Explicit computing the real or complex operator norms for matrices

Explicit computing the real or complex operator norms for matrices

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Abstract

We mainly consider the real or complex operator norms for real or complex matrices on finite dimensional Hilbert spaces. We compute the real operator norm for real matrices directly and explicitly in terms of their entries, and similarly compute the complex operator norm for complex matrices. We also consider the real or complex operator norms of operators on Hilbert spaces, respectively viewed as the supremums of quadratic or Hermitian forms on the spaces.

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1 Introduction

We mainly consider the real or complex operator norms for real or complex matrices on finite dimensional Hilbert spaces. We compute the real operator norm for real matrices directly and explicitly in terms of their entries, and similarly compute the complex operator norm for complex matrices. For computing, we use the polar coordinate in the real or complex finite dimensional, Hilbert (or Euclidean) spaces. It turns out that computing the operator norms of operators is out of checking the spectrums (or in particular, eigenvalues) of operators in general, especially in the nilpotent or non-normal cases. But in fact, computing the operator norms of operators is equivalent to knowing the spectrums (or in particular, eigenvalues) of positive operators as products of operators with their adjoints. However, both of doings are difficult in a sense as computationally or theoretically.

In this section 1, as preliminaries we review several well known elementary general facts on operators on (finite or infinite dimensional) Hilbert spaces as well as real or complex matrices in Linear Algebra. Section 2 is devoted to the 2×2 matrix case. Section 3 is to the 3×3 matrix case. Section 4 is to the general $n \times n$ matrix case. The results obtained in Sections 2 to 4 with direct and explicit computations should be some interesting and useful (as well

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might be some new) as for references and conveniences, and be self-contained for readers. In Section 5, we consider the real or complex operator norms of operators on finite dimensional Hilbert spaces, respectively viewed as the supremums of quadratic or Hermitian forms on the spaces. In Section 6, we consider the case of infinite dimensional Hilbert spaces. The results in Sections 5 and 6 should be known to some extent, but some interesting as for the same reason as above.

Among not a few references, only cited are Bhatia [1] for Matrix Analysis, Conway [2] and Maeda [4] for Functional Analysis, Murphy [6] for C^*-algebras and Operator theory, and Satake [7] for Linear Algebra, and [5]. More references such as Fong, Radjavi, and Rosenthal [3] on related topics can be found in the references of [1] and [2]. Also may refer to [8] for the polar coordinate in real Euclidean spaces.

Let us begin with several elementary general facts on operators on Hilbert spaces. Let H be a finite dimensional, real or complex Hilbert (or Eucledean) space, that is, $\mathbb{R}^n$ or $\mathbb{C}^n$, over $\mathbb{R}$ of reals or $\mathbb{C}$ of complexes respectively. We denote by the same symbol H a separable infinite dimensional, real or complex Hilbert space. The real or complex **operator norm** for A a real or complex linear operator from H to H is defined to be the supremum:

$$\|A\| = \sup_{x\in H, \|x\|=1} \|Ax\| \equiv \sup\{\|Ax\| \mid x \in H, \|x\| = 1\},$$

with the norm squared $\|x\|^2 = \langle x, x\rangle$ the (real or complex) inner product of H. If $\|A\|$ is finite, then A is said to be bounded. We denote by $\mathbb{B}(H)$ the set of all bounded operators on H (either finite or infinite dimensional).

If $H = \mathbb{C}^n$, then $\langle x, x\rangle = \sum_{j=1}^n x_j\overline{x_j}$ for $x = (x_j) \in H$ with $\overline{x_j}$ the complex conjugate of $x_j \in \mathbb{C}$.

Lemma 1.1. *The operator norm $\|A\|$ for $A \in \mathbb{B}(H)$ is invariant under the orthogonal or unitary equivalence in the sense that*

$$\|VAV^t\| = \|A\| \quad \text{or} \quad \|UAU^*\| = \|A\|$$

for any orthogonal operator V or any unitary operator U on H, with transpose $V^t = V^{-1}$ and adjoint $U^* = \overline{U}^t = U^{-1}$ as inverses.

Proof. It is enough to show the unitary case as follows:

$$\begin{aligned}\|UAU^*\|^2 &= \sup_{x\in H, \|x\|=1} \|UAU^*x\|^2 \\ &= \sup_{x\in H, \|U^*x\|=1} \langle AU^*x, AU^*x\rangle = \sup_{y\in H, \|y\|=1} \langle Ay, Ay\rangle = \|A\|^2.\end{aligned}$$

□

Lemma 1.2. *The operator norm of either an orthogonal operator or a unitary operator on a Hilbert space H is one.*

Proof. Let I be the identity operator on H. Then it follows that

$$\begin{aligned}\|V\|^2 &= \sup_{x\in H, \|x\|=1} \langle Vx, Vx\rangle = \sup_{x\in H, \|x\|=1} \langle V^tVx, x\rangle \\ &= \sup_{x\in H, \|x\|=1} \langle Ix, Ix\rangle = \|I\|^2 = 1\end{aligned}$$

and similarly, $\|U\|^2 = \|I\|^2 = 1$. □

The **spectrum** of a bounded operator $T \in \mathbb{B}(H)$ is defined to be the set:

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T \notin GL(H)\},$$

where $GL(H)$ is the group of invertible operators in $\mathbb{B}(H)$.

If $H = \mathbb{C}^n$, then $\sigma(T)$ is equal to the set of all eigenvalues of T. But in general, the set of all eigenvalues of T is strictly contained in $\sigma(T)$, as well known.

The **spectral radius** of T is defined by

$$r(T) = \sup_{\lambda\in\sigma(T)} |\lambda| \equiv \sup\{|\lambda| \mid \lambda \in \sigma(T)\}.$$

Lemma 1.3. *For any $T \in \mathbb{B}(H)$, we always have $r(T) \leq \|T\|$.*

Proof. Suppose that $|\mu| > \|T\|$ with $\mu \in \mathbb{C}$. Then $\mu I - T = \mu(I - \mu^{-1}T)$ with $\|\mu^{-1}T\| = |\mu|^{-1}\|T\| < 1$, and hence $\mu I - T \in GL(H)$. Thus $\mu \notin \sigma(T)$. □

Lemma 1.4. *If A is a diagonal operator of $\mathbb{B}(H)$ with eigenvalues α_j with respect to an orthogonal basis $\{e_j\}$ of H, then $\|A\| = \sup_j |\alpha_j| = r(A)$, with $\sigma(A)$ equal to $\overline{\{\alpha_j\}_j}$ the closure of the set $\{\alpha_j\}$.*

Proof. For $x = \sum_j x_je_j \in H$ with $x_j \in \mathbb{C}$, we have $Ax = \sum_j x_j\alpha_je_j$. Thus,

$$\|Ax\|^2 = \sum_j |x_j\alpha_j|^2 \leq \sup_j |\alpha_j|^2 \sum_j |x_j|^2 = \sup_j |\alpha_j|^2\|x\|^2,$$

and hence $\|A\| \leq \sup_j |\alpha_j|$. On the other hand, $\|Ae_j\| = \|\alpha_je_j\| = |\alpha_j|$. Thus, $|\alpha_j| \leq \|A\|$ for any j.

Also, since $\sigma(A)$ is compact, it contains the closure $\overline{\{\alpha_j\}_j}$.

Suppose now that $\lambda \notin \overline{\{\alpha_j\}_j}$. Then there is an open ball $U(\lambda, \delta) = \{z \in \mathbb{C} \mid |z - \lambda| < \delta\}$ such that $U(\lambda, \delta) \cap \{\alpha_j\}_j = \emptyset$. Then for $\sum_j x_je_j \in H$,

$$(\lambda I - A)(\sum_j x_je_j) = \sum_j (\lambda - \alpha_j)x_je_j$$

a diagonal operator with $|\lambda - \alpha_j| \geq \delta > 0$ and $|\lambda - \alpha_j| \leq |\lambda| + \sup_j |\alpha_j|$. Hence $\lambda I - A$ is invertible in $\mathbb{B}(H)$. Thus, $\lambda \notin \sigma(A)$. Thereore, $\sigma(A) \subset \overline{\{\alpha_j\}_j}$. □

Lemma 1.5. *Let A be a normal $n \times n$ complex matrix on $H = \mathbb{C}^n$, with $AA^* = A^*A$, then $\|A\| = r(A)$, and is equal to the maximum of the set of absolute values of eigenvalues of A.*

Proof. It is known that a normal A is diagonalizable, i.e., there is a unitary U on H such that UAU^* is diagonal, and the set of eigenvalues of UAU^* is the same as that of A. □

Remark. It is known by [6, Theorem 2.4.4] that a compact normal operator in $\mathbb{B}(H)$ is diagonalizable, but normal operators of $\mathbb{B}(H)$ are not necessarily diagonalizable. For instance, the bilateral shift S defined as $Se_j = e_{j+1}$ for $j \in \mathbb{Z}$ of integers with (e_j) an orthogonal basis for H infinite dimensional is unitary, but it has no eigenvalues. ◀

We denote by $M_n(\mathbb{C})$ the set of all $n \times n$ matices over $\mathbb{C}$ and by $M_n(\mathbb{R})$ the set of all $n \times n$ matices over $\mathbb{R}$.

There are several useful **facts** from **Linear Algebra** as follows (cf. [7]).

(**U1**) For any $A \in M_n(\mathbb{C})$, there is a **unitary** $U \in M_n(\mathbb{C})$ such that U^*AU is upper triangular.

(**U2**) For any $A \in M_n(\mathbb{C})$, A is normal as $AA^* = A^*A$ if and only if A is diagonalizable as U^*AU by a unitary U.

(**U3**) For any self-adjoint $A \in M_n(\mathbb{C})$ with $A^* = A$, there is a unitary $U \in M_n(\mathbb{C})$ such that U^*AU is diagonal with real entries.

(**U4**) For any (**real**) symmetric $A \in M_n(\mathbb{R})$ with $A^t = A$, there is an **orthogonal** $V \in M_n(\mathbb{R})$ such that V^tAV is diagonal with real entries.

Remark. It follows from those facts that computing the operator norm for matrices in $M_n(\mathbb{C})$ is reduced to doing those for upper triangular or diagonal matrices in $M_n(\mathbb{C})$. For A as in the facts (U2) to (U4), $\|A\|$ is computable by Lemmas 1.1,1.4, and 1.5. ◀

(**U5**) For any anti-symmetric $A \in M_n(\mathbb{R})$ with $A^t = -A$, there is an orthogonal $V \in M_n(\mathbb{R})$ such that

$$V^tAV = (\oplus_{j=1}^{k_1} 0) \oplus \left(\oplus_{j=1}^{k_2} \begin{pmatrix} 0 & b_j \\ -b_j & 0 \end{pmatrix}\right)$$

the diagonal sum of 1×1 or 2×2 matrices over $\mathbb{R}$, for some $k_1, k_2 \geq 0$ with $k_1 + 2k_2 = n$.

(**U6**) For any orthogonal $A \in M_n(\mathbb{R})$ with $A^t = A^{-1}$, there is an orthogonal $V \in M_n(\mathbb{R})$ such that

$$V^tAV = (\oplus_{j=1}^{k_1} 1) \oplus (\oplus_{j=1}^{k_2} - 1) \left(\oplus_{j=1}^{k_3} \begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix}\right)$$

the diagonal sum of 1×1 or 2×2 matrices over $\mathbb{R}$, for some $k_1, k_2, k_3 \geq 0$ with $k_1 + k_2 + 2k_3 = n$.

Lemma 1.6. *If $A \in \mathbb{B}(H)$ is a diagonal sum $A_1 \oplus A_2$ with respect to the direct sum $H = H_1 \oplus H_2$ of Hilbert spaces, then $\|A\| = \max\{\|A_1\|, \|A_2\|\}$.*

Proof. For $x = x_1 \oplus x_2 \in H_1 \oplus H_2$, we have $\|A_j x_j\| = \|Ax_j\| \leq \|A\|\|x_j\|$ for $j = 1, 2$. Thus $\|A_j\| \leq \|A\|$.

Conversely, by Phthagoras' theorem for Hilbert spaces, $\|A(x_1 \oplus x_2)\|^2 = \|A_1x_1\|^2 + \|A_2x_2\|^2 \leq \max\{\|A_1\|^2, \|A_2\|^2\}(\|x_1\|^2 + \|x_2\|^2)$ with $\|x_1 \oplus x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$. Hence, $\|A\| \leq \max\{\|A_1\|, \|A_2\|\}$. □

Lemma 1.7. *Let A and B be bounded operators on a Hilbert space H. Then $\|AB\| \leq \|A\|\|B\|$.*

Proof. We have $\|ABx\| \leq \|A\|\|Bx\|$ for any $x \in H$, so that

$$\|AB\| = \sup_{x \in H, \|x\|=1} \|ABx\| \leq \|A\| \sup_{x \in H, \|x\|=1} \|Bx\| = \|A\|\|B\|.$$

□

As for the similar equivalence for operators,

Corollary 1.8. *For bounded operators $A, P, P^{-1} \in \mathbb{B}(H)$, we have $\|P^{-1}AP\| \leq \|P^{-1}\|\|A\|\|P\|$ and $\|A\| \leq \|P\|\|P^{-1}AP\|\|P^{-1}\|$. In particular, $1 \leq \|P^{-1}\|\|P\|$.*

(S1) For any $A \in M_n(\mathbb{C})$, there is an invertible $P \in M_n(\mathbb{C})$ such that

$$P^{-1}AP = \oplus_{j=1}^k J_{n_j}(\alpha_j) = \oplus_{j=1}^k \begin{pmatrix} \alpha_j & 1 & & 0 \\ 0 & \alpha_j & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \alpha_j \end{pmatrix} \quad \text{(as similarity)}$$

the diagonal sum of Jordan block $n_j \times n_j$ matrices $J_{n_j}(\alpha_j)$ for some $k \geq 1$, with $n_1 + \cdots + n_k = n$ and each $1 \leq n_j \leq n$.

2 The 2×2 matrix case

Let A be a 2×2 matrix over reals (or $\mathbb{R}$), that is,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}.$$

The **real** operator norm for A on the real 2-dimensional Hilbert (or Euclidean) space $\mathbb{R}^2$ is defined by

$$\|A\| = \sup_{x \in \mathbb{R}^2, \|x\|=1} \|Ax\|$$

with $\|x\| = \sqrt{x_1^2 + x_2^2}$ for $x = (x_1, x_2)^t \in \mathbb{R}^2$, where $(\cdot)^t$ means the vector transpose. Since the real 1-dimensional unit sphere S^1 in $\mathbb{R}^2$ is given by the equation $\cos^2\theta + \sin^2\theta = 1$ for $\theta \in [0, 2\pi]$ by trigonometric functions, with $S^1 \ni x = (\cos\theta, \sin\theta)^t$ the polar coordinate, we compute

$$\begin{aligned}\|A\|^2 &= \sup_{\theta\in[0,2\pi]} \|A(\cos\theta, \sin\theta)^t\|^2 \\ &= \sup_{\theta\in[0,2\pi]} \|(a\cos\theta + b\sin\theta, c\cos\theta + d\sin\theta)^t\|^2\end{aligned}$$

and moreover,

$$\begin{aligned}&\|(a\cos\theta + b\sin\theta, c\cos\theta + d\sin\theta)^t\|^2 \\ &= (a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2 \\ &= (a^2+c^2)\cos^2\theta + (b^2+d^2)\sin^2\theta + 2(ab+cd)\cos\theta\sin\theta \\ &= b^2 + d^2 + (a^2+c^2-b^2-d^2)\cos^2\theta + (ab+cd)\sin 2\theta \\ &= b^2 + d^2 + (a^2+c^2-b^2-d^2)(\frac{1+\cos 2\theta}{2}) + (ab+cd)\sin 2\theta \\ &= \frac{1}{2}(a^2+b^2+c^2+d^2) + \frac{1}{2}(a^2+c^2-b^2-d^2)\cos 2\theta + (ab+cd)\sin 2\theta,\end{aligned}$$

and furthermore, by the additive theorem for trigonometric functions,

$$\begin{aligned}&\frac{1}{2}(a^2+c^2-b^2-d^2)\cos 2\theta + (ab+cd)\sin 2\theta \\ &\equiv \alpha\sin 2\theta + \beta\cos 2\theta = \sqrt{\alpha^2+\beta^2}(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\sin 2\theta + \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\cos 2\theta) \\ &= \sqrt{\alpha^2+\beta^2}\sin(2\theta+\gamma)\end{aligned}$$

where $\cos\gamma = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$ and $\sin\gamma = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$.

It then follows that

Proposition 2.1. *The square of the real operator norm for a 2×2 matrix A with $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$ entries as a, b, c, d in $\mathbb{R}$ is given by*

$$\|A\|^2 = \frac{1}{2}(a^2+b^2+c^2+d^2) + \sqrt{\alpha^2+\beta^2},$$

where $\alpha = ab+cd$ and $\beta = \frac{1}{2}(a^2+c^2-b^2-d^2)$, with

$$\sqrt{\alpha^2+\beta^2} = \frac{1}{2}\sqrt{((a^2+d^2)+(b^2+c^2))^2 - 4(a^2d^2+b^2c^2) + 8abcd}.$$

The supremum for the operator norm is attained at $(\cos\theta, \sin\theta)$ with $2\theta+\gamma = \frac{\pi}{2}$ (mod 2π).

In particular,

Corollary 2.2. *If A is an upper triangular 2×2 matrix with $c = 0$, then*

$$\|A\|^2 = \frac{1}{2}(a^2+b^2+d^2) + \sqrt{a^2b^2+\beta^2},$$

where $\beta = \frac{1}{2}(a^2 - b^2 - d^2)$, *with*

$$\sqrt{a^2b^2 + \beta^2} = \frac{1}{2}\sqrt{((a^2 + d^2) + b^2)^2 - 4a^2d^2}.$$

If A is a nilpotent upper triangular matrix with $a = c = d = 0$, then

$$\|A\|^2 = b^2 \geq 0 = r(A).$$

If A is a diagonal matrix with $b = c = 0$, then

$$\|A\|^2 = \frac{1}{2}(a^2 + d^2) + \frac{1}{2}|a^2 - d^2| = \max\{a^2, d^2\} = r(A)^2.$$

Corollary 2.3. *If A is symmetric as the transpose $A^t = A$ with $b = c$, then*

$$\|A\|^2 = \frac{1}{2}(a^2 + 2b^2 + d^2) + \sqrt{\alpha^2 + \beta^2} = r(A)^2,$$

where $\alpha = (a + d)b$ *and* $\beta = \frac{1}{2}(a^2 - d^2)$, *with*

$$\sqrt{\alpha^2 + \beta^2} = \frac{1}{2}\sqrt{((a^2 + d^2) + 2b^2)^2 - 4(a^2d^2 + b^4) + 8ab^2d}.$$

If A is anti-symmetric as $A^t = -A$ with $a = d = 0$ and $c = -b$, then

$$\|A\|^2 = b^2 = r(A)^2.$$

Moreover, if $A \in M_n(\mathbb{R})$ is anti-symmetric and V^tAV has the form as in (U5) *in Section* 1, *then* $\|A\| = \max_{1\leq j\leq k_2} |b_j|$.

Corollary 2.4. *Suppose that A is an invertible 2×2 real matrix. Then*

$$\|A^{-1}\|^2 = \frac{1}{2(\det A)^2}(a^2 + b^2 + c^2 + d^2) + \frac{1}{|\det A|}\sqrt{(\alpha^{\sim})^2 + (\beta^{\sim})^2},$$

where $\alpha^{\sim} = -(ac + bd)$ *and* $\beta^{\sim} = \frac{1}{2}(d^2 + c^2 - b^2 - a^2)$.

Proof. Note that the inverse A^{-1} of A is given as

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

with $\det A = ad - bc$ the determinant of A. Then use Proposition 2.1. □

Example 2.5. Let

$$A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad A^{-1} = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}$$

for $b \in \mathbb{R}$. Then

$$\|A\|^2 = 1 + \frac{1}{2}b^2 + \frac{1}{2}\sqrt{2b^2 + b^4} = \|A^{-1}\|^2.$$

It follows that

$$\|A\|\|A^{-1}\| = 1 + \frac{1}{2}b^2 + \frac{1}{2}\sqrt{2b^2 + b^4}.$$

Let

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{with} \quad A^{-1} = \begin{pmatrix} \cos\theta & -sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for $\theta \in [0, 2\pi]$. Then

$$\begin{aligned} \|A\|^2 &= 1 + \frac{1}{2}\sqrt{4 - 4(\cos^4\theta + \sin^4\theta) - 8\cos^2\theta\sin^2\theta} \\ &= 1 + \frac{1}{2}\sqrt{4 - 4(\cos^2\theta + \sin^2\theta)^2} = 1 = \|A^{-1}\|^2. \end{aligned}$$

Moreover, if $A \in M_n(\mathbb{R})$ is orthogonal and V^tAV has the form as in (U6) in Section 1, then $\|A\| = 1$.

Let

$$J_2(\alpha) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \in M_2(\mathbb{R}).$$

Then

$$\|J_2(\alpha)\|^2 = \alpha^2 + \frac{1}{2} + \sqrt{\alpha^2 + \frac{1}{4}} \geq \alpha^2 = r(J_2(\alpha))^2.$$

In particular, we have $\|J_2(0)\| = 1 > 0 = r(J_2(0))$. ◀

Lemma 2.6. *An upper triangular 2×2 matrix A over reals or complexes is normal if and only if A is diagonal.*

Proof. We compute

$$AA^t = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} a & 0 \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & bd \\ bd & d^2 \end{pmatrix}$$

and

$$A^tA = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a^2 & ab \\ ab & d^2 \end{pmatrix}.$$

That $AA^t = A^tA$ implies that $b = 0$ from $(1, 1)$ entry. The complex case follows similarly. □

We now consider the **complex** case. Let A be a 2×2 matrix over $\mathbb{C}$, that is,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}.$$

The (complex) operator norm for A on the complex 2-dimensional Hilbert (or Euclidean) space $\mathbb{C}^2$ is defined by

$$\|A\| = \sup_{z \in \mathbb{C}^2, \|z\|=1} \|Az\|$$

with $\|z\| = \sqrt{|z_1|^2 + |z_2|^2}$ for $z = (z_1, z_2)^t = (|z_1|e^{i\rho_1}, |z_2|e^{i\rho_2})^t \in \mathbb{C}^2$ the polar decomposition by polar coordinate for $\rho_1, \rho_2 \in [0, 2\pi]$, where $i^2 = -1$. Since the unit sphere in $\mathbb{C}^2$ is given by the equation $|z_1|^2 + |z_2|^2 = 1$, we set $(|z_1|, |z_2|) = (\cos\theta, \sin\theta)$ for $\theta \in [0, 2\pi]$, and we compute

$$\begin{aligned}\|A\|^2 &= \sup_{z\in\mathbb{C}^2, \|z\|=1} \|A(z_1, z_2)^t\|^2 \\ &= \sup_{z\in\mathbb{C}^2, \|z\|=1} \|(az_1 + bz_2, cz_1 + dz_2)^t\|^2\end{aligned}$$

and moreover,

$$\begin{aligned}&\|(az_1 + bz_2, cz_1 + dz_2)^t\|^2 \\ &= (|a|^2 + |c|^2)|z_1|^2 + 2\mathrm{Re}((a\overline{b} + c\overline{d})z_1\overline{z_2}) + (|b|^2 + |d|^2)|z_2|^2 \\ &= (|a|^2 + |c|^2)\cos^2\theta + 2\mathrm{Re}((a\overline{b} + c\overline{d})e^{i(\rho_1-\rho_2)})\cos\theta\sin\theta + (|b|^2 + |d|^2)\sin^2\theta,\end{aligned}$$

where $\mathrm{Re}(\cdot)$ means the real part and $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$.

By using the explicit computation in the real case and Proposition 2.1, we obtain

Proposition 2.7. *The square of the complex operator norm for a* 2×2 *matrix* A *with entries* a, b, c, d *in* $\mathbb{C}$ *is given by*

$$\|A\|^2 = \frac{1}{2}(|a|^2 + |b|^2 + |c|^2 + |d|^2) + \sup_{\rho_1,\rho_2\in[0,2\pi]} \sqrt{\alpha(\rho_1, \rho_2)^2 + \beta^2},$$

where $\alpha(\rho_1, \rho_2) = \mathrm{Re}((a\overline{b} + c\overline{d})e^{i(\rho_1-\rho_2)})$ *and* $\beta = \frac{1}{2}(|a|^2 + |c|^2 - |b|^2 - |d|^2)$*, and moreover, the supremum of the second term above is given by*

$$\sup_{\rho_1,\rho_2\in[0,2\pi]} \sqrt{\alpha(\rho_1, \rho_2)^2 + \beta^2} = \sqrt{\alpha^2 + \beta^2}, \quad \textit{where } \alpha = |a\overline{b} + c\overline{d}|.$$

Furthermore, this formula contains that formula of Proposition 2.1.

Proof. Just note that

$$|\mathrm{Re}((a\overline{b} + c\overline{d})e^{i(\rho_1-\rho_2)})| \leq |a\overline{b} + c\overline{d}|,$$

so that

$$\sup_{\rho_1,\rho_2\in[0,2\pi]} |\mathrm{Re}((a\overline{b} + c\overline{d})e^{i(\rho_1-\rho_2)})| = |a\overline{b} + c\overline{d}|.$$

□

3 The 3×3 matrix case

Let A be a nilpotent upper triangular 3×3 matrix over reals as

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, \quad a_{ij} \in \mathbb{R} \quad (i \leq j).$$

We compute the **real** operator norm of A via the polar coordinate for the real 2-dimensional unit sphere S^2 in $\mathbb{R}^3$ as:

$$\begin{aligned}\|A\|^2 &= \sup_{x\in\mathbb{R}^3, \|x\|=1} \|Ax\|^2 \\ &= \sup_{\theta\in[0,\pi],\varphi\in[0,2\pi]} \|A(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^t\|^2\end{aligned}$$

and moreover,

$$\begin{aligned}&\|A(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^t\|^2 \\ &= (a_{12}\sin\theta\sin\varphi + a_{13}\cos\theta)^2 + a_{23}^2\cos^2\theta \\ &= a_{12}^2\sin^2\theta\sin^2\varphi + (a_{13}^2 + a_{23}^2)\cos^2\theta + 2a_{12}a_{13}\sin\theta\cos\theta\sin\varphi \\ &= a_{12}^2\sin^2\varphi + (-a_{12}^2\sin^2\varphi + a_{13}^2 + a_{23}^2)\frac{1}{2}(1+\cos 2\theta) + a_{12}a_{13}\sin 2\theta\sin\varphi \\ &= \frac{1}{2}(a_{12}^2\sin^2\varphi + a_{13}^2 + a_{23}^2) + \frac{1}{2}(-a_{12}^2\sin^2\varphi + a_{13}^2 + a_{23}^2)\cos 2\theta + a_{12}a_{13}\sin\varphi\sin 2\theta,\end{aligned}$$

and furthermore, by the additive theorem for trigonometric functions,

$$\begin{aligned}&a_{12}a_{13}\sin\varphi\sin 2\theta + \frac{1}{2}(-a_{12}^2\sin^2\varphi + a_{13}^2 + a_{23}^2)\cos 2\theta \\ &\equiv \alpha(\varphi)\sin 2\theta + \beta(\varphi)\cos 2\theta = \sqrt{\alpha(\varphi)^2+\beta(\varphi)^2}\sin(2\theta + \gamma(\varphi)),\end{aligned}$$

where

$$\begin{aligned}\cos\gamma(\varphi) &= \frac{1}{\sqrt{\alpha(\varphi)^2+\beta(\varphi)^2}}a_{12}a_{13}\sin\varphi, \quad \text{and} \\ \sin\gamma(\varphi) &= \frac{1}{\sqrt{\alpha(\varphi)^2+\beta(\varphi)^2}}\frac{1}{2}(-a_{12}^2\sin^2\varphi + a_{13}^2 + a_{23}^2).\end{aligned}$$

We then compute $\alpha(\varphi)^2 + \beta(\varphi)^2$

$$\begin{aligned}&= a_{12}^2a_{13}^2\sin^2\varphi + \frac{1}{4}((-a_{12}^2\sin^2\varphi + a_{13}^2)^2 + 2(-a_{12}^2\sin^2\varphi + a_{13}^2)a_{23}^2 + a_{23}^4) \\ &= \frac{1}{4}(a_{12}^4\sin^4\varphi + a_{13}^4 + a_{23}^4 + 2a_{12}^2(a_{13}^2 - a_{23}^2)\sin^2\varphi + 2a_{13}^2a_{23}^2).\end{aligned}$$

Since θ and φ are independent, it follows that

Proposition 3.1. *If $A = (a_{ij})$ is a nilpotent upper triangular 3×3 matrix over $\mathbb{R}$ as above, then with $\varphi = \frac{\pi}{2}$,*

$$\|A\|^2 = \frac{1}{2}(a_{12}^2 + a_{13}^2 + a_{23}^2) + \sqrt{\alpha(\frac{\pi}{2})^2 + \beta(\frac{\pi}{2})^2} \geq 0 = r(A),$$

with $\alpha(\frac{\pi}{2}) = a_{12}a_{13}$ and $\beta(\frac{\pi}{2}) = \frac{1}{2}(-a_{12}^2 + a_{13}^2 + a_{23}^2)$, and with

$$\alpha(\frac{\pi}{2})^2 + \beta(\frac{\pi}{2})^2 = \frac{1}{4}(a_{12}^4 + a_{13}^4 + a_{23}^4 + 2a_{12}^2(a_{13}^2 - a_{23}^2) + 2a_{13}^2a_{23}^2).$$

As for the **complex** case, note that for the polar decomposition

$$z = (z_1, z_2, z_3)^t = (|z_1|e^{i\rho_1}, |z_2|e^{i\rho_2}, |z_3|e^{i\rho_3})^t \in \mathbb{C}^3$$

with $\rho_1, \rho_2, \rho_3 \in [0, 2\pi]$, the condition $\|z\| = 1$ is equivalent to the condition $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$. Then we use the polar coordinate for $(|z_1|, |z_2|, |z_3|)^t$ as

$$(|z_1|, |z_2|, |z_3|) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

for $\theta \in [0, \frac{\pi}{2}]$ and $\varphi \in [0, \frac{\pi}{2}]$.

It follows from the similar computation as for Propositions 2.7 and 3.1 that

Proposition 3.2. *If $A = (a_{ij})$ is a nilpotent upper triangular 3×3 matrix over $\mathbb{C}$ as above, then*

$$\|A\|^2 = \frac{1}{2}(|a_{12}|^2 + |a_{13}|^2 + |a_{23}|^2) + \sqrt{\alpha^2 + \beta^2} \geq 0 = r(A),$$

with $\alpha = |\overline{a_{12}}a_{13}|$ and $\beta = \frac{1}{2}(-|a_{12}|^2 + |a_{13}|^2 + |a_{23}|^2)$.

In the **real** case, more generally, let A be an upper triangular 3×3 matrix over reals as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \quad a_{ij} \in \mathbb{R} \quad (i \leq j).$$

We compute the real operator norm of A via the polar coordinate as:

$$\begin{aligned} \|A\|^2 &= \sup_{x \in \mathbb{R}^3, \|x\|=1} \|Ax\|^2 \\ &= \sup_{\theta \in [0,\pi], \varphi \in [0,2\pi]} \|A(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^t\|^2 \end{aligned}$$

and moreover,

$$\begin{aligned}
&\|A(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^t\|^2 \\
&= (a_{11}\sin\theta\cos\varphi + a_{12}\sin\theta\sin\varphi + a_{13}\cos\theta)^2 \\
&\quad + (a_{22}\sin\theta\sin\varphi + a_{23}\cos\theta)^2 + a_{33}^2\cos^2\theta \\
&= a_{11}^2\sin^2\theta\cos^2\varphi + (a_{12}^2 + a_{22}^2)\sin^2\theta\sin^2\varphi + (a_{13}^2 + a_{23}^2 + a_{33}^2)\cos^2\theta \\
&\quad + 2a_{11}a_{12}\sin^2\theta\cos\varphi\sin\varphi + 2a_{11}a_{13}\sin\theta\cos\theta\cos\varphi \\
&\quad + 2(a_{12}a_{13} + a_{22}a_{23})\sin\theta\cos\theta\sin\varphi \\
&= a_{11}^2\cos^2\varphi + (a_{12}^2 + a_{22}^2)\sin^2\varphi \\
&+ [-a_{11}^2\cos^2\varphi - (a_{12}^2 + a_{22}^2)\sin^2\varphi + a_{13}^2 + a_{23}^2 + a_{33}^2 - 2a_{11}a_{12}\cos\varphi\sin\varphi]\cos^2\theta \\
&\quad + 2a_{11}a_{12}\cos\varphi\sin\varphi + [a_{11}a_{13}\cos\varphi + (a_{12}a_{13} + a_{22}a_{23})\sin\varphi]\sin 2\theta \\
&= (a_{12}^2 + a_{22}^2) + (a_{11}^2 - a_{12}^2 - a_{22}^2)\cos^2\varphi + a_{11}a_{12}\sin 2\varphi \\
&+ \frac{1}{2}[(-a_{11}^2 + a_{12}^2 + a_{22}^2)\cos^2\varphi - (a_{12}^2 + a_{22}^2) + a_{13}^2 + a_{23}^2 + a_{33}^2 - a_{11}a_{12}\sin 2\varphi] \\
&+ \frac{1}{2}[(-a_{11}^2 + a_{12}^2 + a_{22}^2)\cos^2\varphi - (a_{12}^2 + a_{22}^2) + a_{13}^2 + a_{23}^2 + a_{33}^2 \\
&\quad - a_{11}a_{12}\sin 2\varphi]\cos 2\theta + [a_{11}a_{13}\cos\varphi + (a_{12}a_{13} + a_{22}a_{23})\sin\varphi]\sin 2\theta,
\end{aligned}$$

and the sum of first two of four lines of the last part of the equality above is equal to

$$\begin{aligned}
&\frac{1}{2}[(a_{12}^2 + a_{13}^2 + a_{22}^2 + a_{23}^2 + a_{33}^2) + (a_{11}^2 - a_{12}^2 - a_{22}^2)\cos^2\varphi + a_{11}a_{12}\sin 2\varphi] \\
&= \frac{1}{2}[(a_{13}^2 + a_{23}^2 + a_{33}^2) + \frac{1}{2}(a_{11}^2 + a_{12}^2 + a_{22}^2) \\
&\quad + \frac{1}{2}(a_{11}^2 - a_{12}^2 - a_{22}^2)\cos 2\varphi + a_{11}a_{12}\sin 2\varphi] \\
&= \frac{1}{2}(a_{13}^2 + a_{23}^2 + a_{33}^2) + \frac{1}{4}(a_{11}^2 + a_{12}^2 + a_{22}^2) \\
&\quad + \frac{1}{4}(a_{11}^2 - a_{12}^2 - a_{22}^2)\cos 2\varphi + \frac{1}{2}a_{11}a_{12}\sin 2\varphi
\end{aligned}$$

with

$$\begin{aligned}
&\frac{1}{4}(a_{11}^2 - a_{12}^2 - a_{22}^2)\cos 2\varphi + \frac{1}{2}a_{11}a_{12}\sin 2\varphi \equiv \alpha\sin 2\varphi + \beta\cos 2\varphi \\
&= \sqrt{\alpha^2 + \beta^2}\sin(2\varphi + \gamma),
\end{aligned}$$

where $\cos\gamma = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$ and $\sin\gamma = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$, and the sum of last two of four lines of the last part of the equality above is denoted by

$$\alpha(\varphi)\sin 2\theta + \beta(\varphi)\cos 2\theta = \sqrt{\alpha(\varphi)^2 + \beta(\varphi)^2}\sin(2\theta + \gamma(\varphi)),$$

where $\cos\gamma(\varphi) = \frac{\alpha(\varphi)}{\sqrt{\alpha(\varphi)^2+\beta(\varphi)^2}}$ and $\sin\gamma(\varphi) = \frac{\beta}{\sqrt{\alpha(\varphi)^2+\beta(\varphi)^2}}$. We then compute

$$\begin{aligned}
\alpha(\varphi) &\equiv a_{11}a_{13}\cos\varphi + (a_{12}a_{13} + a_{22}a_{23})\sin\varphi \\
&= \sqrt{a_{11}^2a_{13}^2 + (a_{12}a_{13} + a_{22}a_{23})^2}\sin(\varphi + \rho),
\end{aligned}$$

where

$$\begin{aligned}
\cos\rho &= \frac{a_{12}a_{13} + a_{22}a_{23}}{\sqrt{a_{11}^2a_{13}^2 + (a_{12}a_{13} + a_{22}a_{23})^2}}, \quad \text{and} \\
\sin\rho &= \frac{a_{11}a_{13}}{\sqrt{a_{11}^2a_{13}^2 + (a_{12}a_{13} + a_{22}a_{23})^2}},
\end{aligned}$$

and we compute $\beta(\varphi)$

$$\begin{aligned}
&\equiv \frac{1}{2}[(-a_{11}^2 + a_{12}^2 + a_{22}^2)\cos^2\varphi - (a_{12}^2 + a_{22}^2) + a_{13}^2 + a_{23}^2 + a_{33}^2 - a_{11}a_{12}\sin 2\varphi] \\
&= \frac{1}{2}[a_{13}^2 + a_{23}^2 + a_{33}^2 - \frac{1}{2}(a_{11}^2 + a_{12}^2 + a_{22}^2) \\
&\quad + \frac{1}{2}(-a_{11}^2 + a_{12}^2 + a_{22}^2)\cos 2\varphi - a_{11}a_{12}\sin 2\varphi] \\
&= \frac{1}{2}(a_{13}^2 + a_{23}^2 + a_{33}^2) - \frac{1}{4}(a_{11}^2 + a_{12}^2 + a_{22}^2) \\
&\quad + \frac{1}{4}(-a_{11}^2 + a_{12}^2 + a_{22}^2)\cos 2\varphi - \frac{1}{2}a_{11}a_{12}\sin 2\varphi,
\end{aligned}$$

with

$$\frac{1}{4}(-a_{11}^2 + a_{12}^2 + a_{22}^2)\cos 2\varphi - \frac{1}{2}a_{11}a_{12}\sin 2\varphi = -(\alpha\sin 2\varphi + \beta\cos 2\varphi)$$
$$= -\sqrt{\alpha^2+\beta^2}\sin(2\varphi+\gamma),$$

where α, β, and γ are the same as above.

Since θ and φ are independent, it then follows that

Proposition 3.3. *If $A = (a_{ij})$ is an upper triangular 3×3 matrix over $\mathbb{R}$ as above, then*

$$\|A\|^2 = \frac{1}{2}(a_{13}^2 + a_{23}^2 + a_{33}^2) + \frac{1}{4}(a_{11}^2 + a_{12}^2 + a_{22}^2) +$$
$$\sup_{\varphi\in[0,2\pi]} \{\sqrt{\alpha^2+\beta^2}\sin(2\varphi+\gamma) + \sqrt{\alpha(\varphi)^2 + \beta(\varphi)^2}\},$$

where $\cos\gamma = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$ *and* $\sin\gamma = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$, *and*

$$\alpha = \frac{1}{2}a_{11}a_{12} \quad \textit{and} \quad \beta = \frac{1}{4}(a_{11}^2 - a_{12}^2 - a_{22}^2)$$

and

$$\alpha(\varphi) = \sqrt{a_{11}^2 a_{13}^2 + (a_{12}a_{13} + a_{22}a_{23})^2}\sin(\varphi+\rho),$$

where

$$\cos\rho = \frac{a_{12}a_{13} + a_{22}a_{23}}{\sqrt{a_{11}^2 a_{13}^2 + (a_{12}a_{13} + a_{22}a_{23})^2}}, \quad \textit{and}$$
$$\sin\rho = \frac{a_{11}a_{13}}{\sqrt{a_{11}^2 a_{13}^2 + (a_{12}a_{13} + a_{22}a_{23})^2}},$$

and

$$\beta(\varphi) = \frac{1}{2}(a_{13}^2 + a_{23}^2 + a_{33}^2) - \frac{1}{4}(a_{11}^2 + a_{12}^2 + a_{22}^2)$$
$$- \sqrt{\alpha^2+\beta^2}\sin(2\varphi+\gamma).$$

Similarly, we obtain

Proposition 3.4. *If $A = (a_{ij})$ is an upper triangular 3×3 matrix over $\mathbb{C}$ as above, then*

$$\|A\|^2 = \frac{1}{2}(|a_{13}|^2 + |a_{23}|^2 + |a_{33}|^2) + \frac{1}{4}(|a_{11}|^2 + |a_{12}|^2 + |a_{22}|^2) +$$
$$\sup_{\varphi\in[0,2\pi]} \{\sqrt{\alpha^2+\beta^2}\sin(2\varphi+\gamma) + \sqrt{\alpha(\varphi)^2 + \beta(\varphi)^2}\},$$

where $\cos\gamma = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$ *and* $\sin\gamma = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$, *and*

$$\alpha = \frac{1}{2}|\overline{a_{11}}a_{12}| \quad \textit{and} \quad \beta = \frac{1}{4}(|a_{11}|^2 - |a_{12}|^2 - |a_{22}|^2)$$

and

$$\alpha(\varphi) = \sqrt{|a_{11}|^2|a_{13}|^2 + (|\overline{a_{12}}a_{13}| + |\overline{a_{22}}a_{23}|)^2}\sin(\varphi+\rho),$$

where

$$\cos\rho = \frac{|\overline{a_{12}}a_{13}| + |\overline{a_{22}}a_{23}|}{\sqrt{|a_{11}a_{13}|^2 + (|\overline{a_{12}}a_{13}| + |\overline{a_{22}}a_{23}|)^2}}, \quad \textit{and}$$
$$\sin\rho = \frac{|a_{11}a_{13}|}{\sqrt{|a_{11}a_{13}|^2 + (|\overline{a_{12}}a_{13}| + |\overline{a_{22}}a_{23}|)^2}},$$

and

$$\beta(\varphi) = \frac{1}{2}(|a_{13}|^2 + |a_{23}|^2 + |a_{33}|^2) - \frac{1}{4}(|a_{11}|^2 + |a_{12}|^2 + |a_{22}|^2) - \sqrt{\alpha^2+\beta^2}\sin(2\varphi+\gamma).$$

Example 3.5. Define a real Jordan block 3×3 matrix as

$$J_3(\alpha) = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix} \in M_3(\mathbb{R}).$$

Then

$$\|J_3(\alpha)\|^2 = \frac{3}{4} + \alpha^2 + \sup_{\varphi\in[0,2\pi]}\{\sqrt{\frac{1}{4}\alpha^2 + \frac{1}{4^2}}\sin(2\varphi+\gamma) + \sqrt{\alpha^2\sin^2(\varphi+\rho) + \left(\frac{1}{4} - \sqrt{\frac{1}{4}\alpha^2 + \frac{1}{4^2}}\sin(2\varphi+\gamma)\right)^2}\}.$$

In particular, we have

$$\|J_3(0)\|^2 = \frac{3}{4} + \sup_{\varphi\in[0,2\pi]}\left\{\frac{1}{4}\sin(2\varphi+\frac{\pi}{2}) + \sqrt{\left(\frac{1}{4} - \frac{1}{4}\sin(2\varphi+\frac{\pi}{2})\right)^2}\right\} = \frac{3}{4} + \sup_{\varphi\in[0,2\pi]}\{\frac{1}{4}\} = 1.$$

We also deduce the following estimate:

$$\|J_3(\alpha)\|^2 \le \frac{3}{4} + \alpha^2 + \sqrt{\frac{1}{4}\alpha^2 + \frac{1}{4^2}} + \sqrt{\alpha^2 + \left(\frac{1}{4} + \sqrt{\frac{1}{4}\alpha^2 + \frac{1}{4^2}}\right)^2}.$$

On the other hand, we may compute the real operator norm of $J_3(\alpha)$ via polar decomposition in another way as:

$$\begin{aligned}\|J_3(\alpha)\|^2 &= \sup_{x\in\mathbb{R}^3,\|x\|=1} \|J_3(\alpha)x\|^2\\ &= \sup_{\theta_1\in[0,\pi],\theta_2\in[0,2\pi]} \|J_3(\alpha)(\cos\theta_1,\sin\theta_1\cos\theta_2,\sin\theta_1\sin\theta_2)^t\|^2\end{aligned}$$

and moreover,

$$\begin{aligned}&\|J_3(\alpha)(\cos\theta_1,\sin\theta_1\cos\theta_2,\sin\theta_1\sin\theta_2)^t\|^2\\ &= (\alpha\cos\theta_1+\sin\theta_1\cos\theta_2)^2+(\alpha\sin\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)^2+\alpha^2\sin^2\theta_1\sin^2\theta_2\\ &= \alpha^2+\sin^2\theta_1+2\alpha\sin\theta_1\cos\theta_1\cos\theta_2+2\alpha\sin^2\theta_1\sin\theta_2\cos\theta_2\\ &= \alpha^2+\frac{1}{2}+\alpha\sin 2\theta_2+(-\frac{1}{2}-\alpha\sin 2\theta_2)\cos 2\theta_1+\alpha\cos\theta_2\sin 2\theta_1\\ &= \alpha^2+\frac{1}{2}+\alpha\sin 2\theta_2+\sqrt{\alpha^2\cos^2\theta_2+(\frac{1}{2}+\alpha\sin 2\theta_2)^2}\sin(2\theta_1+\gamma),\end{aligned}$$

where

$$\begin{aligned}\cos\gamma &= \alpha\cos\theta_2(\alpha^2\cos^2\theta_2+(\frac{1}{2}+\alpha\sin 2\theta_2)^2)^{-\frac{1}{2}},\\ \sin\gamma &= (-\frac{1}{2}-\alpha\sin 2\theta_2)(\frac{1}{2}+\alpha\sin 2\theta_2)^2)^{-\frac{1}{2}}.\end{aligned}$$

Since θ_1 and θ_2 are independent, it follows that

$$\begin{aligned}&\sup_{\theta_1\in[0,\pi]} \|J_3(\alpha)(\cos\theta_1,\sin\theta_1\cos\theta_2,\sin\theta_1\sin\theta_2)^t\|^2\\ &= \alpha^2+\frac{1}{2}+\alpha\sin 2\theta_2+\sqrt{\frac{\alpha^2}{2}(1+\cos 2\theta_2)+(\frac{1}{2}+\alpha\sin 2\theta_2)^2}\end{aligned}$$

and hence that

$$\begin{aligned}&\|J_3(\alpha)\|^2 =\\ &\sup_{\theta_2\in[0,2\pi]}\left\{\alpha^2+\frac{1}{2}+\alpha\sin 2\theta_2+\sqrt{\frac{\alpha^2}{2}(1+\cos 2\theta_2)+(\frac{1}{2}+\alpha\sin 2\theta_2)^2}\right\}\\ &\equiv \sup_{\theta_2\in[0,2\pi]} f(\sin 2\theta_2),\end{aligned}$$

with $\cos 2\theta_2 = \pm\sqrt{1-\sin^2 2\theta_2}$, so that with $t=\sin 2\theta$, when $\cos 2\theta_2 \geq 0$,

$$f(t) = g_+(t) \equiv \alpha^2+\frac{1}{2}+\alpha t+\sqrt{\frac{\alpha^2}{2}(1+\sqrt{1-t^2})+(\frac{1}{2}+\alpha t^2)^2}$$

for $t\in[-1,1]$, and when $\cos 2\theta_2 \leq 0$,

$$f(t) = g_-(t) \equiv \alpha^2+\frac{1}{2}+\alpha t+\sqrt{\frac{\alpha^2}{2}(1-\sqrt{1-t^2})+(\frac{1}{2}+\alpha t^2)^2}$$

for $t \in [-1,1]$. Therefore,

$$\|J_3(\alpha)\|^2 = \max\{\sup_{t\in[-1,1]} g_+(t), \sup_{t\in[-1,1]} g_-(t)\}.$$

But it seems to be difficult to determine the maximum in terms of α. ◀

We next consider the **full real** 3×3 case by encouraged by the success in the non-full partial cases above. Let A be a full 3×3 matrix over reals as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad a_{ij} \in \mathbb{R}.$$

We compute the real operator norm of A via the polar coordinate as:

$$\begin{aligned} \|A\|^2 &= \sup_{x\in\mathbb{R}^3, \|x\|=1} \|Ax\|^2 \\ &= \sup_{\theta\in[0,\pi], \varphi\in[0,2\pi]} \|A(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^t\|^2 \end{aligned}$$

and moreover,

$$\begin{aligned} &\|A(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^t\|^2 \\ &= (a_{11}\sin\theta\cos\varphi + a_{12}\sin\theta\sin\varphi + a_{13}\cos\theta)^2 \\ &\quad + (a_{21}\sin\theta\cos\varphi + a_{22}\sin\theta\sin\varphi + a_{23}\cos\theta)^2 \\ &\quad + (a_{31}\sin\theta\cos\varphi + a_{32}\sin\theta\sin\varphi + a_{33}\cos\theta)^2 \\ &= (a_{11}^2 + a_{21}^2 + a_{31}^2)\sin^2\theta\cos^2\varphi \\ &\quad + (a_{12}^2 + a_{22}^2 + a_{32}^2)\sin^2\theta\sin^2\varphi \\ &\quad + (a_{13}^2 + a_{23}^2 + a_{33}^2)\cos^2\theta \\ &\quad + 2(a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})\sin^2\theta\cos\varphi\sin\varphi \\ &\quad + 2(a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})\sin\theta\cos\theta\cos\varphi \\ &\quad + 2(a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33})\sin\theta\cos\theta\sin\varphi \\ &= (a_{11}^2 + a_{21}^2 + a_{31}^2)\cos^2\varphi + (a_{12}^2 + a_{22}^2 + a_{32}^2)\sin^2\varphi \\ &\quad + [-(a_{11}^2 + a_{21}^2 + a_{31}^2)\cos^2\varphi - (a_{12}^2 + a_{22}^2 + a_{32}^2)\sin^2\varphi \\ &\qquad + a_{13}^2 + a_{23}^2 + a_{33}^2 \\ &\qquad - 2(a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})\cos\varphi\sin\varphi]\cos^2\theta \\ &\quad + 2(a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})\cos\varphi\sin\varphi \\ &\quad + [(a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})\cos\varphi \\ &\qquad + (a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33})\sin\varphi]\sin 2\theta \end{aligned}$$

and then

$$\begin{aligned}
&= (a_{12}^2 + a_{22}^2 + a_{32}^2) + (a_{11}^2 + a_{21}^2 + a_{31}^2 - a_{12}^2 - a_{22}^2 - a_{32}^2)\cos^2\varphi \\
&\quad + (a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})\sin 2\varphi \\
&\quad + \frac{1}{2}[-(a_{11}^2 + a_{21}^2 + a_{31}^2)\cos^2\varphi - (a_{12}^2 + a_{22}^2 + a_{32}^2)\sin^2\varphi + a_{13}^2 + a_{23}^2 + a_{33}^2 \\
&\quad - (a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})\sin 2\varphi] \\
&\quad + \frac{1}{2}[-(a_{11}^2 + a_{21}^2 + a_{31}^2)\cos^2\varphi - (a_{12}^2 + a_{22}^2 + a_{32}^2)\sin^2\varphi + a_{13}^2 + a_{23}^2 + a_{33}^2 \\
&\quad - (a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})\sin 2\varphi]\cos 2\theta \\
&\quad + [(a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})\cos\varphi + (a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33})\sin\varphi]\sin 2\theta
\end{aligned}$$

and the sum from the first line to the fourth line among of seven lines of the last part of the equality above is equal to

$$\begin{aligned}
&(a_{12}^2 + a_{22}^2 + a_{32}^2) + \frac{1}{2}(a_{13}^2 + a_{23}^2 + a_{33}^2) \\
&\quad + (\frac{1}{2}(a_{11}^2 + a_{21}^2 + a_{31}^2) - a_{12}^2 - a_{22}^2 - a_{32}^2)\frac{1}{2}(1 + \cos 2\varphi) \\
&\quad + \frac{1}{2}(a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})\sin 2\varphi - \frac{1}{2}(a_{12}^2 + a_{22}^2 + a_{32}^2)\frac{1}{2}(1 - \cos 2\varphi) \\
&= \frac{1}{4}(a_{12}^2 + a_{22}^2 + a_{32}^2) + \frac{1}{2}(a_{13}^2 + a_{23}^2 + a_{33}^2) + \frac{1}{4}(a_{11}^2 + a_{21}^2 + a_{31}^2) \\
&\quad + \frac{1}{4}(a_{11}^2 + a_{21}^2 + a_{31}^2 - a_{12}^2 - a_{22}^2 - a_{32}^2)\cos 2\varphi \\
&\quad + \frac{1}{2}(a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})\sin 2\varphi
\end{aligned}$$

with letting

$$\alpha = \frac{1}{2}(a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32}) \quad \text{and} \quad \beta = \frac{1}{4}(a_{11}^2 + a_{21}^2 + a_{31}^2 - a_{12}^2 - a_{22}^2 - a_{32}^2),$$

so that

$$\alpha\sin 2\varphi + \beta\cos 2\varphi = \sqrt{\alpha^2 + \beta^2}\sin(2\varphi + \gamma),$$

where $\cos\gamma = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$ and $\sin\gamma = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$. And the sum of last two lines of seven lines of the last part of the equality above is denoted by

$$\alpha(\varphi)\sin 2\theta + \beta(\varphi)\cos 2\theta = \sqrt{\alpha(\varphi)^2 + \beta(\varphi)^2}\sin(2\theta + \gamma(\varphi)),$$

where $\cos\gamma(\varphi) = \frac{\alpha(\varphi)}{\sqrt{\alpha(\varphi)^2+\beta(\varphi)^2}}$ and $\sin\gamma(\varphi) = \frac{\beta}{\sqrt{\alpha(\varphi)^2+\beta(\varphi)^2}}$. We then compute

$$\begin{aligned}
\alpha(\varphi) &\equiv (a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})\cos\varphi + (a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33})\sin\varphi \\
&= \sqrt{(a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})^2 + (a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33})^2}\sin(\varphi + \rho),
\end{aligned}$$

where

$$\cos\rho = \frac{a_{12}a_{13}+a_{22}a_{23}+a_{32}a_{33}}{\sqrt{(a_{11}a_{13}+a_{21}a_{23}+a_{31}a_{33})^2+(a_{12}a_{13}+a_{22}a_{23}+a_{32}a_{33})^2}}, \quad \text{and}$$

$$\sin\rho = \frac{a_{11}a_{13}+a_{21}a_{23}+a_{31}a_{33}}{\sqrt{(a_{11}a_{13}+a_{21}a_{23}+a_{31}a_{33})^2+(a_{12}a_{13}+a_{22}a_{23}+a_{32}a_{33})^2}},$$

and we compute $\beta(\varphi)$

$$\begin{aligned}
&\equiv \frac{1}{2}[-(a_{11}^2+a_{21}^2+a_{31}^2)\cos^2\varphi-(a_{12}^2+a_{22}^2+a_{32}^2)\sin^2\varphi+a_{13}^2+a_{23}^2+a_{33}^2\\
&\quad -(a_{11}a_{12}+a_{21}a_{22}+a_{31}a_{32})\sin 2\varphi]\\
&=\frac{1}{2}[-\frac{1}{2}(a_{11}^2+a_{21}^2+a_{31}^2)-\frac{1}{2}(a_{12}^2+a_{22}^2+a_{32}^2)+a_{13}^2+a_{23}^2+a_{33}^2\\
&\quad -\frac{1}{2}(a_{11}^2+a_{21}^2+a_{31}^2)\cos 2\varphi+\frac{1}{2}(a_{12}^2+a_{22}^2+a_{32}^2)\cos 2\varphi\\
&\quad -(a_{11}a_{12}+a_{21}a_{22}+a_{31}a_{32})\sin 2\varphi]\\
&=-\frac{1}{4}(a_{11}^2+a_{21}^2+a_{31}^2+a_{12}^2+a_{22}^2+a_{32}^2)+\frac{1}{2}(a_{13}^2+a_{23}^2+a_{33}^2)\\
&\quad +\frac{1}{4}[-(a_{11}^2+a_{21}^2+a_{31}^2)+(a_{12}^2+a_{22}^2+a_{32}^2)]\cos 2\varphi\\
&\quad -\frac{1}{2}(a_{11}a_{12}+a_{21}a_{22}+a_{31}a_{32})\sin 2\varphi
\end{aligned}$$

with

$$\begin{aligned}
&\frac{1}{4}[-(a_{11}^2+a_{21}^2+a_{31}^2)+(a_{12}^2+a_{22}^2+a_{32}^2)]\cos 2\varphi\\
&\quad -\frac{1}{2}(a_{11}a_{12}+a_{21}a_{22}+a_{31}a_{32})\sin 2\varphi\\
&=-(\alpha\sin 2\varphi+\beta\cos 2\varphi)=-\sqrt{\alpha^2+\beta^2}\sin(2\varphi+\gamma),
\end{aligned}$$

where α, β, and γ are the same as above.

Since θ and φ are independent, it then follows that

Proposition 3.6. *If $A=(a_{ij})$ is a 3×3 matrix over $\mathbb{R}$, then*

$$\begin{aligned}
\|A\|^2 &= \frac{1}{4}(a_{12}^2+a_{22}^2+a_{32}^2)+\frac{1}{2}(a_{13}^2+a_{23}^2+a_{33}^2)+\frac{1}{4}(a_{11}^2+a_{21}^2+a_{31}^2)\\
&\quad \sup_{\varphi\in[0,2\pi]}\{\sqrt{\alpha^2+\beta^2}\sin(2\varphi+\gamma)+\sqrt{\alpha(\varphi)^2+\beta(\varphi)^2}\},
\end{aligned}$$

where $\cos\gamma=\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$ *and* $\sin\gamma=\frac{\beta}{\sqrt{\alpha^2+\beta^2}}$, *and*

$$\alpha=\frac{1}{2}(a_{11}a_{12}+a_{21}a_{22}+a_{31}a_{32}) \quad \text{and} \quad \beta=\frac{1}{4}(a_{11}^2+a_{21}^2+a_{31}^2-a_{12}^2-a_{22}^2-a_{32}^2),$$

and

$$\alpha(\varphi) = (a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})\cos\varphi + (a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33})\sin\varphi$$
$$= \sqrt{(a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})^2 + (a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33})^2}\sin(\varphi+\rho),$$

where

$$\cos\rho = \frac{a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33}}{\sqrt{(a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})^2 + (a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33})^2}}, \quad \textit{and}$$
$$\sin\rho = \frac{a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33}}{\sqrt{(a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33})^2 + (a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33})^2}},$$

and

$$\beta(\varphi) = -\frac{1}{4}(a_{11}^2 + a_{21}^2 + a_{31}^2 + a_{12}^2 + a_{22}^2 + a_{32}^2) + \frac{1}{2}(a_{13}^2 + a_{23}^2 + a_{33}^2)$$
$$- \sqrt{\alpha^2+\beta^2}\sin(2\varphi+\gamma).$$

Similarly, we obtain

Proposition 3.7. *If $A = (a_{ij})$ is a 3×3 matrix over $\mathbb{C}$, then*

$$\|A\|^2 = \frac{1}{4}(|a_{12}|^2 + |a_{22}|^2 + |a_{32}|^2) + \frac{1}{2}(|a_{13}|^2 + |a_{23}|^2 + |a_{33}|^2)$$
$$+ \frac{1}{4}(|a_{11}|^2 + |a_{21}|^2 + |a_{31}|^2)$$
$$+ \sup_{\varphi\in[0,2\pi]}\{\sqrt{\alpha^2+\beta^2}\sin(2\varphi+\gamma) + \sqrt{\alpha(\varphi)^2+\beta(\varphi)^2}\},$$

where $\cos\gamma = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$ *and* $\sin\gamma = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$, *and*

$$\alpha = \frac{1}{2}(|\overline{a_{11}}a_{12}| + |\overline{a_{21}}a_{22}| + |\overline{a_{31}}a_{32}|) \quad \textit{and}$$
$$\beta = \frac{1}{4}(|a_{11}|^2 + |a_{21}|^2 + |a_{31}|^2 - |a_{12}|^2 - |a_{22}|^2 - |a_{32}|^2),$$

and

$$\alpha(\varphi) = (|\overline{a_{11}}a_{13}| + |\overline{a_{21}}a_{23}| + |\overline{a_{31}}a_{33}|)\cos\varphi + (|\overline{a_{12}}a_{13}| + |\overline{a_{22}}a_{23}| + |\overline{a_{32}}a_{33}|)\sin\varphi$$
$$= \sqrt{(|\overline{a_{11}}a_{13}| + |\overline{a_{21}}a_{23}| + |\overline{a_{31}}a_{33}|)^2 + (|\overline{a_{12}}a_{13}| + |\overline{a_{22}}a_{23}| + |\overline{a_{32}}a_{33}|)^2}\sin(\varphi+\rho),$$

where

$$\cos\rho = \frac{|\overline{a_{12}}a_{13}| + |\overline{a_{22}}a_{23}| + |\overline{a_{32}}a_{33}|}{\sqrt{(|\overline{a_{11}}a_{13}| + |\overline{a_{21}}a_{23}| + |\overline{a_{31}}a_{33}|)^2 + (|\overline{a_{12}}a_{13}| + |\overline{a_{22}}a_{23}| + |\overline{a_{32}}a_{33}|)^2}}, \quad \textit{and}$$
$$\sin\rho = \frac{|\overline{a_{11}}a_{13}| + |\overline{a_{21}}a_{23}| + |\overline{a_{31}}a_{33}|}{\sqrt{(|\overline{a_{11}}a_{13}| + |\overline{a_{21}}a_{23}| + |\overline{a_{31}}a_{33}|)^2 + (|\overline{a_{12}}a_{13}| + |\overline{a_{22}}a_{23}| + |\overline{a_{32}}a_{33}|)^2}},$$

and

$$\beta(\varphi) = -\frac{1}{4}(|a_{11}|^2 + |a_{21}|^2 + |a_{31}|^2 + |a_{12}|^2 + |a_{22}|^2 + |a_{32}|^2) + \frac{1}{2}(|a_{13}|^2 + |a_{23}|^2 + |a_{33}|^2) - \sqrt{\alpha^2 + \beta^2}\sin(2\varphi + \gamma).$$

Lemma 3.8. *An upper triangular* 3×3 *matrix* $A = (a_{ij})$ *over reals or complexes is normal if and only if* A *is diagonal.*

Proof. We compute

$$AA^t = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & 0 & 0 \\ a_{12} & a_{22} & 0 \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11}^2 + a_{12}^2 + a_{13}^2 & a_{12}a_{22} + a_{13}a_{23} & a_{13}a_{33} \\ a_{12}a_{22} + a_{13}a_{23} & a_{22}^2 + a_{23}^2 & a_{23}a_{33} \\ a_{13}a_{33} & a_{23}a_{33} & a_{33}^2 \end{pmatrix}$$

and

$$A^tA = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{12} & a_{22} & 0 \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11}^2 & a_{11}a_{12} & a_{11}a_{13} \\ a_{11}a_{12} & a_{12}^2 + a_{22}^2 & a_{12}a_{13} + a_{22}a_{23} \\ a_{11}a_{13} & a_{12}a_{13} + a_{22}a_{23} & a_{13}^2 + a_{23}^2 + a_{33}^2 \end{pmatrix}.$$

That $AA^t = A^tA$ implies that $a_{12} = a_{13} = 0$ from $(1, 1)$ entry and then $a_{23} = 0$ from $(2, 2)$ entry. The complex case follows similarly. □

4 The general $n \times n$ matrix case

Example 4.1. Consider a special, real Jordan block $n \times n$ matrix as

$$J_n(0) = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & 0 & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in M_n(\mathbb{R}).$$

We compute the **real** operator norm of $J_n(0)$ via polar coordinate for the real $(n-1)$-dimensional unit sphere S^{n-1} in $\mathbb{R}^n$ as:

$$\begin{aligned} \|J_n(0)\|^2 &= \sup_{x\in\mathbb{R}^n, \|x\|=1} \|J_n(0)x\|^2 \\ &= \sup_{\theta_1,\cdots,\theta_{n-2}\in[0,\pi],\theta_{n-1}\in[0,2\pi]} \|J_n(0)(\cos\theta_1, \sin\theta_1\cos\theta_2, \cdots, \\ &\quad \cdots, \sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1})^t\|^2 \end{aligned}$$

and moreover,

$$
\begin{aligned}
&\|J_n(0)(\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \cdots, \\
&\quad \cdots, \sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1})^t\|^2 \\
&= (\sin\theta_1\cos\theta_2)^2 + (\sin\theta_1\sin\theta_2\cos\theta_3)^2 + \\
&\quad \cdots + (\sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1})^2 + (\sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1})^2 \\
&= (\sin\theta_1\cos\theta_2)^2 + (\sin\theta_1\sin\theta_2\cos\theta_3)^2 + \\
&\quad \cdots + (\sin\theta_1\cdots\sin\theta_{n-3}\cos\theta_{n-2})^2 + (\sin\theta_1\cdots\sin\theta_{n-3}\sin\theta_{n-2})^2 \\
&= \cdots = \sin^2\theta_1
\end{aligned}
$$

It then follows that $\|J_n(0)\| = 1 > 0 = r(J_n(0))$. ◄

Example 4.2. Consider a real Jordan block $n \times n$ matrix as

$$
J_n(\alpha) = \begin{pmatrix} \alpha & 1 & & 0 \\ 0 & \alpha & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \alpha \end{pmatrix} \in M_n(\mathbb{R}).
$$

We compute the **real** operator norm of $J_n(0)$ via polar coordinate as:

$$
\begin{aligned}
\|J_n(\alpha)\|^2 &= \sup_{x\in\mathbb{R}^n, \|x\|=1} \|J_n(\alpha)x\|^2 \\
&= \sup_{\theta_1,\cdots,\theta_{n-2}\in[0,\pi], \theta_{n-1}\in[0,2\pi]} \|J_n(\alpha)(\cos\theta_1, \sin\theta_1\cos\theta_2, \cdots, \\
&\quad \cdots, \sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1})^t\|^2
\end{aligned}
$$

and moreover,

$$
\begin{aligned}
&\|J_n(\alpha)(\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \cdots, \\
&\quad \cdots, \sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1})^t\|^2 \\
&= (\alpha\cos\theta_1 + \sin\theta_1\cos\theta_2)^2 + (\alpha\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\theta_3)^2 + \\
&\quad \cdots + (\alpha\sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1} + \sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1})^2 \\
&\quad + (\alpha\sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1})^2 \\
&= \alpha^2 + \sin^2\theta_1 + 2\alpha\sin\theta_1\cos\theta_1\cos\theta_2 + 2\alpha\sin\theta_1^2\sin\theta_2\cos\theta_2\cos\theta_3 + \\
&\quad \cdots + 2\alpha\sin\theta_1^2\cdots\sin^2\theta_{n-2}\sin\theta_{n-1}\cos\theta_{n-1} \\
&= \alpha^2 + \sin^2\theta_1 + 2\alpha\sin\theta_1\cos\theta_1\cos\theta_2 + 2\alpha\sin\theta_1^2\sin\theta_2\cos\theta_2\cos\theta_3 + \\
&\quad \cdots + 2\alpha\sin\theta_1^2\cdots\sin^2\theta_{n-3}\sin\theta_{n-2}\cos\theta_{n-2}\cos\theta_{n-1} \\
&\quad + 2\alpha\sin\theta_1^2\cdots\sin^2\theta_{n-2}\sin\theta_{n-1}\cos\theta_{n-1}.
\end{aligned}
$$

and furthermore,

$$\begin{aligned}
&= \alpha^2 + \frac{1}{2}(1-\cos 2\theta_1) + \alpha \sin 2\theta_1 \cos\theta_2 + \alpha(1-\cos 2\theta_1)\sin\theta_2\cos\theta_2\cos\theta_3 \\
&\quad + \cdots + \alpha(1-\cos 2\theta_1)\sin^2\theta_2\cdots\sin^2\theta_{n-3}\sin\theta_{n-2}\cos\theta_{n-2}\cos\theta_{n-1} \\
&\quad + \alpha(1-\cos 2\theta_1)\sin^2\theta_2\cdots\sin^2\theta_{n-2}\sin\theta_{n-1}\cos\theta_{n-1} \\
&= \alpha^2 + \frac{1}{2} + \alpha\sin 2\theta_1\cos\theta_2 - \cos 2\theta_1[\frac{1}{2} + \alpha\sin\theta_2\cos\theta_2\cos\theta_3 + \\
&\quad \cdots + \alpha\sin^2\theta_2\cdots\sin^2\theta_{n-3}\sin\theta_{n-2}\cos\theta_{n-2}\cos\theta_{n-1} \\
&\quad + \alpha\sin^2\theta_2\cdots\sin^2\theta_{n-2}\sin\theta_{n-1}\cos\theta_{n-1}] \\
&\quad (\text{where we set } [\frac{1}{2} + \cdots + \cdots\cos\theta_{n-1}] \equiv \beta(\theta_2,\cdots,\theta_{n-1}) \text{ for short}) \\
&\quad + \alpha\sin\theta_2\cos\theta_2\cos\theta_3 + \\
&\quad \cdots + \alpha\sin^2\theta_2\cdots\sin^2\theta_{n-3}\sin\theta_{n-2}\cos\theta_{n-2}\cos\theta_{n-1} \\
&\quad + \alpha\sin^2\theta_2\cdots\sin^2\theta_{n-2}\sin\theta_{n-1}\cos\theta_{n-1}
\end{aligned}$$

and then

$$\begin{aligned}
&= \alpha^2 + \frac{1}{2} + \sqrt{\alpha^2\cos^2\theta_2 + \beta(\theta_2,\cdots,\theta_{n-1})^2}\sin(2\theta_1 + \gamma(\theta_2,\cdots,\theta_{n-1})) \\
&\quad + \frac{1}{2}\alpha\sin 2\theta_2\cos\theta_3 + \\
&\quad \cdots + \frac{1}{2}\alpha(1-\cos 2\theta_2)\sin^2\theta_3\cdots\sin^2\theta_{n-3}\sin\theta_{n-2}\cos\theta_{n-2}\cos\theta_{n-1} \\
&\quad + \frac{1}{2}\alpha(1-\cos 2\theta_2)\sin^2\theta_3\cdots\sin^2\theta_{n-2}\sin\theta_{n-1}\cos\theta_{n-1},
\end{aligned}$$

with

$$\begin{aligned}
\cos\gamma(\theta_2,\cdots,\theta_{n-1}) &= \alpha\cos\theta_2(\alpha^2\cos^2\theta_2 + \beta(\theta_2,\cdots,\theta_{n-1})^2)^{-\frac{1}{2}}, \\
\sin\gamma(\theta_2,\cdots,\theta_{n-1}) &= -\beta(\theta_2,\cdots,\theta_{n-1})(\alpha^2\cos^2\theta_2 + \beta(\theta_2,\cdots,\theta_{n-1})^2)^{-\frac{1}{2}}.
\end{aligned}$$

It then follows that

$$\begin{aligned}
&\sup_{\theta_1\in[0,\pi]} \|J_n(\alpha)(\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \cdots, \\
&\quad \cdots, \sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1})^t\|^2 \\
&= \alpha^2 + \frac{1}{2} + \sqrt{\alpha^2\cos^2\theta_2 + \beta(\theta_2,\cdots,\theta_{n-1})^2} \\
&\quad + \frac{1}{2}\alpha\sin 2\theta_2\cos\theta_3 + \\
&\quad \cdots + \frac{1}{2}\alpha(1-\cos 2\theta_2)\sin^2\theta_3\cdots\sin^2\theta_{n-3}\sin\theta_{n-2}\cos\theta_{n-2}\cos\theta_{n-1} \\
&\quad + \frac{1}{2}\alpha(1-\cos 2\theta_2)\sin^2\theta_3\cdots\sin^2\theta_{n-2}\sin\theta_{n-1}\cos\theta_{n-1}.
\end{aligned}$$

Then we can perform the similar computation with respect to θ_2, and inductively we can compute the real operator norm for $J_n(\alpha)$ in a visual way in that sense as before. But it seems to be very difficult to determine the supremum in terms of α. ◀

We next may consider the **general real** $n \times n$ matrix case by encouraged by the (partial) success in the examples above. Let A be a full $n \times n$ matrix over reals as

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad a_{ij} \in \mathbb{R}.$$

We compute the real operator norm of A via the polar coordinate as:

$$\begin{aligned} \|A\|^2 &= \sup_{x \in \mathbb{R}^n, \|x\|=1} \|Ax\|^2 \\ &= \sup_{\theta_1, \cdots, \theta_{n-2} \in [0,\pi], \theta_{n-1} \in [0,2\pi]} \|A(\cos\theta_1, \sin\theta_1\cos\theta_2, \cdots, \\ &\quad \cdots, \sin\theta_1 \cdots \sin\theta_{n-2}\cos\theta_{n-1}, \sin\theta_1 \cdots \sin\theta_{n-2}\sin\theta_{n-1})^t\|^2 \end{aligned}$$

and moreover, by letting as below to simplify the notation

$$\begin{aligned} &\|A(\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \cdots, \\ &\quad \cdots, \sin\theta_1 \cdots \sin\theta_{n-2}\cos\theta_{n-1}, \sin\theta_1 \cdots \sin\theta_{n-2}\sin\theta_{n-1})^t\|^2 \\ &\equiv \|A(f_1(\theta_1), f_2(\theta_1,\theta_2), \cdots, f_{n-1}(\theta_1, \ldots, \theta_{n-1}), f_n(\theta_1, \ldots, \theta_{n-1}, \theta_n = \theta_{n-1}))^t\|^2 \\ &= \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} f_j(\theta_1, \cdots, \theta_j) \right)^2 \\ &= \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}^2 f_j^2(\theta_1, \cdots, \theta_j) + \sum_{j,l=1}^n 2a_{ij}a_{il} f_j(\theta_1, \cdots, \theta_j) f_l(\theta_1, \cdots, \theta_l) \right) \\ &= \sum_{j=1}^n (\sum_{i=1}^n a_{ij}^2) f_j^2(\theta_1, \cdots, \theta_j) + \sum_{j,l=1}^n 2(\sum_{i=1}^n a_{ij}a_{il}) f_j(\theta_1, \cdots, \theta_j) f_l(\theta_1, \cdots, \theta_l). \end{aligned}$$

Note that $|f_j(\theta_1, \cdots, \theta_j)| \le 1$ for every $1 \le j \le n$. Therefore, it follows that

Proposition 4.3. *If $A = (a_{ij})$ is an $n \times n$ matrix over $\mathbb{R}$, then $\|A\|^2$ is equal to the supremum of*

$$\sum_{j=1}^n (\sum_{i=1}^n a_{ij}^2) f_j^2(\theta_1, \cdots, \theta_j) + \sum_{j,l=1}^n 2(\sum_{i=1}^n a_{ij}a_{il}) f_j(\theta_1, \cdots, \theta_j) f_l(\theta_1, \cdots, \theta_l)$$

over $\theta_1, \cdots, \theta_{n-2} \in [0,\pi], \theta_{n-1} \in [0,2\pi]$, where

$$f_1(\theta_1) = \cos\theta_1, \quad f_2(\theta_1,\theta_2) = \sin\theta_1\cos\theta_2, \quad f_3(\theta_1,\theta_2,\theta_3) = \sin\theta_1\sin\theta_2\cos\theta_3,$$
$$\cdots, f_{n-1}(\theta_1, \cdots, \theta_{n-1}) = \sin\theta_1 \cdots \sin\theta_{n-2}\cos\theta_{n-1},$$
$$f_n(\theta_1, \cdots, \theta_{n-1}, \theta_n = \theta_{n-1}) = \sin\theta_1 \cdots \sin\theta_{n-2}\sin\theta_{n-1},$$

and the supremum can be taken inductively from θ_1 to θ_{n-1} by only using the equations such as $\cos^2\theta_j + \sin^2\theta_j = 1$ and $\cos^2\theta_j = \frac{1}{2}(1+\cos 2\theta_j)$ and $\sin^2\theta_j = \frac{1}{2}(1-\cos 2\theta_j)$.

This says that the real operator norm $\|A\|$ is computable in terms of entries of A by taking supremums finitely and inductively with respect to polar coordinate.

We also have the following estimate:

$$\|A\|^2 \le \sum_{i=1}^{n}(\sum_{j=1}^{n}|a_{ij}|)^2.$$

In particular, if A is upper triangular as reduced as in (U1) *in Section* 1, *then*

$$\|A\|^2 \le \sum_{i=1}^{n}(\sum_{j=i}^{n}|a_{ij}|)^2.$$

Similarly, we consider the **complex** case. Note that for the polar decomposition

$$z = (z_1, \cdots, z_n)^t = (|z_1|e^{i\rho_1}, \cdots, |z_n|e^{i\rho_n})^t \in \mathbb{C}^n$$

with $\rho_1, \cdots, \rho_n \in [0, 2\pi]$, the condition $\|z\| = 1$ for $z \in \mathbb{C}^n$ is equivalent to the equation $|z_1|^2 + \cdots + |z_n|^2 = 1$. Then we use the polar coordinate for $(|z_1|, \cdots, |z_n|)^t$ in the unit sphere S^{n-1} of $\mathbb{R}^n$ as in the real case above as

$$\begin{aligned}(|z_1|, |z_2|, \cdots, |z_{n-1}|, |z_n|) = (&\cos\theta_1, \sin\theta_1\cos\theta_2, \cdots,\\ &\cdots, \sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1})\end{aligned}$$

for $\theta_1, \cdots, \theta_{n-1} \in [0, \frac{\pi}{2}]$.

Proposition 4.4. *If $A = (a_{ij})$ is an $n \times n$ matrix over $\mathbb{C}$, then $\|A\|^2$ is equal to the supremum of*

$$\sum_{j=1}^{n}(\sum_{i=1}^{n}|a_{ij}|^2)f_j^2(\theta_1, \cdots, \theta_j) + \sum_{j,l=1}^{n} 2(\sum_{i=1}^{n}|\overline{a_{ij}}a_{il}|)f_j(\theta_1, \cdots, \theta_j)f_l(\theta_1, \cdots, \theta_l)$$

over $\theta_1, \cdots, \theta_{n-1} \in [0, \frac{\pi}{2}]$, where

$$\begin{aligned}&f_1(\theta_1) = \cos\theta_1, \quad f_2(\theta_1, \theta_2) = \sin\theta_1\cos\theta_2, \quad f_3(\theta_1, \theta_2, \theta_3) = \sin\theta_1\sin\theta_2\cos\theta_3,\\ &\cdots, f_{n-1}(\theta_1, \cdots, \theta_{n-1}) = \sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1},\\ &f_n(\theta_1, \cdots, \theta_{n-1}, \theta_n = \theta_{n-1}) = \sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1},\end{aligned}$$

and the supremum can be taken inductively from θ_1 to θ_{n-1} by only using the equations such as $\cos^2\theta_j + \sin^2\theta_j = 1$ and $\cos^2\theta_j = \frac{1}{2}(1+\cos 2\theta_j)$ and $\sin^2\theta_j = \frac{1}{2}(1-\cos 2\theta_j)$.

This says that the complex operator norm $\|A\|$ is computable in terms of entries of A by taking supremums finitely and inductively with respect to polar coordinate.

We also have the following estimate:

$$\|A\|^2 \le \sum_{i=1}^{n}(\sum_{j=1}^{n}|a_{ij}|)^2.$$

In particular, if A is upper triangular as reduced as in (U1) *in Section* 1, *then*

$$\|A\|^2 \le \sum_{i=1}^{n}(\sum_{j=i}^{n}|a_{ij}|)^2.$$

Example 4.5. For a complex Jordan block $n \times n$ matrix $J_n(\alpha) \in M_n(\mathbb{C})$, we obtain

$$\|J_n(\alpha)\|^2 \le (n-1)(|\alpha|+1)^2 + |\alpha|^2.$$

But this estimate is not sharp as in the case where $\alpha = 0$:

$$\|J_n(0)\|^2 = 1 \le (n-1). \quad ◀$$

Corollary 4.6. *For any $A \in M_n(\mathbb{C})$, with the Jordan canonical form as*

$$P^{-1}AP = \oplus_{j=1}^{k} J_{n_j}(\alpha_j)$$

by an invertible $P \in M_n(\mathbb{C})$ as in (S1) *in Section* 1, *we obtain*

$$\|A\|^2 \le \|P\|^2\|P^{-1}\|^2 \max_{1\le j\le k}\{(n_j-1)(|\alpha_j|+1)^2 + |\alpha_j|^2\}.$$

Lemma 4.7. *An upper triangular $n\times n$ matrix $A = (a_{ij})$ over reals or complexes is normal if and only if A is diagonal.*

Proof. We compute

$$AA^t = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ 0 & 0 & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & 0 & 0 \\ \vdots & \ddots & 0 \\ a_{1n} & \cdots & a_{nn} \end{pmatrix} \equiv B = (b_{ij})_{i,j=1}^{n}$$

to have diagonal elements as

$$b_{11} = a_{11}^2 + \cdots + a_{1n}^2, \quad b_{22} = a_{22}^2 + \cdots + a_{2n}^2, \quad \cdots, \quad b_{nn} = a_{nn}^2.$$

We also compute

$$A^tA = \begin{pmatrix} a_{11} & 0 & 0 \\ \cdots & \ddots & 0 \\ a_{1n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ 0 & 0 & a_{nn} \end{pmatrix} \equiv C = (c_{ij})_{i,j=1}^{n}$$

to have diagonal elemets as

$$c_{11} = a_{11}^2, \quad c_{22} = a_{12}^2 + a_{22}^2, \quad \cdots, \quad c_{nn} = a_{1n}^2 + \cdots a_{nn}^2.$$

That $AA^t = A^tA$ implies that $a_{12} = \cdots = a_{1n} = 0$ from $(1,1)$ entry and then $a_{23} = \cdots = a_{2n} = 0$ from $(2,2)$ entry. it follows inductively that odd-diagonal elements $a_{ij} = 0$ for $i < j$.

The complex case follows similarly. □

Remark. As a certainly known fact, we note that for any $A \in \mathbb{B}(H)$,

$$\|A\|^2 = \sup_{x\in H, \|x\|=1} \|Ax\|^2 = \sup_{x\in H, \|x\|=1} \langle A^*Ax, x\rangle,$$

and the inner product $\langle A^*Ax, x\rangle$ is said to be the Hermitian form when $H = \mathbb{C}^n$. Indeed, if $A = (a_{ij}) \in M_n(\mathbb{C})$, then $A^*A = (\sum_{k=1}^n \overline{a_{ki}}a_{kj})_{i,j}$ and

$$\begin{aligned} Q_{A^*A}(x) &\equiv \langle A^*Ax, x\rangle \\ &= \sum_{i=1}^n(\sum_{j=1}^n(\sum_{k=1}^n \overline{a_{ki}}a_{kj})x_j)\overline{x_i} = \sum_{i,j=1}^n(\sum_{k=1}^n \overline{a_{ki}}a_{kj})\overline{x_i}x_j. \end{aligned}$$

Hence, if $H = \mathbb{R}^n$, then $Q_{A^tA}(x)$ is just a real quadratic form on H. In this case, computing the real operator norm is equivalent to solving the conditional extremum problem for $Q_{A^tA}(x)$ with $\|x\| = 1$, and this problem can be solved by Lagrange's method indeterminate coefficients, and in fact, as well known, the maximum is just equal to the maximum of squares of non-negative eigenvalues of A^tA positive as an operator. More details about this method are considered in the next section. But it seems to be difficult in general to determine the conditional extremums or maximum for $Q_{A^tA}(x)$ with $\|x\| = 1$ in terms of entries of A, as carried out so far as above. ◀

5 As the supremums of quadratic or Hermitian forms

We assume that H is a real or complex, finite dimensional Hilbert (or Euclidean) space, that is, $\mathbb{R}^n$ or $\mathbb{C}^n$.

When $H = \mathbb{R}^n$, for any $n \times n$ matrix T over $\mathbb{R}$, the real **quadratic** form Q_{T^tT} for T^tT is defined to be

$$Q_{T^tT}(x) = \langle T^tTx, x\rangle$$

for $x = (x_1, \cdots, x_n)^t \in \mathbb{R}^n$.

When $H = \mathbb{C}^n$, for any $n \times n$ matrix T over $\mathbb{C}$, the **Hermitian** form Q_{T^*T} for T^*T is defined to be

$$Q_{T^*T}(x) = \langle T^*Tx, x\rangle$$

for $x = (x_1, \cdots, x_n)^t \in \mathbb{C}^n$. May refer to [5].

Since T^*T is normal, there is a unitary $n\times n$ matrix U such that $U^*(T^*T)U$ is a diagonal matrix D_T of the **rank** $\mathrm{rk}(T)$ of T with diagonal elements d_j for $1 \le j \le n$, that are non-negative real eigenvalues of T^*T positive as an operator. Therefore,

$$\begin{aligned} Q_{T^*T}(x) &= \langle T^*TUy, Uy\rangle = \langle D_Ty, y\rangle \\ &= Q_{D_T}(y) = \sum_{j=1}^n d_j|y_j|^2 = \sum_{j=1}^{\mathrm{rk}(T)} d_j|y_j|^2 \end{aligned}$$

with $\|y\| = 1$, where we may assume that $d_j > 0$ for $1 \leq j \leq \mathrm{rk}(T)$ and $d_j = 0$ for $\mathrm{rk}(T)+1 \leq j \leq n$ if $\mathrm{rk}(T) < n$, and $d_j > 0$ for $1 \leq j \leq \mathrm{rk}(T)$ if $\mathrm{rk}(T) = n$. .

As a summary,

Proposition 5.1. *Let T be a bounded operator on a real or complex, finite dimensional Hilbert space H. Then*

$$\|T\|^2 = \sup_{y \in H, \|y\|=1} Q_{D_T}(y) = \sup_{y \in H, \|y\|=1} \sum_{j=1}^{\mathrm{rk}(T)} d_j |y_j|^2,$$

where $D_T = U^*(T^*T)U = \oplus_{j=1}^n d_j$ the diagonal sum of eigenvalues d_j of T^*T for some unitary U.

Now may suppose that $y = (y_1, \cdots, y_j) \in H = \mathbb{R}^n$ or $\mathbb{C}^n$ with each y_j real and positive by replacing it with $|y_j|$ if necessary. We then consider the conditional extremum problem for the non-negative, continuous function $Q_{D_T}(y)$ with $g(y) \equiv \|y\|^2 - 1 = 0$. The partial derivative of $Q_{D_T}(y)$ is computed as

$$\frac{\partial}{\partial y_j} Q_{D_T}(y) = 2 d_j y_j$$

for $1 \leq j \leq n$. Thus, $Q_{D_T}(y)$ has stationary points only zero $y = 0$.

Define $F(y, \lambda) = Q_{D_T}(y) - \lambda g(y)$ as a function. By Lagrange's method of indeterminate coefficients, we consider the following equations:

$$\frac{\partial}{\partial y_j} F(y, \lambda) = 2 d_j y_j - \lambda 2 y_j = 2(d_j - \lambda) y_j = 0$$

for $1 \leq j \leq n$ and

$$\frac{\partial}{\partial \lambda} F(y, \lambda) = g(y) = 0.$$

It follows that $y_j = 0$ or $\lambda = d_j$ for each $1 \leq j \leq n$.

Set $d_T = \max_{1 \leq j \leq n} d_j$.

If $d_T = d_j$ for each $1 \leq j \leq n$, then

$$Q_{D_T}(y) = \sum_{j=1}^{n} d_j y_j^2 = d_T = \|T\|^2$$

for y with $\|y\| = 1$.

If $\lambda = d_k$ for some k, then

$$\sup_{y \in H, \|y\|=1} Q_{D_T}(y) = \sup_{y \in H, \|y\|=1} \sum_{j=1}^{n} d_j y_j^2 = \lambda = \|T\|^2,$$

so that $\lambda = d_T$. Note as well that the set of all vectors $y \in \mathbb{R}^n$ with $\|y\| = 1$ is compact, so that $Q_{D_T}(y)$ has maximum (and minimum) on the set. Also, $Q_{D_T}(y)$ takes every d_j at $y = e_j$ the j-th standard basis vector for $\mathbb{R}^n$.

Proposition 5.2. *Let T be a real or complex bounded operator on $H = \mathbb{R}^n$ or $\mathbb{C}^n$. Then*

$$\|T\| = \sqrt{d_T} = \max_{1\leq j\leq n} \sqrt{d_j},$$

the maximum of eigenvalues of T^*T, where $U^*(T^*T)U = \oplus_{j=1}^n d_j$ the diagonal sum of eigenvalues of T^*T for some unitary U, and $d_T = \max_{1\leq j\leq n} d_j$.

Remark. Note that such eigenvalues are not always computatble, because determining these is equivalent to solving eigenvalue (or characteristic) equations for eigenvalue (or characteristic) polynomials, which can not be solved algebraically in general if degrees are more than 4, as shown by Abel. ◀

6 The infinite dimensional case

We assume that H is a complex, infinite dimensional Hilbert space.

Let $T \in \mathbb{B}(H)$. Denote by $\sigma(T)$ the spectrum of T. For any $x \in H$, set

$$Q_{T^*T}(x) = \langle T^*Tx, x\rangle = \|Tx\|^2.$$

If T is compact, then the difference set $\sigma(T) \setminus \{0\}$ consists of eigenvalues λ_n of T with finite multiplicities and $\sigma(T)$ is countable with the sequence $(\lambda_n)_n$ vanishing at infinity (see [4] or [6]).

It is known that if a normal operator T on H is compact, then T is diagonalizable (see [6, Theorem 2.4.4]).

It then follows that

Proposition 6.1. *Let T be a compact, normal operator on H. Then*

$$\|T\|^2 = \sup_{y\in H, \|y\|=1} Q_{D_T}(y) = \sup_{y\in H, \|y\|=1} \sum_{j=1}^{\infty} d_j|y_j|^2 = \max_{y\in H, \|y\|=1} \sum_{j=1}^{\infty} d_j|y_j|^2,$$

where $D_T = U^*(T^*T)U = \oplus_{j=1}^{\infty} d_j$ the diagonal sum of non-negative eigenvalues d_j of T^*T for some unitary U.

Proposition 6.2. *Let T be a compact, normal operator on H. Then*

$$\|T\| = \sqrt{d_T} = \sup_{1\leq j<\infty} \sqrt{d_j} = \max_{1\leq j<\infty} \sqrt{d_j},$$

where $U^*(T^*T)U = \oplus_{j=1}^{\infty} d_j$ for some unitary U, and $d_T = \max_{1\leq j<\infty} d_j$.

Proof. Note that any compact operator T on H is approximated in operator norm by finite rank operators and $\sigma(T)$ is countable. By induction, the statement is proved by Proposition 5.2. □

As proved as Lemma 1.4,

Proposition 6.3. *If T is a bounded diagonal operator on H with d_j on the diagonal with respect to an orthonomal basis of H, then*

$$\|T\|^2 = \sup_{y\in H, \|y\|=1} Q_{T^*T}(y) = \sup_j |d_j|^2.$$

Let $T \in \mathbb{B}(H)$. Since T^*T is normal, there is a unique spectral measure P from Borel subsets in the spectrum $\sigma(T^*T)$ of T^*T to projections of $\mathbb{B}(H)$ such that

$$T^*T = \int_{\sigma(T^*T)} z\,dP,$$

where z is the inclusion map from $\sigma(T^*T)$ to $\mathbb{C}$, in the sense that

$$\langle T^*Tx, x\rangle = \int_{\sigma(T^*T)} z\,dP_{x,x},$$

where $P_{x,x}(S) = \langle P(S)x, x\rangle$ for any Borel subset S of $\sigma(T^*T)$ and any $x \in H$, and each $P_{x,x}$ is a regular Borel measure on $\sigma(T^*T)$, where each $P(S)$ is a projection of $\mathbb{B}(H)$ such that $P(\emptyset) = 0$, $P(\sigma(T^*T)) = 1$, and $P(S_1 \cap S_2) = P(S_1)P(S_2)$ for Borel subsets S_1, S_2 of $\sigma(T^*T)$. See [6] for more details.

It then follows that

Proposition 6.4. *Let $T \in \mathbb{B}(H)$. Then*

$$\|T\|^2 = r(T^*T) \equiv \sup_{z\in\sigma(T^*T)} |z|,$$

the spectral radius for T^*T.

Proof. By functional calculus, the C^*-algebra $C(\sigma(T^*T))$ of all continuous, complex-valued functions on the spectrum $\sigma(T^*T)$ is isomorphic to the C^*-subalgebra of $\mathbb{B}(H)$ via the spectral measure P as:

$$C(\sigma(T^*T)) \ni f(z) \mapsto f(T^*T) = \int_{\sigma(T^*T)} f(z)\,dP \in \mathbb{B}(H).$$

In particular,

$$\|z\|_\infty \equiv \sup_{z\in\sigma(T^*T)} |z| = r(T^*T) = \|T^*T\| = \|T\|^2$$

by the definition of spectral radius and by the C^*-norm condition, where $z : \sigma(T^*T) \to \mathbb{C}$ is the inclusion map, and $\|\cdot\|_\infty$ is the uniform norm for $C(\sigma(T^*T))$. □

Remark. However, that equality in the statement above does not say anything without knowing the spectrum $\sigma(T^*T)$ or just knowing its boundary or maximum.

Added as a note, this paper has been reviewed, but, from which it is some improved for this publication. ◀

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