Generalized Apéry-like numbers arising from the non-commutative harmonic oscillator

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GENERALIZED APÉRY-LIKE NUMBERS ARISING FROM THE NON-COMMUTATIVE HARMONIC OSCILLATOR*

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Abstract

A generalization of the Apéry-like numbers, which is used to describe the special values $\zeta_Q(2)$ and $\zeta_Q(3)$ of the spectral zeta function for the non-commutative harmonic oscillator, are introduced and studied. In fact, we give a recurrence relation for them, which shows a ladder structure among them. Further, we consider the 'rational part' of the generalized Apéry-like numbers. We discuss several kinds of congruence relations among them, which are regarded as an analog of the ones among Apéry numbers.

1 Introduction

The *non-commutative harmonic oscillator* is the system of differential equations defined by the operator

$$Q = Q_{\alpha,\beta} := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 \right) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(x\frac{d}{dx} + \frac{1}{2} \right), \tag{1.1}$$

where α and β are real parameters. In this paper, we always assume that $\alpha > 0$, $\beta > 0$ and $\alpha\beta > 1$. Under these conditions, one can show that the operator Q defines an unbounded, positive, self-adjoint operator on the space $L^2(\mathbb{R};\mathbb{C}^2)$ of $\mathbb{C}^2$-valued square integrable functions which has only a discrete spectrum, and the multiplicities $m(\lambda)$ of the eigenvalues $\lambda \in \mathrm{Spec}(Q)$ are uniformly bounded [27]. Hence, in this case, it is meaningful to define its *spectral zeta function*

$$\zeta_Q(s) = \mathrm{Tr}\, Q^{-s} = \sum_{\lambda \in \mathrm{Spec}(Q)} m(\lambda)\lambda^{-s}.$$

This series converges absolutely if $\Re s > 1$, and hence defines a holomorphic function on the half plane $\Re s > 1$. Further, $\zeta_Q(s)$ is meromorphically continued to the whole complex plane $\mathbb{C}$ which has 'trivial zeros' at $s = 0, -2, -4, \dots$ (see [8], [26]).

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The aim of this paper is to study the *generalized Apéry-like numbers* $J_k(n)$ defined by

$$J_k(n) := 2^k \int_{[0,1]^k} \left(\frac{(1-x_1^4)(1-x_2^4\cdots x_k^4)}{(1-x_1^2\cdots x_k^2)^2} \right)^n \frac{dx_1 dx_2 \cdots dx_k}{1-x_1^2\cdots x_k^2}$$

for $k \geq 2$ and $n \geq 0$, which are a generalization of the Apéry-like numbers $J_2(n)$ and $J_3(n)$ studied in [13]. This object arises from the special values of the spectral zeta function $\zeta_Q(s)$: In [9], the generating functions of the numbers $J_2(n)$ and $J_3(n)$ are used to describe the special values $\zeta_Q(2)$ and $\zeta_Q(3)$ of the spectral zeta function $\zeta_Q(s)$. Similarly, the generalized Apéry-like numbers $J_k(n)$ are closely related to the special values $\zeta_Q(k)$ (see §3.3). Here we should remark that we also study another kind of a generalization of the Apéry-like numbers (which we call 'higher' Apéry-like numbers) in [11, 15, 16].

We first show that $J_k(n)$ satisfy three-term (inhomogeneous) recurrence relations, which is translated to (inhomogeneous) singly confluent Heun differential equations for their generating functions. The point is that these relations or differential equations are connecting $J_k(n)$'s and $J_{k-2}(n)$'s. This fact implies that there could be a certain relation between $\zeta_Q(k)$ and $\zeta_Q(k-2)$. It would be very interesting if one can utilize these relations to understand a modular interpretation of $\zeta_Q(4), \zeta_Q(6), \ldots$ based on that of $\zeta_Q(2)$ (see [14]). We also notice that these recurrence relations quite resemble to those for *Apéry numbers* used to prove the irrationality of $\zeta(2)$ and $\zeta(3)$ (see [2, 31]), and this is why we call $J_k(n)$ the (generalized) Apéry-like numbers.

By a suitable change of variable in the differential equation, we also obtain another kind of recurrence relations, which allow us to define the *rational part* of the generalized Apéry-like numbers (or *normalized generalized Apéry-like numbers*) $\tilde{J}_k(n)$. In fact, each $J_k(n)$ is a linear combination of the Riemann zeta values $\zeta(k), \zeta(k-2), \ldots$ and the coefficients are given by $\tilde{J}_m(n)$'s. Since there are various kind of congruence relations satisfied by Apéry numbers (see, e.g. [5], [6], [1]), it would be natural and interesting to find an analog for our generalized Apéry-like numbers. Actually, we give several congruence relations among $\tilde{J}_2(n)$ and $\tilde{J}_3(n)$ in [14]. We add such congruence relations among $\tilde{J}_k(n)$, and give some conjectural congruences.

2 Apéry numbers for $\zeta(2)$ and $\zeta(3)$

As a quick reference for the readers, we recall the definitions and several properties on the original Apéry numbers.

2.1 Apéry numbers for $\zeta(2)$

Apéry numbers for $\zeta(2)$ are given by

$$A_2(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k},$$

$$B_2(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \left(2\sum_{m=1}^{n} \frac{(-1)^{m-1}}{m^2} + \sum_{m=1}^{k} \frac{(-1)^{n+m-1}}{m^2 \binom{n}{m}\binom{n+m}{m}} \right).$$

These numbers satisfy a recurrence relation of the same form

$$n^2u(n) - (11n^2 - 11n + 3)u(n-1) - (n-1)^2u(n-2) = 0 \quad (n \geq 2) \qquad (2.1)$$

with initial conditions $A_2(0) = 1, A_2(1) = 3$ and $B_2(0) = 0, B_2(1) = 5$. The ratio $B_2(n)/A_2(n)$ converges to $\zeta(2)$, and this convergence is rapid enough to prove the irrationality of $\zeta(2)$. Consider the generating functions

$$\mathcal{A}_2(t) = \sum_{n=0}^{\infty} A_2(n)t^n, \quad \mathcal{B}_2(t) = \sum_{n=0}^{\infty} B_2(n)t^n, \quad \mathcal{R}_2(t) = \mathcal{A}_2(t)\zeta(2) - \mathcal{B}_2(t).$$

It is proved that

$$L_2\mathcal{A}_2(t) = 0, \quad L_2\mathcal{B}_2(t) = -5, \quad L_2\mathcal{R}_2(t) = 5,$$

where L_2 is a differential operator given by

$$L_2 = t(t^2 + 11t - 1)\frac{d^2}{dt^2} + (3t^2 + 22t - 1)\frac{d}{dt} + (t + 3).$$

The function $\mathcal{R}_2(t)$ is also expressed as follows:

$$\mathcal{R}_2(t) = \int_0^1 \int_0^1 \frac{dxdy}{1 - xy + txy(1-x)(1-y)}.$$

The family Q_t^2: $1 - xy + txy(1-x)(1-y) = 0$ of algebraic curves, which comes from the denominator of the integrand, is birationally equivalent to the universal family C_t^2 of elliptic curves having rational 5-torsion. Moreover, the differential equation $L_2\mathcal{A}_2(t) = 0$ is regarded as a Picard-Fuchs equation for this family, and $\mathcal{A}_2(t)$ is interpreted as a period of C_t^2 (see [3]).

2.2 Apéry numbers for $\zeta(3)$

Apéry numbers for $\zeta(3)$ are given by

$$A_3(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$

$$B_3(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}} \right).$$

These numbers satisfy a recurrence relation of the same form

$$n^3u(n) - (34n^3 - 51n^2 + 27n - 5)u(n-1) + (n-1)^3u(n-2) = 0 \quad (n \geq 2)$$

with initial conditions $A_3(0) = 1, A_3(1) = 5$ and $B_3(0) = 0, B_3(1) = 6$. The ratio $B_3(n)/A_3(n)$ converges to $\zeta(3)$ rapidly enough to allow us to prove the irrationality of $\zeta(3)$. Consider the generating functions

$$\mathcal{A}_3(t) = \sum_{n=0}^{\infty} A_3(n)t^n, \quad \mathcal{B}_3(t) = \sum_{n=0}^{\infty} B_3(n)t^n, \quad \mathcal{R}_3(t) = \mathcal{A}_3(t)\zeta(3) - \mathcal{B}_3(t).$$

It is proved that

$$L_3\mathcal{A}_3(t)=0,\quad L_3\mathcal{B}_3(t)=5,\quad L_3\mathcal{R}_3(t)=-5,$$

where L_3 is a differential operator given by

$$L_3=t^2(t^2-34t^2+1)\frac{d^3}{dt^3}+t(6t^2-153t+3)\frac{d^2}{dt^2}+(7t^2-112t+1)\frac{d}{dt}+(t-5).$$

The function $\mathcal{R}_3(t)$ is also expressed as follows:

$$\mathcal{R}_3(t)=\int_0^1\int_0^1\int_0^1\frac{dxdydz}{1-(1-xy)z-txyz(1-x)(1-y)(1-z)}.$$

The family $Q_t^3: 1-(1-xy)z-txyz(1-x)(1-y)(1-z)=0$ of algebraic surfaces coming from the denominator of the integrand is birationally equivalent to a certain family C_t^3 of $K3$ surfaces with Picard number 19. Furthermore, the differential equation $L_3\mathcal{A}_3(t)=0$ is regarded as a Picard-Fuchs equation for this family, and $\mathcal{A}_3(t)$ is interpreted as a period of C_t^3 (see [4]).

2.3 Congruence relations for Apéry numbers

Apéry numbers $A_2(n)$ and $A_3(n)$ have various kind of congruence properties. Here we pick up several of them, for which we will discuss an Apéry-like analog later.

Proposition 2.1. *Let p be a prime and $n=n_0+n_1p+\cdots+n_kp^k$ be the p-ary expansion of $n\in\mathbb{Z}_{\geq 0}$ $(0\leq n_j<p)$. Then it holds that*

$$A_2(n)\equiv\prod_{j=0}^{k}A_2(n_j)\pmod{p},\qquad A_3(n)\equiv\prod_{j=0}^{k}A_3(n_j)\pmod{p}.$$

Proposition 2.2 ([5, Theorems 1 and 2]). *For all odd prime p, it holds that*

$$A_2(mp^r-1)\equiv A_3(mp^{r-1}-1)\pmod{p^r},$$
$$A_3(mp^r-1)\equiv A_3(mp^{r-1}-1)\pmod{p^r}$$

for any $m,r\in\mathbb{Z}_{>0}$. These congruence relations hold modulo p^{3r} if $p\geq 5$ (known and referred to as a supercongruence*).*

We denote by $\eta(\tau)$ the Dedekind eta function

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n),\quad q=e^{2\pi i\tau}\quad(\Im\tau>0). \tag{2.2}$$

Proposition 2.3 ([30, Theorem 13.1]). *For any odd prime p and any $m,r\in\mathbb{Z}_{>0}$ with m odd, it holds that*

$$A_2(\tfrac{mp^r-1}{2})-\lambda_pA_2(\tfrac{mp^{r-1}-1}{2})+(-1)^{(p-1)/2}p^2A_2(\tfrac{mp^{r-2}-1}{2})\equiv 0\pmod{p^r}. \tag{2.3}$$

Here λ_n is defined by

$$\sum_{n=1}^{\infty}\lambda_nq^n=\eta(4\tau)^6=q\prod_{n=1}^{\infty}(1-q^{4n})^6.$$

Proposition 2.4 ([6, Theorem 4]). *For any odd prime p and any $m, r \in \mathbb{Z}_{>0}$ with m odd, it holds that*

$$A_3(\tfrac{mp^r-1}{2}) - \gamma_p A_3(\tfrac{mp^{r-1}-1}{2}) + p^3 A_3(\tfrac{mp^{r-2}-1}{2}) \equiv 0 \pmod{p^r}. \tag{2.4}$$

Here γ_n is defined by

$$\sum_{n=1}^{\infty} \gamma_n q^n = \eta(2\tau)^4 \eta(4\tau)^4 = q \prod_{n=1}^{\infty} (1-q^{2n})^4 (1-q^{4n})^4.$$

3 Apéry-like numbers for $\zeta_Q(2)$ and $\zeta_Q(3)$

We introduce the Apéry like numbers $J_2(n)$ and $J_3(n)$, and give a brief explanation on their basic properties and the connection between the special values $\zeta_Q(2), \zeta_Q(3)$ of the spectral zeta function $\zeta_Q(s)$.

3.1 Definition

We define the *Apéry-like numbers for $\zeta_Q(2)$ and $\zeta_Q(3)$* by

$$J_2(n) := 4 \int_0^1 \int_0^1 \left(\frac{(1-x_1^4)(1-x_2^4)}{(1-x_1^2x_2^2)^2} \right)^n \frac{dx_1 dx_2}{1-x_1^2x_2^2},$$

$$J_3(n) := 8 \int_0^1 \int_0^1 \int_0^1 \left(\frac{(1-x_1^4)(1-x_2^4x_3^4)}{(1-x_1^2x_2^2x_3^2)^2} \right)^n \frac{dx_1 dx_2 dx_3}{1-x_1^2x_2^2x_3^2}.$$

The sequences $\{J_2(n)\}$ and $\{J_3(n)\}$ satisfy the recurrence formula (Propositions 4.11 and 6.4 in [9])

$$4n^2 J_2(n) - (8n^2-8n+3) J_2(n-1) + 4(n-1)^2 J_2(n-2) = 0, \tag{3.1}$$

$$4n^2 J_3(n) - (8n^2-8n+3) J_3(n-1) + 4(n-1)^2 J_3(n-2) = \frac{2^n (n-1)!}{(2n-1)!!} \tag{3.2}$$

with the initial conditions

$$J_2(0) = 3\zeta(2), \quad J_2(1) = \frac{9}{4}\zeta(2); \qquad J_3(0) = 7\zeta(3), \quad J_3(1) = \frac{21}{4}\zeta(3) + \frac{1}{2}.$$

It is notable that the left-hand sides of these relations have the same shape. Since the relations (3.1),(3.2) and the one (2.1) for $A_2(n)$ have quite close shapes, we call the numbers $J_2(n)$ and $J_3(n)$ the *Apéry-like* numbers.

3.2 Generating functions and their differential equations

The generating functions for $J_2(n)$ and $J_3(n)$ are defined by

$$w_2(t) := \sum_{n=0}^{\infty} J_2(n) t^n = 4 \int_0^1 \int_0^1 \frac{1-x_1^2x_2^2}{(1-x_1^2x_2^2)^2 - t(1-x_1^4)(1-x_2^4)} dx_1 dx_2, \tag{3.3}$$

$$w_3(t) := \sum_{n=0}^{\infty} J_3(n) t^n = 8 \int_0^1 \int_0^1 \int_0^1 \frac{1-x_1^2x_2^2x_3^2}{(1-x_1^2x_2^2x_3^2)^2 - t(1-x_1^4)(1-x_2^4x_3^4)} dx_1 dx_2 dx_3. \tag{3.4}$$

By the recurrence relations (3.1) and (3.2), we get the differential equations

$$\mathcal{D}_{\mathrm{H}} w_2(t) = 0, \tag{3.5}$$

$$\mathcal{D}_{\mathrm{H}} w_3(t) = \frac{1}{2}\,{}_2F_1\left(1,1;\frac{3}{2};t\right), \tag{3.6}$$

where $\mathcal{D}_{\mathrm{H}}$ denotes the singly confluent Heun differential operator given by

$$\mathcal{D}_{\mathrm{H}} = t(1-t)^2\frac{d^2}{dt^2} + (1-3t)(1-t)\frac{d}{dt} + t - \frac{3}{4}. \tag{3.7}$$

(3.5) is solved in [23] as

$$w_2(t) = \frac{3\zeta(2)}{1-t}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{t}{t-1}\right).$$

Here ${}_2F_1(a,b;c;z)$ is the Gaussian hypergeometric function. Now it is immediate that

$$J_2(n) = 3\zeta(2)\sum_{j=0}^{n}(-1)^j\binom{-\frac{1}{2}}{j}^2\binom{n}{j}. \tag{3.8}$$

Similarly, (3.6) is solved in [13] as

$$\begin{aligned} w_3(t) = {} & \frac{7\zeta(3)}{1-t}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{t}{t-1}\right) \\ & - 2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k\binom{-\frac{1}{2}}{k}^2\binom{n}{k}\sum_{j=0}^{k-1}\frac{1}{(2j+1)^3}\binom{-\frac{1}{2}}{j}^{-2}\right)t^n. \end{aligned}$$

Therefore it follows that

$$\begin{aligned} J_3(n) = {} & 7\zeta(3)\sum_{j=0}^{n}(-1)^j\binom{-\frac{1}{2}}{j}^2\binom{n}{j} \\ & - 2\sum_{j=0}^{n}(-1)^j\binom{-\frac{1}{2}}{j}^2\binom{n}{j}\sum_{k=0}^{j-1}\frac{1}{(2k+1)^3}\binom{-\frac{1}{2}}{k}^{-2}. \end{aligned} \tag{3.9}$$

Remark 3.1. The function

$$W_2(T) = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;T^2\right) = \frac{1}{3\zeta(2)}(1-t)w_2(t) \qquad \left(T^2 = \frac{t}{t-1}\right)$$

satisfies the differential equation

$$\left(T(T^2-1)\frac{d^2}{dT^2} + (3T^2-1)\frac{d}{dT} + T\right)W_2(T) = 0,$$

which can be regarded as a Picard-Fuchs equation for the universal family of elliptic curves having rational 4-torsion [14]. This is an analog of the result [3] for the Apéry numbers for $\zeta(2)$ (see also Section 2.1). It is natural to ask whether there is such a modular interpretation for $w_3(t)$ (or "$W_3(T)$"). We have not obtained an answer to this question so far.

3.3 Connection to the special values of $\zeta_Q(s)$

We also introduce another kind of generating functions for $J_k(n)$ as

$$g_2(z) := \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} J_2(n) z^n = 4 \int_0^1 \int_0^1 \frac{dx_1 dx_2}{\sqrt{(1-x_1^2x_2^2)^2 + z(1-x_1^4)(1-x_2^4)}},$$

$$g_3(z) := \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} J_2(n) z^n = 8 \int_0^1 \int_0^1 \int_0^1 \frac{dx_1 dx_2 dx_3}{\sqrt{(1-x_1^2x_2^2x_3^2)^2 + z(1-x_1^4)(1-x_2^4x_3^4)}}.$$

The special values of $\zeta_Q(s)$ at $s = 2, 3$ are given as follows.

Theorem 3.2 (Ichinose-Wakayama [9]). *If $\alpha\beta > 2$ (i.e. $0 < 1/(1-\alpha\beta) < 1$), then*

$$\zeta_Q(2) = 2\left(\frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\right)^2 \left(\zeta\Big(2, \frac{1}{2}\Big) + \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 g_2\Big(\frac{1}{\alpha\beta-1}\Big)\right),$$

$$\zeta_Q(3) = 2\left(\frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\right)^3 \left(\zeta\Big(3, \frac{1}{2}\Big) + 3\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 g_3\Big(\frac{1}{\alpha\beta-1}\Big)\right),$$

where $\zeta(s,x) = \sum_{n=0}^{\infty}(n+x)^{-s}$ is the Hurwitz zeta function.

Remark 3.3. We can determine the functions $g_2(x)$ and $g_3(x)$ as follows:

$$g_2(x) = J_2(0)\widetilde{g}_2(x), \qquad g_3(x) = J_3(0)\widetilde{g}_2(x) + \widetilde{g}_3(x),$$

where

$$\widetilde{g}_2(x) := \frac{1}{\sqrt{1+x}}\, {}_2F_1\Big(\frac{1}{4}, \frac{1}{4}; 1; \frac{x}{1+x}\Big)^2 = {}_2F_1\Big(\frac{1}{4}, \frac{3}{4}; 1; -x\Big)^2,$$

$$\widetilde{g}_3(x) := \frac{-2}{\sqrt{1+x}} \sum_{n=1}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n}^3 \left(\frac{x}{1+x}\right)^n \sum_{j=0}^{n-1} \frac{1}{(2j+1)^3} \binom{-\frac{1}{2}}{j}^{-2}.$$

See [23] and [13] for detailed calculation.

4 Generalized Apéry-like numbers

Looking at the definition of $J_2(n)$ and $J_3(n)$, it is natural to introduce the numbers $J_k(n)$ by

$$J_k(n) := 2^k \int_{[0,1]^k} \left(\frac{(1-x_1^4)(1-x_2^4\cdots x_k^4)}{(1-x_1^2\cdots x_k^2)^2}\right)^n \frac{dx_1 dx_2 \cdots dx_k}{1-x_1^2\cdots x_k^2}.$$

We refer to $J_k(n)$ as *generalized Apéry-like numbers.* In fact, the generating function

$$g_k(z) := \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} J_k(n) z^n \tag{4.1}$$

$$= 2^k \int_{[0,1]^k} \frac{dx_1 dx_2 \ldots dx_k}{\sqrt{(1-x_1^2x_2^2\ldots x_k^2)^2 + z(1-x_1^4)(1-x_2^4\ldots x_k^4)}} \tag{4.2}$$

and its further generalizations are used to describe the special values $\zeta_Q(k)$ $(k \geq 4)$ like Theorem 3.2 (see Remark 4.1 below).

It is immediate that $J_k(0) = (2^k - 1)\zeta(k)$. Further, as we mentioned in [13], the formula

$$J_k(1) = \frac{3}{4} \sum_{m=0}^{\lfloor k/2 \rfloor - 1} \frac{1}{4^m} \zeta\Big(k - 2m, \frac{1}{2}\Big) + \frac{1 - (-1)^k}{2^{k-1}} \tag{4.3}$$

holds (see §6.2 for the calculation). It is directly verified that

$$4J_k(1) - 3J_k(0) = J_{k-2}(1) \qquad (k \geq 4).$$

Remark 4.1. We can calculate that

$$\begin{aligned}\zeta_Q(4) = 2\left(\frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\right)^4 &\Bigg(\zeta(4, 1/2) + 4\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 g_4\Big(\frac{1}{\alpha\beta-1}\Big) \\ &+ 2\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 \int_{[0,1]^4} \frac{16dx_1dx_2dx_3dx_4}{\sqrt{(1-x_1^2x_2^2x_3^2x_4^2)^2 + \gamma_1(1-x_1^4x_2^4)(1-x_3^4x_4^4)}} \\ &+ \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^4 \int_{[0,1]^4} \frac{16dx_1dx_2dx_3dx_4}{\sqrt{R(x_1,x_2,x_3,x_4)}}\Bigg),\end{aligned}$$

where $\gamma_1 = 1/(\alpha\beta - 1)$, $\gamma_2 = \alpha\beta/(\alpha\beta-1)^2$ and

$$\begin{aligned}R(x_1, x_2, x_3, x_4) &= (1 - x_1^2x_2^2x_3^2x_4^2)^2 \\ &+ \gamma_1(1-x_1^4x_2^4)(1-x_3^4x_4^4) + \gamma_2(1-x_1^4)(1-x_2^4)(1-x_3^4)(1-x_4^4).\end{aligned}$$

See [11] for the calculation on the special values $\zeta_Q(k)$ for general $k \geq 2$ (see also [15, 16]).

Remark 4.2. In [11, 15, 16], we discuss the 'higher' Apéry-like numbers associated to the special values $\zeta_Q(k)$ for $k \geq 2$, which is slightly different from our generalized Apéry-like numbers. Indeed, our generalized Apéry-like numbers are regarded as a refinement of the higher ones.

Similar to the case of $J_2(n)$ and $J_3(n)$, the generalized Apéry-like numbers $J_k(n)$ also satisfy a three-term recurrence relation as follows.

Theorem 4.3. *The numbers $J_k(n)$ satisfy the recurrence relations*

$$4n^2 J_k(n) - (8n^2 - 8n + 3)J_k(n-1) + 4(n-1)^2 J_k(n-2) = J_{k-2}(n) \tag{4.4}$$

for $n \geq 2$ and $k \geq 4$.

We give the proof of Theorem 4.3 in §5. It is remarkable that the left-hand side of (4.4) has a common shape with those of (3.1) and (3.2), and (4.4) gives a 'vertical' relation among $J_k(n)$'s, i.e. it connects $J_k(n)$'s and $J_{k-2}(n)$'s.

Example 4.4. First several terms of $J_4(n)$ are given by

$$J_4(0)=15\zeta(4),\quad J_4(1)=\frac{45}{4}\zeta(4)+\frac{9}{16}\zeta(2),\quad J_4(2)=\frac{615}{64}\zeta(4)+\frac{807}{1024}\zeta(2),$$
$$J_4(3)=\frac{2205}{256}\zeta(4)+\frac{3745}{4096}\zeta(2),\quad J_4(4)=\frac{129735}{16384}\zeta(4)+\frac{1044135}{1048576}\zeta(2),\dots$$

We also see that

$$\begin{aligned}
4J_4(1)-3J_4(0)&=\frac{9}{4}\zeta(2)=J_2(1),\\
16J_4(2)-19J_4(1)+4J_4(0)&=\frac{123}{64}\zeta(2)=J_2(2),\\
36J_4(3)-51J_4(2)+16J_4(1)&=\frac{441}{256}\zeta(2)=J_2(3),\\
64J_4(4)-99J_4(3)+36J_4(2)&=\frac{25947}{16384}\zeta(2)=J_2(4).
\end{aligned}$$

Define another kind of generating function for $J_k(n)$ by

$$\begin{aligned}
w_k(t)&:=\sum_{n=0}^{\infty}J_k(n)t^n && (4.5)\\
&=2^k\int_0^1\int_0^1\cdots\int_0^1\frac{1-x_1^2\cdots x_k^2}{(1-x_1^2\cdots x_k^2)^2-(1-x_1^4)(1-x_2^4\cdots x_k^4)t}\,dx_1dx_2\cdots dx_k. && (4.6)
\end{aligned}$$

Theorem 4.3 readily implies the

Corollary 4.5. *The differential equation*

$$\mathfrak{D}_{\mathrm{H}}w_k(t)=\frac{w_{k-2}(t)-w_{k-2}(0)}{4t} \tag{4.7}$$

holds for $k\geq 4$. Here $\mathfrak{D}_{\mathrm{H}}$ is the differential operator given in (3.7).

Put

$$\mathbb{J}_0(n):=0,\quad \mathbb{J}_1(n):=\frac{(-1)^n}{n},\quad \mathbb{J}_k(n):=\binom{-\frac{1}{2}}{n}J_k(n)\quad (k\geq 2).$$

By Theorem 4.3, we have

$$\begin{aligned}
8n^3\mathbb{J}_k(n)-(1-2n)(8n^2-8n+3)\mathbb{J}_k(n-1)&\\
+2(n-1)(1-2n)(3-2n)\mathbb{J}_k(n-2)&=2n\mathbb{J}_{k-2}(n)
\end{aligned}$$

for $k\geq 2$ and $n\geq 1$. Hence, if we put

$$\begin{aligned}
\mathfrak{D}_{\mathrm{W}}:=8z^2(1+z)^2\frac{d^3}{dz^3}+24z(1+z)(1+2z)\frac{d^2}{dz^2}&\\
+2(4+27z+27z^2)\frac{d}{dz}+3(1+2z),&\qquad (4.8)
\end{aligned}$$

then we have the following (See also [13, Proposition A.3]).

Corollary 4.6. *The differential equations*

$$\begin{aligned}
\mathcal{D}_{\mathrm{W}} g_2(z) &= 0, \\
\mathcal{D}_{\mathrm{W}} g_3(z) &= -\frac{2}{1+z}, \\
\mathcal{D}_{\mathrm{W}} g_k(z) &= 2z\frac{d}{dz}\left(\frac{g_{k-2}(z)-g_{k-2}(0)}{z}\right) \quad (k \geq 4)
\end{aligned}$$

hold. □

5 Proof of Theorem 4.3

5.1 Setting the stage

Assume $k \geq 2$. We notice that

$$\begin{aligned}
J_k(n) &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{e^{-(t_1+\cdots+t_k)/2}(1-e^{-2t_1})^n(1-e^{-2(t_2+\cdots+t_k)})^n}{(1-e^{-(t_1+\cdots+t_k)})^{2n+1}}\, dt_1\cdots dt_k \\
&= \int_0^\infty \frac{e^{-u/2}}{(1-e^{-u})^{2n+1}}\, du \int_0^u \frac{t^{k-2}}{(k-2)!}(1-e^{-2t})^n(1-e^{-2u+2t})^n\, dt
\end{aligned}$$

for each $n \geq 0$. Let us introduce

$$I_{n,m}^{(k)} = I_{n,m}^{(k)}(u) := \int_0^u \frac{t^{k-2}}{(k-2)!}(1-e^{-2t})^n(1-e^{-2u+2t})^m\, dt$$

for $n, m \geq 0$. We also put

$$\mathbb{I}_{n,m}^{(k)}(u) := \frac{1}{2}(I_{n,m}^{(k)}(u) + I_{m,n}^{(k)}(u)), \qquad \widetilde{\mathbb{I}}_{n,m}^{(k)}(u) := \frac{1}{2}(I_{n,m}^{(k)}(u) - I_{m,n}^{(k)}(u)).$$

$I_{n,m}^{(k)}(u)$ is symmetric in n and m if $k=2$ so that $\widetilde{\mathbb{I}}_{n,m}^{(2)}(u) = 0$, but $\widetilde{\mathbb{I}}_{n,m}^{(k)}(u) \neq 0$ in general.

It is convenient to set $I_{n,m}^{(k)}(u) = 0$ when $k < 2$. We see that

$$J_k(n) = \frac{1}{2^{2n+1}} \int_0^\infty \frac{e^{nu}}{(\sinh\frac{u}{2})^{2n+1}} I_{n,n}^{(k)}(u)\, du.$$

Thus we also set $J_k(n) = 0$ if $k < 2$. Under these convention, the following discussion for $J_k(n)$ is reduced to the one given by Ichinose and Wakayama [9] when $k = 2, 3$.

For later use, we define

$$\begin{aligned}
a_n^{(k)}(u) &:= I_{n,n}^{(k)}(u) = \mathbb{I}_{n,n}^{(k)}(u) \quad (n \geq 0), \\
b_n^{(k)}(u) &:= \frac{1}{2}\left(I_{n,n-1}^{(k)}(u) + I_{n-1,n}^{(k)}(u)\right) = \mathbb{I}_{n,n-1}^{(k)} \quad (n \geq 1), \\
\tilde{b}_n^{(k)}(u) &:= \frac{1}{2}\left(I_{n,n-1}^{(k)}(u) - I_{n-1,n}^{(k)}(u)\right) = \widetilde{\mathbb{I}}_{n,n-1}^{(k)} \quad (n \geq 1), \\
\mathcal{A}_n^{(k)}(u) &:= e^{nu}a_n^{(k)}(u), \quad \mathcal{B}_n^{(k)}(u) := \frac{\mathcal{A}_n^{(k)}(u)}{(\sinh\frac{u}{2})^{2n+1}} \quad (n \geq 0),
\end{aligned}$$

so that

$$J_k(n) = \frac{1}{2^{2n+1}} \int_0^\infty \mathcal{B}_n^{(k)}(u)\, du.$$

5.2 Recurrence formulas for $I_{n,m}^{(k)}(u)$

Integration by parts implies

$$\begin{aligned} I_{n,m}^{(k-1)} &= \int_0^u \left(\frac{d}{dt}\frac{t^{k-2}}{(k-2)!}\right)(1-e^{-2t})^n(1-e^{-2u+2t})^m dt \\ &= -\int_0^u \frac{t^{k-2}}{(k-2)!}\left(\frac{d}{dt}(1-e^{-2t})^n\right)(1-e^{-2u+2t})^m dt \\ &\quad - \int_0^u \frac{t^{k-2}}{(k-2)!}(1-e^{-2t})^n\left(\frac{d}{dt}(1-e^{-2u+2t})^m\right) dt. \end{aligned} \tag{5.1}$$

when $n, m \geq 1$. Since

$$\begin{aligned} \frac{d}{dt}(1-e^{-2t})^n &= 2ne^{-2t}(1-e^{-2t})^{n-1} \\ &= 2n\left((1-e^{-2t})^{n-1} - (1-e^{-2t})^n\right), \\ \frac{d}{dt}(1-e^{-2u+2t})^m &= -2me^{-2u+2t}(1-e^{-2u+2t})^{m-1} \\ &= -2m\left((1-e^{-2u+2t})^{m-1} - (1-e^{-2u+2t})^m\right) \end{aligned}$$

for $n, m \geq 1$, we obtain the

Lemma 5.1. *The following three relations hold:*

$$\frac{1}{2}I_{n,m}^{(k-1)} = (n-m)I_{n,m}^{(k)} - nI_{n-1,m}^{(k)} + mI_{n,m-1}^{(k)} \quad (n, m \geq 1), \tag{5.2}$$

$$\begin{aligned} nI_{n,m}^{(k)} - (2n-1)I_{n-1,m}^{(k)} + (n-1)I_{n-2,m}^{(k)} - me^{-2u}I_{n-1,m-1}^{(k)} \\ = \frac{1}{2}\left(I_{n,m}^{(k-1)} - I_{n-1,m}^{(k-1)}\right) \quad (n \geq 2, m \geq 1), \end{aligned} \tag{5.3}$$

$$\begin{aligned} mI_{n,m}^{(k)} - (2m-1)I_{n,m-1}^{(k)} + (m-1)I_{n,m-2}^{(k)} - ne^{-2u}I_{n-1,m-1}^{(k)} \\ = \frac{1}{2}\left(I_{n,m-1}^{(k-1)} - I_{n,m}^{(k-1)}\right) \quad (n \geq 1, m \geq 2). \end{aligned} \tag{5.4}$$

□

Plugging (5.2) into (5.3), we get

$$I_{n,m}^{(k)} - (I_{n-1,m}^{(k)} + I_{n,m-1}^{(k)}) + (1-e^{-2u})I_{n-1,m-1}^{(k)} = 0 \quad (n \geq 1, m \geq 1), \tag{5.5}$$

which is a generalization of (4.14) in [9]. In particular, if we let $n = m$ in (5.5), then we have

$$I_{n,n}^{(k)} - 2\mathbb{I}_{n,n-1}^{(k)} + (1-e^{-2u})I_{n-1,n-1}^{(k)} = 0. \tag{5.6}$$

Letting $m = n-1$ (or $n = m-1$ and exchanging m by n) in (5.5), we also have another specialization

$$I^{(k)}_{n,n-1} - (I^{(k)}_{n-1,n-1} + I^{(k)}_{n,n-2}) + (1-e^{-2u})I^{(k)}_{n-1,n-2} = 0 \quad (n \geq 2),$$
$$I^{(k)}_{n-1,n} - (I^{(k)}_{n-2,n} + I^{(k)}_{n-1,n-1}) + (1-e^{-2u})I^{(k)}_{n-2,n-1} = 0 \quad (n \geq 2).$$

Adding these equations, we get

$$\mathbb{I}^{(k)}_{n,n-2} = b^{(k)}_n(u) - a^{(k)}_{n-1}(u) + (1-e^{-2u})b^{(k)}_{n-1}(u) \quad (n \geq 2). \tag{5.7}$$

By specializing $m = n$ in (5.3) and (5.4), we have

$$nI^{(k)}_{n,n} - (2n-1)I^{(k)}_{n-1,n} + (n-1)I^{(k)}_{n-2,n} - ne^{-2u}I^{(k)}_{n-1,n-1} = \frac{1}{2}(I^{(k-1)}_{n,n} - I^{(k-1)}_{n-1,n}), \tag{5.8}$$

$$nI^{(k)}_{n,n} - (2n-1)I^{(k)}_{n,n-1} + (n-1)I^{(k)}_{n,n-2} - ne^{-2u}I^{(k)}_{n-1,n-1} = \frac{1}{2}(I^{(k-1)}_{n,n-1} - I^{(k-1)}_{n,n}) \tag{5.9}$$

for $n \geq 2$. Similarly, specializing $m = n-1$ in (5.3) and $n = m-1$ in (5.4) (and exchanging m by n), we have

$$\begin{aligned} nI^{(k)}_{n,n-1} - (2n-1)I^{(k)}_{n-1,n-1} + (n-1)I^{(k)}_{n-2,n-1} - (n-1)e^{-2u}I^{(k)}_{n-1,n-2} \\ = \frac{1}{2}(I^{(k-1)}_{n,n-1} - I^{(k-1)}_{n-1,n-1}), \\ nI^{(k)}_{n-1,n} - (2n-1)I^{(k)}_{n-1,n-1} + (n-1)I^{(k)}_{n-1,n-2} - (n-1)e^{-2u}I^{(k)}_{n-2,n-1} \\ = \frac{1}{2}(I^{(k-1)}_{n-1,n-1} - I^{(k-1)}_{n-1,n}) \end{aligned}$$

for $n \geq 2$. Adding each pair of relations, we obtain

$$2nI^{(k)}_{n,n} - 2(2n-1)\mathbb{I}^{(k)}_{n,n-1} + 2(n-1)\mathbb{I}^{(k)}_{n,n-2} - 2ne^{-2u}I^{(k)}_{n-1,n-1} = \widetilde{\mathbb{I}}^{(k-1)}_{n,n-1}, \tag{5.10}$$
$$2n\mathbb{I}^{(k)}_{n,n-1} - 2(2n-1)I^{(k)}_{n-1,n-1} + 2(n-1)(1-e^{-2u})\mathbb{I}^{(k)}_{n-1,n-2} = \widetilde{\mathbb{I}}^{(k-1)}_{n,n-1}. \tag{5.11}$$

The formulas (5.6), (5.10) and (5.11) are rewritten as follows.

Lemma 5.2. *The equations*

$$a^{(k)}_n(u) + (1-e^{-2u})a^{(k)}_{n-1}(u) = 2b^{(k)}_n(u), \tag{5.12}$$
$$na^{(k)}_n(u) - (2n-1)b^{(k)}_n(u) + (n-1)\mathbb{I}^{(k)}_{n,n-2} - ne^{-2u}a^{(k)}_{n-1}(u) = \frac{1}{2}\tilde{b}^{(k-1)}_n(u), \tag{5.13}$$
$$nb^{(k)}_n(u) - (2n-1)a^{(k)}_{n-1}(u) + (n-1)(1-e^{-2u})b^{(k)}_{n-1}(u) = \frac{1}{2}\tilde{b}^{(k-1)}_n(u) \tag{5.14}$$

hold. □

As a corollary, we also get

Lemma 5.3. *The equation*

$$na_n^{(k)}(u) - (2n-1)(1+e^{-2u})a_{n-1}^{(k)}(u) + (n-1)(1-e^{-2u})^2 a_{n-2}^{(k)}(u) = \tilde{b}_n^{(k-1)}(u) \tag{5.15}$$

holds.

Proof. If we substitute (5.12), then we have

$$\begin{aligned}\tilde{b}_n^{(k-1)}(u) &= 2nb_n^{(k)}(u) - 2(2n-1)a_{n-1}^{(k)}(u) + 2(n-1)(1-e^{-2u})b_{n-1}^{(k)}(u)\\ &= n\left(a_n^{(k)}(u) + (1-e^{-2u})a_{n-1}^{(k)}(u)\right) - 2(2n-1)a_{n-1}^{(k)}(u)\\ &\quad + (n-1)(1-e^{-2u})\left(a_{n-1}^{(k)}(u) + (1-e^{-2u})a_{n-2}^{(k)}(u)\right)\\ &= na_n^{(k)}(u) - (2n-1)(1+e^{-2u})a_{n-1}^{(k)}(u) + (n-1)(1-e^{-2u})^2 a_{n-2}^{(k)}(u),\end{aligned}$$

which is the desired formula. □

Here we give one more useful relation. Using (5.2) twice, we see that

$$\begin{aligned}\frac{1}{4}I_{n,n}^{(k-2)} &= \frac{1}{2}\left(-nI_{n-1,n}^{(k)} + nI_{n,n-1}^{(k)}\right)\\ &= -n\left(-I_{n-1,n}^{(k)} - (n-1)I_{n-2,n}^{(k)} + nI_{n-1,n-1}^{(k)}\right)\\ &\quad + n\left(I_{n,n-1}^{(k)} - nI_{n-1,n-1}^{(k)} + (n-1)I_{n,n-2}^{(k)}\right)\\ &= n\left(2b_n^{(k)}(u) - 2na_{n-1}^{(k)}(u) + 2(n-1)\mathbb{I}_{n,n-2}^{(k)}\right).\end{aligned}$$

Thus we have

$$a_n^{(k-2)}(u) = 8n\left(b_n^{(k)}(u) - na_{n-1}^{(k)}(u) + (n-1)\mathbb{I}_{n,n-2}^{(k)}\right). \tag{5.16}$$

Combining (5.7), (5.16) and (5.14), we obtain

Lemma 5.4. *The equation*

$$a_n^{(k-2)}(u) = 4n\tilde{b}_n^{(k-1)}(u) \tag{5.17}$$

holds. □

In particular, the formula (5.15) is rewritten as

$$na_n^{(k)}(u) - (2n-1)(1+e^{-2u})a_{n-1}^{(k)}(u) + (n-1)(1-e^{-2u})^2 a_{n-2}^{(k)}(u) = \frac{1}{4n}a_n^{(k-2)}(u). \tag{5.18}$$

5.3 Relations for $\mathcal{B}_n^{(k)}(u)$

In view of (5.1), the differential

$$\frac{d}{du}a_n^{(k)}(u) = 2n\int_0^u \frac{t^{k-2}}{(k-2)!}(1-e^{-2t})^n e^{-2u+2t}(1-e^{-2u+2t})^{n-1}dt$$

is written in two ways as

$$\frac{d}{du}a_n^{(k)}(u) = 2n\left(I_{n,n-1}^{(k)} - I_{n,n}^{(k)}\right) = -2n\left(I_{n,n}^{(k)} - I_{n-1,n}^{(k)}\right) + I_{n,n}^{(k-1)}$$

for $n \geq 1$. Hence it follows that

$$\begin{aligned}\frac{d}{du}a_n^{(k)}(u) &= n\left(I_{n,n-1}^{(k)} - I_{n,n}^{(k)}\right) - n\left(I_{n,n}^{(k)} - I_{n-1,n}^{(k)}\right) + \frac{1}{2}I_{n,n}^{(k-1)} \\ &= -na_n^{(k)}(u) + n(1-e^{-2u})a_{n-1}^{(k)}(u) + \frac{1}{2}a_n^{(k-1)}(u).\end{aligned} \tag{5.19}$$

Using this formula, we have

$$\begin{aligned}\frac{d}{du}\mathcal{A}_n^{(k)}(u) - 2n\sinh u\,\mathcal{A}_{n-1}^{(k)}(u) &= e^{nu}\left(\frac{d}{du}a_n^{(k)}(u) + na_n^{(k)}(u) - n(1-e^{-2u})a_{n-1}^{(k)}(u)\right) \\ &= \frac{1}{2}\mathcal{A}_n^{(k-1)}(u).\end{aligned} \tag{5.20}$$

Thus we obtain the

Lemma 5.5. *The equation*

$$2\tanh\frac{u}{2}\frac{d}{du}\mathcal{B}_n^{(k)}(u) = 8n\mathcal{B}_{n-1}^{(k)}(u) - (2n+1)\mathcal{B}_n^{(k)}(u) + \tanh\frac{u}{2}\,\mathcal{B}_n^{(k-1)}(u) \tag{5.21}$$

holds for $n \geq 1$. □

Remark 5.6. The differential of $a_0^{(k)}(u)$ is given by

$$\frac{d}{du}a_0^{(k)}(u) = \frac{u^{k-2}}{(k-2)!}$$

when $k \geq 2$. If $k \geq 3$, this is equal to $a_0^{(k-1)}(u)$.

We also see from (5.18) that

$$\begin{aligned}&n\mathcal{A}_n^{(k)}(u) - 2(2n-1)\cosh u\mathcal{A}_{n-1}^{(k)}(u) + 4(n-1)\sinh^2 u\mathcal{A}_{n-2}^{(k)}(u) \\ &= e^{nu}\tilde{b}_n^{(k-1)}(u) = \frac{1}{4n}\mathcal{A}_n^{(k-2)}(u).\end{aligned} \tag{5.22}$$

This implies the

Lemma 5.7. *The equation*

$$n\left(1-\frac{1}{\cosh^2\frac{u}{2}}\right)\mathcal{B}_n^{(k)}(u)=4(2n-1)\mathcal{B}_{n-1}^{(k)}(u)-\frac{2(2n-1)}{\cosh^2\frac{u}{2}}\mathcal{B}_{n-1}^{(k)}(u)$$
$$-16(n-1)\mathcal{B}_{n-2}^{(k)}(u)+\frac{1}{4n}\left(1-\frac{1}{\cosh^2\frac{u}{2}}\right)\mathcal{B}_n^{(k-2)}(u) \quad (5.23)$$

holds for $n\geq 2$. □

5.4 Recurrence formula for $J_k(n)$

Define

$$K_k(n)=\frac{1}{2^{2n+1}}\int_0^\infty\frac{\mathcal{B}_n^{(k)}(u)}{\cosh^2\frac{u}{2}}du,\quad M_k(n)=\frac{1}{2^{2n+1}}\int_0^\infty\tanh\frac{u}{2}\,\mathcal{B}_n^{(k-1)}(u)\,du. \quad (5.24)$$

By integrating (5.21) and (5.23), we have

$$K_k(n)=(2n+1)J_k(n)-2nJ_k(n-1)-M_k(n), \quad (5.25)$$
$$2n(J_k(n)-K_k(n))=(2n-1)(2J_k(n-1)-K_k(n-1))-2(n-1)J_k(n-2)$$
$$+\frac{1}{2n}(J_{k-2}(n)-K_{k-2}(n)). \quad (5.26)$$

Plugging these equations, we obtain

Lemma 5.8. *Put*

$$L_k(n):=J_{k-2}(n)-J_{k-2}(n-1)+2nM_k(n)-(2n-1)M_k(n-1)-\frac{1}{2n}M_{k-2}(n). \quad (5.27)$$

The recurrence formula

$$4n^2J_k(n)-(8n^2-8n+3)J_k(n-1)+4(n-1)^2J_k(n-2)=L_k(n) \quad (5.28)$$

holds for $k\geq 2$ *and* $n\geq 2$. □

When $k=2$, the inhomogeneous term $L_2(n)$ in (5.28) vanishes and we get (3.1). When $k=3$, we see that $L_3(n)=2nM_3(n)-(2n-1)M_3(n-1)$, which is equal to $\frac{2^n(n-1)!}{(2n-1)!!}$ (Lemma 6.3 in [9]), so we have (3.2).

5.5 Calculation of the inhomogeneous terms

Let us put

$$Q_k(n):=\frac{1}{2^{2n+1}}\int_0^\infty\frac{\mathcal{B}_n^{(k)}(u)}{\tanh\frac{u}{2}}du. \quad (5.29)$$

This definite integral converges if $k\geq 3$.

From (5.21), we have

$$2\frac{d}{du}\mathcal{B}_n^{(k)}(u) = 8n\frac{\mathcal{B}_{n-1}^{(k)}(u)}{\tanh\frac{u}{2}} - (2n+1)\frac{\mathcal{B}_n^{(k)}(u)}{\tanh\frac{u}{2}} + \mathcal{B}_n^{(k-1)}(u).$$

It follows then

$$0 = 8n\cdot 2^{2n-1}Q_k(n-1) - (2n+1)2^{2n+1}Q_k(n) + 2^{2n+1}J_{k-1}(n),$$

and hence

$$J_{k-1}(n) = (2n+1)Q_k(n) - 2nQ_k(n-1) \tag{5.30}$$

for $k \geq 3$ and $n \geq 1$.

From (5.22), we also see that

$$n\tanh\frac{u}{2}\,\mathcal{B}_n^{(k)}(u) - 2(2n-1)\left(\frac{1}{\tanh\frac{u}{2}} + \tanh\frac{u}{2}\right)\mathcal{B}_{n-1}^{(k)}(u) + 16(n-1)\frac{\mathcal{B}_{n-2}^{(k)}(u)}{\tanh\frac{u}{2}} = \frac{1}{4n}\tanh\frac{u}{2}\,\mathcal{B}_n^{(k-2)}(u).$$

Thus we have

$$n2^{2n+1}M_{k+1}(n) - 2(2n-1)2^{2n-1}\left(Q_k(n-1) + M_{k+1}(n-1)\right) + 16(n-1)2^{2n-3}Q_k(n-2) = \frac{1}{4n}2^{2n+1}M_{k-1}(n),$$

which implies

$$\begin{aligned} &2nM_{k+1}(n) - (2n-1)M_{k+1}(n-1) - \frac{1}{2n}M_{k-1}(n) \\ &= (2n-1)Q_k(n-1) - 2(n-1)Q_k(n-2) \end{aligned} \tag{5.31}$$

for $k \geq 3$ and $n \geq 2$.

Using (5.30) and (5.31), we obtain

$$\begin{aligned} &2nM_k(n) - (2n-1)M_k(n-1) - \frac{1}{2n}M_{k-2}(n) \\ &= (2n-1)Q_{k-1}(n-1) - 2(n-1)Q_{k-1}(n-2) = J_{k-2}(n-1) \end{aligned}$$

for $k \geq 4$ and $n \geq 2$. Hence the inhomogeneous term is computed as

$$L_k(n) = J_{k-2}(n) - J_{k-2}(n-1) + J_{k-2}(n-1) = J_{k-2}(n) \tag{5.32}$$

for $k \geq 4$ and $n \geq 2$. This completes the proof of Theorem 4.3.

Remark 5.9. It may be "natural" to assume (or interpret) that

$$J_0(n) = 0, \qquad J_1(n) = 2\int_0^1 (1-x^2)^{n-1}dx = \frac{2^n(n-1)!}{(2n-1)!!}$$

and

$$w_0(t) = 0, \qquad \text{“}w_1(t) - w_1(0)\text{”} = \sum_{n=1}^{\infty}\frac{2^n(n-1)!}{(2n-1)!!}t^n = 2t\,{}_2F_1\left(1,1;\frac{3}{2};t\right).$$

Under this convention, Theorem 4.3 and Corollary 4.5 would include the case where $k = 2, 3$.

6 Infinite series expression

We give an infinite series expression of $J_k(n)$. Using it, we prove the equation (4.3).

6.1 Infinite series expression of $J_k(n)$

Let us put

$$f_n(s,t) := \frac{1}{(1-s^2t^2)}\left(\frac{(1-s^4)(1-t^4)}{(1-s^2t^2)^2}\right)^n = (1-s^4)^n(1-t^4)^n(1-s^2t^2)^{-2n-1}.$$

Then we have

$$J_k(n) = 2^k \int_0^1\int_0^1 \cdots \int_0^1 f_n(x_1, x_2 \cdots x_k) dx_1 \cdots dx_k.$$

Since

$$\begin{aligned} f_n(s,t) &= (1-s^4)^n(1-t^4)^n \sum_{l=0}^{\infty} \binom{-2n-1}{l}(-s^2t^2)^l \\ &= \frac{1}{(2n)!}\sum_{l=0}^{\infty}(l+1)_{2n}\, s^{2l}(1-s^4)^n t^{2l}(1-t^4)^n, \end{aligned}$$

it follows that

$$J_k(n) = \frac{2^k}{(2n)!}\sum_{l=0}^{\infty}(l+1)_{2n}\, I_1(l,n)I_{k-1}(l,n).$$

Here $I_p(l,n)$ is given by

$$I_p(l,n) := \int_0^1\int_0^1 \cdots \int_0^1 (u_1\cdots u_p)^{2l}(1-(u_1\cdots u_p)^4)^n\, du_1\cdots du_p.$$

Notice that

$$\begin{aligned} I_1(l,n) &= \int_0^1 u^{2l}(1-u^4)^n du = \frac{4^n n!}{(2l+1)(2l+5)\cdots(2l+4n+1)}, \\ I_p(l,n) &= \sum_{j=0}^{n}(-1)^j\binom{n}{j}\int_0^1\int_0^1\cdots\int_0^1 (u_1\cdots u_p)^{2l+4j}du_1\cdots du_p \\ &= \sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{1}{(2l+4j+1)^p}. \end{aligned}$$

Thus we obtain the expression

$$J_k(n) = \frac{2^k 4^n n!}{(2n)!}\sum_{l=0}^{\infty}\frac{(l+1)_{2n}}{(2l+1)(2l+5)\cdots(2l+4n+1)}\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j}{(2l+4j+1)^{k-1}}. \tag{6.1}$$

6.2 Example: calculation of $J_k(1)$

When $n=1$, we see that

$$J_k(1) = 2\cdot 2^k \sum_{l=0}^{\infty} \frac{(l+1)(l+2)}{(2l+1)(2l+5)} \left(\frac{1}{(2l+1)^{k-1}} - \frac{1}{(2l+5)^{k-1}} \right)$$
$$= 2\cdot 2^k \sum_{l=0}^{\infty} \frac{(l+1)(l+2)\left((2l+5)^{k-1} - (2l+1)^{k-1}\right)}{(2l+1)^k(2l+5)^k}.$$

Using the identity

$$(l+1)(l+2)\left((2l+5)^{k-1} - (2l+1)^{k-1}\right)$$
$$= ((2l+1)(2l+5) - (2l+1) + (2l+5) - 1) \sum_{j=0}^{k-2} (2l+1)^j (2l+5)^{k-2-j},$$

we have

$$J_k(1) = 2\cdot 2^k \sum_{j=0}^{k-2} \Big\{ S(k-j-1, j+1)$$
$$- S(k-j-1, j+2) + S(k-j, j+1) - S(k-j, j+2)) \Big\},$$

where

$$S(\alpha,\beta) := \sum_{l=0}^{\infty} (2l+1)^{-\alpha}(2l+5)^{-\beta}.$$

Since

$$\sum_{j=1}^{k-1} S(j, k-j) = \sum_{l=0}^{\infty}\sum_{j=1}^{k-1} (2l+1)^{-j}(2l+5)^{j-k}$$
$$= \sum_{l=0}^{\infty} \frac{1}{(2l+5)^k} \frac{2l+5}{2l+1} \frac{1 - \left(\frac{2l+5}{2l+1}\right)^{k-1}}{1 - \left(\frac{2l+5}{2l+1}\right)}$$
$$= \frac{1}{4} \sum_{l=0}^{\infty} \left(\frac{1}{(2l+1)^{k-1}} - \frac{1}{(2l+5)^{k-1}} \right) = \frac{1+3^{1-k}}{4},$$

we have

$$J_k(1) = 2^{k+1} \left(\frac{2}{3^{k+1}} + S(k,1) - S(1,k) + S(k+1,1) + S(1,k+1) \right). \tag{6.2}$$

Let us calculate $S(k,1)$ and $S(1,k)$. By the partial fraction expansion

$$\frac{1}{x(x+\alpha)^k} = \frac{1}{\alpha^k}\left(\frac{1}{x} - \frac{1}{x+\alpha} \right) - \sum_{m=2}^{k} \frac{1}{\alpha^{k-m+1}(x+\alpha)^m},$$

we see that

$$\frac{1}{(2l+1)^k(2l+5)}=-\left(-\frac{1}{4}\right)^k\left(\frac{1}{2l+1}-\frac{1}{2l+5}\right)+\frac{1}{2^k}\sum_{m=2}^{k}\left(-\frac{1}{2}\right)^{k-m+2}\frac{1}{(l+\frac{1}{2})^m},$$
$$\frac{1}{(2l+1)(2l+5)^k}=\left(\frac{1}{4}\right)^k\left(\frac{1}{2l+1}-\frac{1}{2l+5}\right)+\frac{1}{2^k}\sum_{m=2}^{k}\left(\frac{1}{2}\right)^{k-m+2}\frac{1}{(l+2+\frac{1}{2})^m}.$$

Thus it follows that

$$S(k,1)=\frac{1}{2^k}\sum_{m=2}^{k}\left(-\frac{1}{2}\right)^{k-m+2}\zeta\Big(m,\frac{1}{2}\Big)+\frac{1}{3}\left(-\frac{1}{4}\right)^{k-1},$$
$$S(1,k)=-\frac{1}{2^k}\sum_{m=2}^{k}\left(\frac{1}{2}\right)^{k-m+2}\zeta\Big(m,\frac{1}{2}\Big)+\frac{1}{3}\left(\frac{1}{4}\right)^{k-1}+\frac{1}{3}+\frac{1}{3^k}.$$

If we substitute these to (6.2), then we have

$$\begin{aligned}J_k(1)=2^{k+1}\Bigg(&\frac{2}{3^{k+1}}+\frac{1}{2^k}\sum_{m=2}^{k}\left(-\frac{1}{2}\right)^{k-m+2}\zeta\Big(m,\frac{1}{2}\Big)+\frac{1}{3}\left(-\frac{1}{4}\right)^{k-1}\\&+\frac{1}{2^k}\sum_{m=2}^{k}\left(\frac{1}{2}\right)^{k-m+2}\zeta\Big(m,\frac{1}{2}\Big)+\frac{1}{3}\left(\frac{1}{4}\right)^{k-1}-\frac{1}{3}-\frac{1}{3^k}\\&+\frac{1}{2^{k+1}}\sum_{m=2}^{k+1}\left(-\frac{1}{2}\right)^{k-m+3}\zeta\Big(m,\frac{1}{2}\Big)+\frac{1}{3}\left(-\frac{1}{4}\right)^{k}\\&-\frac{1}{2^{k+1}}\sum_{m=2}^{k+1}\left(\frac{1}{2}\right)^{k-m+3}\zeta\Big(m,\frac{1}{2}\Big)-\frac{1}{3}\left(\frac{1}{4}\right)^{k}+\frac{1}{3}+\frac{1}{3^{k+1}}\Bigg).\end{aligned}$$

Now it is straightforward to see that

$$\begin{aligned}J_k(1)&=3\sum_{m=2}^{k}\frac{1+(-1)^{k-m}}{2^{k-m+3}}\zeta\Big(m,\frac{1}{2}\Big)+\frac{1+(-1)^{k-1}}{2^{k-1}}\\&=\frac{3}{4}\sum_{\substack{2\le m\le k\\2\mid k-m}}\frac{2^m-1}{2^{k-m}}\zeta(m)+\frac{1+(-1)^{k-1}}{2^{k-1}}\\&=\frac{3}{4}\sum_{m=0}^{\lfloor k/2\rfloor-1}2^{-2m}\zeta\Big(k-2m,\frac{1}{2}\Big)+\frac{1-(-1)^k}{2^{k-1}}.\end{aligned}$$

7 Differential equations for generating functions

Utilizing the differential equations for the generating functions $w_k(t)$, we give another kind of relations among the generalized Apéry-like numbers $J_k(n)$.

7.1 Equivalent differential equations

Consider the inhomogeneous (singly confluent) Heun differential equation

$$\mathcal{D}_{\mathrm{H}} w(t) = u(t)$$

for a given function $u(t)$. Put $z = \frac{t}{t-1}$ and $v(z) = (1-t)w(t)$. Then we have

$$\mathcal{D}_{\mathrm{O}} v(z) = \frac{1}{z-1} u\left(\frac{z}{z-1}\right).$$

Here $\mathcal{D}_{\mathrm{O}}$ is the *hypergeometric* differential operator given by

$$\mathcal{D}_{\mathrm{O}} = z(1-z)\frac{d^2}{dz^2} + (1-2z)\frac{d}{dz} - \frac{1}{4}.$$

We also remark that this is also the Picard-Fuchs differential operator for the family $y^2 = x(x-1)(x-z)$ of elliptic curves.

7.2 Recurrence formula for $J_k(n)$

Put $z = \frac{t}{t-1}$ and $v_k(z) = (1-t)w_k(t)$. By Theorem 4.3, $v_k(z)$ satisfies the differential equation

$$\begin{aligned}(\mathcal{D}_{\mathrm{O}} v)(z) &= \frac{1}{4(z-1)} \sum_{j=0}^{\infty} J_{k-2}(j+1) \left(\frac{z}{z-1}\right)^j \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{4} \sum_{j=0}^{n} (-1)^{j+1} \binom{n}{j} J_{k-2}(j+1) \right) z^n.\end{aligned} \tag{7.1}$$

The polynomial functions

$$p_n(z) := -\frac{4}{(2n+1)^2} \binom{-\frac{1}{2}}{n}^{-2} \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k}^2 z^k \tag{7.2}$$

satisfy the equation

$$(\mathcal{D}_{\mathrm{O}} p_n)(z) = z^n. \tag{7.3}$$

Hence we can construct a local holomorphic solution to (7.1) as

$$v(z) = \sum_{n=0}^{\infty} \left(\frac{1}{4} \sum_{j=0}^{n} (-1)^{j+1} \binom{n}{j} J_{k-2}(j+1) \right) p_n(z). \tag{7.4}$$

Notice that the difference $v_k(z) - v(z)$ satisfies the homogeneous differential equation

$$(\mathcal{D}_{\mathrm{O}}(v_k - v))(z) = 0. \tag{7.5}$$

Thus it follows that

$$v_k(z) - v(z) = C_k v_2(z), \tag{7.6}$$

where the constant C_k is determined by

$$C_k = \frac{v_k(0) - v(0)}{v_2(0)} = \frac{(2^k - 1)\zeta(k) - v(0)}{3\zeta(2)}, \tag{7.7}$$

and $v(0)$ is given by

$$v(0) = -\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \binom{-\frac{1}{2}}{n}^{-2} \sum_{j=0}^{n} (-1)^{j+1} \binom{n}{j} J_{k-2}(j+1). \tag{7.8}$$

Therefore we have

$$v_k(z) = (2^k - 1)\zeta(k)\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) + \left(v(z) - v(0)\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right)\right). \tag{7.9}$$

Consequently, we obtain the

Theorem 7.1. *When $k \geq 4$, the equation*

$$\begin{aligned} J_k(n) = &\sum_{p=0}^{n} (-1)^p \binom{-\frac{1}{2}}{p}^2 \binom{n}{p} \\ &\times \left((2^k - 1)\zeta(k) - \sum_{i=0}^{p-1} \frac{1}{(2i+1)^2} \binom{-\frac{1}{2}}{i}^{-2} \sum_{j=0}^{i} (-1)^j \binom{i}{j} J_{k-2}(j+1) \right) \end{aligned} \tag{7.10}$$

holds. □

Remark 7.2. If we *formally* put $J_1(n) = \frac{2^n (n-1)!}{(2n-1)!!}$ in (7.10), then we have

$$J_3(n) = \sum_{p=0}^{n} (-1)^p \binom{-\frac{1}{2}}{p}^2 \binom{n}{p} \left(7\zeta(3) - 2\sum_{i=0}^{p-1} \frac{1}{(2i+1)^3} \binom{-\frac{1}{2}}{i}^{-2} \right)$$

since

$$\sum_{j=0}^{i} (-1)^j \binom{i}{j} \frac{2^{j+1} j!}{(2j+1)!!} = \frac{2}{2i+1}.$$

This is nothing but the explicit formula (3.9) for $J_3(n)$.

Example 7.3. Since

$$\begin{aligned} \sum_{j=0}^{i} (-1)^j \binom{i}{j} J_2(j+1) &= 3\zeta(2) \left(\binom{-\frac{1}{2}}{i}^2 - \binom{-\frac{1}{2}}{i+1}^2 \right) \\ &= 3\zeta(2) \binom{-\frac{1}{2}}{i}^2 \left(1 - \frac{(2i+1)^2}{(2i+2)^2} \right), \end{aligned}$$

we have

$$J_4(n) = \sum_{p=0}^{n} (-1)^p \binom{-\frac{1}{2}}{p}^2 \binom{n}{p} \left(15\zeta(4) - 3\zeta(2) \sum_{i=1}^{2p} \frac{(-1)^{i-1}}{i^2} \right).$$

7.3 Normalized generalized Apéry-like numbers

For a given sequence $\{J(n)\}_{n\geq 0}$, we associate a new sequence

$$J(n)^{\sharp} := \sum_{p=0}^{n}(-1)^p\binom{-\frac{1}{2}}{p}^2\binom{n}{p}\left\{\sum_{i=0}^{p-1}\frac{-1}{(2i+1)^2}\binom{-\frac{1}{2}}{i}^{-2}\sum_{j=0}^{i}(-1)^j\binom{i}{j}J(j+1)\right\}. \tag{7.11}$$

Notice that $J(0)^{\sharp} = 0$. It would be natural to extend $J(n)^{\sharp} = 0$ if $n < 0$. By the discussion in the previous subsection, we have the

Lemma 7.4. *Let $\{J(n)\}$ be a given sequence and $\{J(n)^{\sharp}\}$ the one defined by (7.11). Then the equation*

$$4n^2J(n)^{\sharp} - (8n^2-8n+3)J(n-1)^{\sharp} + 4(n-1)^2J(n-2)^{\sharp} = J(n) \tag{7.12}$$

holds for $n \geq 1$.

Let us introduce the *rational* sequences $\tilde{J}_k(n)$ by

$$\tilde{J}_1(n) := \frac{2^n(n-1)!}{(2n-1)!!} \quad (n\geq 1), \qquad \tilde{J}_2(n) := \frac{J_2(n)}{J_2(0)} \quad (n \geq 0),$$
$$\tilde{J}_k(n) := \tilde{J}_{k-2}(n)^{\sharp} \quad (k\geq 3,\ n\geq 0).$$

We see that

$$\tilde{J}_{2k}(1) = \frac{3}{4^k}, \quad \tilde{J}_{2k+1}(1) = \frac{2}{4^k}. \tag{7.13}$$

It is immediate to verify the

Proposition 7.5.

$$J_k(n) = \sum_{m=0}^{\lfloor k/2\rfloor -1}\zeta\Big(k-2m,\frac{1}{2}\Big)\tilde{J}_{2m+2}(n) + \frac{1-(-1)^k}{2}\tilde{J}_k(n). \tag{7.14}$$

□

Based on this fact, we call $\tilde{J}_k(n)$ the *normalized (generalized) Apéry-like numbers.* By definition, $\tilde{J}_k(n)$ for $k \geq 2$ are written in the form

$$\tilde{J}_k(n) = \sum_{p=0}^{n}(-1)^p\binom{-\frac{1}{2}}{p}^2\binom{n}{p}S_k(p), \tag{7.15}$$

where

$$S_2(p) = 1, \qquad S_3(p) = -2\sum_{i=0}^{p-1}\frac{1}{(2i+1)^3}\binom{-\frac{1}{2}}{i}^{-2} = -2\sum_{i=0}^{p-1}\frac{(1/2)_i(1)_i^3}{(3/2)_i^3}\frac{1^i}{i!},$$
$$S_k(p) = \sum_{i=0}^{p-1}\frac{-1}{(2i+1)^2}\binom{-\frac{1}{2}}{i}^{-2}\sum_{j=0}^{i}(-1)^j\binom{i}{j}J_{k-2}(j+1) \qquad (k\geq 4).$$

Thus it is enough to investigate $S_k(p)$ to obtain an explicit expression for normalized Apéry-like numbers.

Lemma 7.6.

$$S_{k+2}(p+1) - S_{k+2}(p) = \frac{S_k(p+1)}{(2p+2)^2} - \frac{S_k(p)}{(2p+1)^2}. \tag{7.16}$$

Proof. By definition, we have

$$S_{k+2}(p+1) - S_{k+2}(p) = \frac{-1}{(2p+1)^2}\binom{-\frac{1}{2}}{p}^{-2}\sum_{j=0}^{p}(-1)^j\binom{p}{j}\tilde{J}_k(j+1). \tag{7.17}$$

The sum in the right hand side is calculated as

$$\begin{aligned}
&\sum_{j=0}^{p}(-1)^j\binom{p}{j}\tilde{J}_k(j+1)\\
&=\sum_{j=0}^{p}(-1)^j\binom{p}{j}\left(\sum_{q=0}^{j}(-1)^q\binom{-\frac{1}{2}}{q}^2\binom{j+1}{q}S_k(q)+(-1)^{j+1}\binom{-\frac{1}{2}}{j+1}^2S_k(j+1)\right)\\
&=\sum_{q=0}^{p}(-1)^q\binom{-\frac{1}{2}}{q}^2S_k(q)\sum_{j=q}^{p}(-1)^j\binom{p}{j}\binom{j+1}{q}-\sum_{j=0}^{p}\binom{p}{j}\binom{-\frac{1}{2}}{j+1}^2S_k(j+1).
\end{aligned}$$

By the elementary identity

$$\sum_{j=p}^{n}(-1)^j\binom{n}{j}\binom{j}{p}=(-1)^p\delta_{np}\quad(n,p\in\mathbb{Z}_{\geq 0}),$$

we get

$$\begin{aligned}
\sum_{j=q}^{p}(-1)^j\binom{p}{j}\binom{j+1}{q}&=\sum_{j=q}^{p}(-1)^j\binom{p}{j}\binom{j}{q}+\sum_{j=q}^{p}(-1)^j\binom{p}{j}\binom{j}{q-1}\\
&=(-1)^q\left(\delta_{pq}+\binom{p}{q-1}\right).
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
\sum_{j=0}^{p}(-1)^j\binom{p}{j}\tilde{J}_k(j+1)&=\binom{-1/2}{p}^2S_k(p)+\sum_{q=0}^{p}\binom{-\frac{1}{2}}{q}^2S_k(q)\binom{p}{q-1}\\
&\quad-\sum_{j=0}^{p}\binom{p}{j}\binom{-\frac{1}{2}}{j+1}^2S_k(j+1)\\
&=\binom{-\frac{1}{2}}{p}^2S_k(p)-\binom{-\frac{1}{2}}{p+1}^2S_k(p+1)\\
&=(2p+1)^2\binom{-\frac{1}{2}}{p}^2\left(\frac{S_k(p)}{(2p+1)^2}-\frac{S_k(p+1)}{(2p+2)^2}\right).
\end{aligned}$$

Therefore we obtain

$$S_{k+2}(p+1) - S_{k+2}(p) = \frac{S_k(p+1)}{(2p+2)^2} - \frac{S_k(p)}{(2p+1)^2}$$

as we desired. □

As a corollary, we readily have the

Lemma 7.7.

$$S_{k+2}(p) = \sum_{q=1}^{p} \left(\frac{S_k(q)}{(2q)^2} - \frac{S_k(q-1)}{(2q-1)^2} \right). \tag{7.18}$$

□

Using this lemma repeatedly, we obtain the

Proposition 7.8. *For each $r \geq 1$,*

$$S_{2r+2}(p) = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq 2p} \frac{(-1)^{i_1+\cdots+i_r}}{i_1^2 \ldots i_r^2} \varepsilon_{i_1,\ldots,i_r}, \tag{7.19}$$

$$S_{2r+3}(p) = \sum_{1 \leq 2j-1 < i_1 \leq \cdots \leq i_r \leq 2p} \frac{1}{(2j-1)^3} \binom{-\frac{1}{2}}{j-1}^{-2} \frac{(-1)^{i_1+\cdots+i_r}}{i_1^2 \ldots i_r^2} \varepsilon_{i_1,\ldots,i_r}, \tag{7.20}$$

where

$$\varepsilon_{i_1,\ldots,i_r} := \begin{cases} 0 & 1 \leq \exists j < r \text{ s.t. } i_j = i_{j+1} \equiv 1 \pmod 2, \\ 1 & \text{otherwise.} \end{cases} \tag{7.21}$$

□

Example 7.9. We have

$$S_4(p) = \sum_{j=1}^{2p} \frac{(-1)^j}{j^2},$$

$$S_6(p) = \sum_{1 \leq i \leq j \leq 2p} \frac{(-1)^{i+j}}{i^2 j^2} \varepsilon_{i,j} = \sum_{1 \leq i < j \leq 2p} \frac{(-1)^{i+j}}{i^2 j^2} + \sum_{i=1}^{p} \frac{1}{(2i)^4},$$

$$S_8(p) = \sum_{1 \leq i \leq j \leq k \leq 2p} \frac{(-1)^{i+j+k}}{i^2 j^2 k^2} \varepsilon_{i,j,k}$$

$$= \sum_{1 \leq i < j < k \leq 2p} \frac{(-1)^{i+j+k}}{i^2 j^2 k^2} + \left(\sum_{1 \leq 2i \leq 2p} \frac{1}{(2i)^4} \right) \left(\sum_{1 \leq k \leq 2p} \frac{(-1)^k}{k^2} \right).$$

Remark 7.10. We see that

$$\lim_{p\to\infty} S_2(p) = 1, \quad \lim_{p\to\infty} S_4(p) = -\frac{\pi^2}{12}, \quad \lim_{p\to\infty} S_6(p) = -\frac{\pi^4}{720}. \tag{7.22}$$

In general, we can prove that

$$\lim_{p\to\infty} S_{2r+2}(p) = -\frac{\zeta(2r)}{2^{2r-1}}. \tag{7.23}$$

See [17] for the proof as well as its generalizations.

8 Congruence relations among Apéry-like numbers

In this section, we study the congruence relation among the *normalized* Apéry-like numbers introduced in the previous section.

8.1 Congruence relations for Apéry-like numbers

We give several congruence relations among Apéry-like numbers.

Proposition 8.1 ([13, Proposition 6.1]). *Let p be a prime and $n = n_0+n_1p+\cdots+n_kp^k$ be the p-ary expansion of $n \in \mathbb{Z}_{\geq 0}$ $(0 \leq n_j < p)$. Then it holds that*

$$\tilde{J}_2(n) \equiv \prod_{j=0}^{k} \tilde{J}_2(n_j) \pmod{p}.$$

The following claim is regarded as an analog of Proposition 2.2.

Proposition 8.2 ([13, Theorem 6.2]). *For any odd prime p and positive integers m, r, the congruence relation*

$$\begin{aligned} \tilde{J}_2(mp^r) &\equiv \tilde{J}_2(mp^{r-1}) \pmod{p^r}, \\ \tilde{J}_3(p^r)p^{3r} &\equiv \tilde{J}_3(p^{r-1})p^{3(r-1)} \pmod{p^r}. \end{aligned}$$

holds.

Proposition 8.3. *For any odd prime p, the congruence relation*

$$\sum_{n=0}^{p-1} \tilde{J}_2(n) \equiv 0 \pmod{p^2} \tag{8.1}$$

holds.

Proof. We see that

$$\begin{aligned} \sum_{n=0}^{p-1} \tilde{J}_2(n) &= \sum_{n=0}^{p-1}\sum_{j=0}^{n}(-1)^j 16^{-j}\binom{2j}{j}^2\binom{n}{j} = \sum_{j=0}^{p-1}(-1)^j 16^{-j}\binom{2j}{j}^2\sum_{n=j}^{p-1}\binom{n}{j} \\ &= \sum_{j=0}^{p-1}(-1)^j 16^{-j}\binom{2j}{j}^2\binom{p}{j+1} \equiv p\sum_{j=0}^{\frac{p-1}{2}} 16^{-j}\binom{2j}{j}^2\binom{p-1}{j}\frac{(-1)^j}{j+1} \\ &\equiv p\sum_{j=0}^{\frac{p-1}{2}} 16^{-j}\binom{2j}{j}^2\frac{1}{j+1} \pmod{p^2} \end{aligned}$$

since $\binom{2j}{j}^2$ is divisible by p^2 if $\frac{p-1}{2} < j < p$. Notice that

$$16^{-j}\binom{2j}{j}^2 \equiv (-1)^j\binom{\frac{p-1}{2}+j}{j}\binom{\frac{p-1}{2}}{j} \pmod{p}$$

for $0 \le j < p$. Hence we have

$$\sum_{n=0}^{p-1} \tilde{J}_2(n) \equiv p \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}+j}{j} \binom{\frac{p-1}{2}}{j} \frac{(-1)^j}{j+1} \pmod{p^2}.$$

By putting $n = \frac{p-1}{2}$ and $m = 0$ in the identity (see [7, Chapter 5.3])

$$\sum_{k \ge 0} \binom{n+k}{k} \binom{n}{k} \frac{(-1)^k}{k+1+m} = (-1)^n \frac{m!n!}{(m+n+1)!} \binom{m}{n}, \tag{8.2}$$

we have

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}+j}{j} \binom{\frac{p-1}{2}}{j} \frac{(-1)^j}{j+1} = \frac{(-1)^{\frac{p-1}{2}}}{(\frac{p+1}{2})!} \binom{0}{\frac{p-1}{2}} = 0. \tag{8.3}$$

Hence we obtain the desired conclusion. □

Proposition 8.4. *For each odd prime p, it holds that*

$$\tilde{J}_2(\tfrac{p-1}{2}) \equiv A_2(\tfrac{p-1}{2}) \pmod{p^2}. \tag{8.4}$$

Here $A_2(n)$ is the Apéry number for $\zeta(2)$.

Proof. It is elementary to check that

$$\binom{\frac{p-1}{2}}{k} \equiv \binom{-\frac{1}{2}}{k} \left\{ 1 - p \sum_{j=1}^{k} \frac{1}{2j-1} \right\} \pmod{p^2},$$

$$\binom{\frac{p-1}{2}+k}{k} \equiv (-1)^k \binom{-\frac{1}{2}}{k} \left\{ 1 + p \sum_{j=1}^{k} \frac{1}{2j-1} \right\} \pmod{p^2}$$

for $k = 0, 1, \ldots, (p-1)/2$. Using these equations, we easily see that both $A_2(\frac{p-1}{2})$ and $\tilde{J}_2(\frac{p-1}{2})$ are congruent to

$$\sum_{k=0}^{(p-1)/2} (-1)^k \binom{-\frac{1}{2}}{k}^3 \left\{ 1 - p \sum_{j=1}^{k} \frac{1}{2j-1} \right\}$$

modulo p^2. □

Remark 8.5. The following *supercongruence*

$$A_2(\tfrac{p-1}{2}) \equiv \lambda_p \pmod{p^2}$$

holds if p is a prime larger than 3 (see [10]; see also [20, 32]).

The following result is conjectured in [13].

Theorem 8.6 (Long-Osburn-Swisher [18]). *For any odd prime p, the congruence relation*

$$\sum_{n=0}^{p-1} \tilde{J}_2(n)^2 \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3} \tag{8.5}$$

holds.

Remark 8.7. The theorem above is quite similar to the Rodriguez-Villegas-type congruence due to Mortenson [19]

$$\sum_{n=0}^{p-1} \binom{-\frac{1}{2}}{n}^2 = \sum_{n=0}^{p-1} \binom{2n}{n}^2 16^{-n} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^2}. \tag{8.6}$$

We also remark that the following "very similar" congruence relation is also obtained in the earlier work [24]:

$$\sum_{n=0}^{(p-1)/2} \binom{2n}{n}^2 16^{-n} + \frac{3}{8}p(-1)^{(p-1)/2} \sum_{i=1}^{(p-1)/2} \binom{2i}{i} \frac{1}{i} \equiv \left(\frac{-1}{p}\right) \pmod{p^3}, \tag{8.7}$$

where p is an arbitrary odd prime number.

8.2 Conjectures

In the final position, we give several conjectures on congruence relations among normalized (generalized) Apéry-like numbers.

The following conjecture is regarded as a "true" analog of Proposition 2.2:

Conjecture 8.8 (Kimoto-Osburn [12]). *For any odd prime p, the congruence relation*

$$\tilde{J}_2(mp^r - 1) \equiv (-1)^{\frac{p-1}{2}} \tilde{J}_2(mp^{r-1} - 1) \pmod{p^r} \tag{8.8}$$

holds for any integers $m, r \geq 1$.

Remark 8.9. When $r = 1$, (8.8) is obtained by using the elementary formulas

$$\binom{-\frac{1}{2}}{kp+j} \equiv \binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{j} \pmod{p},$$

$$(-1)^n \binom{mp^r - 1}{n} \equiv (-1)^{\lfloor \frac{n}{p} \rfloor} \binom{mp^{r-1} - 1}{\lfloor \frac{n}{p} \rfloor} \pmod{p^r}$$

and Mortenson's result (8.6) as follows:

$$\begin{aligned}
\tilde{J}_2(mp-1) &= \sum_{j=0}^{mp-1} (-1)^j \binom{-\frac{1}{2}}{j}^2 \binom{mp-1}{j} = \sum_{j=0}^{p-1}\sum_{k=0}^{m-1} (-1)^{kp+j} \binom{-\frac{1}{2}}{kp+j}^2 \binom{mp-1}{kp+j} \\
&\equiv \sum_{j=0}^{p-1}\sum_{k=0}^{m-1} (-1)^k \binom{-\frac{1}{2}}{k}^2 \binom{-\frac{1}{2}}{j}^2 \binom{m-1}{k} \pmod{p} \\
&\equiv (-1)^{\frac{p-1}{2}} \tilde{J}_2(m-1) \pmod{p}.
\end{aligned}$$

We have

$$\tilde{J}_2(mp^r-1)=\sum_{j=0}^{p-1}\sum_{k=0}^{mp^{r-1}-1}(-1)^{kp+j}\binom{-\frac{1}{2}}{kp+j}^2\binom{mp^r-1}{kp+j}$$
$$\equiv\sum_{j=0}^{p-1}\sum_{k=0}^{mp^{r-1}-1}(-1)^{k}\binom{-\frac{1}{2}}{kp+j}^2\binom{mp^{r-1}-1}{k}\pmod{p^r}$$

and hence

$$\tilde{J}_2(mp^r-1)-(-1)^{\frac{p-1}{2}}\tilde{J}_2(mp^{r-1}-1)$$
$$\equiv\sum_{k=0}^{mp^{r-1}-1}(-1)^k\left(\sum_{j=0}^{p-1}\binom{-\frac{1}{2}}{kp+j}^2-(-1)^{\frac{p-1}{2}}\binom{-\frac{1}{2}}{k}^2\right)\binom{mp^{r-1}-1}{k}\pmod{p^r}$$

Lemma 8.10.

$$\sum_{j=0}^{p-1}\binom{-\frac{1}{2}}{kp+j}^2\equiv(-1)^{\frac{p-1}{2}}\binom{-\frac{1}{2}}{k}^2\pmod{p^2}.$$

Conjecture 8.11. *For any odd prime p and $m,r\in\mathbb{Z}_{>0}$ with m odd, it holds that*

$$\tilde{J}_2(\tfrac{mp^r-1}{2})-\lambda_p\tilde{J}_2(\tfrac{mp^{r-1}-1}{2})+(-1)^{p(p-1)/2}p^2\tilde{J}_2(\tfrac{mp^{r-2}-1}{2})\equiv 0\pmod{p^r},\tag{8.9}$$

where λ_n is given by

$$\sum_{n=1}^{\infty}\lambda_n q^n=q\prod_{n=1}^{\infty}(1-q^{4n})^6=\eta(4\tau)^6.$$

Further, the congruence (8.9) *holds modulo p^{2r} if $p\geq 5$.*

Notice that (8.4) is a special case of the conjecture above (see [20, 32]). It is remarkable that both $A_2(\frac{mp^r-1}{2})$ and $\tilde{J}_2(\frac{mp^r-1}{2})$ satisfy exactly the same congruence relation ((2.3) and (8.9)), though they are *not* congruent modulo p^r in general.

Conjecture 8.12. *For any odd prime p, the congruence relation*

$$\sum_{n=0}^{p-1}\tilde{J}_{2k}(n)\equiv -1\pmod{p^2}\tag{8.10}$$

holds for any $k\geq 2$.

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