AN EXTREMAL VALUE PROBLEM CONCERNING THE INSCRIBED SPHERE OF PYRAMIDS

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Abstract

Consider the following question: *In a circular cone, with the sum of the radius of the base circle and the length of the bus line being* 1, *the inscribed sphere is to be maximal. How much is the radius of the base circle?* It is easy to see that the answer is $\frac{1}{3}$, which is geometrically interpreted as follows: Consider the section of a cone by a plane which contains the apex and is perpendicular to the base circle. Then the answer corresponds to the case that the section is an equilateral triangle. In this paper, we generalize the question to the case that the base circle is generalized to regular polygons.

1 Introduction

We consider the following question:

Question 1. Let $\Gamma_n(x)$ be a regular n-gon in $\mathbb{R}^2$, with vertices $v_1, \dots, v_{n-1}$ and v_n, such that the radius of the circumscribed circle is x. Let $P_n(x,y)$ be the pyramid of $\Gamma_n(x)$ with apex p such that $|pv_i| = y$ for $1 \le i \le n$. Let $r_n(x,y)$ be the radius of the inscribed sphere of $P_n(x,y)$. (See Fig.1.)

Assume that a subset $J \subset \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0}$ is given. We set $R_{n,J} := \max_{(x,y)\in J} r_n(x,y)$ and let $L_{n,J}$ be the element $(x,y) \in J$ which attains $R_{n,J}$. What is $L_{n,J}$? Moreover, if we set $L_{\infty,J} := \lim_{n\to\infty} L_{n,J}$, then what is $L_{\infty,J}$?

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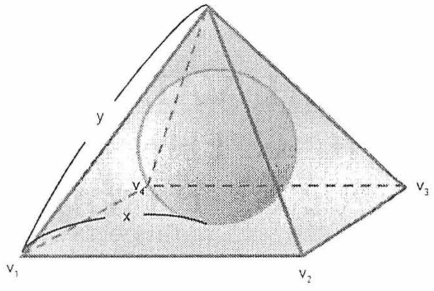

Figure 1: $P_4(x, y)$.

Typical examples for J are given as follows:

(i) $J_1 := \{(1, y) \mid 1 \leq y < \infty\}$.

(ii) $J_2 := \{(x, 1) \mid 0 \leq x \leq 1\}$.

(iii) $J_3 := \{(x, 1-x) \mid 0 \leq x \leq \frac{1}{2}\}$.

About J_2, the condition $0 \leq x \leq 1$ guarantees that $P_n(x, 1) \neq \emptyset$. The conditions for J_1 and J_3 are explained similarly.

The answer to (i) is easy: $r_n(1, y)$ is an increasing function on $y \in [1, \infty)$ such that $r_n(1, 1) = 0$ and $\lim_{y \to \infty} r_n(1, y) = \cos\left(\frac{\pi}{n}\right)$. Here $\lim_{y \to \infty} r_n(1, y)$ is computed from the radius of the inscribed circle of $\Gamma_n(1)$. In particular, although $\sup_{(1,y) \in J_1} r_n(1, y) = \cos\left(\frac{\pi}{n}\right)$, L_{n,J_1} does not exist.

The case (ii) is studied in [3]: The sequence L_{n,J_2} was described explicitly. In particular, we have $L_{\infty,J_2} = \frac{\sqrt{5}-1}{2}$. The result is interesting because this is the reciprocal of the golden section.

The purpose of this paper is to study the case (iii). We determine the sequence L_{n,J_3} explicitly. In particular, we have $L_{\infty,J_3} = \frac{1}{3}$. The result can also be proved by the isoperimetric theorem of triangles. (See Remark 2.)

This paper is organized as follows. In §2 we state our main results and in §3 we prove them.

2 Statement of the main results

Hereafter we always consider J_3 for J in Question 1 and write $P_n(x, 1-x)$ and $r_n(x, 1-x)$ as $P_n(x)$ and $r_n(x)$, respectively. We also abbreviate L_{n,J_3} as L_n.

Theorem A . *We define a sequence λ_n as follows:*

$$\lambda_n = \begin{cases} 1, & n \leq 18, \\ -1, & n \geq 19. \end{cases}$$

We also define sequences a_n, b_n and c_n as follows:

$$a_n = \sqrt{16\sec^4\left(\frac{\pi}{n}\right) - 16\sec^2\left(\frac{\pi}{n}\right) - 3\cdot 2^{2/3}\tan^{\frac{2}{3}}\left(\frac{\pi}{n}\right)\sec^{\frac{2}{3}}\left(\frac{\pi}{n}\right) + 1},$$

$$b_n = \frac{4\left(64\sec^6\left(\frac{\pi}{n}\right) - 96\sec^4\left(\frac{\pi}{n}\right) + 30\sec^2\left(\frac{\pi}{n}\right) + 1\right)}{a_n}$$

and

$$c_n = -96\sec^2\left(\frac{\pi}{n}\right) + \left(\cos\left(\frac{2\pi}{n}\right) + 9\right)^2 \sec^4\left(\frac{\pi}{n}\right) + 6\cdot 2^{2/3}\tan^{\frac{2}{3}}\left(\frac{\pi}{n}\right)\sec^{\frac{2}{3}}\left(\frac{\pi}{n}\right) + \lambda_n b_n.$$

Then we have

$$L_n = \frac{1}{12}\left(2 + 8\sec^2\left(\frac{\pi}{n}\right) - \sqrt{2c_n} + 2\lambda_n a_n\right).$$

Remark 2. For the reason why the sequence λ_n appears in L_n, see Remark 9 in §3.

Corollary B . *We have*

$$L_\infty = \frac{1}{3}.$$

Remark 3. We claim that Corollary B also follows from the isoperimetric theorem of triangles: If we cut $P_\infty(x, 1-x)$ by a plane which contains the apex and is perpendicular to the base circle, then the section is a triangle whose edge lengths are $1-x, 1-x$ and $2x$. Note that the inradius $r_\infty(x, 1-x)$ coincides with the area of the triangle.

The isoperimetric theorem states that among all triangles of given perimeter, the equilateral one has largest area. (See, for example, [2].) Hence $r_\infty(x, 1-x)$ is maximal when x satisfies that $1 - x = 2x$, that is, $x = \frac{1}{3}$. This shows that $L_\infty = \frac{1}{3}$.

Example 4 (Some graphs).

(i) In §3, we determine $r_n(x)$ explicitly. (See Lemma 7.) Using this, we draw the graph of $r_8(x)$ in the left of Fig.3.

(ii) We draw the graph of L_n in the right of Fig.3. Note that L_n is a decreasing sequence. On the other hand, we can check that $2nL_n \sin\left(\frac{\pi}{n}\right)$, the circumference of $P_n(L_n)$, is an increasing sequence.

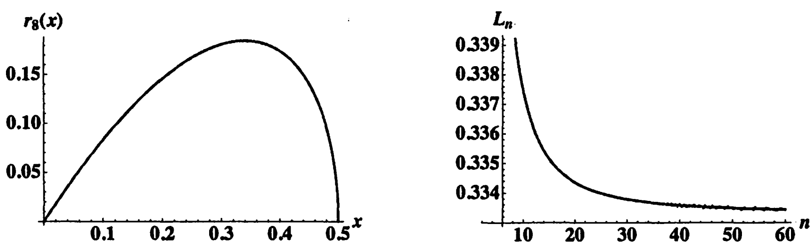

Figure 2: The left: $r_8(x)$. The right: L_n.

Next, we state numerical results when n is small or large.

Example 5 (The result for small n). (i) Although L_n is complicated for general n, only L_4 is slightly simple:

$$L_4 = \frac{1}{2}\left(3+\sqrt{3}-\sqrt{8+\frac{14}{\sqrt{3}}}\right).$$

(ii) The approximate values of L_n for $3 \leq n \leq 10$ are given by the following table:

n	L_n
3	0.3853
4	0.3609
5	0.3505
6	0.3451
7	0.3419
8	0.3399
9	0.3385
10	0.3375
$\vdots$	$\vdots$
∞	$\frac{1}{3}=0.3333$

Table 1: L_n for $3 \leq n \leq 10$

In order to describe the behavior of L_n for large n, we write L_n by the Laurent series:

$$L_n = \sum_{i=0}^{\infty} \frac{\alpha_i}{n^i}.$$

More precisely, in L_n, we replace $\frac{1}{n}$ by t and define a function $K(t)$. Note that $K(t)$ is defined for $t \in (0, \frac{1}{3}]$. Then we set

$$\alpha_i = \frac{1}{i!}\lim_{t \to +0} \frac{d^i K}{dt^i}.$$

Example 6 (The result for large n). (i) We have $a_i = 0$ for odd i and

$$L_n = \frac{1}{3} + \frac{\pi^2}{24n^2} + \frac{5\pi^4}{1152n^4} + \frac{167\pi^6}{276480n^6} + \frac{781\pi^8}{6193152n^8} + \frac{186383\pi^{10}}{5573836800n^{10}} + \cdots.$$

(ii) We set

$$M_n := \sum_{i=0}^{10} \frac{\alpha_i}{n^i}.$$

Then the following inequalities hold:

$$L_n > M_n > L_\infty.$$

Moreover, consider the case for $n = 10^j$, where $1 \leq j \in \mathbb{N}$. Then we have the following results:

$$L_{10^j} - L_\infty \simeq 4.1 \times 10^{-(2j+1)} \quad \text{and} \quad L_{10^j} - M_{10^j} \simeq 9.1 \times 10^{-12j}.$$

3 Proofs of the main results

First we prove Theorem A.

Lemma 7. *We have*

$$r_n(x) = \frac{2x\sqrt{1-2x}\cos\left(\frac{\pi}{n}\right)}{\sqrt{2}\sqrt{x^2\cos\left(\frac{2\pi}{n}\right) + (x-4)x + 2 + 2x\cos\left(\frac{\pi}{n}\right)}}.$$

Proof. The following formula is well-known. (See, for example, [1, Theorem 1].) Let $V(P_n(x))$ and $S(P_n(x))$ be the volume and the surface sea of $P_n(x)$, respectively. Then we have

$$r_n(x) = \frac{3V(P_n(x))}{S(P_n(x))}. \tag{1}$$

Let F be one of the n sides of $P_n(x)$. Since

$$S(F) = \frac{nx\sin\left(\frac{\pi}{n}\right)\sqrt{x^2\cos\left(\frac{2\pi}{n}\right) + (x-4)x + 2}}{\sqrt{2}}$$

and

$$S(\Gamma_n(x)) = \frac{nx^2\sin\left(\frac{2\pi}{n}\right)}{2},$$

we have

$$S(P_n(x)) = \frac{nx\sin\left(\frac{\pi}{n}\right)}{\sqrt{2}}\left(\sqrt{x^2\cos\left(\frac{2\pi}{n}\right) + (x-4)x + 2} + \sqrt{2}x\cos\left(\frac{\pi}{n}\right)\right). \tag{2}$$

On the other hand, since height h of $P_n(x)$ is given by $h = \sqrt{1-2x}$, we have

$$V(P_n(x)) = \frac{nx^2\sqrt{1-2x}\sin\left(\frac{2\pi}{n}\right)}{6}. \tag{3}$$

Now substituting (2) and (3) in (1), Lemma 7 follows. □

Simple computations show that

$$\frac{dr_n}{dx} = \frac{-2\sqrt{2}\cos\left(\frac{\pi}{n}\right)\left(x^3\left(1+\cos\left(\frac{2\pi}{n}\right)\right) - 8x^2 + 8x - 2 + \sqrt{2}x^2\cos\left(\frac{\pi}{n}\right)\sqrt{\phi_n(x)}\right)}{\sqrt{1-2x}\sqrt{\phi_n(x)}\left(\sqrt{2\phi_n(x)} + 2x\cos\left(\frac{\pi}{n}\right)\right)^2}, \tag{4}$$

where we set

$$\phi_n(x) := x^2\left(1+\cos\left(\frac{2\pi}{n}\right)\right) - 4x + 2.$$

We study the graph of $\frac{dr_n}{dx}$ in the range $0 \le x \le \frac{1}{2}$. Since the graphs are similar, we give the case for $n = 10$ in Fig.3.

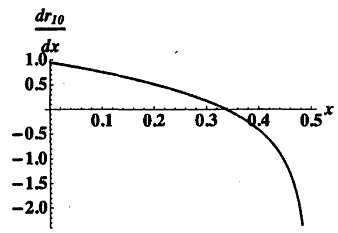

Figure 3: The graph of $\frac{dr_{10}}{dx}$.

Note that the graph has an unique root, which is L_n. Moreover, the first derivative test shows that the graph of $r_n(x)$ is as given in the left of Fig.2.

In order to complete the proof of Theorem A, we need to determine the root in $0 \le x \le \frac{1}{2}$ of the equation "the numerator of (4) = 0".

Lemma 8. *We set*

$$f_n(x) = x^3\left(1+\cos\left(\frac{2\pi}{n}\right)\right) - 8x^2 + 8x - 2 + x^2\cos\left(\frac{\pi}{n}\right)\sqrt{2\phi_n(x)}.$$

Regard that the variable x moves in $\mathbb{R}$. Then the following results hold:

(i) *For all $n \ge 3$, the number of the real roots of the equation $f_n(x) = 0$ is two. We write them by t_1 and t_2, where we take $t_1 < t_2$.*

(ii) *We have $0 < t_1 < 1 < t_2$. It follows that $t_1 = L_1$. (See Fig.4.)*

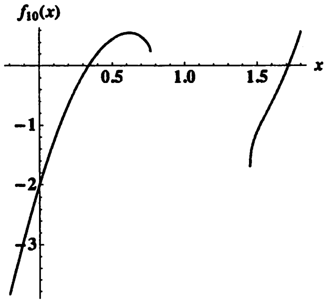

Figure 4: The graph of $f_{10}(x)$.

Proof. We set

$$u_n(x) = x^2 \cos\left(\frac{\pi}{n}\right)\sqrt{2\phi_n(x)} \quad \text{and} \quad v_n(x) = f_n(x) - u_n(x).$$

We set

$$g_n(x) = 3x^4\left(1 + \cos\left(\frac{2\pi}{n}\right)\right) - 2x^3\left(9 + \cos\left(\frac{2\pi}{n}\right)\right) + 24x^2 - 12x + 2.$$

Then we have

$$u_n(x)^2 - v_n(x)^2 = 2(2x-1)g_n(x).$$

Since $\deg g_n(4) = 4$, we can solve the equation $g_n(s) = 0$ by Ferrari's method. (See, for example, [4].) We define a_n and b_n as in Theorem A. Let (μ, ν) be any element of $\{-1, 1\} \times \{-1, 1\}$. We define sequences $c_n(\mu, \nu)$ and $L_n(\mu, \nu)$ as follows:

$$\begin{aligned} c_n(\mu) = &- 96\sec^2\left(\frac{\pi}{n}\right) + \left(\cos\left(\frac{2\pi}{n}\right) + 9\right)^2 \sec^4\left(\frac{\pi}{n}\right) \\ &+ 6 \cdot 2^{2/3} \tan^{\frac{2}{3}}\left(\frac{\pi}{n}\right) \sec^{\frac{2}{3}}\left(\frac{\pi}{n}\right) + \mu b_n \end{aligned}$$

and

$$L_n(\mu, \nu) = \frac{1}{12}\left(2 + 8\sec^2\left(\frac{\pi}{n}\right) + \nu\sqrt{2c_n(\mu)} + 2\mu a_n\right).$$

Then $L_n(\mu, \nu)$ are the solutions of the equation $g_n(x) = 0$. From the direct computations, we can check the following results:

(a) *The case for* $3 \leq n \leq 18$. Only $L_n(1, 1)$ and $L_n(1, -1)$ are real numbers such that

$$0 < L_n(1, -1) < 1 < L_n(1, 1).$$

Moreover, $L_n(1, 1)$ and $L_n(1, -1)$ are the roots of the equation $f_n(x) = 0$, but $L_n(-1, 1)$ and $L_n(-1, -1)$ are not.

(b) *The case for* $n \geq 19$. Only $L_n(-1,1)$ and $L_n(-1,-1)$ are real numbers such that

$$0 < L_n(-1,-1) < 1 < L_n(-1,1).$$

Moreover, $L_n(-1,1)$ and $L_n(-1,-1)$ are the roots of the equation $f_n(x) = 0$, but $L_n(1,1)$ and $L_n(1,-1)$ are not. (The fact that the equation $g_n(x) = 0$ has exactly two real roots can be seen from Fig.5.)

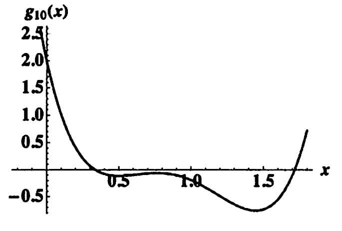

Figure 5: The graph of $g_{10}(x)$.

Now Lemma 8 follows from (a) and (b). □

Proof of Theorem A. From (a) and (b) in the proof of Lemma 8, we see that

$$L_n = \begin{cases} L_n(1,-1), & 3 \leq n \leq 18, \\ L_n(-1,-1), & n \geq 19. \end{cases}$$

By definition, these $L_n(\mu, \nu)$ coincide with L_n in Theorem A. □

Remark 9. We explain why L_n in Theorem A has a different form according as $3 \leq n \leq 18$ or $n \geq 19$. The essential reason is the term $\sqrt{2c_n(\mu)}$. The following figure shows that the signature of $c_n(1)$ changes from plus to minus when n changes from 18 to 19. Moreover, a similar result holds for $c_n(-1)$.

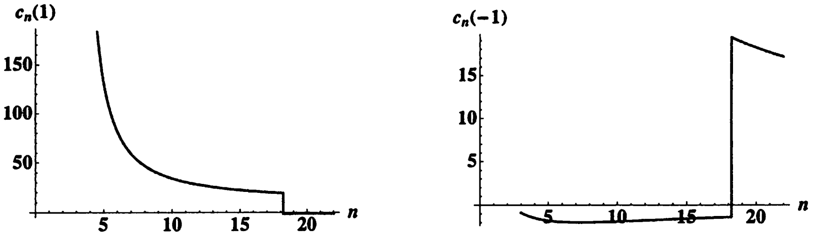

Figure 6: The left: $c_n(1)$. The right: $c_n(-1)$.

Here the vertical line in the graph corresponds to such $n \in \mathbb{R}$ for which $a_n = 0$, hence b_n is not defined. The precise value is

$$n = \frac{\pi}{\cos^{-1}\left(2\sqrt{3\sqrt{2}-4}\right)} \simeq 18.2199.$$

But if we allow a more complicated form, then we can also write L_n by a formula without case statement.

Proof of Corollary B. We give a proof without using an answer to Question 1. (See Remark 3 for the proof which uses the answer.) It is possible to compute L_∞ from Theorem A. But it is more simple to compute $\lim_{n\to\infty} \frac{dr_n}{dx}$ from (4). It is easy to see that

$$\lim_{n\to\infty} \frac{dr_n}{dx} = \frac{1-3x}{\sqrt{1-2x}}.$$

Hence Corollary B follows. □

Proofs of Examples 4, 5 and 6. With the aid of Mathematica, we can deduce these examples from Theorem A. □

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