UNITARY HIGHEST WEIGHT MODULES OF A JACOBI GROUP

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1. Introduction

Let $H_n(\mathbf{R})$ be the $2n+1$ dimensional Heisenberg group over the real number field and G_o the natural semi-direct product of $SL_2(\mathbf{R})$ and $H_n(\mathbf{R})$. This type of group is called a Jacobi group ([1], [3]). Let $\mathfrak{g}_o$ be the Lie algebra of G_o and $\mathfrak{g}$ its complexification. In this note, we classify the irreducible unitarizable highest weight $\mathfrak{g}$-modules. We also give the submodule structure of the Verma modules of $\mathfrak{g}$.

To state our results more precisely, we introduce some notations. Let $K_o = SO(2)$ be the maximal compact subgroup of $SL_2(\mathbf{R})$ and $\mathfrak{k}_o$ the Lie algebra of K_o. We can choose an element $c \in \mathfrak{k}_o$ so that the eigenvalues of the adjoint action of c on $\mathfrak{g}$ are $\pm\sqrt{-1}$, $\pm 2\sqrt{-1}$ and 0. For $l \in \mathbf{Z}$, let $\mathfrak{g}(l)$ be the $l\sqrt{-1}$ eigenspace of $ad(c)$. Then we have a direct sum decomposition:

$$\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2).$$

Put $\mathfrak{n}^- = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1)$, $\mathfrak{h} = \mathfrak{g}(0)$, $\mathfrak{n}^+ = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Let z be a nonzero element of the 1-dimensional center of $\mathfrak{g}_o$. Then $\mathfrak{h} = \mathbf{C}c \oplus \mathbf{C}z$.

For a complex Lie algebra $\mathfrak{a}$, we denote by $U(\mathfrak{a})$ its universal enveloping algebra. For an $\mathfrak{h}$-module V and $\eta \in \mathbf{C}$, we put $V^\eta = \{x \in V : c.x = \eta x\}$.

Definition 1.1. Let χ be a 1-dimensional representation of $\mathfrak{h}$. A $U(\mathfrak{g})$-module V is called a highest wight module with highest weight χ if there exists a nonzero vector v such that $a.v = \chi(a)v$ for $a \in \mathfrak{h}$, $\mathfrak{n}^+.v = 0$ and V is generated by v as a $U(\mathfrak{g})$-module. Moreover if V admits a $\mathfrak{g}_o$-invariant positive definite Hermitian inner product, we say V is unitarizable.

Definition 1.2. Let χ be a 1-dimensional representation of $\mathfrak{h}$ and $\mathbf{C}_\chi$ its representation space. We extend χ to $\mathfrak{b}$ trivially. We define a $U(\mathfrak{g})$-module $M(\chi)$ by

$$M(\chi) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbf{C}_\chi$$

and call it a Verma module.

We denote the irreducible quotient of $M(\chi)$ by $L(\chi)$. We prove the following theorems:

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Theorem 1.3. *The Verma module $M(\chi)$ is reducible if and only if $\chi(z) = 0$ or $\chi(c) = (-n/2 + l)$ for some nonnegative integer l.*

Theorem 1.4. *Assume that $L(\chi)$ is unitarizable. Then $\chi(c) \in \sqrt{-1}\mathbf{R}$, $\chi(z) \in \sqrt{-1}\mathbf{R}$ and $\chi(z)/\sqrt{-1} \leq 0$. Moreover,*
(1) if $\chi(z)/\sqrt{-1} < 0$, then $\chi(c)/\sqrt{-1} \leq -n/2$,
(2) if $\chi(z) = 0$, then $\chi(c)/\sqrt{-1} \leq 0$.
Conversely, if χ satisfies the above conditions, $L(\chi)$ is unitarizable.

Remark. In fact, Theorem 1.4 makes sense only after a particular choice of the element z has been made. See the beginning of section 3 for this.

For the proof of the above theorems, we introduce contravariant sesquilinear forms on $U(\mathfrak{g})$ (Definition 2.1) and on $M(\chi)$ (Definition 2.3). We investigate their fundamental properties in Section 2. The key theorem is Theorem 3.4, which gives a diagonalization of the contravariant sesquilinear form on certain subspaces of $U(\mathfrak{g})$. By this Theorem, in Section 4, we deduce the submodule structures of $M(\chi)$ and the unitarizability criterion of $L(\chi)$.

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2. Contravariant sesquilinear form

In this section, we introduce sesquilinear forms on $U(\mathfrak{g})$ and $M(\chi)$, and describe their fundamental properties. Such a form was first introduced by Shapovalov [5] in the cases of complex semisimple Lie algebras. See also Enright, Howe and Wallach [2]. Let σ be the sesquilinear anti-involution on $\mathfrak{g}$ defined by $\sigma(X) = -\overline{X}$, where the bar is the complex conjugation with respect to $\mathfrak{g}_0$. We extend σ to $U(\mathfrak{g})$ and denote it by the same letter. By the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and the Poincaré - Birkhoff - Witt Theorem, we have :

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+). \tag{2.1}$$

Let $\pi : U(\mathfrak{g}) \to U(\mathfrak{h})$ be the projection to the first component.

Definition 2.1. We define a $U(\mathfrak{h})$-valued form B on $U(\mathfrak{g}) \times U(\mathfrak{g})$ by

$$B(X, Y) = \pi(\sigma(X)Y) \qquad X, Y \in U(\mathfrak{g}). \tag{2.2}$$

Proposition 2.2. (1) $B(Y, X) = \sigma(B(X, Y))$.
(2) *B is sesquilinear* :

$$B(aX + bX', Y) = \overline{a}B(X, Y) + \overline{b}B(X, Y'),$$
$$B(X, aY + bY') = aB(X, Y) + bB(X, Y')$$
$$\textit{for} \quad a, b \in \mathbf{C} \quad \textit{and} \quad X, X', Y, Y' \in U(\mathfrak{g}).$$

(3) *B is contravariant* :

$$B(AX, Y) = B(X, \sigma(A)Y) \quad \textit{for} \quad A, X, Y \in U(\mathfrak{g}).$$

In particular, B is $\mathfrak{g}_o$-invariant :

$$B(A'X,Y)+B(X,A'Y)=0 \quad \textit{for} \quad A'\in\mathfrak{g}_o \quad \textit{and} \quad X,Y\in U(\mathfrak{g}).$$

(4) *For $\eta\in\mathbf{Z}$, let $U(\mathfrak{g})^{\eta}=\{x\in U(\mathfrak{g}):[c,x]=\eta x\}$. Then*

$$B(U(\mathfrak{g})^{\eta},U(\mathfrak{g})^{\xi})=0, \quad \textit{if} \quad \eta\neq\xi.$$

Proof. (1) Note that $\sigma(\mathfrak{n}^+)=\mathfrak{n}^-,\sigma(\mathfrak{n}^-)=\mathfrak{n}^+$ and $\sigma(\mathfrak{h})=\mathfrak{h}$. Hence $\pi\sigma=\sigma|_{U(\mathfrak{h})}\pi$ by (2.1). Thus

$$B(Y,X)=\pi(\sigma(Y)X)=\pi(\sigma(\sigma(X)Y))=\sigma\pi(\sigma(X)Y)=\sigma(B(X,Y)).$$

(2) This is an immediate consequence of Definition 2.1.
(3) For $A,X,Y\in U(\mathfrak{g})$,

$$B(AX,Y)=\pi(\sigma(AX)Y)=\pi(\sigma(X)\sigma(A)Y)=B(X,\sigma(A)Y).$$

(4) Since $\sigma(c)=-c$, we have, for $X\in U(\mathfrak{g})^{\eta}$ and $Y\in U(\mathfrak{g})^{\xi}$,

$$[c,\sigma(X)Y]=\sigma([c,X])Y+\sigma(X)[c,Y]=(\overline{\eta}+\xi)\sigma(X)Y.$$

This means $\sigma(X)Y\in U(\mathfrak{g})^{\overline{\eta}+\xi}$. Since $U(\mathfrak{g})\supset U(\mathfrak{h})$ and $\overline{\eta}=-\eta$, we have

$$B(X,Y)=\pi(\sigma(X)Y)=0, \qquad \textit{if} \qquad \eta\neq\xi.$$

According to the above Proposition 2.2 (2) and (3), we call B a contravariant sesquilinear form on $U(\mathfrak{g})$.

Since $\mathfrak{h}$ is commutative, we can identify $U(\mathfrak{h})$ with the symmetric algebra $S(\mathfrak{h})$, which is the ring of polynomial functions on the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$. Let $\chi\in\mathfrak{h}^*$. We want to define $B_\chi: M(\chi)\times M(\chi)\to\mathbf{C}$ by

(2.3) $$B_\chi(X.v,Y.v)=B(X,Y)(\chi) \quad \text{for} \quad X,Y\in U(\mathfrak{g}),$$

where $v\in M(\chi)$ is a fixed highest weight vector. But, in general, B_χ is not well defined.

Lemma 2.3. *B_χ is well defined if and only if $\chi(c)\in\sqrt{-1}\mathbf{R}$ and $\chi(z)\in\sqrt{-1}\mathbf{R}$.*

Proof. Let $I(\chi)$ be the left ideal of $U(\mathfrak{g})$ generated by the elements:

$$\{a-\chi(a):a\in\mathfrak{h}\}\cup\mathfrak{n}^+.$$

Then $M(\chi)$ is isomorphic to $U(\mathfrak{g})/I(\chi)$ as a left $U(\mathfrak{g})$-module. By Proposition 2.2 (1), the well-definedness of B_χ is equivalent to the condition $\chi(B(I(\chi),U(\mathfrak{g})))=0$. By (2.1) and (2.2), we have $\chi(B(\mathfrak{n}^+,U(\mathfrak{g})))=0$. Hence it is enough to consider the condition:

(2.4) $$\chi(\sigma(a-\chi(a)))=0 \quad \text{for any} \quad a\in\mathfrak{h}.$$

We write $a=x+\sqrt{-1}y$, $x,y\in\mathfrak{g}_o$. Then

$$\begin{aligned}\chi(\sigma(a-\chi(a))&=\chi(-x+\sqrt{-1}y)-\overline{\chi}(x+\sqrt{-1}y)\\&=-(\chi(x)+\overline{\chi}(x))-\sqrt{-1}(\chi(y)+\overline{\chi}(y)).\end{aligned}$$

Hence (2.4) is equivalent to $\chi(c)\in\sqrt{-1}\mathbf{R}$ and $\chi(z)\in\sqrt{-1}\mathbf{R}$.

Definition 2.4. If $\chi \in \mathfrak{h}^*$ satisfies $\chi(c) \in \sqrt{-1}\mathbf{R}$ and $\chi(z) \in \sqrt{-1}\mathbf{R}$, we call B_χ the contravariant sesquilinear form on $M(\chi)$.

The following Proposition is a direct consequence of Proposition 2.2:

Proposition 2.5. *Suppose* $\chi \in \mathfrak{h}^*$ *satisfies* $\chi(c) \in \sqrt{-1}\mathbf{R}$ *and* $\chi(c) \in \sqrt{-1}\mathbf{R}$. *Then*
(1) B_χ *is Hermitian* :

$$B_\chi(u, w) = \overline{B_\chi(w, u)} \quad for \quad u, w \in M(\chi).$$

(2) B_χ *is* $\mathfrak{g}_o$*-invariant* :

$$B_\chi(A.u, w) + B_\chi(u, A.w) = 0 \quad for \quad A \in \mathfrak{g}_o \quad and \quad u, w \in M(\chi).$$

Since we shall discuss the unitarizability of irreducible highest weight modules, we give some properties of $\mathfrak{g}_o$-invariant sesquilinear forms on $M(\chi)$ for general $\chi \in \mathfrak{h}^*$.

Lemma 2.6. *Let* B' *be a* $\mathfrak{g}_o$*-invariant sesquilinear form on* $M(\chi)$.
(1) $B'(M(\chi)^\eta, M(\chi)^\xi) = 0$ *for* $\eta \neq \xi$.
(2) *If* $M(\chi)$ *admits a well defined nonzero* $\mathfrak{g}_o$*-invariant sesquilinear form B', then* $\chi(c) \in \sqrt{-1}\mathbf{R}$ *and* $\chi(z) \in \sqrt{-1}\mathbf{R}$. *In this case,* B' *is a scalar multiple of* B_χ.

Proof. (1) Since $M(\chi)$ is a highest weight module, there exits nonnegative integers i and j such that $\eta(c) = \chi(c) - \sqrt{-1}i$ and $\xi(c) = \chi(c) - \sqrt{-1}j$. Hence if $\eta \neq \xi$, $\overline{\eta}(c) + \xi(c) \neq 0$. On the other hand, for $x \in M(\chi)^\eta$ and $y \in M(\chi)^\xi$,

$$0 = B'(c.x, y) + B'(x, c.y)) = (\overline{\eta}(c) + \xi(c))B'(x, y).$$

This proves (1).
(2) If B' is $\mathfrak{g}_o$-invariant, we have

$$B'(A.u, w) = B'(u, \sigma(A).w) \quad \text{for} \quad u, w \in M(\chi) \quad \text{and} \quad A \in U(\mathfrak{g}). \tag{2.5}$$

Hence by (1), for any $X, Y \in U(\mathfrak{g})$,

$$B'(X.v, Y.v) = B'(v, \sigma(X)Y.v) = \chi(\pi(\sigma(X))Y)B'(v, v).$$

Hence, by the proof of Lemma 2.3, if B' is well defined and nonzero, then $\chi(c) \in \sqrt{-1}\mathbf{R}$ and $\chi(z) \in \sqrt{-1}\mathbf{R}$. Moreover, in this case,

$$B'(X.v, Y.v) = B'(v, v)B_\chi(X.v, Y.v).$$

For a highest weight module V with highest weight vector v, let $pr : V \to \mathbf{C}v$ be the projection map.

Proposition 2.7. (1) *If for any $w \in V$, there exits a $X \in U(\mathfrak{g})$ such that $pr(X.w) \neq 0$, then V is irreducible.*
(2) *Suppose $\chi \in h^*$ satisfies $\chi(c) \in \sqrt{-1}\mathbf{R}$ and $\chi(z) \in \sqrt{-1}\mathbf{R}$, then for $\eta \in \mathbf{C}$,*

$$\mathrm{rank} B_\chi|_{M(\chi)^\eta} = \dim L(\chi)^\eta.$$

Proof. (1) Since V is a highest weight module, every $U(\mathfrak{g})$-submodule of V is a direct sum of its weight spaces. Hence, in particular, $pr(X.w) \in U(\mathfrak{g}).w$. Therefore, if $pr(X.w) \neq 0$, then $U(\mathfrak{g}).w = V$. This proves (1).
(2) By Lemma 2. 6 (1), it is enough to show

$$\mathrm{Rad} B_\chi = \{x \in M(\chi) : B_\chi(x, y) = 0 \quad \text{for any} \quad y \in M(\chi)\}$$

is a proper maximal submodule of $M(\chi)$. Since $B_\chi(v, v) = 1$, $\mathrm{Rad}B_\chi$ is proper. By (2.5) $\mathrm{Rad}B_\chi$ is a $U(\mathfrak{g})$-submodule. If $w = X.v \in \mathrm{Rad}B_\chi$, $X \in U(\mathfrak{g})$, then there exits a $u = Y.v, Y \in U(\mathfrak{g})$ such that $B_\chi(u, w) = \chi(\pi(\sigma(Y)X)) \neq 0$. Since $B_\chi(u, w)v = pr(\sigma(Y)w)$, $U(\mathfrak{g}).w = M(\chi)$ by (1). Hence $\mathrm{Rad}B_\chi$ is maximal.

3. Diagonalization of the contravariant sesquilinear form

In this section, we diagonalize the contravariant sesquilinear form B on certain subspaces of $U(\mathfrak{g})$. For this purpose, we fix a basis of g. Let $h_n(\mathbf{R})$ be the Lie algebra of $H_n(\mathbf{R})$ and $\{p_i, q_i, z\}_{i=1,\cdots,n}$ its canonical basis. That is:

$$[p_i, q_j] = \delta_{i,j} z, \quad [p_i, z] = [q_i, z] = 0. \tag{3.1}$$

This is the choice of the element z in Theorem 1.4. The action of $sl_2(\mathbf{R})$ on $h_n(\mathbf{R})$ is given by

$$[A, x] = \sum_{i=1}^{n} \{(\alpha s_i + \beta t_i)p_i + (\gamma s_i + \delta t_i)q_i\}, \qquad [A, z] = 0 \tag{3.2}$$

$$\text{for} \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in sl_2(\mathbf{R}) \quad \text{and} \quad x = \sum_{i=1}^{n} (s_i p_i + t_i q_i) \in h_n(\mathbf{R}).$$

We choose $c \in k_o$ as $c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Set $E = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & -1 \end{pmatrix}$, $H = -\sqrt{-1}c$, $F = \overline{E}$, $X_i = \frac{1}{\sqrt{2}}(p_i + \sqrt{-1}q_i)$, $Y_i = \frac{1}{\sqrt{2}}(p_i - \sqrt{-1}q_i)$ and $Z = -\sqrt{-1}c$. Then the set $\{E, H, F, X_i, Y_i, Z\}_{i=1,\cdots,n}$ forms a basis of $\mathfrak{g}$.

Lemma 3.1. (1) *In the above notations,*

$$\sigma(E) = -F, \quad \sigma(F) = -E, \quad \sigma(X_i) = -Y_i,$$
$$\sigma(Y_i) = -X_i, \quad \sigma(Z) = Z, \quad \sigma(H) = H.$$

(2) *The above basis satisfies the following bracket relations :*

$$[Z, A] = 0, \quad \textit{for any} \quad A \in \mathfrak{g},$$
$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H, \quad [X_i, Y_j] = \delta_{i,j} Z,$$
$$[H, X_i] = X_i, \quad [H, Y_i] = -Y_i, \quad [E, Y_i] = X_i, \quad [F, X_i] = Y_i,$$
$$[E, X_i] = [F, Y_i] = 0.$$

In particular, $\mathfrak{n}^- = \mathbf{C}F \oplus (\oplus_{i=1}^n \mathbf{C}Y_i)$, $\mathfrak{h} = \mathbf{C}H \oplus \mathbf{C}Z$ *and* $\mathfrak{n}^+ = \mathbf{C}E \oplus (\oplus_{i=1}^n \mathbf{C}X_i)$.

Lemma 3.2. (1) $[X_i, \sum_{j=1}^n Y_j^2 + 2ZF] = [Y_i, \sum_{j=1}^n X_j^2 - 2ZE] = 0.$
(2) $X_i^p Y_i^q = (D_{Y_i} R_Z + R_{X_i})^p Y_i^q = \sum_{j=0}^p \binom{p}{j} \left(\frac{d^j}{dY_i^j} Y_i^q\right) Z^j X_i^{p-j}.$
Here $R_u, u \in \mathfrak{g}$, denote the right multiplication by u:

$$R_u x = xu \quad \textit{for} \quad x \in U(\mathfrak{g})$$

and D_{Y_i} is the differentiation by Y_i.

Proof. (1) By Lemma 3. 1 (2), we have

$$[X_i, \sum_{j=1}^n Y_j^2 + 2ZF] = \sum_{j=1}^n ([X_i, Y_j]Y_j + Y_j[X_i, Y_j]) + 2Z[X_i, F] = 0.$$

Similarly, we get $[Y_i, \sum_{j=1}^n X_j^2 - 2ZE] = 0$.
(2) If $p = 1$, then

$$X_i Y_i^q = \sum_{j=1}^q Y_i^{j-1}[X_i, Y_i]Y_i^{q-j} + Y_i^q X_i = qY_i^{q-1}Z + Y_i^q X = (D_{Y_i} R_Z + R_{X_i})Y_i^q.$$

Since the operators D_{Y_i}, R_Z and R_{X_i} are mutually commutative, we get the proof of (2).

Lemma 3.3. $\pi(E^p(\sum_{j=1}^n Y_j^2 + 2ZF)^p) = p! Z^p \prod_{j=1}^p (2H + n - 2j + 2).$

Proof. First , we prove the following formula by induction on p:

$$(3.3) \quad [E, \sum_{j=1}^n (Y_j^2 + 2ZF)^p]$$
$$= p(\sum_{j=1}^n Y_j^2 + 2ZF)^{p-1} Z(2H + n - 2p + 2) + 2p(\sum_{j=1}^n Y_j^2 + 2ZF)^{p-1} Y_j X_j.$$

In fact , if $p = 1$,

$$\begin{aligned}[E, \sum_{j=1}^n (Y_j^2 + 2ZF)] &= \sum_{j=1}^n ([E, Y_j]Y_j + Y_j[E, Y_j]) + 2Z[E, F] \\ &= \sum_{j=1}^n (X_j Y_j + Y_j X_j) + 2ZH = \sum_{j=1}^n (Z + 2Y_j X_j) + 2ZH \\ &= \sum_{j=1}^n Z(2H + n) + 2Y_j X_j.\end{aligned}$$

Assume (3.3) holds if p is replaced by $p-1$. Then

$$
\begin{aligned}
&[E, \sum_{j=1}^{n}(Y_j^2 + 2ZF)^p] \\
&= [E, \sum_{j=1}^{n}(Y_j^2 + 2ZF)^{p-1}](\sum_{j=1}^{n} Y_j^2 + 2ZF) \\
&\quad + \sum_{j=1}^{n}(Y_j^2 + 2ZF)^{p-1}[E, \sum_{j=1}^{n}(Y_j^2 + 2ZF)] \\
&= (p-1)(\sum_{j=1}^{n} Y_j^2 + 2ZF)^p - 2Z(2H + n - 2p + 4)(\sum_{j=1}^{n} Y_j^2 + 2ZF) \\
&\quad + 2(p-1)(\sum_{j=1}^{n} Y_j^2 + 2ZF)^{p-2}(\sum_{j=1}^{n} Y_j X_j)(\sum_{j=1}^{n} Y_j^2 + 2ZF) \\
&\quad + (\sum_{j=1}^{n} Y_j^2 + 2ZF)^{p-1}(Z(2H + n) + 2\sum_{j=1}^{n} Y_j X_j) \\
&= (p-1)(\sum_{j=1}^{n} Y_j^2 + 2ZF)^{p-1} Z(2H + n - 2p) \\
&\quad + (\sum_{j=1}^{n} Y_j^2 + 2ZF)^{p-1} Z(2H + n) + 2p(\sum_{j=1}^{n} Y_j^2 + 2ZF)^{p-1} Y_j X_j \\
&= p(\sum_{j=1}^{n} Y_j^2 + 2ZF)^{p-1} Z(2H + n - 2p + 2) + 2p(\sum_{j=1}^{n} Y_j^2 + 2ZF)^{p-1} Y_j X_j.
\end{aligned}
$$

Hence (3.3) holds for any p. Now we prove the lemma by induction on p. Assume the lemma holds if p is replaced by $p-1$. Then

$$
\begin{aligned}
&\pi(E^p(\sum_{j=1}^{n} Y_j^2 + 2ZF)^p) = \pi(E^{p-1}E(\sum_{j=1}^{n} Y_j^2 + 2ZF)^p) \\
&= \pi(pE^{p-1}(\sum_{j=1}^{n} Y_j^2 + 2ZF)^{p-1} Z(2H + n - 2p + 2)) \\
&\quad + \pi(2pE^{p-1}(\sum_{j=1}^{n} Y_j^2 + 2ZF)^{p-1} Y_j X_j) + \pi(E^{p-1}(\sum_{j=1}^{n} Y_j^2 + 2ZF)^p E) \\
&= \pi(pE^{p-1}(\sum_{j=1}^{n} Y_j^2 + 2ZF)^{p-1} Z(2H + n - 2p + 2)) \\
&= p! Z^p \prod_{j=1}^{p}(2H + n - 2j + 2).
\end{aligned}
$$

Let $\mathbf{Z}_{\geq 0}$ be the set of nonnegative integers. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbf{Z}^n_{\geq 0}$, we set $|\alpha| = \alpha_1 + ... + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. We also set $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and $Y^\alpha = Y_1^{\alpha_1} \cdots Y_n^{\alpha_n}$. For $m \in \mathbf{Z}_{\geq 0}$, consider the following subspace U^m of $U(\mathfrak{n}^-) \otimes U(\mathfrak{h})$:

$$U^m = \mathbf{C} - \text{linear span of } \{Y^\alpha(\sum_{j=1}^n Y_j^2 + 2ZF)^p : |\alpha| + 2p = m\}.$$

Theorem 3.4. *The restriction $B|_{U^m \times U^m}$ of B to the subspace U^m is given by a diagonal matrix whose diagonal elements are*

$$2^p p! \alpha! (-Z)^m \prod_{j=1}^p (-2H - n + 2j - 2), |\alpha| + 2p = m.$$

Proof. Suppose $\alpha_i < \beta_i$ for some i. Set $\alpha' = (\alpha_1, \cdots, \alpha_{i-1}, 0, \alpha_{i+1}, \cdots, \alpha_n)$ and $\beta' = (\beta_1, \cdots, \beta_{i-1}, 0, \beta_{i+1}, \cdots, \beta_n)$. Then by Lemma 3.2,

$$\begin{aligned}
&\{\sigma(Y^\alpha(\sum_{j=1}^n Y_j^2 + 2ZF)^p)\}\{Y^\beta(\sum_{j=1}^n Y_j^2 + 2ZF)^q\} \\
&= (-1)^{|\alpha|} X_i^{\alpha_i} Y_i^{\beta_i} (\sum_{j=1}^n X_j^2 - 2ZE)^p X^{\alpha'} Y^{\beta'} (\sum_{j=1}^n Y_j^2 + 2ZF)^q \\
&= (-1)^{|\alpha|} \sum_{r=0}^{\alpha_i} \binom{\alpha_i}{r} \left(\frac{d^r}{dY_i^r} Y_i^{\beta_i}\right) Z^r X_i^{\alpha_i - r} (\sum_{j=1}^n X_j^2 - 2ZE)^p X^{\alpha'} Y^{\beta'} (\sum_{j=1}^n Y_j^2 + 2ZF)^q \\
&\in \mathfrak{n}^- U(\mathfrak{g}).
\end{aligned}$$

Similarly, if $\alpha_i > \beta_i$ for some i, we can prove :

$$\{\sigma(Y^\alpha(\sum_{j=1}^n Y_j^2 + 2ZF)^p)\}\{Y^\beta(\sum_{j=1}^n Y_j^2 + 2ZF)^q\} \in U(\mathfrak{g})\mathfrak{n}^+.$$

Hence if $\alpha \neq \beta$,

$$B(Y^\alpha(\sum_{j=1}^n Y_j^2 + 2ZF)^p), Y^\beta(\sum_{j=1}^n Y_j^2 + 2ZF)^q) = 0.$$

By Lemma 3. 2 (1) and Lemma 3. 3,

$$
\begin{aligned}
&B(Y^\alpha(\sum_{j=1}^n Y_j^2+2ZF)^p, Y^\alpha(\sum_{j=1}^n Y_j^2+2ZF)^p)\\
&=\pi((-1)^{|a|}((\sum_{j=1}^n X_j^2-2ZE)^p X^\alpha Y^\alpha(\sum_{j=1}^n Y_j^2+2ZF)^p)\\
&=\alpha!(-Z)^{|a|}\pi((\sum_{j=1}^n X_j^2-2ZE)^p(\sum_{j=1}^n Y_j^2+2ZF)^p)\\
&=\alpha!(-Z)^{|a|}\pi(\sum_{r=0}^p \binom{p}{r}(-2ZE)^r\sum_{j=1}^n n(X_j^2)^{p-r}(\sum_{j=1}^n Y_j^2+2ZF)^p)\\
&=\alpha!(-Z)^{|a|}\pi((-2ZE)^p(\sum_{j=1}^n Y_j^2+2ZF)^p)\\
&=2^p\alpha!(-Z)^{|a|+p}\pi(E^p(\sum_{j=1}^n Y_j^2+2ZF)^p)\\
&=2^p p!\alpha!(-Z)^m\prod_{j=1}^p(-2H-n+2j-2).
\end{aligned}
$$

4. Structure of Verma modules and unitarizability of irreducible highest weight modules

In this section, we describe the structure of the Verma modules $M(\chi)$ and unitarizability condition for $L(\chi)$. First we consider the case $\chi(Z)\neq 0$. For $\alpha\in\mathbf{Z}_{\geq 0}^n$ and $p\in\mathbf{Z}_{\geq 0}$, we set

$$v_{a,p}=Y^\alpha(\sum_{j=1}^n Y_j^2+2ZF)^p.v\in M(\chi).$$

If $\chi(Z)\neq 0$, then the set of the elements

$$\{v_{a,p}:\alpha\in\mathbf{Z}_{\geq 0}^n, p\in\mathbf{Z}_{\geq 0}\}$$

forms a basis of $M(\chi)$.

Thorem 4.1. *Let $\chi\in\mathfrak{h}^*$ and assume $\chi(Z)\neq 0$.*
(1) If $\chi(H)+(n/2)\notin\mathbf{Z}_{\geq 0}$, then the Verma module $M(\chi)$ is irreducible.
(2) If $\chi(H)=-(n/2)+l, l\in\mathbf{Z}_{\geq 0}$, then the proper maximal submodule N of $M(\chi)$ is isomorphic to $M(\chi-2\sqrt{-1}(l+1))$ and given by

$$N=U(\mathfrak{g})(\sum_{j=1}^n Y_j^2+2ZF)^{l+1}.v.$$

Here $\chi - 2\sqrt{-1}(l+1)$ *is an element of* $\mathfrak{h}^*$ *defined by* $(\chi - 2\sqrt{-1}(l+1))(H) = \chi(H) - 2(l+1)$ *and* $(\chi - 2\sqrt{-1}(l+1))(Z) = \chi(Z)$. *Moreover* N *is irreducible. Hence the composition series of* $M(\chi)$ *is given by* $M(\chi) \supset N \supset \{0\}$.

Proof. (1) Let $w = \sum_{i=1}^{q} c_{\alpha_i,p_i} v_{\alpha_i,p_i} \in M(\chi)$. Assume, for example, $c_{\alpha_1,p_1} \neq 0$. Then by the proof of Theorem 3.4,

$$pr(\sigma(Y^{\alpha_1}(\sum_{j=1}^{n} Y_j^2 + 2ZF)^{p_1}).w)$$

$$= 2^{p_1} p_1! \alpha_1! c_{\alpha_i,p_i} \chi(-Z)^{|\alpha_1|+p_1} \prod_{j=1}^{p_1} (-2\chi(H) - n + 2j - 2).v \neq 0$$

Hence by Lemma 2.7 (1), $M(\chi)$ is irreducible. (2) By Lemma 3.2 (1) ,

$$X_i(\sum_{j=1}^{n} Y_j^2 + 2ZF)^{l+1}.v = (\sum_{j=1}^{n} Y_j^2 + 2ZF)^{l+1} X_i.v = 0.$$

Also by (3.3) in the proof of Lemma 3.3,

$$E(\sum_{j=1}^{n} Y_j^2 + 2ZF)^{l+1}.v = [E, (\sum_{j=1}^{n} Y_j^2 + 2ZF)^{l+1}].v$$

$$= (l+1)\left((\sum_{j=1}^{n} Y_j^2 + 2ZF)^{l} \chi(Z)(2\chi(H) + n - 2l) + 2(\sum_{j=1}^{n} Y_j^2 + 2ZF)^{l} \sum_{j=1}^{n} Y_j X_j\right).v$$

$$= 0.$$

Hence $N = U(\mathfrak{g})(\sum_{j=1}^{n} Y_j^2 + 2ZF)^{l+1}.v$ is a proper $U(\mathfrak{g})$-submodule of $M(\chi)$. It is easy to check that N is isomorphic to $M(\chi - 2(l+1)\sqrt{-1})$. The irreducibility of N is easily follows form (1). Let $\rho : M(\chi) \to M(\chi)/N$ be the natural projection. Then the elements

$$\{\rho(v_{\alpha,p}) : \alpha \in \mathbf{Z}_{\geq 0}^n, 0 \leq p \leq l\}$$

forms a basis of $M(\chi)/N$. By the same argument as in the proof of (1), the assumption in Lemma 2. 7 (1) holds for $M(\chi)/N$. Hence $M(\chi)/N$ is irreducible. This implies the maximality of N.

We next discuss the unitarizability of $L(\chi)$. By Lemma 2.6 (2), if $L(\chi)$ is unitarizable, then $\chi(Z) \in \mathbf{R}$ and $\chi(H) \in \mathbf{R}$.

Theorem 4.2. *If* $\chi \in \mathfrak{h}^*$ *satisfies* $\chi(Z) \neq 0$, *then* $L(\chi)$ *is unitarizable if and only if* $\chi(Z) \in \mathbf{R}$, $\chi(H) \in \mathbf{R}$, $\chi(Z) < 0$ *and* $\chi(H) \leq -n/2$.

Proof. For $m \in \mathbf{Z}_{\geq 0}$, we denote the restriction of B_χ to the weight space $M(\chi)^{\chi - m}$ by B_χ^m. Since $B_\chi(v, v) = 1$, $L(\chi)$ is unitarizable if and only if the Hermitian form B_χ^m is positive semi-definite for any $m \in \mathbf{Z}_{\geq 0}$. By Theorem 3.4, if we choose

$\{v_{\alpha,p} : \alpha \in \mathbf{Z}^m_{\geq 0}, p \in \mathbf{Z}_{\geq 0}, |\alpha| + p = m\}$ as the basis of $M(\chi)^{\chi - m}$, B^m_χ is given by a diagonal matrix whose diagonal elements are

$$2^p p! \alpha! \chi(-Z)^m (-2\chi(H) - n + 2j - 2), \quad |\alpha| + 2p = m.$$

Hence B^m_χ is positive semi- definite if and only if $\chi(Z) < 0$ and $-2\chi(H) - n + 2j - 2 \geq 0$ for any positive integer j. This prove the theorem.

If $\chi(Z) = 0$, then $W = \sum_{i=1}^n U(\mathfrak{g}) Y_i.v$ is a nonzero proper $U(\mathfrak{g})$-submodule of $M(\chi)$. Hence the Verma module $M(\chi)$ is reducible in this case.

Lemma 4.3. *In the above notations, $h_n(\mathbf{C})$ acts trivially on the quotient module $M(\chi)/W$. Here $h_n(\mathbf{C})$ is the complexification of the Heisenberg Lie algebra $h_n(\mathbf{R})$.*

Proof. As a vector space, $M(\chi)$ is a direct sum $\oplus_{\alpha,q} \mathbf{C} Y^\alpha F^q.v$. Hence it is enough to show $h_n(\mathbf{C}) F^j.v \in W$ for $j \in \mathbf{Z}_{\geq 0}$. Obviously, $ZF^j.v = X_i.v = 0$ and $Y_i F^j.v \in W$, for $i = 1, \cdots, n$. Also we have, for $j \geq 1$,

$$X_i F^j.v = \sum_{p=0}^{j-1} F^p [X_i, F] F^{j-p-1}.v = j F^{j-1} Y_i.v \in W.$$

By Lemma 4.3, if $\chi(Z) = 0$, the unitarizability of $L(\chi)$ reduces to the $sl_2(\mathbf{R})$-theory. (See, for example, [2].)

Theorem 4.4. *If $\chi \in \mathfrak{h}^*$ satisfies $\chi(Z) = 0$, then*
(1) The Verma module $M(\chi)$ is reducible.
(2) $L(\chi)$ is unitarizable if and only if $\chi(H) \in \mathbf{R}$ and $\chi(H) \leq 0$.

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