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On $\mathfrak{F}$-Normalizers and $\mathfrak{F}$-Covering Subgroups

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1. Introduction

In finite solvable groups, the system normalizers and the Carter subgroups are two important classes of conjugate subgroups. These classes of subgroups take natural forms in the theory of formation which was first introduced by Gaschütz[2].

All groups in this note are finite and solvable. A formation is a class $\mathfrak{F}$ of groups closed under taking homomorphisms and subdirect products. If formations $\mathfrak{F}(p)$, one for each prime p, are given, the local formation $\mathfrak{F}$ locally defined by $\{\mathfrak{F}(p)\}$ is the class of all groups G such that $G \in \mathfrak{F}$ if and only if for each prime p diving $|G|$ and each p-chief factor H/K of G, $A_G(H/K) \in \mathfrak{F}(p)$, where $A_G(H/K)$ $(\cong G/C_G(H/K))$ denotes the group of automorphism .induced by G on H/K.

For a formation $\mathfrak{F}$, an $\mathfrak{F}$-covering subgroup of the group G is a subgroup C such that $C \in \mathfrak{F}$ and whenever $C \leqq H \leqq G$, $K \lhd H$ and $H/K \in \mathfrak{F}$, then $H=KC$. Gaschütz [2] showed that if $\mathfrak{F}$ is a local formation, then $\mathfrak{F}$-covering subgroups of the group always exist.

Carter and Hawkes [1] introduced, for each local formation $\mathfrak{F}$, the concept of an $\mathfrak{F}$-normalizer of the group. Let $\mathfrak{F}$ be a local formation defined by $\{\mathfrak{F}(p)\}$. For each prime p diving $|G|$, let $G_{\mathfrak{F}(p)}$ be the unique minimal normal subgroup of G such that $G/G_{\mathfrak{F}(p)} \in \mathfrak{F}(p)$. Let T^p be a p-complement of $G_{\mathfrak{F}(p)}$. The $D=\bigcap_p N_G(T^p)$ is an $\{\mathfrak{F}(p)\}$-normalizer of G. If $\mathfrak{F}(p) \subseteqq \mathfrak{F}$ for each prime p, then the $\{\mathfrak{F}(p)\}$-normalizers depend only on $\mathfrak{F}$, and not on $\mathfrak{F}(p)$, and are called $\mathfrak{F}$-normalizers.

The $\mathfrak{F}$-normalizers and the $\mathfrak{F}$-covering subgroups coincide with the system normalizers and the Carter subgroups, respectively when $\mathfrak{F}$ is the formation of nilpotent groups. In this note, we shall show some results on the $\mathfrak{F}$-normalizers and the $\mathfrak{F}$-covering subgroups of a group which are generalizations of the known results on the system normalizers and the Carter subgroups.

2. On $\mathfrak{F}$-SC-group

A finite solvable group is called an *SC*-group, if its system normalizers coincide with its Carter subgrous. Some results on the *SC*-group can be found in Huppert [3]. Now we say that a finite solvable group is an $\mathfrak{F}$-*SC*-group, if its $\mathfrak{F}$-normalizers coincide with its $\mathfrak{F}$-covering subgroups. We shall show two results (Therem 1 and Theorem 2) on the $\mathfrak{F}$-*SC*-group which are generalizations of Satz 13.2 and Satz 13.3 in Kapitel VI of Huppert [3].

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THEOREM 1. *Let $\mathfrak{F}$ be a local formation defined by $\{\mathfrak{F}(p)\}$ with $\mathfrak{F}(p) \subseteqq \mathfrak{F}$ for each prime p. If G is an $\mathfrak{F}$-SC-group and N is a normal subgroup of G, then G/N is an $\mathfrak{F}$-SC-group.*

PROOF. Let $\tilde{D}/N$ be an $\mathfrak{F}$-normalizer of G/N. Then by the Corollary 2 of Theorem 4.1 in [1], there exists an $\mathfrak{F}$-normalizer D of G such that $\tilde{D}/N = DN/N$. Since G is an $\mathfrak{F}$-SC-group, there exists an $\mathfrak{F}$-covering subgroup C of G such that $D=C$. Thus we have $DN/N = CN/N$. But CN/N is an $\mathfrak{F}$-covering subgroup of G/N by Hilfssatz 2.2 in [2]. Hence G/N is an $\mathfrak{F}$-SC-group.

THEOREM 2. *Let $\mathfrak{F}$ be a local formation defined by $\{\mathfrak{F}(p)\}$ with $\mathfrak{F}(p) \neq \phi$ and $\mathfrak{F}(p) \subseteqq \mathfrak{F}$ for each prime p. Let G be an $\mathfrak{F}$-SC-group and D be an $\mathfrak{F}$-normalizer of G. Then, for each $\mathfrak{F}$-eccentric chief factor H/K of G, if $h^d K = hK$ for some $h \in H$ and for all $d \in D$, then $h \in K$.*

PROOF. The methods of our proof are similar to those in Satz 13.3 in [3, VI]. We consider an $\mathfrak{F}$-eccentric chief factor H/K of G. Let

$$H' = \{h \in H;\ [h, D] \leqq K\} \text{ and } H' \neq \phi.$$

Then to prove the theorem it suffices to show that $H' \leqq K$.

For $h \in H'$ and $d \in D$, we have

$$d^h = d\,[d, h] = d\,[h, d]^{-1}.$$

Therefore d^h is in $KD(=DK)$. Now K is a normal subgroup of G. Hence $H' \leqq N_G(KD)$. Since G is an $\mathfrak{F}$-SC-group, D is an $\mathfrak{F}$-covering subgroup of G. Hence by Hilfssatz 7.11 in [3, VI], $D \leqq KD \leqq G$ implies that $N_G(KD) = KD$. Thus $H' \leqq KD$. By Theorem 4.1 in [1], D avoids H/K, that is, $H \cap D \leqq K$. Therefore

$$H' \leqq KD \cap H = K(D \cap H) = K$$

and this concludes the proof.

3. On Rose's Lemma

Rose [5] showed two useful facts (Lemma 3.1 and Lemma 3.2) on the system normalizers and the Carter subgroups. Huppert [4] showed that Rose's result on the Carter subgroups holds for the $\mathfrak{F}$-covering subgroups. We shall show that Rose's result on the system normalizers also holds for $\mathfrak{F}$-normalizers (Theorem 3). Our proof of Theorem 3 is similar to that of Lemma 3.1 in Rose [5].

THEOREM 3. *Let $\mathfrak{F}$ be a local formation defined by $\{\mathfrak{F}(p)\}$ with $\mathfrak{F}(p) \subseteqq \mathfrak{F}$ for each prime p. Let D be an $\mathfrak{F}$-normalizer of a finite solvable group G and N_1, N_2 be normal subgroups of G. Then $DN_1 \cap DN_2 = D(N_1 \cap N_2)$.*

PROOF. We may assume, without loss of generality, that $N_1 > 1$, $N_2 > 1$ and $N_1 \cap N_2 = 1$. We use induction on $|G|$. Let N_1' be a minimal normal subgroup of G contained in N_1. Then, since DN_1'/N_1' is an $\mathfrak{F}$-normalizer of G/N_1' by the Corollary 2 of Theorem 4.1 in [1], the induction hypothesis implies that

$$(DN_1'/N_1')(N_1/N_1') \cap (DN_1'/N_1')(N_1'N_2/N_1') = (DN_1'/N_1')(N_1/N_1' \cap N_1'N_2/N_1').$$

so that

$$(DN_1/N_1') \cap (DN_1'N_2/N_1') = DN_1'/N_1'.$$

Hence $DN_1 \cap DN_2. = DN_1' \cap DN_2$ Therefore we may suppose that $N_1 = N_1'$ that is that N_1 is a minimal normal subgroup of G. Similarly we may suppose that N_2 is a minimal normal subgroup of G.

If $N_1/1$ or $N_2/1$ is $\mathfrak{F}$-central chief factor of G, then by Theorem 4.1 in [1] D covers $N_1/1$ or $N_2/1$, that is, $N_1 \leqq D$ or $N_2 \leqq D$. Thus we have

$$DN_1 \cap DN_2 = D \cap DN_2 = D = D(N_1 \cap N_2)$$

or

$$DN_1 \cap DN_2 = DN_1 \cap D = D = D(N_1 \cap N_2).$$

Therefore we may assume that both $N_1/1$ and $N_2/1$ are $\mathfrak{F}$-eccentric chief factor of G.

Now we show that $C_G(N_1N_2/N_1)$ is a subgroup of $C_G(N_2/1)$. Let g be an element in $C_G^*(N_1N_2/N_1)$. For each element n_2 in N_2 and some element n_1 in N_1, we have, since $N_1 \lhd G$, $N_2 \lhd G$ and $N_1 \cap N_2 = 1$,

$$\begin{aligned} g^{-1}(n_1n_2)^{-1}g(n_1n_2) &= g^{-1}n_2^{-1}n_1^{-1}gn_1n_2 \\ &= g^{-1}n_2^{-1}gn_1'n_1n_2 \\ &= g^{-1}n_2^{-1}gn_2n_1'n_1 \end{aligned}$$

where n_1' is in N_1, such that $n_1^{-1}g = gn_1'$. Since $g \in C_G(N_1N_2/N_1)$, we have $g^{-1}n_2^{-1}gn_2n_1'n_1 \in N_1$ and therefore $g^{-1}n_2^{-1}gn_2 \in N_1$. On the other hand, we have $g^{-1}n_2^{-1}gn_2 \in N_2$ since $N_2 \lhd G$. Hence $g^{-1}n_2^{-1}gn_2 \in N_1 \cap N_2 = 1$. Thus $g \in C_G(N_2/1)$. Therefore $C_G(N_2/1)$ containg $C_G(N_1N_2/N_1)$ and this show that $C_G(N_1N_2/N_1)$ is subgroup of $C_G(N_2/1)$, since $C_G(N_1N_2/N_1)$ and $C_G(N_2/1)$ are subgroups of G.

Suppose that N_1N_2/N_1 is an $\mathfrak{F}$-central p-chief factor of G for some prime p. Then we have $A_G(N_1N_2/N_1) \in \mathfrak{F}(p)$, that is, $G/C_G(N_1N_2/N_1) \in \mathfrak{F}(p)$. Since $C_G(N_1N_2/N_1)$ is a subgroup of $C_G(N_2/1)$, $G/C_G(N_2/1)$ is isomorphic to a factor group of $G/C_G(N_1N_2/N_1)$ and so $G/C_G(N_2/1) \in \mathfrak{F}(p)$ since $\mathfrak{F}(p)$ is a formation. Thus $N_2/1$ is an $\mathfrak{F}$-central p-chief factor of G, a contradiction. Therefore N_1N_2/N_1 is an $\mathfrak{F}$-eccentric chief factor. By Theorem 4.1 in [1], D avoids N_1N_2/N_1 and $N_1/1$, that is, $D \cap N_1N_2 \leqq N_1$ and $D \cap N_1 \leqq 1$, and therefore

$$D \cap N_1N_2 = D \cap (D \cap N_1N_2) \leqq D \cap N_1 = 1.$$

Hence $DN_1 \cap DN_2 \leqq D$ and therefore $DN_1 \cap DN_2 = D$. This concludes the proof.

References

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