K-theory ranks and index for C*-algebras

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K-THEORY RANKS AND INDEX FOR C^*-ALGEBRAS

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Dedicated to Professor Hiroshi Takai on his sixtieth birthday

Abstract

We introduce K-theory ranks and index for C^*-algebras (considered as their Euler characteristic) and establish the fundamental theory for the ranks and index. Furthermore, we consider similarly KK-theory ranks and index for C^*-algebras. Also, the ranks and index for equivariant K- and KK-theory are considered as well.

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Introduction

This paper is primarily based on a (reviewed) part of the author's doctoral dissertation [13], in which some basic results on Euler characteristics for C^*-algebras, due to Takai (superviser) [15], are just given without proofs. It was discussed a decade ago, but his original paper [15] has not been published (as a full paper) (cf. a report [14]). Therefore, this time I would like to give a powered account for this certainly important concept with some detailed proofs and some (not a few) additional (new) results by us. It seems Takai is not the first to introduce such a notion in the literature, but he should be the first to do so in the C^*-algebra setting. Furthermore, it should be (potentially) useful for classification of C^*-algebras in the future.

Our first motivation for this paper is to find a (more) suitable notion for dimension for C^*-algebras. It is Rieffel [11] who first introduced the stable rank for C^*-algebras. Brown and Pedersen [5] also defined the real rank for C^*-algebras. The stable rank can be regarded as a noncommutative counterpart to complex dimension for spaces in some sense, and the

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real rank be as to real dimension. However, they are not much satisfactory in some sense. Indeed, for instance, the noncommutative tori $\mathbb{T}^n_\Theta$ $(n \geq 2)$ generated by n unitaries u_j such that $u_k u_j = e^{2\pi i \theta_{jk}} u_j u_k$ for $1 \leq j, k \leq n$, where $\Theta = (\theta_{jk})$ is a skew-adjoint $n \times n$ matrix over $\mathbb{R}$, should have the same rank (or dimension) as the commutative C^*-algebra $C(\mathbb{T}^n)$ generated by commuting n unitaries, or of all continuous functions on the tori $\mathbb{T}^n$ since $\mathbb{T}^n_\Theta$ are deformed in a sense to $C(\mathbb{T}^n)$ with respect to the parameter Θ. It is known that simple $\mathbb{T}^n_\Theta$ have stable rank one and real rank zero but $C(\mathbb{T}^n)$ $(n \geq 2)$ have both ranks ≥ 2. This idea and situation have led us to define K-theory rank(s) for C^*-algebras as well as their K-index as noncommutative Euler characteristic of Takai. We shall use this terminology since we deal with K-theory mainly. Our K-theory rank and index are new invariants for C^*-algebras in this formulation and would become important notions in the future.

The author is benefited very much from some stimulating conversations with Professor Hiroshi Takai on visiting(s) at Tokyo.

This paper is organized as follows. In Section 1 we define K-theory ranks and index for C^*-algebras and study their fundamental and important properties closely related with K-theory formulae in the literature (see [1]). In Section 2 we consider equivariant K-theory ranks and index for C^*-algebras and study their properties similarly as in Section 1. Furthermore, presented are the tables as classification for nuclear and non-nuclear examples by our K-theory ranks and index, and another table for group C^*-algebras. Collected data in those tables would be helpful for further research on this topic. Moreover, in Section 3 we define KK-theory ranks and index (certainly new invariants) for C^*-algebras and study their fundamental properties closely related with KK-theory formulae ([1] and [12]) and our K-theory ranks and index. These frameworks and results would be useful for classification of (some types of) C^*-algebras, especially, group C^*-algebras.

Finally, as Appendix, (roughly) reviewed in some detail from Brown's text book [4] are cohomological dimension, vitrual cohomological dimension, Euler characteristic for groups with or without torsion, and preliminaries for these notions and some (not a few) facts. The contents (minimally and variously picked up) are divided into two sections: Preliminaries and facts; Cohomological dimensions and Euler characteristics. These sections would be far from being (fully) self-contained, but we made an effort to make a concise approach to what we would like to know for further research, so that some remarkable (and yet elementary or classical) facts and definitions are assembled suitably in a way, as desired, and as further extended in a demand.

1 K-theory ranks and index

Let $\mathfrak{A}$ be a unital C^*-algebra. The K_0-group $K_0(\mathfrak{A})$ of $\mathfrak{A}$ is defined to be the abelian (or Grothendieck) group generated by stably equivalent classes of projections of $n \times n$ matrix algebras $M_n(\mathfrak{A})$ over $\mathfrak{A}$ ($n \geq 1$), where addition is defined by $[p] + [q] = [p \oplus q] \in K_0(\mathfrak{A})$ for projections $p, q \in M_n(\mathfrak{A})$ for some n, where $\oplus$ means diagonal sum. Note that if p, q are projections of $M_n(\mathfrak{A})$ and there exists a continuous path of projections of $M_n(\mathfrak{A})$ between p and q, then $[p] = [q]$. The isomorphism $K_0(\mathbb{C}) \cong \mathbb{Z}$ for $\mathbb{C}$ of complex numbers is given by sending $[p]$ to the rank of p.

The K_1-group $K_1(\mathfrak{A})$ of a unital C^*-algebra $\mathfrak{A}$ is defined to be the abelian group generated by homotopy equivalent classes of unitaries of $M_n(\mathfrak{A})$ ($n \geq 1$), where multiplication is defined by $[u][v] = [uv] = [u \oplus v] \in K_1(\mathfrak{A})$ for unitaries $u, v \in M_n(\mathfrak{A})$ for some n.

For a nonunital C^*-algebra $\mathfrak{A}$, its K_0-group is defined to be the kernel of the natural group homomorphism from $K_0(\mathfrak{A}^+)$ to $K_0(\mathbb{C})$, where $\mathfrak{A}^+$ is the unitization of $\mathfrak{A}$ by $\mathbb{C}$. Then we have the following short exact sequence:

$$0 \to K_0(\mathfrak{A}) \to K_0(\mathfrak{A}^+) \to K_0(\mathbb{C}) \to 0$$

and $K_0(\mathfrak{A}^+) \cong K_0(\mathfrak{A}) \oplus \mathbb{Z}$. The K_1-group of $\mathfrak{A}$ is defined similarly so that $K_1(\mathfrak{A}^+) \cong K_1(\mathfrak{A})$ since $K_1(\mathbb{C}) \cong 0$.

Definition 1.1 Let $\mathfrak{A}$ be a C^*-algebra. We define the K_j-rank of $\mathfrak{A}$ to be the $\mathbb{Z}$-rank of the K_j-group of $\mathfrak{A}$ ($j = 0, 1$), and denote it by

$$\mathrm{Kr}_j(\mathfrak{A}) = \mathrm{rank}_{\mathbb{Z}} K_j(\mathfrak{A}) \in \{0, 1, 2, \cdots, +\infty\}.$$

We define the (Euler-Takai) K-index of $\mathfrak{A}$ to be the following difference:

$$\mathrm{index}_K(\mathfrak{A}) = \mathrm{Kr}_0(\mathfrak{A}) - \mathrm{Kr}_1(\mathfrak{A}) \in \mathbb{Z} \cup \{\pm\infty\}.$$

Remark. As usual, $\infty - \infty$ is not allowed. (However, we may allow this case formally to distinguish it from other cases.) Since K-theory groups are homotopy invariants, the ranks Kr_j and index index_K are also so, where two C^*-algebras are homotopy equivalent if there exists a homotopy (or a continuous deformation) between them. The original notation adopted by Takai [15] for $\mathrm{index}_K(\cdot)$ was $\chi(\cdot)$ as the usual Euler characteristics. On the other hand, the Fredholm index for Fredholm operators is defined by another difference: dimension of kernel minus dimension of cokernel for the operators. Since our K-index is a certainly analytic property, and rank is assumed to be dimension, its naming as index is fit well in this sense.

We define the K-index of a compact space X as:

$$\mathrm{index}^K(X) = \mathrm{rank}_{\mathbb{Z}} K^0(X) - \mathrm{rank}_{\mathbb{Z}} K^1(X),$$

where $K^0(X) = K(X)$ is the Grothendieck group of the semigroup of stable isomorphism classes of $\mathbb{C}$-vector bundles over X, and $K^1(X) = K(SX)$ with $SX = \mathbb{R} \times X$.

Denote by $C_0(X)$ the C^*-algebra of continuous functions on a locally compact Hausdorff space X vanishing at infinity. Set $C_0(X) = C(X)$ when X is compact. Let $\mathbb{K}$ be the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space.

Proposition 1.2 (1). *If C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are stably isomorphic, i.e., $\mathfrak{A} \otimes \mathbb{K} \cong \mathfrak{B} \otimes \mathbb{K}$, then* $\mathrm{index}_K(\mathfrak{A}) = \mathrm{index}_K(\mathfrak{B})$. *In particular,*

$$\mathrm{index}_K(\mathfrak{A} \otimes M_n(\mathbb{C})) = \mathrm{index}_K(\mathfrak{A} \otimes \mathbb{K}) = \mathrm{index}_K(\mathfrak{A}).$$

Moreover, if $\mathfrak{A}$ and $\mathfrak{B}$ are homotopic, then $\mathrm{index}_K(\mathfrak{A}) = \mathrm{index}_K(\mathfrak{B})$.

(2). *For the direct sum $\mathfrak{A} \oplus \mathfrak{B}$ of C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$,*

$$\mathrm{index}_K(\mathfrak{A} \oplus \mathfrak{B}) = \mathrm{index}_K(\mathfrak{A}) + \mathrm{index}_K(\mathfrak{B}).$$

(3). *For the suspension $S\mathfrak{A} = C_0(\mathbb{R}) \otimes \mathfrak{A}$ for a C^*-algebra $\mathfrak{A}$,*

$$\mathrm{index}_K(S\mathfrak{A}) = -\mathrm{index}_K(\mathfrak{A}).$$

(4). *If X is a compact space, then* $\mathrm{index}_K(C(X)) = \mathrm{index}^K(X)$.

(5). *If $\mathfrak{A}$ is contractible, then* $\mathrm{index}_K(\mathfrak{A}) = 0$.

(6). *For a C^*-algebra $\mathfrak{A}$,* $\mathrm{index}_K(\mathfrak{A}^+) = \mathrm{index}_K(\mathfrak{A}) + 1$, *where $\mathfrak{A}^+$ is the unitization of $\mathfrak{A}$ by $\mathbb{C}$.*

Remark. For (1), (2), and (5), $\mathrm{index}_K(\cdot)$ can be replaced with $\mathrm{Kr}_j(\cdot)$.

Proof. For (1), note that $K_j(\mathfrak{A} \otimes M_n(\mathbb{C})) \cong K_j(\mathfrak{A} \otimes \mathbb{K}) \cong K_j(\mathfrak{A})$.

For (2), we have $K_j(\mathfrak{A} \oplus \mathfrak{B}) \cong K_j(\mathfrak{A}) \oplus K_j(\mathfrak{B})$.

For (3), we have $K_j(S\mathfrak{A}) \cong K_{j+1}(\mathfrak{A})$ (Bott periodicity). Hence we obtain $\mathrm{Kr}_j(S\mathfrak{A}) = \mathrm{Kr}_{j+1}(\mathfrak{A})$.

For (4), $K_0(C(X)) \cong K^0(X)$ (: Swan's theorem which says that a stable isomorphism class of a complex vector bundle over X can be associated with a stably equivalent class of a finitely generated, projective module over $C(X)$ (that corresponds to a projection of a matrix algebra over $C(X)$)), and $K_1(C(X)) \cong K_0(SC(X)) \cong K^0(SX) = K^1(X)$.

For (5), if $\mathfrak{A}$ is contractible (to $\{0\}$), then its K_0-group is trivial and its K_1 is also trivial since $S\mathfrak{A}$ is contractible when $\mathfrak{A}$ is so.

For (6), note that $K_0(\mathfrak{A}^+) \cong K_0(\mathfrak{A}) \oplus K_0(\mathbb{C})$ and $K_1(\mathfrak{A}^+) \cong K_1(\mathfrak{A})$. $\square$

Example 1.3 (1). Since $K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K_1(\mathbb{C}) \cong 0$, we have

$$\mathrm{index}_K(\mathbb{C}) = \mathrm{index}_K(M_n(\mathbb{C})) = \mathrm{index}_K(\mathbb{K}) = 1.$$

Let $\mathfrak{A}$ be a finite dimensional C^*-algebra so that $\mathfrak{A} \cong \oplus_{j=1}^n M_{n_j}(\mathbb{C})$ for some $n \geq 1$ and $n_j \geq 1$. Then $\mathrm{index}_K(\mathfrak{A}) = n$.

(2). For any AF-algebra $\mathfrak{A}$ (that is an inductive limit of finite dimensional C^*-algebras), $\mathrm{index}_K(\mathfrak{A}) = \mathrm{Kr}_0(\mathfrak{A})$ since K_1 for AF is trivial. This can be infinite. Indeed, let M_{n^∞} be the UHF algebra of type n^∞, that is an inductive limit of tensor products $\otimes^k M_n(\mathbb{C})$ ($k \geq 1$). Then $\mathrm{index}_K(M_{n^\infty}) = +\infty$ while $\mathrm{index}_K(\otimes^k M_n(\mathbb{C})) = 1$. This shows that K-index index_K as well as K-ranks Kr_j are not continuous with respect to inductive limits, but K-theory groups $K_j(\cdot)$ are continuous. This is a serious lack in our theory, but a little bit strange (see a discussion after the tables given below). There exists a simple AF algebra such that its K_0-group is isomorphic to $\theta_1\mathbb{Z}+\theta_2\mathbb{Z}+\cdots+\theta_n\mathbb{Z}$, where θ_j ($1 \leq j \leq n$) in $\mathbb{R}$ are rationally independent. Denote it by AF_n. Hence $\mathrm{index}_K(\mathrm{AF}_n) = n$.

(3). Let $C(\mathbb{T}^n)$ be the C^*-algebra of continuous functions on the n-torus $\mathbb{T}^n$. Then

$$\mathrm{index}_K(C(\mathbb{T}^n)) = \mathrm{Kr}_0(C(\mathbb{T}^n)) - \mathrm{Kr}_1(C(\mathbb{T}^n)) = 2^{n-1} - 2^{n-1} = 0.$$

For S^n the n-dimensional sphere, the Bott periodicity implies

$$\mathrm{index}_K(C(S^n)) = \mathrm{index}_K(C_0(\mathbb{R}^n)^+) = \mathrm{index}_K(C_0(\mathbb{R}^n)) + 1 = \begin{cases} 2 & \text{if } n \text{ even,} \\ 0 & \text{if } n \text{ odd.} \end{cases}$$

(4). Let $C\mathfrak{A} = C([0,1]) \otimes \mathfrak{A}$ be the cone of a C^*-algebra $\mathfrak{A}$. Since $C\mathfrak{A}$ is contractible, we have $\mathrm{index}_K(C\mathfrak{A}) = 0$.

Proposition 1.4 *Let $\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$ be the pullback of C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$ with $*$-homomorphisms from $\mathfrak{A}$ and $\mathfrak{B}$ onto a C^*-algebra $\mathfrak{C}$. Then*

$$\mathrm{index}_K(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) = \mathrm{index}_K(\mathfrak{A}) + \mathrm{index}_K(\mathfrak{B}) - \mathrm{index}_K(\mathfrak{C}).$$

Furthermore, let

$$\mathfrak{A} = (\cdots((\mathfrak{B}_1 \oplus_{\mathfrak{C}_1} \mathfrak{B}_2) \oplus_{\mathfrak{C}_2} \mathfrak{B}_3)\cdots) \oplus_{\mathfrak{C}_n} \mathfrak{B}_{n+1}$$

be a successive pullback of C^*-algebras $\mathfrak{B}_j$ $(1 \leq j \leq n+1)$ and $\mathfrak{C}_j$ $(1 \leq j \leq n)$. If there exist $*$-homomorphisms from $\mathfrak{A}_j = (\cdots(\mathfrak{B}_1 \oplus_{\mathfrak{C}_1} \mathfrak{B}_2)\cdots) \oplus_{\mathfrak{C}_{j-1}} \mathfrak{B}_j$ and $\mathfrak{B}_{j+1}$ onto $\mathfrak{C}_j$ for $1 \leq j \leq n$, then

$$\mathrm{index}_K(\mathfrak{A}) = \sum_{j=1}^{n+1} \mathrm{index}_K(\mathfrak{B}_j) - \sum_{j=1}^{n} \mathrm{index}_K(\mathfrak{C}_j).$$

Proof. We have the Mayer-Vietoris sequence (MV):

$$0 \to K_j(\mathfrak{C}) \to K_j(\mathfrak{A} \oplus \mathfrak{B}) \to K_j(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \to 0.$$

For the second claim, we use this sequence repeatedly for the pullback C^*-algebras defined inductively. □

Recall from [1] (or [12]) that C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are KK-equivalent if their KK-group $KK(\mathfrak{A}, \mathfrak{B})$ contains an invertible element, i.e., there exist an element $x \in KK(\mathfrak{A}, \mathfrak{B})$ and an element $y \in KK(\mathfrak{B}, \mathfrak{A})$ (called KK-equivalences) such that $xy = 1_{\mathfrak{A}}$ and $yx = 1_{\mathfrak{B}}$ under the Kasparov product, where $1_{\mathfrak{A}}$ and $1_{\mathfrak{B}}$ are the classes corresponding to the identity maps on $\mathfrak{A}$ and $\mathfrak{B}$ respectively. A C^*-algebra is K-abelian if it is KK-equivalent to an abelian C^*-algebra. The UCT class N is defined to be the family of all separable K-abelian C^*-algebras. The subclass of nuclear C^*-algebras in N is closed under taking closed ideals, quotients, and extensions and under constructing crossed products of those C^*-algebras by $\mathbb{Z}$ or $\mathbb{R}$.

Theorem 1.5 *Let $\mathfrak{A}$, $\mathfrak{B}$ be C^*-algebras in the UCT class N that are KK-equivalent. Then* $\mathrm{index}_K(\mathfrak{A}) = \mathrm{index}_K(\mathfrak{B})$, *where* $\in \mathbb{Z} \cup \{\pm\infty\}$ *is assumed.*

Proof. The universal coefficient theorem (see [12]) implies that $K_j(\mathfrak{A}) \cong K_j(\mathfrak{B})$ under the assumption. □

Corollary 1.6 *For $\mathfrak{A}, \mathfrak{B}$ in the class N, if* $\mathrm{index}_K(\mathfrak{A}) \neq \mathrm{index}_K(\mathfrak{B})$, *then they are not KK-equivalent.*

Remark. This consequence might be useful to show non-KK-equivalence. The theorem above is very powerful as shown in examples below ([1]), but somewhat tautological in a sense since KK-equivalence is deduced by K-theory isomorphisms in many cases.

Example 1.7 Let $\mathfrak{A}$ be a C^*-algebra. All $\mathfrak{A}$, $M_n(\mathfrak{A}) \cong \mathfrak{A} \otimes M_n(\mathbb{C})$, $\mathfrak{A} \otimes \mathbb{K}$ are KK-equivalent. Homotopy equivalent C^*-algebras are KK-equivalent. AF-algebras are KK-equivalent if and only if their dimension groups (that are inductive limits of free abelian groups $\mathbb{Z}^k$) are isomorphic as groups. Some other interesting examples are given below.

Recall from [1] that the bootstrap category X is defined as the smallest class of separable C^*-algebras such that $\mathbb{C} \in X$ and X is closed under taking inductive limits, extensions, quotients, closed ideals, and KK-equivalence. A C^*-algebra in X is not necessarily nuclear.

Theorem 1.8 *If $\mathfrak{A}$ or $\mathfrak{B}$ are nuclear C^*-algebras in the bootstrap category X (or in the UCT class N) and their K-groups $K_j(\mathfrak{A})$ or $K_j(\mathfrak{B})$ are torsion free, then*

$$\mathrm{index}_K(\mathfrak{A} \otimes \mathfrak{B}) = \mathrm{index}_K(\mathfrak{A})\, \mathrm{index}_K(\mathfrak{B}).$$

Proof. The Künneth formula implies the isomorphisms:

$$\begin{aligned} K_0(\mathfrak{A} \otimes \mathfrak{B}) &\cong (K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})) \oplus (K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})), \\ K_1(\mathfrak{A} \otimes \mathfrak{B}) &\cong (K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})) \oplus (K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \mathrm{Kr}_0(\mathfrak{A} \otimes \mathfrak{B}) &= \mathrm{Kr}_0(\mathfrak{A})\, \mathrm{Kr}_0(\mathfrak{B}) + \mathrm{Kr}_1(\mathfrak{A})\, \mathrm{Kr}_1(\mathfrak{B}), \\ \mathrm{Kr}_1(\mathfrak{A} \otimes \mathfrak{B}) &= \mathrm{Kr}_0(\mathfrak{A})\, \mathrm{Kr}_1(\mathfrak{B}) + \mathrm{Kr}_1(\mathfrak{A})\, \mathrm{Kr}_0(\mathfrak{B}). \end{aligned}$$

It follows that

$$\begin{aligned} \mathrm{index}_K(\mathfrak{A} \otimes \mathfrak{B}) &= (\mathrm{Kr}_0(\mathfrak{A}) - \mathrm{Kr}_1(\mathfrak{A}))(\mathrm{Kr}_0(\mathfrak{B}) - \mathrm{Kr}_1(\mathfrak{B})) \\ &= \mathrm{index}_K(\mathfrak{A})\, \mathrm{index}_K(\mathfrak{B}). \end{aligned}$$

□

Extensions of C^*-algebras

Theorem 1.9 *Let $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$ be a short exact sequence of C^*-algebras. Its six-term exact sequence is*

$$\begin{array}{ccccc} K_0(\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{A}) & \longrightarrow & K_0(\mathfrak{A}/\mathfrak{I}) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(\mathfrak{A}/\mathfrak{I}) & \longleftarrow & K_1(\mathfrak{A}) & \longleftarrow & K_1(\mathfrak{I}). \end{array}$$

If the index maps ∂ are both zero, then

$$\mathrm{Kr}_j(\mathfrak{A}) = \mathrm{Kr}_j(\mathfrak{I}) + \mathrm{Kr}_j(\mathfrak{A}/\mathfrak{I}), \quad \mathrm{index}_K(\mathfrak{A}) = \mathrm{index}_K(\mathfrak{I}) + \mathrm{index}_K(\mathfrak{A}/\mathfrak{I}).$$

In particular, this is the case if the short exact sequence splits.

Moreover, if $\mathrm{index}_K(\mathfrak{A})$, $\mathrm{index}_K(\mathfrak{I})$, *and* $\mathrm{index}_K(\mathfrak{A}/\mathfrak{I})$ *are finite, then*

$$\mathrm{index}_K(\mathfrak{A}) = \mathrm{index}_K(\mathfrak{I}) + \mathrm{index}_K(\mathfrak{A}/\mathfrak{I}),$$

where it is enough to assume that two of those are finite.

Proof. The first part of the statement is clear.

For the second, let $l_j = \mathrm{Kr}_j(\mathfrak{I})$, $m_j = \mathrm{Kr}_j(\mathfrak{A})$, and $n_j = \mathrm{Kr}_j(\mathfrak{A}/\mathfrak{I})$. Furthermore, we assume by half exactness of the six term diagram above that $l_j = l'_j + l''_j$, $m_j = m'_j + m''_j$, and $n_j = n'_j + n''_j$ such that generators corresponding to x' are mapped to zero and those to x'' are not, i.e, we have $m''_0 = n'_0$, $n''_0 = l'_1$, $l''_1 = m'_1$, $m''_1 = n'_1$, $n''_1 = l'_0$, and $l''_0 = m'_0$. Therefore, we obtain

$$\begin{aligned}
\mathrm{index}_K(\mathfrak{A}) &= m_0 - m_1 = (m'_0 + m''_0) - (m'_1 + m''_1) \\
&= (l''_0 + n'_0) - (l''_1 + n'_1) = (l''_0 - l''_1) + (n'_0 - n'_1) \\
&= (l''_0 - l''_1) + (n'_0 - n'_1) + (l'_0 - n''_1) + (n''_0 - l'_1) \\
&= ((l'_0 + l''_0) - (l'_1 + l''_1)) + ((n'_0 + n''_0) - (n'_1 + n''_1)) \\
&= \mathrm{index}_K(\mathfrak{I}) + \mathrm{index}_K(\mathfrak{A}/\mathfrak{I}).
\end{aligned}$$

□

Proposition 1.10 *Let $0 \to \mathfrak{A} \to E \to \mathfrak{B} \to 0$ be a split extension of C^*-algebras. If E is in the UCT class, then*

$$\mathrm{index}_K(E) = \mathrm{index}_K(\mathfrak{A} \oplus \mathfrak{B}).$$

Proof. Such an extension E is KK-equivalent to $\mathfrak{A} \oplus \mathfrak{B}$. Thus, the UCT assumption implies the conclusion. □

Example 1.11 Let X be a locally compact Hausdorff space and Y its closed subspace. Set $U = X \setminus Y$. Then we have the following short exact sequence:

$$0 \to C_0(U) \to C_0(X) \to C_0(Y) \to 0.$$

Hence, it follows that

$$\mathrm{index}_K(C_0(X)) = \mathrm{index}_K(C_0(U)) + \mathrm{index}_K(C_0(Y))$$

where finiteness of these indexes is assumed, which is equivalent to

$$\mathrm{index}^K(X) = \mathrm{index}^K(U) + \mathrm{index}^K(Y).$$

Indeed, we have the following diagram:

$$\begin{array}{ccccc}
K^0(U) & \xrightarrow{q^*} & K^0(X) & \xrightarrow{i^*} & K^0(Y) \\
\partial \uparrow & & & & \downarrow \partial \\
K^1(Y) & \xleftarrow{i^*} & K^1(X) & \xleftarrow{q^*} & K^1(U)
\end{array}$$

where i is the inclusion from Y to X and q is the map from X^+ to U^+ which is the identity on U and which sends $X^+ \setminus U$ to the point at infinity.

Let $\mathfrak{F}$ be the Toeplitz algebra generated by an isometry. Then

$$0 \to \mathbb{K} \to \mathfrak{F} \to C(\mathbb{T}) \to 0$$

and $\mathrm{index}_K(\mathfrak{F}) = 1 - 0 = 1$, $\mathrm{index}_K(\mathbb{K}) = 1$, and $\mathrm{index}_K(C(\mathbb{T})) = 0$ (see [17]). The Toeplitz algebra $\mathfrak{F}$ is KK-equivalent to $\mathbb{C}$. Furthermore, any C^*-algebra $\mathfrak{A}$ and $\mathfrak{A} \otimes \mathfrak{F}$ are KK-equivalent.

Let $\mathbb{B}$ be the C^*-algebra of bounded operators on a Hilbert space. Then

$$0 \to \mathbb{K} \to \mathbb{B} \to \mathbb{B}/\mathbb{K} \to 0$$

and $\mathrm{index}_K(\mathbb{B}) = 0 - 0 = 0$ and $\mathrm{index}_K(\mathbb{B}/\mathbb{K}) = 0 - 1 = -1$ ([17]).

Let A_2 be the real 2-dimensional $ax + b$ group and $C^*(A_2)$ its group C^*-algebra. Then we have

$$0 \to \mathbb{K} \oplus \mathbb{K} \to C^*(A_2) \to C_0(\mathbb{R}) \to 0.$$

Therefore, we obtain $\mathrm{index}_K(C^*(A_2)) = \mathrm{index}_K(\mathbb{K} \oplus \mathbb{K}) + \mathrm{index}_K(C_0(\mathbb{R})) = 2 + (-1) = 1$.

Let $H_3^{\mathbb{R}}$ be the real 3-dimensional Heisenberg Lie group and $C^*(H_3^{\mathbb{R}})$ its group C^*-algebra. Then we have

$$0 \to C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K} \to C^*(H_3^{\mathbb{R}}) \to C_0(\mathbb{R}^2) \to 0$$

([8]). Therefore, we obtain

$$\begin{aligned} \mathrm{index}_K(C^*(H_3^{\mathbb{R}})) &= \mathrm{index}_K(C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K}) + \mathrm{index}_K(C_0(\mathbb{R}^2)) \\ &= 2\,\mathrm{index}_K(C_0(\mathbb{R})) + \mathrm{index}_K(\mathbb{C}) = -2 + 1 = -1. \end{aligned}$$

Proposition 1.12 *Let $M(\mathfrak{A})$ be the multiplier algebra of a (nonunital) C^*-algebra $\mathfrak{A}$. Then*

$$\mathrm{index}_K(M(\mathfrak{A})) = \mathrm{index}_K(\mathfrak{A}) + \mathrm{index}_K(M(\mathfrak{A})/\mathfrak{A}).$$

For any C^*-algebra $\mathfrak{A}$ and a unital C^*-algebra $\mathfrak{B}$,

$$\mathrm{index}_K(M(\mathfrak{A} \otimes \mathbb{K}) \otimes \mathfrak{B}) = 0.$$

In particular, $\mathrm{index}_K(\mathbb{B} \otimes \mathfrak{B}) = 0$, *and if $\mathfrak{A}$ is stable, then* $\mathrm{index}_K(M(\mathfrak{A})) = 0$. *Note that $\otimes$ with $\mathfrak{B}$ is any C^*-tensor product.*

Furthermore, for any C^*-algebra $\mathfrak{A}$,

$$\mathrm{index}_K(\mathfrak{A}) = -\mathrm{index}_K(M(\mathfrak{A} \otimes \mathbb{K})/(\mathfrak{A} \otimes \mathbb{K})).$$

In particular, if $\mathfrak{A}$ is stable, then $\mathrm{index}_K(\mathfrak{A}) = -\mathrm{index}_K(M(\mathfrak{A})/\mathfrak{A})$.

Proof. It is known that for any C^*-algebra $\mathfrak{A}$ and a unital C^*-algebra $\mathfrak{B}$, both K_0 and K_1-groups of $M(\mathfrak{A} \otimes \mathbb{K}) \otimes \mathfrak{B}$ are trivial (see [17, 10]).

Furthermore, it is known that for any C^*-algebra $\mathfrak{A}$, we have $K_j(\mathfrak{A}) \cong K_{j+1}(M(\mathfrak{A} \otimes \mathbb{K})/(\mathfrak{A} \otimes \mathbb{K}))$. □

Proposition 1.13 *For any extension $0 \to \mathfrak{I} \to \mathfrak{B} \to \mathfrak{C} \to 0$ of C^*-algebras and a C^*-algebra $\mathfrak{A}$, we have*

$$\mathrm{index}_K(\mathfrak{A} \otimes_{\max} \mathfrak{B}) = \mathrm{index}_K(\mathfrak{A} \otimes_{\max} \mathfrak{I}) + \mathrm{index}_K(\mathfrak{A} \otimes_{\max} \mathfrak{C})$$

provided that both sides are finite, where $\otimes_{\max}$ is the maximal C^*-tensor product. Furthermore, if $\mathfrak{A}$ is exact, then

$$\mathrm{index}_K(\mathfrak{A} \otimes_{\min} \mathfrak{B}) = \mathrm{index}_K(\mathfrak{A} \otimes_{\min} \mathfrak{I}) + \mathrm{index}_K(\mathfrak{A} \otimes_{\min} \mathfrak{C})$$

provided that they are finite, where $\otimes_{\min}$ is the minimal C^*-tensor product.

Proof. We always have the exact sequence:

$$0 \to \mathfrak{A} \otimes_{\max} \mathfrak{I} \to \mathfrak{A} \otimes_{\max} \mathfrak{B} \to \mathfrak{A} \otimes_{\max} \mathfrak{C} \to 0.$$

If $\mathfrak{A}$ is exact,

$$0 \to \mathfrak{A} \otimes_{\min} \mathfrak{I} \to \mathfrak{A} \otimes_{\min} \mathfrak{B} \to \mathfrak{A} \otimes_{\min} \mathfrak{C} \to 0$$

holds. □

Example 1.14 In particular, for $0 \to \mathbb{K} \to \mathbb{B} \to Q = \mathbb{B}/\mathbb{K} \to 0$,

$$\mathrm{index}_K(\mathfrak{A} \otimes_{\max} \mathbb{B}) = \mathrm{index}_K(\mathfrak{A}) + \mathrm{index}_K(\mathfrak{A} \otimes_{\max} Q),$$
$$\mathrm{index}_K(\mathfrak{A} \otimes_{\min} \mathbb{B}) = \mathrm{index}_K(\mathfrak{A}) + \mathrm{index}_K(\mathfrak{A} \otimes_{\min} Q),$$

where $\mathfrak{A}$ is exact for the second. If $\mathfrak{A} = \mathbb{B}$, then $\mathrm{index}_K(\mathbb{B} \otimes_{\max} \mathbb{B}) = \mathrm{index}_K(\mathbb{B} \otimes_{\max} Q) = 0$, but $\mathbb{B}$ is not exact. If $\mathfrak{A} = Q$, then $\mathrm{index}_K(Q \otimes_{\max} \mathbb{B}) = 0 = -1 + \mathrm{index}_K(Q \otimes_{\max} Q)$, but Q is not exact. Hence, $\mathrm{index}_K(Q \otimes_{\max} Q) = 1$. We also have $\mathrm{index}_K(\mathbb{B} \otimes_{\min} \mathbb{B}) = 0 = \mathrm{index}_K(\mathbb{B} \otimes_{\min} Q)$.

Proposition 1.15 *Let $\mathfrak{A}$ be a C^*-algebra. If $\mathfrak{A}$ has a finite composition series of closed ideal $\mathfrak{I}_j$ with $\mathfrak{I}_0 = \{0\}$ and $\mathfrak{I}_n = \mathfrak{A}$, then*

$$\mathrm{index}_K(\mathfrak{A}) = \sum_{j=1}^{n} \mathrm{index}_K(\mathfrak{I}_j/\mathfrak{I}_{j-1})$$

where finiteness of each K-index is assumed.

Example 1.16 Suppose in the above proposition that each subquotient $\mathfrak{J}_j/\mathfrak{J}_{j-1} \cong C_0(X_j) \otimes \mathbb{K}$ for X_j a locally compact Hausdorff space. Then

$$\mathrm{index}_K(\mathfrak{A}) = \sum_{j=1}^{n} \mathrm{index}_K(C_0(X_j)) = \sum_{j=1}^{n} \mathrm{index}^K(X_j)$$

where finiteness of each K-index is assumed.

Proposition 1.17 *Let $\mathfrak{A}$ be a C^*-algebra with a strictly positive element, or with countable approximate units. If $\mathfrak{B}$ is a full corner of $\mathfrak{A}$, i.e., the hereditary C^*-subalgebra $p\mathfrak{A}p$ of $\mathfrak{A}$ for a projection p of $\mathfrak{A}$ (or the multiplier algebra of $\mathfrak{A}$), that is not contained in any closed ideal of $\mathfrak{A}$, then*

$$\mathrm{Kr}_j(\mathfrak{A}) = \mathrm{Kr}_j(\mathfrak{B}), \quad \textit{and} \quad \mathrm{index}_K(\mathfrak{A}) = \mathrm{index}_K(\mathfrak{B}).$$

Proof. Under the assumptions above, it is obtained by [4] that $\mathfrak{A}$ is stably isomorphic to $\mathfrak{B}$, i.e., $\mathfrak{A} \otimes \mathbb{K} \cong \mathfrak{B} \otimes \mathbb{K}$. □

Remark. If $\mathfrak{A}$ is a simple C^*-algebra, the corner $p\mathfrak{A}p$ is always full and simple.

Crossed products of C^*-algebras

Theorem 1.18 *Let $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ be the crossed product of a C^*-algebra $\mathfrak{A}$ by an action α of $\mathbb{Z}$. Its Pimsner-Voiculescu sequence is*

$$\begin{array}{ccccc} K_0(\mathfrak{A}) & \xrightarrow{(\mathrm{id}-\alpha)_*} & K_0(\mathfrak{A}) & \xrightarrow{i_*} & K_0(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) & \xleftarrow{i_*} & K_1(\mathfrak{A}) & \xleftarrow{(\mathrm{id}-\alpha)_*} & K_1(\mathfrak{A}) \end{array}$$

where $(\mathrm{id} - \alpha)_*$ and i_* are the maps induced from the identity action id *on $\mathfrak{A}$ and the canonical inclusion i from $\mathfrak{A}$ to $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ respectively. If two $(\mathrm{id} - \alpha)_*$ are zero, then*

$$\mathrm{Kr}_0(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) = \mathrm{Kr}_0(\mathfrak{A}) + \mathrm{Kr}_1(\mathfrak{A}) = \mathrm{Kr}_1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}).$$

Hence, $\mathrm{index}_K(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) = 0$.

Moreover, if both $\mathrm{index}_K(\mathfrak{A})$ *and* $\mathrm{index}_K(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$ *are finite, then*

$$\mathrm{index}_K(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) = 0,$$

where it is enough to assume that $\mathrm{index}_K(\mathfrak{A})$ *is finite. Even if* $\mathrm{index}_K(\mathfrak{A})$ *is infinite, when the maps* $(\mathrm{id} - \alpha)_*$ *are isomorphisms, we have* $\mathrm{index}_K(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) = 0$, *Proof.* The first part of the statement is clear.

For the second, let $l_j = \mathrm{Kr}_j(\mathfrak{A})$ (the first one), $m_j = \mathrm{Kr}_j(\mathfrak{A})$ (the second one), and $n_j = \mathrm{Kr}_j(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$. Of course, $l_j = m_j$. Furthermore, we assume by half exactness of the six term diagram above that $l_j = l'_j + l''_j$, $m_j = m'_j + m''_j$, and $n_j = n'_j + n''_j$ such that generators corresponding to x' are mapped to zero and those to x'' are not, i.e, we have $m''_0 = n'_0$, $n''_0 = l'_1$, $l''_1 = m'_1$, $m''_1 = n'_1$, $n''_1 = l'_0$, and $l''_0 = m'_0$. Therefore, we obtain

$$\begin{aligned}
\mathrm{index}_K(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) &= n_0 - n_1 = (n'_0 + n''_0) - (n'_1 + n''_1) \\
&= (m''_0 + l'_1) - (m''_1 + l'_0) = (m''_0 - m''_1) - (l'_0 - l'_1) \\
&= (m''_0 - m''_1) - (l'_0 - l'_1) + (m'_0 - l''_0) - (m'_1 - l''_1) \\
&= ((m'_0 + m''_0) - (m'_1 + m''_1)) - ((l'_0 + l''_0) - (l'_1 + l''_1)) \\
&= \mathrm{index}_K(\mathfrak{A}) - \mathrm{index}_K(\mathfrak{A}) = 0.
\end{aligned}$$

□

Remark. Now consider the crossed product $C_0(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}$, where the action α is the shift. It is isomorphic to $\mathbb{K}$. Hence $\mathrm{index}_K(C_0(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}) = 1$ and $\mathrm{index}_K(C_0(\mathbb{Z})) = +\infty$. Thus, the finiteness condition for the proposition above is required.

In the statement, we can replace $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ with the mapping torus M_α for α defined to be the C^*-algebra of continuous $\mathfrak{A}$-valued functions f on $\mathbb{R}$ such that $f(x+1) = \alpha(f(x))$ for $x \in \mathbb{R}$. Indeed, we have the exact sequence: $0 \to S\mathfrak{A} \to M_\alpha \to \mathfrak{A} \to 0$. Its six-term exact sequence implies the same conclusions as above.

Corollary 1.19 *Let $\mathfrak{A}$ be a C^*-algebra. If* $\mathrm{index}_K(\mathfrak{A})$ *is finite, then*

$$\mathrm{index}_K(\mathfrak{A} \rtimes \mathbb{Z} \rtimes \mathbb{Z} \cdots \rtimes \mathbb{Z}) = 0$$

for any successive crossed product $\mathfrak{A} \rtimes \mathbb{Z} \rtimes \mathbb{Z} \cdots \rtimes \mathbb{Z}$ by $\mathbb{Z}$.

Remark. This is a (not many) merit of our K-index because we can determine it without knowing K-theory data of the successive crossed product which may be difficult to compute in general.

Example 1.20 If α is trivial, then $\mathfrak{A} \rtimes_\alpha \mathbb{Z} \cong \mathfrak{A} \otimes C^*(\mathbb{Z})$, and $C^*(\mathbb{Z}) \cong C(\mathbb{T})$. If $\mathfrak{A}$ is a nuclear C^*-algebra in the category X and its K-groups are torsion free, then $\mathrm{index}_K(\mathfrak{A} \otimes C(\mathbb{T})) = \mathrm{index}_K(\mathfrak{A})\, \mathrm{index}_K(C(\mathbb{T})) = 0$.

Let $G \times \mathbb{Z}$ be the direct product of a locally compact group G with $\mathbb{Z}$, and $C^*(G)$, $C^*_r(G)$ the full and reduced group C^*-algebras of G respectively. Then the full and reduced group C^*-algebras of $G \times \mathbb{Z}$ are isomorphic to tensor products $C^*(G) \otimes C(\mathbb{T})$ and $C_r^*(G) \otimes C(\mathbb{T})$ respectively. Hence $\mathrm{index}_K(C^*(G \times \mathbb{Z})) = 0 = \mathrm{index}_K(C_r^*(G \times \mathbb{Z}))$.

Consider the crossed product $C_0(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}$ with α the shift. Then $C_0(\mathbb{Z}) \rtimes_\alpha \mathbb{Z} \cong \mathbb{K}$. The Pimsner-Voiculescu six-term exact sequence is:

$$\begin{array}{ccccc} \mathbb{Z}^\infty & \xrightarrow{(\mathrm{id}-\alpha)_*} & \mathbb{Z}^\infty & \xrightarrow{i_*} & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \xleftarrow{(\mathrm{id}-\alpha)_*} & 0 & \xleftarrow{i_*} & 0 \end{array}$$

where $i_*([p_n]) = [p]$ and $(\mathrm{id}-\alpha)_*([p_n]) = [p_n] - [p_{n+1}]$, where p is a rank 1 projection of $\mathbb{K}$ and p_n is just the characteristic function at $n \in \mathbb{Z}$. In this case, we have $\mathrm{index}_K(C_0(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}) = 1 \neq 0$ and $(\mathrm{id}-\alpha)_*$ is not zero on K_0.

Let $H_3^{\mathbb{Z}}$ be the discrete Heisenberg group of rank 3 and $C^*(H_3^{\mathbb{Z}})$ its group C^*-algebra. It can be written as the crossed product $C(\mathbb{T}^2) \rtimes \mathbb{Z}$ since $H_3^{\mathbb{Z}} \cong \mathbb{Z}^2 \rtimes \mathbb{Z}$ a semi-direct product. Hence $\mathrm{index}_K(C^*(H_3^{\mathbb{Z}})) = 0$. Furthermore, let G be an amenable discrete group that can be written as a successive semi-direct product by $\mathbb{Z}$, i.e., $G = \mathbb{Z} \rtimes \cdots \rtimes \mathbb{Z}$. Then $C^*(G)$ is isomorphic to a successive crossed product by $\mathbb{Z}$: $C^*(\mathbb{Z}) \rtimes \mathbb{Z} \cdots \rtimes \mathbb{Z}$. Therefore, $\mathrm{index}_K(C^*(G)) = 0$ by induction.

Let $\mathbb{T}_\Theta^n$ be a noncommutative n-torus, that is defined to be the universal C^*-algebra generated by n unitaries U_j such that $U_jU_i = e^{2\pi i\theta_{ij}}U_iU_j$ for $1 \leq i,j \leq n$, where $\Theta = (\theta_{ij})$ is a skew-adjoint $n \times n$ matrix over $\mathbb{R}$. It can be viewed as an n successive crossed product by $\mathbb{Z}$, i.e., $\mathbb{T}_\Theta^n \cong (\cdots(C^*(\mathbb{Z}) \rtimes_{\alpha_2} \mathbb{Z})\cdots) \rtimes_{\alpha_n} \mathbb{Z}$, where the action α_j are induced from the commutation relations. It is known that $K_j(\mathbb{T}_\Theta^n) \cong \mathbb{Z}^{2^{n-1}}$ by using the Pimsner-Voiculesce exact sequence. Hence $\mathrm{index}_K(\mathbb{T}_\Theta^n) = 0$. Furthermore, if $\mathrm{index}_K(\mathfrak{A})$ is finite, then $\mathrm{index}_K(\mathfrak{A} \rtimes \mathbb{Z} \cdots \rtimes \mathbb{Z}) = 0$ by induction, where $\mathfrak{A} \rtimes \mathbb{Z} \cdots \rtimes \mathbb{Z}$ is a successive crossed product of $\mathfrak{A}$ by $\mathbb{Z}$.

Furthermore, it is known that both any inductive limit of finitely generated, abelian free groups $\mathbb{Z}^{n_k}$ and a countable, torsion free, abelian group can be realized as K_0 and K_1-groups of a unital simple AT algebra, i.e., an iductive limit of finite direct sums of matrix algebras over $C(\mathbb{T})$ (see, for instance, [9, 4.7]). Denote by $\mathrm{AT}_{n,m}$ such an AT algebra with

$$\mathrm{index}_K(\mathrm{AT}_{n,m}) = n - m$$

any integer or $\pm\infty$, where $\mathrm{Kr}_0(\mathrm{AT}_{n,m}) = n$ and $\mathrm{Kr}_1(\mathrm{AT}_{n,m}) = m$ finite or infinite. Also, it is known that a simple noncommutative 2 (or 3)-torus and some special simple noncommutative n-torus written as a crossed product $C(\mathbb{T}^{n-1}) \rtimes \mathbb{Z}$ is an AT algebra.

Let O_n denote the Cuntz algebra, defined to be the universal C^*-algebra generated by n isometries S_j ($n \geq 2$) such that $\sum_{j=1}^n S_j S_j^* = 1$. It is known that $K_0(O_n) \cong \mathbb{Z}_{n-1} = \mathbb{Z}/(n-1)\mathbb{Z}$ and $K_1(O_n) \cong 0$. Hence $\mathrm{index}_K(O_n) = 0$. Moreover, it is known that $O_n \otimes \mathbb{K}$ is isomorphic to the crossed product $(M_{n^\infty} \otimes \mathbb{K}) \rtimes \mathbb{Z}$, where M_{n^∞} is the UHF algebra that is an inductive limit of tensor products of $M_n(\mathbb{C})$ (or an infinite tensor product of $M_n(\mathbb{C})$). Then $\mathrm{index}_K(O_n \otimes \mathbb{K}) = 0$ and $\mathrm{index}_K(M_{n^\infty} \otimes \mathbb{K}) = \infty$. Since $K_0(M_{n^\infty}) \cong \mathbb{Z}[\frac{1}{n}]$ and $K_1(M_{n^\infty}) \cong 0$, we have

$$\begin{array}{ccccc}
\mathbb{Z}[\frac{1}{n}] \cong \mathbb{Z}^\infty & \xrightarrow{(\mathrm{id}-\alpha)_*} & \mathbb{Z}[\frac{1}{n}] & \xrightarrow{i_*} & K_0(O_n) \cong \mathbb{Z}/(n-1)\mathbb{Z} \\
\uparrow & & & & \downarrow \\
K_1(O_n) \cong 0 & \xleftarrow{i_*} & 0 & \xleftarrow{(\mathrm{id}-\alpha)_*} & 0
\end{array}$$

where

$$(\mathrm{id} - \alpha)_*(\sum_{j=1}^k a_j \frac{1}{n^j}) = (1 - \frac{1}{n}) \sum_{j=1}^k a_j \frac{1}{n^j} = \sum_{j=1}^k a_j (n-1) \frac{1}{n^{j+1}}$$

for $a_j \in \mathbb{Z}$, and

$$i_*(\sum_{j=1}^k a_j \frac{1}{n^j}) = \sum_{j=1}^k a_j + (n-1)\mathbb{Z}.$$

Also, $K_0(O_\infty) \cong \mathbb{Z}$ and $K_1(O_\infty) \cong 0$. Hence $\mathrm{index}_K(O_\infty) = 1$.

Let $A = (a_{ij})$ be an $n \times n$ matrix with entries 0 or 1. The Cuntz-Krieger algebra O_A is defined to be the universal C^*-algebra generated by n partial isometries s_j ($1 \leq j \leq n$) such that $s_i^* s_i = \sum_{j=1}^n a_{ij} s_j s_j^*$. The tensor product $O_A \otimes \mathbb{K}$ can be written as $\mathfrak{B} \rtimes_\alpha \mathbb{Z}$ for a stable AF algebra $\mathfrak{B}$. Furthermore, the Pimsner-Voiculescu exact sequence implies that $K_0(O_A) \cong \mathbb{Z}^n/(1 - A^t)\mathbb{Z}^n$ and $K_1(O_A) \cong \ker(1 - A^t)$. Therefore,

$$\mathrm{index}_K(O_A) = n - \mathrm{rank}_\mathbb{Z}(1 - A^t)\mathbb{Z}^n - \mathrm{rank}_\mathbb{Z}\ker(1 - A^t) = 0.$$

Corollary 1.21 *Let $\mathfrak{B}$ be a C^*-algebra with $\mathrm{index}_K(\mathfrak{B}) \neq 0$. Then $\mathfrak{B}$ cannot be written as a crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ for a C^*-algebra $\mathfrak{A}$ either with $\mathrm{index}_K(\mathfrak{A})$ finite or with $(\mathrm{id} - \alpha)_*$ isomorphisms.*

Remark. This consequence should be a some interesting criterion. Also, an if-and-only-if characterization for $\mathrm{index}_K(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$ to be zero has been considered by Takai (but still remains open), and his point of view is quite deep ([14]).

Example 1.22 A directed graph E consists of a vertex set E^0 and an edge set E^1 with range and source maps $r, s : E^1 \to E^0$. A directed graph E is called row finite if at most finitely many edges emits from any vertex. Let $C^*(E)$ be the C^*-algebra of a row finite directed graph E. Then

$$K_0(C^*(E)) \cong \text{coker}(I - A_E^t), \quad K_1(C^*(E)) \cong \ker(I - A_E^t),$$

where $I - A_E^t : \mathbb{Z}E^1 \to \mathbb{Z}E^1$, where $\mathbb{Z}E^1$ is the free abelian group of finitely supported functions from E^1 to $\mathbb{Z}$, and $A_E = (A_E(e,f))_{e,f \in E^1}$ is the edge matrix of E defined by $A_E(e,f) = 1$ if $r(e) = s(f)$ and $= 0$ if $r(e) \neq s(f)$. Hence, we obtain

$$\begin{aligned}
\text{index}_K(C^*(E)) &= \text{rank}_{\mathbb{Z}}\text{coker}(I - A_E^t) - \text{rank}_{\mathbb{Z}}\ker(I - A_E^t) \\
&= \text{rank}_{\mathbb{Z}}(\mathbb{Z}E^1/(I - A_E^t)\mathbb{Z}E^1) - \text{rank}_{\mathbb{Z}}\ker(I - A_E^t) \\
&= \text{rank}_{\mathbb{Z}}(\mathbb{Z}E^1) - \text{rank}_{\mathbb{Z}}(I - A_E^t)\mathbb{Z}E^1 - \text{rank}_{\mathbb{Z}}\ker(I - A_E^t),
\end{aligned}$$

where the last equation is conventional. However, if we allow this formal convention, then

$$\begin{aligned}
\text{index}_K(C^*(E)) &= \text{rank}_{\mathbb{Z}}(\mathbb{Z}E^1) - \text{rank}_{\mathbb{Z}}(I - A_E^t)\mathbb{Z}E^1 - \text{rank}_{\mathbb{Z}}\ker(I - A_E^t) \\
&= \text{rank}_{\mathbb{Z}}(\mathbb{Z}E^1) - \text{rank}_{\mathbb{Z}}\mathbb{Z}E^1 = 0.
\end{aligned}$$

But this is not correct in general. Indeed, if we take A as

$$A = \begin{pmatrix}
1 & 0 & 1 & & & & & \\
1 & 1 & 0 & 0 & & & & \\
 & 1 & 1 & 0 & 1 & & & \\
 & & 1 & 1 & 0 & 0 & & \\
 & & & 1 & 1 & 0 & 1 & \\
 & & & & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}$$

then $K_0(C^*(E)) \cong 0$ and $K_1(C^*(E)) \cong \mathbb{Z}$. Therefore, $\text{index}_K(C^*(E)) = -1$. Refer to [12].

Proposition 1.23 *Let $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ be the crossed prdouct of a C^*-algebra $\mathfrak{A}$ by an action α of $\mathbb{R}$. Then*

$$\text{index}_K(\mathfrak{A} \rtimes_\alpha \mathbb{R}) = -\text{index}_K(\mathfrak{A}).$$

Proof. Use the Connes' Thom isomorphism $K_j(\mathfrak{A} \rtimes_\alpha \mathbb{R}) \cong K_{j+1}(\mathfrak{A})$. □

Corollary 1.24 *Let $\mathfrak{A}$ be a C^*-algebra. Then*

$$\mathrm{index}_K(\mathfrak{A} \rtimes \mathbb{R} \rtimes \mathbb{R} \cdots \rtimes \mathbb{R}) = (-1)^k \,\mathrm{index}_K(\mathfrak{A})$$

for any successive crossed product $\mathfrak{A} \rtimes \mathbb{R} \rtimes \mathbb{R} \cdots \rtimes \mathbb{R}$ (k times) by $\mathbb{R}$.

Example 1.25 Let G be a simply connected solvable Lie group with dimension n. It can be viewed as an n successive semi-direct product by $\mathbb{R}$. Thus, the group C^*-algebra $C^*(G)$ is isomorphic to an n successive crossed product by $\mathbb{R}$. Hence

$$\mathrm{index}_K(C^*(G)) = \begin{cases} 1 & \text{if } \dim G \text{ even,} \\ -1 & \text{if } \dim G \text{ odd.} \end{cases}$$

In particular, $\mathrm{index}_K(C^*(A_2)) = 1$ and $\mathrm{index}_K(C^*(H_3^{\mathbb{R}})) = -1$ as shown before.

Proposition 1.26 *The Takai duality is the isomorphism:*

$$(\mathfrak{A} \rtimes_\alpha G) \rtimes_{\alpha^\wedge} G^\wedge \cong \mathfrak{A} \otimes \mathbb{K}(L^2(G))$$

for any C^*-dynamical system $(\mathfrak{A}, G, \alpha)$ with G abelian (and finite or infinite) and for the dual action $\alpha^\wedge$ of the dual group $G^\wedge$ of G defined by $\alpha^\wedge(g) = \gamma(g)g$ for $g \in G$, $\gamma \in G^\wedge$ and $\alpha^\wedge$ trivial on $\mathfrak{A}$. Thus,

$$\mathrm{index}_K((\mathfrak{A} \rtimes_\alpha G) \rtimes_{\alpha^\wedge} G^\wedge) = \mathrm{index}_K(\mathfrak{A}).$$

Remark. Such a duality (or periodicity) for K-index might be true for other (extended) cases.

Free products and amalgams of C^*-algebras

Proposition 1.27 (1). *Let $\mathfrak{A}$, $\mathfrak{B}$ be C^*-algebras and $\mathfrak{A} * \mathfrak{B}$ their full free product. Then*

$$\mathrm{index}_K(\mathfrak{A} * \mathfrak{B}) = \mathrm{index}_K(\mathfrak{A}) + \mathrm{index}_K(\mathfrak{B}).$$

(2). *Let $\mathfrak{A}$, $\mathfrak{B}$ be unital C^*-algebras and $\mathfrak{A} *_{\mathbb{C}} \mathfrak{B}$ their unital full free product. Then*

$$\mathrm{index}_K(\mathfrak{A} *_{\mathbb{C}} \mathfrak{B}) = \mathrm{index}_K(\mathfrak{A}) + \mathrm{index}_K(\mathfrak{B}) - 1.$$

In particular, we obtain

$$\mathrm{index}_K(\mathfrak{A} *_{\mathbb{C}} C(\mathbb{T})) = \mathrm{index}_K(\mathfrak{A}) - 1.$$

(3). *Let $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ be an amalgam of C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$ over a common C^*-subalgebra $\mathfrak{C}$. Suppose that there exist $*$-homomorphisms from $\mathfrak{A}$ and $\mathfrak{B}$ onto $\mathfrak{C}$. Then*

$$\mathrm{index}_K(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) = \mathrm{index}_K(\mathfrak{A}) + \mathrm{index}_K(\mathfrak{B}) - \mathrm{index}_K(\mathfrak{C}).$$

Furthermore, let

$$\mathfrak{A} = (\cdots((\mathfrak{B}_1 *_{\mathfrak{C}_1} \mathfrak{B}_2) *_{\mathfrak{C}_2} \mathfrak{B}_3) \cdots) *_{\mathfrak{C}_n} \mathfrak{B}_{n+1}$$

be a successive amalgam of C^*-algebras $\mathfrak{B}_j$ $(1 \le j \le n+1)$ and $\mathfrak{C}_j$ $(1 \le j \le n)$. If there exist $*$-homomorphisms from $\mathfrak{A}_j = (\cdots(\mathfrak{B}_1 *_{\mathfrak{C}_1} \mathfrak{B}_2)\cdots) *_{\mathfrak{C}_{j-1}} \mathfrak{B}_j$ and $\mathfrak{B}_{j+1}$ onto $\mathfrak{C}_j$ for $1 \le j \le n$, then

$$\mathrm{index}_K(\mathfrak{A}) = \sum_{j=1}^{n+1} \mathrm{index}_K(\mathfrak{B}_j) - \sum_{j=1}^{n} \mathrm{index}_K(\mathfrak{C}_j).$$

(4). *Let Γ be a K-amenable group, and let $C^*(\Gamma)$ and $C_r^*(\Gamma)$ be its full and reduced group C^*-algebras. Then $\mathrm{index}_K(C^*(\Gamma)) = \mathrm{index}_K(C_r^*(\Gamma))$.*

Proof. For (1), we have $K_j(\mathfrak{A} * \mathfrak{B}) \cong K_j(\mathfrak{A}) \oplus K_j(\mathfrak{B})$ (see [1, 10.11.11]).

For (2), we have $K_0(\mathfrak{A} *_{\mathbb{C}} \mathfrak{B}) \cong (K_j(\mathfrak{A}) \oplus K_j(\mathfrak{B}))/\mathbb{Z}$ and $K_1(\mathfrak{A} *_{\mathbb{C}} \mathfrak{B}) \cong (K_1(\mathfrak{A}) \oplus K_1(\mathfrak{B}))$ (see [1, 10.11.11]).

For (3), we have $K_j(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \cong K_j(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B})$ (see [1, 10.11.11]). We use this isomorphism repeatedly for the successive amalgam C^*-algebras defined inductively.

For (4), we have $K_j(C^*(\Gamma)) \cong K_j(C_r^*(\Gamma))$. □

Remark. The class of K-amenable discrete groups is closed under extensions, direct limits, direct products, and free products, and under taking closed subgroups (see [6]). K-index amenability for Γ may be defined as that $\mathrm{index}_K(C^*(\Gamma)) = \mathrm{index}_K(C_r^*(\Gamma))$ holds.

Corollary 1.28 *If we have $\mathrm{index}_K(C^*(\Gamma)) \neq \mathrm{index}_K(C_r^*(\Gamma))$, then Γ is not K-amenable.*

Remark. Such an application might be usuful and interesting, but it has not yet explored.

Example 1.29 Let F_n be the free group of n generators. Since F_n is isomorphic to the n-fold free product $*^n\mathbb{Z}$, we have $C^*(F_n)$ is isomorphic to the unital n-fold full free product $*^n_{\mathbb{C}} C^*(\mathbb{Z}) \cong *^n_{\mathbb{C}} C(\mathbb{T})$. It follows that

$K_0(C^*(F_n)) \cong \mathbb{Z}$ and $K_1(C^*(F_n)) \cong \mathbb{Z}^n$. Therefore, $\mathrm{index}_K(C^*(F_n)) = 1-n$. Since F_n is K-amenable, $\mathrm{index}_K(C_r^*(F_n)) = 1-n$.

Let $F_2 \rtimes_\sigma \mathbb{Z}$ be the semi-direct product for the action σ defined by $\sigma(a) = b$ and $\sigma(b) = a$ for generators a, b of F_2. Then $C^*(F_2 \rtimes_\sigma \mathbb{Z}) \cong C^*(F_2) \rtimes_\sigma \mathbb{Z}$ and $C_r^*(F_2 \rtimes_\sigma \mathbb{Z}) \cong C_r^*(F_2) \rtimes_\sigma \mathbb{Z}$. Therefore, $\mathrm{index}_K(C^*(F_2 \rtimes_\sigma \mathbb{Z})) = 0 = \mathrm{index}_K(C_r^*(F_2 \rtimes_\sigma \mathbb{Z}))$. Indeed, the Pimsner-Voiculescu six-term exact sequence implies that their K_0 and K_1-groups are $\mathbb{Z}^2$.

Let F_∞ be the free group of countably infinite generators. Then we have $K_0(C_r^*(F_\infty)) \cong 0$ and $K_1(C_r^*(F_\infty)) \cong \mathbb{Z}^\infty$ (see [1, 10.11.10]). Thus, we obtain $\mathrm{index}_K(C_r^*(F_\infty)) = -\infty$. Since F_∞ is K-amenable, we have $\mathrm{index}_K(C^*(F_\infty)) = -\infty$.

Let $G = \varinjlim F_\infty$ be an inductive limit of F_∞ via the commutator subgroup $[F_\infty, F_\infty] \cong F_\infty$. Then $K_0(C_r^*(G)) \cong \mathbb{Z}$ and $K_1(C_r^*(G)) \cong 0$ (see [1, 10.11.10]). Hence $\mathrm{index}_K(C_r^*(G)) = 1$. Since G is K-amenable, $\mathrm{index}_K(C^*(G)) = 1$.

Let $\mathbb{Z}^m * \mathbb{Z}^n$ be the free product of $\mathbb{Z}^m$ and $\mathbb{Z}^n$. Then $C^*(\mathbb{Z}^m * \mathbb{Z}^n) \cong C^*(\mathbb{Z}^m) *_{\mathbb{C}} C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^m) *_{\mathbb{C}} C(\mathbb{T}^n)$. Therefore,

$$K_0(C^*(\mathbb{Z}^m * \mathbb{Z}^n)) \cong (K_0(C(\mathbb{T}^m) \oplus K_0(C(\mathbb{T}^n))/K_0(\mathbb{C}) \cong \mathbb{Z}^{2^{m-1}+2^{n-1}-1}$$

and $K_1(C^*(\mathbb{Z}^m * \mathbb{Z}^n)) \cong \mathbb{Z}^{2^{m-1}+2^{n-1}}$. Hence $\mathrm{index}_K(C^*(\mathbb{Z}^m * \mathbb{Z}^n)) = -1$. Since $\mathbb{Z}^m * \mathbb{Z}^n$ is K-amenable, we have $\mathrm{index}_K(C_r^*(\mathbb{Z}^m * \mathbb{Z}^n)) = -1$.

Let $\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m$ be an amalgam of finite cyclic groups. Then it is shown in [10] that $C_r^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)$ has K_0-group isomorphic to $\mathbb{Z}^{n+m-l}$ and K_1 zero. Thus, $\mathrm{index}_K(C_r^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)) = n+m-l$. Since $\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m$ is K-amenable, $\mathrm{index}_K(C^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)) = n+m-l$. In particular, it is known that $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. Therefore, $\mathrm{index}_K(C_r^*(SL_2(\mathbb{Z}))) = 8 = \mathrm{index}_K(C^*(SL_2(\mathbb{Z})))$. Since $PSL_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$, we have

$$\mathrm{index}_K(C_r^*(PSL_2(\mathbb{Z}))) = 4 = \mathrm{index}_K(C^*(PSL_2(\mathbb{Z}))).$$

Let Γ_g be the fundamental group of an orientable closed surface M_g with genus ≥ 2. We have $\Gamma_g \cong F_2 *_{\mathbb{Z}} F_{2g-2}$ (see [10]), and it follows that $C^*(\Gamma_g) \cong C^*(F_2) *_{C^*(\mathbb{Z})} C^*(F_{2g-2})$. Hence

$$\begin{aligned}\mathrm{index}_K(C^*(\Gamma_g)) &= \mathrm{index}_K(C^*(F_2)) + \mathrm{index}_K(C^*(F_{2g-2})) - \mathrm{index}_K(C^*(\mathbb{Z})) \\ &= -1 + (3-2g) - 0 = 2-2g.\end{aligned}$$

Since Γ_g is K-amenable, we have $\mathrm{index}_K(C_r^*(\Gamma_g)) = 2-2g$.

Let Σ_k be a closed non-orientable surface with $k \geq 2$ cross-caps and $\pi_1(\Sigma_k)$ its fundamental group. Then $\pi_1(\Sigma_k) \cong \mathbb{Z} *_{\mathbb{Z}} F_{k-1}$ where $\mathbb{Z} \cong 2\mathbb{Z} \subset \mathbb{Z}$

(see [10]), and $C^*(\mathbb{Z} *_{\mathbb{Z}} F_{k-1}) \cong C^*(\mathbb{Z}) *_{C^*(\mathbb{Z})} C^*(F_{k-1})$. Therefore,

$$\begin{aligned}\mathrm{index}_K(C^*(\pi_1(\Sigma_k))) &= \mathrm{index}_K(C^*(\mathbb{Z})) + \mathrm{index}_K(C^*(F_{k-1})) - \mathrm{index}_K(C^*(\mathbb{Z})) \\ &= 2-k.\end{aligned}$$

Since $\pi_1(\Sigma_k)$ is K-amenable, we have $\mathrm{index}_K(C_r^*(\pi_1(\Sigma_k))) = 2-k$.

Furthermore, note that

$$\begin{aligned}K_0(C^*(\mathbb{Z}_n * F_2)) &\cong (\mathbb{Z}^n \oplus \mathbb{Z})/\mathbb{Z} \cong \mathbb{Z}^n, \\ K_1(C^*(\mathbb{Z}_n * F_2)) &\cong 0 \oplus \mathbb{Z}^2 \cong \mathbb{Z}^2.\end{aligned}$$

Hence $\mathrm{index}_K(C^*(\mathbb{Z}_n * F_2)) = n-2 = \mathrm{index}_K(C_r^*(\mathbb{Z}_n * F_2))$ since $\mathbb{Z}_n * F_2$ is K-amenable. Furthermore, let $\varinjlim \mathbb{Z}_{n^k} * F_2$ be an inductive limit of $\mathbb{Z}_{n^k} * F_2$ $(k \geq 1)$ by natural embeddings. Then $C^*(\varinjlim \mathbb{Z}_{n^k} * F_2)$ is an inductive limit of $C^*(\mathbb{Z}_{n^k} * F_2)$ by the induced embeddings. In this case, we have $K_0(C^*(\varinjlim \mathbb{Z}_{n^k} * F_2)) \cong \mathbb{Z}^\infty$ and $K_1(C^*(\varinjlim \mathbb{Z}_{n^k} * F_2)) \cong \mathbb{Z}^2$. Hence $\mathrm{index}_K(C^*(\varinjlim \mathbb{Z}_{n^k} * F_2)) = +\infty$. Similarly, we can apply this argument for $C_r^*(\varinjlim \mathbb{Z}_{n^k} * F_2)$ to have the same K-ranks and K-index as $C^*(\varinjlim \mathbb{Z}_{n^k} * F_2)$.

Example 1.30 Let $G_n = \mathbb{Z} \rtimes \cdots \rtimes \mathbb{Z}$ (n times). Let $G = \varinjlim G_n$. Then $C^*(G) \cong \varinjlim C^*(G_n)$. We have the following Pimsner-Voiculescu six term exact sequence for each n:

$$\begin{tikzcd} K_0(C^*(G_n)) \arrow[r, "(\mathrm{id}-\alpha)_*"] & K_0(C^*(G_n)) \arrow[r, "i_*"] & K_0(C^*(G_{n+1})) \arrow[d, "\partial"] \\ K_1(C^*(G_{n+1})) \arrow[u, "\partial"] & K_1(C^*(G_n)) \arrow[l, "i_*"] & K_1(C^*(G_n)) \arrow[l, "(\mathrm{id}-\alpha)_*"] \end{tikzcd}$$

Note that $(\mathrm{id}-\alpha)_*$ on K_0-groups is always zero.

The case where i_* on K_0-groups for each n is an isomorphism. Then $K_0(C^*(G)) \cong K_0(C^*(G_n))$ for any n, and the map ∂ from K_0 to K_1 is zero. Hence, $(\mathrm{id}-\alpha)_*$ on K_1-groups for each n is an injection, so that it is an isomorphism. Thus, i_* on K_1-groups for each n is zero. Therefore, $K_1(C^*(G)) \cong 0$. In this case, we obtain $\mathrm{index}_K(C^*(G)) = \mathrm{Kr}_0(C^*(G_n))$ for any n. Also, $K_0(C^*(G_n)) \cong K_1(C^*(G_{n+1}))$.

The case where i_* on K_0-groups for each n is zero. Then $K_0(C^*(G)) \cong 0 \cong K_0(C^*(G_{n+1}))$. But this case does not exist since each $K_0(C^*(G_n))$ is non-zero.

The case where i_* on K_0-groups for each n is neither an isomorphism nor zero. This is the most possible case. Then we may assume that $\mathrm{Kr}_0(C^*(G_n)) < \mathrm{Kr}_0(C^*(G_{n+1}))$ for each n. Hence, $\mathrm{Kr}_0(C^*(G)) = +\infty$.

Also, the map ∂ from K_0 to K_1 is non-zero. Hence, $(\mathrm{id} - \alpha)_*$ on K_1-groups for each n is not an injection. Thus, i_* on K_1-groups for each n is non-zero. Moreover, the map ∂ from K_1 to K_0 is onto, so that $\mathrm{Kr}_1(C^*(G_{n+1})) \geq \mathrm{Kr}_0(C^*(G_n))$, which goes to $+\infty$ as $n \to \infty$, Hence $\mathrm{Kr}_1(C^*(G)) = +\infty$. In this case, we obtain $\mathrm{index}_K(C^*(G)) = +\infty - \infty$ (or undefined).

In particular, if $G = \varinjlim \mathbb{Z}^n$ with $G_n = \mathbb{Z}^n$, then $\mathrm{index}_K(C^*(G)) = +\infty - \infty$ (undefined) while $\mathrm{index}_K(C^*(G_n)) = \mathrm{index}_K(C(\mathbb{T}^n)) = 0$. However, if we allow the convention $\infty - \infty = 0$, then $\mathrm{index}_K(C^*(G)) = 0$ in this sense.

Example 1.31 If the amalgam $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ of C^*-algebras has retractions to $\mathfrak{C}$, then it is KK-equivalent to the pullback $\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$. In particular, $C^*(F_n)$ is KK-equivalent to $C(X_n)$, where the space X_n consists of n circles joined at a point, that is the one-point compactification of the disjoint union of n copies of $\mathbb{R}$.

For many locally compact groups G including F_n, the quotient map from $C^*(G)$ to $C_r^*(G)$ gives a KK-equivalence. In particular, $C_r^*(F_n)$ is KK-equivalent to $C(X_n)$.

Proposition 1.32 *Let $\mathfrak{A}$ be a C^*-algebra and α_j $(1 \leq j \leq n)$ its automorphisms. Let $\mathfrak{A} \rtimes_{\alpha,r} F_n$ be the reduced crossed product by the free group F_n for the action α given by $\alpha(a_j) = \alpha_j(a_j)$ for a_j generators of F_n. If* $\mathrm{index}_K(\mathfrak{A})$ *is finite, then*

$$\mathrm{index}_K(\mathfrak{A} \rtimes_{\alpha,r} F_n) = (1-n)\,\mathrm{index}_K(\mathfrak{A}).$$

Proof. We have the following diagram:

$$\begin{array}{ccccc}
\oplus_{j=1}^n K_0(\mathfrak{A}) & \xrightarrow{\sigma} & K_0(\mathfrak{A}) & \longrightarrow & K_0(\mathfrak{A} \rtimes_{\alpha,r} F_n) \\
\uparrow & & & & \downarrow \\
K_1(\mathfrak{A} \rtimes_{\alpha,r} F_n) & \longleftarrow & K_1(\mathfrak{A}) & \xleftarrow{\sigma} & \oplus_{j=1}^n K_1(\mathfrak{A}),
\end{array}$$

where $\sigma = \sum_{j=1}^n (1-\alpha_j)_*$ (see [1, 10.8]). □

Corollary 1.33 *Let $\mathfrak{A}$ be a C^*-algebra. If* $\mathrm{index}_K(\mathfrak{A})$ *is finite, then*

$$\mathrm{index}_K(\mathfrak{A} \rtimes_r F_n \rtimes_r F_n \cdots \rtimes_r F_n) = (1-n)^k\,\mathrm{index}_K(\mathfrak{A})$$

for any successive reduced crossed product $\mathfrak{A} \rtimes_r F_n \rtimes_r F_n \cdots \rtimes_r F_n$ (k times) by F_n.

Example 1.34 Let $F_m \times F_n$ be the direct product of free groups F_m and F_n, and $C_r^*(F_m \times F_n)$ its reduced group C^*-algebra. Then $C_r^*(F_m \times F_n)$ is isomorphic to the minimal tensor product $C_r^*(F_m) \otimes C_r^*(F_n)$, which is isomorphic to the reduced crossed product $C_r^*(F_m) \rtimes_{\alpha,r} F_n$ with α trivial. Hence, it follows that

$$\begin{aligned}\mathrm{index}_K(C_r^*(F_m \times F_n)) &= \mathrm{index}_K(C_r^*(F_m) \otimes C_r^*(F_n)) \\ &= \mathrm{index}_K(C_r^*(F_m) \rtimes_{\alpha,r} F_n) = (1-m)(1-n).\end{aligned}$$

In this case, the maps σ on K-groups are zero. Thus, we obtain

$$\begin{aligned}K_0(C_r^*(F_m) \rtimes_{\alpha,r} F_n) &\cong K_0(C_r^*(F_m)) \oplus (\oplus_{j=1}^n K_1(C_r^*(F_m)) \\ &\cong \mathbb{Z} \oplus (\oplus_{j=1}^n \mathbb{Z}^m) \cong \mathbb{Z}^{mn+1}, \\ K_1(C_r^*(F_m) \rtimes_{\alpha,r} F_n) &\cong K_1(C_r^*(F_m)) \oplus (\oplus_{j=1}^n K_0(C_r^*(F_m)) \\ &\cong \mathbb{Z}^m \oplus (\oplus_{j=1}^n \mathbb{Z}) \cong \mathbb{Z}^{m+n}.\end{aligned}$$

It follows that $(mn+1) - (m+n) = (1-m)(1-n)$. Since $F_m \times F_n$ is K-amenable, we obtain $\mathrm{index}_K(C^*(F_m \times F_n)) = (1-m)(1-n)$, where the full group C^*-algebra $C^*(F_m \times F_n)$ is isomorphic to the maximal tensor product of $C^*(F_m)$ and $C^*(F_n)$, which is isomorphic to the full crossed product $C^*(F_m) \rtimes_\alpha F_n$ with α trivial.

Furthermore, if we replace $C_r^*(F_m)$ with $C_r^*(F_\infty)$, then $K_0(C_r^*(F_\infty \times F_2)) \cong \oplus^2 \mathbb{Z}^\infty$ and $K_1(C_r^*(F_\infty \times F_2)) \cong \mathbb{Z}^\infty$. Hence $\mathrm{index}_K(C_r^*(F_\infty \times F_2)) = \infty - \infty$.

K-index conjectures

Proposition 1.35 *Let $\Gamma = G_1 *_H G_2$ be an amalgam of discrete groups. If we have the following six-term exact sequence (Conjecture) :*

$$\begin{array}{ccccc} K_0(\mathfrak{A} \rtimes_\alpha H) & \longrightarrow & \oplus_{j=1}^2 K_0(\mathfrak{A} \rtimes_\alpha G_j) & \longrightarrow & K_0(\mathfrak{A} \rtimes_\alpha \Gamma) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{A} \rtimes_\alpha \Gamma) & \longleftarrow & \oplus_{j=1}^2 K_1(\mathfrak{A} \rtimes_\alpha G_j) & \longleftarrow & K_1(\mathfrak{A} \rtimes_\alpha H) \end{array}$$

for any these full (or reduced) crossed products, then

$$\mathrm{index}_K(\mathfrak{A} \rtimes_\alpha \Gamma) = \sum_{j=1}^2 \mathrm{index}_K(\mathfrak{A} \rtimes_\alpha G_j) - \mathrm{index}_K(\mathfrak{A} \rtimes_\alpha H).$$

In particular, if we take $\mathfrak{A} = \mathbb{C}$, then

$$\begin{array}{ccccc} K_0(C^*(H)) & \longrightarrow & \oplus_{j=1}^2 K_0(C^*(G_j)) & \longrightarrow & K_0(C^*(\Gamma)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(\Gamma)) & \longleftarrow & \oplus_{j=1}^2 K_1(C^*(G_j)) & \longleftarrow & K_1(C^*(H)) \end{array}$$

for full group C^*-algebras, so that we obtain

$$\mathrm{index}_K(C^*(\Gamma)) = \sum_{j=1}^{2} \mathrm{index}_K(C^*(G_j)) - \mathrm{index}_K(C^*(H)),$$

and furthermore, those full group C^*-algebras in this formula and the diagram can be replaced with their reduced group C^*-algebras.

Remark. The conjecture has be proved to be affirmative in some cases as given above and is expected to be true in full generality.

Proposition 1.36 *Let Γ be a discrete group with torsion free. The Baum-Connes conjecture* (BC) *is the isomorphism from $K_j(C_r^*(\Gamma))$ to $K^j(B\Gamma)$, where $B\Gamma$ is the classifying space for Γ. If this conjecture is ture, then*

$$\mathrm{Kr}_j(C_r^*(\Gamma)) = \mathrm{Kr}^j(B\Gamma), \quad \textit{and} \quad \mathrm{index}_K(C_r^*(\Gamma)) = \mathrm{index}^K(B\Gamma).$$

Remark. This famous conjecture is known to be true for such Γ in a large class. Indeed, such Γ can be taken as any amenable group, F_n, word-hyperbolic groups such as $\pi_1(M)$ for M a compact Riemannian manifold with negative curvature. Furthermore, the conjecture to be true gives a way to identity our K-index with the Euler characteristic for groups by (co)homology theory. Thus, our K-index might have a potential contribution even to this conjecture (and this vision is of Takai). That connection will be discussed in more details somewhere.

Now let X be a compact space (or a finite CW-complex). The Chern characters from topological K-theory to cohomology theory for spaces are the isomorphisms given by

$$\mathrm{Ch}^0 : K^0(X) \otimes \mathbb{Q} \to \bigoplus_{n:even} H^n(X, \mathbb{Q}) \equiv H^{even}(X, \mathbb{Q}),$$
$$\mathrm{Ch}^1 : K^1(X) \otimes \mathbb{Q} \to \bigoplus_{n:odd} H^n(X, \mathbb{Q}) \equiv H^{odd}(X, \mathbb{Q}),$$

where $H^n(X, \mathbb{Q})$ denotes the n-th (Alexander or Čech) cohomology group of X with coefficients in $\mathbb{Q}$. (As a note, for a complex vector bundle V over (a manifold) X, its Chern characteristic is defined in a way as $\mathrm{Ch}^0(V) = \mathrm{Ch}(V) = \sum_k \mathrm{Ch}_k(V)$, where

$$\mathrm{Ch}_k(V) = (k!)^{-1}[\mathrm{tr}((2\pi)^{-1}\sqrt{-1}K_V)^k] \in H^{2k}(X),$$
$$\mathrm{Ch}(V) = [\mathrm{tr}\exp((2\pi)^{-1}\sqrt{-1}K_V)] \in H^{even}(X) = \bigoplus_{n:even} H^n(X),$$

where K_V is a curvature associated with a connection over V and $H^{2k}(X)$ means the $(2k)$-th de Rham cohomology of X.) Write

$$\mathrm{Ch}^* : K^*(X) \otimes \mathbb{Q} \to H^*(X, \mathbb{Q}),$$

where $K^* = K^0 \oplus K^1$ and $H^* = H^{even} \oplus H^{odd}$. The Euler characteristic for X is defined by the following first (or second) equality:

$$\begin{aligned}\chi(X) &= \sum_{j=0}^{\dim X} \dim_{\mathbb{Q}} H^j(X, \mathbb{Q}) \\ &= \dim_{\mathbb{Q}} H^{even}(X, \mathbb{Q}) - \dim_{\mathbb{Q}} H^{odd}(X, \mathbb{Q}) \\ &= \dim_{\mathbb{Q}} K^0(X) \otimes_{\mathbb{Z}} \mathbb{Q} - \dim_{\mathbb{Q}} K^1(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{index}^K(X).\end{aligned}$$

In particular, for $X = B\Gamma$ the classifying space of a group Γ,

$$\mathrm{Ch}^* : K^*(B\Gamma) \otimes \mathbb{Q} \to H^*(B\Gamma, \mathbb{Q})$$

(an isomorphism). Furthermore, we have the following isomorphism:

$$H^*(B\Gamma, \mathbb{Q}) \cong H^*(\Gamma, \mathbb{Q})$$

where the right hand side means the cohomology for a group Γ (torsion free) with coefficients in $\mathbb{Q}$. Therefore, note that

$$\begin{aligned}\mathrm{index}^K(B\Gamma) &= \dim_{\mathbb{Q}} K^0(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} - \dim_{\mathbb{Q}} K^1(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \\ &= \dim_{\mathbb{Q}} H^{even}(B\Gamma, \mathbb{Q}) - \dim_{\mathbb{Q}} H^{odd}(B\Gamma, \mathbb{Q}) \\ &= \dim_{\mathbb{Q}} H^{even}(\Gamma, \mathbb{Q}) - \dim_{\mathbb{Q}} H^{odd}(\Gamma, \mathbb{Q}) \\ &= \dim_{\mathbb{Z}} H^{even}(\Gamma) - \dim_{\mathbb{Z}} H^{odd}(\Gamma) \equiv \chi(\Gamma),\end{aligned}$$

which is the Euler characteristic for Γ (torsion free), and it also may defined as the following alternative sum:

$$\chi(\Gamma) = \sum_{j \geq 0} (-1)^j \mathrm{rank}_{\mathbb{Z}} H^j(\Gamma, \mathbb{Z})$$

with $H^j(\Gamma, \mathbb{Z}) = H^j(\Gamma)$, where $\mathbb{Q}$ for $\otimes_{\mathbb{Z}}\mathbb{Q}$ and $\dim_{\mathbb{Q}}$ may be replaced with $\mathbb{R}$ or $\mathbb{C}$. Also, for a C^∞-manifold M, its Euler characteristic can be defined as the following alternative sum:

$$\chi(M) = \sum_{j \geq 0}^{\dim M} (-1)^j \dim H^j(M, \mathbb{R})$$

where $H^j(M, \mathbb{R})$ is the j-th (de Rham) cohomology of M with coefficients in $\mathbb{R}$. On the other hand,

$$\mathrm{index}_K(C_r^*(\Gamma)) = \dim_{\mathbb{Q}} K_0(C_r^*(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{Q} - \dim_{\mathbb{Q}} K_1(C_r^*(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

where $C_r^*(\Gamma)$ can be replaced with a C^*-algebra. Therefore, we obtain

Proposition 1.37 *Under the same assumption as above, the Baum-Connes conjecture implies*

$$\mathrm{index}_K(C_r^*(\Gamma)) = \mathrm{index}^K(B\Gamma) = \chi(B\Gamma) = \chi(\Gamma).$$

Remark. Also see below Appendix for $\chi(\Gamma)$ in some details. For the Connes-Chern character from K-theory to Connes' cyclic cohomology theory (for C^*-algebras), its relation with K-index has been discussed by Takai [15].

2 Equivariant K-theory ranks and index

The representation ring $R(G)$ of a compact group G is defined by formal differences of equivalence classes of finite dimensional representations of G, under the direct sum and tensor product as ring operations. The trivial 1-dimensional representation of G is the identity of $R(G)$. Note that $R(G)$ can be identified with $K_0(C^*(G))$ as an additive group.

Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system with $\mathfrak{A}$ unital and G compact. A (finitely generated) projective $(\mathfrak{A}, G, \alpha)$-module is a pair (E, λ) for a (finitely generated) projective $\mathfrak{A}$-module E and a strongly continuous homomorphism λ from G to the group of invertible elements of the set $L(E)$ of bounded linear operators on E such that $\lambda_g(ea) = \lambda_g(e)\alpha_g(a)$ for $g \in G$, $e \in E$, and $a \in \mathfrak{A}$. Let π be a representation of G on a finite dimensional vector space V. Then $V \otimes \mathfrak{A}$ becomes a projective $(\mathfrak{A}, G, \alpha)$-module under the diagonal action of G, and is regarded as a free module. Every projective $(\mathfrak{A}, G, \alpha)$-module is a direct summand of such a free module so that there exists a G-invariant projection p of $L(V) \otimes \mathfrak{A}$ such that $p(V \otimes \mathfrak{A})$ is a projective $(\mathfrak{A}, G, \alpha)$-module.

The G-equivariant K_0-group $K_0^G(\mathfrak{A})$ of a unital C^*-algebra $\mathfrak{A}$ is defined to be the abelian (or Grothendieck) group generated by equivalence classes of projective $(\mathfrak{A}, G, \alpha)$-modules under the direct sum. Moreover, $K_0^G(\mathfrak{A})$ can be viewed as an $R(G)$-module in the way that $p(V \otimes \mathfrak{A})$ is send to $(1 \otimes p)(W \otimes V \otimes \mathfrak{A})$ by the action of $[W] \in R(G)$. Note that $K_0^G(\mathbb{C}) \cong R(G)$ under the trivial G-action on $\mathbb{C}$. For a nonunital C^*-algebra $\mathfrak{A}$, its $K_0^G(\mathfrak{A})$ is defined to be the kernel of the map from $K_0^G(\mathfrak{A}^+)$ to $K_0^G(\mathbb{C})$, where the action on the unitization $\mathfrak{A}^+$ by $\mathbb{C}$ is induced by that of $\mathfrak{A}$ and trivial on $\mathbb{C}$. Refer to [1, Section 11] for the equivariant K-theory for C^*-algebras.

Set $K_1^G(\mathfrak{A}) = K_0^G(SA)$. There is a definition for this in terms of invertible elements, due to N. C. Phillips.

Definition 2.1 Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system, where $\mathfrak{A}$ is a C^*-algebra, G is a compact group, and α is an action of G on $\mathfrak{A}$ by automorphisms. We define the K_j^G-rank of $\mathfrak{A}$ to be the $\mathbb{Z}$-rank of the G-equivariant K-group $K_j^G(\mathfrak{A})$ of $\mathfrak{A}$ ($j = 0, 1$), and denote it by

$$\mathrm{Kr}_j^G(\mathfrak{A}) = \mathrm{rank}_{\mathbb{Z}} K_j(\mathfrak{A}) \in \{0, 1, 2, \cdots, +\infty\}.$$

We define the G-equivariant (Euler-Takai) K-index (or K^G-index) of $\mathfrak{A}$ to be the following difference:

$$\mathrm{index}_K^G(\mathfrak{A}) = \mathrm{Kr}_0^G(\mathfrak{A}) - \mathrm{Kr}_1^G(\mathfrak{A}) \in \mathbb{Z} \cup \{\pm\infty\}.$$

Proposition 2.2 *Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system with G compact and $\mathfrak{A} \rtimes_\alpha G$ its crossed product. Then*

$$\mathrm{index}_K^G(\mathfrak{A}) = \mathrm{index}_K(\mathfrak{A} \rtimes_\alpha G).$$

Proof. We have $K_j^G(\mathfrak{A}) \cong K_j(\mathfrak{A} \rtimes_\alpha G)$ for $j = 0, 1$ (P. Julg). □

Proposition 2.3 (1). *If C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$ are G-stably isomorphic, then we have* $\mathrm{index}_K^G(\mathfrak{A}) = \mathrm{index}_K^G(\mathfrak{B})$. *In particular,*

$$\mathrm{index}_K^G(\mathfrak{A} \otimes M_n(\mathbb{C})) = \mathrm{index}_K^G(\mathfrak{A} \otimes \mathbb{K}) = \mathrm{index}_K^G(\mathfrak{A}),$$

where the actions of G on $\mathfrak{A} \otimes M_n(\mathbb{C})$ and $\mathfrak{A} \otimes \mathbb{K}$ are of the form $\alpha \otimes \mathrm{id}$. Moreover, if $\mathfrak{A}$ and $\mathfrak{B}$ are G-homotopic, then $\mathrm{index}_K^G(\mathfrak{A}) = \mathrm{index}_K^G(\mathfrak{B})$.

(2). *If the action of G on the direct sum $\mathfrak{A} \oplus \mathfrak{B}$ of C^*-algebras is of the form $\alpha \oplus \beta$, then*

$$\mathrm{index}_K^G(\mathfrak{A} \oplus \mathfrak{B}) = \mathrm{index}_K^G(\mathfrak{A}) + \mathrm{index}_K^G(\mathfrak{B}).$$

(3). *If* $\mathfrak{A} \rtimes_\alpha G$, $\mathfrak{B} \rtimes_\beta H$ *with* G, H *compact are nuclear* C^**-algebras in the bootstrap category* X *and their* K*-groups are torsion free, then*

$$\mathrm{index}_K^{G\times H}(\mathfrak{A}\otimes\mathfrak{B}) = \mathrm{index}_K^G(\mathfrak{A})\,\mathrm{index}_K^H(\mathfrak{B}),$$

where the action of $G\times H$ *on* $\mathfrak{A}\otimes\mathfrak{B}$ *is* $\alpha\otimes\beta$.

(4). *For* $S\mathfrak{A} = C_0(\mathbb{R})\otimes\mathfrak{A}$, *and if the action of* G *on* $C_0(\mathbb{R})$ *is trivial, then* $\mathrm{index}_K^G(S\mathfrak{A}) = -\mathrm{index}_K^G(\mathfrak{A})$.

(5). *Let* X *be a compact space with a* G*-action. Then*

$$\mathrm{index}_K^G(C(X)) = \mathrm{index}_G^K(X) \equiv \mathrm{rank}_{\mathbb{Z}}K_G^0(X) - \mathrm{rank}_{\mathbb{Z}}K_G^1(X),$$

where $K_G^*(X)$ *is the* G*-equivariant* K*-theory for* X *defined to be the abelian group(s) generated by the classes of* G*-vector bundles over* X *(and* $\mathbb{R}\times X$*).*

Proof. For (1), note that $K_j^G(\mathfrak{A}\otimes M_n(\mathbb{C})) \cong K_j^G(\mathfrak{A}\otimes\mathbb{K}) \cong K_j^G(\mathfrak{A})$.

For (2), we have the splitting exact sequence:

$$0 \to \mathfrak{A}\rtimes_\alpha G \to (\mathfrak{A}\oplus\mathfrak{B})\rtimes_{\alpha\oplus\beta} G \to \mathfrak{B}\rtimes_\beta G \to 0.$$

For (3), we have $(\mathfrak{A}\otimes\mathfrak{B})\rtimes_{\alpha\otimes\beta}(G\times H) \cong (\mathfrak{A}\rtimes_\alpha G)\otimes(\mathfrak{B}\rtimes_\beta H)$. Using the Künneth formula we obtain

$$\begin{aligned}\mathrm{index}_K^{G\times H}(\mathfrak{A}\otimes\mathfrak{B}) &= \mathrm{index}_K((\mathfrak{A}\rtimes_\alpha G)\otimes(\mathfrak{B}\rtimes_\beta H))\\ &= \mathrm{index}_K(\mathfrak{A}\rtimes_\alpha G)\,\mathrm{index}_K(\mathfrak{B}\rtimes_\beta H)\\ &= \mathrm{index}_K^G(\mathfrak{A})\,\mathrm{index}_K^H(\mathfrak{B}).\end{aligned}$$

For (4), $K_j^G(S\mathfrak{A}) \cong K_{j+1}^G(\mathfrak{A})$ (Bott periodicity). Hence $\mathrm{Kr}_j^G(S\mathfrak{A}) = \mathrm{Kr}_{j+1}^G(\mathfrak{A})$.

For (5), there exists a (Swan) isomorphism between the classes of G-vector bundles E over X and the classes of projective $(C(X), G; \alpha)$-modules $\Gamma(E)$ of continuous sections with a natural induced G-action, where the action α is induced from that of G on X. Hence $K_*^G(C(X)) \cong K_G^*(X)$. □

Example 2.4 (1). $\mathrm{index}_K^{\mathbb{T}^n}(\mathbb{C}) = \infty$ since $\mathbb{C}\rtimes\mathbb{T}^n \cong C^*(\mathbb{T}^n) \cong C_0(\mathbb{Z}^n)$.

(2). For the dual crossed product $(\mathbb{C}\rtimes_\alpha\mathbb{Z}^n)\rtimes\mathbb{T}^n$ for α trivial so that $\mathbb{C}\rtimes_\alpha\mathbb{Z}^n = C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$, we have $\mathrm{index}_K^{\mathbb{T}^n}(C(\mathbb{T}^n)) = \mathrm{index}_K(\mathbb{K}) = 1$ because $(\mathbb{C}\rtimes_\alpha\mathbb{Z}^n)\rtimes\mathbb{T}^n \cong \mathbb{C}\otimes\mathbb{K}$ by Takai duality.

Theorem 2.5 *Let* $0\to\mathfrak{J}\to\mathfrak{A}\to\mathfrak{A}/\mathfrak{J}\to 0$ *be a short exact sequence of* C^**-algebras invariant under an action of a compact group* G*. Its six term* *exact sequence is*

$$\begin{array}{ccccc} K_0^G(\mathfrak{J}) & \longrightarrow & K_0^G(\mathfrak{A}) & \longrightarrow & K_0^G(\mathfrak{A}/\mathfrak{J}) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1^G(\mathfrak{A}/\mathfrak{J}) & \longleftarrow & K_1^G(\mathfrak{A}) & \longleftarrow & K_1^G(\mathfrak{J}). \end{array}$$

If the index maps ∂ are both zero, then

$$\mathrm{Kr}_j^G(\mathfrak{A}) = \mathrm{Kr}_j^G(\mathfrak{J}) + \mathrm{Kr}_j^G(\mathfrak{A}/\mathfrak{J}), \quad \mathrm{index}_K^G(\mathfrak{A}) = \mathrm{index}_K^G(\mathfrak{J}) + \mathrm{index}_K^G(\mathfrak{A}/\mathfrak{J}).$$

In particular, this is the case if the short exact sequence splits.

Moreover, if $\mathrm{index}_K^G(\mathfrak{A})$, $\mathrm{index}_K^G(\mathfrak{J})$, *and* $\mathrm{index}_K^G(\mathfrak{A}/\mathfrak{J})$ *are finite, then*

$$\mathrm{index}_K^G(\mathfrak{A}) = \mathrm{index}_K^G(\mathfrak{J}) + \mathrm{index}_K^G(\mathfrak{A}/\mathfrak{J}),$$

where it is enough to assume that two of those are finite.

Proof. We use similarly the argument for G trivial as shown in Section 1. □

Theorem 2.6 *Let $\mathfrak{A}\rtimes_\alpha\mathbb{Z}$ be the crossed product of a C^*-algebra $\mathfrak{A}$ by an action α of $\mathbb{Z}$. For a C^*-dynamical system $(\mathfrak{A}, G, \beta)$ with G compact, suppose that the action β commutes with α. If both* $\mathrm{index}_K^G(\mathfrak{A})$ *and* $\mathrm{index}_K^G(\mathfrak{A}\rtimes_\alpha\mathbb{Z})$ *are finite, then*

$$\mathrm{index}_K^G(\mathfrak{A}\rtimes_\alpha\mathbb{Z}) = 0,$$

where it is enough to assume that $\mathrm{index}_K^G(\mathfrak{A})$ *is finite.*

Proof. By assumption, $(\mathfrak{A}\rtimes_\alpha\mathbb{Z})\rtimes_\beta G \cong (\mathfrak{A}\rtimes_\beta G)\rtimes_\alpha\mathbb{Z}$. Then we use the Pimsner-Voiculescu exact sequence. □

Proposition 2.7 *Let $(\mathfrak{A}, \mathbb{T}, \alpha)$ be a C^*-dynamical system by the torus $\mathbb{T}$. If* $\mathrm{index}_K^{\mathbb{T}}(\mathfrak{A})$ *and* $\mathrm{index}_K(\mathfrak{A})$ *are finite, then* $\mathrm{index}_K(\mathfrak{A}) = 0$, *where it is enough to assume that* $\mathrm{index}_K^{\mathbb{T}}(\mathfrak{A})$ *is finite.*

Proof. Let $\mathfrak{A}\rtimes_\alpha\mathbb{T}$ be the crossed product and $(\mathfrak{A}\rtimes_\alpha\mathbb{T})\rtimes_\beta\mathbb{Z}$ its dual crossed product with β the dual action since $\mathbb{Z}$ is the dual group of $\mathbb{T}$. The Takai duality implies that the dual crossed product is isomorphic to $\mathfrak{A}\otimes\mathbb{K}$. Thus, the Pimsner-Voiculescu exact sequence implies

$$\begin{array}{ccccc} K_0(\mathfrak{A}\rtimes_\alpha\mathbb{T}) & \longrightarrow & K_0(\mathfrak{A}\rtimes_\alpha\mathbb{T}) & \longrightarrow & K_0(\mathfrak{A}) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{A}) & \longleftarrow & K_1(\mathfrak{A}\rtimes_\alpha\mathbb{T}) & \longleftarrow & K_1(\mathfrak{A}\rtimes_\alpha\mathbb{T}). \end{array}$$

The same argument as before deduces the conclusion. □

Corollary 2.8 *Let $\mathfrak{A}$ be a C^*-algebra with* $\mathrm{index}_K(\mathfrak{A}) \neq 0$. *Then* $\mathrm{index}_K^{\mathbb{T}}(\mathfrak{A})$ *is not finite.*

Proposition 2.9 *Let $(\mathfrak{A}, \mathbb{Z}_n, \alpha)$ be a C^*-dynamical system by a finite cyclic group $\mathbb{Z}_n$. There exists the following exact sequence:*

$$0 \to S(\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n) \to \mathfrak{A} \rtimes_\alpha \mathbb{Z} \to \mathfrak{A} \rtimes_\alpha \mathbb{Z}_n \to 0,$$

where $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ with the action α extended by that of $\mathbb{Z}_n$ is isomorphic to the mapping torus on $\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n$.

If $\mathrm{index}_K^{\mathbb{Z}_n}(\mathfrak{A})$ *and* $\mathrm{index}_K(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$ *are finite, then* $\mathrm{index}_K(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) = 0$, *where it is enough to assume that* $\mathrm{index}_K^{\mathbb{Z}_n}(\mathfrak{A})$ *is finite.*

Example 2.10 Since $(M_{n^\infty} \otimes \mathbb{K}) \rtimes \mathbb{Z} \cong O_n \otimes \mathbb{K}$, consider its dual crossed product $(O_n \otimes \mathbb{K}) \rtimes_\beta \mathbb{T} \cong M_{n^\infty} \otimes \mathbb{K} \otimes \mathbb{K}$. Thus, $\mathrm{index}_K(O_n \otimes \mathbb{K}) = 0$ but $\mathrm{index}_K^{\mathbb{T}}(O_n \otimes \mathbb{K}) = \infty$.

Let $\mathbb{T}_\theta^2 = C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ be a noncommutative 2-torus and $\mathbb{T}_\theta^2 \rtimes_\beta \mathbb{T}$ its dual crossed product. Since $\mathbb{T}_\theta^2 \rtimes_\beta \mathbb{T} \cong C(\mathbb{T}) \otimes \mathbb{K}$ by Takai duality, we obtain $\mathrm{index}_K^{\mathbb{T}}(\mathbb{T}_\theta^2) = \mathrm{index}_K(C(\mathbb{T})) = 0$ and $\mathrm{index}_K(\mathbb{T}_\theta^2) = 0$.

Let $\mathbb{T}_\theta^2 \rtimes_\sigma \mathbb{Z}_2$ be the crossed product of a simple noncommutative 2-torus $\mathbb{T}_\theta^2$ by the flip, i.e., $\sigma(u_j) = u_j^*$ $(j = 1, 2)$ for u_j generating unitaries of $\mathbb{T}_\theta^2$. It is known by [3] that $K_0(\mathbb{T}_\theta^2 \rtimes_\sigma \mathbb{Z}_2) \cong \mathbb{Z}^6$ and $K_1(\mathbb{T}_\theta^2 \rtimes_\sigma \mathbb{Z}_2) \cong 0$. Hence $\mathrm{index}_K(\mathbb{T}_\theta^2 \rtimes_\sigma \mathbb{Z}_2) = 6$. It follows that $\mathrm{index}_K(\mathbb{T}_\theta^2 \rtimes_\sigma \mathbb{Z}) = 0$. Furthermore, let $(\mathbb{T}_\theta^2)^\sigma$ be the fixed point algebra under the flip σ. Since $(\mathbb{T}_\theta^2)^\sigma$ is a corner of a simple C^*-algebra $\mathbb{T}_\theta^2 \rtimes_\sigma \mathbb{Z}_2$, the K-theory of $(\mathbb{T}_\theta^2)^\sigma$ is the same as that of $\mathbb{T}_\theta^2 \rtimes_\sigma \mathbb{Z}_2$. Hence, $\mathrm{index}_K((\mathbb{T}_\theta^2)^\sigma) = 6$. Note that $(\mathbb{T}_\theta^2)^\sigma$ is said to be a noncommutative sphere because $C(\mathbb{T}^2)^\sigma$ is isomorphic to $C(S^2)$. Set $S_\theta^2 = (\mathbb{T}_\theta^2)^\sigma$, but whose K-index is not equal 2 of $C(S^2)$. This is a little bit strange in our point of view. Thus, S_θ^2 might be not a right deformation for $C(S^2)$. Refer to [3] for more details.

Let $\mathfrak{A} = M_{2^\infty} = \otimes^\infty M_2(\mathbb{C})$. The gauge action α of $\mathbb{T}$ on $\mathfrak{A}$ is defined by $\alpha_z = \otimes^\infty \mathrm{Ad}(z \oplus 1)$ for $z \in \mathbb{T}$, where $\mathrm{Ad}(\cdot)$ is the adjoint action by the diagonal unitary matrix $z \oplus 1 \in M_2(\mathbb{C})$. Since $\mathfrak{A} \rtimes_\alpha \mathbb{T} \rtimes_{\alpha^\wedge} \mathbb{Z} \cong \mathfrak{A} \otimes \mathbb{K}$,

$$\begin{tikzcd}
K_0(\mathfrak{A} \rtimes_\alpha \mathbb{T}) \arrow[r, "(\mathrm{id}-\beta)_*"] & K_0(\mathfrak{A} \rtimes_\alpha \mathbb{T}) \arrow[r] & \mathbb{Z}^\infty \arrow[d] \\
0 \arrow[u] & K_1(\mathfrak{A} \rtimes_\alpha \mathbb{T}) \arrow[l] & K_1(\mathfrak{A} \rtimes_\alpha \mathbb{T}). \arrow[l, "(\mathrm{id}-\beta)_*"']
\end{tikzcd}$$

where β is the dual action of $\mathbb{Z}$ on $\mathfrak{A} \rtimes_\alpha \mathbb{T}$. Since the map $(1-\beta)_*$ on K_0-groups is zero, we have $K_0(\mathfrak{A} \rtimes_\alpha \mathbb{T}) \cong 0$. Indeed, note that $\mathfrak{A}$ is contained in $\mathfrak{A} \rtimes_\alpha \mathbb{T}$ and the dual action β on $\mathfrak{A}$ is trivial. Also, $C^*(\mathbb{T}) \cong C_0(\mathbb{Z})$ by Fourier transform. Furthermore, it follows that $K_1(\mathfrak{A} \rtimes_\alpha \mathbb{T}) \cong \mathbb{Z}^\infty$. Therefore, $\mathrm{index}_K^{\mathbb{T}}(\mathfrak{A}) = -\infty$ and $\mathrm{index}_K(\mathfrak{A}) = \infty$.

Corollary 2.11 *Let $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ be the crossed product for a C^*-algebra $\mathfrak{A}$ with $\alpha^n = 1$ and $\mathrm{index}_K(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \neq 0$. Then $\mathrm{index}_K^{\mathbb{Z}_n}(\mathfrak{A})$ is not finite.*

Proposition 2.12 *Let $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ be the crossed prdouct of a C^*-algebra $\mathfrak{A}$ by an action α of $\mathbb{R}$. For a C^*-dynamical system $(\mathfrak{A}, G, \beta)$ with G compact, suppose that the action β commutes with α. If both $\mathrm{index}_K^G(\mathfrak{A})$ and $\mathrm{index}_K^G(\mathfrak{A} \rtimes_\alpha \mathbb{R})$ are finite, then*

$$\mathrm{index}_K^G(\mathfrak{A} \rtimes_\alpha \mathbb{R}) = -\mathrm{index}_K^G(\mathfrak{A}).$$

Example 2.13 For the crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n$ for a C^*-algebra $\mathfrak{A}$, if α is trivial, then

$$\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n \cong \mathfrak{A} \otimes C^*(\mathbb{Z}_n) \cong \mathfrak{A} \otimes \mathbb{C}^n \cong \oplus^n \mathfrak{A}.$$

Thus, $\mathrm{index}_K(\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n) = n\, \mathrm{index}_K(\mathfrak{A})$.

For $H_3^{\mathbb{R}} \cong \mathbb{R}^2 \rtimes \mathbb{R}$ and a semi-direct product $H_3^{\mathbb{R}} \rtimes \mathbb{Z}_2$ and $(H_3^{\mathbb{R}} \times H_3^{\mathbb{R}}) \rtimes \mathbb{Z}_2$, assuming commutativity of the actions by $\mathbb{R}$ and $\mathbb{Z}_2$ we have

$$\begin{aligned}
&\mathrm{index}_K(C^*(H_3^{\mathbb{R}}) \rtimes \mathbb{Z}_2) = \mathrm{index}_K^{\mathbb{Z}_2}(C^*(H_3^{\mathbb{R}})) \\
&= -\mathrm{index}_K^{\mathbb{Z}_2}(\mathbb{C}) = 2(-1) = 2\, \mathrm{index}_K(C^*(H_3^{\mathbb{R}})), \\
&\mathrm{index}_K(C^*(H_3^{\mathbb{R}} \times H_3^{\mathbb{R}}) \rtimes \mathbb{Z}_2) = \mathrm{index}_K^{\mathbb{Z}_2}(C^*(H_3^{\mathbb{R}} \times H_3^{\mathbb{R}})) \\
&= \mathrm{index}_K^{\mathbb{Z}_2}(\mathbb{C}) = 2(+1) = 2\, \mathrm{index}_K(C^*(H_3^{\mathbb{R}} \times H_3^{\mathbb{R}})).
\end{aligned}$$

Moreover, for a simply connected solvable Lie group $N \cong \mathbb{R} \rtimes \cdots \rtimes \mathbb{R}$ a successive semi-direct product by $\mathbb{R}$,

$$\mathrm{index}_K(C^*(N) \rtimes \mathbb{Z}_n) = \mathrm{index}_K^{\mathbb{Z}_n}(C^*(N)) = \begin{cases} n(+1) & \dim N \text{ even}, \\ n(-1) & \dim N \text{ odd} \end{cases}$$

which is equal to $n\, \mathrm{index}_K(C^*(N))$, where commutativity of successive actions by $\mathbb{R}$ and an action of $\mathbb{Z}_n$ is assumed. It is expected to have the following reasonable formula:

$$\mathrm{index}_K(\mathfrak{A} \rtimes \mathbb{Z}_n) = n\, \mathrm{index}_K(\mathfrak{A})$$

for a C^*-algebra $\mathfrak{A}$ in a certain class. Those examples support this formula, but it is false in general. Indeed, for instance, as a non-trivial example,

$$\mathrm{index}_K(\mathbb{T}_\theta^2 \rtimes_\sigma \mathbb{Z}_2) = 6 \neq 2\, \mathrm{index}_K(\mathbb{T}_\theta^2) = 2 \cdot 0 = 0.$$

Example 2.14 Let G be a simply connected amenable Lie group with its radical R. Then G is isomorphic to the semi-direct product $R \rtimes S$ for a simply connected compact Lie group S such that $G/R \cong S$. Thus, the group C^*-algebra $C^*(G)$ is isomorphic to the crossed product $C^*(R) \rtimes S$. Hence

$$\mathrm{index}_K(C^*(G)) = \mathrm{index}_K^S(C^*(R)) = \mathrm{index}_K(C^*(S))\,(-1)^{\dim R},$$

where we assume that the action of S commutes with successive actions of $\mathbb{R}$ accosiated with the decomposition $R \cong \mathbb{R} \rtimes \mathbb{R} \cdots \rtimes \mathbb{R}$. Furthermore, $C^*(S) \cong \oplus_{\pi_j \in S^\wedge} M_{n_j}(\mathbb{C})$ by Fourier transform, where $S^\wedge$ is the dual group of S and n_j are the dimensions for $\pi_j \in S^\wedge$ (the unitary equivalnce classes of) irreducible representations of S. Therefore,

$$\mathrm{index}_K(C^*(S)) = \mathrm{index}_K(\oplus_{\pi_j \in S^\wedge} M_{n_j}(\mathbb{C})) = |S^\wedge|,$$

where $|S^\wedge|$ is the cardinal number of the discrete space $S^\wedge$. Therefore, a discovery under the assumption for actions is the following formula:

$$\mathrm{index}_K(C^*(R) \rtimes S) = |S^\wedge|\,\mathrm{index}_K(C^*(R)).$$

Example 2.15 Let G be a noncompact connected real semi-simple Lie group with real rank 1 and finite center. It is obtained by [16] that $K_p(C_r^*(G)) \cong \mathbb{Z}^\infty$ and $K_{p+1}(C_r^*(G)) \cong 0$, where p is the dimension of G/K for a maximal compact subgroup K of G. It follows that $\mathrm{index}_K(C_r^*(G)) = +\infty$ if $\dim G/K$ even, and $= -\infty$ if $\dim G/K$ odd. It is known that G is locally isomorphic to one of the following: $SO_0(n,1)$ (connected component), $SU(n,1)$, $Sp(n,1)$, and $F_{4(-20)}$ for $n \geq 2$ (see [7]). Furthermore, for $G = SO_0(n,1)$, $SU(n,1)$, and $Sp(n,1)$, its maximal compact subgroup K is given by $SO(n)$, $(U(n) \times U(1)) \cap SL_{n+1}(\mathbb{C})$, and $Sp(n) \times Sp(1)$ respectively, and G/K is identified with the hyperbolic space $H_n(\mathbb{F})$ with dimension n for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ respectively.

In particular, let $G = SL_2(\mathbb{R})$. Then $G = KAN$ the Iwasawa decomposition with $K = SO(2)$, $A \cong \mathbb{R}$ (so that G has real rank 1), and $N \cong \mathbb{R}$. Therefore, $\mathrm{index}_K(C_r^*(SL_2(\mathbb{R}))) = +\infty$.

3 Classification by K-ranks and K-index

We give the following seven tables, six ones of which are given for classification for nuclear and non-nuclear examples by our K-ranks and K-index and the last one for classification for group C^*-algebras by K-index. These tables presented below with collected examples, some of which are very important in C^*-algebra theory, could be useful and helpful for further classification for C^*-algebras by K-ranks and K-index.

Table 1: Classification for nuclear examples by Kr_0

Kr_0	Non-simple	Simple
0	$C_0(\mathbb{R}^{2n+1})$ $C^*(H_3^{\mathbb{R}})$, $C^*(\mathbb{R} \rtimes \cdots \rtimes \mathbb{R})$ (odd) $C_r^*(SO(2n+1,1))$ $C_r^*(SU(2n+1,1))$, $C_r^*(Sp(2n+1,1))$	O_2, O_n $O_n \otimes \mathbb{K}$ $O_2 \otimes \mathbb{T}_\Theta^n$ (simple) $M_{2^\infty} \rtimes_\alpha \mathbb{T}$
1	$C(S^{2n+1})$, $C_0(\mathbb{R}^{2n})$ $C^*(A_2)$, $C^*(\mathbb{R} \rtimes \cdots \rtimes \mathbb{R})$ (even) $\mathfrak{F}$	$\mathbb{C}$, $M_n(\mathbb{C})$ $\mathbb{K}$ O_∞
2	$C(S^{2n})$, $C(\mathbb{T}^2)$	$\mathbb{T}_\Theta^2$ (simple)
6	$\mathbb{C}^6 \cong C^*(\mathbb{Z}_6)$	$\mathbb{T}_\theta^2 \rtimes_\sigma \mathbb{Z}_2$ (simple) $S_\theta^2 = (\mathbb{T}_\theta^2)^\sigma$ (simple)
n	$\mathbb{C}^n \cong C^*(\mathbb{Z}_n)$	AF_n, $\mathrm{AT}_{n,m}$
2^{n-1}	$C(\mathbb{T}^n)$, $\mathbb{T}_\Theta^n$ (non-simple)	$\mathbb{T}_\Theta^n$ (simple)
$+\infty$	$C^*(\mathbb{T}^n)$, $C^*(K)$ (K compact, $K^\wedge$ infinite) $C_r^*(SO(2n,1))$, $C_r^*(SL_2(\mathbb{R}))$ $C_r^*(SU(2n,1))$, $C_r^*(Sp(2n,1))$	M_{2^∞} UHF AF_∞ $\mathrm{AT}_{\infty,m}$

Table 2: Classification for nuclear examples by Kr_1

Kr_1	Non-simple	Simple
0	$\mathbb{C}^n$, $C(S^{2n})$, $C_0(\mathbb{R}^{2n})$ $\mathfrak{F}$, $C^*(A_2)$ $C^*(\mathbb{R} \rtimes \cdots \rtimes \mathbb{R})$ (even) $C^*(\mathbb{Z}_n)$, $C^*(\mathbb{T}^n)$, $C^*(K)$ $C_r^*(SO(2n,1))$, $C_r^*(SL_2(\mathbb{R}))$ $C_r^*(SU(2n,1))$, $C_r^*(Sp(2n,1))$	$M_n(\mathbb{C})$, $\mathbb{K}$ M_{2^∞}, UHF AF_n O_n, O_∞ $\mathbb{T}_\theta^2 \rtimes_\sigma \mathbb{Z}_2$ $S_\theta^2 = (\mathbb{T}_\theta^2)^\sigma$
1	$C(S^{2n+1})$, $C_0(\mathbb{R}^{2n+1})$ $C^*(H_3^{\mathbb{R}})$, $C^*(\mathbb{R} \rtimes \cdots \rtimes \mathbb{R})$ (odd)	$\mathrm{AT}_{n,1}$
n	$\oplus^n C_0(\mathbb{R})$	$\mathrm{AT}_{n,n}$
2^{n-1}	$C(\mathbb{T}^n)$, $\mathbb{T}_\Theta^n$ (non-simple)	$\mathbb{T}_\Theta^n$ (simple)
$+\infty$	$C_r^*(SO(2n+1,1))$ $C_r^*(SU(2n+1,1))$ $C_r^*(Sp(2n+1,1))$	$M_{2^\infty} \rtimes_\alpha \mathbb{T}$ $\mathrm{AT}_{n,\infty}$

Table 3: Classification for nuclear examples by index_K

index_K	Non-simple	Simple
$\infty - \infty$	$C_0(\mathbb{R}) \otimes \mathrm{AT}_{\infty,\infty}$	$\mathrm{AT}_{\infty,\infty}$
$+\infty$	$C^*(\mathbb{T}^n)$, $C^*(K)$ $C_r^*(SO(2n,1))$, $C_r^*(SL_2(\mathbb{R}))$ $C_r^*(SU(2n,1))$ $C_r^*(Sp(2n,1))$	M_{2^∞}, UHF $(O_n \otimes \mathbb{K}) \rtimes_\beta \mathbb{T}$ AF_∞ $\mathrm{AT}_{\infty,m}$
n	$\mathbb{C}^n \cong C^*(\mathbb{Z}_n)$	AF_n, $\mathrm{AT}_{n+m,m}$
6	$\mathbb{C}^6 \cong C^*(\mathbb{Z}_6)$	$\mathbb{T}_\theta^2 \rtimes_\sigma \mathbb{Z}_2$, $S_\theta^2 = (\mathbb{T}_\theta^2)^\sigma$
2	$C(S^{2n})$	AF_2
1	$C_0(\mathbb{R}^{2n})$ $\mathfrak{F}$, $\otimes^k \mathfrak{F}$ $C^*(A_2)$ $C^*(\mathbb{R} \rtimes \cdots \rtimes \mathbb{R})$ (even)	$\mathbb{C}$ $M_n(\mathbb{C})$ $\mathbb{K}$ O_∞
0	$C(\mathbb{T}^n)$, $C(S^{2n+1})$ $\mathbb{T}_\Theta^n$ (non-simple) $C^*(H_3^{\mathbb{Z}})$ $C^*(\mathbb{Z} \rtimes \cdots \rtimes \mathbb{Z})$ $\mathfrak{A} \otimes C(\mathbb{T})$ (nuclear) $\mathfrak{A} \rtimes \mathbb{Z} \cdots \rtimes \mathbb{Z}$ (non-simple) ($\mathfrak{A}$ nuclear, $\mathrm{index}_K(\mathfrak{A})$ finite)	O_n, O_A $\mathbb{T}_\Theta^n$ (simple) $O_n \otimes \mathbb{K}$ $O_2 \otimes \mathbb{T}_\Theta^n$ $\mathbb{T}_\theta^2 \rtimes_\sigma \mathbb{Z}$ $\mathfrak{A} \rtimes \mathbb{Z} \cdots \rtimes \mathbb{Z}$ (simple) ($\mathfrak{A}$ nuclear, $\mathrm{index}_K(\mathfrak{A})$ finite)
-1	$C_0(\mathbb{R}^{2n+1})$ $C^*(H_3^{\mathbb{R}})$ $C^*(\mathbb{R} \rtimes \cdots \rtimes \mathbb{R})$ (odd)	$\mathrm{AT}_{n,n+1}$
$-n$	$S\mathbb{C}^n \cong \oplus^n C_0(\mathbb{R})$ $S\mathrm{AF}_n$, $S\mathrm{AT}_{n+m,m}$	$\mathrm{AT}_{m,n+m}$
$-\infty$	$C_r^*(SO(2n+1,1))$ $C_r^*(SU(2n+1,1))$ $C_r^*(Sp(2n+1,1))$	$M_{2^\infty} \rtimes_\alpha \mathbb{T}$ $\mathrm{AT}_{n,\infty}$

Table 4: Classification for non-nuclear examples by Kr_0

Kr_0	Non-simple	Simple
0	$\mathbb{B}$, $M(\mathfrak{A} \otimes \mathbb{K}) \otimes \mathfrak{B}$ $C^*(F_\infty)$	$\mathbb{B}/\mathbb{K}$ $C_r^*(F_\infty)$
1	$C^*(F_n)$ $C^*(\varinjlim F_\infty)$ via $F_\infty \cong [F_\infty, F_\infty]$ $C^*(\pi_1(M_g))$, $\pi_1(M_g) \cong F_2 *_\mathbb{Z} F_{2g-2}$ $C^*(\pi_1(\Sigma_k))$, $\pi_1(\Sigma_k) \cong \mathbb{Z} *_\mathbb{Z} F_{k-1}$	$C_r^*(F_n)$ $C_r^*(\varinjlim F_\infty)$ $C_r^*(\pi_1(M_g))$ $C_r^*(\pi_1(\Sigma_k))$
2	$C^*(F_2 \rtimes_\sigma \mathbb{Z})$, $C_r^*(F_2 \rtimes_\sigma \mathbb{Z})$	$C_r^*(\mathbb{Z}_2 * F_2)$
4	$C^*(PSL_2(\mathbb{Z}))$, $PSL_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$	$C_r^*(PSL_2(\mathbb{Z}))$
8	$C^*(SL_2(\mathbb{Z}))$, $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$	$C_r^*(SL_2(\mathbb{Z}))$
$mn+1$	$C^*(F_m \times F_n)$	$C_r^*(F_m \times F_n)$
$2^{m-1}+2^{n-1}-1$	$C^*(\mathbb{Z}^m * \mathbb{Z}^n)$	$C_r^*(\mathbb{Z}^m * \mathbb{Z}^n)$
$n+m-l$	$C^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)$	$C_r^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)$
$+\infty$	$C^*(F_\infty \times F_2)$	$C_r^*(F_\infty \times F_2)$

Table 5: Classification for non-nuclear examples by Kr_1

Kr_1	Non-simple	Simple
0	$C^*(\varinjlim F_\infty)$ $C^*(SL_2(\mathbb{Z}))$, $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ $C^*(PSL_2(\mathbb{Z}))$, $PSL_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ $C^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)$ $\mathbb{B}$, $M(\mathfrak{A} \otimes \mathbb{K}) \otimes \mathfrak{B}$	$C_r^*(\varinjlim F_\infty)$ $C_r^*(SL_2(\mathbb{Z}))$ $C_r^*(PSL_2(\mathbb{Z}))$ $C_r^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)$
1	$C^*(\pi_1(\Sigma_2))$	$\mathbb{B}/\mathbb{K}$
2	$C^*(F_2 \rtimes_\sigma \mathbb{Z})$, $C_r^*(F_2 \rtimes_\sigma \mathbb{Z})$	$C_r^*(\pi_1(\Sigma_3))$
n	$C^*(F_n)$	$C_r^*(F_n)$
$m+n$	$C^*(F_m \times F_n)$	$C_r^*(F_m \times F_n)$
$2^{m-1}+2^{n-1}$	$C^*(\mathbb{Z}^m * \mathbb{Z}^n)$	$C_r^*(\mathbb{Z}^m * \mathbb{Z}^n)$
$2g-1$	$C^*(\pi_1(M_g))$, $\pi_1(M_g) \cong F_2 *_\mathbb{Z} F_{2g-2}$	$C_r^*(\pi_1(M_g))$
$k-1$	$C^*(\pi_1(\Sigma_k))$, $\pi_1(\Sigma_k) \cong \mathbb{Z} *_\mathbb{Z} F_{k-1}$	$C_r^*(\pi_1(\Sigma_k))$
$+\infty$	$C^*(F_\infty)$	$C_r^*(F_\infty)$

Table 6: Classification for non-nuclear examples by index_K

index_K	Non-simple	Simple
$\infty - \infty$	$C^*(F_\infty \times F_2)$	$C^*_r(F_\infty \times F_2)$
$+\infty$	$\varinjlim C^*(\mathbb{Z}_{n^k} * F_2)$	$\varinjlim C^*_r(\mathbb{Z}_{n^k} * F_2)$
$n+m-l$	$C^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)$	$C^*_r(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)$
$(1-m)(1-n)$	$C^*(F_m \times F_n)$	$C^*_r(F_m \times F_n)$
$n-2$	$C^*(\mathbb{Z}_n * F_2)$	$C^*_r(\mathbb{Z}_n * F_2)$
8	$C^*(SL_2(\mathbb{Z}))$, $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$	$C^*_r(SL_2(\mathbb{Z}))$
4	$C^*(PSL_2(\mathbb{Z}))$, $PSL_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$	$C^*_r(PSL_2(\mathbb{Z}))$
1	$C^*(\varinjlim F_\infty)$ via $F_\infty \cong [F_\infty, F_\infty]$	$C^*_r(\varinjlim F_\infty)$
0	$\mathbb{B}$, $M(\mathfrak{A} \otimes \mathbb{K}) \otimes \mathfrak{B}$ $C^*(F_2 \rtimes_\sigma \mathbb{Z})$, $C^*_r(F_2 \rtimes_\sigma \mathbb{Z})$ $\mathfrak{A} \otimes C(\mathbb{T})$ (non-nuclear) $\mathfrak{A} \rtimes \mathbb{Z} \cdots \rtimes \mathbb{Z}$ (non-simple), ($\mathfrak{A}$ non-nuclear), ($\mathrm{index}_K(\mathfrak{A})$ finite)	$C^*_r(F_m) \otimes \mathbb{T}^n_\Theta$ (simple) $\mathfrak{A} \rtimes \mathbb{Z} \cdots \rtimes \mathbb{Z}$ (simple), ($\mathfrak{A}$ non-nuclear), ($\mathrm{index}_K(\mathfrak{A})$ finite)
-1	$C^*(F_2)$ $C^*(\mathbb{Z}^m * \mathbb{Z}^n)$	$C^*_r(F_2)$ $C^*_r(\mathbb{Z}^m * \mathbb{Z}^n)$ $\mathbb{B}/\mathbb{K}$
$1-n$	$C^*(F_n)$	$C^*_r(F_n)$
$2-2g$	$C^*(\pi_1(M_g))$, $\pi_1(M_g) \cong F_2 *_{\mathbb{Z}} F_{2g-2}$	$C^*_r(\pi_1(M_g))$
$2-k$	$C^*(\pi_1(\Sigma_k))$, $\pi_1(\Sigma_k) \cong \mathbb{Z} *_{\mathbb{Z}} F_{k-1}$	$C^*_r(\pi_1(\Sigma_k))$
$-\infty$	$C^*(F_\infty)$	$C^*_r(F_\infty)$

Table 7: Classification for group C^*-algebras by index_K

index_K	Non-simple	Simple
$\infty - \infty$	$C^*(F_\infty \times F_2)$	$C_r^*(F_\infty \times F_2)$
$+\infty$	$C^*(\varinjlim \mathbb{Z}_{n^k} * F_2)$ $C^*(\mathbb{T}^n)$, $C^*(K)$ (K compact, $K^\wedge$ infinite) $C_r^*(SO(2n,1))$, $C_r^*(SL_2(\mathbb{R}))$ $C_r^*(SU(2n,1))$, $C_r^*(Sp(2n,1))$	$C_r^*(\varinjlim \mathbb{Z}_{n^k} * F_2)$
$n+m-l$	$C^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)$	$C_r^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)$
$(1-m)(1-n)$	$C^*(F_m \times F_n)$	$C_r^*(F_m \times F_n)$
n	$C^*(\mathbb{Z}_n)$ $C^*(\mathbb{Z}_{n+2} * F_2)$	$C_r^*(F_2 \times F_{n+1})$ $C_r^*(\mathbb{Z}_{n+2} * F_2)$
8	$C^*(SL_2(\mathbb{Z}))$, $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$	$C_r^*(SL_2(\mathbb{Z}))$
4	$C^*(PSL_2(\mathbb{Z}))$, $PSL_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$	$C_r^*(PSL_2(\mathbb{Z}))$
2	$C^*(\mathbb{R}^{2n})^+$ (unitization) $C^*(\mathbb{R} \rtimes \cdots \rtimes \mathbb{R})^+$ (even)	$C_r^*(F_2 \times F_3)$
1	$C^*(\varinjlim F_\infty)$ via $F_\infty = [F_\infty, F_\infty]$	$C_r^*(\varinjlim F_\infty)$
0	$C^*(F_2 \rtimes_\sigma \mathbb{Z})$, $C_r^*(F_2 \rtimes_\sigma \mathbb{Z})$ $C^*(\mathbb{Z}^n)$, $C^*(\mathbb{Z}_2 * F_2)$ $C^*(G \times \mathbb{Z})$, $C_r^*(G \times \mathbb{Z})$ $C^*(\mathbb{Z} \rtimes \cdots \rtimes \mathbb{Z})$ $C^*(\mathbb{R}^{2n+1})^+$ (unitization) $C^*(\mathbb{R} \rtimes \cdots \rtimes \mathbb{R})^+$ (odd)	$C_r^*(\pi_1(\Sigma_2))$ $C_r^*(\mathbb{Z}_2 * F_2)$
-1	$C^*(F_2)$ $C^*(\mathbb{Z}^m * \mathbb{Z}^n)$	$C_r^*(F_2)$ $C_r^*(\mathbb{Z}^m * \mathbb{Z}^n)$
$1-n$	$C^*(F_n)$	$C_r^*(F_n)$
$2-2g$	$C^*(\pi_1(M_g))$, $\pi_1(M_g) \cong F_2 *_{\mathbb{Z}} F_{2g-2}$	$C_r^*(\pi_1(M_g))$
$2-k$	$C^*(\pi_1(\Sigma_k))$, $\pi_1(\Sigma_k) \cong \mathbb{Z} *_{\mathbb{Z}} F_{k-1}$	$C_r^*(\pi_1(\Sigma_k))$
$-\infty$	$C^*(F_\infty)$ $C_r^*(SO(2n+1,1))$ $C_r^*(SU(2n+1,1))$ $C_r^*(Sp(2n+1,1))$	$C_r^*(F_\infty)$

Remark. In general, it is complicated to compute K-groups of crossed products of C^*-algebras by $\mathbb{Z}$ or by successive actions by $\mathbb{Z}$. However, as the tables above suggests, the (most) merit for our K-index is that the K-index for those crossed prdocuts is (almost) always zero without knowing their K-theory groups explicitly.

The K-ranks are finer than the K-index index_K in a sense that they can distinguish noncommtative n-tori for n different while index_K can not. However, the K-index is more computable in a sense than the K-ranks as explained above.

Problem. Determine the classes of C^*-algebras that have K-index positive, zero, and negative respectively.

Remark. This major problem has been considered, but not yet solved completely. This is also a noncommutative analogue to the classical classification result for closed orientable Riemann surfaces M: if $\chi(M) > 0$, then $M \sim S^2$; if $\chi(M) = 0$, then $M_2 \sim \mathbb{T}^2$; if $\chi(M) < 0$, then $M \sim M_g$ ($g \geq 2$), where $\chi(\cdot)$ is the Euler characteristic for spaces, and $\sim$ is homotopy equivalence (see Example below). Hopf's theorem says that for a manifold M, its Euler characteristic is zero if and only if there exists a non-singular vector field on M. Its noncommutative version could be considered.

As mentioned before, homotopy equivalent C^*-algebras have the same K-index. Therefore, the problem we want to consider is to show its converse in some subclasses of C^*-algebras such as certain classes of simple, nuclear (or purely infinite) C^*-algebras.

Example 3.1 Let S^n be the n-dimensional sphere ($n \geq 1$). Let $H^j(S^n, \mathbb{Z})$ be the j-th cohomology of S^n with coefficients in $\mathbb{Z}$. Then we have

$$H^0(S^n, \mathbb{Z}) = \mathbb{Z}, \quad H^j(S^2, \mathbb{Z}) = 0 \quad (1 \leq j \leq n-1), \quad H^n(S^n, \mathbb{Z}) = \mathbb{Z},$$

$$\chi(S^n) = \sum_{j=0}^{n} (-1)^j \mathrm{rank}_{\mathbb{Z}} H^j(S^n, \mathbb{Z}) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

In particular, $\chi(S^2) = 2 > 0$. For the 2-torus $\mathbb{T}^2$, we have

$$H^0(\mathbb{T}^2, \mathbb{Z}) = \mathbb{Z}, \quad H^1(\mathbb{T}^2, \mathbb{Z}) = \mathbb{Z}^2, \quad H^2(\mathbb{T}^2, \mathbb{Z}) = \mathbb{Z},$$

$$\chi(\mathbb{T}^2) = \sum_{j=0}^{2} (-1)^j \mathrm{rank}_{\mathbb{Z}} H^j(\mathbb{T}^2, \mathbb{Z}) = 0.$$

Furthermore, we have $\chi(\mathbb{T}^n) = 0$ ($n \geq 1$). Let M_g be a closed Riemann surface with genus g (≥ 2). Then we have

$$H^0(M_g, \mathbb{Z}) = \mathbb{Z}, \quad H^1(M_g, \mathbb{Z}) = \mathbb{Z}^{2g}, \quad H^2(M_g, \mathbb{Z}) = \mathbb{Z},$$
$$\chi(M_g) = \sum_{j=0}^{n} (-1)^j \mathrm{rank}_{\mathbb{Z}} H^j(M_g, \mathbb{Z}) = 2 - 2g < 0.$$

Let $\Gamma_g = \pi_1(M_g)$ be the fundamental group of M_g and $B\Gamma_g$ its classifying space. Since M_g is homotopic to $B\Gamma_g$, it follows

$$H^j(M_g, \mathbb{Q}) \cong H^j(B\Gamma_g, \mathbb{Q}) \cong H^j(\Gamma_g, \mathbb{Q})$$

(Cartan-Eilenberg). See also Appendix below.

Definition 3.2 We say that a C^*-algebra $\mathfrak{A}$ is K-index positive, K-index zero, and K-index negative if

$$\mathrm{index}_K(\mathfrak{A}) > 0, \quad \mathrm{index}_K(\mathfrak{A}) = 0, \quad \mathrm{index}_K(\mathfrak{A}) < 0$$

respectively. A C^*-algebra $\mathfrak{A}$ is K-index unital if $\mathrm{index}_K(\mathfrak{A}) = 1$.

Remark. Some examples are given in those tables. Both K-index zero and K-index unit are not unique at all.

Problem. Without taking inductive limits such as AF or AT, find examples of C^*-algebras with (explicit) generators and relations that fill the boxes in those tables.

Remark. Since finiteness (in structure or steps to construct C^*-algebras) is crucial in computing our K-index, this question is some reasonable. As mentioned above, it is known that both any inductive limit of finitely generated, abelian free groups $\mathbb{Z}^{n_k}$ and a countable, torsion free, abelian group can be realized as K_0 and K_1-groups of a unital simple AT algebra respectively (see, for instance, [9, 4.7]).

As mentioned early above, our K-index do non behave well with inductive limits of C^*-algebras. For recovering this, we introduce

Definition 3.3 Let $\mathfrak{A}$ be a C^*-algebra that is an inductive limit of C^*-algebras $\mathfrak{A}_n$. Define the forgetful K-index of $\mathfrak{A}$ to be the following limit:

$$\text{f-index}_K(\mathfrak{A}) = \lim_{n\to\infty} \mathrm{index}_K(\mathfrak{A}_n)$$

if it exists (or the limit may be replaced with limit infimum).

Remark. This definition depends on the choice of inductive limit systems so that it, precisely, should be defined as the minimum of such limits (or limit infimums). For instance, (certain) AH algebras that are inductive limits of homogeneous C^*-algebras can be written as AT algebras through some known classification theorems (see [9]).

Example 3.4 Let $\mathfrak{A}$ be an AF algebra. Then $\text{f-index}_K(\mathfrak{A}) = 1$. This consequence comes from that the index f-index_K is forgetful about positions of units in inductive limit systems for $\mathfrak{A}$ (which are crucial to determine K-theory of $\mathfrak{A}$). Such an important information is lost, but so that we can get consistency for continuity with respect to inductive limits. This is the reason for naming the index f-index_K.

Let $\mathfrak{A}$ be an AT algebra. Then $\text{f-index}_K(\mathfrak{A}) = 0$. It is known that simple noncommutative 2-tori $\mathbb{T}^2_\theta$ are AT. This fact is consistent with $\text{index}_K(\mathbb{T}^2_\theta) = 0 = \text{f-index}_K(\mathbb{T}^2_\theta)$.

However, surprisingly, it is shown by [3] that $\mathbb{T}^2_\theta \rtimes_\sigma \mathbb{Z}_2$ and $(\mathbb{T}^2_\theta)^\sigma$ are AF-algebras. Hence they have f-index_K 1, but they have index_K 6.

As a summary, a (partial) permanence result for our K-index is

Theorem 3.5 *The K-index is homotopy invariant and functorial in some senses as follows:*

$$
\begin{aligned}
&\text{(Additive)} && \text{index}_K(\mathfrak{A} \oplus \mathfrak{B}) = \text{index}_K(\mathfrak{A}) + \text{index}_K(\mathfrak{B}),\\
&\text{(Multiplicative)} && \text{index}_K(\mathfrak{A} \otimes \mathfrak{B}) = \text{index}_K(\mathfrak{A})\,\text{index}_K(\mathfrak{B}),\\
&\text{(Stability)} && \text{index}_K(\mathfrak{A} \otimes M_n(\mathbb{C})) = \text{index}_K(\mathfrak{A})\\
& && = \text{index}_K(\mathfrak{A} \otimes \mathbb{K}) = \text{index}_K(\mathfrak{A} \otimes O_\infty),\\
&\text{(Periodicity)} && \text{index}_K(S\mathfrak{A}) = -\text{index}_K(\mathfrak{A}) = \text{index}_K(\mathfrak{A} \rtimes_\alpha \mathbb{R}),\\
&\text{(Vanishing)} && \text{index}_K(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) = 0 = \text{index}_K(\mathfrak{A} \otimes O_2)\\
& && = \text{index}_K(\mathfrak{A} \otimes \mathbb{B}) = \text{index}_K(\mathfrak{A} \otimes M(\mathfrak{B} \otimes \mathbb{K})),\\
&\text{(Dividity)} && \text{index}_K(\mathfrak{A}) = \text{index}_K(\mathfrak{J}) + \text{index}_K(\mathfrak{A}/\mathfrak{J}),\\
&\text{(K-index MV)} && \text{index}_K(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B})\\
& && = \text{index}_K(\mathfrak{A}) + \text{index}_K(\mathfrak{B}) - \text{index}_K(\mathfrak{C})\\
& && = \text{index}_K(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) \quad (\textit{with retractions to } \mathfrak{C}),\\
&\text{(Shifting)} && \text{index}_K(\mathfrak{A}^+) = \text{index}_K(\mathfrak{A}) + 1 = \text{index}_K(\mathfrak{A} \oplus \mathbb{C}),\\
& && \text{index}_K(\mathfrak{A} *_{\mathbb{C}} C(\mathbb{T})) = \text{index}_K(\mathfrak{A}) - 1 \qquad (\mathfrak{A}\ \textit{unital}),\\
&\text{(Scaling)} && \text{index}_K(\mathfrak{A} \rtimes_{\alpha,r} F_n) = (1-n)\,\text{index}_K(\mathfrak{A}),\\
&\text{(K-index BC)} && \text{index}_K(C^*_r(\Gamma)) = \text{index}^K(B\Gamma),\\
&\text{(Classical)} && \text{index}_K(C(X)) = \text{index}^K(X),
\end{aligned}
$$

where some restrictive assumptions such as being in the UCT or the class X, K-index finiteness, and some K-theory conditions are necessary as discussed above.

Also, the K^G-index is homotopy invariant and functorial similarly in (some of) those senses.

Remark. In particular, the multiplicativity for K^G is just

$$\mathrm{index}_K^{G\times G}(\mathfrak{A}\otimes\mathfrak{B}) = \mathrm{index}_K^G(\mathfrak{A})\,\mathrm{index}_K^G(\mathfrak{B})$$

as showen before.

As for classification of group C^*-algebras by K-index,

Theorem 3.6 *Let G a simply connected solvable Lie group. Then*

$$\mathrm{index}_K(C^*(G)^+) = \begin{cases} 2 & \text{if } \dim G \text{ even}, \\ 0 & \text{if } \dim G \text{ odd}. \end{cases}$$

Let G be a solvable discrete group that can be written as a successive semi-direct product by $\mathbb{Z}$. Then

$$\mathrm{index}_K(C^*(G)) = 0.$$

In addition, for a locally compact group G,

$$\mathrm{index}_K(C^*(G\times\mathbb{Z})) = 0 = \mathrm{index}_K(C_r^*(G\times\mathbb{Z})).$$

Proof. Any simply connected solvable Lie group can be written as a successive semi-direct product by $\mathbb{R}$. Hence $C^*(G)$ is isomorphic to a successive crossed product by $\mathbb{R}$. The other statements are proved in Examples above. □

Remark. Those vanishing formulae are of some interest.

Problem. Let G be a solvable (or amenable) Lie (or locally compact) group. Determine G such that $\mathrm{index}_K(C^*(G)^+) \geq 0$, i.e., $C^*(G)^+$ is K-index positive.

Remark. It should be right to consider the unitization of $C^*(G)$ as in the theorem above. Note that $C^*(G)$ is (almost) non-unital if G is non-discrete, and solvable locally compact groups are always amenable. The K-index inequality $\mathrm{index}_K(C^*(G)^+) \geq 0$ is easily false in general. In fact, let G be a connected commutative Lie group so that $G \cong \mathbb{R}^s \times \mathbb{T}^t$ for some $s,t \geq 0$. Then $C^*(G) \cong C_0(\mathbb{R}^s) \otimes C_0(\mathbb{Z}^t)$ by the Fourier transform. Thus $\mathrm{index}_K(C^*(G)) = (-1)^s\,\mathrm{index}_K(C_0(\mathbb{Z}^t)) = -\infty$ if s is odd and $t \geq 1$, and $= +\infty$ if s is even and $t \geq 1$. This alternative is discussed in Examples above for simply connected amenable Lie groups and non-compact connected semi-simple Lie groups.

In general, we obtain

Proposition 3.7 *Let G be a locally compact group. If $\mathrm{Kr}_0(C^*(G))$ is finite and $\mathrm{Kr}_1(C^*(G)) = 0$, then*

$$\mathrm{index}_K(C^*(G \times \mathbb{T})) = +\infty,$$

and if $\mathrm{Kr}_1(C^*(G))$ is finite and $\mathrm{Kr}_0(C^*(G)) = 0$, then

$$\mathrm{index}_K(C^*(G \times \mathbb{T})) = -\infty,$$

and $C^*(G)$, $C^*(G \times \mathbb{T})$ can be replaced with $C_r^*(G)$, $C_r^*(G \times \mathbb{T})$ respectively. Furthermore, $\mathbb{T}$ can be replaced with a compact group whose dual discrete group is infinite.

Proof. Note that $C^*(G \times \mathbb{T}) \cong C^*(G) \otimes C_0(\mathbb{Z})$ and $C_r^*(G \times \mathbb{T}) \cong C_r^*(G) \otimes C_0(\mathbb{Z})$. The statement follows from the Künneth formula. □

Also, at this moment, it is likely that

Conjecture. If Γ is a non-amenable, highly non-commutative discrete group without torsion, then $\mathrm{index}_K(C_r^*(\Gamma)) < 0$, i.e., $C_r^*(\Gamma)$ is K-index negative.

Remark. Torsion freeness is necessary as the examples such as $\Gamma = SL_2(\mathbb{Z})$ and $PSL_2(\mathbb{Z})$ in the tables given above. Highly non-commutativeness for discrete groups should be defined as that such groups are a kind of free groups F_n but they are not a sort of products $F_n \times F_m$ as required.

One of the classification results by Kirchberg (see [12]) says that the tensor product $\mathfrak{A} \otimes O_2$ is isomorphic to O_2 if and only if $\mathfrak{A}$ is a simple, separable, unital, and nuclear C^*-algebra. If $\mathfrak{A}$ is in the category X, then $\mathrm{index}_K(\mathfrak{A} \otimes O_2) = 0 = \mathrm{index}_K(O_2)$ since $\mathrm{Kr}_j(\mathfrak{A} \otimes O_2) = 0$. Another by Kirchberg is that for a simple, separable, and nuclear C^*-algebra $\mathfrak{A}$, we have $\mathfrak{A} \cong \mathfrak{A} \otimes O_\infty$ if and only if $\mathfrak{A}$ is purely infinite. If $\mathfrak{A}$ is in the category X, then $\mathrm{index}_K(\mathfrak{A} \otimes O_\infty) = \mathrm{index}_K(\mathfrak{A})$ since $\mathrm{Kr}_j(\mathfrak{A} \otimes O_\infty) = \mathrm{Kr}_j(\mathfrak{A})$.

Theorem 3.8 (Kirchberg-Phillips), ([12, 8.4.1 and 8.4.7]) *Let $\mathfrak{A}$, $\mathfrak{B}$ be Kirchberg algebras, i.e., purely infinite, simple, nuclear separable C^*-algebras. Then*

(1). *$\mathfrak{A} \cong \mathfrak{B}$ if and only if they are KK-equivalent.*

(2). *If $\mathfrak{A}$, $\mathfrak{B}$ are in the UCT class, then $\mathfrak{A} \cong \mathfrak{B}$ if and only if $K_j(\mathfrak{A}) \cong K_j(\mathfrak{B})$ $(j = 0, 1)$.*

(3). *$\mathfrak{A} \cong \mathfrak{B}$ if and only if they are homotopy equivalent.*

Remark. Note that a Kirchberg algebra is either stable or unital. In each case above, it implies $\mathrm{index}_K(\mathfrak{A}) = \mathrm{index}_K(\mathfrak{B})$.

Proposition 3.9 ([12, 8.4.4]) *Every Kirchberg algebra in the UCT class is isomorphic to an inductive limit of C^*-algebras of the form:*

$$(O_{n_1} \oplus O_{n_2} \oplus \cdots \oplus O_{n_r}) \otimes C(\mathbb{T}), \quad n_j \in \{2, 3, \cdots, \infty\}.$$

Every stable Kirchberg algebra in the UCT class is isomorphic to a crossed product $\mathfrak{B} \rtimes_\alpha \mathbb{Z}$, where $\mathfrak{B}$ is a simple, real rank zero AT algebra, i.e., an inductive limit of matrix algebras over $C(\mathbb{T})$. If $K_1(\mathfrak{A})$ is torsion-free, then $\mathfrak{B}$ can be a simple AF algebra.

Remark. From this it might be likely to show that every Kirchberg algebra (not isomorphic to O_∞ whose K-index is 1) has K-index zero. If so, our K-index is not useful in this case, and just distinguishes O_∞ from others. At least, any Kirchberg algebra in the UCT has f-index_K zero since $\mathrm{index}_K((O_{n_1} \oplus O_{n_2} \oplus \cdots \oplus O_{n_r}) \otimes C(\mathbb{T})) = 0$.

However, that expectation is false by

Proposition 3.10 [12, 4.3.3] *Let G_0, G_1 be countable abelian groups and let $g_0 \in G_0$ be an element. There exist a unital simple AT algebra $\mathfrak{B}$ of real rank zero, a proper projection $p \in \mathfrak{B}$, and an isomorphism $\rho : \mathfrak{B} \to p\mathfrak{B}p$ such that $\mathfrak{A} = \mathfrak{B} \rtimes_\rho \mathbb{N}$ is a unital Kirchberg algebra in the UCT and*

$$(K_0(\mathfrak{A}), [1]_0, K_1(\mathfrak{A})) \cong (G_0, g_0, G_1).$$

If G_1 is torsion-free, then $\mathfrak{B}$ can be chosen to be a unital simple AF algebra.

Also, there exist a unital simple AT algebra $\mathfrak{B}$ of real rank zero and an automorphism ρ of $\mathfrak{B}$ such that $\mathfrak{A} = \mathfrak{B} \rtimes_\rho \mathbb{Z}$ is a stable Kirchberg algebra in the UCT and

$$(K_0(\mathfrak{A}), K_1(\mathfrak{A})) \cong (G_0, G_1).$$

If G_1 is torsion-free, then $\mathfrak{B}$ can be chosen to be a stable simple AF algebra.

Remark. For n, m non-negative integers or ∞, we denote by $\mathrm{uK}_{n,m}$ such a unital Kirchberg algebra such that $\mathrm{index}_K(\mathrm{uK}_{n,m}) = n - m$, where $n = \mathrm{Kr}_0(\mathrm{uK}_{n,m})$ and $m = \mathrm{Kr}_1(\mathrm{uK}_{n,m})$, and by $\mathrm{sK}_{n,m}$ such a stable Kirchberg algebra such that $\mathrm{index}_K(\mathrm{sK}_{n,m}) = n - m$. Hence, all AT algebras $\mathrm{AT}_{n,m}$ in those tables 1 to 3 can be replaced with either $\mathrm{uK}_{n,m}$ or $\mathrm{sK}_{n,m}$.

4 KK-theory ranks and index

Let $\mathfrak{A}$ and $\mathfrak{B}$ be (graded) C^*-algebras. The KK^0-group $KK^0(\mathfrak{A}, \mathfrak{B})$ of $\mathfrak{A}$ and $\mathfrak{B}$ is defined to be the abelian group of homotopy classes of Kasparov $\mathfrak{A}$-$\mathfrak{B}$-modules, or of homotopy equivalent classes of (Cuntz) quasi-homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$, where a quasi-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is a pair of $*$-homomorphisms $\varphi_\pm : \mathfrak{A} \to M(\mathfrak{B} \otimes \mathbb{K})$ such that $\varphi_+(a) - \varphi_-(a) \in \mathfrak{B} \otimes \mathbb{K}$ for $a \in \mathfrak{A}$.

Set $KK^1(\mathfrak{A}, \mathfrak{B}) = KK^0(\mathfrak{A}, S\mathfrak{B})$ (or $KK^0(S\mathfrak{A}, \mathfrak{B})$). See [1] or [12] for more details in KK-theory of C^*-algebras.

Definition 4.1 Let $\mathfrak{A}$, $\mathfrak{B}$ be (graded) C^*-algebras. We define the KK^j-rank of $\mathfrak{A}$ and $\mathfrak{B}$ to be the $\mathbb{Z}$-rank of their KK^j-group ($j = 0, 1$), and denoted it by

$$\mathrm{KKr}^j(\mathfrak{A}, \mathfrak{B}) = \mathrm{rank}_{\mathbb{Z}} KK^j(\mathfrak{A}, \mathfrak{B}) \in \{0, 1, 2, \cdots, +\infty\}$$

where $KK^0 = KK$ of Kasparov. Define the KK-index of $\mathfrak{A}$ and $\mathfrak{B}$ to be the following difference:

$$\mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}) = \mathrm{KKr}^0(\mathfrak{A}, \mathfrak{B}) - \mathrm{KKr}^1(\mathfrak{A}, \mathfrak{B}) \in \mathbb{Z} \cup \{\pm\infty\}.$$

Proposition 4.2 *For a C^*-algebra $\mathfrak{A}$ in the UCT class N and a separable C^*-algebra $\mathfrak{B}$, we have*

$$\mathrm{KKr}^0(\mathfrak{A}, \mathfrak{B}) \geq \sum_{j=0}^{1} \mathrm{rank}_{\mathbb{Z}} \mathrm{Hom}(K_j(\mathfrak{A}), K_j(\mathfrak{B})),$$

and in addition, if either $K_*(\mathfrak{A})$ is free or $K_*(\mathfrak{B})$ is divisible, then

$$\mathrm{KKr}^0(\mathfrak{A}, \mathfrak{B}) = \sum_{j=0}^{1} \mathrm{rank}_{\mathbb{Z}} \mathrm{Hom}(K_j(\mathfrak{A}), K_j(\mathfrak{B})).$$

Proof. The first inequality follows from the universal coefficient theorem (UCT): under the first assumption there exists the following split exact sequence:

$$\begin{aligned} 0 &\to \oplus_{j=0}^{1} \mathrm{Ext}_{\mathbb{Z}}^1(K_j(\mathfrak{A}), K_{j+1}(\mathfrak{B})) \to KK^0(\mathfrak{A}, \mathfrak{B}) \\ &\to \oplus_{j=0}^{1} \mathrm{Hom}(K_j(\mathfrak{A}), K_j(\mathfrak{B})) \to 0. \end{aligned}$$

See Appendix below for the functor Ext. The second assumption induces $KK^0(\mathfrak{A}, \mathfrak{B}) \cong \oplus_{j=0}^{1} \mathrm{Hom}(K_j(\mathfrak{A}), K_j(\mathfrak{B}))$. □

Corollary 4.3 *Under the same assumption as above, if* $\mathrm{Kr}_j(\mathfrak{A}) = \infty$ *and* $\mathrm{Kr}_j(\mathfrak{B}) \neq 0$ $(j = 0$ *or* $1)$*, then* $\mathrm{KKr}^0(\mathfrak{A}, \mathfrak{B}) = +\infty$*, so that* $\mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}) = +\infty$ *(or undefined).*

Example 4.4 Note that $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$, where it consists of the maps $n\times : \mathbb{Z} \to n\mathbb{Z} \subset \mathbb{Z}$ with $n \times m = nm$ for $m \in \mathbb{Z}$. Also, $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}^k) \cong \oplus^k \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}^k$ and $\mathrm{Hom}(\mathbb{Z}^l, \mathbb{Z}) \cong \oplus^l \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}^l$. Hence, we have $\mathrm{Hom}(\mathbb{Z}^l, \mathbb{Z}^k) \cong \mathbb{Z}^{lk}$.

It follows from UCT that $KK^0(\mathfrak{A}, \mathfrak{B}) \cong \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{B}))$ for AF algebras $\mathfrak{A}$ and $\mathfrak{B}$ because their K_1-groups are trivial. Hence

$$\mathrm{KKr}^0(\mathfrak{A}, \mathfrak{B}) = \mathrm{rank}_{\mathbb{Z}} \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{B})).$$

More generally, these hold for C^*-algebras with K_1 trivial and UCT.

Proposition 4.5 *For any* C^**-algebras* $\mathfrak{A}$ *and* $\mathfrak{B}$*,*

$$\begin{aligned}\mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}) &= \mathrm{index}^{KK}(\mathfrak{A} \otimes M_n(\mathbb{C}), \mathfrak{B} \otimes M_m(\mathbb{C})) \\ &= \mathrm{index}^{KK}(\mathfrak{A} \otimes \mathbb{K}, \mathfrak{B}) = \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{K}) \\ &= \mathrm{index}^{KK}(\mathfrak{A} \otimes \mathbb{K}, \mathfrak{B} \otimes \mathbb{K}),\end{aligned}$$

and index^{KK} *can be replaced with* KKr^j*.*

Proof. We have

$$\begin{aligned}KK^j(\mathfrak{A}, \mathfrak{B}) &\cong KK^j(\mathfrak{A} \otimes M_n(\mathbb{C}), \mathfrak{B} \otimes M_m(\mathbb{C})) \\ &\cong KK^j(\mathfrak{A} \otimes \mathbb{K}, \mathfrak{B}) \cong KK^j(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{K}) \\ &\cong KK^j(\mathfrak{A} \otimes \mathbb{K}, \mathfrak{B} \otimes \mathbb{K}).\end{aligned}$$

□

Proposition 4.6 *For any* C^**-algebras* $\mathfrak{A}$, $\mathfrak{A}_j$*, and* $\mathfrak{B}$, $\mathfrak{B}_j$ $(j = 1, 2)$*,*

$$\begin{aligned}\mathrm{index}^{KK}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \mathfrak{B}) &= \mathrm{index}^{KK}(\mathfrak{A}_1, \mathfrak{B}) + \mathrm{index}^{KK}(\mathfrak{A}_2, \mathfrak{B}), \\ \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}_1 \oplus \mathfrak{B}_2) &= \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}_1) + \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}_2),\end{aligned}$$

and index^{KK} *can be replaced with* KKr^j*.*

Proof. This follows from the additivity:

$$\begin{aligned}KK^j(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \mathfrak{B}) &\cong KK^j(\mathfrak{A}_1, \mathfrak{B}) \oplus KK^j(\mathfrak{A}_2, \mathfrak{B}), \\ KK^j(\mathfrak{A}, \mathfrak{B}_1 \oplus \mathfrak{B}_2) &\cong KK^j(\mathfrak{A}, \mathfrak{B}_1) \oplus KK^j(\mathfrak{A}, \mathfrak{B}_2).\end{aligned}$$

□

Proposition 4.7 *Let $\mathfrak{A} = \oplus_j \mathfrak{A}_j$ be a countable direct sum of separable C^*-algebras $\mathfrak{A}_j$. For any C^*-algebra $\mathfrak{B}$,*

$$\mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}) = \sum_j \mathrm{index}^{KK}(\mathfrak{A}_j, \mathfrak{B}).$$

If $\mathfrak{A}$ is in the UCT class and $K_*(\mathfrak{A})$ is finitely generated, and $\mathfrak{B} = \oplus_j \mathfrak{B}_j$ of separable C^*-algebras, then

$$\mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}) = \sum_j \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}_j).$$

Proof. Under the assumption, the coordinate inclusions from $\mathfrak{A}_j$ to $\mathfrak{A}$ induce an isomorphism:

$$KK^0(\mathfrak{A}, \mathfrak{B}) \cong \Pi_j KK^0(\mathfrak{A}_j, \mathfrak{B})$$

by [1, Theorem 19.7.1]. Also, $KK^0(\mathfrak{A}, \mathfrak{B}) \cong \Pi_j KK^0(\mathfrak{A}, \mathfrak{B}_j)$ by [1, 23.15.5]. □

Remark. KK-theory $KK^0(\cdot, \cdot)$ is countably additive in the first variable in that sense, but not so in the second variable in general. If $\mathfrak{B} = \oplus_j \mathfrak{B}_j$, then for any C^*-algebra $\mathfrak{A}$ there exists a natural map from $\oplus_j KK^0(\mathfrak{A}, \mathfrak{B}_j)$ to $KK^0(\mathfrak{A}, \mathfrak{B})$, however, this is not surjective in general. For example, if $\mathfrak{A} = C_0(\mathbb{N}) = \mathfrak{B}$ with $\mathfrak{B}_j = \mathbb{C}$, then the unit of $KK^0(\mathfrak{A}, \mathfrak{B})$ is not in the image under the map.

Proposition 4.8 *For any C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$,*

$$\mathrm{index}^{KK}(\mathfrak{A}, S\mathfrak{B}) = \mathrm{index}^{KK}(S\mathfrak{A}, \mathfrak{B}) = -\mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}).$$

Proof. This follows from the Bott periodicity:

$$KK^0(\mathfrak{A}, \mathfrak{B}) \cong KK^1(\mathfrak{A}, S\mathfrak{B}) \cong KK^1(S\mathfrak{A}, \mathfrak{B}),$$
$$KK^1(\mathfrak{A}, \mathfrak{B}) \cong KK^0(\mathfrak{A}, S\mathfrak{B}) \cong KK^0(S\mathfrak{A}, \mathfrak{B}).$$

□

Example 4.9 We have $\mathrm{index}^{KK}(\mathbb{C}, \mathbb{C}) = 1$ because

$$KK^0(\mathbb{C}, \mathbb{C}) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}, \quad \text{and}$$
$$KK^1(\mathbb{C}, \mathbb{C}) \cong KK^0(\mathbb{C}, S\mathbb{C}) \cong \mathrm{Hom}(\mathbb{Z}, 0) \oplus \mathrm{Hom}(0, \mathbb{Z}) \cong 0.$$

Furthermore, we have

$$\begin{aligned}\operatorname{index}^{KK}(\mathbb{C},\mathbb{C}) = 1 &= \operatorname{index}^{KK}(M_n(\mathbb{C}), M_m(\mathbb{C})) \\ &= \operatorname{index}^{KK}(\mathbb{K},\mathbb{C}) = \operatorname{index}^{KK}(\mathbb{C},\mathbb{K}) = \operatorname{index}^{KK}(\mathbb{K},\mathbb{K}),\end{aligned}$$

and $\operatorname{index}^{KK}$ can be replaced with KKr^0, and with KKr^1 where 1 is replaced with 0. By additivity, $\operatorname{index}^{KK}(\mathbb{C}^n, \mathbb{C}^m) = nm$.

A C^*-algebra $\mathfrak{A}$ is called K-contractible if $KK^0(\mathfrak{A},\mathfrak{A}) = 0$.

Proposition 4.10 *If a C^*-algebra $\mathfrak{A}$ is K-contractible, then*

$$\operatorname{index}^{KK}(\mathfrak{A},\mathfrak{B}) = 0 = \operatorname{index}^{KK}(\mathfrak{B},\mathfrak{A})$$

for any C^*-algebra $\mathfrak{B}$.

Proof. The assumption implies that $KK^0(\mathfrak{A},\mathfrak{B}) = 0 = KK^0(\mathfrak{B},\mathfrak{A})$ for any C^*-algebra $\mathfrak{B}$. Hence

$$KK^1(\mathfrak{A},\mathfrak{B}) = KK^0(\mathfrak{A}, S\mathfrak{B}) = 0 = KK^1(\mathfrak{B},\mathfrak{A}) = KK^0(S\mathfrak{B},\mathfrak{A}).$$

□

Example 4.11 Any contractible C^*-algebra is K-contractible. In particular, $C\mathfrak{B} = C([0,1]) \otimes \mathfrak{B}$ and $C_0([0,1)) \otimes \mathfrak{B} \equiv C'\mathfrak{B}$ for any C^*-algebra $\mathfrak{B}$ are K-contractible. Therefore,

$$\begin{aligned}\operatorname{index}^{KK}(\mathfrak{A}, C\mathfrak{B}) = 0 = \operatorname{index}^{KK}(C\mathfrak{B},\mathfrak{A}),\\ \operatorname{index}^{KK}(\mathfrak{A}, C'\mathfrak{B}) = 0 = \operatorname{index}^{KK}(C'\mathfrak{B},\mathfrak{A})\end{aligned}$$

for any C^*-algebra $\mathfrak{A}$.

Proposition 4.12 *Let $\mathfrak{A}$ be a C^*-algebra. Then*

$$\operatorname{index}_K(\mathfrak{A}) = \operatorname{index}^{KK}(\mathbb{C},\mathfrak{A}).$$

Proof. Note that $K_0(\mathfrak{A}) \cong KK^0(\mathbb{C},\mathfrak{A})$ and $K_1(\mathfrak{A}) \cong KK^0(S\mathbb{C},\mathfrak{A}) \cong KK^1(\mathbb{C},\mathfrak{A})$. □

By definition, the K-homology (group) $K^j(\mathfrak{A})$ $(j = 0, 1)$ of a C^*-algebra $\mathfrak{A}$ is defined to be $KK^j(\mathfrak{A},\mathbb{C})$.

Definition 4.13 Let $\mathfrak{A}$ be a C^*-algebra. Define the K^j-homology rank of $\mathfrak{A}$ to be the $\mathbb{Z}$-rank of the K-homology $K^j(\mathfrak{A}) = KK^j(\mathfrak{A}, \mathbb{C})$, and denote it by

$$\mathrm{Kr}^j(\mathfrak{A}) = \mathrm{rank}_{\mathbb{Z}} K^j(\mathfrak{A}) \in \{0, 1, 2, \cdots, +\infty\}.$$

We define the K-homology index of $\mathfrak{A}$ to be the following difference

$$\mathrm{index}^K(\mathfrak{A}) = \mathrm{Kr}^0(\mathfrak{A}) - \mathrm{Kr}^1(\mathfrak{A}) \in \mathbb{Z} \cup \{\pm\infty\}.$$

Recall that two extensions E_j of C^*-algebras $\mathfrak{A}$ and (stable) $\mathfrak{B}$: $0 \to \mathfrak{B} \to E_j \to \mathfrak{A} \to 0$ identified with the Busby invariants $\tau_j : \mathfrak{A} \to M(\mathfrak{B})/\mathfrak{B}$ $(j = 1, 2)$ are strongly equivalent if there exists a unitary $u \in M(\mathfrak{B})$ such that $\tau_2(a) = \pi(u)\tau_1(a)\pi(u)^*$ for $a \in \mathfrak{A}$, where $\pi : M(\mathfrak{B}) \to M(\mathfrak{B})/\mathfrak{B}$. The sum $\tau_1 \oplus \tau_2$ is defined by $\tau_1 \oplus \tau_2 : (M(\mathfrak{B})/\mathfrak{B}) \oplus (M(\mathfrak{B})/\mathfrak{B}) \subset M_2(M(\mathfrak{B})/\mathfrak{B}) \cong M(\mathfrak{B})/\mathfrak{B}$. An extension E of $\mathfrak{A}$ by $\mathfrak{B}$ with $\tau : \mathfrak{A} \to M(\mathfrak{B})/\mathfrak{B}$ is trivial if it lifts to a $*$-homomorphism from $\mathfrak{A}$ to $M(\mathfrak{B})$, which is the case if and only if there exists a $*$-homomorphism $s : \mathfrak{A} \to E$ such that $q \circ s = \mathrm{id}_{\mathfrak{A}}$ the identity map on $\mathfrak{A}$, where $q : E \to \mathfrak{A} \to 0$. The extension theory $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B})$ of $\mathfrak{A}$ and $\mathfrak{B}$ is defined to be the commutative group of strong equivalence classes of extensions of $\mathfrak{A}$ by $\mathfrak{B}$ modulo trivial extensions. Set $\mathrm{Ext}(\mathfrak{A}) = \mathrm{Ext}(\mathfrak{A}, \mathbb{K})$.

Proposition 4.14 *For any C^*-algebras $\mathfrak{A}$ and (stable) $\mathfrak{B}$,*

$$\mathrm{rank}_{\mathbb{Z}}\mathrm{Ext}(\mathfrak{A}, \mathfrak{B}) = \mathrm{KKr}^1(\mathfrak{A}, \mathfrak{B}).$$

In particular, $\mathrm{rank}_{\mathbb{Z}}\mathrm{Ext}(\mathfrak{A}) = \mathrm{KKr}^1(\mathfrak{A}, \mathbb{C}) = \mathrm{Kr}^1(\mathfrak{A})$.

Proof. We have

$$\mathrm{Ext}(\mathfrak{A}, \mathfrak{B}) \cong KK^0(\mathfrak{A}, S\mathfrak{B}) \cong KK^1(\mathfrak{A}, \mathfrak{B}).$$

In particular, $\mathrm{Ext}(\mathfrak{A}) = \mathrm{Ext}(\mathfrak{A}, \mathbb{K}) \cong KK^0(\mathfrak{A}, S\mathbb{C}) \cong KK^1(\mathfrak{A}, \mathbb{C}) = K^1(\mathfrak{A})$. □

Example 4.15 We have $\mathrm{Ext}(M_n(\mathbb{C}), \mathbb{K}) \cong 0$. Also, $\mathrm{Ext}(\mathbb{C}, \mathfrak{B}) \cong K_1(\mathfrak{B})$. In particular, $\mathrm{Ext}(\mathbb{C}, C_0(\mathbb{R})) \cong \mathbb{Z}$. If X is a locally compact Hausdorff space, then $\mathrm{Ext}(X) = \mathrm{Ext}(C_0(X)) = \mathrm{Ext}(C_0(X), \mathbb{C}) = KK^1(C_0(X), \mathbb{C}) = K^1(C_0(X))$.

Example 4.16 If we take $\mathfrak{A}$ in the UCT class N and $\mathfrak{B} = \mathbb{C}$, then $\mathrm{Ext}(\mathfrak{A}) \cong KK^0(\mathfrak{A}, S\mathbb{C})$ so that UCT becomes

$$\begin{aligned} 0 &\to \oplus_{j=0}^1 \mathrm{Ext}_{\mathbb{Z}}^1(K_j(\mathfrak{A}), K_j(\mathbb{C})) \to KK^0(\mathfrak{A}, S\mathbb{C}) \\ &\to \oplus_{j=0}^1 \mathrm{Hom}(K_j(\mathfrak{A}), K_{j+1}(\mathbb{C})) \to 0, \end{aligned}$$

which implies immediately that

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), \mathbb{Z}) \to \mathrm{Ext}(\mathfrak{A}) \to \mathrm{Hom}(K_1(\mathfrak{A}), \mathbb{Z}) \to 0.$$

This is Brown's universal coefficient theorem.

Theorem 4.17 *The six-term exact sequence for KK: Let*

$$0 \longrightarrow \mathfrak{J} \xrightarrow{j} \mathfrak{A} \xrightarrow{q} \mathfrak{A}/\mathfrak{J} \longrightarrow 0$$

be a semi-split exact sequence of σ-unital (graded) C^*-algebras (i.e,, there exists a completely positive, norm decreasing, grading-preserving section for q). For any separable (graded) C^*-algebra $\mathfrak{D}$,

$$\begin{array}{ccccc}
KK^0(\mathfrak{D},\mathfrak{J}) & \xrightarrow{j_*} & KK^0(\mathfrak{D},\mathfrak{A}) & \xrightarrow{q_*} & KK^0(\mathfrak{D},\mathfrak{A}/\mathfrak{J}) \\
\uparrow & & & & \downarrow \\
KK^1(\mathfrak{D},\mathfrak{A}/\mathfrak{J}) & \xleftarrow{q_*} & KK^1(\mathfrak{D},\mathfrak{A}) & \xleftarrow{j_*} & KK^1(\mathfrak{D},\mathfrak{J}),
\end{array}$$

and if $\mathfrak{A}$ is separable, then for any σ-unital graded C^*-algebra $\mathfrak{D}$,

$$\begin{array}{ccccc}
KK^0(\mathfrak{J},\mathfrak{D}) & \xleftarrow{j^*} & KK^0(\mathfrak{A},\mathfrak{D}) & \xleftarrow{q^*} & KK^0(\mathfrak{A}/\mathfrak{J},\mathfrak{D}) \\
\downarrow & & & & \uparrow \\
KK^1(\mathfrak{A}/\mathfrak{J},\mathfrak{D}) & \xrightarrow{q_*} & KK^1(\mathfrak{A},\mathfrak{D}) & \xrightarrow{j_*} & KK^1(\mathfrak{J},\mathfrak{D}).
\end{array}$$

If $\mathrm{index}^{KK}(\mathfrak{D},\mathfrak{J})$, $\mathrm{index}^{KK}(\mathfrak{D},\mathfrak{A})$, *and* $\mathrm{index}^{KK}(\mathfrak{D},\mathfrak{A}/\mathfrak{J})$ *are finite, then*

$$\mathrm{index}^{KK}(\mathfrak{D},\mathfrak{A}) = \mathrm{index}^{KK}(\mathfrak{D},\mathfrak{J}) + \mathrm{index}^{KK}(\mathfrak{D},\mathfrak{A}/\mathfrak{J}),$$

and if $\mathrm{index}^{KK}(\mathfrak{J},\mathfrak{D})$, $\mathrm{index}^{KK}(\mathfrak{A},\mathfrak{D})$, *and* $\mathrm{index}^{KK}(\mathfrak{A}/\mathfrak{J},\mathfrak{D})$ *are finite, then*

$$\mathrm{index}^{KK}(\mathfrak{A},\mathfrak{D}) = \mathrm{index}^{KK}(\mathfrak{J},\mathfrak{D}) + \mathrm{index}^{KK}(\mathfrak{A}/\mathfrak{J},\mathfrak{D}),$$

where it is enough to assume that two of those are finite in each case.

Proof. This can be proved as shown in the K-theory case above. □

Theorem 4.18 *Let $\mathfrak{A}$ be a (trivially graded) σ-unital C^*-algebra and $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ its crossed product by $\mathbb{Z}$. For any separable (graded) C^*-algebra $\mathfrak{D}$,*

$$\begin{array}{ccccc}
KK^0(\mathfrak{D},\mathfrak{A}) & \xrightarrow{(1-\alpha)_*} & KK^0(\mathfrak{D},\mathfrak{A}) & \longrightarrow & KK^0(\mathfrak{D},\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \\
\uparrow & & & & \downarrow \\
KK^1(\mathfrak{D},\mathfrak{A} \rtimes_\alpha \mathbb{Z}) & \longleftarrow & KK^1(\mathfrak{D},\mathfrak{A}) & \xleftarrow{(1-\alpha)_*} & KK^1(\mathfrak{D},\mathfrak{A}),
\end{array}$$

and if $\mathfrak{A}$ *is separable, then for any* σ*-unital (graded)* C^**-algebra* $\mathfrak{D}$*,*

$$\begin{array}{ccccc} KK^0(\mathfrak{A},\mathfrak{D}) & \xleftarrow{(1-\alpha)_*} & KK^0(\mathfrak{A},\mathfrak{D}) & \longleftarrow & KK^0(\mathfrak{A}\rtimes_\alpha \mathbb{Z},\mathfrak{D}) \\ \downarrow & & & & \uparrow \\ KK^1(\mathfrak{A}\rtimes_\alpha \mathbb{Z},\mathfrak{D}) & \longrightarrow & KK^1(\mathfrak{A},\mathfrak{D}) & \xrightarrow{(1-\alpha)_*} & KK^1(\mathfrak{A},\mathfrak{D}). \end{array}$$

If $\mathrm{index}^{KK}(\mathfrak{D},\mathfrak{A})$ *and* $\mathrm{index}^{KK}(\mathfrak{D},\mathfrak{A}\rtimes_\alpha \mathbb{Z})$ *are finite, then*

$$\mathrm{index}^{KK}(\mathfrak{D},\mathfrak{A}\rtimes_\alpha \mathbb{Z}) = 0,$$

and if $\mathrm{index}^{KK}(\mathfrak{A},\mathfrak{D})$ *and* $\mathrm{index}^{KK}(\mathfrak{A}\rtimes_\alpha \mathbb{Z},\mathfrak{D})$ *are finite, then*

$$\mathrm{index}^{KK}(\mathfrak{A}\rtimes_\alpha \mathbb{Z},\mathfrak{D}) = 0,$$

where it is enough to assume that $\mathrm{index}^{KK}(\mathfrak{D},\mathfrak{A})$ *or* $\mathrm{index}^{KK}(\mathfrak{A},\mathfrak{D})$ *is finite in each case.*

Proof. The proof is the same as given in Section 1 for $\mathrm{index}_K(\mathfrak{A}\rtimes_\alpha \mathbb{Z}) = 0$. □

Proposition 4.19 *Let* $\mathfrak{A}$ *be a (trivially graded)* C^**-algebra and* $\mathfrak{A}\rtimes_\alpha \mathbb{R}$ *its crossed product. For any* C^**-algebra* $\mathfrak{B}$*, we have*

$$\mathrm{index}^{KK}(\mathfrak{A}\rtimes_\alpha \mathbb{R},\mathfrak{B}) = -\mathrm{index}^{KK}(\mathfrak{A},\mathfrak{B}),$$
$$\mathrm{index}^{KK}(\mathfrak{B},\mathfrak{A}\rtimes_\alpha \mathbb{R}) = -\mathrm{index}^{KK}(\mathfrak{B},\mathfrak{A}).$$

Proof. This follows from the Thom isomorphism for KK:

$$KK^j(\mathfrak{A}\rtimes_\alpha \mathbb{R},\mathfrak{A}) \cong KK^{j+1}(\mathfrak{A},\mathfrak{B}), \quad KK^j(\mathfrak{B},\mathfrak{A}\rtimes_\alpha \mathbb{R}) \cong KK^{j+1}(\mathfrak{B},\mathfrak{A}).$$

□

Theorem 4.20 *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be separable* C^**-algebras and* $\mathfrak{A}$ *in the UCT class* N*. If* $K_*(\mathfrak{A})$ *or* $K_*(\mathfrak{B})$ *is finitely generated, then*

$$\mathrm{index}^{KK}(\mathfrak{A},\mathfrak{B}) \geq \mathrm{index}^{K}(\mathfrak{A})\,\mathrm{index}_K(\mathfrak{B}),$$

and in addition, if $K^*(\mathfrak{A})$ *or* $K_*(\mathfrak{B})$ *is torsion-free, then*

$$\mathrm{index}^{KK}(\mathfrak{A},\mathfrak{B}) = \mathrm{index}^{K}(\mathfrak{A})\,\mathrm{index}_K(\mathfrak{B}).$$

Proof. The Künneth theorem is

$$0 \to K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to KK^*(\mathfrak{A}, \mathfrak{B}) \to \mathrm{Tor}_1^{\mathbb{Z}}(K^*(\mathfrak{A}), K_*(\mathfrak{B})) \to 0$$

under the first assumption. In particular, the second assumption implies that

$$\begin{aligned} KK^*(\mathfrak{A}, \mathfrak{B}) &\cong K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B}), \quad \text{i.e.,} \\ KK^0(\mathfrak{A}, \mathfrak{B}) &\cong (K^0(\mathfrak{A}) \otimes K_0(\mathfrak{B})) \oplus (K^1(\mathfrak{A}) \otimes K_1(\mathfrak{B})), \\ KK^1(\mathfrak{A}, \mathfrak{B}) &\cong (K^0(\mathfrak{A}) \otimes K_1(\mathfrak{B})) \oplus (K^1(\mathfrak{A}) \otimes K_0(\mathfrak{B})). \end{aligned}$$

Therefore;

$$\begin{aligned} \mathrm{KKr}^0(\mathfrak{A}, \mathfrak{B}) &= \mathrm{Kr}^0(\mathfrak{A})\,\mathrm{Kr}_0(\mathfrak{B}) + \mathrm{Kr}^1(\mathfrak{A})\,\mathrm{Kr}_1(\mathfrak{B}), \\ \mathrm{KKr}^1(\mathfrak{A}, \mathfrak{B}) &= \mathrm{Kr}^0(\mathfrak{A})\,\mathrm{Kr}_1(\mathfrak{B}) + \mathrm{Kr}^1(\mathfrak{A})\,\mathrm{Kr}_0(\mathfrak{B}). \end{aligned}$$

It follows that

$$\begin{aligned} \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}) &= (\mathrm{Kr}^0(\mathfrak{A}) - \mathrm{Kr}^1(\mathfrak{A}))(\mathrm{Kr}_0(\mathfrak{B}) - \mathrm{Kr}_1(\mathfrak{B})) \\ &= \mathrm{index}^K(\mathfrak{A})\,\mathrm{index}_K(\mathfrak{B}). \end{aligned}$$

□

Remark. Note that $\mathrm{Tor}_1^{\mathbb{Z}}(M, N) = H_1(M \otimes_{\mathbb{Z}} P_*)$ by definition is the first twisted product of $\mathbb{Z}$-modules M and N, where P_* means a projective resolution of N:

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to N \to 0$$

from which the following becomes a complex of $\mathbb{Z}$-modules:

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0$$

so that $M \otimes_{\mathbb{Z}} P_*$ means the following complex of abelian groups:

$$\cdots \to M \otimes_{\mathbb{Z}} P_n \to M \otimes_{\mathbb{Z}} P_{n-1} \to \cdots \to M \otimes_{\mathbb{Z}} P_1 \to M \otimes_{\mathbb{Z}} P_0 \to 0$$

and therefore, $H_1(M \otimes_{\mathbb{Z}} P_*)$ is the first homology group of this complex.

Theorem 4.21 *For a pullback diagram of separable nuclear C^*-algebras:*

$$\begin{array}{ccc} P & \xrightarrow{g_1} & \mathfrak{A}_1 \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ \mathfrak{A}_2 & \xrightarrow{f_2} & \mathfrak{D} \end{array}$$

the Mayer-Vietoris six-term exact sequence (MV) *is*:

$$\begin{array}{ccccc}
KK(\mathfrak{D},\mathfrak{B}) & \xrightarrow{(-f_1^*,f_2^*)} & KK(\mathfrak{A}_1,\mathfrak{B})\oplus KK(\mathfrak{A}_2,\mathfrak{B}) & \xrightarrow{g_1^*+g_2^*} & KK(P,\mathfrak{B}) \\
\uparrow & & & & \downarrow \\
KK^1(P,\mathfrak{B}) & \xleftarrow{g_1^*+g_2^*} & KK^1(\mathfrak{A}_1,\mathfrak{B})\oplus KK^1(\mathfrak{A}_2,\mathfrak{B}) & \xleftarrow{(-f_1^*,f_2^*)} & KK^1(\mathfrak{D},\mathfrak{B})
\end{array}$$

for any σ-unital C^*-algebra $\mathfrak{B}$, from which if follows that

$$\mathrm{index}^{KK}(P,\mathfrak{B}) = \mathrm{index}^{KK}(\mathfrak{A}_1,\mathfrak{B}) + \mathrm{index}^{KK}(\mathfrak{A}_2,\mathfrak{B}) - \mathrm{index}^{KK}(\mathfrak{D},\mathfrak{B}),$$

where finiteness of these indexes is assumed.

For a pullback diagram of σ-unital nuclear C^*-algebras:

$$\begin{array}{ccc}
P & \xrightarrow{g_1} & \mathfrak{B}_1 \\
{\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle f_1} \\
\mathfrak{B}_2 & \xrightarrow{f_2} & \mathfrak{D}
\end{array}$$

the Mayer-Vietoris six-term exact sequence is:

$$\begin{array}{ccccc}
KK(\mathfrak{A},P) & \xrightarrow{(g_1^*,g_2^*)} & KK(\mathfrak{A},\mathfrak{B}_1)\oplus KK(\mathfrak{A},\mathfrak{B}_2) & \xrightarrow{f_2^*-f_1^*} & KK(\mathfrak{A},\mathfrak{D}) \\
\uparrow & & & & \downarrow \\
KK^1(\mathfrak{A},\mathfrak{D}) & \xleftarrow{f_2^*-f_1^*} & KK^1(\mathfrak{A},\mathfrak{B}_1)\oplus KK^1(\mathfrak{A},\mathfrak{B}_2) & \xleftarrow{(g_1^*,g_2^*)} & KK^1(\mathfrak{A},P)
\end{array}$$

for any separable C^*-algebra $\mathfrak{A}$, from which if follows that

$$\mathrm{index}^{KK}(\mathfrak{A},P) = \mathrm{index}^{KK}(\mathfrak{A},\mathfrak{B}_1) + \mathrm{index}^{KK}(\mathfrak{A},\mathfrak{B}_2) - \mathrm{index}^{KK}(\mathfrak{A},\mathfrak{D}),$$

where finiteness of these indexes is assumed.

Equivariant KK-theory ranks and index

Definition 4.22 Let $\mathfrak{A}$ and $\mathfrak{B}$ be (graded) G-C^*-algebras for G a (compact) group. We define the G-equivariant KK-rank of $\mathfrak{A}$ and $\mathfrak{B}$ to be the $\mathbb{Z}$-rank of the G-equivariant KK (abelian) group $KK_G^j(\mathfrak{A},\mathfrak{B})$ ($j = 0,1$), and denote it by

$$\mathrm{KKr}_G^j(\mathfrak{A},\mathfrak{B}) = \mathrm{rank}_{\mathbb{Z}} KK_G^j(\mathfrak{A},\mathfrak{B}) \in \{0,1,2,\cdots,+\infty\},$$

where $KK_G^0 = KK_G$. Define the G-equivariant KK-index (or KK_G-index) of $\mathfrak{A}$ and $\mathfrak{B}$ to be the following difference

$$\mathrm{index}_G^{KK}(\mathfrak{A},\mathfrak{B}) = \mathrm{KKr}_G^0(\mathfrak{A},\mathfrak{B}) - \mathrm{KKr}_G^1(\mathfrak{A},\mathfrak{B}) \in \mathbb{Z}\cup\{\pm\infty\}.$$

Proposition 4.23 *For any G-C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$,*

$$\mathrm{index}_G^{KK}(\mathfrak{A}, S\mathfrak{B}) = \mathrm{index}_G^{KK}(S\mathfrak{A}, \mathfrak{B}) = -\mathrm{index}_G^{KK}(\mathfrak{A}, \mathfrak{B}),$$

where the action of G on $C_0(\mathbb{R})$ is trivial.

Proof. This follows from the Bott periodicity for KK_G^j:

$$KK_G^1(\mathfrak{A}, \mathfrak{B}) \cong KK_G^0(\mathfrak{A}, S\mathfrak{B}) \cong KK_G^0(S\mathfrak{A}, \mathfrak{B}),$$
$$KK_G^0(\mathfrak{A}, \mathfrak{B}) \cong KK_G^1(\mathfrak{A}, S\mathfrak{B}) \cong KK_G^1(S\mathfrak{A}, \mathfrak{B}).$$

□

Theorem 4.24 *The six-term exact sequence for KK_G : Let*

$$0 \longrightarrow \mathfrak{J} \xrightarrow{j} \mathfrak{A} \xrightarrow{q} \mathfrak{A}/\mathfrak{J} \longrightarrow 0$$

be a semi-split exact sequence of σ-unital (graded) G-C^*-algebras with G compact. For any separable (graded) G-C^*-algebra $\mathfrak{D}$,

$$\begin{array}{ccccc} KK_G^0(\mathfrak{D}, \mathfrak{J}) & \xrightarrow{j_*} & KK_G^0(\mathfrak{D}, \mathfrak{A}) & \xrightarrow{q_*} & KK_G^0(\mathfrak{D}, \mathfrak{A}/\mathfrak{J}) \\ \uparrow & & & & \downarrow \\ KK_G^1(\mathfrak{D}, \mathfrak{A}/\mathfrak{J}) & \xleftarrow{q_*} & KK_G^1(\mathfrak{D}, \mathfrak{A}) & \xleftarrow{j_*} & KK_G^1(\mathfrak{D}, \mathfrak{J}), \end{array}$$

and if $\mathfrak{A}$ is separable, then for any σ-unital graded G-C^*-algebra $\mathfrak{D}$,

$$\begin{array}{ccccc} KK_G^0(\mathfrak{J}, \mathfrak{D}) & \xleftarrow{j^*} & KK_G^0(\mathfrak{A}, \mathfrak{D}) & \xleftarrow{q^*} & KK_G^0(\mathfrak{A}/\mathfrak{J}, \mathfrak{D}) \\ \downarrow & & & & \uparrow \\ KK_G^1(\mathfrak{A}/\mathfrak{J}, \mathfrak{D}) & \xrightarrow{q_*} & KK_G^1(\mathfrak{A}, \mathfrak{D}) & \xrightarrow{j_*} & KK_G^1(\mathfrak{J}, \mathfrak{D}). \end{array}$$

If $\mathrm{index}_G^{KK}(\mathfrak{D}, \mathfrak{J})$, $\mathrm{index}_G^{KK}(\mathfrak{D}, \mathfrak{A})$, *and* $\mathrm{index}_G^{KK}(\mathfrak{D}, \mathfrak{A}/\mathfrak{J})$ *are finite, then*

$$\mathrm{index}_G^{KK}(\mathfrak{D}, \mathfrak{A}) = \mathrm{index}_G^{KK}(\mathfrak{D}, \mathfrak{J}) + \mathrm{index}_G^{KK}(\mathfrak{D}, \mathfrak{A}/\mathfrak{J}),$$

and if $\mathrm{index}_G^{KK}(\mathfrak{J}, \mathfrak{D})$, $\mathrm{index}_G^{KK}(\mathfrak{A}, \mathfrak{D})$, *and* $\mathrm{index}_G^{KK}(\mathfrak{A}/\mathfrak{J}, \mathfrak{D})$ *are finite, then*

$$\mathrm{index}_G^{KK}(\mathfrak{A}, \mathfrak{D}) = \mathrm{index}_G^{KK}(\mathfrak{J}, \mathfrak{D}) + \mathrm{index}_G^{KK}(\mathfrak{A}/\mathfrak{J}, \mathfrak{D}),$$

where it is enough to assume that two of those are finite in each case.

Proof. This can be proved as shown in the KK-theory case above. □

Set $K_j^G(\mathfrak{B}) = KK_G^j(\mathbb{C}, \mathfrak{B})$ and $K_G^j(\mathfrak{A}) = KK_G^j(\mathfrak{A}, \mathbb{C})$ for C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$.

Proposition 4.25 *For a C^*-dynamical system $(\mathfrak{B}, \beta, G)$ with G compact,*

$$\mathrm{index}_G^{KK}(\mathbb{C}, \mathfrak{B}) = \mathrm{index}_K(\mathfrak{B} \rtimes_\beta G),$$

and for a C^*-dynamical system $(\mathfrak{A}, \alpha, G)$ with G discrete,

$$\mathrm{index}_G^{KK}(\mathfrak{A}, \mathbb{C}) = \mathrm{index}^K(\mathfrak{A} \rtimes_\alpha G).$$

where index_G^{KK}, index_K, *and* index^K *can be replaced with* KKr_G^j, Kr_j, *and* Kr^j *respectively.*

Proof. It is shown under the assumptions that

$$K_j^G(\mathfrak{B}) \cong K_j(\mathfrak{B} \rtimes_\beta G), \quad K_G^j(\mathfrak{A}) \cong K^j(\mathfrak{A} \rtimes_\alpha G).$$

$\square$

Example 4.26 In particular, if G is compact, then

$$\mathrm{index}_G^{KK}(\mathbb{C}, \mathbb{C}) = \mathrm{index}_K(C^*(G)) = |G^\wedge|,$$

where $G^\wedge$ is the dual group of G and is discrete.

Example 4.27 In general, there exists a homomorphism from $KK_G(\mathfrak{A}, \mathfrak{B})$ to $KK(\mathfrak{A} \rtimes_\alpha G, \mathfrak{B} \rtimes_\beta G)$, so that there exists a homomorphism from $KK_G^1(\mathfrak{A}, \mathfrak{B})$ to $KK^1(\mathfrak{A} \rtimes_\alpha G, \mathfrak{B} \rtimes_\beta G)$. In particular, there exists a homomorphism from $KK_G(\mathbb{C}, \mathbb{C})$ to $KK(C^*(G), C^*(G))$, so that there exists a homomorphism from $KK_G^1(\mathbb{C}, \mathbb{C})$ to $KK^1(C^*(G), C^*(G))$. Compute

$$\begin{aligned}
\mathrm{index}^{KK}(C^*(G), C^*(G)) &= \mathrm{index}^{KK}(\oplus_{\pi \in G^\wedge} M_{n_\pi}(\mathbb{C}), \oplus_{\pi \in G^\wedge} M_{n_\pi}(\mathbb{C})) \\
&= \sum_{\pi \in G^\wedge} \sum_{\pi \in G^\wedge} \mathrm{index}^{KK}(M_{n_\pi}(\mathbb{C}), M_{n_\pi}(\mathbb{C})) \\
&= \sum_{\pi \in G^\wedge} \sum_{\pi \in G^\wedge} \mathrm{index}^{KK}(\mathbb{C}, \mathbb{C}) = |G^\wedge|^2.
\end{aligned}$$

Hence, if $G^\wedge$ is non-trivial and finite, then

$$\mathrm{index}_G^{KK}(\mathbb{C}, \mathbb{C}) \neq \mathrm{index}^{KK}(C^*(G), C^*(G))$$

but the non-equality becomes equality if $G^\wedge$ is infinite.

As a summary, a (partial) permanence result for our KK-index is

Theorem 4.28 *The KK-index is homotopy invariant and functorial in some senses as follows:*

$$\begin{aligned}
&\text{(Additive)} && \mathrm{index}^{KK}(\oplus_j \mathfrak{A}_j, \mathfrak{B}) = \sum_j \mathrm{index}^{KK}(\mathfrak{A}_j, \mathfrak{B}),\\
& && \mathrm{index}^{KK}(\mathfrak{A}, \oplus_j \mathfrak{B}_j) = \sum_j \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}_j),\\
&\text{(Multiplicative)} && \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}) = \mathrm{index}^{K}(\mathfrak{A})\,\mathrm{index}_{K}(\mathfrak{B}),\\
& && \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}_1 \otimes \mathfrak{B}_2) = \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}_1)\,\mathrm{index}_{K}(\mathfrak{B}_2),\\
& && \mathrm{index}^{KK}(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{B}) = \mathrm{index}^{K}(\mathfrak{A}_1)\mathrm{index}^{KK}(\mathfrak{A}_2, \mathfrak{B}_2),\\
&\text{(Stability)} && \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}) = \mathrm{index}^{KK}(\mathfrak{A} \otimes \mathbb{K}, \mathfrak{B})\\
& && = \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{K}) = \mathrm{index}^{KK}(\mathfrak{A} \otimes \mathbb{K}, \mathfrak{B} \otimes \mathbb{K})\\
& && = \mathrm{index}^{KK}(\mathfrak{A} \otimes M_n(\mathbb{C}), \mathfrak{B} \otimes M_m(\mathbb{C})),\\
&\text{(Periodicity)} && \mathrm{index}^{KK}(\mathfrak{A}, S\mathfrak{B}) = -\mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}) = \mathrm{index}^{KK}(S\mathfrak{A}, \mathfrak{B}),\\
& && \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B} \rtimes_\beta \mathbb{R}) = -\mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B})\\
& && = \mathrm{index}^{KK}(\mathfrak{A} \rtimes_\alpha \mathbb{R}, \mathfrak{B}),\\
&\text{(Vanishing)} && \mathrm{index}^{KK}(\mathfrak{A}, C\mathfrak{B}) = 0 = \mathrm{index}^{KK}(C\mathfrak{A}, \mathfrak{B}),\\
& && \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B} \rtimes_\beta \mathbb{Z}) = 0 = \mathrm{index}^{KK}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}, \mathfrak{B}),\\
&\text{(Dividity)} && \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}) = \mathrm{index}^{KK}(\mathfrak{J}, \mathfrak{B}) + \mathrm{index}^{KK}(\mathfrak{A}/\mathfrak{J}, \mathfrak{B}),\\
& && \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}) = \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{D}) + \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}/\mathfrak{D}),\\
&\text{(KK-index MV)} && \mathrm{index}^{KK}(\mathfrak{A}_1 \oplus_{\mathfrak{D}} \mathfrak{A}_2, \mathfrak{B}) =\\
& && \mathrm{index}^{KK}(\mathfrak{A}_1, \mathfrak{B}) + \mathrm{index}^{KK}(\mathfrak{A}_2, \mathfrak{B}) - \mathrm{index}^{KK}(\mathfrak{D}, \mathfrak{B}),\\
& && \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}_1 \oplus_{\mathfrak{D}} \mathfrak{B}_2) =\\
& && \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}_1) + \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}_2) - \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{D}),\\
&\text{(K-index)} && \mathrm{index}^{KK}(\mathbb{C}, \mathfrak{B}) = \mathrm{index}_{K}(\mathfrak{B}),\\
&\text{(K-hom index)} && \mathrm{index}^{KK}(\mathfrak{A}, \mathbb{C}) = \mathrm{index}^{K}(\mathfrak{A}),\\
&\text{(Hom rank)} && \mathrm{index}^{KK}(\mathfrak{A}, \mathfrak{B}) = (\sum_{j=0}^{1} \mathrm{rank}_{\mathbb{Z}}\mathrm{Hom}(K_j(\mathfrak{A}), K_j(\mathfrak{B})))\\
& && - (\sum_{j=0}^{1} \mathrm{rank}_{\mathbb{Z}}\mathrm{Hom}(K_j(\mathfrak{A}), K_{j+1}(\mathfrak{B}))),\\
& && \textit{with} \quad KK^1(\mathfrak{A}, \mathfrak{B}) = KK^0(\mathfrak{A}, S\mathfrak{B}) = \mathrm{Ext}(\mathfrak{A}, \mathfrak{B}),
\end{aligned}$$

where some restrictive assumptions such as being in the UCT or the class

X, KK-index finiteness, and some K-theory conditions are necessary as discussed above.

Also, the KK_G-index is homotopy invariant and functorial similarly in (some of) those senses.

5 Appendix: Preliminaries and facts

Chain complexes

Let R be a ring. A graded R-module is a sequence $C = (C_n)_{n\in\mathbb{Z}}$ of R-modules. If $x \in C_n$, then x has degree $n = \deg(x)$. A map of degree p from a graded R-module C to another C' is a sequence $f = (f_n : C_n \to C'_{n+p})_{n\in\mathbb{Z}}$ of R-module homomorphisms. A chain complex over R is a graded R-module $C = (C, d)$ with $d = (d_n : C_n \to C_{n-1})_{n\in\mathbb{Z}}$ a map of degree -1 such that $d^2 = 0$, i.e., $d_n \circ d_{n+1} = 0$ for each n, called the differential or boundary map of C. Define the cycles $Z(C)$, boundaries $B(C)$, and homology $H(C)$ by $Z(C) = \ker(d) = (\ker(d_n))_{n\in\mathbb{Z}}$ the kernel, $B(C) = \mathrm{im}(d) = (\mathrm{im}(d_{n+1}))_{n\in\mathbb{Z}}$ the image, and $H(C) = Z(C)/B(C) = (H_n(C) = \ker(d_n)/\mathrm{im}(d_{n+1}))_{n\in\mathbb{Z}}$.

A cochain complex is a graded R-module $C = (C^n)_{n\in\mathbb{Z}}$ with $d = (d^n : C^n \to C^{n+1})_{n\in\mathbb{Z}}$ a map of degree 1 such that $d^2 = 0$, called the coboundary map of C. Define the cocycles $Z(C)$, coboundaries $B(C)$, and cohomology $H(C) = (H^n(C))_{n\in\mathbb{Z}}$ similarly as above.

A chain map from (C, d) to (C', d') is a graded module homomorphism $f : C \to C'$ of degree 0 such that $d'f = fd$. A homotopy from a chain map f to another g is a graded module homomorphism $h : C \to C'$ of degree 1 such that $d'h + hd = f - g$, and we then say that f is homotopic to g.

Proposition 5.1 *A chain map $f : C \to C'$ induces a map $H(f) : H(C) \to H(C')$, and $H(f) = H(g)$ if f is homotopic to g.*

We denote by $\mathcal{H}om_R(C, C')_n$ the set of graded module homomorphisms of degree n from a chain complex (C, d) over R to another (C', d'). Then we have $\mathcal{H}om_R(C, C')_n = \Pi_{q\in\mathbb{Z}}\mathrm{Hom}_R(C_q, C'_{q+n})$. The boundary map $D_n : \mathcal{H}om_R(C, C')_n \to \mathcal{H}om_R(C, C')_{n-1}$ is defined by $D_n(f) = d'f - (-1)^n fd$, so that $D^2 = 0$. Note that 0-cycles are the chain maps $C \to C'$, and 0-boundaries are the null-homotopic chain maps. Thus, $H_0(\mathcal{H}om_R(C, C'))$ is the abelian group of homotopy classes of the chain maps. There is an interpretation of $H_n(\mathcal{H}om_R(C, C'))$ in terms of homotopy for any n.

A chain map $f : C \to C'$ is called a homotopy equivalence if there is a chain map $f' : C' \to C$ such that $f' \circ f$ and $f \circ f'$ are homotopic to the identity maps on C and C' respectively. A chain map is called a weak equivalence if $H(f) : H(C) \to H(C')$ is an isomorphism. Any homotopy equivalence is a weak equivalence.

A chain complex is called contractible if it is homotopy equivalent to the zero complex. Any contractible chain complex is acyclic, i.e., $H(C) = 0$.

Proposition 5.2 *A short exact sequence of chain complexes:*

$$0 \longrightarrow C' \xrightarrow{\ i\ } C \xrightarrow{\ \pi\ } C'' \longrightarrow 0$$

gives rise to a long exact sequence in homology:

$$\longrightarrow H_n(C') \xrightarrow{H(i)} H_n(C) \xrightarrow{H(\pi)} H_n(C'') \xrightarrow{\ \partial\ } H_{n-1}(C') \longrightarrow$$

where the connecting homomorphism ∂ is natural in the sense that a commutative diagram of chain complexes:

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & E' \arrow[r] & E \arrow[r] & E'' \arrow[r] & 0 \end{tikzcd}$$

yields a commutative square in homology:

$$\begin{tikzcd} H_n(C'') \arrow[r] \arrow[d] & H_{n-1}(C') \arrow[d] \\ H_n(E'') \arrow[r] & H_{n-1}(E'). \end{tikzcd}$$

The tensor product $C \otimes_R C'$ of chain complexes (C, d) and (C', d') of R-modules is defined by $(C \otimes_R C')_n = \oplus_{p+q=n} C_p \otimes_R C'_q$ and with differential D given by $D(c \otimes c') = d(c) \otimes c' + (-1)^{\deg(c)} c \otimes d'(c')$ for $c \in C$ and $c' \in C'$. Note that $C \otimes_R C'$ is a complex of abelian groups in general, and it is a complex of R-modules if R is commutative.

Proposition 5.3 (Künneth Formula) *Let R be a principal ideal domain and let C and C' be chain complexes over R such that C is dimension-wise free. Then there exist natural exact sequences:*

$$0 \to \bigoplus_{p \in \mathbb{Z}} H_p(C) \otimes_R H_{n-p}(C') \to H_n(C \otimes_R C')$$
$$\to \bigoplus_{p \in \mathbb{Z}} \mathrm{Tor}_1^R(H_p(C), H_{n-p-1}(C')) \to 0,$$

$$0 \to \prod_{p \in \mathbb{Z}} \mathrm{Ext}^1_R(H_p(C), H_{p+n+1}(C')) \to H_n(\mathcal{H}om_R(C, C'))$$
$$\to \prod_{p \in \mathbb{Z}} \mathrm{Hom}_R(H_p(C), H_{p+n}(C')) \to 0,$$

and these sequences split.

In particular, if C' consists of a single module M, i.e., $C'_0 = M$ and $C'_n = 0$ for $n \neq 0$, then the exact sequences become the universal coefficient theorem:

$$0 \to H_n(C) \otimes_R M \to H_n(C \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(C), M) \to 0,$$
$$0 \to \mathrm{Ext}_R^1(H_{n-1}(C), M) \to H^n(\mathcal{H}om_R(C, M)) \to \mathrm{Hom}_R(H_n(C), M) \to 0,$$

where we regard $\mathcal{H}om_R(C, M)$ as a cochain complex with $\mathcal{H}om_R(C, M)^n = \mathcal{H}om_R(C, M)_{-n} = \mathrm{Hom}_R(C_n, M)$.

Resolutions

Let R be a ring and M an R-module. A resolution of M is an exact sequence of R-modules:

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

If each F_j is free, then it is called a free resolution. Free resolutions always exist for any module M by a step-by-step construction: Choose a surjection ε from a free F_0 to M, then choose a surjection from a free F_1 to the kernel of ε, etc. If there is an integer n such that $F_i = 0$ for $i \geq n+1$, then we say that the resolution has length $\leq n$.

Example 5.4 (1). A free module F admits the free resolution: $0 \to F \xrightarrow{\mathrm{id}} F \to 0$ of length 0.

(2). If $R = \mathbb{Z}$, then any submodule of a free module is free. Hence any module M admits a free resolution

$$0 \to F_1 \to F_0 \to M \to 0$$

of length ≤ 1. For example, we have

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2 \to 0.$$

(3). Let $M = \mathbb{Z}_2$ and $R = \mathbb{Z}[x]$ the polynomial ring. Then we have the free resolution of length 2:

$$0 \to R \xrightarrow{\partial_2} R \oplus R \xrightarrow{\partial_1} R \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0$$

where the map ε from R to $\mathbb{Z}_2$ is defined by $\varepsilon(f) = f(0) \bmod 2$, and the maps ∂_1 and ∂_2 are given by the matrix $(x, 2)$ and the transpose of the matrix $(2, -x)$ respectively.

Let G be a group. Let $\mathbb{Z}G$ (or $\mathbb{Z}[G]$) be the free $\mathbb{Z}$-module generated by elements of G, which is called the integral group ring of G with the natural ring structure. The augmentation map $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ is the ring homomorphism defined by $\varepsilon(g) = 1$ for $g \in G$.

A G-complex is a CW-complex X with an action of G on X which permutes cells of X. If X is a G-complex, then the action of G on X induces that of G on the cellular chain complex $C_*(X)$, which becomes a chain complex of G-modules, where $C_n(X)$ is the free $\mathbb{Z}$-module generated by the $(n+1)$-tuples of elements of X and has a $\mathbb{Z}$-basis which is freely permuted by G, hence $C_n(X)$ is a free $\mathbb{Z}G$-module with one basis element for every G-orbit of cells. If X is contractible, $H_*(X) \cong H_*(\text{point})$, so that the following is exact:

$$\cdots \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

Therefore, we have

Proposition 5.5 *Let X be a contractible free G-complex. Then the augmented cellular chain complex of X is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

An (Eilenberg-MacLane) $K(G,1)$-complex is a CW-complex Y such that Y is connected, $\pi_1(Y) = G$, and the universal cover X of Y is contractible. The last condition can be replaced with $H_i(X) = 0$ for $i \geq 2$, or $\pi_i(Y) = 0$ for $i \geq 2$.

Proposition 5.6 *If Y is a $K(G,1)$-complex, then the augmented celluar chain complex of the universal cover of Y is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

Recall that an R-module P is projective if the functor $\mathrm{Hom}_R(P, \cdot)$ is exact. This is equivalent to that every exact sequence $0 \to M' \to M \to P \to 0$ splits, or to that P is a direct summand of a free module.

Recall that an R-module F is flat if the functor $(\cdot) \otimes_R F$ is exact. Free modules are flat, so that projective ones are flat.

Homology for groups

Let G be a group and M a G-module. The group M_G of co-invariants of M is the quotient of M by the subgroup generated by the elements $gm - m$ for $g \in G$ and $m \in M$. Note that M_G is the largest quotient of M on which G acts trivially, whereas the group M^G of invariants (under a G-action) is the largest submodule of M on which G acts trivially. Note also that $M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M$ via the maps $[m] \mapsto 1 \otimes m = g \otimes m = 1 \otimes gm$ and $a \otimes m \mapsto a[m]$.

For an exact sequence of G-modules: $M' \to M \to M'' \to 0$, we have the induced exact sequence: $M'_G \to M_G \to M''_G \to 0$.

Let G be a group and F a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Define the homology groups of G by

$$H_i(G) = H_i(F \otimes_{\mathbb{Z}G} \mathbb{Z}) = H_i(F_G),$$

which is independent of the choice of a resolution.

If $G = \mathbb{Z}_n$ with t a generator, there is a resolution;

$$\cdots \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $N = 1 + t + \cdots + t^{n-1} \in \mathbb{Z}G$, and F_G is given by

$$\cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Thus, $H_0(G) \cong \mathbb{Z}$, $H_i(G) \cong \mathbb{Z}_n$ for i odd and $\cong 0$ for $i \geq 1$ even.

Proposition 5.7 *If Y is a $K(G,1)$-complex, then $H_*(G) \cong H_*(Y)$.*

The Hurewicz theorem says that if $\pi_i(X) = 0$ for $i \leq n-1$ $(n \geq 2)$, then $H_i(X) = 0$ for $1 \leq i \leq n-1$ and $\pi_n(X) \cong H_n(X)$.

Theorem 5.8 (Hopf) *For any connected CW-complex Y, there is a canonical map from $H_*(Y)$ to $H_*(\pi_1(Y))$. If $\pi_i(Y) = 0$ for $2 \leq i \leq n-1$ $(n \geq 2)$, there is an isomorphism from $H_i(Y)$ to $H_i(\pi_1(Y))$ for $i \leq n-1$, and the following sequence is exact:*

$$\pi_n(Y) \to H_n(Y) \to H_n(\pi_1(Y)) \to 0.$$

Theorem 5.9 (Hopf) *If $G = F/R$ where F is free, then $H_2(G) \cong (R \cap [F,F])/[F,R]$.*

Theorem 5.10 (Seifert-van Kampen) *Let X be a CW-complex that is the union of two connected subcomplexes X_1 and X_2 whose intersection Y is connected and non-empty. Then there is an diagram:*

$$\begin{array}{ccc} \pi_1(Y) & \longrightarrow & \pi_1(X_2) \\ \downarrow & & \downarrow \\ \pi_1(X_1) & \longrightarrow & \pi_1(X) \end{array}$$

so that $\pi_1(X) \cong \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$.

This theorem says that π_1 is a functor from connected, pointed complexes to groups, preserving amalgamation.

Theorem 5.11 (Whitehead) *The amalgamation diagram for groups:*

$$\begin{tikzcd} H \arrow[r, "\alpha_2"] \arrow[d, "\alpha_1"'] & G_2 \arrow[d, "\beta_2"] \\ G_1 \arrow[r, "\beta_1"] & G_1 *_H G_2 \end{tikzcd}$$

with α_1, α_2 injective can be realized by a diagram:

$$\begin{tikzcd} Y = X_1 \cap X_2 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_1 \arrow[r] & X = X_1 \cup X_2 \end{tikzcd}$$

for $K(\pi_1(W), 1)$-complexes for $W = Y, X_1, X_2$, and X.

Note that since α_1 and α_2 are injective, then so are β_1 and β_2. The proof requires the following two lemmas:

Lemma 5.12 *Let $X' \hookrightarrow X$ be an inclusion of connected CW-complexes such that the induced maps $\pi_1(X') \to \pi_1(X)$ of fundamental groups is injective. Let $p : X^\sim \to X$ be the universal cover of X. Then each connected component of $p^{-1}(X')$ is simply connected (hence it is a copy of the universal cover of X'). Moreover, these components are permuted transitively by the action of $\pi_1(X)$ on $X^\sim$, and $\pi_1(X')$ is the isotropy group of one of them, i.e., $\pi_0(p^{-1}(X')) \cong \pi_1(X)/\pi_1(X')$.*

Sketch of Proof. We have the following diagram:

$$\begin{tikzcd} \pi_1(p^{-1}(X')) \arrow[r] \arrow[d] & \pi_1(X^\sim) \arrow[d] \\ \pi_1(X') \arrow[r] & \pi_1(X) \end{tikzcd}$$

where the vertical maps are induced by the map p, and the horizontal maps are by inclusions. Since $\pi_1(X^\sim)$ is trivial, the first assertion follows. □

Lemma 5.13 *A diagram of groups: $G_1 \leftarrow H \to G_2$ can be realized by a diagram of $K(\pi, 1)$-complexes: $X_1 \leftarrow Y = X_1 \cap X_2 \to X_2$.*

Sketch of Proof. Since $K(\pi,1)$-complexes can be constructed functorially, we can realize the group homomorphisms by cellular maps of $K(\pi,1)$-complexes: $X_1 \leftarrow Y \rightarrow X_2$. Taking mapping cylinders if necessary, we can make these maps inclusions. □

Sketch of Proof for the theorem. Take $X_1 \leftarrow Y \rightarrow X_2$ as in the lemma above. Let $X = X_1 \cup_Y X_2$ be the adjunction space obtained from the disjoint union of X_1 and X_2 by identifying two copies of Y. Then $\pi_1(X) = G_1 *_H G_2 = G$, so we need only show that $H_i(X^\sim) = 0$ $(i \geq 2)$ for the universal cover $X^\sim$ of X. Let $X_1^\sim$, $X_2^\sim$, and $Y^\sim$ be the inverse images of X_1, X_2, and Y in $X^\sim$ respectively. Since X_1, X_2, and Y have acyclic universal covers, it follows that $X_1^\sim$, $X_2^\sim$, and $Y^\sim$ have trivial homology in positive dimensions. The Mayer-Vietoris sequence for the diagram:

$$\begin{tikzcd} Y^\sim \arrow[r] \arrow[d] & X_2^\sim \arrow[d] \\ X_1^\sim \arrow[r] & X^\sim \end{tikzcd}$$

shows that $H_i(X^\sim) = 0$ for $i \geq 2$. □

Corollary 5.14 *For $G = G_1 *_H G_2$ an amalgam of groups, there is the Mayer-Vietoris sequence:*

$$\cdots \rightarrow H_n(H) \rightarrow H_n(G_1) \oplus H_n(G_2) \rightarrow H_n(G) \rightarrow H_{n-1}(H) \rightarrow \cdots$$

As a bi-product, we obtain

$$0 \rightarrow \mathbb{Z}[G/H] \rightarrow \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2] \rightarrow \mathbb{Z} \rightarrow 0.$$

This is the low-dimensional part of the Mayer-Vietoris sequence that we used in the proof above.

Homology and cohomology with coefficients

Recall that for a right R-module M and a left R-module N, the tensor product $M \otimes_R N$ (over a ring R) is defined to be the quotient of $M \otimes N = M \otimes_{\mathbb{Z}} N$ obtained by assuming the relations: $mr \otimes n = m \otimes rn$ for $m \in M$, $n \in N$, and $r \in R$. For a group G, $M \otimes_G N$ is obtained from $M \otimes N$ by assuming the relations: $g^{-1}m \otimes n = m \otimes gn$. Since $m \otimes n = gm \otimes gn$, we have $M \otimes_G N \cong (M \otimes N)_G$, where $g(m \otimes n) = gm \otimes gn$.

Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and let M be a G-module. Define the homology of G with coefficients in M by

$$H_*(G, M) = H_*(F \otimes_G M).$$

If we take $M = \mathbb{Z}$, then $H_*(G, \mathbb{Z}) = H_*(G)$.

Define the cohomology of G with coefficients in M by

$$H^*(G, M) = H^*(\mathcal{H}om_G(F, M)),$$

where note that $\mathcal{H}om_G(F, M)^n = \mathcal{H}om_G(F, M)_{-n} = \mathrm{Hom}_G(F_n, M)$ and $\mathcal{H}om_G(F, M)_n = \mathrm{Hom}_G(F_{-n}, M)$.

Note that an exact sequence $F_1 \to F_0 \to \mathbb{Z} \to 0$ yields the following:

$$0 \to \mathrm{Hom}_G(\mathbb{Z}, M) \to \mathrm{Hom}_G(F_0, M) \to \mathrm{Hom}_G(F_1, M).$$

Example 5.15 (1). If $G = \mathbb{Z}$ with generator t, then we have a resolution

$$0 \to \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \to \mathbb{Z} \to 0.$$

Hence $H_*(G, M)$ is the homology of

$$\cdots \to M \xrightarrow{t-1} M$$

and $H^*(G, M)$ is the cohomology of

$$M \xrightarrow{t-1} M \to 0 \to \cdots .$$

Thus, we have

$$H_0(G, M) = H^1(G, M) = M_G = \mathbb{Z} \otimes_{\mathbb{Z}G} M,$$
$$H_1(G, M) = H^0(G, M) = M^G = \{m \in M : gm = m \text{ for all } g \in G\},$$

and $H_i(G, M) = H^i(G, M) = 0$ for $i \geq 2$.

Now take projective resolutions $F \to M$ and $P \to N$ of two G-modules M and N. Set

$$\mathrm{Tor}_*^G(M, N) = H_*(F \otimes_G N) = H_*(F \otimes_G P) = H_*(M \otimes_G P).$$

Note that $H_*(G, \cdot)$ is recovered as $\mathrm{Tor}_*^G(\mathbb{Z}, \cdot)$. Set

$$\mathrm{Ext}_G^*(M, N) = H^*(\mathcal{H}om_G(F, N)).$$

Note that $H^*(G, \cdot)$ is recovered as $\mathrm{Ext}_G^*(\mathbb{Z}, \cdot)$.

Proposition 5.16 *Let M and N be G-modules. If M is $\mathbb{Z}$-torsion-free,*

$$\mathrm{Tor}_*^G(M, N) \cong H_*(G, M \otimes N),$$

where G acts diagonally on $M \otimes N$. If M is $\mathbb{Z}$-free,

$$\mathrm{Ext}_G^*(M, N) \cong H^*(G, \mathrm{Hom}(M, N)),$$

where G acts diagonally on $\mathrm{Hom}(M, N)$.

Proof. Let $\varepsilon : F \to \mathbb{Z}$ be a projective resolution, and consider the resolution $\varepsilon \otimes M : F \otimes M \to M$. This is flat if M is $\mathbb{Z}$-torsion-free, and is projective if M is $\mathbb{Z}$-free. Therefore, if M is $\mathbb{Z}$-torsion-free, then

$$\begin{aligned}\mathrm{Tor}_*^G(M,N) &\cong H_*((F\otimes M)\otimes_G N)\\ &= H_*((F\otimes M\otimes N)_G)\\ &= H_*(F\otimes_G (M\otimes N)) = H_*(G, M\otimes N),\end{aligned}$$

and if M is $\mathbb{Z}$-free, then

$$\begin{aligned}\mathrm{Ext}_G^*(M,N) &= H^*(\mathcal{H}om_G(F\otimes M, N))\\ &= H^*(\mathcal{H}om(F\otimes M, N)^G)\\ &= H^*(\mathcal{H}om(F, \mathrm{Hom}(M,N))^G)\\ &= H^*(\mathcal{H}om_G(F, \mathrm{Hom}(M,N))) = H^*(G, \mathrm{Hom}(M,N)).\end{aligned}$$

□

Proposition 5.17 (Shapiro's lemma) *If H is a subgroup of a group G and M is an H-module, then*

$$H_*(H,M) \cong H_*(G, \mathrm{ind}_H^G M), \quad H^*(H,M) \cong H^*(G, \text{co-ind}_H^G M),$$

where $\mathrm{ind}_H^G M = \mathbb{Z}G \otimes_{\mathbb{Z}H} M$ *and* $\text{co-ind}_H^G M = \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$ *are the induced and co-induced modules respectively.*

Proof. Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Then F can be viewed as a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$, so that $H_*(H,M) \cong H_*(F\otimes_{\mathbb{Z}H} M)$. Also,

$$F\otimes_{\mathbb{Z}H} M \cong F\otimes_{\mathbb{Z}G} (\mathbb{Z}G\otimes_{\mathbb{Z}H} M) \cong F\otimes_G (\mathrm{ind}_H^G M),$$

which implies the first isomorphism. The second isomorphism follows from the universal property of co-induction, which implies that $\mathcal{H}om_H(F,M) \cong \mathcal{H}om_G(F, \text{co-ind}_H^G M)$. □

If we take $M = \mathbb{Z}$, then

$$H_*(H) \cong H_*(G, \mathbb{Z}[G/H]).$$

If $[G:H] = |G/H|$ is finite, then $\text{co-ind}_H^G M \cong \mathrm{ind}_H^G M$, so that we have

$$H^*(H,\mathbb{Z}) \cong H^*(G, \mathbb{Z}[G/H]),$$

and also $H^*(H, \mathbb{Z}H) \cong H^*(G, \mathbb{Z}G)$ since $\mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z}H \cong \mathbb{Z}G$.

Cohomology and group extensions

An extension of a group G by a group N is a short exact sequence of groups: $1 \to N \to E \to G \to 1$. Extensions E and E' of G by N are equivalent if there is a map $\varphi : E \to E'$ such that the diagram

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, equal] & N \arrow[r] \arrow[d, equal] & E \arrow[r, "\pi"] \arrow[d, "\varphi"'] & G \arrow[r] \arrow[d, equal] & 1 \arrow[d, equal] \\ 1 \arrow[r] & N \arrow[r] & E' \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

commutes so that φ is an isomorphism.

Now assume that N is an abelian group M. Then M becomes a G-module; for E acts on M by conjugation and the conjugation action of M on itself is trivial, so there is an induced action of G on M.

An extension E of G by M a G-module splits if there is a homomorphism $s : G \to E$ (called a splitting) such that $\pi \circ s = \mathrm{id}_G$. This is equivalent to that the extension E is equivalent to the extension: $0 \to M \to M \rtimes G \to G \to 1$, where $M \rtimes G$ is the semi-direct product of G and M with product given by $(a, g)(b, g) = (a + gb, gh)$ for $a, b \in M$ and $g, h \in G$.

Derivations are functions $d : G \to M$ such that $d(gh) = d(g) + gd(h)$ for $g, h \in G$. Note that a splitting $s : G \to M \rtimes G$ has the form $s(g) = (d(g), g)$ for a derivation d.

Two splittings s_1 and s_2 are said to be M-conjugate if there is an element $a \in M$ such that $s_1(g) = as_2(g)a^{-1}$ for $g \in G$. This relation becomes $d_1(g) = a + d_2(g) - ga$ in terms of the corresponding derivations d_1 and d_2. Thus, s_2 and s_2 are M-conjugate if and only if the difference $d_2 - d_1$ has the form $g \mapsto ga - a$ for some fixed $a \in M$ and is called a principal derivation.

Proposition 5.18 *For any G-module M, there is a one-to-one correspondence between $H^1(G, M)$ and the set M-conjugacy classes of splittings for the split extension:*

$$0 \to M \to M \rtimes G \to G \to 1.$$

Proof. The M-conjugacy classes of splittings of a split extension of G by M correspond to the elements of the quotient group of the abelian group of derivations from G to M by the group of principal derivations, that is just the quotient of the group of 1-cocycles by the group of 1-coboundaries. □

Proposition 5.19 *There is a bijection between $H^2(G, M)$ for any G-module M and the set of equivalent classes of extensions E of G by $M : 0 \to M \to E \to G \to 1$.*

Sketch of Proof. Choose a set-theoretic cross-section of $\pi : E \to G$, i.e., a function $s : G \to E$ such that $\pi \circ s = \mathrm{id}_G$. Assume that $s(1) = 1$ (normalization condition). There is a function $f : G \times G \to M$ that measures the failure of s to be a homomorphism and is defined by $s(g)s(h) = f(g,h)s(gh)$ for $g, h \in G$. Note that f is normalized if $f(g,1) = 0 = f(1,g)$ for $g \in G$. It is known that there is an bijections between extensions with a normalized section s and normalized 2-cocycles f of G with coefficients in M. $\square$

Let E and N be groups. Suppose that we are given an action β of E on N and a homomorphism $\alpha : N \to E$ such that $\beta_{\alpha(n)}(n') = nn'n^{-1}$ and $\alpha(\beta_x(n)) = x\alpha(n)x^{-1}$ for $n, n' \in N$ and $x \in E$. We then say that N is a crossed module over E.

Theorem 5.20 *There is a bijection between $H^3(G, M)$ and the set of equivalent classes of 4-term exact sequences as:*

$$0 \to M \to N \xrightarrow{\alpha} E \to G \to 1,$$

where N is a crossed module over E, M is the kernel of α, and G is the cokernel of α, and the equivalence means that the diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\
\| & & \| & & \downarrow & & \downarrow & & \| & & \| \\
0 & \longrightarrow & M & \longrightarrow & N' & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

commutes for these 4-term exact sequences.

Sketch of Proof. Choose a set-theoretic cross-section $s : G \to E$ for the quotient map $\pi : E \to G$. Its failure to be multiplicative is measured by a function $f : G \times G \to \ker(\pi)$ such that $s(g)s(h) = f(g,h)s(gh)$ for $g, h \in G$. Associativity of the product in E forces a cocycle condition on f such that $f(g,h)f(gh,k) = s(g)f(h,k)s(g)^{-1}f(g,hk)$ for $g, h, k \in G$. Since $\ker(\pi) = \mathrm{im}(\alpha)$, we can lift f to a function $F : G \times G \to N$. The failure of F to satisfy the analogue of the cocycle condition is measured by a function $c : G^3 \to M$ such that

$$s(g)F(h,k)s(g)^{-1}F(g,hk) = c(g,h,k)F(g,h)F(gh,k).$$

Then the function c is a 3-cocycle, whose cohomology class is independent of the choices of s and F, and is the desired element of $H^3(G, M)$. $\square$

Spectral sequences

A double complex is a bigraded module $C = (C_{pq})_{p,q\in\mathbb{Z}}$ with a horizontal differential ∂' of bidegree $(-1,0)$ and a vertical differential ∂'' of bidegree $(0,-1)$ such that $\partial'\partial'' = \partial''\partial'$:

$$\begin{array}{ccc} C_{p-1,q} & \xleftarrow{\partial'} & C_{pq} \\ {\scriptstyle\partial''}\Big\downarrow & & \Big\downarrow{\scriptstyle\partial''} \\ C_{p-1,q-1} & \xleftarrow{\partial'} & C_{p,q-1}. \end{array}$$

For each q we have a horizontal chain complex $C_{*,q}$ with differential ∂', and we are given chain maps $\partial'' : C_{*,q} \to C_{*,q-1}$ such that $\partial'' \circ \partial'' = 0$. Similarly, for each p we have a vertical chain complex $C_{p,*}$ with differential ∂'', and we are given chain maps $\partial' : C_{p,*} \to C_{p-1,*}$ with $\partial' \circ \partial' = 0$.

A double complex C gives rise to a chain complex TC called the total complex: $(TC)_n = \oplus_{p+q=n} C_{pq}$ with differential ∂ given by $\partial|_{C_{pq}} = \partial' + (-1)^p \partial''$.

For two chain complexes C' and C'', we have a double complex C with $C_{pq} = C'_p \otimes C''_q$, and TC is the tensor product $C' \otimes C''$ of chain complexes.

Now assume that $C_{pq} = 0$ when $p < 0$ or $q < 0$. Then we have a spectral sequence $\{E^r\}$ converging to $H_*(TC)$. By definition, $E^0_{pq} = C_{pq}$ with $d^0 = \pm\partial''$, and $E^1_{pq} = H_q(C_{p,*})$ with the differential $d^1 : E^1_{pq} \to E^1_{p-1,q}$ induced by the chain map $\partial' : C_{p,*} \to C_{p-1,*}$, and E^2 can be described as the horizontal homology of the vertical homology of C. There is another spectral sequence converging to $H_*(TC)$ such that $E^0_{pq} = C_{qp}$, $E^1_{pq} = H_q(C_{*,p})$ with $d^1 : E^1_{pq} \to E^1_{p-1,q}$ induced by $\partial'' : C_{*,p} \to C_{*,p-1}$.

Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and C a non-negative chain complex of G-modules. We set $H_*(G,C) = H_*(F \otimes_G C)$. Since $F \otimes_G C$ is the total complex of the double complex of abelian groups $(F_p \otimes_G C_q)$, we have two spectral sequences converging to $H_*(G,C)$. The first spectral sequence has $E^1_{pq} = H_q(F_p \otimes_G C_*) = F_p \otimes_G H_q(C)$ and $E^2_{pq} = H_p(G, H_q(C))$, so that

$$E^2_{pq} = H_p(G, H_q(C)) \Rightarrow H_{p+q}(G,C)$$

The second spectral sequence has $E^1_{pq} = H_q(F_* \otimes_G C_p) = H_q(G, C_p)$ so that

$$E^1_{pq} = H_q(G, C_p) \Rightarrow H_{p+q}(G,C)$$

and the group E^2_{pq} is described as the p-th homology group of the complex obtained from C by applying the functor $H_q(G,\cdot)$ dimension-wise. Both spectral sequences can be thought of as giving approximations to $H_*(G,C)$ in terms of homology groups $H_*(G,M)$.

If G acts trivially on C, then $F \otimes_G C \cong F_G \otimes C$, so there is a Künneth formula:

$$0 \to \bigoplus_{p+q=n} H_p(G) \otimes H_q(C) \to H_n(G, C) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(G), H_q(C)) \to 0.$$

If $\mathfrak{k}$ is a field and C is a complex of $\mathfrak{k}$-ventor spaces with the trivial G-action, then $H_*(G, C) \cong H_*(G, \mathfrak{k}) \otimes_{\mathfrak{k}} H_*(C)$.

Proposition 5.21 *Let C be a non-negative chain complex of G-modules such that each C_n is H_*-acyclic. Then there is a spectral sequence of the form:*

$$E^2_{pq} = H_p(G, H_q(C)) \Rightarrow H_{p+q}(C_G).$$

Sketch of Proof. The assumption implies that the E^1-term is concentrated on the line $q = 0$, and $E^1_{p,0} = (C_p)_G$, so that the spectral sequence collapses at E^2 to yield $H_*(G, C) \cong H_*(C_G)$. □

Theorem 5.22 (Hochshild-Serre) *For $1 \to H \to G \to Q \to 1$ an extension of gruups and M any G-module, there is a spectral sequence denoted by $E^2_{pq} = H_p(Q, H_q(H, M)) \Rightarrow H_{p+q}(G, M)$, which implies the following 5-term exact sequence of low-dimensional homology groups:*

$$H_2(G, M) \to H_2(Q, M_H) \to H_1(H, M)_Q \to H_1(G, M) \to H_1(Q, M_H) \to 0.$$

In particular,

$$H_2(G) \to H_2(Q) \to H_1(N)_Q \to H_1(G) \to H_1(Q) \to 0,$$

which is deduced from Hopf's formula.

Sketch of Proof. If F is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, and M is a G-module, then $F \otimes_G M = (F \otimes M)_G$ can be computed by first dividing out by the action of H on $F \otimes M$ and then dividing out by the action of Q:

$$F \otimes_G M = ((F \otimes M)_H)_Q = (F \otimes_H M)_Q.$$

Thus, $H_*(G, M) = H_*(C_Q)$, where $C = F \otimes_H M$. Note also that we have a Q-module isomorphism: $H_*(H, M) \cong H_*(C)$. Claim that the Q-modules $C_p = (F_p \otimes M)_H$ are H_*-acyclic. In fact, it suffices to show that $(\mathbb{Z}G \otimes M)_H$

is H_*-acyclic. For this one need only observe that $(\mathbb{Z}G \otimes M)_H$ is an induced Q-module $\mathbb{Z}Q \otimes A$. Now apply the above proposition to the Q-complex C.

Furthermore, note that there is an exact sequence:

$$0 \to E^{\infty}_{2,0} \to E^2_{2,0} \to E^2_{0,1} \to H_1(G, M) \to E^{\infty}_{1,0} \to 0$$

where $E^2_{1,0} = E^{\infty}_{1,0}$ and $E^2_{pq} = H_p(Q, H_q(H, M))$ and $E^{\infty}_{2,0}$ is a quotient of $H_2(C, M)$. □

Equivariant Homology

Let X be a G-complex and $C_*(X)$ the cellular chain complex of X. The equivariant homology groups of (G, X) are defined by

$$H^G_*(X) = H_*(G, C_*(X)).$$

More generally, if M is a G-module, there is a diagonal G-action on $C_*(X, M) = C_*(X) \otimes M$, and set

$$H^G_*(X, M) = H_*(G, C_*(X, M)).$$

Similarly, the equivariant cohomology groups are defined by

$$H^*_G(X, M) = H^*(G, C^*(X, M)).$$

Note that $H^G_*(\{\text{point}\}, M) = H_*(G, M)$. Since any G-complex X admits a G-map to a point, there is a canonical map from $H^G_*(X, M)$ to $H_*(G, M)$. Furthermore,

Proposition 5.23 *If $f : X \to Y$ is a cellular map of G-complexes such that $f_* : H_*(X) \to H_*(Y)$ is an isomorphism, then f induces an isomorphism $H^G_*(X, M) \cong H^G_*(Y, M)$ for any G-module M.*

In particular, if X is acyclic, then there is an isomorphism $H^G_*(X, M) \cong H_*(G, M)$ induced by the canonical map.

We have the spectral sequence:

$$E^2_{pq} = H_p(G, H_q(X, M)) \Rightarrow H^G_{p+q}(X, M)$$

where the E^2-term involves the diagonal action of G on $H_*(X, M)$ induced by the action of G on X and M.

For each p-cell σ of X, we have a G_σ-module $\mathbb{Z}_\sigma$ which is isomorphic to $\mathbb{Z}$. Let $M_\sigma = \mathbb{Z}_\sigma \otimes M$, which is a G_σ-module isomorphic to M. Let X_p be the set of p-cells of X and let Σ_p be a set of representatives for X_p/G. Then

$$C_p(X, M) = C_p(X) \otimes M = \oplus_{\sigma \in X_p} M_\sigma,$$

from which we obtain $C_p(X,M) \cong \oplus_{\sigma\in\Sigma_p}\mathrm{ind}_{G_\sigma}^G M_\sigma$. Shapiro's lemma yields

$$H_q(G, C_p(X,M)) \cong \oplus_{\sigma\in\Sigma_p} H_q(G_\sigma, M_\sigma).$$

so that

$$E^1_{pq} = \oplus_{\sigma\in\Sigma_p} H_q(G_\sigma, M_\sigma) \Rightarrow H^G_{p+q}(X,M).$$

Theorem 5.24 (Cartan-Leray) *If X is a free G-complex, then there is a spectral sequence of the form: $E^2_{pq} = H_p(G, H_q(X)) \Rightarrow H_{p+q}(X/G)$.*

Sketch of Proof. Since the G-action is free, we have $G_\sigma = \{1\}$. The above spectral sequence for $M = \mathbb{Z}$ collapses at E^2 to yield

$$H^G_*(X) \cong H_*(C(X)_G) = H_*(X/G).$$

□

Example 5.25 Consider an amalgam $G = G_1 *_H G_2$. Let X be the tree associated to G such that G acts on X with no inversions, i.e., no elements of G interchanges the vertices of a 1-simplex of X, where a tree is a graph (or a 1-dimensional CW-complex) such that it is either contractible, simply connected, acyclic, or connected and contains no non-trivial reduced loops, equivalently. There is a single 1-simplex e which maps isomorphically onto the quotient graph X/G, and the isotropy groups of e and its vertices v and w are given by $G_v = G_1$, $G_w = G_2$, and $G_e = H$. Therefore, for any G-module M we have a spectral sequence converging to $H_*(G,M)$ with

$$E^1_{0,*} = H_*(G_1,M) \oplus H_*(G_2,M), \quad E^1_{1,*} = H_*(H,M),$$

and $E^1_{p,*} = 0$ for $p \geq 2$. The spectral sequence collapses at E^2 to yield a Mayer-Vietoris sequence:

$$\to H_n(H,M) \to H_n(G_1,M) \oplus H_n(G_2,M) \to H_n(G,M) \to H_{n-1}(H,M) \to .$$

A graph of groups is a connected graph Y with groups G_v and G_e for vertices v and edges e of Y such that there are injections $G_e \hookrightarrow G_v$ for v vertices of every edge e. As in the case of amalgams, there is a tree X associated to Y such that the fundamental group G of Y acts on X without inversion. Then $Y = X/G$ and the groups G_v and G_e are the isotropy subgroups of G. Consequently, we obtain a Mayer-Vietoris sequence:

$$\begin{aligned} \cdots \to \oplus_{e\in Y_1} H_n(G_e,M) \to \oplus_{v\in Y_0} H_n(G_v,M) \to H_n(G,M) \\ \to \oplus_{e\in Y_1} H_{n-1}(G_e,M) \to \cdots \end{aligned}$$

where Y_i is the set of i-cells of Y. In particular, the amalgam $G = G_1 *_H G_2$ is the fundamental group of the graph Y of groups with $Y_0 = \{G_1 = G_v, G_2 = G_w\}$ and $Y_1 = \{H = G_e\}$, where v, w are vertexes of the single edge e. Also, for a group H, a subgroup K, and an injection $\theta : K \to H$, an HNN extension $G = H*_K$ is defined by adjoining an element t to H subject to the relations $t^{-1}at = \theta(a)$ for $a \in K$. Then G is regarded as the fundamental group of the graph Y of groups with $Y_0 = \{H = G_v = G_w\}$ and $Y_1 = \{K = G_e\}$ to make a circle and with the inclusion $K \hookrightarrow H$ and the map $\theta : K \hookrightarrow H$ as injections. In this case, we have

$$\cdots \to H_n(K, M) \to H_n(H, M) \to H_n(G, M) \to H_{n-1}(K, M) \to \cdots .$$

6 Appendix: Cohomological dimensions and Euler characteristics

Cohomological dimension

If R is a ring and M is an R-module, then the projective dimension of M, denoted by $\operatorname{proj\,dim}_R M$, is defined to be the least non-negative integer n such that M admits a projective resolution

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

of length n. Recall that the Ext functors are defined by

$$\operatorname{Ext}^i_R(M, \cdot) = H^i(\mathcal{H}om_R(P, \cdot)),$$

where P is a projective resolution of M. In particular, $\operatorname{Ext}^i_{\mathbb{Z}\Gamma}(\mathbb{Z}, \cdot) = H^i(\Gamma, \cdot)$.

Lemma 6.1 *The following conditions are equivalent:*

$$\begin{aligned} &(1). \quad \operatorname{proj\,dim}_R M \leq n. \\ &(2). \quad \operatorname{Ext}^i_R(M, \cdot) = 0 \quad \textit{for } i \geq n+1. \\ &(3). \quad \operatorname{Ext}^{n+1}_R(M, \cdot) = 0. \end{aligned}$$

(4). *If* $0 \to K \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ *is any exact sequence of* R*-modules with each* P_i *projective, then* K *is projective.*

Sketch of Proof. It is obvious that (4) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3). We need only prove (3) $\Rightarrow$ (4). Given a partial resolution as in (4), complete it to a projective resolution such that

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r] \arrow[d] & P_{n-1} \arrow[r] & \cdots \\
& K \arrow[r, equal] \arrow[d] & K_n \arrow[u] & \\
& 0 & 0 \arrow[u] &
\end{tikzcd}$$

where we just set $K = K_n$. For any R-module N, an $(n+1)$-cocycle in $\mathcal{H}om_R(P, N)$ is a map $P_{n+1} \to N$ whose composition with the map $P_{n+2} \to P_{n+1}$ is zero. Such a cocycle can be regarded as a map $\varphi : K_{n+1} \to N$, where K_{n+1} is defined as K_n by replacing n with $n+1$. The cocycle is a cobundary if and only if φ extends to a map $P_n \to N$. Thus, (3) implies that every map on K_{n+1} extends to P_n. In particular, the identity map on K_{n+1} extends to P_n, so that $P_n \cong K_{n+1} \oplus K_n$. Hence K_n is projective. □

Consider the special case of $R = \mathbb{Z}\Gamma$ for a group Γ and $M = \mathbb{Z}$. The cohomological dimension of a group Γ is defined to be the integer:

$$\begin{aligned}
\mathrm{cd}(\Gamma) &= \operatorname{proj\,dim}_{\mathbb{Z}\Gamma} \mathbb{Z} \\
&= \inf\{n \geq 0 : \mathbb{Z} \text{ admits a projective resolution of length } n\} \\
&= \inf\{n \geq 0 : H^j(\Gamma, \cdot) = 0 \text{ for } j \geq n+1\} \\
&= \sup\{n \geq 0 : H^n(\Gamma, M) \neq 0 \text{ for some } \Gamma\text{-module } M\}.
\end{aligned}$$

The geometric dimension of Γ is defined to be the minimal dimension of a $K(\Gamma, 1)$-complex, denoted by $\operatorname{geom\,dim} \Gamma$. Since the cellular chain complex of the universal cover of a $K(\Gamma, 1)$-complex Y yields a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ of length equal to the dimension of Y, we have

Proposition 6.2 $\mathrm{cd}(\Gamma) \leq \operatorname{geom\,dim} \Gamma$.

Example 6.3 (1). $\mathrm{cd}(\Gamma) = 0$ if and only if Γ is trivial.

(2). If Γ is free and non-trivial, then $\mathrm{cd}(\Gamma) = 1$. Its converse is a deep theorem of Stallings and Swan. In other words, if Γ is a group with no non-split extension with abelian kernel, then Γ has no non-split extension.

(3). If Γ is the fundamental group of a connected closed surface Y other than S^2 or P^2, then Y is a 2-dimensional $K(\Gamma, 1)$, so $\mathrm{cd}(\Gamma) \leq 2$. Since $H^2(\Gamma, \mathbb{Z}_2) \cong H^2(Y, \mathbb{Z}_2) \neq 0$, thus $\mathrm{cd}(\Gamma) = 2$. If Γ is a one-relator group whose relator is not a proper power, then $\mathrm{cd}(\Gamma) \leq 2$ by Lyndon's theorem.

(4). If $\Gamma = \mathbb{Z}^n$, then the n-torus $\mathbb{T}^n$ is a $K(\Gamma, 1)$ with $H^n(\mathbb{T}^n, \mathbb{Z}) \cong \mathbb{Z} \neq 0$, hence $\mathrm{cd}(\Gamma) = n$.

(5). Let $\Gamma = H_3^{\mathbb{Z}}$ be the discrete Heisenberg group of rank 3. Then $H^3(\Gamma, \mathbb{Z}) \cong \mathbb{Z}$, hence $\mathrm{cd}(\Gamma) = 3$. More generally, if Γ is a finitely generated, torsion free, nilpotent group, then

$$\mathrm{cd}(\Gamma) = \mathrm{rank}(\Gamma),$$

which is the rank (or Hirsch number) of Γ that is defined to be the sum of ranks of free abelian subquotients associated with central series of Γ, which is independent of the choise of central series.

Proposition 6.4 *If* $\mathrm{cd}(\Gamma) < \infty$, *then*

$$\mathrm{cd}(\Gamma) = \sup\{n \geq 0 : H^n(\Gamma, F) \neq 0 \textit{ for some free } \mathbb{Z}\Gamma\textit{-module } F\}.$$

Proof. Let $n = \mathrm{cd}(\Gamma)$. In view of the long exact cohomology sequence, the functor $H^n(\Gamma, \cdot)$ is right exact. Since $H^n(\Gamma, M) \neq 0$ for some M, it follows that $H^n(\Gamma, F) \neq 0$ for any free module F which maps onto M. □

Proposition 6.5 *If* Γ' *is a subgroup of a group* Γ, *then*

$$\mathrm{cd}(\Gamma') \leq \mathrm{cd}(\Gamma)$$

where equality holds if $\mathrm{cd}(\Gamma) < \infty$ *and* $[\Gamma : \Gamma'] < \infty$.

If $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ *is a short exact sequence of groups, then*

$$\mathrm{cd}(\Gamma) \leq \mathrm{cd}(\Gamma') + \mathrm{cd}(\Gamma'').$$

If $\Gamma = \Gamma_1 *_H \Gamma_2$ *is an amalgam of groups, then*

$$\mathrm{cd}(\Gamma) \leq \max\{\mathrm{cd}(\Gamma_1), \mathrm{cd}(\Gamma_2), 1 + \mathrm{cd}(H)\}.$$

Sketch of Proof. The first inequality follows from Shapiro's lemma, or from the fact that a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ can be regarded as a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma'$. If $\mathrm{cd}(\Gamma) = n < \infty$, there is a free $\mathbb{Z}\Gamma$-module F with $H^n(\Gamma, F) \neq 0$. If F' is a free $\mathbb{Z}\Gamma'$-module of the same rank, then $F \cong \mathrm{ind}_{\Gamma'}^{\Gamma} F' = \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma'} F'$, so Shapiro's lemma yields $H^n(\Gamma', F') \cong H^n(\Gamma, F) \neq 0$. Thus $\mathrm{cd}(\Gamma') \geq n$.

The second follows from a consequence of the Hochschild-Serre spectral sequence. The third follows from the cohomology version of the Mayer-Vietoris sequence. □

Corollary 6.6 *If* $\mathrm{cd}(\Gamma) < \infty$, *then* Γ *is torsion-free.*

Proof. If Γ is not torsion-free, then Γ contains a nontrivial finite cyclic subgroup Γ'. Then $\mathrm{cd}(\Gamma') = \infty$ since $H^{2k}(\Gamma', \mathbb{Z}) \neq 0$ for all k, which implies $\mathrm{cd}(\Gamma) = \infty$. □

If Γ is a group with torsion, then $\mathrm{cd}(\Gamma) = \infty$.

Proposition 6.7 *For any group* Γ, *there is a free resolution of* $\mathbb{Z}$ *over* $\mathbb{Z}\Gamma$ *of length equal to* $\mathrm{cd}(\Gamma)$.

Note that for a projective module P over a ring, there is a free module F such that $P \oplus F \cong F$ (Eilenberg's trick). In fact, F is taken as a countable direct sum of $P \oplus Q$ that is free for a module Q.

Proof. Let $n = \mathrm{cd}(\Gamma)$ for some $0 < n < \infty$. Choose a partial free resolution $F_{n-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0$ of length $n-1$. Let P is the kernel of the map $\partial : F_{n-1} \to F_{n-2}$, where $F_{-1} = \mathbb{Z}$. Then P is projective, so there is a free module F such that $P \oplus F$ is free. Replacing F_{n-1} with $F_{n-1} \oplus F$ and setting $\partial|_F = 0$, we obtain a partial free resolution of length $n-1$ with $\ker(\partial)$ free. □

Theorem 6.8 (Serre) *If* Γ *is a torsion-free group and* Γ' *is a subgroup of finite index, then* $\mathrm{cd}(\Gamma') = \mathrm{cd}(\Gamma)$.

Sketch of Proof. In view of Proposition 6.5 above, we need only show that if $\mathrm{cd}(\Gamma') < \infty$, then $\mathrm{cd}(\Gamma) < \infty$. Suppose that $\mathrm{cd}(\Gamma') < \infty$. Then there is a finite dimensional $K(\Gamma', 1)$-complex whose universal cover X' is a finite dimensional, contractible free Γ'-complex. To prove $\mathrm{cd}(\Gamma) < \infty$, we construct from X' a finite dimensional, contractible free Γ-complex X. The construction is a straightforward analogue of the co-induction construction for modules, where it is omitted. To complete the proof, it is shown that Γ acts freely on X as follows. There is a canonical map $X \to X'$ given by evaluation at $1 \in \Gamma$. This map is Γ'-equivalent and takes cells to cells. Since Γ' acts freely on X', it follows that Γ' acts freely on X. For any cell σ of X, we have $\Gamma_\sigma \cap \Gamma' = \{1\}$, hence Γ_σ is finite. Since Γ is torsion-free, these finite isotropy groups Γ_σ are trivial. □

Example 6.9 The group $SL_n(\mathbb{Z})$ $(n \geq 2)$ has torsion. Hence we have $\mathrm{cd}(SL_n(\mathbb{Z})) = \infty$. It has torsion-free subgroups Γ of finite index. The intersection of Γ with the strict upper triangular group N has finite index in N and has $\mathrm{cd} = n(n-1)/2$. Thus, $\mathrm{cd}(\Gamma) \geq n(n-1)/2$. In fact, equality holds.

Virtual cohomological dimension

A group Γ is virtually torsion-free if Γ has a torsion-free subgroup of finite index. In this case Serre's theorem implies that all such subgroups have the same cd; for if Γ' and Γ'' are two torsion-free subgroups of finite index, then $\Gamma' \cap \Gamma''$ has finite index in both Γ' and Γ'', so that

$$\mathrm{cd}(\Gamma') = \mathrm{cd}(\Gamma' \cap \Gamma'') = \mathrm{cd}(\Gamma'').$$

This common cohomological dimension is called the virtual cohomological dimension of Γ and is denoted by $\mathrm{vcd}(\Gamma)$.

Theorem 6.10 *Let Γ be a virtually torsion-free group. Then* $\mathrm{vcd}(\Gamma) < \infty$ *if and only if there exists a finite dimensional, contractible proper* Γ-*complex.*

Example 6.11 (1). $\mathrm{vcd}(\Gamma) = 0$ if and only if Γ is finite.

(2). If $\Gamma = \Gamma_1 *_H \Gamma_2$ where Γ_1 and Γ_2 are finite, then $\mathrm{vcd}(\Gamma) \leq 1$. Moreover, if Γ is the fundamental group of a finite graph of finite groups, then $\mathrm{vcd}(\Gamma) \leq 1$, which follows from Serre's theory of groups acting on trees, providing a contractible 1-complex on which Γ acts properly, with finite quotient. If $\mathrm{vcd}(\Gamma) \leq 1$, then Stallings-Swan theorem shows that Γ has a free subgroup of finite index, and it is then known that Γ is the fundamental group of a graph of finite groups.

(3). If Γ is a finitely generated one-relator group, then $\mathrm{vcd}(\Gamma) \leq 2$. Indeed, Γ is virtually torsion-free by Fischer-Karrass-Solitar, and Lyndon showed the following exact sequence:

$$0 \to \mathbb{Z}[\Gamma/H] \to F \to \mathbb{Z}\Gamma \to \mathbb{Z} \to 0,$$

where F is a free $\mathbb{Z}\Gamma$-module of finite rank and H is a finite cyclic subgroup of Γ. Note that $\mathbb{Z}[\Gamma/H]$ is a free $\mathbb{Z}\Gamma'$-module of finite rank for any torsion-free subgroup Γ' of Γ with finite index. Also, we have a contractible proper 2-dimensional Γ-complex with finite quotient.

(4). Let G be a Lie group and X its homogeneous space G/K, where K is a maximal compact subgroup. Let Γ be a discrete subgroup of G, and assume that Γ is virtually torsion-free. This is automatic if, for instance, Γ is a subgroup of $GL_n(\mathbb{Z})$. It follows that

$$\mathrm{vcd}(\Gamma) \leq \dim X,$$

where equality holds if and only if Γ is co-compact in G. For instance,

$$\mathrm{vcd}(SL_n(\mathbb{Z})) = n(n-1)/2.$$

Proposition 6.12 *If* Γ *is a virtually torsion-free group and* Γ' *is a subgroup, then*

$$\mathrm{vcd}(\Gamma') \leq \mathrm{vcd}(\Gamma),$$

where equality holds if $[\Gamma : \Gamma'] < \infty$.

Euler characteristic for complexes

Let G be a finitely generated abelian group. The rank (or torsion-free rank) of G is defined by

$$\mathrm{rank}_{\mathbb{Z}}(G) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} G).$$

In particular, $\mathrm{rank}_{\mathbb{Z}}(G) = 0$ if and only if G is finite.

Let C be a non-negative chain complex of abelian groups. We say that C is finite dimensional if $C_i = 0$ for sufficiently large i. If, in addition, each C_i is finitely generated, then C is called finite.

Suppose that C is finite dimensional and $H_*(C)$ is finitely generated. Define the Euler characteristic of C by

$$\chi(C) = \sum_{i \geq 0} (-1)^i \mathrm{rank}_{\mathbb{Z}} H_i(C).$$

If C is the cellular chain complex $C_*(X)$ of a finite dimensional CW-complex X, set $\chi(X) = \chi(C_*(X))$, so that

$$\chi(X) = \sum_{i \geq 0} (-1)^i \mathrm{rank}_{\mathbb{Z}} H_i(X).$$

If X is finite, then $\chi(X) = \sum_i (-1)^i n_i$ the classical Euler characteristic for X, where n_i is the number of i-cells of X. Indeed,

Proposition 6.13 *If* C *is a finite chain complex, then*

$$\chi(C) = \sum_i (-1)^i \mathrm{rank}_{\mathbb{Z}}(C_i).$$

Proposition 6.14 *Let* C *be a finite dimensional free chain complex over* $\mathbb{Z}$ *such that* $H_*(C)$ *is finitely generated. Let* p *be a prime number. Then*

$$\chi(C) = \sum_i (-1)^i \dim_{\mathbb{Z}_p} H_i(C \otimes \mathbb{Z}_p).$$

Proof. Let $r_i = \dim_{\mathbb{Z}_p} H_i(C)_p$ and $s_i = \dim_{\mathbb{Z}_p} {}_pH_i(C)$, where for G an abelian group, $G_p = G \otimes \mathbb{Z}_p = G/_pG$ for ${}_pG = \{g \in G : pg = 0\} = \mathrm{Tor}(G, \mathbb{Z}_p)$. Using the universal coefficient theorem (or the long exact homology sequence associated to the short exact sequence: $0 \to C \xrightarrow{p} C \to C \otimes \mathbb{Z}_p \to 0$), one finds $\dim_{\mathbb{Z}_p} H_i(C \otimes \mathbb{Z}_p) = r_i + s_{i-1}$. On the other hand, since $H_i(C)$ is a direct sum of cyclic groups, one sees that $r_i = \mathrm{rank}_{\mathbb{Z}} H_i(C) + s_i$. Thus, $\dim_{\mathbb{Z}_p} H_i(C \otimes \mathbb{Z}_p) = \mathrm{rank}_{\mathbb{Z}} H_i(C) + s_i + s_{i-1}$. $\square$

Theorem 6.15 *Let G be a finite group and let C be a finite dimensional chain complex of projective $\mathbb{Z}G$-modules. If $H_*(C)$ is finitely generated, then so is $H_*(C_G)$, and*

$$\chi(C) = |G|\,\chi(C_G).$$

Corollary 6.16 *Let G be a finite group and let X be a finite dimensional free G-CW-complex with $H_*(X)$ finitely generated. Then $H_*(X/G)$ is finitely generated, and $\chi(X) = |G|\,\chi(X/G)$.*

If Γ is a group such that $H_i(\Gamma)$ is finitely generated for all i and finite for sufficiently large i, then set

$$\chi^{\sim}(\Gamma) = \sum_i (-1)^i \mathrm{rank}_{\mathbb{Z}} H_i(\Gamma),$$

which is a sort of Euler characteristic for Γ, but not always right in a sense as explained below and that this Euler characteristic is ignorant of torsion data of Γ (or $H_*(\Gamma)$) and is in the same spirit as defining our K-index.

Corollary 6.17 *Let Γ be a group with $\mathrm{cd}(\Gamma) < \infty$ and let Γ' be a normal subgroup of finite index. If $H_*(\Gamma')$ is finitely generated, then so is $H_*(\Gamma)$, and $\chi^{\sim}(\Gamma') = [\Gamma; \Gamma']\,\chi^{\sim}(\Gamma)$.*

Proof. Let P be a projective resolution of finite length of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$, let $G = \Gamma/\Gamma'$, and $C = P_{\Gamma'} = \mathbb{Z}G \otimes_{\mathbb{Z}\Gamma} P$. Then C is a finite dimensional complex of projective $\mathbb{Z}G$-modules such that $H_*(C) = H_*(\Gamma')$ and $H_*(C_G) = H_*(\Gamma)$. $\square$

Euler characteristic for groups without torision

A group Γ is said to be of finite homological type if $\mathrm{vcd}(\Gamma) < \infty$, and for every Γ-module M which is finitely generated as an abelian group, $H_j(\Gamma, M)$ is finitely generated for all j. If Γ is torsion-free, then $\mathrm{vcd}(\Gamma) < \infty$ implies $\mathrm{cd}(\Gamma) < \infty$.

Lemma 6.18 *If Γ is a group and Γ' is a subgroup of finite index, then Γ is of finite homological type if and only if Γ' is so.*

Suppose that Γ is of finite homological type and torsion-free. Define the (Brown's) Euler characteristic of Γ by

$$\chi(\Gamma) = \sum_j (-1)^j \mathrm{rank}_{\mathbb{Z}} H_j(\Gamma).$$

In this case, $\chi(\Gamma) = \chi^{\sim}(\Gamma)$. Also, if P is a finite projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$, then

$$\chi(\Gamma) = \sum_{j \geq 0} (-1)^j \mathrm{rank}_{\mathbb{Z}} (\mathbb{Z} \otimes_{\mathbb{Z}\Gamma} P_j)$$

(Serre's Euler characteristic of Γ).

Example 6.19 Let F_n be the free group with n generators. There exists a $K(F_n, 1)$-complex with one vertex and n 1-cells. Hence $\chi(F_n) = 1 - n$.

The n-torus $\mathbb{T}^n$ is a $K(\mathbb{Z}^n, 1)$-complex with Euler chracteristic $\chi(\mathbb{T}^n) = \chi(\mathbb{T}) \cdots \chi(\mathbb{T}) = 0$. Hence $\chi(\mathbb{Z}^n) = 0$.

Let Γ be the commutator subgroup of $SL_2(\mathbb{Z})$. Then $\Gamma \cong F_2$. Hence $\chi(\Gamma) = 1 - 2 = -1$. Note that Γ is of index 12 in $SL_2(\mathbb{Z})$.

Theorem 6.20 *If Γ is torsion-free and of finite homological type and Γ' is a subgroup of finite index, then*

$$\chi(\Gamma') = [\Gamma : \Gamma']\, \chi(\Gamma).$$

Corollary 6.21 *For a group extension $1 \to \Gamma \to E \to G \to 1$ such that Γ is torsion-free and of finite homological type and G is of prime order p, if p does not divide $\chi(\Gamma)$, then the extension splits. (If $\Gamma = F_n$, then such an extension do split whenever p does not divide $n - 1$.)*

Proof. The group E necessarily has torsion; for if it were torsion-free, then p must devide $\chi(\Gamma)$, contrary to the hypothesis. Let H be a non-trivial finite subgroup of E. Since Γ is torsion-free, $H \cap \Gamma = \{1\}$. Thus H is maped injectively to G. Since $|G|$ is prime, H is maped isomorphically and it provides a splitting. □

Euler characteristic for groups with torision

If Γ is a group of finite homological type, then we choose a torsion-free subgroup Γ' of finite index and set

$$\chi(\Gamma) = [\Gamma : \Gamma']^{-1} \chi(\Gamma') \in \mathbb{Q}$$

where it is shown that the right hand side is independent of the choice of Γ'. Indeed, suppose that Γ'' is another such subgroup and let $\Gamma_0 = \Gamma' \cap \Gamma''$. Then

$$\frac{\chi(\Gamma')}{[\Gamma : \Gamma']} = \frac{\chi(\Gamma_0)}{[\Gamma : \Gamma'][\Gamma' : \Gamma_0]} = \frac{\chi(\Gamma_0)}{[\Gamma : \Gamma_0]}$$

and similarly $[\Gamma : \Gamma'']^{-1}\chi(\Gamma'') = [\Gamma : \Gamma_0]^{-1}\chi(\Gamma_0)$.

Example 6.22 If Γ is finite, then we can take $\Gamma' = \{1\}$ and

$$\chi(\Gamma) = |\Gamma|^{-1}.$$

If $\Gamma = SL_2(\mathbb{Z})$, then we take Γ' as the commutator subgroup and

$$\chi(SL_2(\mathbb{Z})) = \frac{\chi(\Gamma')}{[\Gamma, \Gamma']} = \frac{-1}{12}.$$

Proposition 6.23 (1). *If Γ is torsion-free and of finite homological type, then*

$$\chi(\Gamma) = \chi^{\sim}(\Gamma) = \sum_i (-1)^i \dim_{\mathbb{Z}_p} H_i(\Gamma, \mathbb{Z}_p)$$

for any prime p.

(2). *Let $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ be a short exact sequence of groups with Γ' and Γ'' of finite homological type. If Γ is virtually torsion-free, then Γ is of finite homological type and*

$$\chi(\Gamma) = \chi(\Gamma')\,\chi(\Gamma'').$$

(3). *Let $\Gamma = \Gamma_1 *_H \Gamma_2$ be an amalgam of groups, where Γ_1, Γ_2, and H are of finite homological type. If Γ is virtually torsion-free, then Γ is of finite homological type and*

$$\chi(\Gamma) = \chi(\Gamma_1) + \chi(\Gamma_2) - \chi(H).$$

Sketch of Proof. (1). The first equality follows by definition and the second equality follows from Proposition 6.14 given above.

(2). Let Γ_0 be a torsion-free subgroup of Γ with finite index whose image Γ_0'' in Γ'' is torsion-free. Let $\Gamma_0' = \Gamma_0 \cap \Gamma'$. Claim that

$$[\Gamma : \Gamma_0] = [\Gamma' : \Gamma_0'] \cdot [\Gamma'' : \Gamma_0''];$$

for $[\Gamma : \Gamma_0] = [\Gamma : \Gamma'\Gamma_0] \cdot [\Gamma'\Gamma_0 : \Gamma_0]$ and the isomorphism laws of group theory imply that $[\Gamma : \Gamma'\Gamma_0] = [\Gamma'' : \Gamma_0'']$ and $[\Gamma'\Gamma_0 : \Gamma_0] = [\Gamma' : \Gamma_0']$. Therefore, we replace the given exact sequence by $1 \to \Gamma_0' \to \Gamma_0 \to \Gamma_0'' \to 1$, so that we may assume that Γ', Γ, and Γ'' are torsion-free. Then $\mathrm{cd}(\Gamma) < \infty$, and the Hochschild-Serre spectral sequence:

$$E^2_{pq} = H_p(\Gamma'', H_q(\Gamma', M)) \Rightarrow H_{p+q}(\Gamma, M)$$

shows that Γ is of finite homological type. Now take $M = \mathbb{Z}_2$. Since $H_*(\Gamma', \mathbb{Z}_2)$ is finite, there is a subgroup Γ''_0 of Γ'' with finite index which acts trivially on it. Replacing Γ'' by Γ''_0 and Γ by the inverse image of Γ''_0, we may assume that Γ'' acts trivially on $H_*(\Gamma', \mathbb{Z}_2)$. Then $E^2_{pq} = H_p(\Gamma'', \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H_q(\Gamma', \mathbb{Z}_2)$. Computing Euler characteristics from this spectral sequence, we obtain $\chi(\Gamma) = \chi(\Gamma')\,\chi(\Gamma'')$.

(3). This follows from the statement (3) in the next proposition, applied to the tree associated to $\Gamma_1 *_H \Gamma_2$. $\square$

Example 6.24 Since $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ and $PSL_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$,

$$\begin{aligned} \chi(SL_2(\mathbb{Z})) &= \frac{1}{4} + \frac{1}{6} - \frac{1}{2} = -\frac{1}{12}, \\ \chi(PSL_2(\mathbb{Z})) &= \frac{1}{2} + \frac{1}{3} - 1 = -\frac{1}{6}. \end{aligned}$$

Let M_g be a closed Riemann surface with genus g (≥ 2) and $\Gamma_g = \pi_1(M_g)$. Since $\Gamma_g \cong F_2 *_{\mathbb{Z}} F_{2g-2}$, it follows that

$$\begin{aligned} \chi(\Gamma_g) &= \chi(F_2 *_{\mathbb{Z}} F_{2g-2}) = \chi(F_2) + \chi(F_{2g-2}) - \chi(\mathbb{Z}) \\ &= (1-2) + (1+2-2g) + 0 = 2 - 2g = \chi(M_g). \end{aligned}$$

Remark. It seems not clear to extend the definition of Euler characteristic (or K-index) for C^*-algebras with torsion-free K-groups to those with K-groups with torsion, without deleting torsion data. Our K-index is ignorant of torsion data. That point of view for extending the notion will be discussed somewhere in the future.

Now suppose that X is a Γ-complex such that every isotropy group Γ_σ is of finite homological type and X has only finitely many cells mod Γ. Define the equivariant Euler characteristic $\chi_\Gamma(X)$ for such X by setting:

$$\chi_\Gamma(X) = \sum_{\sigma \in \mathfrak{C}} (-1)^{\dim(\sigma)} \chi(\Gamma_\sigma),$$

where $\mathfrak{C}$ is a set of representatives for the cells of X mod Γ. Note that $\chi(\Gamma) = \chi_\Gamma(\text{point})$. Thus, the equivariant Euler characteristic can be viewed as a generalization of $\chi(\cdot)$.

Proposition 6.25 (1). *If $\chi_\Gamma(X)$ is defined and Γ' is a subgroup of Γ with finite index, then $\chi_{\Gamma'}(X)$ is defined, and*

$$\chi_{\Gamma'}(X) = [\Gamma : \Gamma']\, \chi_\Gamma(X).$$

(2). *If $\chi_\Gamma(X)$ is defined and each Γ_σ is torsion-free, then the equivariant homology $H_*^\Gamma(X)$ is finitely generated, and*

$$\chi_\Gamma(X) = \chi_\Gamma^{\sim}(X) \equiv \sum_i (-1)^i \mathrm{rank}_{\mathbb{Z}} H_i^\Gamma(X).$$

(3). *Suppose that X is a contractible Γ-complex such that $\chi_\Gamma(X)$ is defined. If Γ is virtually torsion-free, then Γ is of finite homological type and $\chi(\Gamma) = \chi_\Gamma(X)$.*

Sketch of Proof. To prove (1), fix a cell σ of X. Consider the set $\Gamma\sigma$ of cells which are equivalent to σ mod Γ. There is a 1-1 correspondence between $\Gamma\sigma$ and Γ/Γ_σ, so that it decomposes into finitely many Γ'-orbits, represented by the cells $\gamma\sigma$ where γ ranges over a set S of representatives for $\Gamma'\backslash\Gamma/\Gamma_\sigma$. Note that $\Gamma'_{\gamma\sigma} = \Gamma' \cap \Gamma_{\gamma\sigma} = \Gamma' \cap \gamma\Gamma_\sigma\gamma^{-1}$, which is conjugate in Γ to $\gamma^{-1}\Gamma'\gamma \cap \Gamma_\sigma$. Therefore,

$$\begin{aligned}\sum_{\gamma\in S}(-1)^{\dim(\gamma\sigma)}\chi(\Gamma'\cap\gamma\Gamma_\sigma\gamma^{-1}) &= (-1)^{\dim(\sigma)}\sum_{\gamma\in S}\chi(\gamma^{-1}\Gamma'\gamma\cap\Gamma_\sigma)\\ &= (-1)^{\dim(\sigma)}\sum_{\gamma\in S}[\Gamma_\sigma : \gamma^{-1}\Gamma'\gamma\cap\Gamma_\sigma]\,\chi(\Gamma_\sigma)\\ &= (-1)^{\dim(\sigma)}[\Gamma : \Gamma']\,\chi(\Gamma_\sigma).\end{aligned}$$

To prove (2), consider the equivariant homology spectral sequence:

$$E^1_{pq} = \oplus_{p\in\mathfrak{C}_p} H_q(\Gamma_\sigma, \mathbb{Z}_\sigma) \Rightarrow H^\Gamma_{p+q}(X),$$

where $\mathfrak{C}_p$ is a set of representatives for the p-cells of X mod Γ. Since Γ_σ is torsion-free and of finite homological type, $H_*(\Gamma_\sigma)$ is finitely generated. Hence $H_*^\Gamma(X)$ is finitely generated, and we can compute $\chi_\Gamma^{\sim}(X)$ from the E^1-term of the spectral sequence. Now assume that Γ_σ acts trivially on $\mathbb{Z}_\sigma$ for all σ, so that

$$\begin{aligned}\chi_\Gamma^{\sim}(X) &= \sum_{p,q}\sum_{\sigma\in\mathfrak{C}_p}(-1)^{p+q}\mathrm{rank}_{\mathbb{Z}} H_q(\Gamma_\sigma)\\ &= \sum_p\sum_{\sigma\in\mathfrak{C}_p}(-1)^p\chi(\Gamma_\sigma) = \chi_\Gamma(X).\end{aligned}$$

In the general case, note that all Euler characteristics can be computed from homology with $\mathbb{Z}_2$-coefficients. Since Γ_σ acts trivially on $(\mathbb{Z}_2)_\sigma$, the result follows from as above from the equivariant homology spectral sequence with $\mathbb{Z}_2$-coefficients.

It suffices to prove (3) when Γ is torsion-free. Since X is contractible, we have $H_*(\Gamma, M) \cong H_*^\Gamma(X, M)$ for any Γ-module M, and similarly for cohomology. A spectral sequence argument then shows that $H^i(\Gamma, M) = 0$ for $i \geq \dim X + \max\{\mathrm{cd}(\Gamma_\sigma)\} + 1$ and that $H_*(\Gamma, M)$ is finitely generated if M is. Hence Γ is of finite homological type. Moreover,

$$\chi(\Gamma) = \chi^\sim(\Gamma) = \chi_\Gamma^\sim(X) = \chi_\Gamma(X)$$

because Γ is torsion-free, $H_*(\Gamma) \cong H_*^\Gamma(X)$, and (2) in the statement. □

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