STRUCTURE OF THE C*-ALGEBRAS OF SOLVABLE LIE GROUPS OF FULL TRIANGULAR MATRICES

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STRUCTURE OF THE C^*-ALGEBRAS OF SOLVABLE LIE GROUPS OF FULL TRIANGULAR MATRICES

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ABSTRACT. We study the algebraic structure of group C^*-algebras of connected solvable Lie groups of full triangular matrices to obtain their finite composition series and determine the structure of their subquotients, and give its applications.

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1. INTRODUCTION

This paper is a continuation of the study on the (global) algebraic structure of group C^*-algebras of connected solvable Lie groups (cf. [Sd1, Sd5], and refer to [Sd6, Sd8] for some disconnected solvable Lie group cases). Specifically, we consider the group C^*-algebras of connected solvable Lie groups of all the full triangular matrices. The main purpose of this paper is to construct finite composition series of the C^*-algebras of these solvable Lie groups and to determine the structure of their subquotients. We start with the two by two matrix case in Section 2 (cf. [Sd7]). We next analyze the three by three matrix case in details in Section 3, and consider the four by four matrix case in Section 4 and the five by five matrix case in Section 5 without full details. The main tool for analysis of subquotients of those composition series is Green's result [Gr2, Corollary 10] (a consequence of the imprimitivity theorem). The results in Sections 4 and 5 would be helpful to understanding the general case considered

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in Section 6. By induction and reduction to lower matrix size cases, the above purpose for the general case is achieved. Finally, a few applications of the main result (Theorem 6.1) are given in Section 7. In particular, we estimate the topological stable rank and connected stable rank of those group C^*-algebras in terms of groups. These results are reasonable but not so surprising from the previous works ([Sh], [ST1-2] and [Sd1-8]). However, our formulation by induction is elementary and explicit, and it would not be found in the literature. As an appendix, for the convenience to readers we briefly review Green's imprimitivity theorem and its consequences ([Gr1], [Gr2]).

Notation. For G a connected Lie group, we denote by $C^*(G)$ the group C^*-algebra of G (cf. [Dx]). For X a locally compact Hausdorff space, let $C_0(X)$ be the C^*-algebra of all continuous functions on X vanishing at infinity. Set $C_0(X) = C(X)$ when X is compact. Let $\mathbb{K}$ be the C^*-algebra of all compact operators on a separable, infinite dimensional Hilbert space. Denote by $\mathfrak{A} \rtimes_\alpha G$ the (full) crossed product of a C^*-algebra $\mathfrak{A}$ by a Lie group G with α an action (the symbol α is often omitted when it is specified) (cf. [Pd]).

2. The two by two matrix case

Let $T_2(\mathbb{C})$ be the connected solvable Lie group of all the 2×2 triangular matrices:

$$T_2(\mathbb{C}) \ni (a_{12}, z_2, z_1) = \begin{pmatrix} z_1 & a_{12} \\ 0 & z_2 \end{pmatrix} \in GL_2(\mathbb{C})$$

for $a_{12} \in \mathbb{C}$, $z_1, z_2 \in \mathbb{C}_\times$ (the multiplicative group of $\mathbb{C}$, which is isomorphic to the direct product $\mathbb{R} \times \mathbb{T}$). Under the above identification, $T_2(\mathbb{C})$ is isomorphic to the semi-direct product $\mathbb{C} \rtimes_\alpha \mathbb{C}_\times^2$ with the action α defined by $\alpha_{(z_2,z_1)}(a_{12}) = z_1 \bar{z}_2 a_{12}$. Then

Theorem 2.1. *Let $T_2(\mathbb{C})$ be as above. Then $C^*(T_2(\mathbb{C}))$ has the exact sequence*: $0 \to C_0(\mathbb{R} \times \mathbb{Z}) \otimes \mathbb{K} \to C^*(T_2(\mathbb{C})) \to C_0(\mathbb{R}^2 \times \mathbb{Z}^2) \to 0$.

Proof. Since $T_2(\mathbb{C}) \cong \mathbb{C} \rtimes_\alpha \mathbb{C}_\times^2$, then $C^*(T_2(\mathbb{C})) = C^*(\mathbb{C}) \rtimes_\alpha \mathbb{C}_\times^2$. By the Fourier transform, $C^*(\mathbb{C}) \rtimes_\alpha \mathbb{C}_\times^2 \cong C_0(\mathbb{C}) \rtimes_{\hat{\alpha}} \mathbb{C}_\times^2$, where $\hat{\alpha}_{(z_2,z_1)}(b_{12}) = \bar{z}_1 z_2 b_{12}$ for $b_{12} \in \mathbb{C}$. Since the origin of $\mathbb{C}$ is fixed under $\hat{\alpha}$, the following exact sequence is obtained:

$$0 \to C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{C}_\times^2 \to C_0(\mathbb{C}) \rtimes_{\hat{\alpha}} \mathbb{C}_\times^2 \to C^*(\mathbb{C}_\times^2) \to 0$$

and $C^*(\mathbb{C}^2_\times) \cong C^*(\mathbb{R}^2 \times \mathbb{T}^2) \cong C_0(\mathbb{R}^2 \times \mathbb{Z}^2)$. Since the action $\hat{\alpha}$ is transitive on $\mathbb{C} \setminus \{0\}$, it is obtained by [Gr2] that

$$C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{C}^2_\times \cong C_0(\mathbb{C}^2_\times/(\mathbb{C}^2_\times)_1) \rtimes \mathbb{C}^2_\times \cong C^*((\mathbb{C}^2_\times)_1) \otimes \mathbb{K}$$

where $(\mathbb{C}^2_\times)_1$ is the stabilizer of $1 \in \mathbb{C} \setminus \{0\}$ and equal to the set $\{(z_2, z_1) \in \mathbb{C}^2_\times \mid \bar{z}_1 z_2 = 1\}$. Then $(\mathbb{C}^2_\times)_1$ is isomorphic to $\mathbb{C}_\times$. Therefore, we obtain $C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{C}^2_\times \cong C_0(\mathbb{R} \times \mathbb{Z}) \otimes \mathbb{K}$. □

Next let $T_2(\mathbb{R}, \mathbb{C})$ be the connected solvable Lie group of all the 2×2 triangular matrices:

$$T_2(\mathbb{R}, \mathbb{C}) \ni (a_{12}, t_2, t_1) = \begin{pmatrix} e^{z_1 t_1} & a_{12} \\ 0 & e^{z_2 t_2} \end{pmatrix} \in GL_2(\mathbb{C})$$

for $a_{12} \in \mathbb{C}$, $t_1, t_2 \in \mathbb{R}$ and fixed nonzero, non purely imaginary $z_1, z_2 \in \mathbb{C}$. When z_1 or z_2 are purely imaginary, we let $T_2(\mathbb{R}, \mathbb{C})$ consist of all elements (a_{12}, t_2, t_1), and it has a quotient map to the above matrices.

Remark. We often use this convention of the real case in the following sections. Since the group of those matrices is a quotient of $T_2(\mathbb{R}, \mathbb{C})$, the structure of its group C^*-algebra is deduced from that of the C^*-algebra of $T_2(\mathbb{R}, \mathbb{C})$. This also holds for the cases with matrix size larger in the following sections.

Then $T_2(\mathbb{R}, \mathbb{C})$ is isomorphic to the semi-direct product $\mathbb{C} \rtimes_\alpha \mathbb{R}^2$ with α defined by $\alpha_{(t_2,t_1)}(a_{12}) = (e^{(z_1 t_1 - z_2 t_2)} a_{12})$.

Theorem 2.2. *Let $T_2(\mathbb{R}, \mathbb{C})$ be as above. Then $C^*(T_2(\mathbb{R}, \mathbb{C}))$ has the exact sequence:* $0 \to \mathfrak{I} \to C^*(T_2(\mathbb{R}, \mathbb{C})) \to C_0(\mathbb{R}^2) \to 0$, *and the closed ideal $\mathfrak{I}$ is isomorphic to either $C_0(\mathbb{R}^2 \times \mathbb{T}) \otimes \mathbb{K}$, $C_0(\mathbb{R} \times \mathbb{T}) \otimes \mathbb{K}$, or $C(\mathbb{T}) \otimes \mathbb{K}$.*

Proof. By the Fourier transform, $C^*(\mathbb{C} \rtimes_\alpha \mathbb{R}^2) \cong C^*(\mathbb{C}) \rtimes \mathbb{R}^2 \cong C_0(\mathbb{C}) \rtimes_{\hat{\alpha}} \mathbb{R}^2$, where $\hat{\alpha}_{(t_2,t_1)}(b_{12}) = e^{(\bar{z}_1 t_1 - \bar{z}_2 t_2)} b_{12}$ for $b_{12} \in \mathbb{C}$. Since the origin of $\mathbb{C}$ is fixed under $\hat{\alpha}$, it follows that $0 \to C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{R}^2 \to C_0(\mathbb{C}) \rtimes_{\hat{\alpha}} \mathbb{R}^2 \to C^*(\mathbb{R}^2) \to 0$.

(Case 1): Suppose that z_1, z_2 are purely imaginary. Then we have $C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{R}^2 \cong C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{R}^2)$. Since $\hat{\alpha}$ is transitive on $\mathbb{T}$, it is obtained by [Gr2] that $C(\mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{R}^2 \cong C(\mathbb{R}^2/\mathbb{R}^2_1) \rtimes \mathbb{R}^2 \cong$

$C^*(\mathbb{R}^2_1) \otimes \mathbb{K}$, where $\mathbb{R}^2_1$ is the stabilizer of $1 \in \mathbb{T}$ and equal to the set $\{(t_2, t_1) \in \mathbb{R}^2 \mid \bar{z}_1 t_1 - \bar{z}_2 t_2 = 2n\pi i,\ n \in \mathbb{Z}\}$. Then $\mathbb{R}^2_1$ is isomorphic to the product group $\mathbb{R} \times \mathbb{Z}$. Hence $C^*(\mathbb{R}^2_1) \cong C_0(\mathbb{R} \times \mathbb{T})$.

(Case 2): Suppose that z_1 or z_2 are not purely imaginary.

(Case 2_1): If z_1, z_2 are linearly dependent, any orbit of $\mathbb{C} \setminus \{0\}$ is homeomorphic to $\mathbb{R}$. Then $C_0(\mathbb{C}\setminus\{0\}) \rtimes \mathbb{R}^2 \cong C(\mathbb{T}) \otimes (C_0(\mathbb{R}_+) \rtimes_{\hat{\alpha}} \mathbb{R}^2)$. Since $\hat{\alpha}$ is transitive on $\mathbb{R}_+$, it is obtained by [Gr2] that $C_0(\mathbb{R}_+) \rtimes \mathbb{R}^2 \cong C_0(\mathbb{R}^2/\mathbb{R}^2_1) \rtimes \mathbb{R}^2 \cong C^*(\mathbb{R}^2_1) \otimes \mathbb{K}$, where $\mathbb{R}^2_1$ is the stabilizer of $1 \in \mathbb{R}_+$ and isomorphic to $\mathbb{R}$.

(Case 2_2): If z_1, z_2 are linearly independent, then the action of $\mathbb{R}^2$ on $\mathbb{C} \setminus \{0\}$ is transitive. By [Gr2], it follows that $C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{R}^2 \cong C_0(\mathbb{R}^2/\mathbb{R}^2_{1_2}) \rtimes \mathbb{R}^2 \cong C^*(\mathbb{R}^2_{1_2}) \otimes \mathbb{K}$, where $\mathbb{R}^2_{1_2}$ is the stabilizer of $1_2 = (1,1) \in \mathbb{C} \setminus \{0\}$ and equal to the set $\{(t_2, t_1) \in \mathbb{R}^2 \mid \mathrm{Re}(z_1)t_1 = \mathrm{Re}(z_2)t_2$ and $\mathrm{Im}(z_1)t_1 - \mathrm{Im}(z_2)t_2 = 2n\pi i,\ n \in \mathbb{Z}\}$. Then $\mathbb{R}^2_{1_2} \cong \mathbb{Z}$. □

Next let $T_2(\mathbb{T}, \mathbb{C})$ be the connected solvable Lie group of all the triangular matrices:

$$T_2(\mathbb{T}, \mathbb{C}) \ni (a_{12}, w_2, w_1) = \begin{pmatrix} w_1 & a_{12} \\ 0 & w_2 \end{pmatrix} \in GL_2(\mathbb{C})$$

for $a_{12} \in \mathbb{C}$ and $w_1, w_2 \in \mathbb{T}$. Then $T_2(\mathbb{T}, \mathbb{C})$ is isomorphic to the semi-direct product $\mathbb{C} \rtimes_\alpha \mathbb{T}^2$ with the action α defined by $\alpha_{(w_2, w_1)}(a_{12}) = w_1 \bar{w}_2 a_{12}$. Then

Theorem 2.3. *The group C^*-algebra $C^*(T_2(\mathbb{T}, \mathbb{C}))$ has the exact sequence*: $0 \to C_0(\mathbb{R} \times \mathbb{Z}) \otimes \mathbb{K} \to C^*(T_2(\mathbb{T}, \mathbb{C})) \to C_0(\mathbb{Z}^2) \to 0$.

Proof. The sequence: $0 \to C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{T}^2 \to C^*(T_2(\mathbb{T}, \mathbb{C})) \to C^*(\mathbb{T}^2) \to 0$ is exact, and $C^*(\mathbb{T}^2) \cong C_0(\mathbb{Z}^2)$. Moreover, $C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{T}^2 \cong C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{T}^2)$. Since $\hat{\alpha}$ is transitive on $\mathbb{T}$, it is obtained that $C(\mathbb{T}) \rtimes \mathbb{T}^2 \cong C(\mathbb{T}^2/\mathbb{T}^2_1) \rtimes \mathbb{T}^2 \cong C^*(\mathbb{T}^2_1) \otimes \mathbb{K}$, where $\mathbb{T}^2_1$ is the stabilizer of $1 \in \mathbb{T}$ and isomorphic to $\{(w_2, w_1) \in \mathbb{T}^2 \mid \bar{w}_1 w_2 = 1\}$. Then $\mathbb{T}^2_1$ is isomorphic to the diagonal of $\mathbb{T}^2$. Hence $\mathbb{T}^2_1 \cong \mathbb{T}$. □

3. The three by three matrix case

Let $T_3(\mathbb{C})$ be the connected solvable Lie group of all the 3×3 triangular matrices:

$$T_3(\mathbb{C}) \ni (a_{13}, a_{23}, z_3, a_{12}, z_2, z_1) = \begin{pmatrix} z_1 & a_{12} & a_{13} \\ 0 & z_2 & a_{23} \\ 0 & 0 & z_3 \end{pmatrix} \in GL_3(\mathbb{C})$$

for $z_1, z_2, z_3 \in \mathbb{C}_\times$, $a_{12}, a_{13}, a_{23} \in \mathbb{C}$. From the above identification, $T_3(\mathbb{C})$ is isomorphic to $\mathbb{C}^2 \rtimes_\alpha (\mathbb{C}_\times \times T_2(\mathbb{C}))$ where the action α is defined by

$$\begin{aligned}
\alpha_g(a_{13}, a_{23}) &= g \begin{pmatrix} 1 & 0 & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} g^{-1} \\
&= \begin{pmatrix} h & 0 \\ 0 & z_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h^{-1} & 0 \\ 0 & \bar{z}_3 \end{pmatrix} \\
&= \bar{z}_3(z_1 a_{13} + a_{12}a_{23}, z_2 a_{23}) \in \mathbb{C}^2,
\end{aligned}$$

for $g \in \mathbb{C}_\times \times T_2(\mathbb{C})$ and $h \in T_2(\mathbb{C})$. Then, $C^*(T_3(\mathbb{C})) \cong C^*(\mathbb{C}^2) \rtimes_\alpha (\mathbb{C}_\times \times T_2(\mathbb{C})) \cong C_0(\mathbb{C}^2) \rtimes_{\hat{\alpha}} (\mathbb{C}_\times \times T_2(\mathbb{C}))$, where $\hat{\alpha}_g(l_{13}, l_{23}) = \bar{z}_3 h^* l = \bar{z}_3(\bar{z}_1 l_{13}, \bar{a}_{12} l_{13} + \bar{z}_2 l_{23})$ for $l = (l_{13}, l_{23}) \in \mathbb{C}^2$. Since the subspace $\{0\} \times \mathbb{C}$ of $\mathbb{C}^2$ is invariant under $\hat{\alpha}$, the following exact sequence is obtained:

$$\begin{aligned}
(M) : 0 &\to C_0((\mathbb{C} \setminus \{0\}) \times \mathbb{C}) \rtimes (\mathbb{C}_\times \times T_2(\mathbb{C})) \\
&\to C_0(\mathbb{C}^2) \rtimes (\mathbb{C}_\times \times T_2(\mathbb{C})) \\
&\to C_0(\{0\} \times \mathbb{C}) \rtimes (\mathbb{C}_\times \times T_2(\mathbb{C})) \to 0.
\end{aligned}$$

The quotient C^*-algebra of (M) has the following decomposition:

$$\begin{aligned}
(Q) : 0 \to\ & C_0(\mathbb{C} \setminus \{0\}) \rtimes (\mathbb{C}_\times \times T_2(\mathbb{C})) \\
&\to C_0(\{0\} \times \mathbb{C}) \rtimes (\mathbb{C}_\times \times T_2(\mathbb{C})) \to C^*(\mathbb{C}_\times \times T_2(\mathbb{C})) \to 0.
\end{aligned}$$

Moreover, by [Gr2] $C_0(\mathbb{C} \setminus \{0\}) \rtimes (\mathbb{C}_\times \times T_2(\mathbb{C}))$ of (Q) is isomorphic to

$$\begin{aligned}
& C_0((\mathbb{C}_\times \times T_2(\mathbb{C}))/(\mathbb{C}_\times \times T_2(\mathbb{C}))_{(0,1)}) \rtimes (\mathbb{C}_\times \times T_2(\mathbb{C})) \\
& \cong C^*((\mathbb{C}_\times \times T_2(\mathbb{C}))_{(0,1)}) \otimes \mathbb{K}
\end{aligned}$$

with the stabilizer $(\mathbb{C}_\times \times T_2(\mathbb{C}))_{(0,1)}$ of $(0,1) \in \{0\} \times \mathbb{C} \setminus \{0\}$ isomorphic to $T_2(\mathbb{C})$. On the other hand, $C_0((\mathbb{C} \setminus \{0\}) \times \mathbb{C}) \rtimes (\mathbb{C}_\times \times T_2(\mathbb{C}))$ of (M) is isomorphic to

$$\begin{aligned}
& C_0((\mathbb{C}_\times \times T_2(\mathbb{C}))/(\mathbb{C}_\times \times T_2(\mathbb{C}))_{(1,1)}) \rtimes (\mathbb{C}_\times \times T_2(\mathbb{C})) \\
& \cong C^*((\mathbb{C}_\times \times T_2(\mathbb{C}))_{(1,1)}) \otimes \mathbb{K}
\end{aligned}$$

with the stabilizer $(\mathbb{C}_\times \times T_2(\mathbb{C}))_{(1,1)} \cong \mathbb{C}_\times^2$.

Summing up the above argument, it is obtained that

Theorem 3.1. *Let $T_3(\mathbb{C})$ be as above. Then $C^*(T_3(\mathbb{C}))$ has the decomposition:* $0 \to C^*((\mathbb{C}_\times \times T_2(\mathbb{C}))_{(1,1)}) \otimes \mathbb{K} \to C^*(T_3(\mathbb{C})) \to \mathfrak{D} \to 0$, *and*

$$0 \to C^*((\mathbb{C}_\times \times T_2(\mathbb{C}))_{(0,1)}) \otimes \mathbb{K} \to \mathfrak{D} \to C^*(\mathbb{C}_\times \times T_2(\mathbb{C})) \to 0.$$

Moreover, we have $C^*((\mathbb{C}_\times \times T_2(\mathbb{C}))_{(1,1)}) \cong C_0(\mathbb{R}^2 \times \mathbb{Z}^2)$ *and* $C^*((\mathbb{C}_\times \times T_2(\mathbb{C}))_{(0,1)}) \cong C^*(T_2(\mathbb{C}))$.

Next let $T_3(\mathbb{R}, \mathbb{C})$ be the solvable Lie group of all the 3×3 triangular matrices:

$$T_3(\mathbb{R}, \mathbb{C}) \ni (a_{13}, a_{23}, t_3, a_{12}, t_2, t_1)$$
$$= \begin{pmatrix} e^{z_1 t_1} & a_{12} & a_{13} \\ 0 & e^{z_2 t_2} & a_{23} \\ 0 & 0 & e^{z_3 t_3} \end{pmatrix} \in GL_3(\mathbb{C})$$

for $a_{12}, a_{13}, a_{23} \in \mathbb{C}$, $t_1, t_2, t_3 \in \mathbb{R}$ and fixed non-zero, non purely imaginary $z_1, z_2, z_3 \in \mathbb{C}$. When z_1, z_2 or z_3 are purely imaginary, we let $T_3(\mathbb{R}, \mathbb{C})$ consist of all the above tuples, and it has a quotient map to the above matrices. Then $T_3(\mathbb{R}, \mathbb{C}) \cong \mathbb{C}^2 \rtimes_\alpha (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))$, where $\alpha_g(a_{13}, a_{23}) = e^{-z_3 t_3}(e^{z_1 t_1} a_{13} + a_{12} a_{23}, e^{z_2 t_2} a_{23})$ for $g \in \mathbb{R} \times T_2(\mathbb{R}, \mathbb{C})$. Then $C^*(T_3(\mathbb{R}, \mathbb{C})) \cong C^*(\mathbb{C}^2) \rtimes_\alpha (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C})) \cong C_0(\mathbb{C}^2) \rtimes_{\hat{\alpha}} (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))$, where

$$\hat{\alpha}_g(l_{13}, l_{23}) = e^{-\bar{z}_3 t_3} h^* l = e^{-\bar{z}_3 t_3}(e^{\bar{z}_1 t_1} l_{13}, \bar{a}_{12} l_{13} + e^{\bar{z}_2 t_2} l_{23}) \in \mathbb{C}^2$$

for $l = (l_{13}, l_{23}) \in \mathbb{C}^2$. Moreover, it follows that

$$\begin{aligned} (M) : 0 &\to C_0((\mathbb{C} \setminus \{0\}) \times \mathbb{C}) \rtimes (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C})) \\ &\to C_0(\mathbb{C}^2) \rtimes (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C})) \\ &\to C_0(\{0\} \times \mathbb{C}) \rtimes (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C})) \to 0. \end{aligned}$$

We first examine the structure of the quotient $C_0(\mathbb{C}) \rtimes (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))$ of (M) in the following. The following exact sequence is obtained:

$$\begin{aligned} 0 &\to C_0(\mathbb{C} \setminus \{0\}) \rtimes (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C})) \\ &\to C_0(\mathbb{C}) \rtimes (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C})) \to C^*(\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C})) \to 0. \end{aligned}$$

(Case 1_1): If z_2, z_3 are linearly dependent, then

$$C_0(\mathbb{C}\setminus\{0\}) \rtimes (\mathbb{R}\times T_2(\mathbb{R},\mathbb{C})) \cong \begin{cases} C_0(\mathbb{R}_+)\otimes (C(\mathbb{T}) \rtimes (\mathbb{R}\times T_2(\mathbb{R},\mathbb{C}))) \\ C(\mathbb{T})\otimes (C_0(\mathbb{R}_+) \rtimes (\mathbb{R}\times T_2(\mathbb{R},\mathbb{C}))) \end{cases}$$

where alternative cases correspond to whether or not z_2, z_3 are purely imaginary. Since the action $\hat{\alpha}$ of $\mathbb{R}\times T_2(\mathbb{R},\mathbb{C})$ is transitive on $\mathbb{T}$ and $\mathbb{R}_+$ respectively, it is obtained by [Gr2] that

$$\begin{cases} C(\mathbb{T}) \rtimes (\mathbb{R}\times T_2(\mathbb{R},\mathbb{C})) \cong C^*((\mathbb{R}\times T_2(\mathbb{R},\mathbb{C}))_{1\in\mathbb{T}})\otimes\mathbb{K} \\ C_0(\mathbb{R}_+) \rtimes (\mathbb{R}\times T_2(\mathbb{R},\mathbb{C})) \cong C^*((\mathbb{R}\times T_2(\mathbb{R},\mathbb{C}))_{1\in\mathbb{R}_+})\otimes\mathbb{K} \end{cases}$$

where the stabilizers are isomorphic to the following:

$$\begin{cases} (\mathbb{R}\times T_2(\mathbb{R},\mathbb{C}))_{1\in\mathbb{T}} \cong \mathbb{Z}\times T_2(\mathbb{R},\mathbb{C}) & \text{with } z_2, z_3 \in i\mathbb{R} \\ (\mathbb{R}\times T_2(\mathbb{R},\mathbb{C}))_{1\in\mathbb{R}_+} \cong T_2(\mathbb{R},\mathbb{C}) & \text{with } z_2, z_3 \notin i\mathbb{R}. \end{cases}$$

(Case 1_2): If z_2, z_3 are linearly independent, then the action on $\mathbb{C}\setminus\{0\}$ is transitive. Thus $C_0(\mathbb{C}\setminus\{0\}) \rtimes (\mathbb{R}\times T_2(\mathbb{R},\mathbb{C})) \cong C^*((\mathbb{R}\times T_2(\mathbb{R},\mathbb{C}))_{1\in\mathbb{C}\setminus\{0\}})\otimes\mathbb{K}$ by [Gr2], and

$$(\mathbb{R}\times T_2(\mathbb{R},\mathbb{C}))_{1\in\mathbb{C}\setminus\{0\}} \cong \begin{cases} \mathbb{Z}\times(\mathbb{C}\rtimes\mathbb{R}) & \text{if } z_3\in i\mathbb{R}, \\ \mathbb{C}\rtimes(\mathbb{R}\times\mathbb{Z}) & \text{otherwise} \end{cases}$$

where $\mathbb{C}\rtimes\mathbb{R}$ and $\mathbb{C}\rtimes(\mathbb{R}\times\mathbb{Z})$ are regarded as closed subgroups of $T_2(\mathbb{R},\mathbb{C})$.

Next examine the structure of the ideal $C_0((\mathbb{C}\setminus\{0\})\times\mathbb{C}) \rtimes (\mathbb{R}\times T_2(\mathbb{R},\mathbb{C}))$ of (M).

(Case 2_1): If z_1, z_3 are linearly dependent, then

$$\begin{aligned} &C_0((\mathbb{C}\setminus\{0\})\times\mathbb{C}) \rtimes (\mathbb{R}\times T_2(\mathbb{R},\mathbb{C})) \\ &\cong \begin{cases} C_0(\mathbb{R}_+)\otimes(C_0(\mathbb{T}\times\mathbb{C}) \rtimes (\mathbb{R}\times T_2(\mathbb{R},\mathbb{C}))) \\ C(\mathbb{T})\otimes(C_0(\mathbb{R}_+\times\mathbb{C}) \rtimes (\mathbb{R}\times T_2(\mathbb{R},\mathbb{C}))) \end{cases} \end{aligned}$$

where alternative cases correspond to whether or not z_1, z_3 are purely imaginary. Since the action $\hat{\alpha}$ of $\mathbb{R}\times T_2(\mathbb{R},\mathbb{C})$ is transitive on $\mathbb{T}\times\mathbb{C}$ and $\mathbb{R}_+\times\mathbb{C}$ respectively, it is obtained by [Gr2] that

$$\begin{cases} C_0(\mathbb{T}\times\mathbb{C}) \rtimes (\mathbb{R}\times T_2(\mathbb{R},\mathbb{C})) \cong C^*((\mathbb{R}\times T_2(\mathbb{R},\mathbb{C}))_{(1,1)\in\mathbb{T}\times\mathbb{C}})\otimes\mathbb{K} \\ C_0(\mathbb{R}_+\times\mathbb{C}) \rtimes (\mathbb{R}\times T_2(\mathbb{R},\mathbb{C})) \\ \cong C^*((\mathbb{R}\times T_2(\mathbb{R},\mathbb{C}))_{(1,1)\in\mathbb{R}_+\times\mathbb{C}})\otimes\mathbb{K} \end{cases}$$

where the stabilizers are isomorphic to the following:

$$\begin{cases} (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))_{(1,1) \in \mathbb{T} \times \mathbb{C}} \cong \mathbb{Z} \times \mathbb{R}^2 & \text{with } z_1, z_3 \in i\mathbb{R} \\ (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))_{(1,1) \in \mathbb{R}_+ \times \mathbb{C}} \cong \mathbb{R}^2 & \text{with } z_1, z_3 \notin i\mathbb{R}. \end{cases}$$

(Case 2_2): If z_1, z_3 are linearly independent, then $\hat{\alpha}$ is transitive on $(\mathbb{C} \setminus \{0\}) \times \mathbb{C}$. Thus by [Gr2]

$$\begin{aligned} &C_0(\mathbb{C} \setminus \{0\} \times \mathbb{C}) \rtimes (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C})) \\ &\cong C^*((\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))_{(1,1) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}}) \otimes \mathbb{K}, \\ &(\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))_{(1,1) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}} \cong \mathbb{Z} \times \mathbb{R}. \end{aligned}$$

Summing up the above argument, it is obtained that

Theorem 3.2. *Let $T_3(\mathbb{R}, \mathbb{C})$ be as above. Then $C^*(T_3(\mathbb{R}, \mathbb{C}))$ is decomposed as follows: $0 \to \mathfrak{K}_2 \to C^*(T_3(\mathbb{R}, \mathbb{C})) \to \mathfrak{D} \to 0$, and $0 \to \mathfrak{K}_1 \to \mathfrak{D} \to C^*(\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C})) \to 0$. Moreover,*

$$\mathfrak{K}_1 \cong \begin{cases} C_0(\mathbb{R}_+) \otimes C^*((\mathbb{R} \times T_2(\mathbb{C}))_{1 \in \mathbb{T}}) \otimes \mathbb{K}, & \textit{or} \\ C(\mathbb{T}) \otimes C^*((\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{R}_+}) \otimes \mathbb{K}, & (\textit{Case } 1_1) \\ C^*((\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{C} \setminus \{0\}}) \otimes \mathbb{K} & (\textit{Case } 1_2) \end{cases}$$

$$\textit{where} \begin{cases} (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{T}} \cong \mathbb{Z} \times T_2(\mathbb{R}, \mathbb{C}), \quad \textit{with } z_2, z_3 \in i\mathbb{R} \\ (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{R}_+} \cong T_2(\mathbb{R}, \mathbb{C}), \quad \textit{with } z_2, z_3 \notin i\mathbb{R} \\ (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{C} \setminus \{0\}} \cong \begin{cases} \mathbb{Z} \times (\mathbb{C} \rtimes \mathbb{R}) & \textit{if } z_3 \in i\mathbb{R}, \\ \mathbb{C} \rtimes (\mathbb{R} \times \mathbb{Z}) & \textit{otherwise}, \end{cases} \end{cases}$$

$$\mathfrak{K}_2 \cong \begin{cases} C_0(\mathbb{R}_+) \otimes C^*((\mathbb{R} \times T_2(\mathbb{C}))_{(1,1) \in \mathbb{T} \times \mathbb{C}}) \otimes \mathbb{K}, & \textit{or} \\ C(\mathbb{T}) \otimes C^*((\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))_{(1,1) \in \mathbb{R}_+ \times \mathbb{C}}) \otimes \mathbb{K}, & (\textit{Case } 2_1) \\ C^*((\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))_{(1,1) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}}) \otimes \mathbb{K} & (\textit{Case } 2_2) \end{cases}$$

$$\textit{where} \begin{cases} (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))_{(1,1) \in \mathbb{T} \times \mathbb{C}} \cong \mathbb{Z} \times \mathbb{R}^2, \quad \textit{with } z_1, z_3 \in i\mathbb{R} \\ (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))_{(1,1) \in \mathbb{R}_+ \times \mathbb{C}} \cong \mathbb{R}^2, \quad \textit{with } z_1, z_3 \notin i\mathbb{R} \\ (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))_{(1,1) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}} \cong \mathbb{Z} \times \mathbb{R}. \end{cases}$$

Remark. One can take a composition series $\{\mathfrak{J}_j\}_{j=1}^3$ of $C^*(T_3(\mathbb{R}, \mathbb{C}))$ such that $\mathfrak{J}_1 = \mathfrak{K}_2$ and $\mathfrak{J}_2/\mathfrak{J}_1 \cong \mathfrak{K}_1$ and $\mathfrak{J}_3/\mathfrak{J}_2 \cong C^*(\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C}))$ as constructed in the following sections. We skip the case of $T_3(\mathbb{T}, \mathbb{C})$ defined as $T_3(\mathbb{R}, \mathbb{C})$ or $T_2(\mathbb{T}, \mathbb{C})$, and consider the cases with the matrix size lager than $T_3(\mathbb{R}, \mathbb{C})$ in the following sections.

4. The four by four matrix case

Define $T_4(\mathbb{R}, \mathbb{C})$ to be the connected solvable Lie group of all the 4×4 triangular matrices:

$$\begin{pmatrix} e^{z_1 t_1} & & (a_{ij})_{1 \le i < j \le 4} \\ & \ddots & \\ 0 & & e^{z_4 t_4} \end{pmatrix}$$

for $t_i \in \mathbb{R}$, $a_{ij} \in \mathbb{C}$ and fixed nonzero, non purely imaginary $z_i \in \mathbb{C}$. When z_i for some i is purely imaginary, we let $T_4(\mathbb{R}, \mathbb{C})$ consist of all the tuples $((a_{ij})_{1 \le i < j \le 4}, t_1, \cdots, t_4)$, and it has a quotient map to the above matrices. Then $T_4(\mathbb{R}, \mathbb{C}) \cong \mathbb{C}^3 \rtimes_\alpha (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))$, where

$$\alpha_g(b_{14}, b_{24}, b_{34}) = g(b_{14}, b_{24}, b_{34})g^{-1} = gbg^{-1} = \begin{pmatrix} 1_3 & e^{-z_4 t_4} hb \\ 0 & 1 \end{pmatrix}$$

$$= e^{-z_4 t_4}((e^{z_1 t_1} b_{14} + \sum_{i=2}^{3} a_{1i} b_{i4}), (e^{z_2 t_2} b_{24} + a_{23} b_{34}), e^{z_3 t_3} b_{34}),$$

for $g = e^{z_4 t_4} \oplus h$ (the diagonal sum) of $\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C})$ and $h \in T_3(\mathbb{R}, \mathbb{C})$.

Theorem 4.1. *The group C^*-algebra $C^*(T_4(\mathbb{R}, \mathbb{C}))$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^4$ such that $\mathfrak{I}_4/\mathfrak{I}_3 \cong C^*(\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))$ and $\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong C_0(X_{4-j}) \rtimes_{\hat\alpha} (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))$ for $j = 1, 2, 3$, where $X_j = (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{j-1}$. Moreover,*

$$C_0(X_1) \rtimes_{\hat\alpha} (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C})) \cong \begin{cases} C_0(\mathbb{R}_+) \otimes C^*((\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{T}}) \otimes \mathbb{K} \\ C(\mathbb{T}) \otimes C^*((\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{R}_+}) \otimes \mathbb{K} \\ C^*((\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{C} \setminus \{0\}}) \otimes \mathbb{K}, \end{cases}$$

where the stabilizers are given by

$$\begin{cases} (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{T}} \cong \mathbb{Z} \times T_3(\mathbb{R}, \mathbb{C}) & \text{with } z_3, z_4 \in i\mathbb{R} \\ (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{R}_+} \cong T_3(\mathbb{R}, \mathbb{C}) & \text{with } z_3, z_4 \notin i\mathbb{R} \\ (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{C} \setminus \{0\}} \cong \begin{cases} \mathbb{Z} \times (\mathbb{C}^2 \rtimes T_2(\mathbb{R}, \mathbb{C})) & \text{if } z_4 \in i\mathbb{R}, \\ (\mathbb{C}^2 \rtimes (\mathbb{Z} \times T_2(\mathbb{R}, \mathbb{C})) & \text{otherwise.} \end{cases} \end{cases}$$

Furthermore, for $i = 2, 3$, $C_0(X_i) \rtimes_{\hat{\alpha}} (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))$ *is isomorphic to the following*:

$$\begin{cases} C_0(\mathbb{R}) \otimes C^*((\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{(1,1_{i-1})\in\mathbb{T}\times\mathbb{C}^{i-1}}) \otimes \mathbb{K} \\ C(\mathbb{T}) \otimes C^*((\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{(1,1_{i-1})\in\mathbb{R}_+\times\mathbb{C}^{i-1}}) \otimes \mathbb{K} \\ C^*((\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{(1,1_{i-1})\in\mathbb{C}\setminus\{0\}\times\mathbb{C}^{i-1}}) \otimes \mathbb{K} \end{cases}$$

$$\textit{where} \quad \begin{cases} (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{(1,1)\in\mathbb{T}\times\mathbb{C}} \cong \mathbb{C} \rtimes (\mathbb{R} \times \mathbb{Z} \times T_2(\mathbb{R}, \mathbb{C})), \\ (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{(1,1)\in\mathbb{R}_+\times\mathbb{C}} \cong \mathbb{C} \rtimes (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C})), \\ (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{(1,1)\in(\mathbb{C}\setminus\{0\})\times\mathbb{C}} \\ \cong \begin{cases} \mathbb{Z} \times (\mathbb{C} \rtimes (\mathbb{R} \times (\mathbb{C} \rtimes \mathbb{R}))), \\ \mathbb{C} \rtimes (\mathbb{R} \times (\mathbb{C} \rtimes (\mathbb{Z} \times \mathbb{R}))), \end{cases} \end{cases}$$

$$\textit{and} \quad \begin{cases} (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{(1,1_2)\in\mathbb{T}\times\mathbb{C}^2} \cong \mathbb{C} \rtimes (\mathbb{R}^3 \times \mathbb{Z}), \\ (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{(1,1_2)\in\mathbb{R}_+\times\mathbb{C}^2} \cong \mathbb{C} \rtimes \mathbb{R}^3, \\ (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{(1,1_2)\in(\mathbb{C}\setminus\{0\})\times\mathbb{C}^2} \cong \mathbb{C} \rtimes (\mathbb{R}^2 \times \mathbb{Z}), \end{cases}$$

where the following decompositions of $T_3(\mathbb{R}, \mathbb{C})$ *are used respectively*:

$$T_3(\mathbb{R}, \mathbb{C}) \cong \begin{cases} \mathbb{C}^2 \rtimes (\mathbb{R} \times T_2(\mathbb{R}, \mathbb{C})), \\ (\mathbb{C}^2 \rtimes \mathbb{C}) \rtimes \mathbb{R}^3. \end{cases}$$

Proof. Note that $C^*(T_4(\mathbb{R}, \mathbb{C})) \cong C^*(\mathbb{C}^3) \rtimes_\alpha (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C})) \cong C_0(\mathbb{C}^3) \rtimes_{\hat{\alpha}} (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))$, where

$$\begin{aligned} \hat{\alpha}_g(l) &= e^{-\bar{z}_n t_n} h^* l \\ &= e^{-\bar{z}_4 t_4} (e^{\bar{z}_1 t_1} l_{14}, \bar{a}_{12} l_{14} + e^{\bar{z}_2 t_2} l_{24}, \sum_{j=1}^{2} \bar{a}_{j3} l_{j4} + e^{\bar{z}_3 t_3} l_{34}). \end{aligned}$$

Consider the following decomposition of $\mathbb{C}^3$ into $\hat{\alpha}$-invariant subsets:

$$\mathbb{C}^3 = \{0_3\} \sqcup (\sqcup_{i=1}^3 \{0_{3-i}\} \times (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{i-1}) \equiv \sqcup_{i=0}^3 X_i.$$

Then $C_0(\mathbb{C}^3) \rtimes_{\hat{\alpha}} (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^4$ such that $\mathfrak{I}_j / \mathfrak{I}_{j-1} \cong C_0(X_{4-j}) \rtimes_{\hat{\alpha}} (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))$.

By the same way as Cases 1_1 and 1_2 in Section 3, the structure of $C_0(X_1) \rtimes_{\hat{\alpha}} (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))$ is obtained as in the statement.

Since the restrictions of $\hat{\alpha}$ to X_2 and X_3 are given by

$$\begin{cases} \hat{\alpha}_g(l_{24}, l_{34}) = e^{-\bar{z}_4 t_4}(e^{\bar{z}_2 t_2} l_{24}, \bar{a}_{23} l_{24} + e^{\bar{z}_3 t_3} l_{34}), \\ \hat{\alpha}_g(l_{14}, l_{24}, l_{34}) = \\ e^{-\bar{z}_4 t_4}(e^{\bar{z}_1 t_1} l_{14}, \bar{a}_{12} l_{14} + e^{\bar{z}_2 t_2} l_{24}, \sum_{j=1}^{2} \bar{a}_{j3} l_{j4} + e^{\bar{z}_3 t_3} l_{34}), \end{cases}$$

it follows that the quotient space $(\mathbb{C} \setminus \{0\} \times \mathbb{C}^{i-1})/(\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))$ is homeomorphic to either $\mathbb{R}$, $\mathbb{T}$ or $\{\text{point}\}$. Thus, for $i = 2, 3$, $C_0(X_i) \rtimes_{\hat{\alpha}} (\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))$ is isomorphic to the following:

$$\begin{cases} C_0(\mathbb{R}) \otimes C^*((\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{(1,1_{i-1}) \in \mathbb{T} \times \mathbb{C}^{i-1}}) \otimes \mathbb{K} \\ C(\mathbb{T}) \otimes C^*((\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{(1,1_{i-1}) \in \mathbb{R}_+ \times \mathbb{C}^{i-1}}) \otimes \mathbb{K} \\ C^*((\mathbb{R} \times T_3(\mathbb{R}, \mathbb{C}))_{(1,1_{i-1}) \in \mathbb{C} \setminus \{0\} \times \mathbb{C}^{i-1}}) \otimes \mathbb{K}. \end{cases} \qquad \square$$

Remark. We may consider the orbits of $(1, 0_{i-1})$ of either $\mathbb{T} \times \mathbb{C}^{i-1}$, $\mathbb{R}_+ \times \mathbb{C}^{i-1}$ or $\mathbb{C} \setminus \{0\} \times \mathbb{C}^{i-1}$ instead of those of generic points $(1, 1_{i-1})$ since $\hat{\alpha}$ is transitive on these invariant subspaces. Then it is quite easy to check the isomorphisms of the stabilizers as given in the above statement, and this situation is the same in the following sections.

5. The five by five matrix case

Denote by $T_5(\mathbb{R}, \mathbb{C})$ the connected solvable Lie group of all the 5×5 triangular matrices:

$$\begin{pmatrix} e^{z_1 t_1} & & (a_{ij})_{1 \le i < j \le 5} \\ & \ddots & \\ 0 & & e^{z_5 t_5} \end{pmatrix}$$

for $t_i \in \mathbb{T}$, $a_{ij} \in \mathbb{C}$ and fixed nonzero, non purely imaginary $z_i \in \mathbb{C}$. When z_i for some i is purely imaginary, we let $T_5(\mathbb{R}, \mathbb{C})$ consist of all the tuples $((a_{ij})_{1 \le i < j \le 5}, t_1, \cdots, t_5)$, and it has a quotient map to the above matrices. Then it is obtained that

Theorem 5.1. *The group C^*-algebra $C^*(T_5(\mathbb{R},\mathbb{C}))$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^5$ such that $\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong C_0(X_{5-j}) \rtimes_{\hat{\alpha}} (\mathbb{R} \times T_4(\mathbb{R},\mathbb{C}))$ for $1 \le j \le 5$, where $\mathbb{C}^4 = \{0_4\} \sqcup (\sqcup_{i=1}^4 (\mathbb{C}\setminus\{0\}) \times \mathbb{C}^{i-1}) \equiv X_0 \sqcup (\sqcup_{i=1}^4 X_i)$. Moreover,*

$$C_0(X_1) \rtimes_{\hat{\alpha}} (\mathbb{R}\times T_4(\mathbb{R},\mathbb{C})) \cong \begin{cases} C_0(\mathbb{R}) \otimes C^*((\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{1\in\mathbb{T}}) \otimes \mathbb{K} \\ C(\mathbb{T}) \otimes C^*((\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{1\in\mathbb{R}_+}) \otimes \mathbb{K} \\ C^*((\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{1\in\mathbb{C}\setminus\{0\}}) \otimes \mathbb{K}, \end{cases}$$

where the stabilizers are given by

$$\begin{cases} (\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{1\in\mathbb{T}} \cong \mathbb{Z}\times T_4(\mathbb{R},\mathbb{C}) \quad \text{with } z_4, z_5 \in i\mathbb{R} \\ (\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{1\in\mathbb{R}_+} \cong T_4(\mathbb{R},\mathbb{C}) \quad \text{with } z_4, z_5 \notin i\mathbb{R} \\ (\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{1\in\mathbb{C}\setminus\{0\}} \cong \begin{cases} \mathbb{Z}\times(\mathbb{C}^3 \rtimes T_3(\mathbb{R},\mathbb{C})) & \text{if } z_5 \in i\mathbb{R}, \\ \mathbb{C}^3 \rtimes (\mathbb{Z}\times T_3(\mathbb{R},\mathbb{C})) & \text{otherwise.} \end{cases} \end{cases}$$

For $i = 2,3,4$, $C_0(X_i)\rtimes_{\hat{\alpha}}(\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))$ is isomorphic to the following:

$$\begin{cases} C_0(\mathbb{R}) \otimes C^*((\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{(1,1_{i-1})\in\mathbb{T}\times\mathbb{C}^{i-1}}) \otimes \mathbb{K} \\ C(\mathbb{T}) \otimes C^*((\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{(1,1_{i-1})\in\mathbb{R}_+\times\mathbb{C}^{i-1}}) \otimes \mathbb{K} \\ C^*((\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{(1,1_{i-1})\in\mathbb{C}\setminus\{0\}\times\mathbb{C}^{i-1}}) \otimes \mathbb{K}, \end{cases}$$

where the stabilizers for $i = 2,3,4$ are given respectively as follows:

$$\begin{cases} (\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{(1,1)\in\mathbb{T}\times\mathbb{C}} \cong \mathbb{C}^2 \rtimes (\mathbb{R}\times\mathbb{Z}\times T_3(\mathbb{R},\mathbb{C})), \quad z_3 \in i\mathbb{R}, \\ (\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{(1,1)\in\mathbb{R}_+\times\mathbb{C}} \cong \mathbb{C}^2 \rtimes (\mathbb{R}\times T_3(\mathbb{R},\mathbb{C})), \quad z_3 \notin i\mathbb{R}, \\ (\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{(1,1)\in(\mathbb{C}\setminus\{0\})\times\mathbb{C}} \cong \\ \begin{cases} \mathbb{Z}\times(\mathbb{C}^2 \rtimes (\mathbb{R}\times(\mathbb{C}^2 \rtimes T_2(\mathbb{R},\mathbb{C})))), \\ \mathbb{C}^2 \rtimes (\mathbb{R}\times(\mathbb{C}^2 \rtimes (\mathbb{Z}\times T_2(\mathbb{R},\mathbb{C})))), \end{cases} \end{cases}$$

$$\begin{cases} (\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{(1,1_2)\in\mathbb{T}\times\mathbb{C}^2} \cong (\mathbb{C}^2 \rtimes \mathbb{C}) \rtimes (\mathbb{R}^2\times\mathbb{Z}\times T_2(\mathbb{R},\mathbb{C})), \\ (\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{(1,1_2)\in\mathbb{R}_+\times\mathbb{C}^2} \cong (\mathbb{C}^2 \rtimes \mathbb{C}) \rtimes (\mathbb{R}^2\times T_2(\mathbb{R},\mathbb{C})), \\ (\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{(1,1_2)\in(\mathbb{C}\setminus\{0\})\times\mathbb{C}^2} \cong \\ \begin{cases} \mathbb{Z}\times((\mathbb{C}^2 \rtimes \mathbb{C}) \rtimes (\mathbb{R}^2\times(\mathbb{C}\rtimes\mathbb{R}))), \\ (\mathbb{C}^2 \rtimes \mathbb{C}) \rtimes (\mathbb{R}^2\times(\mathbb{C}\rtimes(\mathbb{Z}\times\mathbb{R}))), \end{cases} \end{cases}$$

$$\text{and}\quad \begin{cases} (\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{(1,1_3)\in\mathbb{T}\times\mathbb{C}^3} \cong (\mathbb{C}^2 \rtimes \mathbb{C}) \rtimes (\mathbb{R}^4\times\mathbb{Z}), \\ (\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{(1,1_3)\in\mathbb{R}_+\times\mathbb{C}^3} \cong (\mathbb{C}^2 \rtimes \mathbb{C}) \rtimes \mathbb{R}^4, \\ (\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{(1,1_3)\in(\mathbb{C}\setminus\{0\})\times\mathbb{C}^3} \cong (\mathbb{C}^2 \rtimes \mathbb{C}) \rtimes (\mathbb{R}^3\times\mathbb{Z}), \end{cases}$$

where the following decompositions of $T_4(\mathbb{R},\mathbb{C})$ are used respectively:

$$T_4(\mathbb{R},\mathbb{C}) \cong \begin{cases} \mathbb{C}^3 \rtimes (\mathbb{R}\times T_3(\mathbb{R},\mathbb{C})), \\ (\mathbb{C}^3 \rtimes \mathbb{C}^2) \rtimes (\mathbb{R}^2\times T_2(\mathbb{R},\mathbb{C})), \\ (\mathbb{C}^3 \rtimes \mathbb{C}^2 \rtimes \mathbb{C}) \rtimes \mathbb{R}^4. \end{cases}$$

Remark. Each product group of the form: $\mathbb{Z}\times$ (Semi-direct product) in the upper line of the two alternatives of the third cases of the stabilizers $(\mathbb{R}\times T_4(\mathbb{R},\mathbb{C}))_{(1,1_{i-1})}$ for $i=2,3$ as above can be written as a semi-direct product without $\mathbb{Z}$ as a direct factor as in its lower line. This convention is also used in some other cases and in other theorems.

6. The general n by n matrix case

Define $T_n(\mathbb{R},\mathbb{C})$ to be the connected solvable Lie group of all the n by n triangular matrices:

$$\begin{pmatrix} e^{z_1t_1} & & (a_{ij})_{1\le i<j\le n} \\ & \ddots & \\ 0 & & e^{z_nt_n} \end{pmatrix}$$

with $t_i\in\mathbb{R}$, $a_{ij}\in\mathbb{C}$ and fixed nonzero, non purely imaginary $z_i\in\mathbb{C}$. When z_i for some i is purely imaginary, we let $T_n(\mathbb{R},\mathbb{C})$ consist of all the tuples $((a_{ij})_{1\le i<j\le n}, t_1,\cdots,t_n)$, and it has a quotient map to the above matrices. Then $T_n(\mathbb{R},\mathbb{C}) \cong \mathbb{C}^{n-1}\rtimes_\alpha(\mathbb{R}\times T_{n-1}(\mathbb{R},\mathbb{C}))$, where the action α is defined by, for $b=(b_{1n},\cdots,b_{n-1n})\in\mathbb{C}^{n-1}$

$$\begin{aligned} \alpha_g(b) &= gbg^{-1} \\ &= \begin{pmatrix} h & 0 \\ 0 & e^{z_nt_n}\end{pmatrix}\begin{pmatrix} 1_{n-1} & b \\ 0 & 1\end{pmatrix}\begin{pmatrix} h^{-1} & 0 \\ 0 & e^{-z_nt_n}\end{pmatrix} = e^{-z_nt_n}hb \\ &= e^{-z_nt_n}((e^{z_1t_1}b_{1n}+\sum_{i=2}^{n-1}a_{1i}b_{in}), (e^{z_2t_2}b_{2n}+\sum_{i=3}^{n-1}a_{2i}b_{in}), \\ &\quad \cdots, e^{z_{n-1}t_{n-1}}b_{n-1\,n}) \end{aligned}$$

with $h \in T_{n-1}(\mathbb{R}, \mathbb{C})$. Then $C^*(T_n(\mathbb{R}, \mathbb{C})) \cong C_0(\mathbb{C}^{n-1}) \rtimes_{\hat{\alpha}} (\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))$, where

$$\hat{\alpha}_g(l) = e^{-\bar{z}_n t_n} h^* l = e^{-\bar{z}_n t_n}(e^{\bar{z}_1 t_1} l_{1n}, \bar{a}_{12} l_{1n} + e^{\bar{z}_2 t_2} l_{2n},$$
$$\cdots, \sum_{j=1}^{n-2} \bar{a}_{j\,n-1} l_{jn} + e^{\bar{z}_{n-1} t_{n-1}} l_{n-1\,n})$$

for $l = (l_{1n}, \cdots, l_{n-1\,n}) \in \mathbb{C}^{n-1}$. Then it is obtained that

Theorem 6.1. *The group C^*-algebra $C^*(T_n(\mathbb{R}, \mathbb{C}))$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^n$ such that $\mathfrak{I}_n/\mathfrak{I}_{n-1} \cong C^*(\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))$ and*

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong C_0(X_{n-j}) \rtimes_{\hat{\alpha}} (\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))$$

for $1 \leq j \leq n-1$ and the decomposition into $\hat{\alpha}$-invariant subsets:

$$\mathbb{C}^{n-1} \setminus \{0_{n-1}\} = \sqcup_{i=1}^{n-1} \{0_{n-i}\} \times (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{i-1} \equiv \sqcup_{i=1}^{n-1} X_i.$$

Moreover, $C_0(X_1) \rtimes_{\hat{\alpha}} (\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))$ is isomorphic to the following:

$$\begin{cases} C_0(\mathbb{R}) \otimes C^*((\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{T}}) \otimes \mathbb{K} \\ C(\mathbb{T}) \otimes C^*((\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{R}_+}) \otimes \mathbb{K} \\ C^*((\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{C} \setminus \{0\}}) \otimes \mathbb{K}, \end{cases}$$

where the stabilizers are given by

$$\begin{cases} (\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{T}} \cong \mathbb{Z} \times T_{n-1}(\mathbb{R}, \mathbb{C}) & \text{with } z_{n-1}, z_n \in i\mathbb{R} \\ (\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{R}_+} \cong T_{n-1}(\mathbb{R}, \mathbb{C}) & \text{with } z_{n-1}, z_n \notin i\mathbb{R} \\ (\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))_{1 \in \mathbb{C} \setminus \{0\}} \cong & \\ \begin{cases} \mathbb{Z} \times (\mathbb{C}^{n-2} \rtimes T_{n-2}(\mathbb{R}, \mathbb{C})) & \text{if } z_n \in i\mathbb{R}, \\ \mathbb{C}^{n-2} \rtimes (\mathbb{Z} \times T_{n-2}(\mathbb{R}, \mathbb{C})) & \text{otherwise.} \end{cases} & \end{cases}$$

For $2 \leq i \leq n$, $C_0(\mathbb{C} \setminus \{0\} \times \mathbb{C}^{i-1}) \rtimes_{\hat{\alpha}} (\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))$ is isomorphic to the following:

$$\begin{cases} C_0(\mathbb{R}) \otimes C^*((\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))_{(1,1_{i-1}) \in \mathbb{T} \times \mathbb{C}^{i-1}}) \otimes \mathbb{K} \\ C(\mathbb{T}) \otimes C^*((\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))_{(1,1_{i-1}) \in \mathbb{R}_+ \times \mathbb{C}^{i-1}}) \otimes \mathbb{K} \\ C^*((\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))_{(1,1_{i-1}) \in \mathbb{C} \setminus \{0\} \times \mathbb{C}^{i-1}}) \otimes \mathbb{K}, \end{cases}$$

where the stabilizers are given by

$$\begin{cases} (\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))_{(1,1_{j-1}) \in \mathbb{T} \times \mathbb{C}^{j-1}} \\ \cong (\mathbb{C}^{n-3} \rtimes \mathbb{C}^{n-4} \rtimes \cdots \rtimes \mathbb{C}^{n-j-1}) \rtimes (\mathbb{R}^{j-1} \times \mathbb{Z} \times T_{n-j}(\mathbb{R}, \mathbb{C})), \\ (\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))_{(1,1_{j-1}) \in \mathbb{R}_+ \times \mathbb{C}^{j-1}} \\ \cong (\mathbb{C}^{n-3} \rtimes \mathbb{C}^{n-4} \rtimes \cdots \rtimes \mathbb{C}^{n-j-1}) \rtimes (\mathbb{R}^{j-1} \times T_{n-j}(\mathbb{R}, \mathbb{C})), \\ (\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))_{(1,1_{j-1}) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{j-1}} \\ \cong \begin{cases} \mathbb{Z} \times ((\mathbb{C}^{n-3} \rtimes \mathbb{C}^{n-4} \rtimes \cdots \rtimes \mathbb{C}^{n-j-1}) \\ \quad \rtimes (\mathbb{R}^{j-1} \times (\mathbb{C}^{n-j-1} \rtimes T_{n-j-1}(\mathbb{R}, \mathbb{C})))), \\ (\mathbb{C}^{n-3} \rtimes \mathbb{C}^{n-4} \rtimes \cdots \rtimes \mathbb{C}^{n-j-1}) \\ \quad \rtimes (\mathbb{R}^{j-1} \times (\mathbb{C}^{n-j-1} \rtimes (\mathbb{Z} \times T_{n-j-1}(\mathbb{R}, \mathbb{C})))), \end{cases} \end{cases}$$

for $2 \leq j \leq n-2$, *and*

$$\begin{cases} (\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))_{(1,1_{n-1}) \in \mathbb{T} \times \mathbb{C}^{n-1}} \\ \cong (\mathbb{C}^{n-3} \rtimes \mathbb{C}^{n-4} \rtimes \cdots \rtimes \mathbb{C}) \rtimes (\mathbb{R}^{n-1} \times \mathbb{Z}), \\ (\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))_{(1,1_{n-1}) \in \mathbb{R}_+ \times \mathbb{C}^{n-1}} \\ \cong (\mathbb{C}^{n-3} \rtimes \mathbb{C}^{n-4} \rtimes \cdots \rtimes \mathbb{C}) \rtimes \mathbb{R}^{n-1}, \\ (\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))_{(1,1_{n-1}) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{n-1}} \\ \cong (\mathbb{C}^{n-3} \rtimes \mathbb{C}^{n-4} \rtimes \cdots \rtimes \mathbb{C}) \rtimes (\mathbb{R}^{n-2} \times \mathbb{Z}), \end{cases}$$

where the following decompositions of $T_{n-1}(\mathbb{R}, \mathbb{C})$ *are used respectively:*

$$T_{n-1}(\mathbb{R}, \mathbb{C}) \cong \begin{cases} (\mathbb{C}^{n-2} \rtimes \mathbb{C}^{n-3} \rtimes \cdots \rtimes \mathbb{C}^{n-j}) \rtimes (\mathbb{R}^{j-1} \times T_{n-j}(\mathbb{R}, \mathbb{C})), \\ (\mathbb{C}^{n-2} \rtimes \mathbb{C}^{n-3} \rtimes \cdots \rtimes \mathbb{C}) \rtimes \mathbb{R}^{n-1}. \end{cases}$$

Proof. Note that the action $\hat{\alpha}$ of $C_0(X_i) \rtimes_{\hat{\alpha}} (\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))$ with $X_i = \mathbb{C} \setminus \{0\} \times \mathbb{C}^{i-1}$ $(2 \leq i \leq n)$ is given by

$$\hat{\alpha}_g(l_{n-i,n}, \cdots, l_{n-1,n}) = e^{-\bar{z}_n t_n}(e^{\bar{z}_{n-i} t_{n-i}} l_{n-i,n}, \bar{a}_{n-i,n-i+1} l_{n-i,n} + e^{\bar{z}_{n-i+1} t_{n-i+1}} l_{n-i+1,n}, \cdots, \sum_{j=n-i}^{n-2} \bar{a}_{j,n-1} l_{jn} + e^{\bar{z}_{n-1} t_{n-1}} l_{n-1,n}).$$

Since $l_{n-i,n} \neq 0$, then the quotient space $((\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{i-1})/(\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C}))$ is homeomorphic to either $\mathbb{R}$, $\mathbb{T}$ or $\{\text{point}\}$. $\square$

Remark. As an important note, the stabilizers of $\mathbb{R} \times T_{n-1}(\mathbb{R}, \mathbb{C})$ as above are regarded as either certain closed subgroups of $T_{n-1}(\mathbb{R}, \mathbb{C})$ which are regarded as semi-direct products similar with $T_j(\mathbb{R}, \mathbb{C})$ for some $j \leq n-1$ or their direct products with $\mathbb{Z}$, whose off-diagonal parts have dimension reduced than that of $T_{n-1}(\mathbb{R}, \mathbb{C})$. Therefore, our methods work for group C^*-algebras of these stabilizers similarly. Thus induction on n about the structure of $C^*(T_n(\mathbb{R}, \mathbb{C}))$ also works.

7. Applications

The following theorem is deduced from Theorem 6.1:

Theorem 7.1. *The group C^*-algebra $C^*(T_n(\mathbb{R}, \mathbb{C}))$ has a finite composition series $\{\mathfrak{D}_j\}_{j=1}^{l}$ with subquotients given by $\mathfrak{D}_l/\mathfrak{D}_{l-1} \cong C_0(\mathbb{R}^n)$ and $\mathfrak{D}_j/\mathfrak{D}_{j-1} \cong \mathfrak{X}_j \otimes \mathbb{K}$ for $1 \leq j \leq l-1$, where $\mathfrak{X}_j$ are group C^*-algebras of stabilizers tensored with either $C_0(\mathbb{R})$, $C(\mathbb{T})$ or $\mathbb{C}$ (non-tensored), which are obtained inductively from the structure of $C^*(T_n(\mathbb{R}, \mathbb{C}))$ as given in Theorem* 6.1.

Similarly, it follows that

Theorem 7.2. *The group C^*-algebra $C^*(T_n(\mathbb{C}))$ has a finite composition series $\{\mathfrak{L}_j\}_{j=1}^{s}$ with subquotients $\mathfrak{L}_s/\mathfrak{L}_{s-1} = C^*(\mathbb{C}_\times^n) \cong C_0(\mathbb{R}^n \times \mathbb{Z}^n)$ and $\mathfrak{L}_j/\mathfrak{L}_{j-1} \cong \mathfrak{Y}_j \otimes \mathbb{K}$ for $1 \leq j \leq s-1$, where $T_n(\mathbb{C})$ is defined similarly as $T_n(\mathbb{R}, \mathbb{C})$, and $\mathfrak{Y}_j$ are group C^*-algebras of stabilizers tensored with either $C_0(\mathbb{R})$, $C_0(\mathbb{Z})$ or $\mathbb{C}$, which are obtained inductively as $\mathfrak{X}_j$ in Theorem* 7.1.

Moreover, it is obtained that

Theorem 7.3. *The group C^*-algebra $C^*(T_n(\mathbb{T}, \mathbb{C}))$ has a finite composition series $\{\mathfrak{M}_j\}_{j=1}^{t}$ with subquotients $\mathfrak{M}_t/\mathfrak{M}_{t-1} \cong C_0(\mathbb{Z}^n)$ and $\mathfrak{M}_j/\mathfrak{M}_{j-1} \cong \mathfrak{Z}_j \otimes \mathbb{K}$ for $1 \leq j \leq t-1$, where $T_n(\mathbb{T}, \mathbb{C})$ is defined similarly as $T_n(\mathbb{R}, \mathbb{C})$, and $\mathfrak{Z}_j$ are group C^*-algebras of stabilizers tensored with either $C_0(\mathbb{R})$, $C_0(\mathbb{Z})$ or $\mathbb{C}$, which are obtained inductively as $\mathfrak{X}_j$ in Theorem* 7.1.

As a corollary,

Corollary 7.4. *The group C^*-algebras $C^*(T_n(\mathbb{R},\mathbb{C}))$, $C^*(T_n(\mathbb{C}))$ and $C^*(T_n(\mathbb{T},\mathbb{C}))$ are of type I (or GCR).*

Proof. The subquotients of finite composition series of $C^*(T_n(\mathbb{R},\mathbb{C}))$, $C^*(T_n(\mathbb{C}))$ and $C^*(T_n(\mathbb{T},\mathbb{C}))$ in Theorems 7.1, 7.2 and 7.3 can be decomposed into finite composition series with subquotients liminary (or CCR) by induction as Remark of Theorem 6.1. Since C^*-algebras are of type I if and only if they are composed finitely or infinitely by liminary subquotients (cf. [Dx]), the conclusion is obtained. □

Remark. This result could be deduced from a point of view of the unitary representation theory of those Lie groups. However, it could not be found in the literature explicitly.

Now denote by $\mathrm{sr}(\mathfrak{A})$ and $\mathrm{csr}(\mathfrak{A})$ the (topological) stable rank and connected stable rank of a C^*-algebra $\mathfrak{A}$ respectively (cf. [Rf]). Let $\dim X$ be the covering dimension of a topological space X. Set $\dim_{\mathbb{C}} X = [\dim X/2]+1$, where $[x]$ means the maximum integer $\leq x$. Denote by $G_1^\wedge$ the space of all 1-dimensional representations of a Lie group G. Note that $G_1^\wedge$ is homeomorphic to the dual of $G/[G,G]$, where $[G,G]$ is the commutator of G (cf. [ST2]). Under this setting, it is obtained that

Theorem 7.5. *The topological stable rank of the group C^*-algebras $C^*(T_n(\mathbb{R},\mathbb{C}))$, $C^*(T_n(\mathbb{C}))$ and $C^*(T_n(\mathbb{T},\mathbb{C}))$ is estimated as follows:*

$$\dim_{\mathbb{C}} T_n(\mathbb{R},\mathbb{C})_1^\wedge = [n/2]+1 \leq \mathrm{sr}(C^*(T_n(\mathbb{R},\mathbb{C}))) \leq [(n+1)/2]+1,$$
$$\dim_{\mathbb{C}} T_n(\mathbb{C})_1^\wedge = [n/2]+1 \leq \mathrm{sr}(C^*(T_n(\mathbb{C}))) \leq [(n+1)/2]+1,$$
$$\mathrm{sr}(C^*(T_n(\mathbb{T},\mathbb{C}))) = \begin{cases} 1 = \dim_{\mathbb{C}} T_2(\mathbb{T},\mathbb{C})_1^\wedge, & n=2, \\ 2 = 1+\dim_{\mathbb{C}} T_n(\mathbb{T},\mathbb{C})_1^\wedge, & n \geq 3 \end{cases}$$

where $T_n(\mathbb{R},\mathbb{C})_1^\wedge = \mathbb{R}^n$, $T_n(\mathbb{C})_1^\wedge = \mathbb{R}^n \times \mathbb{Z}^n$ and $T_n(\mathbb{T},\mathbb{C})_1^\wedge = \mathbb{Z}^n$. Also, the connected stable rank of those group C^*-algebras is estimated as

$$\mathrm{csr}(C^*(T_n(\mathbb{R},\mathbb{C}))) \leq [(n+1)/2]+1,$$
$$\mathrm{csr}(C^*(T_n(\mathbb{C}))) \leq [(n+1)/2]+1,$$
$$\mathrm{csr}(C^*(T_n(\mathbb{T},\mathbb{C}))) \leq 2.$$

Proof. Apply the following basic formulas of stable and connected stable ranks to the finite composition series in Theorems 7.1, 7.2 and 7.3 inductively:

$$\begin{aligned}
&\mathrm{sr}(\mathfrak{J}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{J}) \leq \mathrm{sr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{J}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{J}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{J}),\\
&\mathrm{csr}(\mathfrak{A}) \leq \mathrm{csr}(\mathfrak{J}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{J}),\\
&\mathrm{sr}(\mathfrak{A} \otimes \mathbb{K}) = 2 \wedge \mathrm{sr}(\mathfrak{A}), \quad \mathrm{csr}(\mathfrak{A} \otimes \mathbb{K}) \leq 2 \wedge \mathrm{csr}(\mathfrak{A}),
\end{aligned}$$

for a C^*-algebra $\mathfrak{A}$, and for an exact sequence $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{J} \to 1$ of C^*-algebras, where $\vee$ is the maximum and $\wedge$ is the minimum, and

$$\begin{aligned}
&\mathrm{sr}(C_0(\mathbb{R}^n)) = [n/2] + 1, \quad \mathrm{csr}(C_0(\mathbb{R})) = 2, \quad \mathrm{csr}(C_0(\mathbb{R}^2)) = 1,\\
&\mathrm{csr}(C_0(\mathbb{R}^m)) = [(m+1)/2] + 1 \, (m \geq 3),\\
&\mathrm{sr}(C_0(\mathbb{Z}^n)) = 1, \quad \mathrm{csr}(C_0(\mathbb{Z}^n)) = 1
\end{aligned}$$

([Rf, Proposition 1.7, Theorems 3.6, 4.3, 4.4, 4.11 and 6.4], [Sh, Theorem 3.9 and p. 381], [Ns1]). As for the exact sequence of $C^*(T_2(\mathbb{T}, \mathbb{C}))$ in Theorem 2.3, note that the index map from K_1-group $K_1(C_0(\mathbb{Z}^2)) \cong \{0\}$ to K_0-group $K_0(C_0(\mathbb{R} \times \mathbb{Z}) \otimes \mathbb{K})$ is trivial. By [Ng] or [Ns2], $\mathrm{sr}(C^*(T_2(\mathbb{T}, \mathbb{C}))) = 1$. □

Remark. Note that $G/[G, G]$ is isomorphic to $\mathbb{R}^n$, $\mathbb{C}_\times^n \cong \mathbb{R}^n \times \mathbb{T}^n$ and $\mathbb{T}^n$ (each is the diagonal part) when $G = T_n(\mathbb{R}, \mathbb{C})$, $T_n(\mathbb{C})$ and $T_n(\mathbb{T}, \mathbb{C})$ respectively. Thus, Theorem 7.5 suggests that the action by the compact part of the diagonals of $T_n(\mathbb{C})$ and $T_n(\mathbb{T}, \mathbb{C})$ will not affect on the stable rank and connected stable rank of those group C^*-algebras. Also, the theorem suggests that the stable ranks of those group C^*-algebras can be estimated by the dimension of the spaces of their all 1-dimensional representations (cf. [Sh], [ST1-2] and [Sd2-6, 8]). Compare Theorem 7.5 with the similar result of [ST2] by using another method.

Remark. We can show that $\mathrm{csr}(C^*(T_2(\mathbb{T}, \mathbb{C}))) = 2$. In fact,

$$\begin{aligned}
K_1(C^*(T_2(\mathbb{T}, \mathbb{C}))) &= K_1(C_0(\mathbb{C} \rtimes \mathbb{T}^2))\\
&\cong K_1^{\mathbb{T}^2}(C_0(\mathbb{C})) \cong K_1^{\mathbb{T}^2}(\mathbb{C}) \cong R(\mathbb{T}^2)
\end{aligned}$$

(the representation ring) (cf. [Bl]), which is non trivial. Thus, $\mathrm{csr}(C^*(T_2(\mathbb{T}, \mathbb{C}))) \geq 2$ by [Eh]. If one can show that K_1-group of

$C^*(T_n(\mathbb{T}, \mathbb{C}))$ for $n \geq 3$ is non trivial, then the same conclusion is obtained.

Remark. Now define $T_n(\mathbb{R})$ to be the connected solvable Lie group of all the $n \times n$ triangular matrices:

$$\begin{pmatrix} e^{s_1 t_1} & & (x_{ij})_{1 \leq i < j \leq n} \\ & \ddots & \\ 0 & & e^{s_n t_n} \end{pmatrix}$$

for $t_i, x_{ij} \in \mathbb{R}$ and fixed nonzero $s_i \in \mathbb{R}$. Then we can obtain the similar theorems of $C^*(T_n(\mathbb{R}))$ as Theorems 6.1, 7.1, 7.4 and 7.5 since $C^*(T_n(\mathbb{R}))$ is regarded as a quotient C^*-algebra of a certain $C^*(T_n(\mathbb{R}, \mathbb{C}))$ when $T_n(\mathbb{R})$ is regarded as a closed subgroup of $T_n(\mathbb{R}, \mathbb{C})$.

APPENDIX

In this section we briefly review Green's imprimitivity theorem and its consequences ([Gr2], cf. [Gr1], [RW]).

G1. The Green imprimitivity theorem (for crossed products). *Let G be a locally compact group, H a closed subgroup of G and $\mathfrak{A}$ a C^*-algebra. Then*

$$(\mathfrak{A} \otimes C_0(G/H)) \rtimes_{\alpha \otimes \lambda} G \cong (\mathfrak{A} \rtimes_\alpha H) \otimes \mathbb{K}(L^2(G/H)),$$

where $\alpha \otimes \lambda$ is the diagonal action for α an action of G on $\mathfrak{A}$ and λ the left translation action of G on $C_0(G/H)$, and $\mathbb{K}(L^2(G/H))$ is the C^*-algebra of compact operators on the Hilbert space $L^2(G/H)$.

Remark. The proof is as follows. Let $C_c(G, \mathfrak{A})$ be the algebra of continuous functions from G to $\mathfrak{A}$ with compact supports, and let $C_c(H, \mathfrak{A})$ define similarly. Then $C_c(H, \mathfrak{A})$ acts on $C_c(G, \mathfrak{A})$ on the right by

$$(xf)(g) = \int_H x(gh)\alpha_{gh}(f(h^{-1}))d\mu_H(t)$$

for $x \in C_c(G, \mathfrak{A})$, $f \in C_c(H, \mathfrak{A})$, $g \in G$ and μ_H the left Haar measure on H. Define a $C_c(H, \mathfrak{A})$-valued inner product on $C_c(G, \mathfrak{A})$ by

$$\langle x, y \rangle (h) = \int_G \alpha_g(x(g^{-1}))^* \alpha_g(y(g^{-1}h))d\mu_G(g)$$

for $x, y \in C_c(G, \mathfrak{A})$, $h \in H$ and $d\mu_G$ the left Haar measure on G. Then the space $X_{G,\mathfrak{A}}$ (imprimitivity bimodule) is defined to be the completion of $C_c(G, \mathfrak{A})$ by the norm $\|\langle x, x\rangle\|^{1/2}$. Now let $Y_{H,\mathfrak{A}}$ denote the space defined by the completion of $C_c(H, \mathfrak{A}) \otimes C_c(G/H)$ with the right $C_c(H, \mathfrak{A})$-action by the right convolution on the tensor factor $C_c(H, \mathfrak{A})$ and the $C_c(H, \mathfrak{A})$-valued inner product defined by

$$\langle f_1 \otimes g_1, f_2 \otimes g_2\rangle = (\int_{G/H} \bar{g}_1 g_2 d\mu_{G/H}) f_1^* * f_2$$

for $f_j \otimes g_j \in C_c(H, \mathfrak{A}) \otimes C_c(G/H)$. When a measurable cross section from G/H to G is fixed, it is shown that $X_{G,\mathfrak{A}}$ is isomorphic to $Y_{H,\mathfrak{A}}$ as a space with the structure so that their imprimitivity algebras $(\mathfrak{A} \otimes C_0(G/H)) \rtimes_{\alpha\otimes\lambda} G$ and $(\mathfrak{A} \rtimes_\alpha H) \otimes \mathbb{K}(L^2(G/H))$ are isomorphic.

Remark. This theorem also says that a $*$-representation of the C^*-dynamical system $(\mathfrak{A}, G, \alpha)$ is induced from the system $(\mathfrak{A}, H, \alpha)$ if and only if it is equivalent to a covariant representation of the system $(\mathfrak{A} \otimes C_0(G/H), G, \alpha \otimes \lambda)$.

G2. The Takai duality theorem. *When G is abelian and H is trivial in* G1, *we obtain*

$$(\mathfrak{A} \otimes C_0(G)) \rtimes_{\alpha\otimes\lambda} G \cong \mathfrak{A} \otimes \mathbb{K}(L^2(G)).$$

Moreover,

$$(\mathfrak{A} \otimes C_0(G)) \rtimes_{\alpha\otimes\lambda} G \cong (\mathfrak{A} \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G},$$

where $\hat{G}$ is the dual group of G, and the dual action $\hat{\alpha}$ is defined by

$$\hat{\alpha}_\chi(x)(g) = \overline{\chi(g)} x(g)$$

for $\chi \in \hat{G}$, $g \in G$ and $x \in L^1(G, \mathfrak{A})$ which is a dense $*$-subalgebra of $\mathfrak{A} \rtimes_\alpha G$ consisting of all integrable functions on G to $\mathfrak{A}$ with α-twisted convolution and involution. Hence we obtain the following duality:

$$(\mathfrak{A} \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \cong \mathfrak{A} \otimes \mathbb{K}(L^2(G)).$$

Remark. This theorem says that a $*$-representation of $\mathfrak{A} \rtimes_\alpha G$ (or $(\mathfrak{A}, G, \alpha)$) is induced from $\mathfrak{A}$ if and only if it is equivalent to a covariant representation of the dual dynamical system $(\mathfrak{A} \rtimes_\alpha G, \hat{G}, \hat{\alpha})$.

G3. The Mackey-Rieffel imprimitivity theorem. *When $\mathfrak{A}$ is trivial in* G1, *we obtain*

$$C_0(G/H) \rtimes_\lambda G \cong C^*(H) \otimes \mathbb{K}(L^2(G/H)).$$

Remark. This theorem says that a $*$-representation of G (or $C^*(G)$) is induced from H (or $C^*(H)$) if and only if it is equivalent to a covariant representation of the dynamical system $(C_0(G/H), G, \lambda)$.

G4. The Stone-von Neumann theorem. *When H is trivial in* G3, *we obtain*

$$C_0(G) \rtimes_\lambda G \cong \mathbb{K}(L^2(G)).$$

Remark. The named theorem in fact says that a covariant representation of $(C_0(G), G, \lambda)$ corresponding to the multiplication representation of $C_0(G)$ on $L^2(G)$ and the left regular representation of G on $L^2(G)$ gives a faithful representation of $C_0(G) \rtimes_\lambda G$ to $\mathbb{K}(L^2(G))$.

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