(SL_2(\mathbb{R})) \to 0, \quad \text{and } \mathrm{sr}(C^*(G)) = 2,\ \mathrm{csr}(C^*(G)) \leq 2
\end{aligned}$$

where $0 \to \mathfrak{I} \to C^*(M_5 \times M_5) \to C_0(\mathbb{R}^2) \to 0$, and $\hat{\beta}$ means the dual action of β. Then $\mathfrak{I}$ is of non type I, so is $C^*(G)$ (cf.[Sd5]). Since $SL_2(\mathbb{R})$ is minimally almost periodic (cf.[Sg]), we deduce from the above structure that $\hat{G} = \{1\} \cup \hat{G}_\infty$.

§4. Solvable Lie semi-direct products of non type I

Lie semi-direct products of $\mathbb{C}^n$ by $\mathbb{R}$.

Theorem [Sd5]. *Let G be a Lie semi-direct product $\mathbb{C}^n \rtimes \mathbb{R}$. Then $C^*(G)$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^K$ such that its subquotients are given by*

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \begin{cases} C_0(\hat{G}_1) = C_0(\mathbb{C}^{n_0+u} \times \mathbb{R}), & j = K, \\ C_0(\mathbb{C}^{n_0+s_j} \times (\mathbb{C}\setminus\{0\})^{t_j} \times \mathbb{T}) \otimes \mathbb{K}, & \text{or} \\ C_0(\mathbb{C}^{n_0+s_j} \times (\mathbb{C}\setminus\{0\})^{t_j} \times \mathbb{R}) \otimes \mathbb{K}, & \text{or} \\ C_0(\mathbb{C}^{n_0+s_j} \times \mathbb{R}_+^{u_j}) \otimes \mathfrak{A}_{\Theta_j} \otimes \mathbb{K}, & \text{for } 1 \leq j \leq K-1, \end{cases}$$

where $\mathfrak{A}_{\Theta_j}$ is a noncommutative torus of the form $C(\mathbb{T}^{u_j-1}) \rtimes \mathbb{Z}$, and $0 \leq n_0 \leq n$ and $0 \leq s_j, t_j \leq n - n_0$ and $2 \leq u_j \leq n - n_0$ and $s_j + t_j + 1 \leq n - n_0$ and $s_j + u_j \leq n - n_0$.

Moreover, using the structure theorem above, we have that

Theorem [Sd5]. *Let G be a Lie semi-direct product $\mathbb{C}^n \rtimes \mathbb{R}$. Then*

$$\begin{cases} 2 \vee \dim_{\mathbb{C}} \hat{G}_1 \leq \mathrm{sr}(C^*(G)) \leq \dim_{\mathbb{C}} \hat{G}_1 + 1, \\ 2 \leq \mathrm{csr}(C^*(G)) \leq \dim_{\mathbb{C}} \hat{G}_1 + 1. \end{cases}$$

Remark. For Lie semi-direct products of the form $G = \mathbb{R}^n \rtimes \mathbb{R}$, we can obtain the similar structure theorem for $C^*(G)$ and its rank estimations as above. See [Sd5] for details.

In particular, we obtain that

Example 4.1. Let M_{2n+1} be the generalized Mautner group of the form $\mathbb{C}^n \rtimes_\alpha \mathbb{R}$ with α a multi-rotation on $\mathbb{C}^n$ by $\alpha_t = (e^{2\pi i t\theta_j})_{j=1}^n$ with $\theta_j \in \mathbb{R}$ rationally independent. Then $C^*(M_{2n+1})$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^K$ with $K = 1 + \sum_{k=1}^n \binom{n}{n-k+1}$ and its subquotients given by

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \begin{cases} C_0((M_{2n+1})_1^\wedge) = C_0(\mathbb{R}), & j = K, \\ C_0(\mathbb{R}_+^{u_{n-l+1}}) \otimes \mathfrak{A}_{\Theta_{n-l+1}} \otimes \mathbb{K} \end{cases}$$

for $\sum_{k=1}^{l-1} \binom{n}{n-k+1} + 1 \leq j \leq \sum_{k=1}^{l} \binom{n}{n-k+1}$ $(1 \leq l \leq n)$ as in the above structure theorem, and we obtain that

$$\begin{cases} \mathrm{sr}(C^*(M_{2n+1})) = 2 = \dim_{\mathbb{C}} \hat{G}_1 + 1, \\ \mathrm{csr}(C^*(M_{2n+1})) = 2 = \dim_{\mathbb{C}} \hat{G}_1 + 1. \end{cases}$$

The following is easily deduced from Example 4.1:

Theorem 4.2. *The group C^*-algebra $C^*(M_{2n+1})$ has no nontrivial projections.*

Generalized Dixmier groups.

We recall that the real $(6n+1)$-dimensional, generalized Dixmier group D_{6n+1} is defined by the semi-direct product:

$$D_{6n+1} = \mathbb{C}^{2n} \rtimes_\alpha H_{2n+1}$$

where H_{2n+1} is the real $(2n+1)$-dimensional, generalized Heisenberg group, and the action α is a multi-rotation on $\mathbb{C}^{2n}$ such that

$$\begin{aligned} &(z_1, \cdots, z_n, w_1, \cdots, w_n) \\ &\mapsto (e^{2\pi i a_1} z_1, \cdots, e^{2\pi i a_n} z_n, e^{2\pi i b_1} w_1, \cdots, e^{2\pi i b_1} w_n) \end{aligned}$$

for $z_j, w_j \in \mathbb{C}$ $(1 \leq j \leq n)$ and $(c, b, a) \in H_{2n+1}$.

Theorem [Sd6]. *Let D_{6n+1} be the generalized Dixmier group. Then there exists a finite composition series $\{\mathfrak{K}_j\}_{j=1}^K$ of $C^*(D_{6n+1})$ with its subquotients $\mathfrak{K}_j/\mathfrak{K}_{j-1}$ given by*

$$\begin{cases} C_0((D_{6n+1})_1^\wedge) = C_0(\mathbb{R}^{2n}) & \text{for } j = K, \\ C_0(\mathbb{R}) \otimes \mathbb{K} \quad \text{or} \\ C_0(\mathbb{T}^k \times \mathbb{R}^{2n}) \otimes \mathbb{K} \quad \text{or} \\ C_0(\mathbb{T}^k \times \mathbb{R}^{k+1}) \otimes \mathbb{K} \quad \text{or} \\ C_0(\mathbb{R}_+^k) \otimes \mathbb{K} \otimes C_0(\mathbb{R} \setminus \{0\}, \cup_{\theta \in \mathbb{R}\setminus\{0\}}((\otimes^{s_1}\mathfrak{A}_\theta) \otimes C(\mathbb{T}^{s_2}) \otimes \mathbb{K})) \end{cases}$$

for $1 \le j \le K-1$ with $1 \le k \le 2n$, $s_1 \ge 1$, $s_2 \ge 0$, $2s_1 + s_2 = k$, where $\mathfrak{A}_\theta$ is the rotation algebra $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$, and the third tensor factor in the last case means a C^*-algebra of continuous fields on $\mathbb{R} \setminus \{0\}$ with fibers the tensor products $(\otimes^{s_1}\mathfrak{A}_\theta) \otimes C(\mathbb{T}^{s_2}) \otimes \mathbb{K}$ for $\theta \in \mathbb{R} \setminus \{0\}$.

Corollary [Sd6]. *For the group C^*-algebra $C^*(D_{6n+1})$, the following holds:*

$$\begin{cases} \mathrm{sr}(C^*(D_{6n+1})) = n+1 = \dim_{\mathbb{C}}(D_{6n+1})_1^\wedge \\ 2 \le \mathrm{csr}(C^*(D_{6n+1})) \le n+1. \end{cases}$$

Corollary [Sd6]. *The group C^*-algebra $C^*(D_{6n+1})$ has no nontrivial projections.*

§5. Lie groups of type R or non type R

This section is an appendix. In this section, we first check that semi-direct products of the form $G = \mathbb{C}^n \rtimes_\alpha \mathbb{R}$ are generally of non type R in the sense of [AM] as follows. We note that any element g of G is identified with the following matrix:

$$\begin{pmatrix} \alpha_t & w^t \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(\mathbb{C}), \quad t \in \mathbb{R}, w \in \mathbb{C}^n$$

with $\alpha_t \in GL_n(\mathbb{C})$, where w^t is the transpose of w. We now suppose that α_t for $t \in \mathbb{R}$ is diagonal. Then the Lie algebra $\mathfrak{G}$ of G consists of the following matrices:

$$X = \begin{pmatrix} (a_1 + 2\pi i b_1)t & & 0 & w_1 \\ & \ddots & & \vdots \\ & & (a_n + 2\pi i b_n)t & w_n \\ 0 & & & 0 \end{pmatrix} \in M_{n+1}(\mathbb{C})$$
$$= (w_1, \cdots, w_n, t) \in \mathbb{C}^n \times \mathbb{R}$$

for some $a_j, b_j \in \mathbb{R}$ $(1 \leq j \leq n)$. Then the adjoint representation of $\mathfrak{G}$ is given by

$$\begin{aligned} \mathrm{ad}(X)(Y) &= [X, Y] = XY - YX \\ &= \begin{pmatrix} 0 & & 0 & (a_1 + 2\pi i b_1)tw'_1 - (a_1 + 2\pi i b_1)t'w_1 \\ & \ddots & & \vdots \\ & & 0 & (a_n + 2\pi i b_n)tw'_n - (a_1 + 2\pi i b_1)t'w_n \\ 0 & & & 0 \end{pmatrix} \end{aligned}$$

for $X, Y \in \mathfrak{G}$ with Y defined by replacing t, w_j of X with t', w'_j. In particular, we have that for Y with $t' = 0$,

$$(\exp \mathrm{ad}(X))(Y) = \begin{pmatrix} 0 & & 0 & z_1 w'_1 \\ & \ddots & & \vdots \\ & & 0 & z_n w'_n \\ 0 & & & 0 \end{pmatrix}$$

where $z_j = e^{(a_j + 2\pi i b_j)t}$ for $1 \leq j \leq n$, and for Y with $w'_j = 0$ for all $1 \leq j \leq n$,

$$(\exp \mathrm{ad}(X))(Y) = \begin{pmatrix} (a_1 + 2\pi i b_1)t' & & 0 & z'_1 w_1 \\ & \ddots & & \vdots \\ & & (a_n + 2\pi i b_n)t' & z'_n w_n \\ 0 & & & 0 \end{pmatrix}$$

where $z'_j = e^{-(a_n + 2\pi i b_n)t'}$ for $1 \leq j \leq n$. Therefore, the adjoint representation Ad of G has all the eigenvalues absolute value one if any, that is, G of type R, if and only if $a_j = 0$ for $1 \leq j \leq n$, since $\mathrm{Ad}(\exp X) = \exp \mathrm{ad}(X)$ where exp is the exponential map, which implies that G is of non type R in general. In particular, the real extended $ax+b$ group A_{n+1} is of non type R, and the generalized Mautner group M_{2n+1} is of type R.

Note that a Lie group G is of type R (or completely solvable) if and only if all the eigenvalues of $\mathrm{ad}(X)$ for $X \in \mathfrak{G}$ are real, where $\mathfrak{G}$ is the Lie algebra of G. Any Lie group of type R is solvable.

On the other hand, the real Heisenberg Lie group H_{2n+1} is of type R since any connected solvable Lie group of CCR is of type R ([AM]). In fact, the Lie algebra $\mathfrak{H}_{2n+1}$ of H_{2n+1} consists of the following matrices:

$$X = (c, b, a) = \begin{pmatrix} 0 & a & c \\ & 0_n & b^t \\ 0 & & 0 \end{pmatrix} \in M_{n+2}(\mathbb{R})$$

Then for $Y = (c', b', a') \in \mathfrak{H}_{2n+1}$ and $O = (0, \cdots, 0) \in \mathbb{R}^n$,

$$\mathrm{ad}(X)(Y) = \begin{pmatrix} 0 & O & \sum_{j=1}^{n} a_j b'_j - \sum_{j=1}^{n} a'_j b_j \\ & 0_n & O^t \\ 0 & & 0 \end{pmatrix}, \qquad \mathrm{ad}(X)^2 = 0.$$

Therefore,

$$(\exp \mathrm{ad}(X))(Y) = \begin{pmatrix} 0 & a' & c' + \sum_{j=1}^{n} a_j b'_j - \sum_{j=1}^{n} a'_j b_j \\ & 0_n & b' \\ 0 & & 0 \end{pmatrix}$$

Moreover, we can show that the generalized Dixmier group D_{6n+1} is of type R.

Finally, we note that any simply connected, solvable Lie group of non type R has a quotient isomorphic to one of the following:

$$A_2, \quad S_c = \mathbb{R}^2 \rtimes_{\alpha^c} \mathbb{R}, \quad S = \mathbb{R}^2 \rtimes_{\beta} \mathbb{R}^2$$

where the actions α, β are defined by

$$\alpha_t^c = e^{ct} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad \beta_{(s,t)} = e^s \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

for $s, t \in \mathbb{R}$ and $c \in \mathbb{R} \setminus \{0\}$ ([AM]). Then we have that

$$\begin{aligned} \mathrm{sr}(C^*(S_c)) &= 2 = \mathrm{sr}(C^*(S)), \\ \mathrm{csr}(C^*(S_c)) &= 2 = \mathrm{csr}(C^*(S)) \end{aligned}$$

while $(S_c)_1^\wedge = \mathbb{R}$ and $\hat{S}_1 = \mathbb{R}^2$, and respectively,

$$0 \to C(\mathbb{T}) \otimes \mathbb{K} \to \left\{ \begin{array}{l} C^*(S_c) \\ C^*(S) \end{array} \right. \to \left\{ \begin{array}{l} C_0(\mathbb{R}) \\ C_0(\mathbb{R}^2) \end{array} \right. \to 0.$$

As a remark, the direct products $A_2 \times M_{2n+1}$, $A_2 \times D_{6n+1}$ are of non type R and non type I.

§6. Tables of examples of Lie groups

This section is an appendix. We summarize some examples considered in this paper into the following tables:

Table of solvable Lie groups

	Type I	Non type I
Type R	$\mathbb{R}^n \times \mathbb{T}^k$, H_{2n+1}	M_{2n+1}, D_{6n+1}
Non type R	A_{n+1}, $A_2 \times H_{2n+1}$	$A_2 \times M_{2n+1}$

Table of non-solvable, amenable Lie groups

Type I	Non type I
$SO(n)$, $\mathbb{R}^n \rtimes SO(n)$ $(n \geq 2)$, $H_3 \rtimes SO(2)$	$(M_5 \times M_5) \rtimes SO(2)$, $D_7 \rtimes SO(2)$

where the actions of $SO(2)$ on H_3, $M_5 \times M_5$ and D_7 are similar with that on $M_5 \times M_5$ as in Example 3.4, so that their dual actions of $SO(2)$ are given by the matrix multiplication on $(H_3)_1^\wedge$, $(M_5 \times M_5)_1^\wedge$ and $(D_7)_1^\wedge$, and trivial elsewhere.

Table of non-amenable Lie groups

Type I	Non type I
$SL_n(\mathbb{R})$, $GL_n(\mathbb{R})$ $(n \geq 2)$, $\mathbb{R}^2 \rtimes SL_2(\mathbb{R})$, $H_3 \rtimes SL_2(\mathbb{R})$	$(M_5 \times M_5) \rtimes SL_2(\mathbb{R})$, $D_7 \rtimes SL_2(\mathbb{R})$

where the actions of $SL_2(\mathbb{R})$ on $\mathbb{R}^2$, H_3 and D_7 are similar with that on $M_5 \times M_5$ as in Example 3.4 and as above.

Acknowledgment. The author would like to thank the staff of Department of Mathematical Science, University of the Ryukyus for sharing the time and floor during this research. He also would like to thank his wife Jingrong Fang for her encouragement and deep care.

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