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The K-theory for the C^*-algebras of nilpotent discrete groups by examples

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Dedicated to the memory of Ryuji Maehara

Abstract

We study the K-theory for the group C^*-algebras of some semi-direct products of discrete abelian groups. We especially consider nilpotent semi-direct products of finitely generated free abelian groups and several typical examples such as the discrete Heisenberg group and its subgroups and their higher dimensional generalizations and more other ones.

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1 Introduction

In this paper, we study the K-theory for the group C^*-algebras of some semi-direct products of discrete (free or torsion) abelian groups. We especially consider nilpotent semi-direct products of finitely generated, free abelian groups and several typical examples such as the discrete Heisenberg group of rank three ([1]) and its subgroups with the same rank and their higher dimensional generalizations defined as the generalized discrete Heisenberg groups of odd ranks ([3]) and the generalized (weighted) discrete Heisenberg subgroups of the same ranks, defined this time, but certainly known, and more other typical examples defined as (weighted) shifted ones from those Heisenberg groups and subgroups as applications, and direct products of these and those examples as further applications.

As results, we compute explicitly the K-theory groups of the group C^*-algebras of those nilpotent semi-direct products as examples. In the computation we use the Künneth formula in the K-theory for tensor products of C^*-algebras and the Pimsner-Voiculescu six-term exact sequence for the K-theory groups of crossed products of C^*-algebras by actions of integers and perform the

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diagram chasing to obtain the case by case results, with some rather considerable effort. The full exact and explicit, computations and expositions with some (known or renewed, and new) results obtained as the K-theory group formulae for those group C^*-algebras with more enough information about the K-theory class generators, should become useful and helpful, as a reference for further study in this direction.

As notes, the case of the discrete Heisenberg group of rank three is considered by [1], and its generalization to the generalized discrete Heisenberg groups of rank odd is done by [3]. This time we take a different method from [3] as mentioned above to obtain another result and also consider similarly the case of the generalized discrete Heisenberg subgroups of rank odd, but to obtain another result involving torsion (i.e., with some corrections in the literature as a purpose and as a motivation). Compare to [5] without torsion. Also, some cases of non-nilpotent, solvable discrete groups are considered by [6].

This paper starts with Section 2 Preliminaries, with some basics for convenience to readers, and is then organized of subsequent four sections of: 3 The abelian case, 4 The nilpotent case, 5 More examples, and 6 Further applications, with some basics for convenience.

Refer to textbooks [2] or [8] for some facts in K-theory of C^*-algebras and as well in the general theory of C^*-algebras, used (with or without mentioning) below in the text. Also refer to [4] for computing the K-theory groups of tensor products of C^*-algebras.

A few (early) versions of this paper with slightly different titles have been reviewed, from which it is some improved for this publication.

2 Preliminaries

Let $\mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$ denote the semi-direct product of the additive n-fold direct product group $\mathbb{Z}^n$ of $\mathbb{Z}$ by an action α of the group $\mathbb{Z}$ of integers. Then for $t \in \mathbb{Z}$, $\alpha_t \in GL_n(\mathbb{Z})$ the group of $n \times n$ invertible matrices over $\mathbb{Z}$, with $\alpha_0 = 1_n$ the $n \times n$ identity matrix. Let $\alpha = \alpha_1 = A = (a_{ij})_{i,j=1}^n \in GL_n(\mathbb{Z})$, with the determinant $\det A$ nonzero. Then, for $x = (x_1, \cdots, x_n) \in \mathbb{Z}^n$, also viewed as a column vector $x^{\mathbf{t}}$,

$$\alpha(x) = \alpha_1(x) = Ax^{\mathbf{t}} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n a_{1j}x_j \\ \vdots \\ \sum_{j=1}^n a_{nj}x_j \end{pmatrix} \in \mathbb{Z}^n$$

and $\alpha_s(x) = \alpha^s(x) = A^s x^{\mathbf{t}}$ for $s \in \mathbb{Z}$. For $(x,t), (y,s) \in \mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$, their product is given by

$$(x,t) \cdot (y,s) = (x + \alpha_t(y), t+s) \in \mathbb{Z}^n \rtimes_\alpha \mathbb{Z}.$$

The inverse $(x,t)^{-1}$ for (x,t) is $(\alpha_{-t}(-x), -t)$. We have

$$(0,t) \cdot (x,0) \cdot (0,-t) = (\alpha_t(x), 0).$$

There is the following short exact sequence of groups:

$$0 \to \mathbb{Z}^n \to \mathbb{Z}^n \rtimes_\alpha \mathbb{Z} \to \mathbb{Z} \to 0.$$

Note that $(x,t)\cdot(y,0)\cdot(x,t)^{-1} = (\alpha_t(y),0) \in \mathbb{Z}^n$, and hence $\mathbb{Z}^n$ is a normal subgroup of $\mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$. The sequence splits by a section $s : \mathbb{Z} \to \mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$ given by $s(t) = (0,t)$ for $t \in \mathbb{Z}$. Since a group obtained by an extension by solvable groups, in particular, by abelian groups is solvable, the semi-direct product is always solvable.

Let $C^*(\mathbb{Z}^n \rtimes_\alpha \mathbb{Z})$ denote the group C^*-algebra of $\mathbb{Z}^n \rtimes_\alpha \mathbb{Z} = G$. Let e_j $(1 \le j \le n)$ be the canonical coordinate generators of the group $\mathbb{Z}^n$, with $e_1 = (1,0,\cdots,0),\cdots,e_n = (0,\cdots,0,1)$ in $\mathbb{Z}^n$. Let $e_{n+1} = ((0,\cdots,0),1) = 1$ be the generator of $\mathbb{Z}$ in G. Let u_j $(1 \le j \le n)$, v be unitaries corresponding to e_j $(1 \le j \le n)$, e_{n+1}, on the Hilbert space $l^2(\mathbb{Z}^n \rtimes_\alpha \mathbb{Z})$ of all square summable, complex valued functions on G, via the left regular representation λ. Namely, for $\xi \in l^2(\mathbb{Z}^n \rtimes_\alpha \mathbb{Z})$,

$$\begin{aligned} u_j\xi(x,t) &= \lambda_{e_j}\xi(x,t) = \xi(e_j^{-1}(x,t)), \\ v\xi(x,t) &= \lambda_{e_{n+1}}\xi(x,t) = \xi(e_{n+1}^{-1}(x,t)) \in \mathbb{C}, \end{aligned}$$

and also

$$\lambda_{(x,t)\cdot(y,s)}\xi(z,l) = \xi((y,s)^{-1}(x,t)^{-1}(z,l)) = \lambda_{(x,t)}(\lambda_{(y,s)}\xi)(z,l) \in \mathbb{C}.$$

Since G is solvable, and thus amenable, then the group C^*-algebra $C^*(\mathbb{Z}^n \rtimes_\alpha \mathbb{Z})$ is viewed as the C^*-algebra generated by the unitaries u_j $(1 \le j \le n)$ and v such that u_j are mutually commuting and

$$\begin{aligned} vu_jv^*\xi(x,t) &= \xi(e_{n+1}e_j^{-1}e_{n+1}^{-1}(x,t)) \\ &= \lambda_{e_{n+1}e_je_{n+1}^{-1}}\xi(x,t) = \lambda_{\alpha_1(e_j)}\xi(x,t), \end{aligned}$$

so that $vu_jv^* = \lambda_{\alpha_1(e_j)} \equiv \alpha_1(u_j) \equiv \alpha(u_j)$. This definition implies an action α (the same symbol) of $\mathbb{Z}$ on the C^*-algebra $C^*(\mathbb{Z}^n)$ generated by u_j $(1 \le j \le n)$. Therefore, the group C^*-algebra $C^*(\mathbb{Z}^n \rtimes_\alpha \mathbb{Z})$ is viewed as the crossed product C^*-algebra $C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}$ with α the induced action.

We now denote by $K_0(\mathfrak{A})$ and $K_1(\mathfrak{A})$ the K-theory groups of a unital (or non-unital) C^*-algebra $\mathfrak{A}$. Note that $K_0(\mathfrak{A})$ is an additive abelian group of (stably) equivalence classes of projections of matrix algebras over $\mathfrak{A}$ unital, while $K_1(\mathfrak{A})$ is a multiplicative abelian group (but viewed as an additive group) of (unitary) equivalence classes of unitaries of matrix algebras over $\mathfrak{A}$ unital, in general.

There is the following **Pimsner-Voiculescu six-term exact sequence** of K-theory groups ([2] or [8]):

$$\begin{array}{ccccc} K_0(C^*(\mathbb{Z}^n)) & \xrightarrow{(\mathrm{id}-\alpha)_*} & K_0(C^*(\mathbb{Z}^n)) & \xrightarrow{i_*} & K_0(C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1(C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}) & \xleftarrow{i_*} & K_1(C^*(\mathbb{Z}^n)) & \xleftarrow{(\mathrm{id}-\alpha)_*} & K_1(C^*(\mathbb{Z}^n)) \end{array}$$

where id is the identity map on $C^*(\mathbb{Z}^n)$ and i is the inclusion map from $C^*(\mathbb{Z}^n)$ to $C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}$, and $(\mathrm{id}-\alpha)_*$ and i_* are induced map on K-theory groups, and the up and down arrows ∂ are index and exponential maps respectively. Since the unitaries u_j are mutually commuting, $C^*(\mathbb{Z}^n)$ is isomorphic to the tensor product $\otimes^n C^*(\mathbb{Z})$ and is isomorphic to $\otimes^n C(\mathbb{T}) \cong C(\mathbb{T}^n)$ the C^*-algebra of all complex-valued, continuous functions on the n-dimensional torus $\mathbb{T}^n$, via the Fourier transform, we then have

$$K_j(C^*(\mathbb{Z}^n)) \cong K_j(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}$$

for $j = 0, 1$ and $n \geq 1$, which can be deduced from the Pimsner-Voiculescu six-term exact sequence since $C^*(\mathbb{Z}^n)$ is viewed as the successive crossed product $C^*(\mathbb{Z}) \rtimes_{\mathrm{id}} \mathbb{Z} \cdots \rtimes_{\mathrm{id}} \mathbb{Z}$ with trivial actions id of $\mathbb{Z}$ (see also the lemma given below).

Furthermore, the group C^*-algebra $C^*(\mathbb{Z}^n)$ is also defined to be the C^*-completion of the Banach $*$-algebra $l^1(\mathbb{Z}^n)$ of all summable functions on $\mathbb{Z}^n$ with convolution product and the l^1-norm as $\|f\|_1 = \sum_{x\in\mathbb{Z}^n} |f(x)|$ for $f \in l^1(\mathbb{Z}^n)$. The **Fourier transform** $\mathfrak{F}$ from Banach $*$-algebra $l^1(\mathbb{Z}^n)$ to the C^*-algebra $C(\mathbb{T}^n)$ with pointwise operations and the supremum norm is a $*$-homomorphism and extends to an $*$-isomorphism from $C^*(\mathbb{Z}^n)$ to $C(\mathbb{T}^n)$. where

$$\mathfrak{F}(f)(z) = \sum_{x\in\mathbb{Z}^n} f(x) z^{-x}$$

for $z = (z_1, \cdots, z_n) \in \mathbb{T}^n$ and $z^{-x} = z_1^{-x_1} \cdots z_n^{-x_n}$. Indeed, $f, g \in l^1(\mathbb{Z})$, their convolution product is defined by

$$(f * g)(x) = \sum_{y\in\mathbb{Z}^n} f(x-y) g(y)$$

and

$$\begin{aligned}\mathfrak{F}(f*g)(z) &= \sum_{x\in\mathbb{Z}^n} (f*g)(x) z^{-x} = \sum_{x\in\mathbb{Z}^n} \left(\sum_{y\in\mathbb{Z}^n} f(x-y) g(y) \right) z^{-x} \\ &= \sum_{y\in\mathbb{Z}^n} g(y) z^{-z} \left(\sum_{x\in\mathbb{Z}^n} f(x-y) z^{-(x-y)} \right) = \mathfrak{F}(f)(z) \cdot \mathfrak{F}(g)(z).\end{aligned}$$

As well, the $*$-operation is defined by $f^*(x) = \overline{f(-x)}$ (complex conjugate) and

$$\mathfrak{F}(f^*)(z) = \sum_{x\in\mathbb{Z}^n} \overline{f(-x)} z^{-x} = \overline{\sum_{x\in\mathbb{Z}^n} f(-x) z^{x}} = \overline{\mathfrak{F}(f)}(z).$$

Then the following commutative diagram is induced:

$$\begin{array}{ccc} C^*(\mathbb{Z}^n) & \xrightarrow{\mathfrak{F}} & C(\mathbb{T}^n) \\ {\scriptstyle \alpha_t}\downarrow & & \downarrow{\scriptstyle \alpha_t^\wedge} \\ C^*(\mathbb{Z}^n) & \xrightarrow{\mathfrak{F}} & C(\mathbb{T}^n) \end{array}$$

where $\alpha_t^\wedge = \mathfrak{F} \circ \alpha_t \circ \mathfrak{F}^{-1}$. Then

$$\alpha_t^\wedge(\mathfrak{F}(f))(z) = (\mathfrak{F} \circ \alpha_t)(f)(z) = \mathfrak{F}(\mathrm{Ad}(v^t)(f))(z) = \sum_{x \in \mathbb{Z}^n} \mathrm{Ad}(v^t)(f)(x) z^{-x},$$

where $\mathrm{Ad}(v^t)(f) = v^t f (v^t)^*$ with $v^t = v \cdots v$ the t-times multiple of v (not the transpose).

In particular, we then have

$$\mathfrak{F}(1)(z) = \mathfrak{F}(\chi_0)(z) = \sum_{x \in \mathbb{Z}^n} \chi_0(x) z^{-x} = z^0 = 1 = 1(z) \in C(\mathbb{T}^n)$$

and that

$$\begin{aligned} \alpha_t^\wedge(1)(z) &= \sum_{x \in \mathbb{Z}^n} \mathrm{Ad}(v^t) \chi_0(x) z^{-x} \\ &= \sum_{x \in \mathbb{Z}^n} \chi_0(x) z^{-x} = z^0 = 1 = 1(z) \in C(\mathbb{T}^n). \end{aligned}$$

Also, since each u_j may be (or not) identified with the characteristic function $\chi_{e_j^{-1}}$ at e_j^{-1} (not e_j), with e_j the j-th coordinate, canonical generator of $\mathbb{Z}^n$, we have

$$\mathfrak{F}(u_j)(z) = \mathfrak{F}(\chi_{e_j^{-1}})(z) = \sum_{x \in \mathbb{Z}^n} \chi_{e_j^{-1}}(x) z^{-x} = z^{e_j} = z_j = z_j(z) \in C(\mathbb{T}^n)$$

and that

$$\begin{aligned} \alpha_t^\wedge(z_j)(z) &= \sum_{x \in \mathbb{Z}^n} \mathrm{Ad}(v^t) u_j(x) z^{-x} = \sum_{x \in \mathbb{Z}^n} \lambda_{\alpha_t(e_j)}(x) z^{-x} \\ &= \sum_{x \in \mathbb{Z}^n} \chi_{\alpha_t(e_j^{-1})}(x) z^{-x} = z^{\alpha_t(e_j)} \in C(\mathbb{T}^n), \end{aligned}$$

from which, one can define the induced action $\alpha^\wedge$ (the same symbol) on the space $\mathbb{T}^n$, the spectrum of $C(\mathbb{T}^n)$.

Note also that the convolution of $\chi_{e_j^{-1}}$ with χ_{e_j} is computed as

$$\begin{aligned} (\chi_{e_j^{-1}} * \chi_{e_j})(x) &= \sum_{y \in \mathbb{Z}^n} \chi_{e_j^{-1}}(x-y) \chi_{e_j}(y) = \chi_{e_j^{-1}}(x - e_j) = \chi_0(x), \\ (\chi_{e_j} * \chi_{e_j^{-1}})(x) &= \sum_{y \in \mathbb{Z}^n} \chi_{e_j}(x-y) \chi_{e_j^{-1}}(y) = \chi_{e_j}(x + e_j) = \chi_0(x), \end{aligned}$$

where χ_0 is the characteristic function at the zero identity element $0 \in \mathbb{Z}^n$, which is the unit of $C^*(\mathbb{Z}^d)$ with convolution. Indeed, for any $f \in l^1(\mathbb{Z}^n)$ in $C^*(\mathbb{Z}^n)$,

$$\begin{aligned} (f * \chi_0)(x) &= \sum_{y \in \mathbb{Z}^n} f(x-y) \chi_0(y) = f(x - 0) = f(x), \\ (\chi_0 * f)(x) &= \sum_{y \in \mathbb{Z}^n} \chi_0(x-y) f(y) = f(x). \end{aligned}$$

Note that the action α on $C^*(\mathbb{Z}^n)$ extends pointwise to α_* on $K_j(C^*(\mathbb{Z}^n))$ $(j = 0, 1)$ in the sense that if the K-theory class $[p] \in K_j(C^*(\mathbb{Z}^n))$ $(j = 0$ or $1)$ with $p = (p_{ij}) = (p_{ij})_{i,j=1}^k$ either a projection or a unitary of some $k \times k$ matrix algebra over $C^*(\mathbb{Z}^n)$, respectively, then $\alpha_*([p]) = [\alpha(p)] = [(\alpha(p_{ij}))]$. In particular, we then compute

$$\begin{aligned}(\mathrm{id} - \alpha)_*[1] &= [1] - [\alpha(1)] = [\chi_0] - [\alpha(\chi_0)] = [0] \in K_0(C^*(\mathbb{Z}^n)),\\ &= [1] - [\alpha^\wedge(1)] = [1] - [1] = [0] \in K_0(C(\mathbb{T}^n)),\end{aligned}$$

where $\alpha(1) = \alpha(\chi_0) = \lambda_{\alpha_1(0)} = v\chi_0 v^* = 1 = \chi_0$ in $C^*(\mathbb{Z}^n)$ and $1 = \alpha^\wedge(1) = 1(z)$ in $C(\mathbb{T}^n)$ for $z \in \mathbb{T}^n$.

As usual, we say that the action α is **free** if $\alpha_t \neq \alpha_0$ for any nonzero $t \in \mathbb{Z}$. Also, the action α has **period** k (a positive integer) if $\alpha_k = \alpha_0$ and $\alpha_t \neq \alpha_0$ for $t \notin k\mathbb{Z}$. In particular, α has period 1 if and only if α is trivial.

When α is trivial, $C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z} = C^*(\mathbb{Z}^{n+1}) \cong C(\mathbb{T}^{n+1})$.

When α has period k, we have the following short exact sequence of C^*-algebras ([2]):

$$0 \to S(C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}_k) \xrightarrow{\ i\ } C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z} \xrightarrow{\ q\ } C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}_k \to 0,$$

with $S(\cdot) = C_0(\mathbb{R}) \otimes (\cdot)$ the suspension and $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$, by [2, Proposition 10.3.3], where $C_0(\mathbb{R})$ is the C^*-algebra of all continuous, complex-valued functions on $\mathbb{R}$ vanishing at infinity. Indeed, the quotient map from $\mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$ to $\mathbb{Z}^n \rtimes_\alpha \mathbb{Z}_k$ induces the quotient map from $C^*(\mathbb{Z}^n \rtimes_\alpha \mathbb{Z})$ to $C^*(\mathbb{Z}^n \rtimes_\alpha \mathbb{Z}_k)$ by universality of (full) group C^*-algebras. Moreover, the crossed product C^*-algebra $C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}$ is viewed as the mapping torus M_β for the dual action β of the dual group $\mathbb{Z}_k^\wedge \cong \mathbb{Z}_k$ (extended to $\mathbb{Z}$ periodically and corresponding to the action α of $\mathbb{Z}_k$) on $C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}_k$, defined by:

$$M_\beta = \{f \in C([0,1], C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}_k) \mid f(1) = \beta(f(0))\}$$

of $C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}_k$-valued continuous functions on $[0, 1]$ with the periodic condition above, which has $C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}_k$ as a quotient as the evaluaion at $0 \in [0, 1]$. Note also that $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ in $C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}$ is mapped to $C^*(\mathbb{Z}_k) \cong \mathbb{C}^k$ in $C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}_k$ by the quotient map q, whose kernel is viewed as $C^*(\mathbb{Z}_k) \otimes C_0(\mathbb{R})$. It follows from the K-theory diagram for the short exact sequence above that

$$\begin{array}{ccccc} K_1(C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}_k) & \xrightarrow{\ i_*\ } & K_0(C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}) & \xrightarrow{\ q_*\ } & K_0(C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}_k) \\ \partial \Big\uparrow (\mathrm{id}-\beta)_* & & & & (\mathrm{id}-\beta)_* \Big\downarrow \partial \\ K_1(C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}_k) & \xleftarrow{\ q_*\ } & K_1(C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}) & \xleftarrow{\ i_*\ } & K_0(C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}_k). \end{array}$$

Furthermore, there is an $\mathbb{R}$-action γ on the mapping torus M_β such that the crossed product $M_\beta \rtimes_\gamma \mathbb{R}$ is stably isomorphic to $(C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z}_k) \rtimes_\beta \mathbb{Z}$ (see [8, 9.K]), so that the diagram above becomes indeed the Pimsner-Voiculesu sequence, via the Connes' Thom isomorphism.

On the other hand, the **Künneth formula** for K-theory of tensor products of C^*-algebras ([2] or [8]) is that there are the following short exact sequences of groups, which have the torsion products $\mathrm{Tor}_1^{\mathbb{Z}}$ as splitting quotients:

$$\begin{aligned} 0 &\to [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \to K_0(\mathfrak{A} \otimes \mathfrak{B}) \\ &\to \mathrm{Tor}_1^{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(K_1(\mathfrak{A}), K_0(\mathfrak{B})) \to 0 \end{aligned}$$

and

$$\begin{aligned} 0 &\to [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \to K_1(\mathfrak{A} \otimes \mathfrak{B}) \\ &\to \mathrm{Tor}_1^{\mathbb{Z}}(K_0(\mathfrak{A}), K_0(\mathfrak{B})) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(K_1(\mathfrak{A}), K_1(\mathfrak{B})) \to 0 \end{aligned}$$

if C^*-algebras $\mathfrak{A}$ or $\mathfrak{B}$ is in the bootstrap category, which contain separable abelian C^*-algebras and separable nuclear KK-abelian C^*-algebras. The converse also holds, so that the tensor product $\otimes$ in the formula may be the minimal, maximal, or any C^*-tensor product by being nuclear. Also, in the bootstrap category, being KK-abelian is equivalent to being K-abelian, where a KK-abelian or K-abelian C^*-algebra is a C^*-algebra that is KK-theory equivalent or K-theory equivalent to an abelian C^*-algebra, respectively (cf. [7]).

The Künneth formula implies that

$$\begin{aligned} K_0(\mathfrak{A} \otimes \mathfrak{B}) &\cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})], \\ K_1(\mathfrak{A} \otimes \mathfrak{B}) &\cong [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \end{aligned}$$

if $\mathfrak{A}$ or $\mathfrak{B}$ is in the bootstrap category and if K-theory groups of $\mathfrak{A}$ or $\mathfrak{B}$ are torsion free, because the torsion products are zero in this case. For instance,

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^k, G) \cong \oplus^k \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, G) \cong 0$$

and $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, G) \cong \mathrm{Tor}_1^{\mathbb{Z}}(G, \mathbb{Z})$ for any G. We may call the isomorphisms for K_0 and K_1 of $\mathfrak{A} \otimes \mathfrak{B}$ under the assumption as the **torsion free Künneth formula**.

As well, under the same conditions in each step for $(\mathfrak{A} \otimes \mathfrak{B}) \otimes \mathfrak{C}$, we have

$$\begin{aligned} K_0(\mathfrak{A} \otimes \mathfrak{B} \otimes \mathfrak{C}) &\cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B} \otimes \mathfrak{C})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B} \otimes \mathfrak{C})], \\ &\cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B}) \otimes K_0(\mathfrak{C})] \oplus [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B}) \otimes K_1(\mathfrak{C})] \oplus \\ &\quad [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B}) \otimes K_1(\mathfrak{C})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B}) \otimes K_0(\mathfrak{C})], \\ K_1(\mathfrak{A} \otimes \mathfrak{B} \otimes \mathfrak{C}) &\cong [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B} \otimes \mathfrak{C})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B} \otimes \mathfrak{C})] \\ &\cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B}) \otimes K_1(\mathfrak{C})] \oplus [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B}) \otimes K_0(\mathfrak{C})] \oplus \\ &\quad [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B}) \otimes K_0(\mathfrak{C})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B}) \otimes K_1(\mathfrak{C})]. \end{aligned}$$

More generally, inductively we obtain

Lemma 2.1. *Let $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ be C^*-algebras such that the torsion free Künneth formula holds for multi-tensor products $\mathfrak{A}_j \otimes \mathfrak{A}_{j+1} \cdots \otimes \mathfrak{A}_{j+k}$ for any $1 \le j < j+k \le n$, viewed as the tensor products $(\mathfrak{A}_j \otimes \cdots \otimes \mathfrak{A}_{j+l}) \otimes (\mathfrak{A}_{j+l+1} \otimes \cdots \otimes \mathfrak{A}_{j+k})$ of the tensor products with $0 \le l \le k-1$. Then*

$$\begin{aligned} K_0(\mathfrak{A}_1 \otimes \cdots \otimes \mathfrak{A}_n) &\cong \oplus_{j_1+\cdots+j_n=0} [K_{j_1}(\mathfrak{A}_1) \otimes \cdots \otimes K_{j_n}(\mathfrak{A}_n)], \\ K_1(\mathfrak{A}_1 \otimes \cdots \otimes \mathfrak{A}_n) &\cong \oplus_{j_1+\cdots+j_n=1} [K_{j_1}(\mathfrak{A}_1) \otimes \cdots \otimes K_{j_n}(\mathfrak{A}_n)], \end{aligned}$$

where the summations run over the equations $j_1+\cdots+j_n=0$ *and* $j_1+\cdots+j_n=1$ *in* $\{0,1\}=\mathbb{Z}_2=\mathbb{Z}/2\mathbb{Z}$*, respectively.*

Remark. Note that there are 2^{n-1} cases of $(j_1,\cdots,j_n)$, for which either $j_1+\cdots+j_n=0$ or $j_1+\cdots+j_n=1$, respectively. Note also that the lemma holds for abelian C^*-algebras, and this is mainly the case applied in the following sections. $\lhd$ (which means the end in what follows.)

3 The abelian case

Let $\mathbb{Z}^n$ be the free abelian group with $\mathbb{Z}$-rank n. As in the previous section, we have

$$C^*(\mathbb{Z}^n)\cong\otimes^n C^*(\mathbb{Z})\cong\otimes^n C(\mathbb{T})\cong C(\mathbb{T}^n).$$

Well known is that $K_j(C(\mathbb{T}^n))\cong\mathbb{Z}^{2^{n-1}}$ for $j=0,1$. What we need in what follows is to determine generators of the K-theroy groups.

Example 3.1. The case where $n=1$. We have

$$K_0(C(\mathbb{T}))\cong\mathbb{Z}[1]\quad\text{and}\quad K_1(C(\mathbb{T}))\cong\mathbb{Z}[z],$$

where 1 is the identity element of $C(\mathbb{T})$ and $z:z\to z\in\mathbb{T}$ the coordinate function on $\mathbb{T}$.

The case where $n=2$. We have

$$K_0(C(\mathbb{T}^2))\cong\mathbb{Z}[1]\oplus\mathbb{Z}[b]^\sim\quad\text{and}\quad K_1(C(\mathbb{T}^2))\cong\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_2],$$

where $z_1:z\to z_1$ and $z_2:z\to z_2$ are coordinate functions on $\mathbb{T}\times\mathbb{T}$, with $z=(z_1,z_2)\in\mathbb{T}^2$, and $b=b(z_1,z_2)$ is the **Bott projection**, a $2^2\times 2^2$ projection matrix valued, continuous function on $\mathbb{T}^2$ (but exactly on $(\mathbb{R}\times\mathbb{T})^2$), defined (by us) as in the following:

$$\begin{aligned}b(z_1,z_2)&=B(t,z_1)\otimes B(t,z_2),\\ [b(z_1,z_2)]^\sim&=[B(t,z_1)]^\sim\otimes[B(t,z_2)]^\sim\in K_0(SC(\mathbb{T}))\otimes K_0(SC(\mathbb{T}))\\ &\cong K_1(C(\mathbb{T}))\otimes K_1(C(\mathbb{T}))\subset K_0(C(\mathbb{T})\otimes C(\mathbb{T})),\end{aligned}$$

where $SC(\mathbb{T})\cong C_0(\mathbb{R})\otimes C(\mathbb{T})$ the suspension of $C(\mathbb{T})$, with $C_0(\mathbb{R})$ the C^*-algebra of all continuous functions on $\mathbb{R}$ vanishing at infinity, and with $B(t,z_j)$ the (standard) Bott projections defined as (and modified from [8])

$$B(t,z_j)=u(t)u(z_j)p_1u(z_j)^*u(t)^*\equiv\mathrm{Ad}(u(t))\mathrm{Ad}(u(z_j))\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

for $t\in\mathbb{R}$ and $z_j\in\mathbb{T}$, where

$$u(x)=\frac{1}{\sqrt{1+|x|^2}}\begin{pmatrix}1&-\bar{x}\\x&1\end{pmatrix},\quad x\in\mathbb{C}.$$

Note that the crucial isomorphism above between $K_1(C(\mathbb{T}))$ and $K_0(SC(\mathbb{T}))$ is obtained by sending the class of z_j to the associated class of $B(t, z_j)$. Namely, the class $[z_j]$ is mapped to the following difference class, which we may call the **Bott class**:

$$[B(t, z_j)]^{\sim} \equiv [B(t, z_j)] - [p_1] \in K_0(SC(\mathbb{T})),$$

with $[B(t, z_j)] \notin K_0(SC(\mathbb{T}))$ but $\in K_0((SC(\mathbb{T}))^+)$, where $(SC(\mathbb{T}))^+$ is the unitization by $\mathbb{C}$, and p_1 is the standard rank 1 projection. Note as well that $SC(\mathbb{T})$ is non-unital, so that its K_0-theory classes are defined to be such differences.

As a contrast to the Künneth formula, we have

$$\begin{aligned} K_0(C(\mathbb{T}^2)) &\cong [K_0(C(\mathbb{T})) \otimes K_0(C(\mathbb{T}))] \oplus [K_1(C(\mathbb{T}) \otimes K_1(C(\mathbb{T}))] \\ &\cong [\mathbb{Z}[1] \otimes \mathbb{Z}[1]] \oplus [\mathbb{Z}[z_1] \otimes \mathbb{Z}[z_2]] \cong \mathbb{Z}[1] \oplus \mathbb{Z}[b(z_1, z_2)]^{\sim}, \\ K_1(C(\mathbb{T}^2)) &\cong [K_1(C(\mathbb{T})) \otimes K_0(C(\mathbb{T}))] \oplus [K_0(C(\mathbb{T}) \otimes K_1(C(\mathbb{T}))] \\ &\cong [\mathbb{Z}[z_1] \otimes \mathbb{Z}[1]] \oplus [\mathbb{Z}[1] \otimes \mathbb{Z}[z_2]] \cong \mathbb{Z}[z_1] \oplus \mathbb{Z}[z_2], \end{aligned}$$

This picture is quite convenient and is generalized as follows.

The case where $n = 3$. We have $K_0(C(\mathbb{T}^3))$ isomorphic to

$$\begin{aligned} &[K_0(C(\mathbb{T})) \otimes K_0(C(\mathbb{T})) \otimes K_0(C(\mathbb{T}]) \oplus [K_0(C(\mathbb{T})) \otimes K_1(C(\mathbb{T})) \otimes K_1(C(\mathbb{T}))] \oplus \\ &[K_1(C(\mathbb{T})) \otimes K_0(C(\mathbb{T})) \otimes K_1(C(\mathbb{T}))] \oplus [K_1(C(\mathbb{T})) \otimes K_1(C(\mathbb{T})) \otimes K_0(C(\mathbb{T}))] \\ &\cong [\mathbb{Z}[1] \otimes \mathbb{Z}[1] \otimes \mathbb{Z}[1]] \oplus [\mathbb{Z}[1] \otimes \mathbb{Z}[z_2] \otimes \mathbb{Z}[z_3]] \oplus \\ &\quad [\mathbb{Z}[z_1] \otimes \mathbb{Z}[1] \otimes \mathbb{Z}[z_3]] \oplus [\mathbb{Z}[z_1] \otimes \mathbb{Z}[z_2] \otimes \mathbb{Z}[1]] \\ &\cong \mathbb{Z}[1] \oplus \mathbb{Z}[b(z_2, z_3)]^{\sim} \oplus \mathbb{Z}[b(z_1, z_3)]^{\sim} \oplus \mathbb{Z}[b(z_1, z_2)]^{\sim} \in K_0(C(\mathbb{T}^3)), \end{aligned}$$

and $K_1(C(\mathbb{T}^3))$ isomorphic to

$$\begin{aligned} &[K_0(C(\mathbb{T})) \otimes K_0(C(\mathbb{T})) \otimes K_1(C(\mathbb{T}))] \oplus [K_0(C(\mathbb{T})) \otimes K_1(C(\mathbb{T})) \otimes K_0(C(\mathbb{T}))] \oplus \\ &[K_1(C(\mathbb{T})) \otimes K_0(C(\mathbb{T})) \otimes K_0(C(\mathbb{T}))] \oplus [K_1(C(\mathbb{T})) \otimes K_1(C(\mathbb{T})) \otimes K_1(C(\mathbb{T}))] \\ &\cong [\mathbb{Z}[1] \otimes \mathbb{Z}[1] \otimes \mathbb{Z}[z_3]] \oplus [\mathbb{Z}[1] \otimes \mathbb{Z}[z_2] \otimes \mathbb{Z}[1]] \oplus \\ &\quad [\mathbb{Z}[z_1] \otimes \mathbb{Z}[1] \otimes \mathbb{Z}[1]] \oplus [\mathbb{Z}[z_1] \otimes \mathbb{Z}[z_2] \otimes \mathbb{Z}[z_3]] \\ &\cong \mathbb{Z}[z_3] \oplus \mathbb{Z}[z_2] \oplus \mathbb{Z}[z_3] \oplus \mathbb{Z}[u(b(z_1, z_2), z_3)], \end{aligned}$$

where

$$u(b(z_1, z_2), z_3) = \begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} + (z_3 - 1) \otimes b(z_1, z_2) \in M_{2^2}(C(\mathbb{T}^3)),$$

where the hidden, real variables t in $b(z_1, z_2)$ are omitted, but to make sense. Note that

$$\begin{aligned} &u(b(z_1, z_2), z_3)^* u(b(z_1, z_2), z_3) = u(b(z_1, z_2), z_3) u(b(z_1, z_2), z_3)^* = \\ &1_4 + (z_3 - 1) \otimes b(z_1, z_2) + (z_3^* - 1) \otimes b(z_1, z_2) + (z_3^* - 1)(z_3 - 1) \otimes b(z_1, z_2) = 1_4. \end{aligned}$$

Consequently, we may choose other representatives such as $u(b(z_2, z_3), u_1)$ and $u(b(z_1, z_3), u_2)$ as in the same definition as above. $\triangleleft$

In general, we obtain

Proposition 3.2. *The K_0-group $K_0(C(\mathbb{T}^n))$ is isomorphic to*

$$\oplus_{j_1+\cdots+j_n=0}\,[K_{j_1}(C(\mathbb{T}))\otimes\cdots\otimes K_{j_n}(C(\mathbb{T}))]$$

with $K_{j_k}(C(\mathbb{T}))\cong\mathbb{Z}[1]$ if $j_k=0$ and $K_{j_k}(C(\mathbb{T}))\cong\mathbb{Z}[z_k]$ if $j_k=1$ for $1\le k\le n$, and the direct factor $\mathbb{Z}[1]\otimes\cdots\otimes\mathbb{Z}[1]$ is replaced by $\mathbb{Z}[1]$, and if

$$K_{j_1}(C(\mathbb{T}))\otimes\cdots\otimes K_{j_n}(C(\mathbb{T}))\cong\mathbb{Z}[z_{k_1}]\otimes\cdots\otimes\mathbb{Z}[z_{k_{2l}}]$$

for some $1\le k_1<\cdots<k_{2l}\le n$, then the direct factor is replaced by $\mathbb{Z}[b(z_{k_1},\cdots,z_{k_{2l}})]^\sim$ with $b(z_{k_1},\cdots,z_{k_{2l}})$ the **generalized Bott projection** *defined (by us) as*

$$b(z_{k_1},\cdots,z_{k_{2l}})=B(t,z_{k_1})\otimes\cdots\otimes B(t,z_{k_{2l}}),$$

so that the **generalized Bott class** *is defined by*

$$[b(z_{k_1},\cdots,z_{k_{2l}})]^\sim=([B(t,z_{k_1})]-[p_1])\otimes\cdots\otimes([B(t,z_{k_{2l}})]-[p_1]).$$

Also, the K_1-group $K_1(C(\mathbb{T}^n))$ is isomorphic to

$$\oplus_{j_1+\cdots+j_n=1}\,[K_{j_1}(C(\mathbb{T}))\otimes\cdots\otimes K_{j_n}(C(\mathbb{T}))],$$

and if $K_{j_1}(C(\mathbb{T}))\otimes\cdots\otimes K_{j_n}(C(\mathbb{T}))\cong\mathbb{Z}[z_{k_1}]\otimes\cdots\otimes\mathbb{Z}[z_{k_{2l+1}}]$ for some $1\le k_1<\cdots<k_{2l+1}\le n$, and if $l=0$, then the direct factor is replaced by $\mathbb{Z}[z_{k_1}]$, and if $l\ge 1$, then replaced by $\mathbb{Z}[u(b(z_{k_1},\cdots,z_{k_{2l}}),z_{k_{2l+1}})]$ with $u(b(z_{k_1},\cdots,z_{k_{2l}},z_{k_{2l+1}})$ **the unitary** *involving the generalized Bott projection defined as*

$$u(b(z_{k_1},\cdots,z_{k_{2l}}),z_{k_{2l+1}})=1_{2^{2l}}+[b(z_{k_1},\cdots,z_{k_{2l}})\otimes(z_{k_{2l+1}}-1)].$$

Furthermore, for the class of the unitary we may choose other representatives obtained from changing variables $z_{k_1},\cdots,z_{k_{2l+1}}$ in $u(b(z_{k_1},\cdots,z_{k_{2l}}),z_{k_{2l+1}})$ combinatoricly.

Proof. One can check that those representatives viewed as generators of the K-theory groups $K_j(C(\mathbb{T}^n))$ are mutually inequivalent, as given in the Example above, and note as well that differences in coordinates also implies the inequivalence. □

Remark. The similar result has already obtained by the author in [3], in which the similar, generalized Bott projection on $\mathbb{T}^n$ is defined, and based on the similar, Bott projection on $\mathbb{T}^2$ of [1]. But that projection by them is rather complicated to be defined and its generalization seems to need to be corrected as in the statement. Anyhow, this time we take a rather, simple and clear definition for that by us, based on [8]. The formulation of the Bott projection to be higher dimensional and of the unitaries involving higher dimensional Bott projections is slightly but rather different from that as condidered in [3]. We may say that such unitaries involving generalized Bott proejctions are (generalized) **Bott unitaries**. ◁

Note that for convenience, we identify classes of $K_j(C^*(\mathbb{Z}^n))$ with corresponding classes of $K_j(C(\mathbb{T}^n))$, in what follows in the next section.

4 The nilpotent case

Let $\mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$ be the semi-direct product as defined above. We denote by $\mathfrak{Z} = \mathfrak{Z}(G)$ the center of a group G.

Lemma 4.1. *The center $\mathfrak{Z}$ of $G = \mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$ is isomorphic to the direct product $F \times k\mathbb{Z}$, where F is the fixed point subgroup of $\mathbb{Z}^n$ under the action α and k is a positive integer.*

In particular, if the action α is free, then $\mathfrak{Z}$ is isomorphic to F.

Proof. For an element (x, t) and any $(y, s) \in \mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$, if

$$(x,t)\cdot(y,s) = (x + \alpha_t(y), t + s) \quad \text{and} \quad (y,s)\cdot(x,t) = (y + \alpha_s(x), s + t)$$

are equal, then $x + \alpha_t(y) = y + \alpha_s(x)$. In particular, if $y = 0$, then $\alpha_s(x) = x$ for any $s \in \mathbb{Z}$. Thus, $\alpha_t(y) = y$ for any $y \in \mathbb{Z}^n$ as well, which implies that $\alpha_t = \alpha_0$. Therefore, an element (x, t) belongs to the center $\mathfrak{Z}$ if and only if x is a fixed point under the action α and $\alpha_t = \alpha_0$. In this case, we have

$$(x,0)\cdot(0,t) = (x,t) = (\alpha_t(x), t) = (0,t)\cdot(x,t).$$

Thus, $\mathfrak{Z}$ is isomorphic to $F \times k\mathbb{Z}$ for some k, just when α has period k. □

By definition, a group G is **nilpotent** if there is a central composition series of finitely many, increasing subgroups G_j of G such that G_{j-1} is normal in G_j with $G_0 = \{1\}$ trivial and $G_m = G$ for some positive integer m, and that the quotient group G_j/G_{j-1} is in the center of G/G_{j-1}.

Corollary 4.2. *If the fixed point subgroup F of the group $G = \mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$ under the action α is trivial, then the group is not nilpotent.*

Proof. When the action α is free, $\mathfrak{Z} = F$ is trivial. Hence, the group G has no central composition series.

When the action α has period k, there is the following short exact sequence of groups:

$$0 \to \mathbb{Z}^n \times k\mathbb{Z} \to \mathbb{Z}^n \rtimes_\alpha \mathbb{Z} \to \mathbb{Z}^n \rtimes_\alpha (\mathbb{Z}/k\mathbb{Z}) \to 0.$$

It follows that the quotient group with the reduced action α (the same symbol) of $\mathbb{Z}/k\mathbb{Z}$ to be free is not nilpotent by the same reason as above, which implies that G is not nilpotent. □

When the $n \times n$ matrix $\beta = \alpha_1 - \alpha_0$ over $\mathbb{Z}$ in $M_n(\mathbb{Z})$ is nilpotent, namely, $\beta^k = 0$ the $n \times n$ zero matrix for some k (such the least positive integer), we say that the action α is **nilpotent**.

Lemma 4.3. *If the action α is nilpotent, then $1 \in \mathbb{Z}$ is an eigen-value of α_1, so that F is not trivial.*

Proof. Since $\beta^k = 0$, we have

$$0 = \det \beta^k = \det(\alpha_1 - \alpha_0)^k = [\det(\alpha_1 - 1_n)]^k,$$

so that $\det(\alpha_1 - 1_n) = 0$. This says that $1 \in \mathbb{Z}$ is an eigen-value for α_1. Hence there is a non-zero element of $\mathbb{Z}^n$ invariant under the action α. □

Proposition 4.4. *If the action α is nilpotent, then $\mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$ is nilpotent. The converse also holds.*

Proof. It follows from Lemma 4.3 above that there is a non-zero subgroup F_1 of $\mathbb{Z}^n$ invariant under the action α. Then there is the following short exact sequence of groups:

$$0 \to F_1 \times \mathbb{Z} \to \mathbb{Z}^n \rtimes_\alpha \mathbb{Z} \to (\mathbb{Z}^n/F_1) \rtimes_\alpha \mathbb{Z} \to 0.$$

Since the reduced action α on the quotient group $\mathbb{Z}^n/F_1$ is still nilpotent, then one can find an invariant subgroup F_2 of $\mathbb{Z}^n/F_1$, for which the following is exact:

$$0 \to F_2 \times \mathbb{Z} \to (\mathbb{Z}^n/F_1) \rtimes_\alpha \mathbb{Z} \to [(\mathbb{Z}^n/F_1)/F_2] \rtimes_\alpha \mathbb{Z} \to 0.$$

Note that $(\mathbb{Z}^n/F_1)/F_2 \cong \mathbb{Z}^n/(F_1 \times F_2)$.

Moreover, we can continue this process inductively to find $F_1, F_2, \cdots, F_m$ viewed as subgroups of $\mathbb{Z}^n$ with $G_j = F_1 \times F_2 \times \cdots \times F_j \times \mathbb{Z}$ such that $G_m = G = \mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$ and each $G_j/G_{j-1} \cong F_j$ is in the center of G/G_{j-1}. Therefore, the sequence $\{G_j\}_{j=1}^n$ is a central composition series of G.

The converse follows by taking this argument reversely. Note that a central composition series of G in this case has the same form as above. Since each component F_j is invariant under the action α by restriction in the inductive steps, so that the matrix α_1 may be assumed to be upper triangular with 1 on the diagonal. □

Remark. Note also as a well known fact in linear algebra that any nilpotent $n \times n$ matrix A over $\mathbb{C}$ (not $\mathbb{Z}$, however) is similar to a Jordan canonical form as a finite number (say l) of diagonal sums of Jordan blocks J_k with zero on the diagonal and some sizes. Namely, there is an invertible $n \times n$ matrix P over $\mathbb{C}$ with inverse P^{-1} such that

$$P^{-1}AP = \oplus_{k=1}^l J_k = \left[\oplus_{k=1}^l \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \right] \in M_n(\mathbb{C}).$$

This **standard** case with only one block is considered in the section 5, but the case with more than one block is not this time. However, the case of products in the standard case is done in the section 6. ◁

Before these,

Example 4.5. Let H_3 denote the **discrete Heisenberg** group with $\mathbb{Z}$-rank 3, consisting of the following 3×3 matrices:

$$\begin{pmatrix} 1 & t & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{Z}), \quad t, a, b \in \mathbb{Z}.$$

The group H_3 is viewed as the semi-direct product $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$ of tuples (a, b, t) with $(a, b) \in \mathbb{Z}^2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and $t \in \mathbb{Z}$, where the action α induced by matrix multiplication is defined by $\alpha_t(a, b) = (a + tb, b) \in \mathbb{Z}^2$, where

$$\alpha_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z}),$$

so that $\alpha_1 - \alpha_0$ is nilpotent. In this case, $F = \mathbb{Z}e_1 = \mathfrak{Z}$ and the following is exact:

$$0 \to F = \mathbb{Z}e_1 \to H_3 \to H_3/F \cong \mathbb{Z}^2 \to 0.$$

This is a central composition series for H_3. We have the following isomorphisms:

$$C^*(H_3) \cong C^*(\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}) \cong C^*(\mathbb{Z}^2) \rtimes_\alpha \mathbb{Z} \cong C(\mathbb{T}^2) \rtimes_{\alpha^\wedge} \mathbb{Z}.$$

The associated, Pimsner-Voiculescu six-term exact sequence becomes:

$$\begin{array}{ccccc} \mathbb{Z}^2 & \xrightarrow{(\mathrm{id}-\alpha)_*} & \mathbb{Z}^2 & \xrightarrow{i_*} & K_0(C^*(H_3)) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(C^*(H_3)) & \xleftarrow{i_*} & \mathbb{Z}^2 & \xleftarrow{(\mathrm{id}-\alpha)_*} & \mathbb{Z}^2 \end{array}$$

with $K_0(C^*(\mathbb{Z}^2)) \cong K_0(C(\mathbb{T}^2)) \cong \mathbb{Z}[1] \oplus \mathbb{Z}[b]^\sim$ and $K_1(C^*(\mathbb{Z}^2)) \cong \mathbb{Z}[u_1] \oplus \mathbb{Z}[u_2]$ as a multiplicative group for $u_j = \lambda_{e_j} = \chi_{e_j}$. We then compute

$$\begin{aligned} (\mathrm{id} - \alpha)_*[u_1] &= [u_1] \cdot [(vu_1v^*)^{-1}] = [u_1] \cdot [\lambda^{-1}_{\alpha_1(e_1)}] \\ &= [u_1] \cdot [\lambda^{-1}_{e_1}] = [u_1] \cdot [u_1^*] = [u_1u_1^*] = [1], \\ (\mathrm{id} - \alpha)_*[u_2] &= [u_2] \cdot [(vu_2v^*)^{-1}] = [u_2] \cdot [\lambda^{-1}_{\alpha_1(e_2)}] \\ &= [u_2] \cdot [\lambda^{-1}_{e_1+e_2}] = [u_2u_2^*u_1^*] = [u_1^*]. \end{aligned}$$

Moreover, for convenience, as a partial computation,

$$\begin{aligned} (\mathrm{id} - \alpha)_*[b] &= [b] - [\alpha^\wedge(b)] = [b] - [\alpha^\wedge(B(t, z_1)) \otimes \alpha^\wedge(B(t, z_2))] \\ &= [b] - [\otimes_{j=1}^2 \{\mathrm{Ad}(\alpha^\wedge(u(t)))\mathrm{Ad}(\alpha^\wedge(u(z_j)))\alpha^\wedge(p_1)\}] \\ &= [b] - [\{\mathrm{Ad}(u(t))\mathrm{Ad}(u(z_1))p_1\} \otimes \{\mathrm{Ad}(u(t))\mathrm{Ad}(u(z_1z_2))p_1\}] \\ &= [b] - [B(t, z_1) \otimes B(t, z_2)] = [b] - [b] = [0], \end{aligned}$$

where note that the projection $\mathrm{Ad}(u(t))\mathrm{Ad}(u(z_1z_2))p_1$ is unitarily equivalent to $B(t, z_2)$ via

$$\mathrm{Ad}(u(t))\mathrm{Ad}(u(z_2))\mathrm{Ad}(u(t)^*)\mathrm{Ad}(u(z_1z_2)^*)$$

and the unitary equivalence implies the K-theory equivalence in K_0 in general and that the class $[b]$ is out of the K-theroy group $K_0(C(\mathbb{T}^2))$. It then follows that

$$(\mathrm{id}-\alpha)_*[b]^\sim = (\mathrm{id}-\alpha)_*([b[-[p_1]) = ([b]-[b]) - ([p_1]-[p_1]) = [0]-[0] = [0].$$

Therefore, the map $(\mathrm{id}-\alpha)_*$ on $K_0(C^*(\mathbb{Z}^2))$ is zero.

Therefore, it follows from exactness of the diagram above that

$$K_0(C^*(H_3)) \cong \mathbb{Z}^3 \quad \text{and} \quad K_1(C^*(H_3)) \cong \mathbb{Z}^3.$$

Note that

$$K_0(C^*(H_3)) \cong K_0(C(\mathbb{T}^2)) \oplus [K_1(C(\mathbb{T}^2))/\mathbb{Z}[u_2]] \cong \mathbb{Z}[1] \oplus \mathbb{Z}[b]^\sim \oplus \mathbb{Z}[u_1],$$
$$K_1(C^*(H_3)) \cong K_0(C(\mathbb{T}^2)) \oplus [K_1(C(\mathbb{T}^2))/\mathbb{Z}[u_1]] \cong \mathbb{Z}[1] \oplus \mathbb{Z}[b]^\sim \oplus \mathbb{Z}[u_2],$$

where the generators of the K-theory groups in the left hand sides correspond to those in the right hand sides represented by the identity element 1 of $C(\mathbb{T}^2)$, the Bott projection $b = b(z_1, z_2)$ on $\mathbb{T}^2$, and the unitary generators u_1, u_2 of $C(\mathbb{T}^2)$. $\lhd$

Example 4.6. Let H_{2n+1} denote the **generalized discrete Heisenberg** group with $\mathbb{Z}$-rank $2n+1$, consisting of the following $(n+2)\times(n+2)$ matrices:

$$\begin{pmatrix} 1 & t & a \\ 0 & 1_n & b^{\mathbf{t}} \\ 0 & 0 & 1 \end{pmatrix} \in GL_{n+2}(\mathbb{Z}), \quad a \in \mathbb{Z}, t, b \in \mathbb{Z}^n,$$

where elements of $\mathbb{Z}^n$ are viewed as row vectors and $b^{\mathbf{t}}$ is the transpose of b. The group H_{2n+1} is viewed as the semi-direct product $\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n$ of tuples (a, b, t) with $(a, b) \in \mathbb{Z}^{n+1} = \mathbb{Z}e_1 \oplus [\mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e_{n+1}]$ and $t = (t_i) \in \mathbb{Z}^n$, where the action α induced by matrix multiplication is defined by

$$\alpha_t(a, b) = (a + tb^{\mathbf{t}}, b) = (a + \sum_{j=1}^{n} t_j b_j, b) \in \mathbb{Z}^{n+1},$$

where

$$\alpha_t = \begin{pmatrix} 1 & t \\ 0 & 1_n \end{pmatrix} \in GL_{n+1}(\mathbb{Z}),$$

so that $\alpha_t - \alpha_0$ is nilpotent. In this case, $F = \mathbb{Z}e_1 = \mathfrak{Z}$ and the following is exact:

$$0 \to F = \mathbb{Z}e_1 \to H_{2n+1} \to H_{2n+1}/F \cong \mathbb{Z}^{2n} \to 0.$$

This is a central composition series for H_{2n+1}. We have the following isomorphisms:

$$C^*(H_{2n+1}) \cong C^*(\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n) \cong C^*(\mathbb{Z}^{n+1}) \rtimes_\alpha \mathbb{Z}^n$$

and the right hand side is a successive crossed product by $\mathbb{Z}$ as

$$C^*(\mathbb{Z}^{n+1}) \rtimes_\alpha \mathbb{Z}^n \cong C^*(\mathbb{Z}^{n+1}) \rtimes_{\alpha(1)} \mathbb{Z} \cdots \rtimes_{\alpha(n)} \mathbb{Z}$$

with $\alpha(j)$ the restriction of α to the j-th coordinate.

As the first step, the associated, Pimsner-Voiculescu six-term exact sequence becomes:

$$\begin{array}{ccccc} \mathbb{Z}^{2^n} & \xrightarrow{(\mathrm{id}-\alpha(1))_*} & \mathbb{Z}^{2^n} & \xrightarrow{i_*} & K_0(C^*(\mathbb{Z}^{n+1}) \rtimes_{\alpha(1)} \mathbb{Z}) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1(C^*(\mathbb{Z}^{n+1}) \rtimes_{\alpha(1)} \mathbb{Z}) & \xleftarrow{i_*} & \mathbb{Z}^{2^n} & \xleftarrow{(\mathrm{id}-\alpha(1))_*} & \mathbb{Z}^{2^n}. \end{array}$$

As a note, $K_1(C^*(\mathbb{Z}^{n+1}))$ contains $\mathbb{Z}[u_1] \oplus \cdots \oplus \mathbb{Z}[u_{n+1}]$ (as a multiplicative group) for $u_j = \lambda_{e_j} = \chi_{e_j}$. Let $v_i = \lambda_{t_i}$ for $1 \leq i \leq n$. Let $1_i \in \mathbb{Z}^n$ with the i-th component 1 and other components zero. We then compute

$$\begin{aligned} (\mathrm{id}-\alpha(1))_*[u_1] &= [u_1] \cdot [(v_1 u_1 v_1^*)^{-1}] = [u_1] \cdot [\lambda^{-1}_{\alpha_{1_1}(e_1)}] \\ &= [u_1] \cdot [\lambda^{-1}_{e_1}] = [u_1] \cdot [u_1^*] = [u_1 u_1^*] = [1], \\ (\mathrm{id}-\alpha(1))_*[u_2] &= [u_j] \cdot [(v_1 u_j v_1^*)^{-1}] = [u_2] \cdot [\lambda^{-1}_{\alpha_{1_1}(e_2)}] \\ &= [u_2] \cdot [\lambda^{-1}_{e_1+e_2}] = [u_2 u_2^* u_1^*] = [u_1^*] \end{aligned}$$

and $(\mathrm{id}-\alpha(1))_*[u_j] = [1]$ for $2 \leq j \leq n+1$ by the same computation.

Moreover, the map $(\mathrm{id}-\alpha(1))_*$ on $K_0(C^*(\mathbb{Z}^{n+1})) \cong K_0(C(\mathbb{T}^{n+1}))$ is zero, because $K_0(C(\mathbb{T}^{n+1}))$ is generated by the class of the unit and the classes of generalized Bott projections, which are invariant under $\alpha^\wedge$ as shown in the example above. Indeed, with $2k \leq n+1$, as a partial computation,

$$\begin{aligned} &[\alpha(1)^\wedge(b(z_1, z_2, \cdots, z_{2k}))] \\ &= [\alpha(1)^\wedge(B(t, z_1)) \otimes \cdots \otimes \alpha(1)^\wedge(B(t, z_{2k}))] \\ &= [\otimes_{j=1}^{2k}\{\mathrm{Ad}(\alpha(1)^\wedge(u(t)))\mathrm{Ad}(\alpha(1)^\wedge(u(z_j)))p_1\}] \\ &= [\{\mathrm{Ad}(u(t))\mathrm{Ad}(u(z_1))p_1\} \otimes \{\mathrm{Ad}(u(t))\mathrm{Ad}(u(z_1 z_2))p_1\} \\ &\quad \otimes_{j=3}^{2k} \{\mathrm{Ad}(u(t))\mathrm{Ad}(u(z_j))p_1\}] \\ &= [B(t, z_1) \otimes B(t, z_2) \otimes \cdots \otimes B(t, z_{2k})] = [b(z_1, \cdots, z_{2k})]. \end{aligned}$$

By the Fourier transform, we have

$$C^*(\mathbb{Z}^{n+1}) \rtimes_{\alpha(1)} \mathbb{Z} \cong C(\mathbb{T}^{n+1}) \rtimes_{\alpha(1)^\wedge} \mathbb{Z}.$$

By Lemma 2.1, we have

$$K_1(C(\mathbb{T}^{n+1})) \cong \oplus_{j_1+\cdots+j_{n+1}=1}[K_{j_1}(C(\mathbb{T})) \otimes \cdots \otimes K_{j_{n+1}}(C(\mathbb{T}))].$$

By Proposition 3.2, each direct factor in the direct sum above is assumed to be generated by either the K_1-classes $[z_j]$ $(1 \leq j \leq n+1)$ or the K_1-classes of generalized Bott unitaries $u(b(z_{k_1}, \cdots, z_{k_{2l}}), z_{k_{2l+1}})$ with $1 \leq k_1 < \cdots < k_{2l+1} \leq n+1$.

We then compute that if $j \neq 2$, then

$$\begin{aligned}(\mathrm{id} - \alpha(1)^\wedge)_*[z_j] &= [z_j] \cdot [\alpha(1)^\wedge(z_j)^{-1}] \\ &= [z_j] \cdot [(z^{\alpha(1)(e_j)})^{-1}] = [z_j] \cdot [(z^{e_j})^{-1}] = [z_j] \cdot [z_j^{-1}] = [1],\end{aligned}$$

and if $j = 2$, then

$$\begin{aligned}(\mathrm{id} - \alpha(1)^\wedge)_*[z_2] &= [z_2] \cdot [\alpha(1)^\wedge(z_2)^{-1}] \\ &= [z_2] \cdot [(z^{\alpha(1)(e_2)})^{-1}] = [z_2] \cdot [(z^{e_1+e_2})^{-1}] = [z_1^{-1}],\end{aligned}$$

Moreover, the K_1-classes of the generalized Bott unitaries are mapped to the identity K_1-class under the map $(\mathrm{id} - \alpha(1)^\wedge)_*$ because the K_0-classes of the generalized Bott projections are mapped to zero K_0-class under the same map. Indeed,

$$\begin{aligned}&(\mathrm{id} - \alpha(1)^\wedge)_*[u(b(z_1, \cdots, z_{2k}), z_{2k+1})] \\ &= [u(b(z_1, \cdots, z_{2k}), z_{2k+1})] \cdot [\alpha(1)^\wedge(u(b(z_1, \cdots, z_{2k}), z_{2k+1}))^{-1}] \\ &= [u(b(z_1, \cdots, z_{2k}), z_{2k+1})] \cdot [u(b(z_1, \cdots, z_{2k}), z_{2k+1})^{-1}] = [1],\end{aligned}$$

where in more general we can assume that $2k + 1 \neq 2$ for z_{2k+1} added to make a generalized Bott unitary, if necessary, by changing variables. Consequently, the image of $\mathbb{Z}^{2^n} \cong K_1(C(\mathbb{T}^{n+1}))$ is isomorphic to $\mathbb{Z}$.

Therefore, it follows from exactness of the diagram above that

$$K_0(C^*(\mathbb{Z}^{n+1}) \rtimes_{\alpha(1)} \mathbb{Z}) \cong \mathbb{Z}^{2^{n+1}-1} \quad \text{and} \quad K_1(C^*(\mathbb{Z}^{n+1}) \rtimes_{\alpha(1)} \mathbb{Z}) \cong \mathbb{Z}^{2^{n+1}-1}.$$

Note that

$$\begin{aligned}K_0(\mathfrak{A}_1) &\cong K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]], \\ K_1(\mathfrak{A}_1) &\cong K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]]\end{aligned}$$

with $\mathfrak{A}_1 = C^*(\mathbb{Z}^{n+1}) \rtimes_{\alpha(1)} \mathbb{Z}$.

As the second step, we consider $\mathfrak{A}_2 = \mathfrak{A}_1 \rtimes_{\alpha(2)} \mathbb{Z}$. The associated, Pimsner-Voiculescu six-term exact sequence becomes:

$$\begin{array}{ccccc}\mathbb{Z}^{2^{n+1}-1} & \xrightarrow{(\mathrm{id}-\alpha(2))_*} & \mathbb{Z}^{2^{n+1}-1} & \xrightarrow{i_*} & K_0(\mathfrak{A}_1 \rtimes_{\alpha(2)} \mathbb{Z}) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(\mathfrak{A}_1 \rtimes_{\alpha(2)} \mathbb{Z}) & \xleftarrow{i_*} & \mathbb{Z}^{2^{n+1}-1} & \xleftarrow{(\mathrm{id}-\alpha(2))_*} & \mathbb{Z}^{2^{n+1}-1}\end{array}$$

and

$$
\begin{aligned}
&(\mathrm{id}-\alpha(2))_* K_0(\mathfrak{A}_1)\\
&\cong (\mathrm{id}-\alpha(2))_* K_0(C(\mathbb{T}^{n+1})) \oplus (\mathrm{id}-\alpha(2))_*[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]],\\
&\cong [0] \oplus \mathbb{Z}[z_1] \cong \mathbb{Z},\\
&(\mathrm{id}-\alpha(2))_* K_1(\mathfrak{A}_1)\\
&\cong (\mathrm{id}-\alpha(2))_* K_0(C(\mathbb{T}^{n+1})) \oplus (\mathrm{id}-\alpha(2))_*[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]]\\
&\cong [0] \oplus [0] \cong [0],
\end{aligned}
$$

where the same computation as given above shows that $(\mathrm{id}-\alpha(2)^\wedge)_*[z_j]=[1]$ if $j \neq 3$ and $(\mathrm{id}-\alpha(2)^\wedge)_*[z_3]=[z_1^{-1}]$. It then follows that

$$
\begin{aligned}
K_0(\mathfrak{A}_2) &\cong \mathbb{Z}^{2^{n+1}-2} \oplus \mathbb{Z}^{2^{n+1}-1} \cong \mathbb{Z}^{2^{n+2}-3}\\
&\cong \{K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_2])]\}\\
&\quad \oplus \{K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]]\},\\
K_1(\mathfrak{A}_2) &\cong \mathbb{Z}^{2^{n+1}-2} \oplus \mathbb{Z}^{2^{n+1}-1} \cong \mathbb{Z}^{2^{n+2}-3}\\
&\cong \{K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2] \oplus \mathbb{Z}[z_3])]\}\\
&\quad \oplus \{K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]]\}.
\end{aligned}
$$

As the third step, we consider $\mathfrak{A}_3 = \mathfrak{A}_2 \rtimes_{\alpha(3)} \mathbb{Z}$. Since we have that $(\mathrm{id}-\alpha(3)^\wedge)_*[z_j]=[1]$ if $j \neq 4$ and $(\mathrm{id}-\alpha(3)^\wedge)_*[z_4]=[z_1^{-1}]$, then

$$
\begin{aligned}
&(\mathrm{id}-\alpha(3)^\wedge)_* K_0(\mathfrak{A}_2)\\
&\cong (\mathrm{id}-\alpha(3)^\wedge)_*[\oplus^2 K_0(C(\mathbb{T}^{n+1}))] \oplus (\mathrm{id}-\alpha(3)^\wedge)_*[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]]\\
&\oplus (\mathrm{id}-\alpha(3)^\wedge)_*[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_2])]\\
&\cong [0] \oplus [0] \oplus [0] \cong [0],\\
&(\mathrm{id}-\alpha(3)^\wedge)_* K_1(\mathfrak{A}_2)\\
&\cong (\mathrm{id}-\alpha(3)^\wedge)_*[\oplus^2 K_0(C(\mathbb{T}^{n+1}))] \oplus (\mathrm{id}-\alpha(3)^\wedge)_*[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]]\\
&\oplus (\mathrm{id}-\alpha(3)^\wedge)_*[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2] \oplus \mathbb{Z}[z_3])]\\
&\cong [0] \oplus [0] \oplus \mathbb{Z}[z_1] \cong \mathbb{Z}.
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
&K_0(\mathfrak{A}_3) \cong \mathbb{Z}^{2^{n+2}-3} \oplus \mathbb{Z}^{2^{n+2}-4} \cong \mathbb{Z}^{2^{n+3}-7}\\
&\cong [\oplus^{2^2} K_0(C(\mathbb{T}^{n+1}))] \oplus [\oplus^2 K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]] \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_2])]\\
&\oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2] \oplus \mathbb{Z}[z_3] \oplus \mathbb{Z}[z_4])],\\
&K_1(\mathfrak{A}_3) \cong \mathbb{Z}^{2^{n+2}-3} \oplus \mathbb{Z}^{2^{n+2}-4} \cong \mathbb{Z}^{2^{n+3}-7}\\
&\cong [\oplus^{2^2} K_0(C(\mathbb{T}^{n+1}))] \oplus [\oplus^2 K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]] \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_2])]\\
&\oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_2] \oplus \mathbb{Z}[z_3])]. \quad \lhd
\end{aligned}
$$

Since the situation as above is the same in the inductive steps, alternatively, we obtain the following:

Theorem 4.7. *Let H_{2n+1} be the generalized discrete Heisenberg group with rank $2n+1$ and $C^*(H_{2n+1})$ the group C^*-algebra of H_{2n+1}. Then*

$$K_0(C^*(H_{2n+1})) \cong \mathbb{Z}^{2^n(2^n-1)+1} \cong K_1(C^*(H_{2n+1})).$$

Moreover, these K-theory groups have the direct sum decompositions: if n is odd, then

$$\begin{aligned}
K_0(C^*(H_{2n+1})) \cong\ & [\oplus^{2^{n-1}} K_0(C(\mathbb{T}^{n+1}))] \oplus [\oplus^{2^{n-2}} K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]] \\
& \oplus [\oplus^{2^{n-3}} K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_2])] \oplus \cdots \\
& \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \cdots \oplus \mathbb{Z}[z_{n-1}])] \\
& \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2] \oplus \cdots \oplus \mathbb{Z}[z_{n+1}])], \\
K_1(C^*(H_{2n+1})) \cong\ & [\oplus^{2^{n-1}} K_0(C(\mathbb{T}^{n+1}))] \oplus [\oplus^{2^{n-2}} K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]] \\
& \oplus [\oplus^{2^{n-3}} K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_2])] \oplus \cdots \\
& \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \cdots \oplus \mathbb{Z}[z_{n-1}])] \\
& \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \cdots \oplus \mathbb{Z}[z_n])],
\end{aligned}$$

and if n is even, then

$$\begin{aligned}
K_0(C^*(H_{2n+1})) \cong\ & [\oplus^{2^{n-1}} K_0(C(\mathbb{T}^{n+1}))] \oplus [\oplus^{2^{n-2}} K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]] \\
& \oplus [\oplus^{2^{n-3}} K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_2])] \oplus \cdots \\
& \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \cdots \oplus \mathbb{Z}[z_{n-1}])] \\
& \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \cdots \oplus \mathbb{Z}[z_n])], \\
K_1(C^*(H_{2n+1})) \cong\ & [\oplus^{2^{n-1}} K_0(C(\mathbb{T}^{n+1}))] \oplus [\oplus^{2^{n-2}} K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]] \\
& \oplus [\oplus^{2^{n-3}} K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_2])] \oplus \cdots \\
& \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \cdots \oplus \mathbb{Z}[z_{n-1}])] \\
& \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2] \oplus \cdots \oplus \mathbb{Z}[z_{n+1}])].
\end{aligned}$$

Furthermore, representatives for K-theory classes of $K_0(C^*(H_{2n+1}))$ and $K_1(C^*(H_{2n+1}))$ are assumed to be corresponding to the identity element of $C^*(H_{2n+1})$ or generalized Bott projections, and to be to the unitary generators of $C^*(H_{2n+1})$ or generalized Bott unitaries in those direct sum decompositions, respectively, up to changing variables among chosen variables.

Remark. Note that for any integer $n \geq 1$, we have

$$3^n \leq 2^n(2^n-1)+1,$$

which can be proved by induction with respect to n. It seems to follow that the corresponding result in [3] is viewed as computing a (generic) sub-basis of the K-theory groups.

Note also that for any integer $n \geq 1$, we have

$$2^n(2^n - 1) + 1 \leq 2^{2n},$$

which can be proved by induction with respect to n, with $K_j(C(\mathbb{T}^{2n+1})) \cong \mathbb{Z}^{2n}$ $(j = 0, 1)$. It follows that $K_j(C^*(H_{2n+1}))$ is viewed as a subgroup (or a quotient in more general as the next example below shows) of $K_j(C^*(\mathbb{Z}^{2n+1}))$. It should be an interesting question to determine what class of more general (nilpotent) discrete groups, for which the same holds. $\triangleleft$

Example 4.8. Let k be a positive integer (as weight). We define $H_3(k)$ to be the (**weighted**) discrete **Heisenberg subgroup** of H_3 of rank 3, consisting of the following matrices:

$$\begin{pmatrix} 1 & tk & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{Z}), \quad t, a, b \in \mathbb{Z}.$$

Note that $H_3(1) = H_3$. Let $k \geq 2$ in what follows. The group $H_3(k)$ is viewed as the semi-direct product $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$ of tuples (a, b, t) with $(a, b) \in \mathbb{Z}^2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and $t \in \mathbb{Z}$, where the action α induced by matrix multiplication is defined by $\alpha_t(a, b) = (a + tkb, b) \in \mathbb{Z}^2$, where

$$\alpha_t = \begin{pmatrix} 1 & tk \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z}),$$

so that $\alpha_1 - \alpha_0$ is nilpotent. In this case, $F = \mathbb{Z}e_1 = \mathfrak{Z}$ and the following is exact:

$$0 \to F = \mathbb{Z}e_1 \to H_3(k) \to H_3(k)/F \cong \mathbb{Z}^2 \to 0.$$

This is a central composition series for $H_3(k)$. We have the following isomorphisms:

$$C^*(H_3(k)) \cong C^*(\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}) \cong C^*(\mathbb{Z}^2) \rtimes_\alpha \mathbb{Z} \cong C(\mathbb{T}^2) \rtimes_{\alpha^\wedge} \mathbb{Z}.$$

The associated, Pimsner-Voiculescu six-term exact sequence becomes:

$$\begin{array}{ccccc} \mathbb{Z}^2 & \xrightarrow{(\mathrm{id}-\alpha)_*=0} & \mathbb{Z}^2 & \xrightarrow{i_*} & K_0(C^*(H_3(k))) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(C^*(H_3(k))) & \xleftarrow{i_*} & \mathbb{Z}^2 & \xleftarrow{(\mathrm{id}-\alpha)_*} & \mathbb{Z}^2 \end{array}$$

where the map $(\mathrm{id} - \alpha)_*$ on K_0 is proved to be zero by the same way as in the case of H_3, and with $K_1(C^*(\mathbb{Z}^2)) \cong \mathbb{Z}[u_1] \oplus \mathbb{Z}[u_2]$ as a multiplicative group for $u_j = \lambda_{e_j} = \chi_{e_j}$. We then compute

$$\begin{aligned} (\mathrm{id} - \alpha)_*[u_1] &= [u_1] \cdot [(v^k u_1 (v^k)^*)^{-1}] = [u_1] \cdot [\lambda^{-1}_{\alpha_1(e_1)}] \\ &= [u_1] \cdot [\lambda^{-1}_{e_1}] = [u_1] \cdot [u_1^*] = [u_1 u_1^*] = [1], \\ (\mathrm{id} - \alpha)_*[u_2] &= [u_2] \cdot [(v^k u_2 (v^k)^*)^{-1}] = [u_2] \cdot [\lambda^{-1}_{\alpha_1(e_2)}] \\ &= [u_2] \cdot [\lambda^{-1}_{ke_1+e_2}] = [u_2 u_2^* (u_1^*)^k] = [(u_1^*)^k]. \end{aligned}$$

Hence the image of $\mathbb{Z}^2 \cong K_1(C^*(\mathbb{Z}^2))$ under $(\mathrm{id} - \alpha)_*$ is isomorphic to the subgroup $k\mathbb{Z} \times \{0\}(\cong \mathbb{Z})$ in $\mathbb{Z}^2$. Therefore, it follows from exactness of the diagram above that

$$K_0(C^*(H_3(k))) \cong \mathbb{Z}^3 \quad \text{and} \quad K_1(C^*(H_3(k))) \cong \mathbb{Z}^3 \times \mathbb{Z}_k$$

with $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$. Note that

$$\begin{aligned} K_0(C^*(H_3(k))) &\cong K_0(C(\mathbb{T}^2)) \oplus [K_1(C(\mathbb{T}^2))/\mathbb{Z}[u_2]] \\ &\cong \mathbb{Z}[1] \oplus \mathbb{Z}[b]^\sim \oplus \mathbb{Z}[u_1], \\ K_1(C^*(H_3(k))) &\cong K_0(C(\mathbb{T}^2)) \oplus [K_1(C(\mathbb{T}^2))/k\mathbb{Z}[u_1]] \\ &\cong \mathbb{Z}[1] \oplus \mathbb{Z}[b]^\sim \oplus \mathbb{Z}[u_2] \oplus \mathbb{Z}_k[u_1], \end{aligned}$$

where the generators of the K-theory groups in the left hand sides correspond to those in the right hand sides represented by the identity element 1 of $C(\mathbb{T}^2)$, the Bott projection $b = b(z_1, z_2)$ on $\mathbb{T}^2$, and the unitary generators u_1, u_2 of $C(\mathbb{T}^2)$. $\lhd$

Remark. As kindly reviewed and pointed out by Xin Li in Math Review MR3154863, it follows from this example that the K-theory groups for the group C^*-algebras of torsion free, nilpotent discrete groups may have torsion in general. The corresponding result in [5] is viewed as computing a (generic) sub-basis of the K-theory groups (or the basis is full but coefficients should involve torsion in general, as in the example above). $\lhd$

Example 4.9. Let $k_1, \cdots, k_n$ be positive integers (as weights) such that each k_j is a power $p_j^{s_j}$ of a prime number $p_j \geq 2$ for some $s_j \geq 1$ and p_j are mutually relatively prime, for convenience in computation given below. Set $k = (k_1, \cdots, k_n)$. We define $H_{2n+1}(k)$ to be the **generalized (weighted) discrete Heisenberg subgroup** of H_{2n+1} with $\mathbb{Z}$-rank $2n+1$, consisting of the following $(n+2) \times (n+2)$ matrices:

$$\begin{pmatrix} 1 & t \cdot k & a \\ 0 & 1_n & b^{\mathbf{t}} \\ 0 & 0 & 1 \end{pmatrix} \in GL_{n+2}(\mathbb{Z}), \quad a \in \mathbb{Z}, t, b \in \mathbb{Z}^n,$$

where $t \cdot k = (t_1 k_1, \cdots, t_n k_n) \in \mathbb{Z}^n$. The group $H_{2n+1}(k)$ is viewed as the semi-direct product $\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n$ of tuples (a, b, t) with $(a, b) \in \mathbb{Z}^{n+1} = \mathbb{Z}e_1 \oplus [\mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e_{n+1}]$ and $t = (t_i) \in \mathbb{Z}^n$, where the action α induced by matrix multiplication is defined by $\alpha_t(a, b) = (a + (t \cdot k)b^{\mathbf{t}}, b) \in \mathbb{Z}^{n+1}$, where

$$\alpha_t = \begin{pmatrix} 1 & t \cdot k \\ 0 & 1_n \end{pmatrix} \in GL_{n+1}(\mathbb{Z}),$$

so that $\alpha_t - \alpha_0$ is nilpotent. In this case, $F = \mathbb{Z}e_1 = \mathfrak{Z}$ and the following is exact:

$$0 \to F = \mathbb{Z}e_1 \to H_{2n+1}(k) \to H_{2n+1}(k)/F \cong \mathbb{Z}^{2n} \to 0.$$

This is a central composition series for $H_{2n+1}(k)$. We have the following isomorphisms:

$$C^*(H_{2n+1}(k)) \cong C^*(\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n) \cong C^*(\mathbb{Z}^{n+1}) \rtimes_\alpha \mathbb{Z}^n$$

and the right hand side is a successive crossed product by $\mathbb{Z}$ as

$$C^*(\mathbb{Z}^{n+1}) \rtimes_\alpha \mathbb{Z}^n \cong C^*(\mathbb{Z}^{n+1}) \rtimes_{\alpha(1)} \mathbb{Z} \cdots \rtimes_{\alpha(n)} \mathbb{Z}$$

with $\alpha(j)$ the restriction of α to the j-th coordinate.

As the first step, the associated, Pimsner-Voiculescu six-term exact sequence becomes:

$$\begin{array}{ccccc}
\mathbb{Z}^{2^n} & \xrightarrow{(\mathrm{id}-\alpha(1))_*=0} & \mathbb{Z}^{2^n} & \xrightarrow{i_*} & K_0(C^*(\mathbb{Z}^{n+1}) \rtimes_{\alpha(1)} \mathbb{Z}) \\
\partial\uparrow & & & & \downarrow\partial \\
K_1(C^*(\mathbb{Z}^{n+1}) \rtimes_{\alpha(1)} \mathbb{Z}) & \xleftarrow{i_*} & \mathbb{Z}^{2^n} & \xleftarrow{(\mathrm{id}-\alpha(1))_*} & \mathbb{Z}^{2^n}
\end{array}$$

where the map $(\mathrm{id} - \alpha(1))_*$ on K_0 is proved to be zero by the same way as in the case of H_{2n+1}, and with $K_1(C^*(\mathbb{Z}^{n+1}))$ containing $\mathbb{Z}[u_1] \oplus \cdots \oplus \mathbb{Z}[u_{n+1}]$ (as a multiplicative group) for $u_j = \lambda_{e_j} = \chi_{e_j}$. Let $v_i = \lambda_{t_i}$ for $1 \leq i \leq n$. Let $1_i \in \mathbb{Z}^n$ with the i-th component 1 and other components zero. We then compute

$$\begin{aligned}
(\mathrm{id} - \alpha(1))_*[u_1] &= [u_1] \cdot [(v_1^{k_1} u_1 (v_1^*)^{k_1})^{-1}] = [u_1] \cdot [\lambda_{\alpha_{1_1}(e_1)}^{-1}] \\
&= [u_1] \cdot [\lambda_{e_1}^{-1}] = [u_1] \cdot [u_1^*] = [u_1 u_1^*] = [1], \\
(\mathrm{id} - \alpha(1))_*[u_2] &= [u_j] \cdot [(v_1^{k_1} u_j (v_1^*)^{k_1})^{-1}] = [u_2] \cdot [\lambda_{\alpha_{1_1}(e_2)}^{-1}] \\
&= [u_2] \cdot [\lambda_{k_1 e_1 + e_2}^{-1}] = [u_2 u_2^* (u_1^*)^{k_1}] = [(u_1^*)^{k_1}]
\end{aligned}$$

and $(\mathrm{id} - \alpha(1))_*[u_j] = [1]$ for $2 \leq j \leq n+1$ by the same computation.

By the Fourier transform, we have

$$C^*(\mathbb{Z}^{n+1}) \rtimes_{\alpha(1)} \mathbb{Z} \cong C(\mathbb{T}^{n+1}) \rtimes_{\alpha(1)^\wedge} \mathbb{Z}.$$

By Lemma 2.1, we have

$$K_1(C(\mathbb{T}^{n+1})) \cong \oplus_{j_1+\cdots+j_{n+1}=1} [K_{j_1}(C(\mathbb{T})) \otimes \cdots \otimes K_{j_{n+1}}(C(\mathbb{T}))].$$

By proposition 3.2, each direct factor in the direct sum above is assumed to be generated by either the K_1-classes $[z_j]$ $(1 \leq j \leq n+1)$ or the K_1-classes of generalized Bott unitaries $u(b(z_{k_1}, \cdots, z_{k_{2l}}), z_{k_{2l+1}})$ with $1 \leq k_1 < \cdots < k_{2l+1} \leq n+1$.

We then compute that if $j \neq 2$, then

$$\begin{aligned}
(\mathrm{id} - \alpha(1)^\wedge)_*[z_j] &= [z_j] \cdot [\alpha(1)^\wedge(z_j)^{-1}] \\
&= [z_j] \cdot [(z^{\alpha(1)(e_j)})^{-1}] = [z_j] \cdot [(z^{e_j})^{-1}] = [z_j] \cdot [z_j^{-1}] = [1],
\end{aligned}$$

and if $j = 2$, then

$$\begin{aligned}(\mathrm{id} - \alpha(1)^\wedge)_*[z_2] &= [z_2] \cdot [\alpha(1)^\wedge(z_2)^{-1}] \\ &= [z_2] \cdot [(z^{\alpha(1)(e_2)})^{-1}] = [z_2] \cdot [(z^{k_1 e_1 + e_2})^{-1}] = [z_1^{-k_1}],\end{aligned}$$

Moreover, the K_1-classes of the generalized Bott unitaries are mapped to the identity K_1-class under the map $(\mathrm{id} - \alpha(1)^\wedge)_*$ because the K_0-classes of the generalized Bott projections are mapped to zero K_0-class under the same map. Consequently, the image of $\mathbb{Z}^{2^n} \cong K_1(C(\mathbb{T}^{n+1}))$ is isomorphic to $k_1\mathbb{Z}$.

Therefore, it follows from exactness of the diagram above that

$$\begin{aligned}&K_0(C^*(\mathbb{Z}^{n+1}) \rtimes_{\alpha(1)} \mathbb{Z}) \cong \mathbb{Z}^{2^{n+1}-1} \quad \text{and} \\ &K_1(C^*(\mathbb{Z}^{n+1}) \rtimes_{\alpha(1)} \mathbb{Z}) \cong \mathbb{Z}^{2^{n+1}-1} \times \mathbb{Z}_{k_1}[z_1].\end{aligned}$$

Note that

$$\begin{aligned}K_0(\mathfrak{A}_1) &\cong K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]], \\ K_1(\mathfrak{A}_1) &\cong K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/k_1\mathbb{Z}[z_1]]\end{aligned}$$

with $\mathfrak{A}_1 = C^*(\mathbb{Z}^{n+1}) \rtimes_{\alpha(1)} \mathbb{Z}$.

As the second step, we consider $\mathfrak{A}_2 = \mathfrak{A}_1 \rtimes_{\alpha(2)} \mathbb{Z}$. The associated, Pimsner-Voiculescu six-term exact sequence becomes:

$$\begin{array}{ccccc}\mathbb{Z}^{2^{n+1}-1} & \xrightarrow{(\mathrm{id}-\alpha(2))_*} & \mathbb{Z}^{2^{n+1}-1} & \xrightarrow{i_*} & K_0(\mathfrak{A}_1 \rtimes_{\alpha(2)} \mathbb{Z}) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1(\mathfrak{A}_1 \rtimes_{\alpha(2)} \mathbb{Z}) & \xleftarrow{i_*} & \mathbb{Z}^{2^{n+1}-1} \times \mathbb{Z}_{k_1}[z_1] & \xleftarrow{(\mathrm{id}-\alpha(2))_*} & \mathbb{Z}^{2^{n+1}-1} \times \mathbb{Z}_{k_1}[z_1]\end{array}$$

and

$$\begin{aligned}&(\mathrm{id} - \alpha(2))_* K_0(\mathfrak{A}_1) \\ &\cong (\mathrm{id} - \alpha(2))_* K_0(C(\mathbb{T}^{n+1})) \oplus (\mathrm{id} - \alpha(2))_*[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]], \\ &\cong [0] \oplus k_2\mathbb{Z}[z_1] \cong \mathbb{Z}, \\ &(\mathrm{id} - \alpha(2))_* K_1(\mathfrak{A}_1) \\ &\cong (\mathrm{id} - \alpha(2))_* K_0(C(\mathbb{T}^{n+1})) \oplus (\mathrm{id} - \alpha(2))_*[K_1(C(\mathbb{T}^{n+1}))/k_1\mathbb{Z}[z_1]] \\ &\cong [0] \oplus \mathbb{Z}_{k_1}[z_1] \cong \mathbb{Z}_{k_1},\end{aligned}$$

where the same computation as given above shows that $(\mathrm{id} - \alpha(2)^\wedge)_*[z_j] = [1]$ if $j \neq 3$ and $(\mathrm{id} - \alpha(2)^\wedge)_*[z_3] = [z_1^{-k_2}]$, which in the second case above is not the identity in $\mathbb{Z}_{k_1}[z_1]$, so that it must generate $\mathbb{Z}_{k_1}[z_1]$, because k_1 is a power

of a prime so that $\mathbb{Z}_{k_1}$ has no direct factors. It then follows that

$$\begin{aligned}
&K_0(\mathfrak{A}_2) \cong [\mathbb{Z}^{2^{n+1}-2} \times \mathbb{Z}_{k_2}[z_1]] \oplus [\mathbb{Z}^{2^{n+1}-1} \times \mathbb{Z}_{k_1}[z_1]] \cong \mathbb{Z}^{2^{n+2}-3} \times \mathbb{Z}_{k_1}[z_1] \times \mathbb{Z}_{k_2}[z_1] \\
&\cong \{K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/(k_2\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_2])]\} \\
&\quad \oplus \{K_0(C(\mathbb{T}^{n+1})) \oplus ([K_1(C(\mathbb{T}^{n+1}))/(k_1\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_3])] \times k_1\mathbb{Z}[z_3])\}, \\
&K_1(\mathfrak{A}_2) \cong \mathbb{Z}^{2^{n+1}-2} \oplus \mathbb{Z}^{2^{n+1}-1} \cong \mathbb{Z}^{2^{n+2}-3} \\
&\cong \{K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2] \oplus \mathbb{Z}[z_3])]\} \\
&\quad \oplus \{K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]]\}.
\end{aligned}$$

As the third step, we consider $\mathfrak{A}_3 = \mathfrak{A}_2 \rtimes_{\alpha(3)} \mathbb{Z}$. Since we have that $(\mathrm{id} - \alpha(3)^\wedge)_*[z_j] = [1]$ if $j \neq 4$ and $(\mathrm{id} - \alpha(3)^\wedge)_*[z_4] = [z_1^{-k_3}]$, which in the first case above is not the identity in $\mathbb{Z}_{k_2}[z_1]$, then

$$\begin{aligned}
&(\mathrm{id} - \alpha(3)^\wedge)_* K_0(\mathfrak{A}_2) \\
&\cong (\mathrm{id} - \alpha(3)^\wedge)_* \{K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/(k_2\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_2])]\} \\
&\quad \oplus (\mathrm{id} - \alpha(3)^\wedge)_* \{K_0(C(\mathbb{T}^{n+1})) \oplus ([K_1(C(\mathbb{T}^{n+1}))/(k_1\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_3])] \times k_1\mathbb{Z}[z_3])\} \\
&\cong ([0] \oplus \mathbb{Z}_{k_2}[z_1]) \oplus ([0] \oplus \mathbb{Z}_{k_1}[z_1]) \cong \mathbb{Z}_{k_2}[z_1] \oplus \mathbb{Z}_{k_1}[z_1] \cong \mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2}, \\
&(\mathrm{id} - \alpha(3)^\wedge)_* K_1(\mathfrak{A}_2) \\
&\cong (\mathrm{id} - \alpha(3)^\wedge)_* \{K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2] \oplus \mathbb{Z}[z_3])]\} \\
&\quad \oplus (\mathrm{id} - \alpha(3)^\wedge)_* \{K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]]\} \\
&\cong ([0] \oplus k_3\mathbb{Z}[z_1]) \oplus ([0] \oplus [0]) \cong k_3\mathbb{Z}[z_1] \cong \mathbb{Z}.
\end{aligned}$$

It then follows that

$$\begin{aligned}
&K_0(\mathfrak{A}_3) \cong \mathbb{Z}^{2^{n+2}-3} \oplus \mathbb{Z}^{2^{n+2}-4} \cong \mathbb{Z}^{2^{n+3}-7} \\
&\cong \{[\oplus^2 K_0(C(\mathbb{T}^{n+1}))] \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_2])] \\
&\quad \oplus ([K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_3])] \times k_1\mathbb{Z}[z_3])\} \oplus \{[\oplus^2 K_0(C(\mathbb{T}^{n+1}))] \\
&\quad \oplus [K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]] \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2] \oplus \mathbb{Z}[z_3] \oplus \mathbb{Z}[z_4])]\}, \\
&K_1(\mathfrak{A}_3) \cong [\mathbb{Z}^{2^{n+2}-3} \times \mathbb{Z}_{k_1}[z_1] \times \mathbb{Z}_{k_2}[z_1]] \oplus [\mathbb{Z}^{2^{n+2}-4} \times \mathbb{Z}_{k_3}[z_1]] \\
&\cong \mathbb{Z}^{2^{n+3}-7} \times \mathbb{Z}_{k_1}[z_1] \times \mathbb{Z}_{k_2}[z_1] \times \mathbb{Z}_{k_3}[z_1] \\
&\cong \{[\oplus^2 K_0(C(\mathbb{T}^{n+1}))] \oplus ([K_1(C(\mathbb{T}^{n+1}))/(k_2\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_2] \oplus \mathbb{Z}[z_4])] \times k_2\mathbb{Z}[z_4]) \\
&\quad \oplus [K_1(C(\mathbb{T}^{n+1}))/(k_1\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_3] \oplus \mathbb{Z}[z_4])] \times (k_1\mathbb{Z}[z_3] \times k_1\mathbb{Z}[z_4])\} \\
&\quad \oplus \{[\oplus^2 K_0(C(\mathbb{T}^{n+1}))] \\
&\quad \oplus [K_1(C(\mathbb{T}^{n+1}))/(k_3\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_2] \oplus \mathbb{Z}[z_3])] \oplus [K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]]\}.
\end{aligned}$$

As the fourth step, we consider $\mathfrak{A}_4 = \mathfrak{A}_3 \rtimes_{\alpha(4)} \mathbb{Z}$. Since we have that $(\mathrm{id} - \alpha(4)^\wedge)_*[z_j] = [1]$ if $j \neq 5$ and $(\mathrm{id} - \alpha(4)^\wedge)_*[z_5] = [z_1^{-k_4}]$, which in the first case

above is not the identity in $\mathbb{Z}_{k_2}[z_1]$, then similarly as above,

$$\begin{aligned}
&(\mathrm{id}-\alpha(4)^\wedge)_* K_0(\mathfrak{A}_3)\\
&\cong \{(\oplus^2[0])\oplus[0]\oplus[0]\}\oplus\{(\oplus^2[0])\oplus[0]\oplus k_4\mathbb{Z}[z_1]\}\cong k_4\mathbb{Z}[z_1]\cong\mathbb{Z},\\
&(\mathrm{id}-\alpha(4)^\wedge)_* K_1(\mathfrak{A}_3)\\
&\cong \{(\oplus^2[0])\oplus\mathbb{Z}_{k_2}[z_1]\oplus\mathbb{Z}_{k_1}[z_1]\}\oplus\{(\oplus^2[0])\oplus\mathbb{Z}_{k_3}[z_1]\oplus[0]\}.
\end{aligned}$$

It then follows that

$$\begin{aligned}
K_0(\mathfrak{A}_4) &\cong [\mathbb{Z}^{2^{n+3}-7}\times\mathbb{Z}_{k_4}[z_1]]\oplus[\mathbb{Z}^{2^{n+3}-8}\times\mathbb{Z}_{k_1}[z_1]\times\mathbb{Z}_{k_2}[z_1]\times\mathbb{Z}_{k_3}[z_1]]\\
&\cong \mathbb{Z}^{2^{n+4}-2^4+1}\times\mathbb{Z}_{k_1}[z_1]\times\mathbb{Z}_{k_2}[z_1]\times\mathbb{Z}_{k_3}[z_1]\times\mathbb{Z}_{k_4}[z_1]\\
&\cong \{[\oplus^{2^2}K_0(C(\mathbb{T}^{n+1}))]\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_2])]\\
&\quad\oplus([K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_3])]\times k_1\mathbb{Z}[z_3])\oplus[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]]\\
&\quad\oplus[K_1(C(\mathbb{T}^{n+1}))/(k_4\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_2]\oplus\mathbb{Z}[z_3]\oplus\mathbb{Z}[z_4])]\}\\
&\quad\oplus\{[\oplus^{2^2}K_0(C(\mathbb{T}^{n+1}))]\\
&\quad\oplus[K_1(C(\mathbb{T}^{n+1}))/(k_2\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_2]\oplus\mathbb{Z}[z_4]\oplus\mathbb{Z}[z_5])]\times(k_2\mathbb{Z}[z_4]\times k_2\mathbb{Z}[z_4])\\
&\quad\oplus[K_1(C(\mathbb{T}^{n+1}))/(k_1\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_3]\oplus\mathbb{Z}[z_4]\oplus\mathbb{Z}[z_5])]\\
&\qquad\times(k_1\mathbb{Z}[z_3]\times k_1\mathbb{Z}[z_4]\times k_1\mathbb{Z}[z_5])\}\\
&\quad\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2]\oplus\mathbb{Z}[z_3]\oplus k_3\mathbb{Z}[z_4]\oplus\mathbb{Z}[z_5])]\\
&\quad\oplus[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]]\}
\end{aligned}$$

and

$$\begin{aligned}
K_1(\mathfrak{A}_4) &\cong \mathbb{Z}^{2^{n+3}-8}\oplus\mathbb{Z}^{2^{n+3}-7}\cong\mathbb{Z}^{2^{n+4}-2^4+1}\\
&\cong \{[\oplus^{2^2}K_0(C(\mathbb{T}^{n+1}))]\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_2])]\\
&\quad\oplus([K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_3])]\times k_1\mathbb{Z}[z_3])\\
&\quad\oplus[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]]\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2]\oplus\mathbb{Z}[z_3]\oplus\mathbb{Z}[z_4]\oplus\mathbb{Z}[z_5])]\}\\
&\quad\oplus\{[\oplus^{2^2}K_0(C(\mathbb{T}^{n+1}))]\oplus([K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_2]\oplus\mathbb{Z}[z_4])]\times k_2\mathbb{Z}[z_4])\\
&\quad\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_3]\oplus\mathbb{Z}[z_4])]\times(k_1\mathbb{Z}[z_3]\times k_1\mathbb{Z}[z_4])\}\\
&\quad\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_2]\oplus\mathbb{Z}[z_3])]\oplus[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]]\}. \quad\lhd
\end{aligned}$$

Since the situation as above is the same in the inductive steps, alternatively, we obtain the following:

Theorem 4.10. *Let $H_{2n+1}(k)$ be the generalized (weighted) discrete Heisenberg subgroup with rank $2n+1$ with $k=(k_1,\cdots,k_n)$ and $C^*(H_{2n+1}(k))$ the group C^*-algebra of $H_{2n+1}(k)$, where the components $k_1,\cdots,k_n$ are some powers of primes and mutually relatively prime. If $n\geq 1$ is odd, then*

$$\begin{aligned}
K_0(C^*(H_{2n+1}(k))) &\cong \mathbb{Z}^{2^n(2^n-1)+1} \quad \textit{and}\\
K_1(C^*(H_{2n+1}(k)) &\cong \mathbb{Z}^{2^n(2^n-1)+1}\times\mathbb{Z}_{k_1}\times\cdots\times\mathbb{Z}_{k_n},
\end{aligned}$$

and if n is even, then

$$K_0(C^*(H_{2n+1}(k)) \cong \mathbb{Z}^{2^n(2^n-1)+1} \times \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_n} \quad \text{and}$$
$$K_1(C^*(H_{2n+1}(k)) \cong \mathbb{Z}^{2^n(2^n-1)+1},$$

where each $\mathbb{Z}_{k_j} = \mathbb{Z}_{k_j}[z_1]$ $(1 \leq j \leq n)$.

Moreover, these K-theory groups have inductive, direct sum decompositions which involves the direct sum $\oplus^{2^{n-1}} K_0(C(\mathbb{T}^{n+1}))$ and inductively many, mutually distinct 2^{n-1} quotients of $K_1(C(\mathbb{T}^{n+1}))$ as direct summands.

Furthermore, representatives for K-theory classes of $K_0(C^*(H_{2n+1}(k)))$ and $K_1(C^*(H_{2n+1}(k)))$ are assumed to be corresponding to the identity element of $C^*(H_{2n+1}(k))$ or generalized Bott projections, and to be to the unitary generators of $C^*(H_{2n+1}(k))$ or generalized Bott unitaries in the inductive, direct sum decompositions, respectively, up to changing variables among chosen variables.

Remark. We could not write down the more details for the inductive, direct sum decomposition in the general case, but it would be possible to do so if more cases such as $n = 5, 6$ or more than are considered as above. And then the induction is exactly completed, in fact. $\lhd$

5 More examples

As another kind of examples as an application by the same methods as in the previous section, we consider the following:

Example 5.1. We denote by N_{2n+1} $(1 \geq 2)$ the discrete nilpotent group with rank $2n+1$, consisting of the following $(n+2) \times (n+2)$ matrices:

$$\begin{pmatrix} \beta_t & x^{\mathbf{t}} \\ 0 & 1 \end{pmatrix} \in GL_{n+2}(\mathbb{Z}), \quad t \in \mathbb{Z}^n, x \in \mathbb{Z}^{n+1},$$

where $x^{\mathbf{t}}$ is the transpose of x and the matrix β_t is defined by

$$\beta_t = \begin{pmatrix} 1 & t_1 & & 0 \\ 0 & \ddots & \ddots & \\ \vdots & & \ddots & t_n \\ 0 & & & 1 \end{pmatrix} \in GL_{n+1}(\mathbb{Z}).$$

The group N_{2n+1} is viewed as the semi-direct product $\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n$ of tuples (x, t) with $x \in \mathbb{Z}^{n+1} = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{n+1}$ and $t = (t_i) \in \mathbb{Z}^n$, where the action β induced by matrix multiplication is defined by

$$\beta_t(x) = (x_1 + t_1x_2, x_2 + t_2x_3, \cdots, x_n + t_nx_{n+1}, x_{n+1}) \in \mathbb{Z}^{n+1}.$$

Compared to the action α of H_{2n+1}, since β is viewed as shifted from α in the sense that the summation $\sum_{j=1}^n t_jb_j$ in the definition of α_t is shifted componentwise in β_t, so thus we may call N_{2n+1} the **shifted** generalized discrete Heisenberg

group (from H_{2n+1}) (or the semi-direct product by the **standard** nilpotent action).

In this case, $F = \mathbb{Z}e_1 = \mathfrak{Z}$ and the following is exact:

$$0 \to F = \mathbb{Z}e_1 \to N_{2n+1} \to N_{2n+1}/F \cong [\mathbb{Z}^{n+1}/\mathbb{Z}e_1] \rtimes_\alpha \mathbb{Z}^n \to 0.$$

Note that the center of the quotient group is $[\mathbb{Z}e_1 \oplus \mathbb{Z}e_2]/\mathbb{Z}e_1$. Inductively, we obtain a central composition series for N_{2n+1} as

$$\mathbb{Z}e_1 \subset \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \subset \cdots \subset \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{n+1} \subset N_{2n+1}.$$

We have the following isomorphisms:

$$C^*(N_{2n+1}) \cong C^*(\mathbb{Z}^{n+1} \rtimes_\beta \mathbb{Z}^n) \cong C^*(\mathbb{Z}^{n+1}) \rtimes_\beta \mathbb{Z}^n$$

and the right hand side is a successive crossed product by $\mathbb{Z}$ as

$$C^*(\mathbb{Z}^{n+1}) \rtimes_\beta \mathbb{Z}^n \cong C^*(\mathbb{Z}^{n+1}) \rtimes_{\beta(1)} \mathbb{Z} \cdots \rtimes_{\beta(n)} \mathbb{Z}$$

with $\beta(j)$ the restriction of β to the j-th coordinate.

As the first step, the associated, Pimsner-Voiculescu six-term exact sequence becomes:

$$\begin{array}{ccccc}
\mathbb{Z}^{2^n} & \xrightarrow{(\mathrm{id}-\beta(1))_*} & \mathbb{Z}^{2^n} & \xrightarrow{i_*} & K_0(C^*(\mathbb{Z}^{n+1}) \rtimes_{\beta(1)} \mathbb{Z}) \\
\partial \uparrow & & & & \downarrow \partial \\
K_1(C^*(\mathbb{Z}^{n+1}) \rtimes_{\beta(1)} \mathbb{Z}) & \xleftarrow{i_*} & \mathbb{Z}^{2^n} & \xleftarrow{(\mathrm{id}-\beta(1))_*} & \mathbb{Z}^{2^n}.
\end{array}$$

As a note, $K_1(C^*(\mathbb{Z}^{n+1}))$ contains $\mathbb{Z}[u_1] \oplus \cdots \oplus \mathbb{Z}[u_{n+1}]$ (as a multiplicative group) for $u_j = \lambda_{e_j} = \chi_{e_j}$. Let $v_i = \lambda_{t_i}$ for $1 \le i \le n$. Let $1_i \in \mathbb{Z}^{n-1}$ with the i-th component 1 and other components zero. We then compute

$$\begin{aligned}
(\mathrm{id} - \beta(1))_*[u_1] &= [u_1] \cdot [(v_1 u_1 v_1^*)^{-1}] = [u_1] \cdot [\lambda^{-1}_{\beta_{1_1}(e_1)}] \\
&= [u_1] \cdot [\lambda^{-1}_{e_1}] = [u_1] \cdot [u_1^*] = [u_1 u_1^*] = [1], \\
(\mathrm{id} - \beta(1))_*[u_2] &= [u_j] \cdot [(v_1 u_j v_1^*)^{-1}] = [u_2] \cdot [\lambda^{-1}_{\beta_{1_1}(e_2)}] \\
&= [u_2] \cdot [\lambda^{-1}_{e_1+e_2}] = [u_2 u_2^* u_1^*] = [u_1^*]
\end{aligned}$$

and $(\mathrm{id} - \beta(1))_*[u_j] = [1]$ for $2 \le j \le n+1$ by the same computation.

Moreover, the map $(\mathrm{id} - \beta(1))_*$ on $K_0(C^*(\mathbb{Z}^{n+1})) \cong K_0(C(\mathbb{T}^{n+1}))$ is zero, because $K_0(C(\mathbb{T}^{n+1}))$ is generated by the class of the unit and the classes of generalized Bott projections, which are invariant under $\beta^\wedge$.

By the Fourier transform, we have

$$C^*(\mathbb{Z}^{n+1}) \rtimes_{\beta(1)} \mathbb{Z} \cong C(\mathbb{T}^{n+1}) \rtimes_{\beta(1)^\wedge} \mathbb{Z}.$$

By Lemma 2.1, we have

$$K_1(C(\mathbb{T}^{n+1})) \cong \oplus_{j_1+\cdots+j_{n+1}=1}[K_{j_1}(C(\mathbb{T})) \otimes \cdots \otimes K_{j_{n+1}}(C(\mathbb{T}))].$$

By Proposition 3.2, each direct factor in the direct sum above is assumed to be generated by either the K_1-classes $[z_j]$ $(1 \leq j \leq n+1)$ or the K_1-classes of generalized Bott unitaries $u(b(z_{k_1}, \cdots, z_{k_{2l}}), z_{k_{2l+1}})$ with $1 \leq k_1 < \cdots < k_{2l+1} \leq n+1$.

We then compute that if $j \neq 2$, then

$$\begin{aligned}(\mathrm{id} - \beta(1)^\wedge)_*[z_j] &= [z_j] \cdot [\beta(1)^\wedge(z_j)^{-1}] \\ &= [z_j] \cdot [(z^{\beta(1)(e_j)})^{-1}] = [z_j] \cdot [(z^{e_j})^{-1}] = [z_j] \cdot [z_j^{-1}] = [1],\end{aligned}$$

and if $j = 2$, then

$$\begin{aligned}(\mathrm{id} - \beta(1)^\wedge)_*[z_2] &= [z_2] \cdot [\beta(1)^\wedge(z_2)^{-1}] \\ &= [z_2] \cdot [(z^{\beta(1)(e_2)})^{-1}] = [z_2] \cdot [(z^{e_1+e_2})^{-1}] = [z_1^{-1}],\end{aligned}$$

Moreover, the K_1-classes of the generalized Bott unitaries are mapped to the identity K_1-class under the map $(\mathrm{id} - \beta(1)^\wedge)_*$ because the K_0-classes of the generalized Bott projections are mapped to zero K_0-class under the same map. Consequently, the image of $\mathbb{Z}^{2^n} \cong K_1(C(\mathbb{T}^{n+1}))$ is isomorphic to $\mathbb{Z}$.

Therefore, it follows from exactness of the diagram above that

$$K_0(C^*(\mathbb{Z}^{n+1}) \rtimes_{\beta(1)} \mathbb{Z}) \cong \mathbb{Z}^{2^{n+1}-1} \quad \text{and} \quad K_1(C^*(\mathbb{Z}^{n+1}) \rtimes_{\beta(1)} \mathbb{Z}) \cong \mathbb{Z}^{2^{n+1}-1}.$$

Note that

$$\begin{aligned}K_0(\mathfrak{A}_1) &\cong K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]], \\ K_1(\mathfrak{A}_1) &\cong K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]]\end{aligned}$$

with $\mathfrak{A}_1 = C^*(\mathbb{Z}^{n+1}) \rtimes_{\beta(1)} \mathbb{Z}$.

As the second step, we consider $\mathfrak{A}_2 = \mathfrak{A}_1 \rtimes_{\beta(2)} \mathbb{Z}$. The associated, Pimsner-Voiculescu six-term exact sequence becomes:

$$\begin{tikzcd}
\mathbb{Z}^{2^{n+1}-1} \arrow[r, "(\mathrm{id}-\beta(2))_*"] & \mathbb{Z}^{2^{n+1}-1} \arrow[r, "i_*"] & K_0(\mathfrak{A}_1 \rtimes_{\beta(2)} \mathbb{Z}) \arrow[d, "\partial"] \\
K_1(\mathfrak{A}_1 \rtimes_{\beta(2)} \mathbb{Z}) \arrow[u, "\partial"] & \mathbb{Z}^{2^{n+1}-1} \arrow[l, "i_*"'] & \mathbb{Z}^{2^{n+1}-1} \arrow[l, "(\mathrm{id}-\beta(2))_*"']
\end{tikzcd}$$

and

$$\begin{aligned}&(\mathrm{id} - \beta(2))_* K_0(\mathfrak{A}_1) \\ &\cong (\mathrm{id} - \beta(2))_* K_0(C(\mathbb{T}^{n+1})) \oplus (\mathrm{id} - \beta(2))_*[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]], \\ &\cong [0] \oplus [0] \cong [0], \\ &(\mathrm{id} - \beta(2))_* K_1(\mathfrak{A}_1) \\ &\cong (\mathrm{id} - \beta(2))_* K_0(C(\mathbb{T}^{n+1})) \oplus (\mathrm{id} - \beta(2))_*[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_1]] \\ &\cong [0] \oplus \mathbb{Z}[z_2] \cong \mathbb{Z},\end{aligned}$$

where the same computation as given above shows that $(\mathrm{id}-\beta(2)^\wedge)_*[z_j]=[1]$ if $j\neq 3$ and $(\mathrm{id}-\beta(2)^\wedge)_*[z_3]=[z_2^{-1}]$. It then follows that

$$\begin{aligned}
K_0(\mathfrak{A}_2) &\cong \mathbb{Z}^{2^{n+1}-1}\oplus\mathbb{Z}^{2^{n+1}-2}\cong\mathbb{Z}^{2^{n+2}+2(-1)-1}\\
&\cong\{K_0(C(\mathbb{T}^{n+1}))\oplus[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]]\}\\
&\quad\oplus\{K_0(C(\mathbb{T}^{n+1}))\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_3])]\},\\
K_1(\mathfrak{A}_2) &\cong \mathbb{Z}^{2^{n+1}-1}\oplus\mathbb{Z}^{2^{n+1}-2}\cong\mathbb{Z}^{2^{n+2}+2(-1)-1}\\
&\cong\{K_0(C(\mathbb{T}^{n+1}))\oplus[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]]\}\\
&\quad\oplus\{K_0(C(\mathbb{T}^{n+1}))\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_2])]\}.
\end{aligned}$$

As the third step, we consider $\mathfrak{A}_3=\mathfrak{A}_2\rtimes_{\beta(3)}\mathbb{Z}$. Since we have that $(\mathrm{id}-\beta(3)^\wedge)_*[z_j]=[1]$ if $j\neq 4$ and $(\mathrm{id}-\beta(3)^\wedge)_*[z_4]=[z_3^{-1}]$, then

$$\begin{aligned}
&(\mathrm{id}-\beta(3)^\wedge)_*K_0(\mathfrak{A}_2)\\
&\cong(\mathrm{id}-\beta(3)^\wedge)_*[\oplus^2K_0(C(\mathbb{T}^{n+1}))]\oplus(\mathrm{id}-\beta(3)^\wedge)_*[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]]\\
&\oplus(\mathrm{id}-\beta(3)^\wedge)_*[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_3])]\\
&\cong[0]\oplus\mathbb{Z}[z_3]\oplus[0]\cong\mathbb{Z},\\
&(\mathrm{id}-\beta(3)^\wedge)_*K_1(\mathfrak{A}_2)\\
&\cong(\mathrm{id}-\beta(3)^\wedge)_*[\oplus^2K_0(C(\mathbb{T}^{n+1}))]\oplus(\mathrm{id}-\beta(3)^\wedge)_*[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]]\\
&\oplus(\mathrm{id}-\beta(3)^\wedge)_*[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_2])]\\
&\cong[0]\oplus\mathbb{Z}[z_3]\oplus\mathbb{Z}[z_3]\cong\mathbb{Z}^2.
\end{aligned}$$

It then follows that

$$\begin{aligned}
K_0(\mathfrak{A}_3) &\cong \mathbb{Z}^{2^{n+2}-4}\oplus\mathbb{Z}^{2^{n+2}-5}\cong\mathbb{Z}^{2^{n+3}+2(-3)-3}\\
&\cong[\oplus^{2^2}K_0(C(\mathbb{T}^{n+1}))]\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2]\oplus\mathbb{Z}[z_3])]\\
&\quad\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_3])]\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_4])]\\
&\quad\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_2]\oplus\mathbb{Z}[z_4])],\\
K_1(\mathfrak{A}_3) &\cong \mathbb{Z}^{2^{n+2}-4}\oplus\mathbb{Z}^{2^{n+2}-5}\cong\mathbb{Z}^{2^{n+3}+2(-3)-3}\\
&\cong[\oplus^{2^2}K_0(C(\mathbb{T}^{n+1}))]\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2]\oplus\mathbb{Z}[z_4])]\\
&\quad\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_3])]\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2]\oplus\mathbb{Z}[z_3])]\\
&\quad\oplus[K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_2]\oplus\mathbb{Z}[z_3])].
\end{aligned}$$

Similarly, we obtain that the ranks of $K_j(\mathfrak{A}_4)$ $(j=0,1)$ are equal to $2^{n+4}+2(-9)-5$.

Now let $r_k=\mathrm{rank}_{\mathbb{Z}}K_j(\mathfrak{A}_k)-2^{n+k}$ be the minus part of the $\mathbb{Z}$-rank of $K_j(\mathfrak{A}_k)$ in the computation as above. Then we have the following inductive relation:

$$r_k=2r_{k-1}-(2(k-1)-1)$$

for $k \geq 2$, where $r_1 = -1$. Set $s_k = r_k - r_{k-1}$ for $k \geq 2$. Then

$$s_k = 2s_{k-1} - 2, \quad k \geq 3.$$

This equation is converted to $s_k - 2 = 2(s_{k-1} - 2)$. It follows that $s_k - 2 = 2^{k-2}(s_2 - 2)$ for $k \geq 2$, with $s_2 = r_2 - r_1 = -3 + 1 = -2$. Therefore,

$$r_k = r_1 + \sum_{j=2}^{k} s_j = -1 + \sum_{j=2}^{k}[2 - 2^j] = -1 + 2(k-1) - 2^2(2^{k-1} - 1)$$

Hence the ranks of $K_j(C^*(N_{2n+1}))$ $(j = 0, 1)$ are computed as

$$\begin{aligned} 2^{2n} + r_n &= 2^{2n} - 1 + 2(n-1) - 2^2(2^{n-1} - 1) \\ &= 2^{2n} - 2^{n+1} + 2n + 1 = 2^n(2^n - 2) + 2n + 1 \end{aligned}$$

$(n \geq 2)$. $\triangleleft$

We then obtain the following:

Theorem 5.2. *Let $C^*(N_{2n+1})$ be the group C^*-algebra of the shifted, generalized discrete Heisenberg group N_{2n+1} with rank $2n+1$ $(n \geq 1)$. Then*

$$K_0(C^*(N_{2n+1})) \cong \mathbb{Z}^{2^n(2^n-2)+2n+1} \cong K_1(C^*(N_{2n+1})).$$

Moreover, these K-theory groups have the inductive direct sum decompositions whose direct factors are given by $\oplus^{2^{n-1}} K_0(C(\mathbb{T}^{n+1}))$ and some direct sums of quotients of $K_1(C(\mathbb{T}^{n+1}))$ by some direct sums of $\mathbb{Z}[z_j]$ $(1 \leq j \leq n+1)$.

Furthermore, representatives for K-theory classes of $K_0(C^*(N_{2n+1}))$ and $K_1(C^*(N_{2n+1}))$ are assumed to be corresponding to the identity element of $C^*(N_{2n+1})$ or generalized Bott projections, and to be to the unitary generators of $C^*(N_{2n+1})$ or generalized Bott unitaries in those direct sum decompositions, respectively, up to changing variables among chosen variables.

Example 5.3. Let $k_1, \cdots, k_n$ be positive integers (as weights), and set $k = (k_1, \cdots, k_n)$. We define $N_{2n+1}(k)$ $(n \geq 1)$ to be the **(weighted) shifted**, generalized discrete Heisenberg **subgroup** of N_{2n+1} with rank $2n+1$, consisting of the following $(n+2) \times (n+2)$ matrices:

$$\begin{pmatrix} \beta_t & x^{\mathbf{t}} \\ 0 & 1 \end{pmatrix} \in GL_{n+2}(\mathbb{Z}), \quad t \in \mathbb{Z}^n, x \in \mathbb{Z}^{n+1},$$

where $x^{\mathbf{t}}$ is the transpose of x and

$$\beta_t = \begin{pmatrix} 1 & t_1k_1 & & 0 \\ 0 & \ddots & \ddots & \\ \vdots & & \ddots & t_nk_n \\ 0 & & & 1 \end{pmatrix} \in GL_{n+1}(\mathbb{Z}).$$

The group $N_{2n+1}(k)$ is viewed as the semi-direct product $\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n$ of tuples (x,t) with $x \in \mathbb{Z}^{n+1} = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{n+1}$ and $t = (t_i) \in \mathbb{Z}^n$, where the action β induced by matrix multiplication is defined by

$$\beta_t(x) = (x_1 + t_1k_1x_2, x_2 + t_2k_2x_3, \cdots, x_n + t_nk_nx_{n+1}, x_{n+1}) \in \mathbb{Z}^{n+1}.$$

In this case, $F = \mathbb{Z}e_1 = \mathfrak{Z}$ and the following is exact:

$$0 \to F = \mathbb{Z}e_1 \to N_{2n+1}(k) \to N_{2n+1}/F \cong [\mathbb{Z}^{n+1}/\mathbb{Z}e_1] \rtimes_\beta \mathbb{Z}^n \to 0.$$

Note that the center of the quotient group is $[\mathbb{Z}e_1 \oplus \mathbb{Z}e_2]/\mathbb{Z}e_1$. Inductively, we obtain a central composition series for $N_{2n+1}(k)$ as

$$\mathbb{Z}e_1 \subset \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \subset \cdots \subset \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{n+1} \subset N_{2n+1}(k).$$

We have the following isomorphisms:

$$C^*(N_{2n+1}(k)) \cong C^*(\mathbb{Z}^{n+1} \rtimes_\beta \mathbb{Z}^n) \cong C^*(\mathbb{Z}^{n+1}) \rtimes_\beta \mathbb{Z}^n$$

and the right hand side is a successive crossed product by $\mathbb{Z}$ as

$$C^*(\mathbb{Z}^{n+1}) \rtimes_\beta \mathbb{Z}^n \cong C^*(\mathbb{Z}^{n+1}) \rtimes_{\beta(1)} \mathbb{Z} \cdots \rtimes_{\beta(n)} \mathbb{Z}$$

with $\beta(j)$ the restriction of β to the j-th coordinate.

As the first step, the associated, Pimsner-Voiculescu six-term exact sequence becomes:

$$\begin{array}{ccccc}
\mathbb{Z}^{2^n} & \xrightarrow{(\mathrm{id}-\beta(1))_*} & \mathbb{Z}^{2^n} & \xrightarrow{i_*} & K_0(C^*(\mathbb{Z}^{n+1}) \rtimes_{\beta(1)} \mathbb{Z}) \\
\partial \uparrow & & & & \downarrow \partial \\
K_1(C^*(\mathbb{Z}^{n+1}) \rtimes_{\beta(1)} \mathbb{Z}) & \xleftarrow{i_*} & \mathbb{Z}^{2^n} & \xleftarrow{(\mathrm{id}-\beta(1))_*} & \mathbb{Z}^{2^n}.
\end{array}$$

As a note, $K_1(C^*(\mathbb{Z}^{n+1}))$ contains $\mathbb{Z}[u_1] \oplus \cdots \oplus \mathbb{Z}[u_{n+1}]$ (as a multiplicative group) for $u_j = \lambda_{e_j} = \chi_{e_j}$. Let $v_i = \lambda_{t_i}$ for $1 \le i \le n$. Let $1_i \in \mathbb{Z}^n$ with the i-th component 1 and other components zero. We then compute

$$\begin{aligned}
(\mathrm{id} - \beta(1))_*[u_1] &= [u_1] \cdot [(v_1^{k_1} u_1 (v_1^{k_1})^*)^{-1}] = [u_1] \cdot [\lambda^{-1}_{\beta_{1_1}(e_1)}] \\
&= [u_1] \cdot [\lambda^{-1}_{e_1}] = [u_1] \cdot [u_1^*] = [u_1 u_1^*] = [1], \\
(\mathrm{id} - \beta(1))_*[u_2] &= [u_j] \cdot [(v_1^{k_1} u_j (v_1^{k_1})^*)^{-1}] = [u_2] \cdot [\lambda^{-1}_{\beta_{1_1}(e_2)}] \\
&= [u_2] \cdot [\lambda^{-1}_{k_1 e_1 + e_2}] = [u_2 u_2^* (u_1^*)^{k_1}] = [(u_1^*)^{k_1}]
\end{aligned}$$

and $(\mathrm{id} - \beta(1))_*[u_j] = [1]$ for $2 \le j \le n+1$ by the same computation.

Moreover, the map $(\mathrm{id} - \beta(1))_*$ on $K_0(C^*(\mathbb{Z}^{n+1})) \cong K_0(C(\mathbb{T}^{n+1}))$ is zero, because $K_0(C(\mathbb{T}^{n+1}))$ is generated by the class of the unit and the classes of generalized Bott projections, which are invariant under $\beta^\wedge$.

By the Fourier transform, we have

$$C^*(\mathbb{Z}^{n+1}) \rtimes_{\beta(1)} \mathbb{Z} \cong C(\mathbb{T}^{n+1}) \rtimes_{\beta(1)^\wedge} \mathbb{Z}.$$

By Lemma 2.1, we have

$$K_1(C(\mathbb{T}^{n+1})) \cong \oplus_{j_1+\cdots+j_{n+1}=1}[K_{j_1}(C(\mathbb{T})) \otimes \cdots \otimes K_{j_{n+1}}(C(\mathbb{T}))].$$

By Proposition 3.2, each direct factor in the direct sum above is assumed to be generated by either the K_1-classes $[z_j]$ $(1 \leq j \leq n+1)$ or the K_1-classes of generalized Bott unitaries $u(b(z_{k_1}, \cdots, z_{k_{2l}}), z_{k_{2l+1}})$ with $1 \leq k_1 < \cdots < k_{2l+1} \leq n+1$.

We then compute that if $j \neq 2$, then

$$\begin{aligned}(\mathrm{id} - \beta(1)^\wedge)_*[z_j] &= [z_j] \cdot [\beta(1)^\wedge(z_j)^{-1}] \\ &= [z_j] \cdot [(z^{\beta(1)(e_j)})^{-1}] = [z_j] \cdot [(z^{e_j})^{-1}] = [z_j] \cdot [z_j^{-1}] = [1],\end{aligned}$$

and if $j = 2$, then

$$\begin{aligned}(\mathrm{id} - \beta(1)^\wedge)_*[z_2] &= [z_2] \cdot [\beta(1)^\wedge(z_2)^{-1}] \\ &= [z_2] \cdot [(z^{\beta(1)(e_2)})^{-1}] = [z_2] \cdot [(z^{k_1e_1+e_2})^{-1}] = [z_1^{-k_1}],\end{aligned}$$

Moreover, the K_1-classes of the generalized Bott unitaries are mapped to the identity K_1-class under the map $(\mathrm{id} - \beta(1)^\wedge)_*$ because the K_0-classes of the generalized Bott projections are mapped to zero K_0-class under the same map. Consequently, the image of $\mathbb{Z}^{2^n} \cong K_1(C(\mathbb{T}^{n+1}))$ is isomorphic to $k_1\mathbb{Z}[z_1]$.

Therefore, it follows from exactness of the diagram above that

$$\begin{aligned}&K_0(C^*(\mathbb{Z}^{n+1}) \rtimes_{\beta(1)} \mathbb{Z}) \cong \mathbb{Z}^{2^{n+1}-1} \quad \text{and} \\ &K_1(C^*(\mathbb{Z}^{n+1}) \rtimes_{\beta(1)} \mathbb{Z}) \cong \mathbb{Z}^{2^{n+1}-1} \times \mathbb{Z}_{k_1}[z_1].\end{aligned}$$

Note that

$$\begin{aligned}K_0(\mathfrak{A}_1) &\cong K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]], \\ K_1(\mathfrak{A}_1) &\cong K_0(C(\mathbb{T}^{n+1})) \oplus [K_1(C(\mathbb{T}^{n+1}))/k_1\mathbb{Z}[z_1]]\end{aligned}$$

with $\mathfrak{A}_1 = C^*(\mathbb{Z}^{n+1}) \rtimes_{\beta(1)} \mathbb{Z}$.

As the second step, we consider $\mathfrak{A}_2 = \mathfrak{A}_1 \rtimes_{\beta(2)} \mathbb{Z}$. The associated, Pimsner-Voiculescu six-term exact sequence becomes:

$$\begin{array}{ccccc}
\mathbb{Z}^{2^{n+1}-1} & \xrightarrow{(\mathrm{id}-\beta(2))_*} & \mathbb{Z}^{2^{n+1}-1} & \xrightarrow{i_*} & K_0(\mathfrak{A}_1 \rtimes_{\beta(2)} \mathbb{Z}) \\
\partial \uparrow & & & & \downarrow \partial \\
K_1(\mathfrak{A}_1 \rtimes_{\beta(2)} \mathbb{Z}) & \xleftarrow{i_*} & \mathbb{Z}^{2^{n+1}-1} \times \mathbb{Z}_{k_1}[z_1] & \xleftarrow{(\mathrm{id}-\beta(2))_*} & \mathbb{Z}^{2^{n+1}-1} \times \mathbb{Z}_{k_1}[z_1]
\end{array}$$

and

$$
\begin{aligned}
&(\mathrm{id}-\beta(2))_* K_0(\mathfrak{A}_1)\\
&\cong (\mathrm{id}-\beta(2))_* K_0(C(\mathbb{T}^{n+1})) \oplus (\mathrm{id}-\beta(2))_*[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]],\\
&\cong [0]\oplus[0]\cong[0],\\
&(\mathrm{id}-\beta(2))_* K_1(\mathfrak{A}_1)\\
&\cong (\mathrm{id}-\beta(2))_* K_0(C(\mathbb{T}^{n+1})) \oplus (\mathrm{id}-\beta(2))_*[K_1(C(\mathbb{T}^{n+1}))/k_1\mathbb{Z}[z_1]]\\
&\cong [0]\oplus k_2\mathbb{Z}[z_2]\cong\mathbb{Z},
\end{aligned}
$$

where the same computation as given above shows that $(\mathrm{id}-\beta(2)^\wedge)_*[z_j]=[1]$ if $j\neq 3$ and $(\mathrm{id}-\beta(2)^\wedge)_*[z_3]=[z_2^{-k_2}]$. It then follows that

$$
\begin{aligned}
K_0(\mathfrak{A}_2) &\cong \mathbb{Z}^{2^{n+1}-1}\oplus[\mathbb{Z}^{2^{n+1}-2}\times\mathbb{Z}_{k_1}[z_1]]\cong\mathbb{Z}^{2^{n+2}+2(-1)-1}\times\mathbb{Z}_{k_1}[z_1]\\
&\cong\{K_0(C(\mathbb{T}^{n+1}))\oplus[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]]\}\\
&\quad\oplus\{K_0(C(\mathbb{T}^{n+1}))\oplus[K_1(C(\mathbb{T}^{n+1}))/(k_1\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_3])]\},\\
K_1(\mathfrak{A}_2) &\cong \mathbb{Z}^{2^{n+1}-1}\oplus[\mathbb{Z}^{2^{n+1}-2}\times\mathbb{Z}_{k_1}[z_1]\times\mathbb{Z}_{k_2}[z_2]]\\
&\cong\mathbb{Z}^{2^{n+2}+2(-1)-1}\times\mathbb{Z}_{k_1}[z_1]\times\mathbb{Z}_{k_2}[z_2]\\
&\cong\{K_0(C(\mathbb{T}^{n+1}))\oplus[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]]\}\\
&\quad\oplus\{K_0(C(\mathbb{T}^{n+1}))\oplus[K_1(C(\mathbb{T}^{n+1}))/(k_1\mathbb{Z}[z_1]\oplus k_2\mathbb{Z}[z_2])]\}.
\end{aligned}
$$

As the third step, we consider $\mathfrak{A}_3=\mathfrak{A}_2\rtimes_{\beta(3)}\mathbb{Z}$. Since we have that $(\mathrm{id}-\beta(3)^\wedge)_*[z_j]=[1]$ if $j\neq 4$ and $(\mathrm{id}-\beta(3)^\wedge)_*[z_4]=[z_3^{-k_3}]$, then

$$
\begin{aligned}
&(\mathrm{id}-\beta(3)^\wedge)_* K_0(\mathfrak{A}_2)\\
&\cong (\mathrm{id}-\beta(3)^\wedge)_*[\oplus^2 K_0(C(\mathbb{T}^{n+1}))]\oplus(\mathrm{id}-\beta(3)^\wedge)_*[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]]\\
&\oplus(\mathrm{id}-\beta(3)^\wedge)_*[K_1(C(\mathbb{T}^{n+1}))/(k_1\mathbb{Z}[z_1]\oplus\mathbb{Z}[z_3])]\\
&\cong[0]\oplus k_3\mathbb{Z}[z_3]\oplus[0]\cong\mathbb{Z},\\
&(\mathrm{id}-\beta(3)^\wedge)_* K_1(\mathfrak{A}_2)\\
&\cong (\mathrm{id}-\beta(3)^\wedge)_*[\oplus^2 K_0(C(\mathbb{T}^{n+1}))]\oplus(\mathrm{id}-\beta(3)^\wedge)_*[K_1(C(\mathbb{T}^{n+1}))/\mathbb{Z}[z_2]]\\
&\oplus(\mathrm{id}-\beta(3)^\wedge)_*[K_1(C(\mathbb{T}^{n+1}))/(k_1\mathbb{Z}[z_1]\oplus k_2\mathbb{Z}[z_2])]\\
&\cong[0]\oplus k_3\mathbb{Z}[z_3]\oplus k_3\mathbb{Z}[z_3]\cong\mathbb{Z}^2.
\end{aligned}
$$

It then follows that

$$\begin{aligned}
K_0(\mathfrak{A}_3) &\cong [\mathbb{Z}^{2^{n+2}-4} \times \mathbb{Z}_{k_1}[z_1] \times \mathbb{Z}_{k_3}[z_3]] \oplus [\mathbb{Z}^{2^{n+2}-5} \times \mathbb{Z}_{k_1}[z_1] \times \mathbb{Z}_{k_2}[z_2]] \\
&\cong \mathbb{Z}^{2^{n+3}+2(-3)-3} \times (\mathbb{Z}_{k_1}[z_1])^2 \times \mathbb{Z}_{k_2}[z_2] \times \mathbb{Z}_{k_3}[z_3] \\
&\cong [\oplus^{2^2} K_0(C(\mathbb{T}^{n+1}))] \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2] \oplus k_3\mathbb{Z}[z_3])] \\
&\quad \oplus [K_1(C(\mathbb{T}^{n+1}))/(k_1\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_3])] \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_4])] \\
&\quad \oplus [K_1(C(\mathbb{T}^{n+1}))/(k_1\mathbb{Z}[z_1] \oplus k_2\mathbb{Z}[z_2] \oplus \mathbb{Z}[z_4])], \\
K_1(\mathfrak{A}_3) &\cong [\mathbb{Z}^{2^{n+2}-4} \times \mathbb{Z}_{k_1}[z_1]] \oplus [\mathbb{Z}^{2^{n+2}-5} \times \mathbb{Z}_{k_1}[z_1] \times \mathbb{Z}_{k_2}[z_2] \times (\mathbb{Z}_{k_3}[z_3])^2] \\
&\cong \mathbb{Z}^{2^{n+3}+2(-3)-3} \times (\mathbb{Z}_{k_1}[z_1])^2 \times \mathbb{Z}_{k_2}[z_2] \times (\mathbb{Z}_{k_3}[z_3])^2 \\
&\cong [\oplus^{2^2} K_0(C(\mathbb{T}^{n+1}))] \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2] \oplus \mathbb{Z}[z_4])] \\
&\quad \oplus [K_1(C(\mathbb{T}^{n+1}))/(k_1\mathbb{Z}[z_1] \oplus \mathbb{Z}[z_3])] \oplus [K_1(C(\mathbb{T}^{n+1}))/(\mathbb{Z}[z_2] \oplus k_3\mathbb{Z}[z_3])] \\
&\quad \oplus [K_1(C(\mathbb{T}^{n+1}))/(k_1\mathbb{Z}[z_1] \oplus k_2\mathbb{Z}[z_2] \oplus k_3\mathbb{Z}[z_3])]. \quad \lhd
\end{aligned}$$

Inductively, as in the case of $N_{2n+1} = N_{2n+1}(k)$ where $k = (1, \cdots, 1) \in \mathbb{Z}^n$, we then obtain the following:

Theorem 5.4. *Let $C^*(N_{2n+1}(k))$ be the group C^*-algebra of the (weighted) discrete nilpotent subgroup of the shifted N_{2n+1} with rank $2n+1$ $(n \geq 1)$ and $k = (k_1, \cdots, k_n)$ any of positive integers. Then both of the free parts of $K_0(C^*(N_{2n+1}(k)))$ and $K_1(C^*(N_{2n+1}(k)))$ are isomorphic to $\mathbb{Z}^{2^n(2^n-2)+2n+1}$ but $K_0(C^*(N_{2n+1}(k)))$ is not isomorphic to $K_1(C^*(N_{2n+1}(k)))$, where both of the torsion parts of $K_0(C^*(N_{2n+1}(k)))$ and $K_1(C^*(N_{2n+1}(k)))$ are not isomorphic and given inductively by some direct sums of $\mathbb{Z}_{k_j}[z_j]$ with some multiplicities $(1 \leq j \leq n)$, and the direct sum of the torsion parts of K_0 and K_1 has the number of direct product factors $2^{n+1} - 2n - 1$ with some multiplicities.*

In particular, when $n = 1, 2, 3$,

$$K_0(C^*(N_3(k))) \cong \mathbb{Z}^3 \quad \text{and} \quad K_1(C^*(N_3(k))) \cong \mathbb{Z}^3 \times \mathbb{Z}_{k_1}[z_1],$$

with $0 + 1 = 2^2 - 2 - 1$, and

$$\begin{aligned}
&K_0(C^*(N_5(k))) \cong \mathbb{Z}^{13} \times \mathbb{Z}_{k_1}[z_1] \\
&\quad \text{and} \quad K_1(C^*(N_5(k))) \cong \mathbb{Z}^{13} \times \mathbb{Z}_{k_1}[z_1] \times \mathbb{Z}_{k_2}[z_2],
\end{aligned}$$

with $1 + 2 = 2^3 - 4 - 1$, and

$$\begin{aligned}
&K_0(C^*(N_7(k))) \cong \mathbb{Z}^{55} \times (\mathbb{Z}_{k_1}[z_1])^2 \times \mathbb{Z}_{k_2}[z_2] \times \mathbb{Z}_{k_3}[z_3] \\
&\quad \text{and} \quad K_1(C^*(N_7(k))) \cong \mathbb{Z}^{55} \times (\mathbb{Z}_{k_1}[z_1])^2 \times \mathbb{Z}_{k_2}[z_2] \times (\mathbb{Z}_{k_3}[z_3])^2,
\end{aligned}$$

with $4 + 5 = 2^4 - 6 - 1$.

Moreover, both of the free and torsion parts of these K-theory groups have the inductive direct sum decompositions whose direct factors are given by the direct sum $\oplus^{2^{n-1}} K_0(C(\mathbb{T}^{n+1}))$ and some direct sums of quotients of $K_1(C(\mathbb{T}^{n+1}))$ by some direct sums of either $\mathbb{Z}[z_j]$ $(1 \leq j \leq n+1)$ or $k_j\mathbb{Z}[z_j]$ $(1 \leq j \leq n)$.

Furthermore, representatives for K-theory classes of $K_0(C^*(N_{2n+1}(k)))$ and $K_1(C^*(N_{2n+1}(k)))$ are assumed to be corresponding to the identity element of $C^*(N_{2n+1}(k))$ or generalized Bott projections, and to be to the unitary generators of $C^*(N_{2n+1}(k))$ or generalized Bott unitaries in those direct sum decompositions, respectively, up to changing variables among chosen variables.

6 Further applications

As further applications, we consider the following:

Example 6.1. Let H_{2n+1} be the generalized discrete Heisenberg group and N_{2m+1} the shifted of H_{2m+1}. Let $H_{2n+1} \times N_{2m+1}$ be the direct product of H_{2n+1} and N_{2m+1}. The direct product is viewed as the nilpotent discrete group of the following $(n+m+3) \times (n+m+3)$ matrices:

$$\begin{pmatrix} \alpha_t & 0_{n+1,m+1} & x^{\mathbf{t}} \\ 0_{m+1,n+1} & \beta_s & y^{\mathbf{t}} \\ 0_{1,n+1} & 0_{1,m+1} & 1 \end{pmatrix} \in GL_{n+m+3}(\mathbb{Z}), \quad x \in \mathbb{Z}^{n+1}, y \in \mathbb{Z}^{m+1},$$

where $0_{k,l}$ is the $k \times l$ zero matrix and the matrices α_t and β_s for $t \in \mathbb{Z}^n$ and $s \in \mathbb{Z}^m$ correspond to the actions in H_{2n+1} and N_{2m+1} respectively. Let $C^*(H_{2n+1} \times N_{2m+1})$ be the group C^*-algebra of $H_{2n+1} \times N_{2m+1}$. We have

$$C^*(H_{2n+1} \times N_{2m+1}) \cong C^*(H_{2n+1}) \otimes C^*(N_{2m+1}).$$

Since the K-theory groups of $C^*(H_{2n+1})$ and $C^*(N_{2m+1})$ are torsion free, as shown above, then the Künneth formula in K-theory for C^*-algebras implies that

$$\begin{aligned}
&K_0(C^*(H_{2n+1}) \otimes C^*(N_{2m+1})) \\
&\cong [K_0(C^*(H_{2n+1})) \otimes K_0(C^*(N_{2m+1}))] \oplus [K_1(C^*(H_{2n+1})) \otimes K_1(C^*(N_{2m+1}))] \\
&\cong \oplus^2[\mathbb{Z}^{2^n(2^n-1)+1} \otimes \mathbb{Z}^{2^m(2^m-2)+2m+1}] \\
&\cong \oplus^2[\oplus^{(2^n(2^n-1)+1)(2^m(2^m-2)+2m+1)}\mathbb{Z} \otimes \mathbb{Z}] \\
&\cong \oplus^{2(2^n(2^n-1)+1)(2^m(2^m-2)+2m+1)}\mathbb{Z}
\end{aligned}$$

with $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$, where $n \otimes m = 1 \otimes nm = nm(1 \otimes 1)$ for $n, m \in \mathbb{Z}$, and also

$$\begin{aligned}
&K_1(C^*(H_{2n+1}) \otimes C^*(N_{2m+1})) \\
&\cong [K_0(C^*(H_{2n+1})) \otimes K_1(C^*(N_{2m+1}))] \oplus [K_1(C^*(H_{2n+1})) \otimes K_0(C^*(N_{2m+1}))] \\
&\cong \oplus^2[\mathbb{Z}^{2^n(2^n-1)+1} \otimes \mathbb{Z}^{2^m(2^m-2)+2m+1}] \\
&\cong \oplus^2[\oplus^{(2^n(2^n-1)+1)(2^m(2^m-2)+2m+1)}\mathbb{Z} \otimes \mathbb{Z}] \\
&\cong \oplus^{2(2^n(2^n-1)+1)(2^m(2^m-2)+2m+1)}\mathbb{Z}. \quad \lhd
\end{aligned}$$

As a result, and similarly, it follows by the same computation as above that

Corollary 6.2. *We obtain that*

$$\begin{aligned}
K_0(C^*(H_{2n+1} \times N_{2m+1})) &\cong \oplus^{2(2^n(2^n-1)+1)(2^m(2^m-2)+2m+1)}\mathbb{Z} \\
&\cong K_1(C^*(H_{2n+1} \times N_{2m+1})), \\
K_0(C^*(H_{2n+1} \times H_{2m+1})) &\cong \oplus^{2(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z} \\
&\cong K_1(C^*(H_{2n+1} \times H_{2m+1})), \\
K_0(C^*(N_{2n+1} \times N_{2m+1})) &\cong \oplus^{2(2^n(2^n-2)+2n+1)(2^m(2^m-2)+2m+1)}\mathbb{Z} \\
&\cong K_1(C^*(N_{2n+1} \times N_{2m+1})).
\end{aligned}$$

Next,

Example 6.3. We consider the direct product $H_{2n+1} \times H_{2m+1}(k)$. Since $C^*(H_{2n+1})$ has torsion free K-theory groups, then the Künneth formula implies that if m is odd, then

$$\begin{aligned}
&K_0(C^*(H_{2n+1}) \otimes C^*(H_{2m+1}(k))) \\
&\cong [K_0(C^*(H_{2n+1})) \otimes K_0(C^*(H_{2m+1}(k)))] \\
&\quad \oplus [K_1(C^*(H_{2n+1})) \otimes K_1(C^*(H_{2m+1}(k)))] \\
&\cong [\mathbb{Z}^{2^n(2^n-1)+1} \otimes \mathbb{Z}^{2^m(2^m-1)+1}] \\
&\quad \oplus [\mathbb{Z}^{2^n(2^n-1)+1} \otimes (\mathbb{Z}^{2^m(2^m-1)+1} \times \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_m})] \\
&\cong \{\oplus^2[\oplus^{(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z} \otimes \mathbb{Z}]\} \oplus \{\oplus_{j=1}^m[\oplus^{2^n(2^n-1)+1}\mathbb{Z} \otimes \mathbb{Z}_{k_j}] \\
&\cong [\oplus^{2(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z}] \oplus \{\oplus_{j=1}^m[\oplus^{2^n(2^n-1)+1}\mathbb{Z}_{k_j}]\}
\end{aligned}$$

with $\mathbb{Z} \otimes \mathbb{Z}_{k_j} \cong \mathbb{Z}_{k_j}$, where $k_j(1 \otimes 1) = 1 \otimes k_j = 0$, and also

$$\begin{aligned}
&K_1(C^*(H_{2n+1}) \otimes C^*(H_{2m+1}(k))) \\
&\cong [K_0(C^*(H_{2n+1})) \otimes K_1(C^*(H_{2m+1}(k)))] \\
&\quad \oplus [K_1(C^*(H_{2n+1})) \otimes K_0(C^*(H_{2m+1}(k)))],
\end{aligned}$$

which is isomorphic to the same resulting direct sum as above by the same computation. Even if m is even, by symmetry, we obtain the same conclusion as in the case where m is odd. $\lhd$

As a result, and similarly, it follows by the same computation as above that

Corollary 6.4. *We obtain that*

$$\begin{aligned}
&K_0(C^*(H_{2n+1} \times H_{2m+1}(k))) \\
&\cong [\oplus^{2(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z}] \oplus \{\oplus_{j=1}^m[\oplus^{2^n(2^n-1)+1}\mathbb{Z}_{k_j}]\} \\
&\cong K_1(C^*(H_{2n+1} \times H_{2m+1}(k))), \\
&K_0(C^*(N_{2n+1} \times H_{2m+1}(k))) \\
&\cong [\oplus^{2(2^n(2^n-2)+2n+1)(2^m(2^m-1)+1)}\mathbb{Z}] \oplus \{\oplus_{j=1}^m[\oplus^{2^n(2^n-2)+2n+1}\mathbb{Z}_{k_j}]\} \\
&\cong K_1(C^*(N_{2n+1} \times H_{2m+1}(k))).
\end{aligned}$$

As another explicitly computable case,

Example 6.5. We consider the direct product $H_{2n+1}(k) \times H_{2m+1}(l)$, where $k = (k_1, \cdots, k_n)$ and $l = (l_1, \cdots, l_m)$ such that both of components k_i and l_j are some powers of primes and mutually relatively prime, but each pair of k_i and l_j may not be relatively prime.

If both n and m are odd, then

$$\begin{aligned}
&K_0(C^*(H_{2n+1}(k)) \otimes K_0(C^*(H_{2m+1}(l))) \\
&\cong [K_0(C^*(H_{2n+1}(k))) \otimes K_0(C^*(H_{2m+1}(l)))] \\
&\quad \oplus [K_1(C^*(H_{2n+1}(k))) \otimes K_1(C^*(H_{2m+1}(l)))] \\
&\cong [\mathbb{Z}^{2^n(2^n-1)+1} \otimes \mathbb{Z}^{2^m(2^m-1)+1}] \\
&\quad \oplus [(\mathbb{Z}^{2^n(2^n-1)+1} \times \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_n}) \otimes (\mathbb{Z}^{2^m(2^m-1)+1} \times \mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_m})] \\
&\cong \{\oplus^2[\oplus^{(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z} \otimes \mathbb{Z}]\} \oplus \{\oplus_{i=1}^n[\oplus^{2^m(2^m-1)+1}\mathbb{Z} \otimes \mathbb{Z}_{k_i}]\} \\
&\quad \oplus \{\oplus_{j=1}^m[\oplus^{2^n(2^n-1)+1}\mathbb{Z} \otimes \mathbb{Z}_{l_j}]\} \oplus [\oplus_{i=1}^n \oplus_{j=1}^m \mathbb{Z}_{k_i} \otimes \mathbb{Z}_{l_j}] \\
&\cong [\oplus^{2(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z}] \oplus \{\oplus_{i=1}^n[\oplus^{2^m(2^m-1)+1}\mathbb{Z}_{k_i}]\} \\
&\quad \oplus \{\oplus_{j=1}^m[\oplus^{2^n(2^n-1)+1}\mathbb{Z}_{l_j}]\} \oplus [\oplus_{i=1}^n \oplus_{j=1}^m \mathbb{Z}_{k_i \wedge l_j}]
\end{aligned}$$

with $\mathbb{Z}_{k_i} \otimes \mathbb{Z}_{l_j} \cong \mathbb{Z}_{k_i \wedge l_j}$, where $k_i \wedge l_j = \min\{k_i, l_j\}$ and $(k_i \wedge l_j)(1 \otimes 1) = (k_i \wedge l_j) \otimes 1 = 1 \otimes (k_i \wedge l_j) = 0$. Indeed, the Künneth formula implies that the following short exact sequence splits:

$$\begin{aligned}
0 \to\ & [K_0(C^*(H_{2n+1}(k))) \otimes K_0(C^*(H_{2m+1}(l)))] \\
& \oplus [K_1(C^*(H_{2n+1}(k))) \otimes K_1(C^*(H_{2m+1}(l)))] \\
\to\ & K_0(C^*(H_{2n+1}(k)) \otimes K_0(C^*(H_{2m+1}(l))) \\
\to\ & \mathrm{Tor}_1^{\mathbb{Z}}(K_0(C^*(H_{2n+1}(k))), K_1(C^*(H_{2m+1}(l))) \\
& \oplus \mathrm{Tor}_1^{\mathbb{Z}}(K_1(C^*(H_{2n+1}(k))), K_0(C^*(H_{2m+1}(l))) \to 0,
\end{aligned}$$

where the torsion products $\mathrm{Tor}_1^{\mathbb{Z}}$ in the splitting quotient are both zero, because K_0 are torsion free.

Similarly, but $K_1(C^*(H_{2n+1}(k)) \otimes C^*(H_{2m+1}(l)))$ contains the following as a normal subgroup:

$$\begin{aligned}
&[K_0(C^*(H_{2n+1}(k))) \otimes K_1(C^*(H_{2m+1}(l)))] \\
&\quad \oplus [K_1(C^*(H_{2n+1}(k))) \otimes K_0(C^*(H_{2m+1}(l)))] \\
&\cong [\mathbb{Z}^{2^n(2^n-1)+1} \otimes (\mathbb{Z}^{2^m(2^m-1)+1} \times \mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_m})] \\
&\quad \oplus [(\mathbb{Z}^{2^n(2^n-1)+1} \times \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_n}) \otimes \mathbb{Z}^{2^m(2^m-1)+1}] \\
&\cong \{\oplus^2[\oplus^{(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z} \otimes \mathbb{Z}]\} \oplus \{\oplus_{i=1}^n[\oplus^{2^m(2^m-1)+1}\mathbb{Z} \otimes \mathbb{Z}_{k_i}]\} \\
&\quad \oplus \{\oplus_{j=1}^m[\oplus^{2^n(2^n-1)+1}\mathbb{Z} \otimes \mathbb{Z}_{l_j}]\} \\
&\cong [\oplus^{2(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z}] \oplus \{\oplus_{i=1}^n[\oplus^{2^m(2^m-1)+1}\mathbb{Z}_{k_i}]\} \\
&\quad \oplus \{\oplus_{j=1}^m[\oplus^{2^n(2^n-1)+1}\mathbb{Z}_{l_j}]\},
\end{aligned}$$

and the splitting quotient by which is the following torsion products:

$$\begin{aligned}&\mathrm{Tor}_1^{\mathbb{Z}}(K_0(C^*(H_{2n+1}(k))), K_0(C^*(H_{2m+1}(l)))\\&\quad\oplus \mathrm{Tor}_1^{\mathbb{Z}}(K_1(C^*(H_{2n+1}(k))), K_1(C^*(H_{2m+1}(l)))\\&\cong \{0\}\oplus[\oplus_{i=1}^n \oplus_{j=1}^m \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{k_i}, \mathbb{Z}_{l_j})]\end{aligned}$$

and moreover, if k_i and l_j are powers of a common prime, then $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{k_i}, \mathbb{Z}_{l_j}) \cong \mathbb{Z}_{k_i \wedge l_j}$, and if k_i and l_j are relatively prime, then $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{k_i}, \mathbb{Z}_{l_j}) \cong 0$.

Even if n and m are even, by symmetry, we obtain the same conclusion as in the case where m is odd.

On the other hand, if n is odd and m is even, then $K_0(C^*(H_{2n+1}(k)) \otimes K_0(C^*(H_{2m+1}(l)))$ contains the following as a normal subgroup:

$$\begin{aligned}&[K_0(C^*(H_{2n+1}(k))) \otimes K_0(C^*(H_{2m+1}(l)))]\\&\quad\oplus [K_1(C^*(H_{2n+1}(k))) \otimes K_1(C^*(H_{2m+1}(l)))]\\&\cong [\mathbb{Z}^{2^n(2^n-1)+1} \otimes (\mathbb{Z}^{2^m(2^m-1)+1} \times \mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_m})]\\&\quad\oplus [(\mathbb{Z}^{2^n(2^n-1)+1} \times \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_n}) \otimes \mathbb{Z}^{2^m(2^m-1)+1}]\\&\cong \{\oplus^2[\oplus^{(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z} \otimes \mathbb{Z}]\} \oplus \{\oplus_{i=1}^n[\oplus^{2^m(2^m-1)+1}\mathbb{Z} \otimes \mathbb{Z}_{k_i}]\}\\&\quad\oplus \{\oplus_{j=1}^m[\oplus^{2^n(2^n-1)+1}\mathbb{Z} \otimes \mathbb{Z}_{l_j}]\}\\&\cong [\oplus^{2(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z}] \oplus \{\oplus_{i=1}^n[\oplus^{2^m(2^m-1)+1}\mathbb{Z}_{k_i}]\}\\&\quad\oplus \{\oplus_{j=1}^m[\oplus^{2^n(2^n-1)+1}\mathbb{Z}_{l_j}]\}\end{aligned}$$

and the splitting quotient by which is the following torsion products:

$$\begin{aligned}&\mathrm{Tor}_1^{\mathbb{Z}}(K_0(C^*(H_{2n+1}(k))), K_1(C^*(H_{2m+1}(l)))\\&\quad\oplus \mathrm{Tor}_1^{\mathbb{Z}}(K_1(C^*(H_{2n+1}(k))), K_0(C^*(H_{2m+1}(l)))\\&\cong \{0\}\oplus[\oplus_{i=1}^n \oplus_{j=1}^m \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{k_i}, \mathbb{Z}_{l_j})].\end{aligned}$$

Similarly, but $K_1(C^*(H_{2n+1}(k)) \otimes C^*(H_{2m+1}(l)))$ contains the following as a normal subgroup:

$$\begin{aligned}&[K_0(C^*(H_{2n+1}(k))) \otimes K_1(C^*(H_{2m+1}(l)))]\\&\quad\oplus [K_1(C^*(H_{2n+1}(k))) \otimes K_0(C^*(H_{2m+1}(l)))]\\&\cong [\mathbb{Z}^{2^n(2^n-1)+1} \otimes \mathbb{Z}^{2^m(2^m-1)+1}]\\&\quad\oplus [(\mathbb{Z}^{2^n(2^n-1)+1} \times \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_n}) \otimes (\mathbb{Z}^{2^m(2^m-1)+1} \times \mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_m})]\\&\cong \{\oplus^2[\oplus^{(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z} \otimes \mathbb{Z}]\} \oplus \{\oplus_{i=1}^n[\oplus^{2^m(2^m-1)+1}\mathbb{Z} \otimes \mathbb{Z}_{k_i}]\}\\&\quad\oplus \{\oplus_{j=1}^m[\oplus^{2^n(2^n-1)+1}\mathbb{Z} \otimes \mathbb{Z}_{l_j}]\} \oplus [\oplus_{i=1}^n \oplus_{j=1}^m \mathbb{Z}_{k_i} \otimes \mathbb{Z}_{l_j}]\\&\cong [\oplus^{2(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z}] \oplus \{\oplus_{i=1}^n[\oplus^{2^m(2^m-1)+1}\mathbb{Z}_{k_i}]\}\\&\quad\oplus \{\oplus_{j=1}^m[\oplus^{2^n(2^n-1)+1}\mathbb{Z}_{l_j}]\} \oplus [\oplus_{i=1}^n \oplus_{j=1}^m \mathbb{Z}_{k_i \wedge l_j}]\end{aligned}$$

and the splitting quotient by which is the following torsion products:

$$\begin{aligned}&\mathrm{Tor}_1^{\mathbb{Z}}(K_0(C^*(H_{2n+1}(k))), K_0(C^*(H_{2m+1}(l)))\\&\quad\oplus \mathrm{Tor}_1^{\mathbb{Z}}(K_1(C^*(H_{2n+1}(k))), K_1(C^*(H_{2m+1}(l)))\\&\cong \{0\}\oplus\{0\}.\end{aligned}$$

By symmetry, if n is even and m is odd, we obtain the same conclusion as the the case where n is odd and m is even. $\lhd$

Consequently, it follows that

Corollary 6.6. *Let $k=(k_1,\cdots,k_n)$ and $l=(l_1,\cdots,l_m)$ of positive integers and assume that each of k_i $(1\le i\le n)$ and l_j $(1\le j\le m)$ is a power of a prime and k_i are mutually relatively prime and l_j are also so. We then obtain that if both n and m are either odd or even, then*

$$\begin{aligned}&K_0(C^*(H_{2n+1}(k)\times H_{2m+1}(l)))\\&\cong [\oplus^{2(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z}]\oplus\{\oplus_{i=1}^n[\oplus^{2^m(2^m-1)+1}\mathbb{Z}_{k_i}]\}\\&\quad\oplus\{\oplus_{j=1}^m[\oplus^{2^n(2^n-1)+1}\mathbb{Z}_{l_j}]\}\oplus[\oplus_{i=1}^n\oplus_{j=1}^m\mathbb{Z}_{k_i\wedge l_j}]\end{aligned}$$

and

$$\begin{aligned}&K_1(C^*(H_{2n+1}(k)\times H_{2m+1}(l)))\\&\cong [\oplus^{2(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z}]\oplus\{\oplus_{i=1}^n[\oplus^{2^m(2^m-1)+1}\mathbb{Z}_{k_i}]\}\\&\quad\oplus\{\oplus_{j=1}^m[\oplus^{2^n(2^n-1)+1}\mathbb{Z}_{l_j}]\}\oplus[\oplus_{i=1}^n\oplus_{j=1}^m\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{k_i},\mathbb{Z}_{l_j})],\end{aligned}$$

where $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{k_i},\mathbb{Z}_{l_j})$ is zero if k_i and l_j are relatively prime, and if not, it is isomorphic to $\mathbb{Z}_{k_i\wedge l_j}$.

If n is odd and m is even, or if vice versa, then we have

$$\begin{aligned}&K_0(C^*(H_{2n+1}(k)\times H_{2m+1}(l)))\\&\cong [\oplus^{2(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z}]\oplus\{\oplus_{i=1}^n[\oplus^{2^m(2^m-1)+1}\mathbb{Z}_{k_i}]\}\\&\quad\oplus\{\oplus_{j=1}^m[\oplus^{2^n(2^n-1)+1}\mathbb{Z}_{l_j}]\}\oplus[\oplus_{i=1}^n\oplus_{j=1}^m\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{k_i},\mathbb{Z}_{l_j})]\end{aligned}$$

and

$$\begin{aligned}&K_1(C^*(H_{2n+1}(k)\times H_{2m+1}(l)))\\&\cong [\oplus^{2(2^n(2^n-1)+1)(2^m(2^m-1)+1)}\mathbb{Z}]\oplus\{\oplus_{i=1}^n[\oplus^{2^m(2^m-1)+1}\mathbb{Z}_{k_i}]\}\\&\quad\oplus\{\oplus_{j=1}^m[\oplus^{2^n(2^n-1)+1}\mathbb{Z}_{l_j}]\}\oplus[\oplus_{i=1}^n\oplus_{j=1}^m\mathbb{Z}_{k_i\wedge l_j}].\end{aligned}$$

Remark. Similarly and inductively (but surely only finitely), one can compute the K-theory groups of the group C^*-algebras of direct products $H_{2n+1}\times N_{2m+1}(k)$, $N_{2n+1}\times N_{2m+1}(k)$, $H_{2n+1}(k)\times N_{2m+1}(l)$, and $N_{2n+1}(k)\times N_{2m+1}(l)$ involving the shifted, generalized discrete Heisenberg subgroups. $\lhd$

Furthermore,

Example 6.7. Let $n_1, \cdots, n_p$ be positive integers. Let $\Pi_{j=1}^p H_{2n_j+1}$ be the p-fold direct product of the generalized discrete Heisenberg group H_{2n_j+1}. The direct product is also viewed as a group of the block diagonal sum of upper triangular matrices with several components zero and with the diagonal components one. Then we have

$$C^*(\Pi_{j=1}^p H_{2n_j+1}) \cong \otimes_{j=1}^p C^*(H_{2n_j+1})$$

the p-fold tensor product of the group C^*-algebras $C^*(H_{2n_j+1})$. $\lhd$

Inductively using Example 6.1 and Corollary 6.2, and similarly we obtain

Corollary 6.8. *We have*

$$\begin{aligned} K_0(C^*(\Pi_{j=1}^p H_{2n_j+1})) &\cong \oplus^{2^p \Pi_{j=1}^p (2^{n_j}(2^{n_j}-1)+1)} \mathbb{Z} \\ &\cong K_1(C^*(\Pi_{j=1}^p H_{2n_j+1})). \end{aligned}$$

We also have

$$\begin{aligned} K_0(C^*(\Pi_{j=1}^p N_{2n_j+1})) &\cong \oplus^{2^p \Pi_{j=1}^p (2^{n_j}(2^{n_j}-2)+2n_j+1)} \mathbb{Z} \\ &\cong K_1(C^*(\Pi_{j=1}^p N_{2n_j+1})). \end{aligned}$$

Remark. Especially, similarly and inductively, using Example 6.5 and Corollary 6.6 one could obtain the more complicated results as above in the case where $\Pi_{j=1}^p H_{2n_j+1}(k_j)$ with $k_j \in \mathbb{Z}^{n_j}$ and as well, also (finitely or inductively) in the case where $\Pi_{j=1}^p N_{2n_j+1}(k_j)$, but omitted. Several cases of other combinations among those groups or subgroups could also be considered, but omitted. $\lhd$

Remark. As a non-nilpotent case, we have already considered some non-nilpotent, solvable semi-direct products of discrete abelian groups in [6]. More general or other, nilpotent or solvable cases, to be expected to include, but not this time, should be considered in elsewhere in the future. $\lhd$

References

[1] J. Anderson and W. Paschke, *The rotation algebra*, Houston J. Math. **15** (1989), 1-26.

[2] B. Blackadar, *K-theory for Operator Algebras*, Second Edition, Cambridge, (1998).

[3] T. Sudo, *K-theory of continuous fields of quantum tori*, Nihonkai Math. J. **15** (2004), 141-152.

[4] T. Sudo, *Computing K-theory groups for tensor products of C^*-algebras*, Sci. Math. Japon. **75** (2012), no. 2, 203-215 :e-2012, 285-297.

[5] T. Sudo, *K-theory for the group C^*-algebras of nilpotent discrete groups*, Cubo A Math. J. **15** (2013), no. 3, 123-132.

[6] T. Sudo, *K-theory for the group C^*-algebras of certain solvable discrete groups*, Hokkaido Math. J. **43** (2014), no. 2, 209-260.

[7] T. Sudo, *A local study on the KK-theory equivalence and more basics for C^*-algebras*, Ryukyu Math. J., **26** (2014), 19-84.

[8] N. E. Wegge-Olsen, *K-theory and C^*-algebras*, Oxford Univ. Press (1993).

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