WHICH INSCRIBED SPHERE OF PYRAMIDS IS MAXIMAL?

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WHICH INSCRIBED SPHERE OF PYRAMIDS IS MAXIMAL?

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Abstract

Consider the following question: *In a circular cone, with bus line having length 1, the inscribed sphere is to be maximal. How much is the radius of the base circle?* It is easy to see that the answer is $\frac{\sqrt{5}-1}{2}$, which is interesting because this is the reciprocal of the golden section. In this paper, we generalize the question to the case that the base circle is generalized to regular polygons.

1 Introduction

In [3, p.124, Question 76], the following question is posed: *In an isosceles triangle, with sloping sides of a given length, the inradius is to be maximal. How big is this?*

Specifying the length of the sloping sides to be 1, we restate the question as follows:

Question 1. Let $\triangle(x)$ be the triangle in $\mathbb{R}^2$, with vertices A, B and C, such that the side lengths are given by $|AB| = |AC| = 1$ and $|BC| = 2x$. Let $r(x)$ be the radius of the inscribed circle of $\triangle(x)$. (See Fig.1.) We set $R := \max_{0 \leq x \leq 1} r(x)$, where the condition $0 \leq x \leq 1$ is the consequence of the triangle inequality. Now let L be the number x which attains R. What is L?

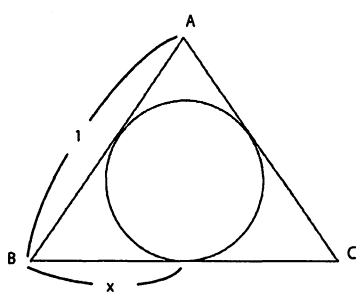

Figure 1: $\triangle(x)$.

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It is easy to see that $L = \frac{\sqrt{5}-1}{2}$. In [4, p.177], a solution using differential is given. Since the answer is the reciprocal of the golden section, it is natural to search for a solution without using differential. But it is stated in [4, p.177] that the translator does not know such a method.

The purpose of this paper is to study one-dimensional higher analogue of Question 1.

Question 2. Let $\Gamma_n(x)$ be a regular n-gon in $\mathbb{R}^2$, with vertices $v_1, \ldots, v_{n-1}$ and v_n, such that the radius of the circumscribed circle is x. Let $P_n(x)$ be the pyramid of $\Gamma_n(x)$ with apex p such that $|pv_i| = 1$ for $1 \leq i \leq n$. Let $r_n(x)$ be the radius of the inscribed sphere of $P_n(x)$. (See Fig.2.) We set $R_n := \max_{0 \leq x \leq 1} r_n(x)$ and let L_n be the number x which attains R_n. What is L_n?

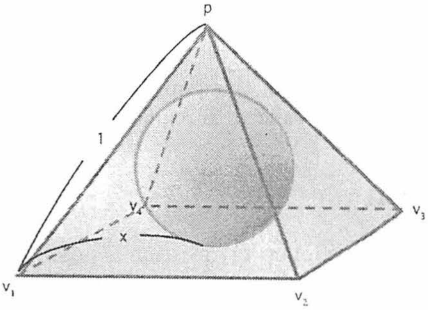

Figure 2: $P_4(x)$.

This paper is organized as follows. In §2 we state our main results and in §3 we prove them.

2 Statement of the main results

The answer to Question 2 is given as follows:

Theorem A . *We set*

$$a_n = \frac{13}{2} - 6\cos\left(\frac{2\pi}{n}\right) - \frac{21}{2}\cos\left(\frac{4\pi}{n}\right) + 2\cos\left(\frac{6\pi}{n}\right) + 3\sin^2\left(\frac{2\pi}{n}\right)\sqrt{33 - 48\cos\left(\frac{2\pi}{n}\right)}.$$

Then we have

$$L_n = \frac{1}{2\sqrt{3}\sin\left(\frac{\pi}{n}\right)}\sqrt{8 - 4\cos\left(\frac{2\pi}{n}\right) - 2\sqrt[3]{a_n} - \frac{4\left(\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right)\right)}{\sqrt[3]{a_n}}}.$$

Here the terms $\sqrt{33-48\cos\left(\frac{2\pi}{n}\right)}$ *in* a_n *and* $\sqrt[3]{a_n}$ *in* L_n *denote the principal values. More precisely, for a given* $\zeta \in \mathbb{C}$, *we write* $\zeta = r\exp(i\theta)$, *where* $0 \le r \in \mathbb{R}$ *and* $-\pi < \theta \le \pi$. *Then for* $p \in \mathbb{N}$, *we set*

$$\sqrt[p]{\zeta} := \sqrt[p]{r}\exp\left(\frac{i\theta}{p}\right),$$

where $\sqrt[p]{r}$ *denotes the real p-th root of* r.

Corollary B . *We set* $L_\infty := \lim_{n\to\infty} L_n$. *Then we have*

$$L_\infty = \frac{\sqrt{5}-1}{2}.$$

Remark 3. We claim that Corollary B is an immediate consequence of Question 1. In fact, if we cut $P_\infty(x)$ by a plane which contains the apex and is perpendicular to the base circle. Then the section is $\Delta(x)$ in Fig.1. Since $L = \frac{\sqrt{5}-1}{2}$, so is L_∞.

Example 4 (Some graphs). (i) In §3, we determine $r_n(x)$ explicitly. (See Lemma 7.) Using this, we draw the graph of $r_8(x)$ in the left of Fig.3.

(ii) We draw the graph of L_n in the right of Fig.3. Note that L_n is a decreasing sequence. On the other hand, we can check that $2nL_n\sin\left(\frac{\pi}{n}\right)$, the circumference of $P_n(L_n)$, is an increasing sequence.

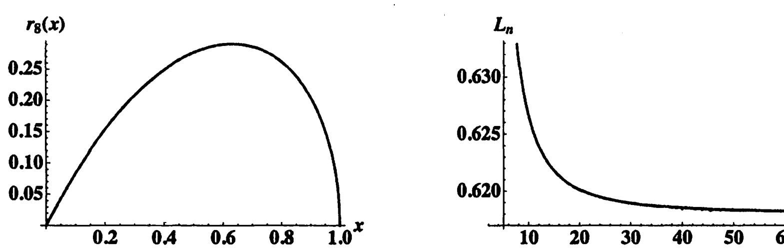

Figure 3: The left: $r_8(x)$. The right: L_n.

Next, we state numerical results when n is small or large.

Example 5 (The result for small n). (i) Although L_n is complicated for general n, we have the following simple results:

$$L_6 = \sqrt{2-2^{2/3}} \quad \text{and} \quad L_8 = \sqrt{2+\frac{1}{\sqrt{2}}-\sqrt{\frac{5}{2}+2\sqrt{2}}}.$$

(ii) The approximate values of L_n for $3 \le n \le 10$ are given by the following table:

n	L_n
3	0.7284
4	0.6755
5	0.6536
6	0.6423
7	0.6357
8	0.6315
9	0.6286
10	0.6266
$\vdots$	$\vdots$
∞	$\frac{\sqrt{5}-1}{2} = 0.6180$

Table 1: L_n for $3 \leq n \leq 10$

In order to describe the behavior of L_n for large n, we write L_n by the Laurent series:

$$L_n = \sum_{i=0}^{\infty} \frac{\alpha_i}{n^i}.$$

More precisely, in L_n, we replace $\frac{1}{n}$ by t and define a function $K(t)$. Note that $K(t)$ is defined for $t \in \left(0, \frac{1}{3}\right]$. Then we set

$$\alpha_i = \frac{1}{i!} \lim_{t \to +0} \frac{d^i K}{dt^i}.$$

Example 6 (The result for large n). (i) We have $a_i = 0$ for odd i and

$$\begin{aligned} L_n =& \frac{\sqrt{5}-1}{2} + \frac{\left(3\sqrt{5}-5\right)\pi^2}{20n^2} + \frac{\left(249\sqrt{5}-550\right)\pi^4}{600n^4} \\ &+ \frac{\left(55383\sqrt{5}-123775\right)\pi^6}{36000n^6} + \frac{\left(2589297\sqrt{5}-5789695\right)\pi^8}{403200n^8} \\ &+ \frac{\left(13078846761\sqrt{5}-29245148500\right)\pi^{10}}{453600000n^{10}} + \cdots . \end{aligned}$$

(ii) We set

$$M_n := \sum_{i=0}^{10} \frac{\alpha_i}{n^i}.$$

Then the following inequalities hold:

$$L_n > M_n > L_\infty.$$

Moreover, consider the case for $n = 10^j$, where $1 \leq j \in \mathbb{N}$. Then we have the following results:

$$L_{10^j} - L_\infty \simeq 8.4 \times 10^{-(2j+1)} \quad \text{and} \quad L_{10^j} - M_{10^j} \simeq 2.4 \times 10^{-12j+1}.$$

3 Proofs of the main results

First we prove Theorem A.

Lemma 7. *In the notation of Question 2, we have*

$$r_n(x) = \frac{x\sqrt{1-x^2}\cos\left(\frac{\pi}{n}\right)}{\sqrt{1-x^2\sin^2\left(\frac{\pi}{n}\right)}+x\cos\left(\frac{\pi}{n}\right)}.$$

Proof. The following formula is well-known. (See, for example, [1, Theorem 1].) Let $V(P_n(x))$ and $S(P_n(x))$ be the volume and the surface sea of $P_n(x)$, respectively. Then we have

$$r_n(x) = \frac{3V(P_n(x))}{S(P_n(x))}. \tag{1}$$

Let F be one of the n sides of $P_n(x)$. Since

$$S(F) = x\sin\left(\frac{\pi}{n}\right)\sqrt{1-x^2\sin^2\left(\frac{\pi}{n}\right)} \quad \text{and} \quad S(\Gamma_n(x)) = \frac{nx^2}{2}\sin\left(\frac{2\pi}{n}\right),$$

we have

$$S(P_n(x)) = nx\sin\left(\frac{\pi}{n}\right)\left(\sqrt{1-x^2\sin^2\left(\frac{\pi}{n}\right)}+x\cos\left(\frac{\pi}{n}\right)\right). \tag{2}$$

On the other hand, since height h of $P_n(x)$ is given by $h=\sqrt{1-x^2}$, we have

$$V(P_n(x)) = \frac{nx^2}{6}\sqrt{1-x^2}\sin\left(\frac{2\pi}{n}\right). \tag{3}$$

Now substituting (2) and (3) in (1), Lemma 7 follows. □

Simple computations show that

$$\frac{dr_n}{dx} = \frac{\cos\left(\frac{\pi}{n}\right)\left(x^4\sin^2\left(\frac{\pi}{n}\right)-x^3\cos\left(\frac{\pi}{n}\right)\sqrt{1-x^2\sin^2\left(\frac{\pi}{n}\right)}-2x^2+1\right)}{\sqrt{1-x^2}\sqrt{1-x^2\sin^2\left(\frac{\pi}{n}\right)}\left(\sqrt{1-x^2\sin^2\left(\frac{\pi}{n}\right)}+x\cos\left(\frac{\pi}{n}\right)\right)^2}. \tag{4}$$

We study the graph of $\frac{dr_n}{dx}$ in the range $0 \le x \le 1$. Since the graphs are similar, we give the case for $n=8$ in Fig.4.

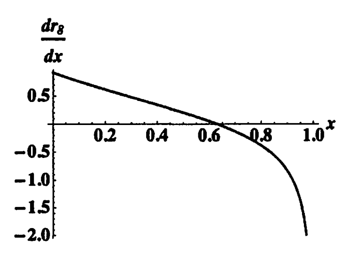

Figure 4: The graph of $\frac{dr_8}{dx}$.

Note that the graph has an unique root, which is L_n. Moreover, the first derivative test shows that the graph of $r_n(x)$ is as given in the left of Fig.3.

In order to complete the proof of Theorem A, we need to determine the root in $0 \leq x \leq 1$ of the equation "the numerator of (4) = 0".

Lemma 8. *We set*

$$f_n(x) = x^4 \sin^2\left(\frac{\pi}{n}\right) - x^3 \cos\left(\frac{\pi}{n}\right) \sqrt{1 - x^2 \sin^2\left(\frac{\pi}{n}\right)} - 2x^2 + 1.$$

Regard that the variable x moves in $\mathbb{R}$. Then the following results hold:

(i) *The number of the real roots of the equation $f_n(x) = 0$ is two or four according as $3 \leq n \leq 7$ or $n \geq 8$.*

(ii) *For all $n \geq 3$, one of the real roots is -1. Moreover, the positive real root of the equation $f_n(x) = 0$ is unique. By definition, the root is L_n. (See Fig.5.)*

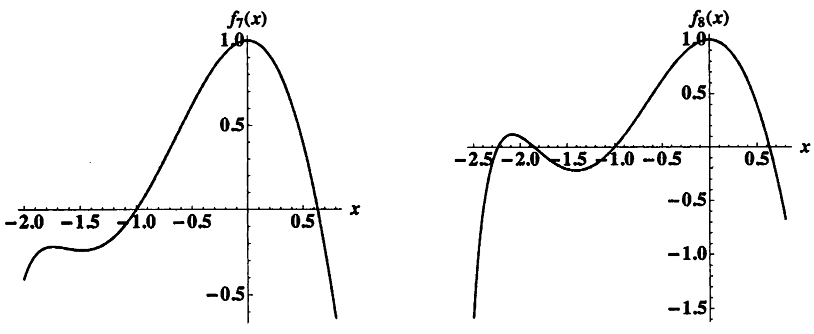

Figure 5: The left: $f_7(x)$. The right: $f_8(x)$.

Proof. We set

$$u_n(x) = x^4 \sin^2\left(\frac{\pi}{n}\right) - 2x^2 + 1 \quad \text{and} \quad v_n(x) = f_n(x) - u_n(x).$$

We set

$$g_n(s) = s^3 \sin^2\left(\frac{\pi}{n}\right) - \left(2\sin^2\left(\frac{\pi}{n}\right) + 1\right) s^2 + 3s - 1.$$

Then we have

$$u_n(x)^2 - v_n(x)^2 = (x^2 - 1) g_n(x^2).$$

Since $\deg g_n(s) = 3$, we can solve the equation $g_n(s) = 0$ by Cardano's formula. (See, for example, [5].) The graph of $g_n(s)$ is given by Fig.6.

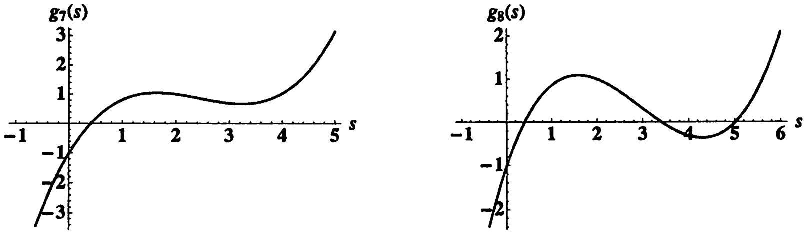

Figure 6: The left: $g_7(s)$. The right: $g_8(s)$.

From the direct computations, we can check the following results:

(a) *The case for* $3 \le n \le 7$. The real root the equation $g_n(s) = 0$ is unique. If we denote this by t_1, then we have $0 < t_1$. Moreover, $x = \sqrt{t_1}$ and $x = -1$ are the solutions of the equation $f_n(x) = 0$. (See Fig.5. Note that if $x = a$ (where $a \neq -1$) satisfies that $f_n(a) = 0$, then we also have $g_n(a^2) = 0$. But not vice versa.)

(b) *The case for* $n \ge 8$. The number of the real roots the equation $g_n(s) = 0$ is three. We denote them by t_1, t_2 and t_3, where we take $t_1 < t_2 < t_3$. Then we have $0 < t_1$. Moreover, $x = \sqrt{t_1}, x = -\sqrt{t_2}$, $x = -\sqrt{t_3}$ and $x = -1$ are the solutions of the equation $f_n(x) = 0$. This completes of proof of Lemma 8. □

Proof of Corollary B. We give a proof without using an answer to Question 1. (See Remark 3 for the proof which uses the answer.) It is possible to compute L_∞ from Theorem A. But it is more simple to compute $\lim_{n\to\infty} \frac{dr_n}{dx}$ from (4). It is easy to see that

$$\lim_{n\to\infty} \frac{dr_n}{dx} = -\frac{x^2 + x - 1}{(x+1)\sqrt{1-x^2}}.$$

Since the positive root of the equation $x^2 + x - 1 = 0$ is $\frac{\sqrt{5}-1}{2}$, Corollary B follows. □

Proofs of Examples 3, 4 and 5. With the aid of Mathematica, we can deduce these examples from Theorem A. □

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