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1. Introduction

Let $h_n(C)$ be the $2n+1$ dimensional Heisenberg Lie algebra over the complex number field C and g the natural semidirect product Lie algebra of $sl_2(C)$ and $h_n(C)$. Explicitly, g is given by the following brackets. Let $\{E, H, F\}$ be the standard basis of $sl_2(C)$ and $\{P_i, Q_i, Z\}_{i=1,\cdots,n}$ the canonical basis of $h_n(C)$, then

$$[Z, X] = 0, \qquad \text{for any } X \in g,$$

$$[P_i, Q_j] = \delta_{ij} Z, \qquad [E, P_i] = [F, Q_i] = 0,$$

$$[H, E] = 2E, \qquad [H, P_i] = P_i,$$

$$[H, Q_i] = -Q_i, \qquad [H, F] = -2F,$$

$$[E, Q_i] = P_i, \qquad [F, P_i] = Q_i,$$

$$[E, F] = H.$$

Let $n^- = CF \oplus (\oplus_{i=1,\cdots,n} CQ_i)$, $h = CH \oplus CZ$, $n^+ = CE \oplus (\oplus_{i=1,\cdots,n} CP_i)$ and $b = h \oplus n^+$. These subspaces are Lie subalgebras of g.

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Let χ be a 1 - dimensional representation of h. We extend it to b trivially and denote by the same letter. A $U(g)$ - module V is called a ***highest weight module*** with ***highest weight*** χ if there exists a nonzero vector v such that $a \cdot v = \chi(a) v$ for $a \in b$ and V is generated by v as a $U(g)$ - module. For any 1 - dimensional representation χ of h there exists a unique, up to isomorphism, *irreducible* highest weight module with highest weight χ.

In this note we give a realization of irreducible highest weight module of g [Theorem 2]. Our method is essentially the same in [S1]. The similar arguments is found in [B]. As an application, we get a reducibility criterion of Verma modules [Proposition 3].

2. Main Theorem

Let $W = C < u, v_i, w, \partial_u, \partial_{v_i}, \partial_w > i = 1,\cdots,n$ be the Weyl algebra of $2n+2$ variables. The following Lemma is easily checked by direct calculations.

LEMMA 1. *For* $\lambda \in C$, *let* $\varphi_\lambda : g \to W$ *be the* C *- linear map defined by the following correspondence. Then* φ_λ *defines an algebra homomorphism between* $U(g)$ *and* W, *where* $U(g)$ *is the universal enveloping algebra of* g.

$$\varphi_\lambda(Z) = 2\partial_u, \qquad \varphi_\lambda(Q_i) = -\partial_{v_i},$$

$$\varphi_\lambda(F) = -\partial_w, \qquad \varphi_\lambda(P_i) = 2v_i\partial_u + w\partial_{v_i},$$

$$\varphi_\lambda(H) = 2w\partial_w + \sum_{i=1}^{n} v_i\partial_{v_i} - \lambda,$$

$$\varphi_\lambda(E) = \sum_{i=1}^{n} v_i^2 \partial_u + w\sum_{i=1}^{n} v_i \partial_{v_i} + w^2\partial_w - \lambda w.$$

REMARK. If $n = 1$ then we can realize g as a subalgebra of $sp_4(C)$ in the following way. We define $sp_4(C)$ by

$$sp_4(C)=\{X\in M_4(C): {}^tXJ+JX=O\}\ ,\ J=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then

$$\iota(aH+bE+cF+xP_1+yQ_1+zZ)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ x & a & 0 & b \\ -2z & y & 0 & -x \\ y & c & 0 & -a \end{pmatrix}$$

defines an injective homomorphism of Lie algebras. The φ_λ in the Lemma is given by the restriction of φ_λ in [S1] to the Lie subalgebra $\iota(g)$ of $sp_4(C)$.

For $\lambda, \mu \in C$, we define a 1-dimensional representation $\chi = \chi(\lambda, \mu)$ of h by $\chi(H) = \lambda$ and $\chi(Z) = \mu$. Let $f_{\lambda,\mu} = f_{\lambda,\mu}(u, v_i, w) = w^\lambda \exp \mu\,(u - \sum_{i=1}^{n} v_i^2/w)$. We consider the $U(g)$-module $L(\chi) = \varphi_\lambda(U(g))\, f_{\lambda,\mu}$. Our main Theorem is:

THEOREM 2. *$L(\chi)$ is an irreducible highest weight module with highest weight $\chi = \chi(\lambda,\mu)$ and a highest weight vector $f_{\lambda,\mu}$.*

Proof. It is easy to check that $\varphi_\lambda(P_i)f_{\lambda,\mu} = 0$, $i = 1,\cdots,n$, $\varphi_\lambda(E)f_{\lambda,\mu} = 0$ and $\varphi_\lambda(a)f_{\lambda,\mu} = \chi(a)\, f_{\lambda,\mu}$ for $a \in h$. Hence $L(\chi)$ is a highest weight module with highest weight $\chi=\chi(\lambda,\mu)$ and a highest weight vector $f_{\lambda,\mu}$. We shall prove the irreducibility.

If $\mu = 0$, then $\varphi_\lambda(h_n(C))$ acts trivially on $L(\chi)$. Hence the Theorem is reduced to the representation theory of $sl_2(C)$. Then the Theorem is well known .

Suppose $\mu \neq 0$. By the Poincaré - Birkhoff - Witt theorem , $U(g) = U(n^-) \otimes U(h) \otimes U(n^+)$, where $U(h)$ and $U(n^+)$ are the universal enveloping algebras of h and n^+ , respectively . Hence

$$L(\chi) = \varphi_\lambda(U(g))f_{\lambda,\mu} = \varphi_\lambda(U(n^-))f_{\lambda,\mu}$$

$$= \sum_{\alpha_i \geq 0, \beta \geq 0} C\, \partial_{v_i}^{\alpha_i} \partial_w^{\beta} f_{\lambda,\mu} .$$

For $x \in L(\chi)$, we consider the condition $\varphi_\lambda(E)\, x = \varphi_\lambda(P_i)\, x = 0$, $i = 1,\cdots,n$. Since

$$\partial_w f_{\lambda,\mu} = (\lambda / w) f_{\lambda,\mu} + (\mu / w^2) \sum_{i=1}^{n} v_i^2 f_{\lambda,\mu} ,$$

$$\partial_{v_i} f_{\lambda,\mu} = -2 (\mu v_i / w) f_{\lambda,\mu} ,$$

we can assume $x = p(v_i , w)f_{\lambda,\mu}$ for some $p(v_i , w) \in C[v_i,w^{-1}]$. By simple calculations ,

$$\varphi_\lambda(P_i)\, x = (w\, \partial_{v_i} p(v_i , w))\, f_{\lambda,\mu} ,$$

$$\varphi_\lambda(E)\, x = \{(w \sum_{i=1}^{n} v_i\, \partial_{v_i} + w^2 \partial_w) p(v_i , w)\} f_{\lambda,\mu} .$$

Hence it is easy to see that if $x = p(v_i,w)f_{\lambda,\mu} \in L(\chi)$ is annihilated by $\varphi_\lambda(E)$ and $\varphi_\lambda(P_i)$, $i = 1,\cdots,n$, then it is a scalar multiple of the highest weight vector $f_{\lambda,\mu}$. Since $L(\chi)$ is a highest weigh module , this implies its irreducibility.

3. A Reducibility criterion of Verma modules

For a one dimensional representation χ of h, we denote its representation space by C_χ. Let $U(b)$ be the universal enveloping algebra of b. We define Verma modules analogous to semisimple Lie algebras.

DEFINITION. We call the g-module $M(\chi)=U(g)\otimes_{U(b)}C_\chi$ Verma module.

$M(\chi)$ is the *universal* highest weight module with highest weight χ. As a vector space, $M(\chi)$ is isomorphic to $U(n^-)$.

PROPOSITION 3. *If $\chi(Z) = 0$ or $\chi(H) = -n/2+k$ for some non-negative integer k, then $M(\chi)$ is reducible.*

Proof. Let $v = 1 \otimes 1$ be the highest weight vector of $M(\chi)$. If $\chi(Z) = 0$, then $U(g)Q_i\,v$ is a nonzero proper submodule of $M(\chi)$ and the Lemma is obvious.

We consider the case $\chi(Z) = \mu \neq 0$. The highest weight vector $f_{\lambda,\mu}$ of $L(\chi)$ satisfies the following differential equation:

$$(\mu\,\partial_w - \frac{1}{4}\sum_{i=1}^{n} \partial_{v_i}^2) f_{\lambda,\mu} = \frac{1}{2}\mu\,(2\lambda+n) f_{\lambda-1,\mu}.$$

Iterating this, we have

$$(\mu\,\partial_w - \frac{1}{4}\sum_{i=1}^{n} \partial_{v_i}^2)^m f_{\lambda,\mu}$$

$$= \frac{1}{2^m}\mu(2\lambda+n)(2\lambda+n-2)\cdots(2\lambda+n-2m+2) f_{\lambda-m,\mu}.$$

Assume $M(\chi)$ is irreducible . Then by Theorem 2 and the uniqueness of irreducible highest weight modules , $M(\chi)$ and $L(\chi)$ are isomorphic . Then , as a vector space , $L(\chi)$ is isomorphic to $U(n^-)$. Then $\partial_{v_i}^{\alpha_i}\partial_w^{\beta} f_{\lambda,\mu}$'s must be linearly independent . But if $\chi(H) = -n/2 + k$ for some non-negative integer k , the right hand side of the above differential equation is equal to zero for large m . This yields a contradiction .

REMARK . In the forthcoming paper [S2] , using the contravariant form we shall prove the converse of the above Proposition 3 . That is , if $M(\chi)$ is reducible then $\chi(Z) = 0$ or $\chi(H) = -n/2 + k$ for some non-negative integer k .

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