The spectrum theory for weighted shift operators and their C^⊠-algebras

The spectrum theory for weighted shift operators and their C^*-algebras

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Dedicated to Professor Jun Tomiyama on his 87th birthday with gratitude and respect

Abstract

We study weighted unilateral and bilateral shift operators and their C^*-algebras in a systematic way. We mainly consider some basic or extended, spectrum theory for those operators and their C^*-algebras. As results we obtain several extended generalizations from certainly known results in some cases.

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1 Introduction

In this paper we would like to study weighted unilateral and bilateral shift operators and their C^*-algebras respectively in some details, beyond the usual (non-weighted) unilateral shift operator and bilateral operator and their C^*-algebras (as an account as in Murphy [11]), although weighted or not, such operators and C^*-algebras are quite well known in the literature in Operator Theory and Operator Algebras. For instance, may refer to Bunce [1], Bunce-Deddens [2], Conway [4], Davidson [6], Ghatage [7], Ghatage-Phillips [8], Halmos [9], and Hiai-Yanagi [10] (and more items). Especially, it has been noticed that the paper [12] by Shields should be refereed as a reference, but this item has not been at hand and so not checked. This is the very first reason for this work as a motivation, before the notice, to make some details by ourselves on this subject for some purpose using them later somewhere suitably.

This paper is organized as follows. In Section 2, we review and study weighted unilateral and bilateral shift operators respectively in a systematic way (cf. [4] and [2]). As a result we determine the spectrums of those operators in terms of, or under conditions of bounded sequences of complex numbers as

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weights of shifts in some cases. In particular, we obtain several extended generalizations from the cases of the usual shift operators ([10]) and hyponormal weighted shift operators ([4]). In Section 3, as an attempt we consider Banach spaces of all weighted shift operators. In Section 4, we consider C^*-algebras generated by weighted shift operators. As a result we determine the algebraic structures of those C^*-algebras in terms of, or under conditions of bounded sequences as weights of shifts in some cases. In particular, we obtain several extended generalizations from the case of C^*-algebras generated by hyponormal weighted shift operators ([1] and [4]).

The results obtained, but containing several or many basic and elementary facts, as accumulation would be some useful as a convenient reference for further studying this topic. More investigations on this subject may be continued and considered in elsewhere.

Added as a note. The (early) versions of this paper with slightly different titles have been reviewed, from which this paper is improved to some extent for this publication.

2 Weighted shift operators

We denote by $L^2(\mathbb{N})$ the Hilbert space of all square summable sequences $a = (a_j) = \sum_{j=1}^\infty a_j e_j$ of complex numbers $a_j \in \mathbb{C}$ over $\mathbb{N}$ of natural numbers, with the 2-norm squared as

$$\|a\|_2^2 = \sum_{j=1}^\infty |a_j|^2 = \sum_{j=1}^\infty a_j \overline{a_j} = \langle a, a\rangle$$

as the inner product, which is linear in the first variable and complex-conjugate linear in the second, where $(e_j)_{j=1}^\infty$ is the canonical orthonormal basis of $L^2(\mathbb{N})$. Similarly, we define the Hilbert space $L^2(\mathbb{Z})$ of all square summable sequences $b = (b_j) = \sum_{j\in\mathbb{Z}} b_j e_j$ of $b_j \in \mathbb{C}$ over $\mathbb{Z}$ of the integers.

Let $w = (w_n)$ be a bounded sequence of complex numbers. We denote by $C^b(\mathbb{N})$ the C^*-algebra of all bounded sequences $w = (w_n), z = (z_n)$ of complex numbers with the pointwise operations such as addition $w + z = (w_n + z_n)$, multiplication $wz = (w_n z_n)$, and involution $w^* = (\overline{w_n})$, and with the uniform norm as $\|w\|_\infty = \sup_{n\in\mathbb{N}} |w_n|$. Similarly, we define the C^*-algebra $C^b(\mathbb{Z})$.

Define the **unilateral weighted shift** (UWS) operator S_w with weight $w \in C^b(\mathbb{N})$, acting on $L^2(\mathbb{N})$ as an $\infty \times \infty$ infinite matrix:

$$S_w = \begin{pmatrix} 0 & 0 & \cdots & \cdots \\ w_1 & 0 & \cdots & \cdots \\ 0 & w_2 & \ddots & \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

so that $S_w(\sum_{j=1}^\infty a_j e_j) = \sum_{j=1}^\infty w_j a_j e_{j+1}$ for $\sum_{j=1}^\infty a_j e_j \in L^2(\mathbb{N})$. Namely,

$$S_w(a_1, a_2, \cdots)^t = (0, w_1 a_1, w_2 a_2, \cdots)^t \in L^2(\mathbb{N}).$$

Because

$$\|S_w(a)\|_2^2 = \sum_{j=1}^\infty |w_j a_j|^2 \leq \|w\|_\infty^2 \sum_{j=1}^\infty |a_j|^2 = \|w\|_\infty^2 \|a\|_2^2 < \infty.$$

If $w = (w_n) = 1 = (1)$ with $w_n = 1$, then S_1 is the usual unilateral shift operator on $L^2(\mathbb{N})$. The adjoint operator S_w^* of S_w is identified with

$$S_w^* = \begin{pmatrix} 0 & \overline{w_1} & & \\ & 0 & \overline{w_2} & \\ & & \ddots & \ddots \end{pmatrix}$$

so that $S_w^* e_1 = 0$, $S_w^* e_j = \overline{w_{j-1}} e_{j-1}$, and $S_w^*(\sum_{j=1}^\infty a_j e_j) = \sum_{j=2}^\infty \overline{w_{j-1}} a_j e_{j-1}$ as

$$S_w^*(a_1, a_2, \cdots)^t = (\overline{w_1} a_2, \overline{w_2} a_3, \cdots)^t \in L^2(\mathbb{N}).$$

Because

$$\|S_w^*(a)\|_2^2 = \sum_{j=1}^\infty |w_j a_{j+1}|^2 \leq \|w\|_\infty^2 \sum_{j=1}^\infty |a_{j+1}|^2 \leq \|w\|_\infty^2 \|a\|_2^2 < \infty.$$

The **bilateral weighted shift** (BWS) operator U_w for $w \in C^b(\mathbb{Z})$, acting on $L^2(\mathbb{Z})$ is defined by $U_w e_j = w_j e_{j+1}$ for $j \in \mathbb{Z}$. If $w = 1$, then U_1 is the usual bilateral shift operator on $L^2(\mathbb{Z})$. The adjoint operator U_w^* of U_w is defined by $U_w^* e_j = \overline{w_{j-1}} e_{j-1}$ for $j \in \mathbb{Z}$. Namely,

$$U_w = \begin{pmatrix} \ddots & \ddots & & \\ \ddots & 0 & 0 & \\ & w_j & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad U_w^* = \begin{pmatrix} \ddots & \ddots & & \\ \ddots & 0 & \overline{w_j} & \\ & 0 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

We denote by D_w the diagonal operator on $L^2(\mathbb{N})$ with $w = (w_n) \in C^b(\mathbb{N})$ as entries on the diagonal. Similarly, we define the diagonal operator D_w on $L^2(\mathbb{Z})$ with $w \in C^b(\mathbb{Z})$.

Let H be a Hilbert space with the 2-norm associated to an inner product, such as $\|\cdot\|_2 = \sqrt{\langle \cdot, \cdot \rangle}$. We denote by $\mathbb{B}(H)$ the C^*-algebra of all bounded (linear) operators on H, with the operator (uniform or supremum) norm

$$\|B\| = \sup_{\xi \in H, \|\xi\|_2 \leq 1} \|B\xi\|_2 \quad \text{finite for any } B \in \mathbb{B}(H).$$

Lemma 2.1. *The product $S_w^* S_w$ is the diagonal operator $D_{w^* w}$ with the bounded sequence $(|w_j|^2)$ on the diagonal, so that the operator norm $\|S_w^* S_w\|$ is equal to $\sup_{j \in \mathbb{N}} |w_j|^2 = \|w\|_\infty^2$.*

The similar as above holds for $U_w^* U_w$ on $L^2(\mathbb{Z})$.

The product $S_w S_w^*$ *is the diagonal sum operator* $0 \oplus D_{w^*w} \equiv D_{(0,w^*w)}$, *so that the operator norm* $\|S_w S_w^*\|$ *is equal to* $\sup_{j\in\mathbb{N}} |w_j|^2 = \|w\|_\infty^2$.

We have $U_w U_w^* = D_{(w^*w)_{-1}}$ *with* $(w^*w)_{-1} = (w_{j-1}^* w_{j-1})_{j\in\mathbb{Z}}$ **defined so**, *so that* $\|U_w U_w^*\| = \|w\|_\infty^2$.

Proof. By computing the infinite matrix multiplication, we have $S_w^* S_w = D_{w^*w}$ and $S_w S_w^* = 0 \oplus D_{w^*w}$.

Since $D_{w^*w} e_j = |w_j|^2 e_j$ with $\|e_j\|_2 = 1$, we have $\|D_{w^*w}\| \geq \|w\|_\infty^2$. Conversely, for $\xi = (\xi_j) \in L^2(\mathbb{N})$,

$$\begin{aligned}\|D_{w^*w}\xi\|^2 = \langle w^*w\xi, w^*w\xi\rangle &= \sum_{j=1}^\infty w_j \overline{w_j \xi_j}\,\overline{w_j} w_j \xi_j \\ &\leq \sup_{j\in\mathbb{N}} |w_j|^4 \sum_{j=1}^\infty |\xi_j|^2 = \|w\|_\infty^4 \|\xi\|_2^2.\end{aligned}$$

Therefore, we obtain $\|D_{w^*w}\| \leq \|w\|_2^2$.

The proof for $U_w^* U_w$ and $U_w U_w^*$ is similar. □

We denote by $\mathbb{T} = S^1$ the real 1-dimensional torus or the circle.

Corollary 2.2. *The product* $S_w S_w^*$ *is not invertible in* $\mathbb{B}(L^2(\mathbb{N}))$.

The product $S_w^* S_w$ *is invertible if and only if the sequence* $w = (w_j)$ *is bounded away from zero, that is, there is* $\varepsilon > 0$ *such that* $|w_j| \geq \varepsilon$ *for any* $j \in \mathbb{N}$.

The products $U_w^* U_w$ *and* $U_w U_w^*$ *are invertible if and only if the sequence* $w = (w_j)$ *is bounded away from zero.*

The unilateral weighted shift S_w *is isometry if and only if each* $w_j \in \mathbb{T}$, *that is,* w *is a unitary of* $C^b(\mathbb{N})$.

The adjoint S_w^* *is a partial isometry if and only if each* $w_j \in \mathbb{T}$, *that is,* w *is a unitary of* $C^b(\mathbb{N})$.

The bilateral weighted U_w *and* U_w^* *are unitaries if and only if each* $w_j \in \mathbb{T}$, *that is,* w *is a unitary of* $C^b(\mathbb{Z})$.

The bilateral weighted U_w *and* U_w^* *are invertibles if and only if* w *is invertible in* $C^b(\mathbb{Z})$. *In this case,*

$$\begin{aligned}U_w^{-1} &= D_{(w^*w)^{-1}} U_w^* = D_{(w^*w)^{-1}} D_{\overline{w}} U_1^* \\ &= U_w^* D_{(w^*w)_{-1}^{-1}} = D_{\overline{w}} U_1^* D_{(w^*w)_{-1}^{-1}}\end{aligned}$$

and

$$\begin{aligned}(U_w^*)^{-1} &= U_w D_{(w^*w)^{-1}} = U_1 D_w D_{(w^*w)^{-1}} \\ &= D_{(w^*w)_{-1}^{-1}} U_w = D_{(w^*w)_{-1}^{-1}} U_1 D_w,\end{aligned}$$

where $(w^*w)^{-1} = ((|w_j|^2)^{-1})$, $(w^*w)_{-1}^{-1} = (|w_{j-1}|^{-2}) \in C^b(\mathbb{Z})$ **defined so**.

Proof. It is clear that the being bounded away from zero implies the invertibility. If not being bounded away from zero, there is a subsequence $(|w_{j(k)}|^2)$ converging to zero. Since each $|w_{j(k)}|^2$ belongs to the spectrum $\sigma(S_w^*S_w)$ of $S_w^*S_w = D_{w^*w}$, that is a compact subset of $\mathbb{C}$, thus the zero point belongs to $\sigma(S_w^*S_w)$. This says that $S_w^*S_w$ is not invertible in $\mathbb{B}(L^2(\mathbb{N}))$.

If w is invertible in $C^b(\mathbb{Z})$, then $U_w^*U_w = D_{w^*w}$ is invertible, and thus $D_{w^*w}^{-1}U_w^* = D_{(w^*w)^{-1}}U_w^*$ is the left inverse for U_w. Also, $U_wU_w^* = D_{(w^*w)_{-1}}$ is invertible, and hence $U_w^*D_{(w^*w)_{-1}}^{-1} = U_w^*D_{(w^*w)_{-1}^{-1}}$ is the right inverse for U_w. The similar holds for $U_wU_w^*$. The converse for these also holds. □

Lemma 2.3. *The weighted shift operator S_w for $w = (w_n) \in C^b(\mathbb{N})$ is bounded, with the operator norm $\|S_w\| = \|w\|_\infty = \|S_w^*\|$. Namely, S_w, $S_w^* \in \mathbb{B}(L^2(\mathbb{N}))$.*

The similar holds for U_w and U_w^*.

Proof. For $\xi = (\xi_n) \in L^2(\mathbb{N})$,

$$\|S_w\xi\|^2 = \langle S_w\xi, S_w\xi\rangle = \langle S_w^*S_w\xi, \xi\rangle = \langle D_{w^*w}\xi, \xi\rangle.$$

The Cauchy-Schwarz inequality implies that

$$\langle D_{w^*w}\xi, \xi\rangle \le \|D_{w^*w}\xi\|\|\xi\| \le \|D_{w^*w}\|\|\xi\|^2 = \|w\|_\infty^2\|\xi\|^2.$$

It thus follows that $\|S_w\| \le \|w\|_\infty$.

Conversely, $\|S_we_j\| = \|w_je_{j+1}\| = |w_j|$. Hence $\|S_w\| \ge \|w\|_\infty$.

Similarly, we obtain that $\|S_w^*\| = \|w\|_\infty$. It is well known that the operator norm of bounded operators preserves the involution $*$ (cf. [11]). □

Remark. The above Lemma 2.1 together with the C^*-norm condition for the operator norm of $\mathbb{B}(H)$ as $\|B^*B\| = \|B\|^2$ implies the preceding lemma.

Proposition 2.4. *For any $w = (w_j) \in C^b(\mathbb{N})$, we have the unique polar decomposition $S_w = (S_1D_{e^{i\arg(w)}})D_{|w|}$, with $S_1D_{e^{i\arg(w)}}$ a proper isometry and $\ker(S_w) = \ker(S_1D_{e^{i\arg(w)}})$, where $|w| = (|w_j|) \in C^b(\mathbb{N})$ and $e^{i\arg(w)} = (e^{i\arg(w_j)}) \in C^b(\mathbb{N})$ with $w_j = e^{i\arg(w_j)}|w_j|$ the polar decomposition of $w_j \in \mathbb{C}$, with each $\arg(w_j) \in [0, 2\pi]$ and $i^2 = -1$.*

Also, we have the polar decomposition $U_w = (U_1D_{e^{i\arg(w)}})D_{|w|}$ on $L^2(\mathbb{Z})$ in the same sense, with $U_1D_{e^{i\arg(w)}}$ a unitary.

Proof. It follows from Lemma 2.1 above that $\sqrt{S_w^*S_w} = D_{|w|}$. Compute that

$$(S_1D_{e^{i\arg(w)}})^*(S_1D_{e^{i\arg(w)}}) = D_{e^{-i\arg(w)}}S_1^*S_1D_{e^{i\arg(w)}} = 1,$$
$$(S_1D_{e^{i\arg(w)}})(S_1D_{e^{i\arg(w)}})^* = S_1D_{e^{i\arg(w)}}D_{e^{-i\arg(w)}}S_1^* = 0 \oplus 1,$$

and hence $S_1D_{e^{i\arg(w)}}$ is an isometry and not a unitary. The uniqueness follows from the equality of the kernels in the statement. □

Remark. This result certainly generalizes a similar statement in the case that each component w_j is non-negative, as mentioned before [2, Lemma 2.1].

As well, we have

Lemma 2.5. *We have* $S_w = S_{|w|}D_{e^{i\arg(w)}}$ *with* $D_{e^{i\arg(w)}}$ *a unitary of* $\mathbb{B}(L^2(\mathbb{N}))$.

Also, $U_w = U_{|w|}D_{e^{i\arg(w)}}$ *with* $D_{e^{i\arg(w)}}$ *a unitary of* $\mathbb{B}(L^2(\mathbb{Z}))$.

Moreover, as mentioned in [2], it holds that

Proposition 2.6. ([4, Proposition 8.1]). *The unilateral weighted shift operator* S_w *for* $w = (w_j) \in C^b(\mathbb{N})$ *is unitarily equivalent to the unilateral weighted shift operator* $S_{|w|}$ *with* $|w| = (|w_j|) \in C^b(\mathbb{N})$.

The similar holds for U_w *bilateral.*

Remark. Constructed in the proof below is a unitary equivalence between S_w and $S_{|w|}$ by a diagonal unitary operator.

Proof. Let V be a unitary operator defined on $L^2(\mathbb{N})$ by $Ve_n = \lambda_n e_n$ with $|\lambda_n| = 1$ for all $n \in \mathbb{N}$. Compute

$$VS_wV^*e_n = \lambda_{n+1}w_n\overline{\lambda_n}e_{n+1}$$

and suppose that

$$S_{|w|}e_n = |w_n|e_{n+1} = \lambda_{n+1}w_n\overline{\lambda_n}e_{n+1}$$

for $n \in \mathbb{N}$. If we take $\lambda_1 = 1$, then $\lambda_2 = e^{-i\arg(w_1)}$, and then $\lambda_3 = e^{-i\arg(w_2)}e^{i\arg(w_1)}$, and inductively, we can determine such a unitary operator V. □

Corollary 2.7. *If* $w \in C^b(\mathbb{N})$ *is a unitary, then* S_w *is unitarily equivalent to* S_1.

If $w \in C^b(\mathbb{Z})$ *is a unitary, then* U_w *is unitarily equivalent to* U_1.

Remark. If $T \in \mathbb{B}(L^2(\mathbb{N}))$ is unitarily equivalent to S_w, then $T = VS_wV^*$ for some unitary V on $L^2(\mathbb{N})$. Then $T^*T = 1$ and $TT^* = V(S_wS_w^*)V^*$. Thus, T is the same as S_w up to the choice of a basis for $L^2(\mathbb{N})$. Namely, for any $j, k \in \mathbb{N}$,

$$\langle Te_j, e_k\rangle = \langle S_w(V^*e_j), V^*e_k\rangle.$$

We denote by $\sigma(B)$ the (full) **spectrum** of a bounded (linear) operator $B \in \mathbb{B}(H)$ on a Hilbert space H. By definition, a complex number $\lambda \in \mathbb{C}$ does not belong to $\sigma(B)$ if and only if $\lambda 1 - B$ is invertible in $\mathbb{B}(H)$.

Corollary 2.8. *We have that* $\sigma(S_w) = \sigma(S_{|w|})$ *and* $\sigma(U_w) = \sigma(U_{|w|})$.

Proof. The unitary equivalence between bounded operators as in Proposition 2.6 above implies the equality of their spectrums. □

Proposition 2.9. ([4, Proposition 8.4]). *The unilateral weighted shift operator* S_w *for* $w = (w_j) \in C^b(\mathbb{N})$ *is unitarily equivalent to* zS_w *for any* $z \in \mathbb{T}$.

The same also holds for U_w *bilateral.*

Proof. Define a unitary operator V on $L^2(\mathbb{N})$ by $Ve_n = z^n e_n$ for $n \in \mathbb{N}$. Then compute

$$\begin{aligned} VS_wV^*e_n &= z^{-n}VS_we_n = z^{-n}w_nVe_{n+1} \\ &= z^{-n}w_nz^{n+1}e_{n+1} = zw_ne_{n+1} = zS_we_n. \end{aligned}$$

□

Corollary 2.10. *For any real $\theta \in \mathbb{R}$, We have the equalities of the spectrums:*

$$\sigma(S_w) = \sigma(e^{i\theta}S_w) = e^{i\theta}\sigma(S_w).$$

Namely, it says that the spectrum of S_w is circular in this sense.

The same holds for U_w bilateral.

Proposition 2.11. *If $w \in C^b(\mathbb{N})$ is a unitary, then we have*

$$\sigma(S_w) = \sigma(S_1) = D = \{z \in \mathbb{C} \,|\, |z| \leq 1\},$$

that is, D is the **closed** *unit disk in $\mathbb{C}$.*

If $w \in C^b(\mathbb{Z})$ is a unitary, then we have

$$\sigma(U_w) = \sigma(U_1) = \mathbb{T}.$$

Proof. The first equality in the first statement follows from the unitary equivalence between S_w and $S_{|w|} = S_1$ as in Corollary 2.7. The second equality in the first statement is well known. Refer to [10].

The second statement follows similarly. May use Corollary 2.10. □

Recall that an element B of $\mathbb{B}(L^2(\mathbb{N}))$ is said to be a **Fredholm** operator if the kernel $\ker(B)$ is finite dimensional and the image $\mathrm{im}(B)$ is finite codimensional. In this case, the **index** of B is defined to be an integer:

$$\mathrm{index}(B) = \dim(\ker(B)) - \dim(L^2(\mathbb{N})/\mathrm{im}(B)),$$

where $L^2(\mathbb{N})/\mathrm{im}(B)$ is the quotient space of $L^2(\mathbb{N})$ by $\mathrm{im}(B)$ closed in this case.

Lemma 2.12. *If $w = (w_n)$ is an invertible bounded sequence of non-zero complex numbers in $C_0(\mathbb{N})$, then S_w is irreducible. The converse also holds.*

If $w = (w_n)$ is an invertible bounded sequence of complex numbers in $C_0(\mathbb{N})$, then S_w is a Fredholm operator with index -1 and S_w^* is a Fredholm operator with index 1. Moreover,

$$\mathrm{index}(S_w) = \mathrm{index}(S_1) = \mathrm{index}(S_{|w|}) = -1.$$

For U_w bilateral with w invertible in $C^b(\mathbb{Z})$, we have

$$\mathrm{index}(U_w) = \mathrm{index}(U_1) = \mathrm{index}(U_{|w|}) = 0 = \mathrm{index}(U_w^*).$$

But if w is unitary in $C^b(\mathbb{Z})$, then U_w is not irreducible.

Proof. In the first case, note that $S_w = S_1 D_w$ with D_w invertible. Hence, the irreducibility of S_w is equivalent to that of S_1. It is known that S_1 is irreducible (cf. [11]). For the converse, if S_w is irreducible, then suppose that w is not invertible. Then we may assume that D_w is a compact operator, so is S_w. But this implies that S_w is not irreducible as a fact in the invariant subspace problem ([10]).

As in the case above, note also that S_w is injective, so that $\ker(S_w) = \{0\}$ and $\mathrm{im}(S_w) \cong L^2(\mathbb{N} \setminus \{1\})$, and that $\ker(S_w^*) = \mathbb{C}e_1$ and $\mathrm{im}(S_w^*) = L^2(\mathbb{N})$. Indeed, as well,

$$\begin{aligned}\mathrm{index}(S_w) &= \mathrm{index}(S_1) + \mathrm{index}(D_w) = \mathrm{index}(S_1) \\ &= \mathrm{index}(S_{|w|}) + \mathrm{index}(D_{e^{i\mathrm{aug}(w)}}) = \mathrm{index}(S_{|w|}).\end{aligned}$$

Note that an invertible bounded operator always has index zero by definition.

As a fact in the invariant subspace problem, it is known that a normal operator such as unitary operators always has a non-trivial invariant closed subspace via functional calculus ([10]). □

Lemma 2.13. *If w_n is the first zero of $w \in C^b(\mathbb{N})$, then $S_w = F_n \oplus S_{w'}$ a diagonal (or direct) sum of the finite rank operator F_n identified with*

$$\begin{pmatrix} 0 & & & \\ w_1 & \ddots & & \\ & \ddots & \ddots & \\ & & w_{n-1} & 0 \end{pmatrix}$$

and the weighted shift operator $S_{w'}$ of $w' = (w'_j)$ with $w'_j = 0$ for $1 \leq j \leq n$ and $w'_j = w_j$ for $j \geq n+1$, identified with its restriction to $L^2(\mathbb{N} \setminus \{1, \cdots, n\})$.

For $w \in C^b(\mathbb{Z})$, if $w_n = 0$ for some $n \in \mathbb{Z}$, then $U_w = U_{w'} \oplus U_{w''}$, where $w' = (w'_j)$ with $w'_j = w_j$ for $j \leq n-1$ and $w'_j = 0$ for $j \geq n$ and $w'' = (w''_j)$ with $w''_j = 0$ for $j \leq n$ and $w''_j = w_j$ for $j \geq n+1$. Namely, $U_{w''}$ is identified with $S_{w''}$ on $L^2(\mathbb{N})$, where each $k \in \mathbb{N}$ is identified with $k' = k+n-1$ of the set $\{k' \in \mathbb{Z} \,|\, k' \geq n\}$, and $U_{w'}$ is identified with $S^*_{\overline{w'}}$ on $L^2(\mathbb{N})$ for $\overline{w'} = (\overline{w'_j})$, where each $k \in \mathbb{N}$ is identified with $k' = n-k$ of the set $\{k' \in \mathbb{Z} \,|\, k' \leq n-1\}$. In other words, U_w is the direct sum of the forward and backward unilateral shift operators $S_{w''}$ and $S^*_{\overline{w'}} = B_{w'}$ so denoted, so that

$$U_w = \begin{pmatrix} \ddots & & \\ \ddots & \ddots & \\ & w_{n-1} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & & \\ w_{n+1} & \ddots & \\ & \ddots & \ddots \end{pmatrix} = B_{w'} \oplus S_{w''},$$

where the last equality is obtained by converting the orthonormal basis for $L^2(\mathbb{N})$.

Proof. The first part of the statement is clear.

For the second, note that $U_{w'}e_{n-1} = 0$ and for $j = n - k \leq n - 2$,

$$U_{w'}e_j = w_{j+1}e_{j+1} = w_{n-(k-1)}e_{n-(k-1)},$$

which is identified with $w_{k-1}e_{k-1} = S^*_{\overline{w'}}e_k$ with e_k identified with e_j. □

We denote by $C_0(\mathbb{N})$ the C^*-algebra of all bounded sequences $w = (w_j)$ of complex numbers vanishing at infinity, so that $\lim_{j\to\infty} w_n = 0$. Denote by $C_0(\mathbb{Z})$ the C^*-algebra of all bounded sequences vanishing at both $\pm\infty$.

We denote by $\mathbb{K}(H)$ the C^*-algebra of all compact operators on a Hilbert space H.

Lemma 2.14. *If $w = (w_j) \in C_0(\mathbb{N})$, then S_w and S_w^* are compact operators.*

If $w = (w_j) \in C_0(\mathbb{Z})$, then U_w and U_w^* belong to $\mathbb{K}(L^2(\mathbb{Z}))$.

Proof. In this case, these operators are norm limits of finite rank operators. □

Lemma 2.15. *If $w = (w_j) \in C^b(\mathbb{N})$ has no or finitely many components w_j not equal to 1, then S_w and S_w^* are not compact, but $1 - S_w^*S_w$ and $1 - S_wS_w^*$ are compact operators.*

The same holds for U_w and U_w^*.

Proof. It is clear. □

A bounded operator $B \in \mathbb{B}(H)$ on a Hilbert space H is said to be **essentially invertible** if $\pi(B)$ is invertible in the Calkin algebra $\mathbb{B}(H)/\mathbb{K}(H) = \mathbb{B}/\mathbb{K}$, where π is the quotient map from $\mathbb{B}$ to $\mathbb{B}/\mathbb{K}$. This is equivalent to say that B is a Fredholm operator on H.

Corollary 2.16. *If $w \in C^b(\mathbb{N})$ has no or finitely many zero components, then S_w and S_w^* are essentially invertible.*

The same holds for U_w and U_w^*.

Lemma 2.17. *For S_w with $w \in C^b(\mathbb{N})$, the (additive) commutator $[S_w^*, S_w] = S_w^*S_w - S_wS_w^*$ is given by*

$$[S_w^*, S_w] = D_{w^*w} - (0 \oplus D_{w^*w}),$$

so that $[S_w^*, S_w]e_1 = |w_1|^2e_1$ and $[S_w^*, S_w]e_n = (|w_n|^2 - |w_{n-1}|^2)e_n$ for $n \geq 2$.

For $w \in C^b(\mathbb{Z})$, we have

$$[U_w^*, U_w] = D_{w^*w} - D_{(w^*w)_{-1}},$$

so that $[U_w^*, U_w]e_n = (|w_n|^2 - |w_{n-1}|^2)e_n$ for each $n \in \mathbb{Z}$.

Corollary 2.18. *The unilaterel S_w is* **normal**, *that is, $[S_w^*, S_w] = 0$ in $\mathbb{B}$, if and only if $w_1 = 0$ and $|w_n| = |w_{n-1}|$ for any $n \geq 2$, so that w is the zero sequence.*

The bilateral U_w is normal if and only if $|w_n| = |w_{n-1}|$ for any $n \in \mathbb{Z}$.

The S_w is **essentially normal**, *that is, $[\pi(S_w)^*, \pi(S_w)] = 0$ in $\mathbb{B}/\mathbb{K}$, if and only if $D_{w^*w} - (0 \oplus D_{w^*w}) \in \mathbb{K}$, if and only if $w^*w - (0, w^*w) \in C_0(\mathbb{N})$.*

In particular, if $w \in C_0(\mathbb{N})$*, then* S_w *is essentially normal.*

The U_w *is essentially normal, that is,* $[\pi(U_w)^*, \pi(U_w)] = 0$ *in* $\mathbb{B}/\mathbb{K}$*, if and only if* $D_{w^*w} - D_{(w^*w)_{-1}} \in \mathbb{K}$*, if and only if* $w^*w - (w^*w)_{-1} \in C_0(\mathbb{Z})$*.*

In particular, if $w \in C_0(\mathbb{Z})$*, then* U_w *is essentially normal.*

A bounded operator $B \in \mathbb{B}(H)$ is said to be **hyponormal** if $B^*B \geq BB^*$, i.e., $\langle B^*B\xi, \xi\rangle \geq \langle BB^*\xi, \xi\rangle$ for every $\xi \in H$.

Lemma 2.19. *A bounded operator* $B \in \mathbb{B}(H)$ *is hyponormal if and only if* $\|B\xi\| \geq \|B^*\xi\|$ *for any* $\xi \in H$*.*

Proof. By definition, $B^*B \geq BB^*$ implies that for any $\xi \in H$,

$$\|B\xi\|^2 = \langle B^*B\xi, \xi\rangle \geq \langle BB^*\xi, \xi\rangle = \|B^*\xi\|^2.$$

□

Remark. Note that B^* is hyponormal if $BB^* \geq B^*B$. Namely, $B^*B \leq BB^*$ as the reverse inequality of the inequality of B hyponormal.

Proposition 2.20. ([4, Proposition 8.6]). *For* $w \in C^b(\mathbb{N})$*, the weighted unilateral shift* S_w *is hyponormal if and only if the sequence* $|w| = (|w_j|) \in C^b(\mathbb{N})$ *is monotone increasing as that* $|w_j| \leq |w_{j+1}|$ *for* $j \in \mathbb{N}$*.*

Also, the similar holds for U_w *with* $w \in C^b(\mathbb{Z})$*.*

Similarly, S_w^* *is hyponormal if and only if the sequence* $|w| = (|w_j|)$ *is monotone decreasing.*

Also, the similar holds for U_w^**.*

Proof. Note that

$$S_w^*S_w - S_wS_w^* = D_{w^*w} - [0 \oplus D_{w^*w}] = |w_1|^2 \oplus (\oplus_{j=1}^{\infty}|w_{j+1}|^2 - |w_j|^2) \geq 0$$

if and only if $|w_{j+1}| \geq |w_j|$ for all $j \in \mathbb{N}$. □

Corollary 2.21. *If* S_w *for* $w \in C^b(\mathbb{N})$ *is hyponormal, then the norm* $\|S_w\| = \lim_{j\to\infty} |w_j|$ *the limit at + infinity.*

If U_w *for* $w \in C^b(\mathbb{Z})$ *is hyponormal, then* $\|U_w\| = \lim_{j\to\infty} |w_j|$ *the limit at + infinity, and the infimum* $\inf_{j\in\mathbb{Z}} |w_j| = \lim_{j\to-\infty} |w_j|$ *the limit at − infinity.*

Similarly, if S_w^* *is hyponormal, then* $\inf_{j\in\mathbb{N}} |w_j| = \lim_{j\to+\infty} |w_j|$*.*

Also, if U_w^* *is hyponormal, then* $\|U_w^*\| = \lim_{j\to-\infty} |w_j|$ *and* $\inf_{j\in\mathbb{Z}} |w_j| = \lim_{j\to+\infty} |w_j|$

We may define $\|w\|_0 = \inf_{j\in\mathbb{Z}} |w_j|$ (or $\inf_{j\in\mathbb{N}} |w_j|$) for $w \in C^b(\mathbb{Z})$ (or $C^b(\mathbb{N})$) and use this notation in what follows, which may be called as **infimum height** (or weight) for w. It is clear that $w \in C^b(\mathbb{Z})$ is invertible if and only if $\|w\|_0 > 0$. It also holds that $\|\alpha w\|_0 = |\alpha|\|w\|_0$ for $\alpha \in \mathbb{C}$ as only one of three axioms of norms, but does not hold by some particular examples that $\|w + w'\|_0 \leq \|w\|_0 + \|w'\|_0$ for some $w, w' \in C^b(\mathbb{Z})$. As well, for any $w, w' \in C^b(\mathbb{Z})$, we have $\|w\|_0\|w'\|_0 \leq \|w \cdot w'\|_0$ as the reverse submultiplicativity.

Proof. For instance, let $w = (0, 1, 1, \cdots)$ and $w' = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ in $C^b(\mathbb{N})$. Then

$$\|w\|_0 + \|w'\|_0 = 0 + 0 < \|w + w'\|_0 = 1.$$

Also, for any $w = (w_j), w' = (w'_j) \in C^b(\mathbb{Z})$, we have $\|w\|_0\|w'\|_0 \leq |w_j w'_j|$ for any $j \in \mathbb{N}$. Hence $\|w\|_0\|w'\|_0 \leq \|w \cdot w'\|_0$. In particular, if $w = (\frac{1}{2}, 1, 1, \cdots)$ and $w' = (2, 1, 1, \cdots)$ in $C^b(\mathbb{N})$, then

$$\|w\|_0\|w'\|_0 = \frac{1}{2} < \|w \cdot w'\|_0 = 1.$$

□

For any $A \in \mathbb{B}(H)$, we denote by $\sigma_p(A)$ the **point** spectrum of A consisting of $\lambda \in \mathbb{C}$ such that the kernel $\ker(\lambda 1 - A)$ is nonzero.

We define that an element $w = (w_j) \in C^b(\mathbb{N})$ or $C^b(\mathbb{Z})$ is **continuous at plus infinity** $+\infty$ to $\alpha \in \mathbb{C}$ if $\lim_{j\to+\infty} w_j = \alpha$. As well, an element $w = (w_j) \in C^b(\mathbb{Z})$ is **continuous at minus infinity** $-\infty$ to β if $\lim_{j\to-\infty} w_j = \beta$. In particular, if $|w| = (|w_j|)$ is continuous at $+\infty$ to $\|S_w\|$, or if $|w| = (|w_j|)$ is continuous at $+\infty$ to $\|U_w\|$ and continuous at $-\infty$ to $\|w\|_0$, then we may define that the corresponding S_w and U_w are **hyponormal-like**, respectively. This condition holds, in particular, if S_w and U_w are hyponormal.

We define that an element $w = (w_j) \in C^b(\mathbb{N})$ or $C^b(\mathbb{Z})$ is **upper boundedly continuous at plus infinity** $+\infty$ to $\alpha \in \mathbb{C}$ if $\lim_{j\to+\infty} w_j = \alpha$ and $|w_j| \leq |\alpha|$ for any $j \geq n_0$ for some $n_0 \in \mathbb{N}$. As well, an element $w = (w_j) \in C^b(\mathbb{Z})$ is **lower boundedly continuous at minus infinity** $-\infty$ to $\beta \in \mathbb{C}$ if $\lim_{j\to-\infty} w_j = \beta$ and $|w_j| \geq |\beta|$ for any $j \geq n_0$ for some $n_0 \in \mathbb{N}$. In this case, we may define that the corresponding S_w and U_w are **less hyponormal-like**. In this definition, we may replace $w = (w_j)$ with $|w| = (|w_j|)$ from the beginning, as in what follows.

Similarly, we may define that S_w^* is **hyponormal-like** if $|w| = (|w_j|)$ is continuous at $+\infty$ to $\|w\|_0$, and U_w^* is **hyponormal-like** if $|w| = (|w_j|)$ is continuous at $-\infty$ to $\|U_w^*\|$ and continuous at $+\infty$ to $\|w\|_0$.

Also, S_w^* is **less hyponormal-like** if $|w| = (|w_j|)$ is lower boundedly continuous at $+\infty$ to β, and U_w^* is **less hyponormal-like** if $|w| = (|w_j|)$ is upper boundedly continuous at $-\infty$ to α and lower boundedly continuous at $+\infty$ to β.

Remark. Under those assumptions as above, we could obtain the similar results on S_w^* and U_w^* as those on S_w and U_w given below in this section, but omitted or only commented. The upper or lower boundedness at $\pm\infty$ is crucial as a technical assumption below. As a problem to be considered, this condition may (or not) be weakened only to the continuity at $\pm\infty$. This remark is also applied for several results given below in this section.

Proposition 2.22. (Extended from [4, Proposition 8.7]). *Suppose that every component w_j of $w \in C^b(\mathbb{N})$ or $C^b(\mathbb{Z})$ is non zero.*

If S_w is hyponormal, or if $|w| = (|w_j|)$ is continuous at $+\infty$ to $\|S_w\|$ (namely, S_w is hyponormal-like), and if $|\lambda| < \|S_w\|$, then $\lambda \in \sigma_p(S_w^*)$ and $\dim(\ker(S_w^* - \lambda 1)) = 1$.

If $|\lambda| \geq \|S_w\|$*, then* $\lambda \notin \sigma_p(S_w^*)$*.*

The first statement also holds by replacing $\|S_w\|$ *with* $\alpha \leq \|S_w\|$ *if* $|w|$ *is continuous at* $+\infty$ *to* α*, and the second holds if* $|w|$ *is upper boundedly continuous at* $+\infty$ *to* α*.*

If U_w *is hyponormal, or if* $|w| = (|w_j|)$ *is continuous at* $\pm\infty$ *to* $\|U_w\|$ *and* $\|w\|_0 = \inf_{j\in\mathbb{Z}} |w_j|$ *respectively (namely,* U_w *is hyponormal-like), and if* $\|w\|_0 = \lim_{j\to-\infty} |w_j| < |\lambda| < \|U_w\|$*, then* $\lambda \in \sigma_p(U_w^*)$ *and* $\dim(\ker(U_w^* - \lambda 1)) = 1$*.*

If $|\lambda| \geq \|U_w\|$ *or* $|\lambda| \leq \|w\|_0 = \lim_{j\to-\infty} |w_j|$*, then* $\lambda \notin \sigma_p(U_w^*)$*.*

The first statement also holds by replacing $\|U_w\|$ *and* $\|w\|_0$ *with* $\alpha \leq \|U_w\|$ *and* $\beta \geq \|w\|_0$ *if* $|w|$ *is continuous at* $+\infty$ *to* α *and continuous at* $-\infty$ *to* β*, and the second holds if* $|w|$ *is upper boundedly continuous at* $+\infty$ *to* α *and lower boundedly continuous at* $-\infty$ *to* β*.*

Proof. Suppose that $S_w^* x = \sum_{j=2}^\infty \overline{w_{j-1}} x_j e_{j-1} = \lambda x$ for some $\lambda \in \mathbb{C}$ and $x = \sum_{j=1}^\infty x_j e_j \in L^2(\mathbb{N})$. Then $\overline{w_{j-1}} x_j = \lambda x_{j-1}$ for $j \geq 2$. It then follows that for $j \geq 2$,

$$x_j = \frac{\lambda}{\overline{w_{j-1}}} x_{j-1} = \cdots = \frac{\lambda^{j-1}}{\overline{w_{j-1}} \cdots \overline{w_1}} x_1,$$

so that

$$\|x\|^2 = \sum_{j=1}^\infty |x_j|^2 = |x_1|^2 + |x_1|^2 \sum_{j=2}^\infty \frac{|\lambda|^{2(j-1)}}{|w_{j-1} \cdots w_1|^2}.$$

Now suppose that $|\lambda| < \rho < \|S_w\|$. Then there is $n_0 \in \mathbb{N}$ such that $|w_j| > \rho$ for $j \geq n_0$. If $j \geq n_0 + 1$, then

$$\frac{|\lambda|^{2(j-1)}}{|w_{j-1} \cdots w_1|^2} \leq \frac{|\lambda|^{2n_0}}{|w_{n_0} \cdots w_1|^2} \left(\frac{\rho}{|w_{n_0}|}\right)^{2(j-1-n_0)}$$

with $\frac{\rho}{|w_{n_0}|} < 1$. Therefore, the series displayed above converges for any $x_1 \in \mathbb{C}$, and then $x = \sum_{j=1}^\infty x_j e_j \in L^2(\mathbb{N})$ is defined with each $x_j = \frac{\lambda}{\overline{w_{j-1}}} x_{j-1}$, to satisfy $(S_w^* - \lambda 1)x = 0$.

If $|\lambda| = \|S_w\|$ (or $|\lambda| > \|S_w\|$), then the above equation for x_j in terms of x_1 implies that $|x_j| \geq |x_1|$. Hence it follows that $x_1 = 0 = x_j$ for every $j \in \mathbb{N}$.

Note as well that the arguments above essentially only depend on the behavior at infinity and can be converted to the case of converging to α if necessary changing the base point x_1 to a suitable point x_k for some k large enough.

For U_w, use the same approach to compute $U_w^* x = \lambda x$ to determine similarly x_j as well as x_{-j} for j positive as

$$x_{-j} = \frac{\overline{w_{-j}}}{\lambda} x_{-j+1} = \cdots = \frac{\overline{w_{-j}} \cdots \overline{w_{-1}}}{\lambda^j} x_0.$$

Then suppose that $\lim_{j\to-\infty} |w_j| < \rho' < |\lambda| < \rho < \|U_w\|$, to deduce the similar estimate in the terms of the series of x over positive and negative integers. □

Proposition 2.23. (Extended from [4, Proposition 8.10]). *Suppose that every component w_j of $w \in C^b(\mathbb{N})$ or $C^b(\mathbb{Z})$ is non zero. Then the point spectrum $\sigma_p(S_w) = \emptyset$ the empty set.*

The same also holds for U_w for $|w| = (|w_j|) \in C^b(\mathbb{Z})$ continuous at infinity $+\infty$ to $\|U_w\|$ or upper boundedly continuous at $+\infty$ to $\alpha \leq \|U_w\|$.

Proof. Suppose that $S_w x = \lambda x$ for some $\lambda \in \mathbb{C}$ and $x = (x_j) \in L^2(\mathbb{N})$. If $\lambda = 0$, then $x = 0$. If $\lambda \neq 0$, then $\lambda x_1 = 0$ and $\lambda x_j = w_{j-1} x_{j-1}$ for $j \geq 2$, so that

$$x_j = \frac{w_{j-1}}{\lambda} x_{j-1} = \cdots = \frac{w_{j-1} \cdots w_1}{\lambda^{j-1}} x_1 = 0,$$

and hence $x = 0$.

If $|\lambda| \geq \|U_w\|$, then for $j > 0$,

$$x_{-j} = \frac{\lambda}{w_{-j}} x_{-j+1} = \cdots = \frac{\lambda^j}{w_{-j} \cdots w_{-1}} x_0$$

and thus,

$$|x_{-j}| \geq \left(\frac{|\lambda|}{\|U_w\|} \right)^j |x_0|.$$

Since $|x_{-j}| \to 0$ as $j \to \infty$, then $|x_0| = 0$. Hence $x_j = 0$ for any $j \in \mathbb{Z}$ follows.

Similarly, if $0 < |\lambda| < \rho < \|U_w\|$, then there is $n_0 \in \mathbb{Z}$ such that $|w_j| \geq \rho$ for $j \geq n_0$. Since $x_j \to 0$ as $j \to \infty$, it follows from the above equation for x_j that $x_{n_0} = 0$. Hence $x = 0$.

The arguments above also valid in the case where $|w|$ is upper boundedly continuous at infinity. □

Recall that for any $A \in \mathbb{B}(H)$, the spectrum $\sigma(A)$ is decomposed into the following disjoint union:

$$\sigma(A) = \sigma_p(A) \sqcup \sigma_r(A) \sqcup \sigma_c(A),$$

where the **residue** spectrum $\sigma_r(A)$ consists of $\lambda \in \mathbb{C}$ such that $\ker(\lambda 1 - A) = \{0\}$ but the closure of the range $(\lambda 1 - A)(H)$ of $\lambda 1 - A$ is not equal to H, and the **continuous** spectrum $\sigma_c(A)$ consists of $\lambda \in \mathbb{C}$ such that $\ker(\lambda 1 - A) = \{0\}$, the closure of the range $(\lambda 1 - A)(H)$ is H, but the closure is not equal to the range. Also, denote by $\sigma_{ap}(A)$ the **approximate** point spectrum of A, consisting of $\lambda \in \mathbb{C}$ such that there is a sequence (ξ_n) of H with norm 1 such that $\|A\xi_n - \lambda \xi_n\| \to 0$ as $n \to \infty$, or equivalently, $\inf\{\|(A - \lambda)\xi\| \mid \xi \in H, \|\xi\| = 1\} = 0$, or $A - \lambda 1$ is not left invertible. As facts,

- If $\lambda \in \sigma_r(A)$, then its complex conjugate $\overline{\lambda} \in \sigma_p(A^*)$.
- If $\lambda \in \sigma_p(A)$, then $\overline{\lambda} \in \sigma_r(A^*) \sqcup \sigma_p(A^*)$.
- It holds that $\lambda \in \sigma_c(A)$ if and only if $\overline{\lambda} \in \sigma_c(A^*)$.
- Note as well that $\sigma(A^*) = \overline{\sigma(A)}$ the complex conjugate of $\sigma(A)$.
- Note that both $\sigma_p(A) \sqcup \sigma_c(A)$ and $\partial\sigma(A) \subset \sigma_{ap}(A)$ contained, and $\sigma_{ap}(A)$ is closed, where $\partial\sigma(A)$ is the boundary of $\sigma(A)$. Note also that $\lambda \in \sigma(A) \setminus \sigma_{ap}(A)$

if and only if the range of $A - \lambda 1$ is closed, but proper, and $\ker(A - \lambda 1) = \{0\}$, so that $\sigma(A) \setminus \sigma_{ap}(A) \subset \sigma_r(A)$. For these facts, may refer to [4] or [10].

We denote by $B(r)$ the closed **ball** in $\mathbb{C}$ with center 0 and radius $r > 0$ and by $B(r_1, r_2)$ the closed (balled) **band** (or annulus) in $\mathbb{C}$ with center 0 and (outer and inner) radii r_1 and r_2 with $0 < r_2 \leq r_1$. Set $B(r, 0) = B(r)$. Let $B^\circ(r)$ and $B^\circ(r_1, r_2)$ be the interiors of $B(r)$ and $B(r_1, r_2)$ respectively. Denote by $\partial B(r)$ and $\partial B(r_1, r_2)$ the boundaries of $B(r)$ and $B(r_1, r_2)$ respectively. As a note, $B(r_1, r_2) = B(r_1) \cap B^\circ(r_2)^c$ with $B^\circ(r_2)^c$ the complement of $B^\circ(r_2)$ in $\mathbb{C}$. Also, $B(r, r) = \partial B(r)$.

Corollary 2.24. (Extended from [10, Example 3.1.21]). *Suppose that every component w_j of $w = (w_j) \in C^b(\mathbb{N})$ or $C^b(\mathbb{Z})$ is non zero.*

If S_w is hyponormal, or if $|w| = (|w_j|)$ is continuous at $+\infty$ to $\|S_w\|$, then we have $\sigma_r(S_w^*) = \emptyset$ and $\sigma(S_w^*) = \{\lambda \in \mathbb{C} \,|\, |\lambda| \leq \|S_w^*\|\}$, so that

$$\sigma(S_w) = \{\lambda \in \mathbb{C} \,|\, |\lambda| \leq \|S_w\|\} \equiv B(\|S_w\|).$$

If $|w| = (|w_j|) \in C^b(\mathbb{Z})$ is continuous at $+\infty$ to $\|U_w\|$ and continuous at $-\infty$ to $\|w\|_0$, then $\sigma_r(U_w^*) = \emptyset$ and

$$\sigma(U_w^*) = \{\lambda \in \mathbb{C} \,|\, \|w\|_0 = \inf_{j \in \mathbb{Z}} |w_j| \leq |\lambda| \leq \|U_w^*\|\},$$

so that

$$\sigma(U_w) = \{\lambda \in \mathbb{C} \,|\, \|w\|_0 \leq |\lambda| \leq \|U_w\|\} \equiv B(\|U_w\|, \|w\|_0).$$

Namely,

$$\begin{aligned} \sigma(S_w^*) &= \sigma_p(S_w^*) \sqcup \sigma_c(S_w^*) = B^\circ(\|S_w^*\|) \sqcup \partial B(\|S_w^*\|) \\ &= \sigma(S_w) = \sigma_r(S_w) \sqcup \sigma_c(S_w) = B^\circ(\|S_w\|) \sqcup \partial B(\|S_w\|), \end{aligned}$$

where $B^\circ(\|S_w^*\|) = \{\lambda \in \mathbb{C} \,|\, |\lambda| < \|S_w^*\|\}$ and $\partial B(\|S_w^*\|) = \{\lambda \in \mathbb{C} \,|\, |\lambda| = \|S_w^*\|\}$, with $\sigma_{ap}(S_w^*) = \sigma(S_w^*) = B(\|S_w^*\|)$ and $\sigma_{ap}(S_w) = \sigma_c(S_w) = \partial B(\|S_w\|)$, and

$$\begin{aligned} \sigma(U_w^*) &= \sigma_p(U_w^*) \sqcup \sigma_c(U_w^*) = B^\circ(\|U_w^*\|, \|w\|_0) \sqcup \partial B(\|U_w^*\|, \|w\|_0) \\ &= \sigma(U_w) = \sigma_r(U_w) \sqcup \sigma_c(U_w) = B^\circ(\|U_w\|, \|w\|_0) \sqcup \partial B(\|U_w\|, \|w\|_0), \end{aligned}$$

where $B^\circ(\|U_w^*\|, \|w\|_0) = \{\lambda \in \mathbb{C} \,|\, \|w\|_0 < |\lambda| < \|U_w^*\|\}$ and $\partial B(\|U_w^*\|, \|w\|_0) = \{\lambda \in \mathbb{C} \,|\, |\lambda| = \|w\|_0$ or $|\lambda| = \|U_w^*\|\}$, with

$$\begin{aligned} \sigma_{ap}(U_w^*) &= \sigma(U_w^*) = B(\|U_w^*\|, \|w\|_0), \\ \sigma_{ap}(U_w) &= \sigma_c(U_w) = \partial B(\|U_w\|, \|w\|_0). \end{aligned}$$

Proof. It follows from that $\sigma_p(S_w)$ and $\sigma_p(U_w)$ are empty sets in this case and that the compact set $\sigma(S_w^*)$ contains the interior $B^\circ(\|S_w^*\|)$ of $B(\|S_w^*\|)$ equal to $\sigma_p(S_w^*)$ and is contained in $B(\|S_w^*\|)$ and that the compact set $\sigma(U_w^*)$ contains the interior $B^\circ(\|U_w^*\|, \|w\|_0)$ of $B(\|U_w^*\|, \|w\|_0)$ equal to $\sigma_p(U_w^*)$ and is contained in $B(\|U_w^*\|, \|w\|_0)$. □

Corollary 2.25. *Suppose that every component w_j of $w = (w_j) \in C^b(\mathbb{Z})$ is non zero.*

If $|w| = (|w_j|) \in C^b(\mathbb{N})$ is upper boundedly continuous at $+\infty$ to α, then

$$\sigma(S_w^*) = \sigma_p(S_w^*) \sqcup \sigma_c(S_w^*), \sigma_p(S_w^*) = B^\circ(\alpha), \sigma_c(S_w^*) \supset \partial B(\alpha),$$
$$\sigma(S_w) = \sigma_r(S_w) \sqcup \sigma_c(S_w), \sigma_r(S_w) = B^\circ(\alpha), \sigma_c(S_w) \supset \partial B(\alpha),$$

with $\sigma_{ap}(S_w^*) = \sigma(S_w^*) \supset B(\alpha)$ and $\sigma(S_w)_{ap} = \sigma_c(S_w) \supset \partial B(\alpha)$.

If $|w| = (|w_j|) \in C^b(\mathbb{Z})$ is upper boundedly continuous at $+\infty$ to α and is lower boundedly continuous at $-\infty$ to β, with $0 \leq \beta < \alpha$, then

$$\sigma(U_w^*) = \sigma_p(U_w^*) \sqcup \sigma_c(U_w^*), \sigma_p(U_w^*) = B^\circ(\alpha, \beta), \sigma_c(U_w^*) \supset \partial B(\alpha, \beta),$$
$$\sigma(U_w) = \sigma_r(U_w) \sqcup \sigma_c(U_w), \sigma_r(U_w) = B^\circ(\alpha, \beta), \sigma_c(U_w) \supset \partial B(\alpha, \beta),$$

with $\sigma_{ap}(U_w^*) = \sigma(U_w^*) \supset B(\alpha, \beta)$ and $\sigma_{ap}(U_w) = \sigma_c(U_w) \supset \partial B(\alpha, \beta)$.

Remark. The assumption on the equality $\beta < \alpha$ is rather restrictive and not automatic. In fact, as noticed in the last moment, the case where $\beta = \alpha$ does hold, and the proof has already done as contained above, and the case as with $B(\alpha, \alpha), B^\circ(\alpha, \alpha)$, and $\partial B(\alpha, \alpha)$ can be contained in the above case. The left case of $\beta > \alpha$ may be considered as a problem left to be considered. As a slightly different case, as a question, if $|w| = (|w_j|)$ is lower boundedly continuous at $+\infty$ to α, then does the same statement hold by (or without) replacing S_w with S_w^*? Philosophically, this should be true. Namely, the limits at $\pm$ infinity determine the boundary (or origin) of the spectrums. Also, if $|w| = (|w_j|)$ is upper boundedly continuous at $-\infty$ to α and lower boundedly continuous at $+\infty$ to β, with $\beta < \alpha$, then the same statement does hold by replacing U_w with U_w^*. It follows from the philosophical point of view that even in the left case where $\beta > \alpha$, the same statement would hold by the same replacing as above. This remark is also applied for several results given below in this section.

Recall from [10] the following **facts**.

Let $A \in \mathbb{B}(H)$. The **essential** spectrum $\sigma_e(A)$ of A is defined to be the set of $\lambda \in \mathbb{C}$ such that $\pi(\lambda 1 - A)$ is not invertible in $\mathbb{B}(H)/\mathbb{K}(H)$. Namely, the complement $\mathbb{C} \setminus \sigma_e(A)$ consists of $\lambda \in \mathbb{C}$ such that $\lambda 1 - A$ is a Fredholm operator on H. The **Weyl** spectrum $\sigma_W(A)$ of A is defined to be the intersection of $\sigma(A + K)$ for any $K \in \mathbb{K}(H)$.

A moment of thought implies that $\sigma_e(A) = \sigma_e(A^*)$ and $\sigma_W(A^*) = \overline{\sigma_W(A)}$ for any $A \in \mathbb{B}(H)$. Also, $\sigma_W(A) = \sigma_W(A + K)$ for any $K \in \mathbb{K}(H)$.

Proof. Fix $K \in \mathbb{K}(H)$. For any $K' \in \mathbb{K}(H)$, $K + K' \in \mathbb{K}(H)$. Thus, $\sigma_W(A) \subset \sigma(A + K + K')$ for any $K' \in \mathbb{K}(H)$. Hence $\sigma_W(A) \subset \sigma_W(A + K)$. Conversely, $\sigma_W(A + K) \subset \sigma(A + K - K + K') = \sigma(A + K')$ for any $K' \in \mathbb{K}(H)$. Therefore, $\sigma_W(A + K) \subset \sigma_W(A)$. □

It is known as a **fact** that for any $A \in \mathbb{B}(H)$, $\lambda \in \mathbb{C} \setminus \sigma_e(A)$ if and only if the range of $\lambda 1 - A$ is closed, the dimension of $\ker(\lambda 1 - A)$ is finite, and the dimension of the orthogonal complement of the range of $\lambda 1 - A$ is finite.

For any $A \in \mathbb{B}(H)$, both $\sigma_e(A)$ and $\sigma_W(A)$ are non-empty closed sets, and

$$\sigma_c(A) \subset \sigma_e(A) \subset \sigma_W(A) \subset \sigma(A).$$

Proof. If $\lambda \in \sigma_c(A)$, then the range of $\lambda 1 - A$ is not closed.

If we take $K = 0$ in the definition, then $\sigma_W(A) \subset \sigma(A)$.

If $\lambda 1 - A$ is invertible in $\mathbb{B}(H)$, then $\pi(\lambda 1 - A)$ is invertible. Thus $\sigma_e(A) \subset \sigma(A)$.

For any $K \in \mathbb{K}(H)$, we have $\sigma_e(A) = \sigma_e(A + K) \subset \sigma(A + K)$ since $\pi(\lambda 1 - A) = \pi(\lambda 1 - (A + K))$. Hence, $\sigma_e(A) \subset \sigma_W(A)$. □

Moreover, for any $A \in \mathbb{B}(H)$, $\lambda \in \mathbb{C} \setminus \sigma_W(A)$ if and only if $\lambda 1 - A$ is a Fredholm operator and the index of $\lambda 1 - A$ is 0, and $\lambda \in \sigma_W(A) \setminus \sigma_e(A)$ if and only if $\lambda 1 - A$ is a Fredholm operator and the index of $\lambda 1 - A$ is not zero.

If $A \in \mathbb{B}(H)$ is normal, then $\sigma_e(A) = \sigma_W(A)$.

Corollary 2.26. (Extended from [4, Proposition 8.13] and [10, Example 6.1.21]). *Suppose that every component w_j of $w = (w_j) \in C^b(\mathbb{N})$ or $C^b(\mathbb{Z})$ is non zero.*

If S_w is hyponormal, or if $|w| = (|w_j|)$ is continuous at $+\infty$ to $\|S_w\|$, then

$$\sigma_e(S_w^*) = \sigma_c(S_w^*) = \partial B(\|S_w^*\|) = \partial B(\|S_w\|) = \sigma_{ap}(S_w) = \sigma_c(S_w) = \sigma_e(S_w)$$

and

$$\sigma_W(S_w^*) = \sigma(S_w^*) = \sigma_{ap}(S_w^*) = B(\|S_w^*\|) = B(\|S_w\|) = \sigma(S_w) = \sigma_W(S_w).$$

If $|w| = (|w_j|) \in C^b(\mathbb{Z})$ is continuous at $+\infty$ to $\|U_w\|$ and continuous at $-\infty$ to $\|w\|_0$, then

$$\begin{aligned}\sigma_e(U_w^*) &= \sigma_c(U_w^*) = \partial B(\|U_w^*\|, \|w\|_0)\\ &= \partial B(\|U_w\|, \|w\|_0) = \sigma_{ap}(U_w) = \sigma_c(U_w) = \sigma_e(U_w)\end{aligned}$$

and

$$\begin{aligned}\sigma_W(U_w^*) &= \sigma(U_w^*) = \sigma_{ap}(U_w^*) = B(\|U_w^*\|, \|w\|_0)\\ &= B(\|U_w\|, \|w\|_0) = \sigma(U_w) = \sigma_W(U_w).\end{aligned}$$

Proof. If $|\lambda| = \|S_w^*\| = \|S_w\|$, then $\lambda \in \sigma_c(S_w^*) = \sigma_c(S_w)$, so that the ranges of $\lambda 1 - S_w^*$ and $\lambda 1 - S_w$ are not closed. Thus, $\lambda \in \sigma_e(S_w^*) = \sigma_e(S_w)$.

If $|\lambda| < \|S_w^*\| = \|S_w\|$, then $\lambda \in \sigma_r(S_w)$ and $\overline{\lambda} \in \sigma_p(S_w^*)$, so that $\ker(\lambda 1 - S_w) = \{0\}$ and $\dim \ker(\overline{\lambda} 1 - S_w^*) = 1$. Moreover, since

$$\|(S_w - \lambda 1)\xi\| \geq (\|S_w\| - |\lambda|)\|\xi\|,$$

it then follows that any Cauchy sequence in the range of $S_w - \lambda 1$ converges in the range, so that the range is closed. Thus, $S_w - \lambda 1$ is a Fredholm operator with index $-1 \neq 0$. Hence $\pi(S_w - \lambda 1)$ is not invertible, so that $\lambda \notin \sigma_e(S_w)$ but $\lambda \in \sigma_W(S_w)$.

Similarly, if $|\lambda| = \|U_w^*\| = \|U_w\|$ or $|\lambda| = \|w\|_0$, then $\lambda \in \sigma_c(U_w^*) = \sigma_c(U_w)$, so that the ranges of $\lambda 1 - U_w^*$ and $\lambda 1 - U_w$ are not closed. Thus, $\lambda \in \sigma_e(U_w^*) = \sigma_e(U_w)$.

If $\|w\|_0 < |\lambda| < \|U_w^*\| = \|U_w\|$, then $\lambda \in \sigma_r(U_w)$ and $\overline{\lambda} \in \sigma_p(U_w^*)$, so that $\ker(\lambda 1 - U_w) = \{0\}$ and $\dim\ker(\overline{\lambda}1 - U_w^*) = 1$. Moreover, since

$$\|(U_w - \lambda 1)\xi\| \geq (\|U_w\| - |\lambda|)\|\xi\|,$$

it then follows that any Cauchy sequence in the range of $U_w - \lambda 1$ converges in the range, so that the range is closed. Thus, $U_w - \lambda 1$ is a Fredholm operator with index $-1 \neq 0$. Hence $\pi(U_w - \lambda 1)$ is not invertible, so that $\lambda \notin \sigma_e(U_w)$ but $\lambda \in \sigma_W(U_w)$. □

Corollary 2.27. *Suppose that every component w_j of $w = (w_j) \in C^b(\mathbb{N})$ or $C^b(\mathbb{Z})$ is non zero.*

If $|w| = (|w_j|)$ is upper boundedly continuous at $+\infty$ to α, then

$$B^\circ(\alpha)^c \supset \sigma_e(S_w^*) \supset \sigma_c(S_w^*) \supset \partial B(\alpha) \subset \sigma_{ap}(S_w) = \sigma_c(S_w) \subset \sigma_e(S_w) \subset B^\circ(\alpha)^c$$

and

$$\sigma_W(S_w^*) = \sigma(S_w^*) = \sigma_{ap}(S_w^*) \supset B(\alpha) \subset \sigma(S_w) = \sigma_W(S_w).$$

If $|w| = (|w_j|) \in C^b(\mathbb{Z})$ is upper boundedly continuous at $+\infty$ to α and lower boundedly continuous at $-\infty$ to β, with $0 \leq \beta < \alpha$, then

$$\begin{aligned} B^\circ(\alpha,\beta)^c &\supset \sigma_e(U_w^*) \supset \sigma_c(U_w^*) \supset \partial B(\alpha,\beta) \\ &\subset \sigma_{ap}(U_w) = \sigma_c(U_w) \subset \sigma_e(U_w) \subset B^\circ(\alpha,\beta)^c \end{aligned}$$

and

$$\sigma_W(U_w^*) = \sigma(U_w^*) = \sigma_{ap}(U_w^*) \supset B(\alpha,\beta) \subset \sigma(U_w) = \sigma_W(U_w).$$

In fact, we obtain

Theorem 2.28. *Under the same assumptions as above, we have*

$$\sigma_W(S_w^*) = \sigma(S_w^*) = \sigma_{ap}(S_w^*) = B(\alpha) = \sigma(S_w) = \sigma_W(S_w)$$

and

$$\sigma_e(S_w^*) = \sigma_c(S_w^*) = \partial B(\alpha) = \sigma_{ap}(S_w) = \sigma_c(S_w) = \sigma_e(S_w).$$

Similarly,

$$\sigma_W(U_w^*) = \sigma(U_w^*) = \sigma_{ap}(U_w^*) = B(\alpha,\beta) = \sigma(U_w) = \sigma_W(U_w)$$

and

$$\sigma_e(U_w^*) = \sigma_c(U_w^*) = \partial B(\alpha,\beta) = \sigma_{ap}(U_w) = \sigma_c(U_w) = \sigma_e(U_w).$$

Proof. Let F_n be a finite rank operator obtained from restricting S_w on $\mathbb{C}^n$ generated by the standard basis vectors $e_1, \cdots, e_n$ of $\mathbb{C}^n$ viewed in $L^2(\mathbb{N})$ canonically. By the assumption, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then the limit $\alpha = \lim_{j\to\infty} |w_j|$ is equal to the norm $\|S_w - F_n\|$. Therefore,

$$\sigma_W(S_w^* - F_n^*) = \sigma(S_w^* - F_n^*) = B(\alpha) = \sigma(S_w - F_n) = \sigma_W(S_w - F_n),$$

with $\sigma_W(S_w^* - F_n^*) = \sigma_W(S_w^*)$ and $\sigma_W(S_w - F_n) = \sigma_W(S_w)$, and

$$\sigma_e(S_w^* - F_n^*) = \sigma_c(S_w^* - F_n^*) = \partial B(\alpha) = \sigma_c(S_w - F_n) = \sigma_e(S_w - F_n).$$

with $\sigma_e(S_w^* - F_n^*) = \sigma_e(S_w^*)$ and $\sigma_e(S_w - F_n) = \sigma_e(S_w)$. It then follows from Corollary 2.27 above that

$$\sigma(S_w^*) = B(\alpha) = \sigma(S_w) \quad \text{and} \quad \sigma_c(S_w^*) = \partial B(\alpha) = \sigma_c(S_w).$$

Similarly, the same argument as above is applied for U_w by using Corollaries 2.26 and 2.27. □

Remark. There may be more results on this subject in the literature, or in the future to be continued. As a summary, given as a non-surprising present (to the experts) are the following tables for the spectrums of UWS S_w and BWS U_w before the last minute or so. As a note, for UWS* S_w^* and BWS* U_w^* hyponormal-like or less hyponormal-like, the same tables could be obtained by just exchanging (or replacing) S_w for (or with) S_w^* and U_w for (or with) U_w^* respectively, where, in the case of U_w^*, α at $+\infty$ and β at $-\infty$, with $\alpha \geq \beta \geq 0$, are respectively replaced with α at $-\infty$ and β at $+\infty$, with $\alpha \geq \beta \geq 0$. The last table (as in [10]) in the next page corresponds to one of the cases where $\alpha = \beta$.

Table 1: The spectrums of the hyponormal-like UWS and BWS

Spectrum	S_w	S_w^*	U_w	U_w^*
Full σ Weyl σ_W	$B(\|S_w\|)$ Ball	$B(\|S_w^*\|)$ Disk	$B(\|U_w\|, \|w\|_0)$ Band	$B(\|U_w^*\|, \|w\|_0)$ Annulus
Point σ_p	$\emptyset$	$B^0(\|S_w^*\|)$	Empty $\emptyset$	$B^0(\|U_w^*\|, \|w\|_0)$
Conti. σ_c Ess. σ_e	$\partial B(\|S_w\|)$ Circle	$\partial B(\|S_w^*\|)$ Boundary	$\partial B(\|U_w\|, \|w\|_0)$ 1 or 2 circles	$\partial B(\|U_w^*\|, \|w\|_0)$ Same as left
Res. σ_r	$B^0(\|S_w\|)$	$\emptyset$	$B^0(\|U_w\|, \|w\|_0)$	No $\emptyset$
App. σ_{ap}	$\sigma_c(S_w)$	$\sigma(S_w^*)$	$\sigma_c(U_w)$	$\sigma(U_w^*)$

Table 2: The spectrums of the less hyponormal-like UWS and BWS

Spectrum	S_w	S_w^*	U_w	U_w^*
Full σ	$B(\alpha)$	$B(\alpha)$	$B(\alpha,\beta)$	$B(\alpha,\beta)$
Weyl σ_W	The	same	as	above
Point σ_p	$\emptyset$	$B^0(\alpha)$	$\emptyset$	$B^0(\alpha,\beta)$
Conti. σ_c	$\partial B(\alpha)$	$\partial B(\alpha)$	$\partial B(\alpha,\beta)$	$\partial B(\alpha,\beta)$
Ess. σ_e	The	same	as	above
Res. σ_r	$B^0(\alpha)$	$\emptyset$	$B^0(\alpha,\beta)$	$\emptyset$
App. σ_{ap}	$\sigma_c(S_w)$	$\sigma(S_w^*)$	$\sigma_c(U_w)$	$\sigma(U_w^*)$

Table 3: The spectrums of the US and BS

Spectrum	S_1	S_1^*	U_1	U_1^*
Full σ	$B(1)$	$D = B(1)$	∂D	$\mathbb{T} = \partial D$
Weyl σ_W	Ball	Disk	Circle	Torus
Point σ_p	$\emptyset$	D^0	$\emptyset$	$\emptyset$
Conti. σ_c	$\partial B(1)$	∂D	∂D	$\mathbb{T}$
Ess. σ_e	The	same	as	above
Res. σ_r	$B^0(1)$	$\emptyset$	$\emptyset$	$\emptyset$
App. σ_{ap}	$\sigma_c(S_1)$	$\sigma(S_1^*)$	$\sigma_c(U_1)$	$\sigma(U_1^*)$

3 Banach spaces of all weighted shift operators

The C^*-algebras $C^b(\mathbb{N})$ and $C^b(\mathbb{Z})$ with the supremum norm may be viewed as only Banach spaces by the same symbols, with forgetting product and involution.

We denote by $S(C^b(\mathbb{N}))$ the (linear) space of all unilateral weighted shift operators S_w corresponding to $w \in C^b(\mathbb{N})$ and by $U(C^b(\mathbb{N}))$ the (linear) space of all bilateral weighted shift operators U_w corresponding to $w \in C^b(\mathbb{Z})$.

Define a linear map $S : C^b(\mathbb{N}) \to S(C^b(\mathbb{N}))$ by $S(w) = S_w$ and a linear map $U : C^b(\mathbb{Z}) \to U(C^b(\mathbb{Z}))$ by $U(w) = U_w$ (for both of which, see the proof below).

Proposition 3.1. *There are Banach space linear isomorphisms between $C^b(\mathbb{N})$ and $S(C^b(\mathbb{N}))$ and between $C^b(\mathbb{Z})$ and $U(C^b(\mathbb{Z}))$ under the maps S and U respectively.*

Proof. Check that for $w, w' \in C^b(\mathbb{N})$ and $k \in \mathbb{C}$,

$$S_{w+w'}e_j = (w_j + w'_j)e_{j+1} = S_w e_j + S_{w'}e_j,$$
$$S_{kw}e_j = kw_j e_{j+1} = kS_w e_j.$$

It is shown in Lemma 2.3 above that $\|S_w\| = \|w\|_\infty$. $\square$

Define a conjugate linear map $S^* : C^b(\mathbb{N}) \to S^*(C^b(\mathbb{N}))$ by $S^*(w) = S_w^*$ and a conjugate linear map $U^* : C^b(\mathbb{Z}) \to U^*(C^b(\mathbb{Z}))$ by $U(w) = U_w^*$ (for both of which, see the proof below).

Define the linear map $S \oplus S^*$ from the direct sum $C^b(\mathbb{N}) \oplus C^b(\mathbb{N}) = \oplus^2 C^b(\mathbb{N})$ to the sequilinear direct sum $S(C^b(\mathbb{N})) \oplus^\sim S^*(C^b(\mathbb{N}))$ by $(S \oplus S^*)(w \oplus w') = S_w \oplus S_{w'}^*$ and define the linear map $U \oplus U^*$ similarly (for both of which, see the proof below). We assume that these linear and sesquiliner direct sums have the maximum norm defined as $\|w \oplus w'\|_\infty = \max\{\|w\|_\infty, \|w'\|_\infty\}$. In details,

Proposition 3.2. *There are Banach space linear isomorphisms between the linear direct sum $\oplus^2 C^b(\mathbb{N})$ and the* **sesquilinear** *direct sum $S(C^b(\mathbb{N})) \oplus^\sim S^*(C^b(\mathbb{N}))$ and between $\oplus^2 C^b(\mathbb{Z})$ and $U(C^b(\mathbb{Z})) \oplus^\sim U^*(C^b(\mathbb{Z}))$ under the linear maps $S \oplus S^*$ and $U \oplus U^*$ respectively.*

Proof. Check that for $w, w' \in C^b(\mathbb{N})$ and $k \in \mathbb{C}$, $S_{w+w'}^* e_1 = 0 = S_w^* e_1 + S_{w'}^* e_1$, $S_{kw}^* e_1 = 0 = \overline{k} S_w^* e_1$, and

$$\begin{aligned} S_{w+w'}^* e_j &= (\overline{w_{j-1} + w'_{j-1}}) e_{j-1} = S_w^* e_j + S_{w'}^* e_j, \\ S_{kw}^* e_j &= \overline{k w_{j-1}} e_{j-1} = \overline{k} S_w^* e_j \end{aligned}$$

for $j \geq 2$. Moreover, for $k \in \mathbb{C}$,

$$(S \oplus S^*)(kw, kw') = S_{kw} \oplus S_{kw'}^* = k S_w \oplus \overline{k} S_{w'}^* \equiv k(S_w \oplus S_{w'}^*),$$

where the last identification is the definition of the component-wise, **sesquilinear** scalar multiplication (which we define so). Note that by Lemma 2.3,

$$\|S_w \oplus S_{w'}^*\| = \max\{\|S_w\|, \|S_{w'}^*\|\} = \max\{\|w\|_\infty, \|w'\|_\infty\} = \|w \oplus w'\|.$$

□

Remark. The sesquilinear scalar multiplication as well as the sequilinear direct sums of linear spaces, which we introduce as an attempt, but only for this, may not be found in the literature so far. But these notions may be natural in that sense and be some useful for some purposes somewhere later.

Lemma 3.3. *For $w, w' \in C^b(\mathbb{N})$, the product $S_w S_{w'}$ is equal to $S_1^2 D_{w_{+1} w'}$ with the pointwise multiplication $w_{+1} w' = (w_{j+1} w'_j) \in C^b(\mathbb{N})$, and is not equal to $S_{ww'} = S_1 D_{ww'}$ if non-zero.*

The operator $S_{\overline{w}}$ is equal to $S_1 D_{\overline{w}}$, and is not equal to $S_w^* = D_{\overline{w}} S_1^*$.

Similarly, $U_w U_{w'} = U_1^2 D_{w_{+1} w'}$ and $U_{\overline{w}} = U_1 D_{\overline{w}}$.

It then follows that the maps S, U, $S \oplus S^*$, and $U \oplus U^*$ can not extend to $*$-homomorphisms of C^*-algebras.

Proof. Compute

$$S_w S_{w'} e_j = S_w w'_j e_{j+1} = w_{j+1} w'_j e_{j+2} = w_{j+1} w'_j S_1^2 e_j.$$

The reason for those maps not to be extended to $*$-homomorphisms is simply in that the C^*-algebras $C^b(\mathbb{N})$ and $C^b(\mathbb{Z})$ are commutative, but the C^*-algebras generated by the images under those maps are non-commutative. □

Corollary 3.4. *For $w \in C^b(\mathbb{N})$, we have $S_w^2 = S_1^2 D_{w_{+1}w}$ and*

$$S_w^{k+1} = S_1^{k+1} D_{w_{+k}\cdots w_{+1}w},$$

where the successive pointwise multiplication $w_{+k}\cdots w_{+1}w = (w_{j+k}\cdots w_{j+1}w_j) \in C^b(\mathbb{N})$.

For $w \in C^b(\mathbb{Z})$, we have $U_w^2 = U_1^2 D_{w_{+1}w}$. Moreover, we have $U_w^{k+1} = U_1^{k+1} D_{w_{+k}\cdots w_{+1}w}$.

A weighted shift operator S_w for $w \in C^b(\mathbb{N})$ is said to be p-**periodic** if there is a positive integer p such that $w_j = w_{j+p}$ for all $j \in \mathbb{N}$, where such p is assumed to be the least period. Similarly, U_w for $w \in C^b(\mathbb{Z})$ is defined to be p-periodic.

Note that if S_w is 1-periodic, then $w = w_1 1$ and $S_w = w_1 S_1$.

Lemma 3.5. *If S_w is p-periodic, then $S_w^p = w_1 w_2 \cdots w_p S_1^p$.*

If U_w is p-periodic, then $U_w^p = w_1 w_2 \cdots w_p U_1^p$.

Proof. Note that in this case

$$S_w^p = S_1^p D_{w_{+(p-1)}\cdots w_{+1}w} = S_1^p w_1 \cdots w_p D_1 = w_1 \cdots w_p S_1^p.$$

□

Remark. There may be more results on this subject in the literature, or in the future to be continued. In fact, the last and several corresponding results in the next section are just the beginning of the advanced theory for certain non-type I C^*-algebras involving the inductive limit structure for C^*-algebras (cf. [2]). Namely, the limit C^*-algebra is the Bunce-Deddens algebra (cf. [6]).

4 C^*-algebras of weighted shift operators

For any $w \in C^b(\mathbb{N})$ (fixed), we denote by $C^*(S_w)$ the (universal) C^*-algebra generated by the unilateral weighted shift operator S_w (and S_w^*), which may be called the unilateral **weighted shift** C^*-algebra. Similarly, we define $C^*(U_w)$ as the bilateral weighted shift C^*-algebra.

Recall that $C^*(S_1)$ is said to be the **Toeplitz** C^*-algebra generated by the non-untary isometry S_1, and the C^*-algebra $C^*(U_1)$ generated by the unitary U_1 is isomorphic to the C^*-algebra $C(\mathbb{T})$ of all continuous, complex-valued functions on the 1-**torus** $\mathbb{T}$, by functional calculus. Indeed, $C^*(U_1) \cong C(\sigma(U_1)) = C(\mathbb{T})$ by the Gelfand transform ([11]).

There is a short exact sequence of C^*-algebras ([3], or for instance [11]):

$$0 \to \mathbb{K} \to C^*(S_1) \to C(\mathbb{T}) \cong C^*(U_1) \to 0.$$

Indeed, the quotient map is induced by the universality of $C^*(S_1)$ as well as $C^*(U_1)$. Note that $1 - S_1^* S_1 = 0$ but $1 - S_1 S_1^*$ is the rank one projection in $\mathbb{K}$. It then follows that $\mathbb{K}$ is contained in $C^*(S_1)$ as a two-sided closed ideal, so that the quotient C^*-algebra $C^*(S_1)/\mathbb{K}$ is isomorphic to $C^*(\mathbb{T})$.

Corollary 4.1. *For any $w \in C^b(\mathbb{N})$, the C^*-algebra $C^*(S_w)$ is isomorphic to $C^*(S_{|w|})$.*

For any $w \in C^b(\mathbb{Z})$, the C^*-algebra $C^*(U_w)$ is isomorphic to $C^*(U_{|w|})$.

Proof. The unitary equivalence between S_w and $S_{|w|}$ as in Proposition 2.6 above extends to a $*$-isomorphism between $C^*(S_w)$ and $C^*(S_{|w|})$.

The proof for U_w is the same as this. □

Corollary 4.2. *If $w \in C^b(\mathbb{N})$ is a unitary, then $C^*(S_w)$ is isomorphic to $C^*(S_1)$.*

If $w \in C^b(\mathbb{Z})$ is a unitary, then $C^*(U_w)$ is isomorphic to $C^*(U_1)$.

Proof. The unitary equivalence between S_w and $S_{|w|} = S_1$ as in Corollary 2.7 above extends to a $*$-isomorphism between $C^*(S_w)$ and $C^*(S_1)$.

The proof for U_w is the same as this. □

Slightly generalizing from [2, Lemma 2.1] we obtain

Lemma 4.3. [2, Lemma 2.1]. *If $w \in C^b(\mathbb{N})$ is invertible, then $C^*(S_w)$ contains the C^*-algebra $\mathbb{K}$ of all compact operators.*

Proof. Since $S_w = S_1 D_w$, then $S_w^* S_w = D_{\overline{w}} D_w = D_{|w|^2}$ is in $C^*(S_w)$. Thus, $D_{|w|} \in C^*(S_w)$.

Suppose now that each component w_j of w is positive. Then we have $D_w = (S_w^* S_w)^{\frac{1}{2}} \in C^*(S_w)$. Since D_w is invertible by hypothesis, then its inverse D_w^{-1} belongs to $C^*(S_w)$. Hence $S_1 = S_w D_w^{-1} \in C^*(S_w)$. It is known that $C^*(S_1)$ contains $\mathbb{K}$, so that $C^*(S_w)$ does also.

In the general case, we use the fact that S_w is unitarily equivalent to $S_{|w|}$ in the sense that there is a unitary V on $L^2(\mathbb{N})$ such that $\mathrm{Ad}(V)S_w = V S_w V^* = S_{|w|}$. The adjoint map $\mathrm{Ad}(V)$ extends to a $*$-isomorphism from $C^*(S_w)$ onto $C^*(S_{|w|})$. Since $C^*(S_{|w|})$ contains $\mathbb{K}$ by hypothesis, then $C^*(S_w)$ contains $\mathrm{Ad}(V^*)\mathbb{K} \cong \mathbb{K}$. □

We denote by $\{e_{ij}\}_{i,j=1}^2$ the matrix unit for $M_2(\mathbb{C})$. We define

$$F_2 = w_1 e_{21} = \begin{pmatrix} 0 & 0 \\ w_1 & 0 \end{pmatrix} \in M_2(\mathbb{C})$$

for $w_1 \in \mathbb{C}$ non-zero. Note that $F_2 F_2 = F_2^2 = 0$ and $\sigma(F_2) = \{0\}$.

Lemma 4.4. *The C^*-algebra $C^*(F_2)$ generated by the nilpotent matrix F_2 is isomorphic to the 2×2 matrix C^*-algebra $M_2(\mathbb{C})$.*

Proof. Since $F_2^* F_2 = |w_1|^2 e_{11}$ and $F_2 F_2^* = |w_1|^2 e_{22}$, then $e_{11}, e_{22} \in C^*(F_2)$. Since $F_2 = w_1 e_{21}$ and $F_2^* = \overline{w_1} e_{12}$, then $e_{21}, e_{12} \in C^*(F_2)$. □

We denote by $\{e_{ij}\}_{i,j=1}^3$ the matrix unit for $M_3(\mathbb{C})$. We define

$$F_3 = w_1 e_{21} + w_2 e_{32} = \begin{pmatrix} 0 & 0 & 0 \\ w_1 & 0 & 0 \\ 0 & w_2 & 0 \end{pmatrix} \in M_3(\mathbb{C})$$

for $w_1, w_2 \in \mathbb{C}$ non-zero. Note that $F_3^2 = w_1 w_2 e_{31}$, $F_3^3 = 0$ and $\sigma(F_3) = \{0\}$.

Lemma 4.5. *The C^*-algebra $C^*(F_3)$ generated by the nilpotent matrix F_3 is isomorphic to the 3×3 matrix C^*-algebra $M_3(\mathbb{C})$.*

Proof. Since $F_3^* F_3 = |w_1|^2 e_{11} + |w_2|^2 e_{22}$ and $F_3 F_3^* = |w_1|^2 e_{22} + |w_2|^2 e_{33}$, it then follows that $e_{11}, e_{22}, e_{33} \in C^*(F_3)$. Since $F_3 = w_1 e_{21} + w_2 e_{32}$, then $e_{22} F_3 = w_1 e_{21}$, $e_{33} F_3 = w_2 e_{32} \in C^*(F_3)$. □

We denote by $\{e_{ij} = e_{i,j}\}_{i,j=1}^n$ the matrix unit for $M_n(\mathbb{C})$. We define

$$F_n = w_1 e_{21} + \cdots + w_{n-1} e_{n,n-1} = \begin{pmatrix} 0 & & & 0 \\ w_1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & w_{n-1} & 0 \end{pmatrix} \in M_n(\mathbb{C})$$

for non-zero $w_1, \cdots, w_{n-1} \in \mathbb{C}$. Note that $F_n^n = 0$ and $\sigma(F_n) = \{0\}$.

Proposition 4.6. *The C^*-algebra $C^*(F_n)$ generated by the nilpotent matrix F_n is isomorphic to the $n \times n$ matrix C^*-algebra $M_n(\mathbb{C})$.*

Proof. Since $F_n^* F_n = |w_1|^2 e_{11} + \cdots + |w_{n-1}|^2 e_{n-1,n-1}$ and $F_n F_n^* = |w_1|^2 e_{22} + \cdots + |w_{n-1}|^2 e_{nn}$, it then follows that $e_{11}, \cdots, e_{nn} \in C^*(F_n)$. Indeed, note that

$$(F_n^2)^* F_n^2 = |w_1 w_2|^2 e_{11} + \cdots + |w_{n-2} w_{n-1}|^2 e_{n-2,n-2},$$
$$F_n^2 (F_n^2)^* = |w_1 w_2|^2 e_{33} + \cdots + |w_{n-2} w_{n-1}|^2 e_{n,n},$$

and we compute inductively $(F_n^k)^*(F_n^k)$ and $(F_n^k)(F_n^k)^*$ and then the products F_n^{n-1} and $(F_n^{n-1})^*(F_n^{n-1})$ implies that $e_{n,1}$ and e_{11} belong to $C^*(F_n)$, and use the equations of those products reversely.

It then also follows that $e_{i,j} \in C^*(F_n)$ for $i \neq j$ by taking products of the matrix unit components e_{jj} on the diagonal with F_n^k and $(F_n^k)^*$. □

Lemma 4.7. *The C^*-algebra generated by $F_n \oplus F_m$ with $n \neq m$ is isomorphic to $C^*(F_n) \oplus C^*(F_m)$ as a direct sum C^*-algebra.*

But the C^*-algebra generated by $F_n \oplus F_n$ is isomorphic to $C^*(F_n)$.

Proof. If $n < m$, then $(F_n \oplus F_m)^n = 0 \oplus F_m^n = F_m^n$. Hence $(F_m^n)^* F_m^n$ belongs to $C^*(F_n \oplus F_m)$. It then follows that $C^*(F_m)$ is contained in $C^*(F_n \oplus F_m)$, so that $C^*(F_n)$ and $C^*(F_n) \oplus C^*(F_m)$ are contained in $C^*(F_n \oplus F_m)$. Its converse also holds. □

Corollary 4.8. *The C^*-algebra generated by $F_{n_1} \oplus \cdots \oplus F_{n_k}$ with $n_1, \cdots, n_k$ mutually distinct is isomorphic to $C^*(F_{n_1}) \oplus \cdots \oplus C^*(F_{n_k})$.*

Proposition 4.9. *If $w \in C^b(\mathbb{N})$ has only one zero component $w_n = 0$, then $C^*(S_w)$ is isomorphic to $C^*(F_n) \oplus C^*(S_{w'})$ with $S_w = F_n \oplus S_{w'}$, where $S_{w'}$ is assumed to be an isometry. Moreover, $C^*(S_w)$ is unital in this case.*

If $w \in C^b(\mathbb{Z})$ has only one zero component $w_n = 0$, then $C^*(U_w)$ is isomorphic to $C^*(B_{w'}) \oplus C^*(S_{w''})$, with $U_w = B_{w'} \oplus S_{w''}$, where $B_{w'}$ is assumed to a co-isometry and $S_{w''}$ is an isometry. Moveover, $C^*(U_w)$ is unital in this case.

Proof. Let $L^2(\mathbb{N}) = \mathbb{C}^n \oplus H'$ be the corresponding direct sum of Hilbert spaces. Then $S_w^* S_w = F_n^* F_n \oplus S_{w'}^* S_{w'}$, with $S_{w'}^* S_{w'} = 1_{H'}$ the identity operator on H' and $F_n^* F_n = |w_1|^2 \oplus \cdots \oplus |w_{n-1}|^2 \oplus 0$. Since $F_n^n = 0$, we have $S_w^n(S_w^* S_w) = 0 \oplus S_{w'}^n$. Hence, $(S_w^*)^n(0 \oplus S_{w'}^n) = 0 \oplus 1_{H'}$ belongs to $C^*(S_w)$. It then follows that $C^*(S_{w'})$ is contained in $C^*(S_w)$, so that $F_n \oplus 0_{H'}$ as well as $C^*(F_n)$ are contained in $C^*(S_w)$. Therefore, the direct sum $C^*(F_n) \oplus C^*(S_{w'})$ is contained in $C^*(S_w)$ and its converse also holds.

Since $B_{w'}$ can be identified with $S_{\overline{w'}}^*$, we may assume that $U_w = S_{\overline{w'}}^* \oplus S_{w''}$. Then $U_w^* U_w = S_{\overline{w'}} S_{\overline{w'}}^* \oplus 1_{H''}$ with $1 = 1_H = 1_{H'} \oplus 1_{H''}$ the corresponding identity operator. Then

$$U_w(U_w^* U_w) - U_w = S_{\overline{w'}}^* S_{\overline{w'}} S_{\overline{w'}}^* \oplus 0_{H''} = S_{\overline{w'}}^* \oplus 0_{H''} \in C^*(U_w).$$

It then follows that $C^*(B_{w'})$ is contained in $C^*(U_w)$, so that $0_{H'} \oplus S_{w'}$ as well as $C^*(S_{w''})$ are contained in $C^*(U_w)$. Hence the direct sum $C^*(B_{w'}) \oplus C^*(S_{w''})$ is contained in $C^*(S_w)$, and its converse also holds. □

Corollary 4.10. *If $w \in C^b(\mathbb{N})$ has finitely many zero components $w_{n_1+\cdots+n_j}$ with $n_j \geq 1$ $(1 \leq j \leq k-1)$ mutually distinct such that $S_w = F_{n_1} \oplus \cdots \oplus F_{n_{k-1}} \oplus S_{w'}$, where $S_{w'}$ is assumed to be an isometry. Then $C^*(S_w)$ is isomorphic to $C^*(F_{n_1}) \oplus \cdots \oplus C^*(F_{n_{k-1}}) \oplus C^*(S_{w'})$.*

If $w \in C^b(\mathbb{Z})$ has finitely many zero components $w_{l+n_0+n_1+\cdots+n_j}$ with $n_0 = 0$ and otherwise $n_j \geq 1$ $(0 \leq j \leq k-1)$ mutually distinct for some $l \in \mathbb{Z}$ such that $U_w = B_{w'} \oplus F_{n_1} \oplus \cdots \oplus F_{n_{k-1}} \oplus S_{w''}$, where $B_{w'}$ is assumed to a co-isometry and $S_{w''}$ is an isometry. Then $C^*(U_w)$ is isomorphic to

$$C^*(B_{w'}) \oplus C^*(F_{n_1}) \oplus \cdots \oplus C^*(F_{n_{k-1}}) \oplus C^*(S_{w''}).$$

Proposition 4.11. ([6, V.3]). *For $p \geq 1$, if S_w is p-periodic with $w \in C^b(\mathbb{N})$ invertible, then $C^*(S_w)$ is isomorphic to the $p \times p$ matrix C^*-algebra $M_p(C^*(S_1))$ over $C^*(S_1)$.*

The same also holds for $C^*(U_w)$ with $w \in C^b(\mathbb{Z})$ p-periodic and invertible.

Proof. It is clear for $p = 1$.

Since $C^*(S_w) \cong C^*(S_{|w|})$ by Corollary 4.1, we may assume that each component w_j of w is positive, and $w_1, \cdots, w_p$ are mutually distinct, and as well that $0 < w_1 < w_2 < \cdots < w_p$.

If $p = 2$, then

$$\begin{aligned} S_w(\sum_{j=1}^{\infty} x_j e_j) &= S_w(\sum_{j\,=\,1 \bmod 2} x_j e_j + \sum_{j\,=\,0 \bmod 2} x_j e_j) \\ &= w_1(\sum_{j\,=\,1 \bmod 2} x_j e_{j+1}) + w_2(\sum_{j\,=\,0 \bmod 2} x_j e_{j+1}) \\ &= w_1 S_1(\sum_{j\,=\,1 \bmod 2} x_j e_j) + w_2 S_1(\sum_{j\,=\,0 \bmod 2} x_j e_j). \end{aligned}$$

If we set $H = H_1 \oplus H_0$ with each H_k the Hilbert space generated by elements $\sum_{j\,=\,k \bmod 2} x_j e_j$, then we have the identification as

$$S_w = \begin{pmatrix} 0 & w_2 S_1 \\ w_1 S_1 & 0 \end{pmatrix}.$$

We then have

$$S_w^* S_w = \begin{pmatrix} |w_1|^2 S_1^* S_1 & 0 \\ 0 & |w_2|^2 S_1^* S_1 \end{pmatrix} = |w_1|^2 1_{H_1} \oplus |w_2|^2 1_{H_0}$$

on $H = H_1 \oplus H_0$. Since the spectrum $\sigma(S_w^* S_w)$ is equal to the set $\{|w_1|^2, |w_2|^2\}$, the functional calculus implies that both $1_{H_1} \oplus 0_{H_0}$ and $0_{H_1} \oplus 1_{H_0}$ are contained in $C^*(S_w^* S_w^*) \subset C^*(S_w)$. It then follows that $M_2(C^*(S_1))$ is contained in $C^*(S_w)$, and its converse also holds.

For p in general,

$$\begin{aligned} S_w(\sum_{j=1}^{\infty} x_j e_j) &= S_w(\sum_{k=0}^{p-1} \left(\sum_{j\,=\,k+1 \bmod p} x_j e_j \right)) \\ &= \sum_{k=0}^{p-1} w_{k+1} \left(\sum_{j\,=\,k+1 \bmod p} x_j e_{j+1} \right) \\ &= \sum_{k=0}^{p-1} w_{k+1} S_1 \left(\sum_{j\,=\,k+1 \bmod p} x_j e_j \right) \end{aligned}$$

If we set $H = H_1 \oplus \cdots \oplus H_{p-1} \oplus H_0$ with each H_k the Hilbert space generated by elements $\sum_{j\,=\,k \bmod p} x_j e_j$, then we have the identification as

$$S_w = \begin{pmatrix} 0 & & & w_p S_1 \\ w_1 S_1 & 0 & & \\ & \ddots & \ddots & \\ & & w_{p-1} S_1 & 0 \end{pmatrix}.$$

We then have

$$S_w^* S_w = \begin{pmatrix} |w_1|^2 S_1^* S_1 & & 0 \\ & \ddots & \\ 0 & & |w_p|^2 S_1^* S_1 \end{pmatrix} = |w_1|^2 1_{H_1} \oplus \cdots \oplus |w_p|^2 1_{H_p}$$

on $H = H_1 \oplus \cdots \oplus H_{p-1} \oplus H_0$. Since the spectrum $\sigma(S_w^* S_w)$ is equal to the set $\{|w_1|^2, \cdots, |w_p|^2\}$, the functional calculus implies that 1_{H_j} for $1 \leq j \leq p$ are contained in $C^*(S_w^* S_w^*) \subset C^*(S_w)$. It then follows that $M_p(C^*(S_1))$ is contained in $C^*(S_w)$, and its converse also holds. □

Corollary 4.12. *For $p \geq 1$, if S_w is p-periodic with $w \in C^b(\mathbb{N})$ invertible, then there is a short exact sequence of C^*-algebras*

$$0 \to \mathbb{K} \to C^*(S_w) \to M_p(C(\mathbb{T})) \to 0.$$

If U_w is p-periodic with $w \in C^b(\mathbb{Z})$ invertible, then $C^*(U_w) \cong M_p(C(\mathbb{T}))$.

Proof. Tensoring the short exact sequence for $C^*(S_1)$ (as given in the first of this section) with $M_p(\mathbb{C})$ implies the statement. □

Lemma 4.13. *For any $w = (w_j) \in C^b(\mathbb{N})$ such that each $w_j \neq 0$ and the limit $\lim_{j\to\infty} |w_j| = 0$, then $C^*(S_w)$ is isomorphic to $\mathbb{K}$.*

The same also holds for $C^*(U_w)$ with $w \in C^b(\mathbb{Z})$ such that each $w_j \neq 0$ and both of the limits $\lim_{j\to\pm\infty} |w_j|$ are zero.

Proof. Since the limit is zero, S_w is a non-zero compact operator, so that $C^*(S_w)$ is contained in $\mathbb{K}$. Conversely, since S_w is irreducible by the non-zero condition of $w = (w_j)$, $C^*(S_w)$ is also irreducible and has non-zero intersection with $\mathbb{K}$, so that $\mathbb{K}$ is contained in $C^*(S_w)$ ([11, Theorem 2.4.9]). □

Proposition 4.14. *For any $w = (w_j) \in C^b(\mathbb{N})$ such that each $w_j \neq 0$ and the limit $\lim_{j\to\infty} w_j \equiv \alpha$ exists and is nonzero, if S_w is non-normal, then $C^*(S_w)$ is isomorphic to $C^*(S_1)$.*

For any $w = (w_j) \in C^b(\mathbb{Z})$ such that each $w_j \neq 0$ and both of the limits $\lim_{j\to\pm\infty} w_j \equiv \alpha$ exist and coincide, and is nonzero, if U_w is non-normal, then $C^*(U_w)$ is isomorphic to $C^*(\mathbb{K}, U_1)$ the C^*-algebra generated by $\mathbb{K}$ and U_1, which is isomorphic to $C^*(S_1)$.

Proof. It holds that $S_w = \alpha S_1 + G$ for G some compact operator. Then compute

$$\begin{aligned} S_w^* S_w - S_w(S_w)^* &= (\overline{\alpha} S_1^* + G^*)(\alpha S_1 + G) - (\alpha S_1 + G)(\overline{\alpha} S_1^* + G^*) \\ &= |\alpha|^2(1 - S_1 S_1^*) + \overline{\alpha}(S_1^* G - G S_1^*) + \alpha(G^* S_1 - S_1 G^*) + G^* G - G G^*, \end{aligned}$$

which belongs to $\mathbb{K}$ and is non-zero by non-normality of S_w. Since $C^*(S_w)$ is irreducible, then $C^*(S_w)$ contains $\mathbb{K}$. Thus S_1 is contained in $C^*(S_w)$, and hence $C^*(S_1)$ is contained in $C^*(S_w)$. Conversely, S_w is contained in $C^*(S_1)$ by the equation, and hence $C^*(S_w)$ is contained in $C^*(S_1)$.

It holds that $U_w = \alpha U_1 + G$ for G some compact operator. Then $U_w^* U_w - U_w U_w^*$ is computed to be a non-zero compact operator. Since $C^*(U_w)$ is irreducible, then $C^*(U_w)$ contains $\mathbb{K}$. Thus U_1 is contained in $C^*(U_w)$, and hence $C^*(U_1)$ is contained in $C^*(U_w)$. Therefore, $C^*(\mathbb{K}, U_1)$ is contained in $C^*(U_w)$. Its converse also holds by the equation. □

As shown in [1] or [4, Proposition 4.14], with some refinement,

Proposition 4.15. ([4, Proposition 4.14]). *Let $A \in \mathbb{B}(H)$ and $C^*(A, 1)$ the C^*-algebra generated by A and 1 the identity operator, where H is a Hilbert space. If there is a $*$-homomorphism φ from $C^*(A, 1)$ to $\mathbb{C}$, i.e. a character of $C^*(A, 1)$ such that $\varphi(A)$ is equal to λ, then $\lambda \in \sigma_{ap}(A)$.*

Proof. We refer to the proof of Conway [4, Proposition 4.14].

Suppose that $\varphi : C^*(A,1) \to \mathbb{C}$ is a character with $\varphi(A) = \lambda$. If we assume that $\lambda \notin \sigma_{ap}(A)$, then there is a constant $c > 0$ such that $\|(A - \lambda 1)\xi\| \geq c\|\xi\|$ for any $\xi \in H$. This implies that

$$\langle (A - \lambda 1)^*(A - \lambda 1)\xi, \xi\rangle = \|(A - \lambda 1)\xi\|^2 \geq c^2\|\xi\|^2 = \langle c^2\xi, \xi\rangle,$$

so that $(A - \lambda 1)^*(A - \lambda 1) - c^2 1$ is a positive operator. Thus,

$$\begin{aligned} 0 &\leq \varphi((A - \lambda 1)^*(A - \lambda 1) - c^2 1) \\ &= (\varphi(A^*) - \lambda^*\varphi(1))\varphi(A) - \lambda\varphi(1)) - c^2\varphi(1) = -c^2 < 0, \end{aligned}$$

a contradiction. Hence $\lambda \in \sigma_{ap}(A)$. □

Conversely, in part,

Proposition 4.16. ([4, Proposition 4.14]). *Let $A \in \mathbb{B}(H)$ and $C^*(A,1)$ the C^*-algebra generated by A and 1, where H is a Hilbert space. If A is hyponormal and $\lambda \in \sigma_{ap}(A)$, then there is a $*$-homomorphism φ from $C^*(A,1)$ to $\mathbb{C}$, i.e. a character of $C^*(A,1)$ such that $\varphi(A)$ is equal to λ.*

Proof. We refer to the proof of Conway [4, Proposition 4.14].

Suppose that $\lambda \in \sigma_{ap}(A)$. Then there is a sequence (ξ_n) of unit vectors in H such that $\|(A - \lambda 1)\xi_n\| \to 0$ as $n \to \infty$. Define a positive linear functional $\varphi : \mathbb{B}(l^2(\mathbb{N})) \to \mathbb{C}$ by $\varphi(b) = B\text{-lim}(\langle b\xi_n, \xi_n\rangle)_n$ for $b \in \mathbb{B}(H)$, where B-lim : $l^\infty(\mathbb{C}) \to \mathbb{C}$ means the Banach limit, defined to be a positive linear functional such that the norm is 1, i.e., φ is a state, B-lim is the usual limit for convergent sequences, and the B-limit is invariant under the shift on $\mathbb{N}$ (see [5, III, 7]).

For any $b \in \mathbb{B}(H)$, we have $\|b(A - \lambda 1)\xi_n\| \to 0$ as $n \to \infty$, so that $\varphi(b(A - \lambda 1)) = 0$. In particular, $\varphi(A) = \lambda$, so that $\varphi(A^*) = \lambda^*$. Since A is hyponormal, then

$$\begin{aligned} (A - \lambda 1)^*(A - \lambda 1) &= A^*A - \lambda A^* - \lambda^* A + |\lambda|^2 \\ &\geq AA^* - \lambda A^* - \lambda^* A + |\lambda|^2 = (A - \lambda 1)(A - \lambda 1)^*, \end{aligned}$$

and hence $\|(A - \lambda 1)^*\xi_n\| \leq \|(A - \lambda 1)\xi_n\|$. Thus, $\varphi(b(A - \lambda 1)^*) = 0$ for any $b \in \mathbb{B}(H)$. Also, $\varphi(1) = B\text{-lim}\,\|\xi_n\|^2 = 1$. Therefore, if $p(A - \lambda 1, A^* - \lambda^* 1)$ is a polynomial without the constant term, then $\varphi(p(A - \lambda 1, A^* - \lambda^* 1) + \alpha 1) = \alpha$ for any $\alpha \in \mathbb{C}$. It then follows that φ is multiplicative on $C^*(A,1)$, because if $q(A - \lambda 1, A^* - \lambda^* 1)$ is another such polynomial, then

$$\begin{aligned} &\varphi(\{p(A - \lambda 1, A^* - \lambda^* 1) + \alpha 1\}\{q(A - \lambda 1, A^* - \lambda^* 1) + \beta 1\}) = \alpha\beta \\ &= \varphi(p(A - \lambda 1, A^* - \lambda^* 1) + \alpha 1)\varphi(q(A - \lambda 1, A^* - \lambda^* 1) + \beta 1). \end{aligned}$$

Note as well that $C^*(A - \lambda 1, 1)$ is isomorphic to $C^*(A,1)$. □

Similarly, but extendedly in part,

Proposition 4.17. *Let $A \in \mathbb{B}(H)$ and $C^*(A,1)$ the C^*-algebra generated by A and 1, where H is a Hilbert space. If $\lambda \in \sigma_{ap}(A)$ and $\overline{\lambda} \in \sigma_{ap}(A^*)$, then there is a $*$-homomorphism $\varphi^\sim$ from $C^*(A,1)$ to $\mathbb{C}$, i.e. a character of $C^*(A,1)$ such that $\varphi^\sim(A)$ is equal to λ.*

Remark. In fact, that $\varphi^\sim$ is defined to be the normalization of φ obtained similarly but slightly differently as in the proof above, without the normalization condition.

Proof. We modify the proof of Conway [4, Proposition 4.14].

Suppose that $\lambda \in \sigma_{ap}(A)$ and $\overline{\lambda} \in \sigma_{ap}(A^*)$. Then there is a sequence (ξ_n) and (η_n) of unit vectors in H such that $\|(A-\lambda 1)\xi_n\| \to 0$ as $n \to \infty$ and that $\|(A^* - \overline{\lambda}1)\eta_n\| \to 0$ as $n \to \infty$. Define a bounded sequence (s_n) as $s_n = s_n(b) = \langle b\xi_n, \eta_n\rangle$ for $n \in \mathbb{N}$ and $b \in \mathbb{B}(H)$. Define a bounded linear functional $\varphi : \mathbb{B}(l^2(\mathbb{N})) \to \mathbb{C}$ by $\varphi(b) = B\text{-}\lim(s_n(b))$ for $b \in \mathbb{B}(H)$. For any $b \in \mathbb{B}(H)$, we have that $\|b(A-\lambda 1)\xi_n\| \to 0$ as $n \to \infty$ and that $\|b^*(A^* - \overline{\lambda}1)\eta_n\| \to 0$ as $n \to \infty$. It then follows that $\varphi(b(A-\lambda 1)) = 0$ and $\varphi((A-\lambda 1)b) = 0$. In particular, $\varphi(A) = \lambda\varphi(1)$. But $\varphi(1) = B\text{-}\lim(\langle \xi_n, \eta_n\rangle)$ may not be equal to 1. We thus need to redefine $\varphi^\sim = \frac{1}{\varphi(1)}\varphi$, so that $\varphi^\sim(1) = 1$ and $\varphi^\sim(A) = \lambda$. It then follows from the same argument in the proof above that $\varphi^\sim$ is multiplicative on $C^*(A,1)$. □

Theorem 4.18. *Suppose that every component w_j of $w = (w_j) \in C^b(\mathbb{N})$ or $C^b(\mathbb{Z})$ is non zero.*

If $|w| = (|w_j|)$ is upper boundedly continuous at $+\infty$ to α, then $\lambda \in \sigma_{ap}(S_w) = \partial B(\alpha)$ if and only if there is a $*$-homomorphism ρ from $C^*(S_w,1)$ to $\mathbb{C}$, i.e. a character of $C^*(S_w,1)$ such that $\rho(S_w)$ is equal to λ, where $C^*(S_w,1)$ is the C^*-algebra generated by S_w and 1.

If $|w| = (|w_j|) \in C^b(\mathbb{Z})$ is upper boundedly continuous at $+\infty$ to α and lower boundedly continuous at $-\infty$ to β, with $\beta < \alpha$, then $\lambda \in \sigma_{ap}(U_w) = \partial B(\alpha,\beta)$ if and only if there is a $*$-homomorphism ρ from $C^*(U_w,1)$ to $\mathbb{C}$, i.e. a character of $C^*(U_w,1)$ such that $\rho(U_w)$ is equal to λ, where $C^*(U_w,1)$ is the C^*-algebra generated by U_w and 1, and $C^*(U_w,1) = C^*(U_w)$ if and only if β is positive in this case.

Proof. It is shown in Theorem 2.28 that $\sigma_{ap}(S_w) = \partial B(\alpha) \subset B(\alpha) = \sigma_{ap}(S_w^*)$.

It is also shown in Theorem 2.28 that $\sigma_{ap}(U_w) = \partial B(\alpha,\beta) \subset B(\alpha,\beta) = \sigma_{ap}(U_w^*)$. □

Lemma 4.19. ([4, Lemma 13.1]). *Let $\mathfrak{A}$ be a unital C^*-algebra. Denote by $\mathfrak{A}_1^\wedge$ the space of all non-zero $*$-homomorphisms (or characters) from $\mathfrak{A}$ to $\mathbb{C}$ (with the weak $*$, or the point-wise convergence topology). Let $\overline{[\mathfrak{A},\mathfrak{A}]}$ denote the closed (commutator) ideal of $\mathfrak{A}$ generated by additive commutators $[a,b] = ab - ba$ for $a,b \in \mathfrak{A}$, as the completion of the algebraic commutator $[\mathfrak{A},\mathfrak{A}]$. Then the following equality holds:*

$$\overline{[\mathfrak{A},\mathfrak{A}]} \equiv \mathfrak{C} = \cap_{\varphi \in \mathfrak{A}_1^\wedge} \ker(\varphi) \equiv \mathfrak{I},$$

the intersection of all kernels of $\varphi \in \mathfrak{A}_1^\wedge$.

Proof. It is clear that $\mathfrak{I}$ is a closed (two-sided) ideal of $\mathfrak{A}$. For any $a, b \in \mathfrak{A}$ and any $\varphi \in \mathfrak{A}_1^\wedge$, we have $\varphi([a,b]) = \varphi(a)\varphi(b) - \varphi(b)\varphi(a) = 0$. Hence, $\mathfrak{C}$ is contained in $\mathfrak{I}$.

Conversely, take $a \in \mathfrak{A} \setminus \mathfrak{C}$. Then the class $a + \mathfrak{C}$ is non zero in the commutative, quotient C^*-algebra $\mathfrak{A}/\mathfrak{C}$. Thus, there is a $*$-homomorphism ρ from $\mathfrak{A}/\mathfrak{C}$ to $\mathbb{C}$ such that $\rho(a+\mathfrak{C}) \neq 0$. Then define an element $\varphi \in \mathfrak{A}_1^\wedge$ as the composition $\varphi = \rho \circ q$, where $q : \mathfrak{A} \to \mathfrak{A}/\mathfrak{C}$ is the quotient homomorphism, so that $\varphi(a) \neq 0$. Hence $a \notin \mathfrak{I}$. Therefore, $\mathfrak{I}$ is contained in $\mathfrak{C}$. □

Proposition 4.20. ([4, Proposition 13.2]). *With the same notation as in the preceding lemma, $\mathfrak{A}_1^\wedge$ is the maximal ideal space of $\mathfrak{A}/\mathfrak{C}$. It then follows that $\mathfrak{A}/\mathfrak{C} \cong C(\mathfrak{A}_1^\wedge)$ the C^*-algebra of all continuous, complex-valued functions on $\mathfrak{A}_1^\wedge$ under the Gelfand transfrom defined as $(a + \mathfrak{C})^\wedge(\varphi) = \varphi(a)$ for $a \in \mathfrak{A}$ and $\varphi \in \mathfrak{A}_1^\wedge$.*

Proof. If $\varphi \in \mathfrak{A}_1^\wedge$, then $\varphi(\mathfrak{C}) = 0$. Thus, we define $\varphi : \mathfrak{A}/\mathfrak{C} \to \mathbb{C}$ by the same symbol as $\varphi(a+\mathfrak{C}) = \varphi(a)$. Hence $\mathfrak{A}_1^\wedge$ is identified with the space of all characters of $\mathfrak{A}/\mathfrak{C}$ with the weak-$*$ topology, which is identified with the maximal ideal space of $\mathfrak{A}/\mathfrak{C}$, that is, the space of all kernels of characters of $\mathfrak{A}/\mathfrak{C}$. Since $\mathfrak{A}/\mathfrak{C}$ is a commutative C^*-algebra, then the C^*-algebra isomorphism in the statement is deduced from the Gelfand transform (for instance, see [11]). □

Since we may have $C^*(A) \neq C^*(A,1)$ in general, with some refinement we obtain

Corollary 4.21. ([4, Corollary 13.3]). *If $A \in \mathbb{B}(H)$ is hyponormal, then*

$$C^*(A,1)/\overline{[C^*(A,1), C^*(A,1)]} \equiv C^*(A,1)/\mathfrak{C} \cong C(C^*(A,1)_1^\wedge) \cong C(\sigma_{ap}(A)),$$

where $A+\mathfrak{C}$ is sent to the coordinate function on $\sigma_{ap}(A)$, identified with elements $\lambda \in \sigma_{ap}(A)$.

Proof. There is a homeomorphism from $C^*(A,1)_1^\wedge$ onto $\sigma_{ap}(A)$ by sending $\varphi \in C^*(A)_1^\wedge$ to $\varphi(A) \in \sigma_{ap}(A)$. It is checked that the map is well defined and is surjective. Since $C^*(A,1)$ is generated by A and 1, then if $\varphi(A) = \psi(A)$, then $\varphi = \psi$ in $C^*(A,1)_1^\wedge$. Thus, the map is injective. It is clear that the map is continuous and so is its inverse as well, because $\sigma_{ap}(A)$ is compact and $C^*(A,1)_1^\wedge$ is a Hausdorff space. Note that $(A+\mathfrak{C})^\wedge(\varphi) = \varphi(A) = \lambda$. □

Similarly, but extendedly in part,

Corollary 4.22. *Let $A \in \mathbb{B}(H)$. Suppose that if $\lambda \in \sigma_{ap}(A)$, then $\overline{\lambda} \in \sigma_{ap}(A^*)$ (which is not automatic). It then follows that*

$$C^*(A,1)/\overline{[C^*(A,1), C^*(A,1)]} \equiv C^*(A,1)/\mathfrak{C} \cong C(C^*(A,1)_1^\wedge) \cong C(\sigma_{ap}(A)).$$

Therefore, we get

Theorem 4.23. *Suppose that every component w_j of $w = (w_j) \in C^b(\mathbb{N})$ or $C^b(\mathbb{Z})$ is non zero.*

If $|w| = (|w_j|)$ is upper boundedly continuous at $+\infty$ to α, then

$$C^*(S_w,1)/\overline{[C^*(S_w,1),C^*(S_w,1)]} \equiv C^*(S_w,1)/\mathfrak{C} \cong C(C^*(S_w,1)_1^\wedge) \cong C(\sigma_{ap}(S_w)).$$

If $|w| = (|w_j|) \in C^b(\mathbb{Z})$ is upper boundedly continuous at $+\infty$ to α and lower boundedly continuous at $-\infty$ to β, then

$$C^*(U_w,1)/\overline{[C^*(U_w,1),C^*(U_w,1)]} \equiv C^*(U_w,1)/\mathfrak{C} \cong C(C^*(U_w,1)_1^\wedge) \cong C(\sigma_{ap}(U_w)),$$

where $C^*(U_w,1) = C^*(U_w)$ if and only if β is positive in this case.

Remark. There may be more results on this subject in the literature, or in the future to be continued. For more advanced details, may refer to, for instance, [2], [7], and [8]. In fact, we could only consider the quotient structure as given above, but not the corresponding ideal structure which does involve a pathological but familiar non-type I representation theory.

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