Around the stable rank of discrete group C^⊠-algebras

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AROUND THE STABLE RANK OF DISCRETE GROUP C^*-ALGEBRAS

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ABSTRACT. We give a report on the stable rank of the group C^*-algebras of discrete groups, and study the stable rank of C^*-algebras of continuous fields involving the operations such as tensor products, crossed products and taking multiplier algebras, and estimate the stable rank of the C^*-algebras of the generalized diamond Lie groups as an application. Furthermore, we review and collect some basic notions and properties of groups and some properties of group C^*-algebras especially for some remarkable known results on the stable rank.

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0. OVERVIEW

CONTENTS

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This paper is multiple of the above 4 sections. The sections are almost independent of each other but are concerned with some each other. Section 1 is based on the paper [NS] by Ping-Wong Ng and myself, and on my talk at the conference of Mathematical Society of Japan in September 2004 ([Sd8]). Sections 2 and 3 are based on our original results. We give brief introductions at the beginnings of Sections 1, 2 and 3. In Section 4, from Margulis [Mg] (book), de la Harpe [dH2] (book), and Bekka-Louvet [BL] including a brief survey on some results by Kirchberg [Kr] we pick up some notions and their properties with sketchy proofs, which have connections with the results of Section 1. Specifically, the first 4 subsections (A to D) of Section 4 consist of A: Groups with properties (T) or (R) (cf. [Mg]), B: Residually finite groups (cf. [dH2]), and C: The factorization property, and D: Residually finite dimensional C^*-algebras (cf. [BL]). Moreover, the last subsection E: Some properties for group C^*-algebras, is based on the remarkable paper by Dykema and de la Harpe [DyH] on the stable rank of the group C^*-algebras for a large class of discrete groups including certain amalgamated free products and certain hyperbolic groups, and in part based on the paper by Jolissaint [Jl] on rapidly decreasing functions in reduced group C^*-algebras.

1. Stable rank of discrete group C^*-algebras

In this section the author would like to give a report about the (topological) stable rank of group C^*-algebras of discrete groups. This is a (undetailed) survey on this topic including some new results by Ping Wong Ng and myself. First of all, recall that discrete groups are divided into two classes. One consists of all amenable discrete groups, and the other consists of all non-amenable discrete groups. Amenability for operator algebras as well as groups is known to be one of very important notions.

Examples. All abelian (discrete) groups and all nilpotent discrete groups are amenable. In particular, the generalized discrete Heisenberg group H_{2n+1}^d of rank $2n+1$ consists of the matrices:

$$\begin{pmatrix} 1 & a & c \\ 0_n^t & 1_n & b^t \\ 0 & 0_n & 0 \end{pmatrix} \in GL_{n+2}(\mathbb{Z}), \quad a, b \in \mathbb{Z}^n, c \in \mathbb{Z},$$

where 1_n the $n \times n$ identity matrix, and $0_n = (0) \in \mathbb{Z}^n$, and b^t, 0_n^t are the transposes of b, 0_n respectively. On the other hand, the free groups are non-amenable. Residually finite discrete groups with Kazhdan's property (T) such as $SL_n(\mathbb{Z})$ for $n \geq 3$ are non-amenable.

Our first result is the following:

Theorem 1.1 [NS], [Sd10]. *Let $C^*(H_{2n+1}^d)$ be the group C^*-algebra of H_{2n+1}^d. Then $\mathrm{sr}(C^*(H_{2n+1}^d)) = n+1$, where $\mathrm{sr}(\cdot)$ is the stable rank for C^*-algebras (see Section 2).*

Sketch of proof. Since H_{2n+1}^d is the semi-direct product $\mathbb{Z}^{n+1} \rtimes \mathbb{Z}^n$ via the identification of the tuples (c, b, a) with elements of H_{2n+1}^d, $C^*(H_{2n+1}^d) \cong C^*(\mathbb{Z}^{n+1}) \rtimes \mathbb{Z}^n$ the crossed product of the group C^*-algebra $C^*(\mathbb{Z}^{n+1})$ of $\mathbb{Z}^{n+1}$ by the adjoint action of $\mathbb{Z}^n$. By the Fourier transform, $C^*(\mathbb{Z}^{n+1}) \rtimes \mathbb{Z}^n \cong C(\mathbb{T}^{n+1}) \rtimes \mathbb{Z}^n$, where $C(\mathbb{T}^{n+1})$ is the C^*-algebra of continuous functions on $\mathbb{T}^{n+1}$. Moreover, $C(\mathbb{T}^{n+1}) \rtimes \mathbb{Z}^n$ can be viewed as the C^*-algebra $\Gamma(\mathbb{T}, \{C(\mathbb{T}^n) \rtimes_z \mathbb{Z}^n\}_{z \in \mathbb{T}})$ of a continuous field on $\mathbb{T}$ with fibers $C(\mathbb{T}^n) \rtimes_z \mathbb{Z}^n \cong \otimes^n C(\mathbb{T}) \rtimes_z \mathbb{Z}$ n-tensor products of rotation algebras $C(\mathbb{T}) \rtimes_z \mathbb{Z}$ for $z \in \mathbb{T}$ (cf. [AP]). Then we use Lemma 1.3 below. □

For the proof of Theorem 1.1, we need the following lemmas:

Lemma 1.2 [NS]. *Let $\mathfrak{A}$ be a unital maximal full algebra of operator fields on a locally compact Hausdorff space X with fibers C^*-algebras $\{\mathfrak{A}_t\}_{t \in X}$. Suppose that there is an integer M such that $M \geq \sup_{t \in X} \mathrm{sr}(\mathfrak{A}_t)$. Let $(a_j) \in \mathfrak{A}^M$. Then for any $\varepsilon > 0$ and $t \in X$, there is an open neighborhood U of t and there exists $(b_j) \in \mathfrak{A}^M$ such that $\|a_j - b_j\| < \varepsilon$ for $1 \leq j \leq M$, and $\sum_{j=1}^M b_j(t)^* b_j(t)$ is invertible in $\mathfrak{A}_t$ for $t \in U$.*

Lemma 1.3 [NS]. *Let $\mathfrak{A}$ be a unital maximal full algebra of operator fields on the interval $[0,1]$ with fibers C^*-algebras $\{\mathfrak{A}_t\}_{t \in [0,1]}$. Then $\mathrm{sr}(\mathfrak{A}) \leq \sup_{t \in [0,1]} \mathrm{sr}(C([0,1]) \otimes \mathfrak{A}_t)$, where $C([0,1])$ is the C^*-algebra of continuous functions on $[0,1]$.*

Moreover, as the second result we have

Theorem 1.4 [Sd8]. *Let G be a finitely generated, countable amenable discrete subgroup of $GL_n(\mathbb{C})$. Suppose that the C^*-algebra*

$C^*(G)$ of G has a continuous separating family of finite dimensional representations. Then $\mathrm{sr}(C^*(G)) = \sup_{n\in\mathbb{N}}(\{[\dim \hat{G}_n/2]/n\} + 1)$, *where $\hat{G}_n$ is the space of all n-dimensional irreducible representations of G up to unitary equivalence, and* $\dim(\cdot)$ *is the covering dimension for spaces, and $[x]$ means the least integer $\leq x$, and $\{y\}$ means the least integer $\geq y$.*

Remark. Under the assumption on G (even if non-amenable), G is residually finite. The assumption of continuity on the separating family $C^*(G)$ might be unnecessary.

On the other hand, as the third one we obtain

Theorem 1.5 [Sd8]. *Let G be a residually finite, countable discrete group with Kazhdan's property (T), and $C_r^*(G)$ the reduced group C^*-algebra of G. Then* $\mathrm{sr}(C_r^*(G)) = 1$.

Sketch. Under the assumption we have the following factorization:

$$C^*(G) \to \oplus_j \pi_j(C^*(G)) \to C_r^*(G) \to 0,$$

where $\oplus_j \pi_j$ is the direct sum representation of finite dimensional representations π_j of $C^*(G)$. $\square$

Examples. As residually finite groups with (T),

$$SL_n(\mathbb{Z}), PSL_n(\mathbb{Z}), \mathbb{Z}^n \rtimes SL_n(\mathbb{Z}) \ (n \geq 3), Sp(n,1)_{\mathbb{Z}} \ (n \geq 2)$$

(see [dHV], [Mg]). On the other hand, as residually finite groups without (T) (see [Mg], [Se]),

$$\begin{cases} SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 & \text{as an amalgam,} \\ PSL_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3, F_n = *^n\mathbb{Z} \ (n \geq 2) & \text{as free products.} \end{cases}$$

To prove Theorems 1.4 and 1.5, we have obtained

Theorem 1.6 [Sd8]. *Let $\mathfrak{A}$ be a residually finite dimensional C^*-algebra with a separating family $\{\pi_j\}_{j\in J}$. Suppose that J is a locally compact Hausdorff space and $\{\pi_j\}_{j\in J}$ is continuous on J, and $n_j =$* $\dim \pi_j$. *Then*

$$\mathrm{sr}(\mathfrak{A}) = \sup_{j\in J} \mathrm{sr}(C_0(\mathfrak{A}^\wedge_{n_j}, \pi_j(\mathfrak{A}))) = \sup_{j\in J}(\{[\dim \mathfrak{A}^\wedge_{n_j}/2]/n_j\} + 1),$$

where $\mathfrak{A}^\wedge_{n_j}$ is the subspace of the spectrum of $\mathfrak{A}$ consisting of irreducible representations with dimension n_j, and $C_0(\mathfrak{A}^\wedge_{n_j}, \pi_j(\mathfrak{A}))$ is the C^*-algebra of continuous $\pi_j(\mathfrak{A})$-valued functions on $\mathfrak{A}^\wedge_{n_j}$ vanishing at infinity.

For a further generalization, we may introduce

Definition. We say that a C^*-algebra $\mathfrak{A}$ is residually elementary if it has a separating family of elementary irreducible representations π_j, that is, $\pi_j(\mathfrak{A})$ is either $M_n(\mathbb{C})$ a matrix algebra over $\mathbb{C}$ or $\mathbb{K}$ the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space.

Remark. At this moment we do not know which virtues this new class of C^*-algebras might have. If the subspace $\mathfrak{A}^\wedge_\infty$ of the spectrum of $\mathfrak{A}$ consisting of infinite dimensional irreducible representations is a Hausdorff space, then one can obtain the same estimates as given in Theorem 1.6 by including $\mathrm{sr}(C_0(\mathfrak{A}^\wedge_\infty, \mathbb{K})) = \min\{2, \mathrm{sr}(C_0(\mathfrak{A}^\wedge_\infty))\}$ by [Rf1, Theorems 3.6 and 6.4], where $C_0(\mathfrak{A}^\wedge_\infty)$ is the C^*-algebra of continuous functions vanishing at infinity on $\mathfrak{A}^\wedge_\infty$. However, in this case $\mathfrak{A}$ is of type I (see [Sd9]).

2. Stable rank of C^*-algebras of continuous fields involving operations

The C^*-algebras of continuous fields regarded as noncommutative counterparts of topological fiber spaces are defined by giving their base spaces, fibers, and some continuous operator fields (cf.[Dx]). Some operations on the C^*-algebras of continuous fields such as tensor products and crossed products have studied by Kirchberg and Wassermann recently [KW1], [KW2]. On the other hand, as a result in the dimension theory for C^*-algebras first studied and developed by Rieffel [Rf1], the author estimated the stable rank and connected stable rank of C^*-algebras of continuous fields in terms of the base spaces and fibers [Sd10]. Also, Ping-Wong Ng and the author [NS] gave a more direct proof of the stable rank estimate of C^*-algebras of continuous fields in the case where the base spaces are product spaces of the intervals.

In this section we estimate the stable ranks of tensor products and crossed products of C^*-algebras of continuous fields by using some results of [KW1], [KW2] and [Sd10], and also estimate those of the multiplier algebras of C^*-algebras of continuous fields by the similar methods as in [Sd10]. These results obtained here complements the results previously obtained in [Sd10], and their applications are carried out in the next Section 3.

Notation. For a C^*-algebra $\mathfrak{A}$ (or its unitization $\mathfrak{A}^+$), its stable rank and connected stable rank are denoted by $\mathrm{sr}(\mathfrak{A})$, $\mathrm{csr}(\mathfrak{A})$ respectively (cf.[Rf1]). By definition, we have that $\mathrm{sr}(\mathfrak{A})$, $\mathrm{csr}(\mathfrak{A}) \in \{1, 2, \cdots, \infty\}$. Recall that $\mathrm{sr}(\mathfrak{A}) \leq n$ if and only if any $(a_j) \in \mathfrak{A}^n$ is approximated by elements $(b_j) \in \mathfrak{A}^n$ with $\sum_{j=1}^n b_j^* b_j$ invertible in $\mathfrak{A}$, and $\mathrm{csr}(\mathfrak{A}) \leq m$ if and only if for any $n \geq m$, the set of all elements $(b_j) \in \mathfrak{A}^n$ with $\sum_{j=1}^n b_j^* b_j$ invertible in $\mathfrak{A}$ is connected.

The rank estimates. We now recall the following estimates of the stable rank and connected stable rank obtained in [Sd10]: for $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ the C^*-algebra of a continuous field (vanishing at infinity) on a paracompact, locally compact Hausdorff space X,

$$(F): \quad \begin{cases} \mathrm{sr}(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})) \leq \sup_{t\in X} \mathrm{sr}(C_0(X, \mathfrak{A}_t)), \\ \mathrm{csr}(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})) \leq \\ \quad \sup_{t\in X}(\mathrm{csr}(C_0(X, \mathfrak{A}_t)) \vee \mathrm{sr}(C_0(X, \mathfrak{A}_t))), \end{cases}$$

where $C_0(X, \mathfrak{A}_t)$ is the C^*-algebra of continuous $\mathfrak{A}_t$-valued functions vanishing at infinity on X, and $\vee$ means the maximum. See [NS] for the case of $X = [0, 1]^k$ for $k \geq 1$. In fact, this case is essential from the argument of [NOP] to reduce a general case of locally compact Hausdorff spaces to the case of the intervals, and see also [Sd9]. Refer to Theorem 2.5 below.

Sketch of proof. Let $\Pi_{t\in X} C_0(X, \mathfrak{A}_t)^+$ be the direct product of the unitization $C_0(X, \mathfrak{A}_t)^+$ for $t \in X$. Consider the evaluation map Φ_t from $C_0(X, \mathfrak{A}_t)^+$ to $\mathfrak{A}_t$ by $\Phi_t(f, \lambda) = f(t)$ for $t \in X$. Then we define a C^*-subalgebra $\mathfrak{D} = \Gamma_0(X, \{C_0(X, \mathfrak{A}_t)^+\}_{t\in X})$ of the direct product $\Pi_{t\in X} C_0(X, \mathfrak{A}_t)^+$ consisting of all elements $(f_t, \lambda_t)_{t\in X}$ such that their images by the quotient map Φ belong to $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$, where $\Phi((f_t, \lambda_t)_{t\in X})$ is defined by the function $X \ni t \mapsto \Phi_t(f_t)$. When X is noncompact, we may replace $\mathfrak{D}$ with $\Gamma(X^+, \{C_0(X, \mathfrak{A}_t)^+\}_{t\in X} \cup \mathbb{C})$ the C^*-algebra of a continuous field on the one-point compactification X^+ of X. Then we show $\mathrm{sr}(\mathfrak{D}) \leq N \equiv \sup_{t\in X} \mathrm{sr}(C_0(X, \mathfrak{A}_t))$.

Indeed, any $f_j = (f_{j,t})_{t\in X} \in \mathfrak{D}$ $(1 \le j \le N)$ can be approximated by $g_j = (g_{j,t})_{t\in X} \in \mathfrak{D}$ $(1 \le j \le N)$ such that $\sum_{j=1}^N g_j^* g_j$ is invertible in $\mathfrak{D}$. For this, we may assume that $\sum_{j=1}^N g_{j,t}^* g_{j,t}$ is invertible in $C_0(X, \mathfrak{A}_t)^+$ for any $t \in X$. To show that the function $X \ni t \mapsto \sum_{j=1}^N g_{j,t}^* g_{j,t}$ belongs to $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ and is invertible, we use a perturbation of $f_{j,t}$ if necessary and that if $\sum_{j=1}^N g_{j,s}^* g_{j,s}$ for some $s \in X$ is invertible, then $\sum_{j=1}^N g_{j,t}^* g_{j,t}$ is invertible for t near s, which is deduced from a direct computation using the continuity $X \ni t \mapsto \|\sum_{j=1}^N g_{j,t}^* g_{j,t}\|$. Use also the estimate: $\mathrm{csr}(\mathfrak{A}/\mathfrak{J}) \le \mathrm{csr}(\mathfrak{A}) \vee \mathrm{sr}(\mathfrak{A})$ for a C^*-algebra $\mathfrak{A}$ and its closed ideal $\mathfrak{J}$ [Eh, Theorem 1.1] for the estimate of connected stable rank. $\square$

Theorem 2.1. *Let $\mathfrak{A} = \Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$, $\mathfrak{B} = \Gamma_0(Y, \{\mathfrak{B}_s\}_{s\in Y})$ be C^*-algebras of continuous fields on paracompact, locally compact Hausdorff spaces X, Y respectively. If $\mathfrak{A}, \mathfrak{B}$ are exact, then*

$$\begin{cases} \mathrm{sr}(\mathfrak{A}\otimes\mathfrak{B}) \le \sup_{t\in X, s\in Y} \mathrm{sr}(C_0(X\times Y, \mathfrak{A}_t\otimes\mathfrak{B}_s)), \\ \mathrm{csr}(\mathfrak{A}\otimes\mathfrak{B}) \le \sup_{t\in X, s\in Y} \\ \quad (\mathrm{csr}(C_0(X\times Y, \mathfrak{A}_t\otimes\mathfrak{B}_s)) \vee \mathrm{sr}(C_0(X\times Y, \mathfrak{A}_t\otimes\mathfrak{B}_s))), \end{cases}$$

where $\otimes$ means the minimal tensor product.

Proof. Since $\mathfrak{A}$ is exact, we have $\mathfrak{A}\otimes\mathfrak{B} \cong \Gamma_0(X, \{\mathfrak{A}_t\otimes\mathfrak{B}\}_{t\in X})$ by [KW1]. Since $\mathfrak{B}$ is exact, we have $\mathfrak{A}_t\otimes\mathfrak{B} \cong \Gamma_0(Y, \{\mathfrak{A}_t\otimes\mathfrak{B}_s\}_{s\in Y})$ again by [KW1]. By using (F),

$$\begin{cases} \mathrm{sr}(\mathfrak{A}\otimes\mathfrak{B}) \le \sup_{t\in X} \mathrm{sr}(C_0(X, \mathfrak{A}_t\otimes\mathfrak{B})), \\ \mathrm{csr}(\mathfrak{A}\otimes\mathfrak{B}) \le \sup_{t\in X} (\mathrm{csr}(C_0(X, \mathfrak{A}_t\otimes\mathfrak{B})) \vee \mathrm{sr}(C_0(X, \mathfrak{A}_t\otimes\mathfrak{B}))) \end{cases}$$

Moreover, $C_0(X, \mathfrak{A}_t\otimes\mathfrak{B}) \cong \Gamma_0(Y, \{C_0(X)\otimes\mathfrak{A}_t\otimes\mathfrak{B}_s\}_{s\in Y})$. By (F),

$$\begin{cases} \mathrm{sr}(C_0(X, \mathfrak{A}_t\otimes\mathfrak{B})) \le \sup_{s\in Y} \mathrm{sr}(C_0(X\times Y, \mathfrak{A}_t\otimes\mathfrak{B}_s)), \\ \mathrm{csr}(C_0(X, \mathfrak{A}_t\otimes\mathfrak{B})) \le \sup_{s\in Y} \\ \quad (\mathrm{csr}(C_0(X\times Y, \mathfrak{A}_t\otimes\mathfrak{B}_s)) \vee \mathrm{sr}(C_0(X\times Y, \mathfrak{A}_t\otimes\mathfrak{B}_s))). \end{cases} \quad \square$$

Let $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \rtimes_\alpha G$ be the (full) crossed product of the continuous field C^*-algebra $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ by a locally compact group G with the action α defined by $\alpha_g(f)(t) = \alpha_g^t(f(t))$ for $f \in \Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$, $g \in G$, where α^t is an action of G on the fiber $\mathfrak{A}_t$ ([KW1], [Pd]). Then

Theorem 2.2. *Let $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ be the C^*-algebras of a continuous field on a paracompact locally compact Hausdorff space X, and G a locally compact, amenable group acting on it fiberwise. Then*

$$\begin{cases} \mathrm{sr}(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \rtimes_\alpha G) \leq \sup_{t\in X} \mathrm{sr}(C_0(X, \mathfrak{A}_t \rtimes_{\alpha^t} G)), \\ \mathrm{csr}(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \rtimes_\alpha G) \leq \\ \quad \sup_{t\in X}(\mathrm{csr}(C_0(X, \mathfrak{A}_t \rtimes_{\alpha^t} G)) \vee \mathrm{sr}(C_0(X, \mathfrak{A}_t \rtimes_{\alpha^t} G))), \end{cases}$$

where $\mathfrak{A}_t \rtimes_{\alpha^t} G$ are the crossed products of the fibers $\mathfrak{A}_t$ by G.

Proof. Since G is amenable, using [KW1] we have

$$\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \rtimes_\alpha G \cong \Gamma_0(X, \{\mathfrak{A}_t \rtimes_{\alpha^t} G\}_{t\in X}, \mathfrak{F})$$

for $\mathfrak{F}$ a family of continuous operator fields on X. By (F),

$$\begin{cases} \mathrm{sr}(\Gamma_0(X, \{\mathfrak{A}_t \rtimes_{\alpha^t} G\}_{t\in X}, \mathfrak{F})) \leq \sup_{t\in X} \mathrm{sr}(C_0(X, \mathfrak{A}_t \rtimes_{\alpha^t} G)) \\ \mathrm{csr}(\Gamma_0(X, \{\mathfrak{A}_t \rtimes_{\alpha^t} G\}_{t\in X}, \mathfrak{F})) \leq \sup_{t\in X} \\ \quad (\mathrm{csr}(C_0(X, \mathfrak{A}_t \rtimes_{\alpha^t} G)) \vee \mathrm{sr}(C_0(X, \mathfrak{A}_t \rtimes_{\alpha^t} G))). \end{cases} \qquad \square$$

Let $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \rtimes_r G$ be the reduced crossed product of the crossed product $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \rtimes_\alpha G$ with an action α preserving fibers (cf. [Pd]). Then

Theorem 2.3. *Let $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ be the C^*-algebra of a continuous field on a locally compact paracompact Hausdorff space X, and G a locally compact, exact group acting on it fiberwise. Then*

$$\begin{cases} \mathrm{sr}(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \rtimes_r G) \leq \sup_{t\in X} \mathrm{sr}(C_0(X, \mathfrak{A}_t \rtimes_r G)), \\ \mathrm{csr}(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \rtimes_r G) \leq \sup_{t\in X} \\ \quad (\mathrm{csr}(C_0(X, \mathfrak{A}_t \rtimes_r G)) \vee \mathrm{sr}(C_0(X, \mathfrak{A}_t \rtimes_r G))), \end{cases}$$

where $\mathfrak{A}_t \rtimes_r G$ means the reduced crossed product of $\mathfrak{A}_t \rtimes_{\alpha^t} G$.

Proof. Since G is exact, using [KW2] we have

$$\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \rtimes_r G \cong \Gamma_0(X, \{\mathfrak{A}_t \rtimes_r G\}_{t\in X}, \mathfrak{F})$$

for $\mathfrak{F}$ a family of continuous operator fields on X. Then we use (F). $\square$

In particular, we obtain

Theorem 2.4. *Let $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ be the C^*-algebra of a continuous field on a locally compact paracompact Hausdorff space X, and G a connected solvable Lie group acting on the C^*-algebra fiberwise. Then*

$$\mathrm{sr}(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \rtimes G) \leq \dim X + \dim G + \sup_{t\in X} \mathrm{sr}(\mathfrak{A}_t).$$

Moreover, if X is compact and contractible,

$$\mathrm{csr}(\Gamma(X, \{\mathfrak{A}_t\}_{t\in X}) \rtimes G) \leq \sup_{t\in X}(\mathrm{csr}(\mathfrak{A}_t \rtimes G) \vee (\dim X + \dim G + \mathrm{sr}(\mathfrak{A}_t))).$$

Proof. By [NOP], $\mathrm{sr}(C_0(X, \mathfrak{A}_t \rtimes G)) \leq \dim X + \mathrm{sr}(\mathfrak{A}_t \rtimes G)$. By [Sd7], we obtain $\mathrm{sr}(\mathfrak{A}_t \rtimes G) \leq \mathrm{sr}(\mathfrak{A}_t) + \dim G$. By [Eh, Corollary 2.12] (cf. [Ns1]), we have $\mathrm{csr}(C(X, \mathfrak{A}_t \rtimes G)) = \mathrm{csr}(\mathfrak{A}_t \rtimes G)$. □

Remark. We can deduce the similar variations in the case where G is an elementary, topological Abelian group using some results of [Sd7].

On the other hand, we have

Theorem 2.5. *Let $M(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}))$ be the multiplier algebra of the continuous field C^*-algebra $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ on a locally compact paracompact Hausdorff space X. Then*

$$\begin{cases} \mathrm{sr}(M(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}))) \leq \sup_{t\in X} \mathrm{sr}(C^b(X) \otimes M(\mathfrak{A}_t)), \\ \mathrm{csr}(M(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}))) \leq \sup_{t\in X} \\ \quad (\mathrm{csr}(C^b(X) \otimes M(\mathfrak{A}_t)) \vee \mathrm{sr}(C^b(X) \otimes M(\mathfrak{A}_t))), \end{cases}$$

where $M(\mathfrak{A}_t)$ means the multiplier algebra of $\mathfrak{A}_t$, and $C^b(X)$ is the C^*-algebra of bounded continuous functions on X.

Proof. We note that $M(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})) \cong C^b(X, \{M(\mathfrak{A}_t)_*\}_{t\in X}; \mathfrak{F})$ for $\mathfrak{F}$ a family of bounded continuous operator fields on X, where the right hand side means the C^*-algebra of (certain) bounded, strictly continuous operator fields on X with fibers $\{M(\mathfrak{A}_t)_*\}_{t\in X}$ in the sense of [APT], where $M(\mathfrak{A}_t)_*$ means the multiplier of $\mathfrak{A}_t$ with the strict topology. Following the idea of [Sd10], we define a C^*-subalgebra $\mathfrak{B} = \Gamma^b(X, \{C^b(X) \otimes M(\mathfrak{A}_t)\}_{t\in X})$ (the C^*-algebra of (certain) bounded continuous operator fields on X) of the direct product $\Pi_{t\in X}(C^b(X) \otimes M(\mathfrak{A}_t))$ such that for any $f \in \mathfrak{B}$ with

$f(t,\cdot) \in C^b(X)\otimes M(\mathfrak{A}_t)$, the function $t \mapsto f(t,t) \in M(\mathfrak{A}_t)$ belongs to $C^b(X, \{M(\mathfrak{A}_t)_*\}_{t\in X}, \mathfrak{F})$. In fact, for any $h \in C^b(X, \{M(\mathfrak{A}_t)_*\}_{t\in X}, \mathfrak{F})$, the function $t \mapsto h_t \otimes h(t)$ belongs to $\mathfrak{B}$ where $h_t \in C^b(X)$ with $h_t(t) = 1$ for $t \in X$, and the norm-valued function: $t \mapsto \|h_t \otimes h(t)\|$ is continuous on X. Then $C^b(X, \{M(\mathfrak{A}_t)_*\}_{t\in X}, \mathfrak{F})$ is a quotient of $\mathfrak{B}$. Hence, by [Rf1, Theorem 4.3] we have

$$\mathrm{sr}(C^b(X, \{M(\mathfrak{A}_t)_*\}_{t\in X}, \mathfrak{F})) \le \mathrm{sr}(\mathfrak{B}).$$

By the similar methods as in [Sd10] (cf.[Eh, Theorem 1.1]), we have

$$\begin{cases} \mathrm{sr}(\mathfrak{B}) \le \sup_{t\in X} \mathrm{sr}(C^b(X) \otimes M(\mathfrak{A}_t)) \equiv M, \\ \mathrm{csr}(C^b(X, \{M(\mathfrak{A}_t)_*\}_{t\in X}, \mathfrak{F})) \le \sup_{t\in X} \\ \quad (\mathrm{csr}(C^b(X) \otimes M(\mathfrak{A}_t)) \vee \mathrm{sr}(C^b(X) \otimes M(\mathfrak{A}_t))). \end{cases}$$

When $M < \infty$ we take $f_j \in \mathfrak{B}$ $(1 \le j \le M)$, and for any $h \in \mathfrak{F}$, $\varepsilon > 0$ and $t_0 \in X$, there exists $l \in \mathfrak{F}$ such that $\|(f_j(t,t) - l(t))h(t)\| \le \varepsilon$ for t in an open neighborhood of t_0. To prove the stable rank estimate, we take $g_j \in \Pi_{t\in X} C^b(X) \otimes M(\mathfrak{A}_t)$ with $\|f_j(t) - g_j(t)\| < \varepsilon_j(t) < \varepsilon$, and $e(t) \equiv \sum_{j=1}^M g_j(t)^* g_j(t)$ invertible in $C^b(X)\otimes M(\mathfrak{A}_t)$. For a large constant $L > 0$, we may assume that $e(t) \ge \varepsilon/L > 0$ if necessary, by taking $\varepsilon_j(t)$ small enough, and replacing $g_j(t)$ with its suitable perturbation, and $\varepsilon_j(t)$ with $\varepsilon_j(t)' < \varepsilon$, when $e(t) \ge \delta(t) > 0$ and $\delta(t) < \varepsilon/L$ for some t.

In general, for a unital C^*-algebra $\mathcal{A}$ we have a continuous map Φ from $L_n(\mathcal{A}) = \{(a_j) \in \mathcal{A}^n \mid \sum_{j=1}^n a_j^* a_j \in \mathcal{A}^{-1}\}$ to the positive part $\mathcal{A}_+$ of $\mathcal{A}$ defined by $(a_j) \mapsto \sum_{j=1}^n a_j^* a_j$. Let $\mathcal{S} = \{b \in \mathcal{A}_+ \mid \|\sum_{j=1}^n a_j^* a_j - b\| < \eta$, and $b - (\sum_{j=1}^n a_j^* a_j + \eta' 1) > 0$ (invertible)$\}$ for some $\eta, \eta' > 0$. Then $\mathcal{S}$ is open in $\mathcal{A}_+$ since for $b' \in \mathcal{A}_+$ with $\|b - b'\|$ small, we can make the distance of their spectrums small. Taking η, η' suitably, we make the distance between $\sum_{j=1}^n a_j^* a_j$ and $\mathcal{S}$ small enough. Then we can find a small open neighborhood of (a_j) such that its image under Φ has the nonzero intersection with $\mathcal{S}$.

Moreover, we can assume that the function $t \mapsto g_j(t,t)$ on X belongs to the C^*-algebra $C^b(X, \{M(\mathfrak{A}_t)_*\}_{t\in X}, \mathfrak{F})$. Indeed, for given $t \in X$, there exists $\{g_j(t)\}_{j=1}^M$ satisfying the above required conditions. By definition of $\mathfrak{B}$, we can find $\{h_j\}_{j=1}^M \in \mathfrak{B}^M$ such that $\|h_j(t) - g_j(t)\|$ $(1 \le j \le M)$ are small enough so that $\sum_{j=1}^M h_j^*(t) h_j(t)$

is invertible, and $\|h_j(s) - f_j(s)\|$ $(1 \leq j \leq M)$ are small enough for s in an open neighborhood of t (cf. [Dx, Lemma 10.1.11 and Proposition 10.2.2]). Note that if $\sum_{j=1}^{M} h_j^*(t)h_j(t)$ is invertible, then $\sum_{j=1}^{M} h_j^*(s)h_j(s)$ is also invertible in an open neighborhood of t, which is deduced from a direct computation using continuity of the norm on fibers. Thus we continue this process inductively for a suitable open covering of X and replace g_j with h_j. Hence, $\mathrm{sr}(\mathfrak{B}) \leq M$.

To prove the above estimate of the connected stable rank, we use [Eh, Theorem 1.1] and the similar approximation as above putting $M = \sup_{t\in X}(\mathrm{csr}(C^b(X) \otimes M(\mathfrak{A}_t)) \vee \mathrm{sr}(C^b(X) \otimes M(\mathfrak{A}_t)))$. $\square$

Remark. The rank estimates in the statement were partially obtained in [Sd10] under the condition that the fibers $\mathfrak{A}_t$ are unital and $\mathfrak{F}$ contains the unit field.

Corollary 2.6. *Let $M(C_0(X,\mathfrak{A}))$ be the multiplier algebra of the C^*-algebra $C_0(X,\mathfrak{A})$ with X a locally compact paracompact Hausdorff space and $\mathfrak{A}$ a C^*-algebra. Then*

$$\begin{cases} \mathrm{sr}(M(C_0(X,\mathfrak{A}))) \leq \mathrm{sr}(C^b(X) \otimes M(\mathfrak{A})) \\ \mathrm{csr}(M(C_0(X,\mathfrak{A}))) \leq \mathrm{csr}(C^b(X) \otimes M(\mathfrak{A})) \vee \mathrm{sr}(C^b(X) \otimes M(\mathfrak{A})). \end{cases}$$

Corollary 2.7. *Let $M(C_0(X,\mathfrak{A}))$ be the multiplier algebra of the C^*-algebra $C_0(X,\mathfrak{A})$ with X a paracompact locally compact, Hausdorff space and $\mathfrak{A}$ a C^*-algebra. Then*

$$\mathrm{sr}(M(C_0(X,\mathfrak{A}))) \leq \dim X + \mathrm{sr}(M(\mathfrak{A})).$$

Proof. By [NOP], $\mathrm{sr}(C^b(X)\otimes M(\mathfrak{A})) \leq \dim X + \mathrm{sr}(M(\mathfrak{A}))$. Note that $C^b(X) \cong C(\beta X)$ where βX is the Stone-Čech compactification of X, and $\dim \beta X = \dim X$ (cf. [Nm], [Wo]). $\square$

Remark. Since the corona algebra $M(\mathfrak{A})/\mathfrak{A}$ of a C^*-algebra $\mathfrak{A}$ is a quotient of the multiplier $M(\mathfrak{A})$, we have the same rank estimates for the corona algebras of the multiplier algebras of Theorem 2.5 and Corollaries 2.6 and 2.7.

3. Stable rank of group C^*-algebras of the generalized diamond Lie groups

The stable rank and connected stable rank of group C^*-algebras of some connected or disconnected Lie groups have been computed in terms of groups by Sheu [Sh], Takai and the author [ST] and [Sd1-7]. On the other hand, the structure of the group C^*-algebra of the diamond Lie group was studied by Diep [Dp] and Vu [Vu] from the point of the K-orbits structure of solvable Lie groups. Refer to [Dp] for more details about group C^*-algebras of the class MD including the diamond Lie group and some advanced topics.

In this section we will apply the stable rank estimates (F) in Section 2 for C^*-algebras of continuous fields obtained in [Sd10] (or [N-S]) to the group C^*-algebras of the generalized diamond Lie groups with the structure as C^*-algebras of continuous fields, and to the C^*-algebras of their disconnected and discrete versions. The stable rank estimate in the case of the generalized diamond Lie groups is in fact included in a result of [ST], but our computing methods here are different from those of [ST]. Moreover, we consider a condition to have the connected stable rank one, and deduce from it that the connected stable rank is controlled by the K_1-group of C^*-algebras in the case of C^*-tensor products of group C^*-algebras of simply connected, solvable Lie groups with the C^*-algebra of compact operators.

Notation. We use the same notation as given in Section 2. In addition, let $\hat{G}_1$ denote the space of all 1-dimensional representations of a Lie group G.

The estimates of stable rank. Recall the following formulas:

$(F0)$: $\mathrm{gsr}(\mathfrak{A}) \leq \mathrm{csr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}) + 1$ for any C^*-algebra $\mathfrak{A}$.

$(F1)$: $\mathrm{sr}(C(X)) = [\dim X/2] + 1 \equiv \dim_{\mathbb{C}} X$, and
$\mathrm{csr}(C(X)) \leq [(\dim X + 1)/2] + 1$.

$(F2)$: $\mathrm{sr}(\mathfrak{A} \otimes \mathbb{K}) = \mathrm{sr}(\mathfrak{A}) \wedge 2$, $\mathrm{csr}(\mathfrak{A} \otimes \mathbb{K}) \leq \mathrm{csr}(\mathfrak{A}) \wedge 2$,
where $\wedge$ means the minimum.

$(F3)$: For a closed ideal $\mathfrak{I}$ of a C^*-algebra $\mathfrak{A}$,
$\mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \leq \mathrm{sr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I})$,
and $\mathrm{csr}(\mathfrak{A}) \leq \mathrm{csr}(\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I})$.

(cf. [Rf1, Proposition 1.7, Theorems 3.6, 4.3, 4.4, 4.11 and 6.4, Corollary 4.10 and p.328], [Ns1] and [Sh, Theorems 3.9 and 3.10]).

THE C^*-ALGEBRA OF THE DIAMOND LIE GROUP

The real 3-dimensional Heisenberg Lie group H_3 consists of

$$(c, b, a) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad a, b, c \in \mathbb{R}.$$

The real 4-dimensional diamond Lie group D_4 is defined to be the Lie semi-direct product $H_3 \rtimes_\alpha \mathbb{R}$, where the action α is defined by $\alpha_x(c, b, a) = (c, e^x b, e^{-x} a)$ for $x \in \mathbb{R}$. Then D_4 is a simply connected, solvable Lie group. It is known by [Le] that the group C^*-algebra $C^*(H_3)$ of H_3 is regarded as $\Gamma_0(\mathbb{R}, \{\mathfrak{A}_t\}_{t\in\mathbb{R}})$ the C^*-algebra of a continuous field on $\mathbb{R}$ with fibers $\mathfrak{A}_t$ given by $\mathfrak{A}_0 = C_0(\mathbb{R}^2)$ and $\mathfrak{A}_t = \mathbb{K}$ for $t \in \mathbb{R} \setminus \{0\}$. In fact, since $H_3 \cong \mathbb{R}^2 \rtimes \mathbb{R}$, $C^*(H_3)$ is decomposed into the crossed product $C_0(\mathbb{R}^2) \rtimes_\beta \mathbb{R}$, where $\beta_a(t, m) = (t, m + at)$ for $(t, m) \in \mathbb{R}^2$, $a \in \mathbb{R}$, and $C_0(\mathbb{R}^2) \rtimes_\beta \mathbb{R} \cong \Gamma_0(\mathbb{R}, \{C_0(\mathbb{R}) \rtimes_{\beta^t} \mathbb{R}\}_{t\in\mathbb{R}}) \equiv \Gamma_0(\mathbb{R}, \{\mathfrak{A}_t\}_{t\in\mathbb{R}})$, where β^t is the restriction of β to $\{t\} \times \mathbb{R}$, and $C_0(\mathbb{R}) \rtimes_{\beta^0} \mathbb{R} \cong C_0(\mathbb{R}^2)$ since β^0 is trivial, and $C_0(\mathbb{R}) \rtimes_{\beta^t} \mathbb{R} \cong \mathbb{K}$ for $t \in \mathbb{R} \setminus \{0\}$ since β^t is the shift. Therefore, for the group C^*-algebra $C^*(D_4)$ of D_4,

$$\begin{aligned} C^*(D_4) &\cong C^*(H_3) \rtimes_\alpha \mathbb{R} \cong (C_0(\mathbb{R}^2) \rtimes \mathbb{R}) \rtimes_\alpha \mathbb{R} \\ &\cong \Gamma_0(\mathbb{R}, \{\mathfrak{A}_t\}_{t\in\mathbb{R}}) \rtimes_\alpha \mathbb{R} \cong \Gamma_0(\mathbb{R}, \{\mathfrak{A}_t \rtimes_{\hat\alpha} \mathbb{R}\}_{t\in\mathbb{R}}) \end{aligned}$$

with $\mathfrak{A}_t \rtimes_{\hat\alpha} \mathbb{R} \cong \mathbb{K} \otimes C_0(\mathbb{R})$ for $t \in \mathbb{R} \setminus \{0\}$, where $\hat\alpha$ means the dual action of α (cf. [KW1]). By (F) in Section 2,

$$\begin{cases} \mathrm{sr}(C^*(D_4)) \leq \mathrm{sr}(C_0(\mathbb{R}, \mathbb{K} \otimes C_0(\mathbb{R}))) \vee \mathrm{sr}(C_0(\mathbb{R}, C_0(\mathbb{R}^2) \rtimes_{\hat\alpha} \mathbb{R})) \\ \mathrm{csr}(C^*(D_4)) \leq \mathrm{csr}(C_0(\mathbb{R}, \mathbb{K} \otimes C_0(\mathbb{R}))) \vee \mathrm{sr}(C_0(\mathbb{R}, \mathbb{K} \otimes C_0(\mathbb{R}))) \\ \qquad \vee \mathrm{csr}(C_0(\mathbb{R}, C_0(\mathbb{R}^2) \rtimes \mathbb{R})) \vee \mathrm{sr}(C_0(\mathbb{R}, C_0(\mathbb{R}^2) \rtimes \mathbb{R})). \end{cases}$$

Moreover, for $t = 0$ we have

$$0 \to C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathbb{R} \to C_0(\mathbb{R}^2) \rtimes_{\hat\alpha} \mathbb{R} \to C_0(\mathbb{R}) \to 0,$$

$$0 \to \oplus^4 (C_0(\mathbb{R}_+^2) \rtimes \mathbb{R}) \to C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathbb{R} \to \oplus^4 (C_0(\mathbb{R}_+) \rtimes \mathbb{R}) \to 0$$

with $C_0(\mathbb{R}_+)\rtimes\mathbb{R}\cong\mathbb{K}$ and $C_0(\mathbb{R}_+^2)\rtimes\mathbb{R}\cong C_0(\mathbb{R})\otimes\mathbb{K}$ by [Gr, Corollary 15], where $\mathbb{R}_+$ is the space of all positive real numbers. Using (F1), (F2) and (F3) we have

$$\operatorname{sr}(C_0(\mathbb{R}, C_0(\mathbb{R}^2)\rtimes\mathbb{R})) = 2, \quad \text{and} \quad \operatorname{csr}(C_0(\mathbb{R}, C_0(\mathbb{R}^2)\rtimes\mathbb{R})) \le 2.$$

Therefore we get $\operatorname{sr}(C^*(D_4)) \le 2$ and $\operatorname{csr}(C^*(D_4)) \le 2$. By [ST, Lemma 3.7], we get $\operatorname{sr}(C^*(D_4)) \ge 2$. Since $C^*(D_4)^+$ is finite, we have $\operatorname{gsr}(C^*(D_4)) = 1$ by (F0) (cf. [Rf2, p.247]). Summing up

Theorem 3.1. *Let $D_4 = H_3\rtimes_\alpha\mathbb{R}$ be the real 4-dimensional diamond Lie group. Then $C^*(D_4)$ is isomorphic to $\Gamma_0(\mathbb{R}, \{\mathfrak{B}_t\}_{t\in\mathbb{R}})$ the C^*-algebra of a continuous field on $\mathbb{R}$ with fibers $\mathfrak{B}_t$ given by $\mathfrak{B}_t = \mathbb{K}\otimes C_0(\mathbb{R})$ for $t\in\mathbb{R}\setminus\{0\}$ and $\mathfrak{B}_0 = C_0(\mathbb{R}^2)\rtimes\mathbb{R}$ which has a composition series $\{\mathfrak{I}_j\}_{j=1}^3$ such that*

$$\mathfrak{I}_1 = \oplus^4 C_0(\mathbb{R})\otimes\mathbb{K}, \quad \mathfrak{I}_2/\mathfrak{I}_1 = \oplus^4\mathbb{K}, \quad \mathfrak{I}_3/\mathfrak{I}_2 = C_0(\mathbb{R}).$$

Moreover, $\operatorname{sr}(C^*(D_4)) = 2$, $\operatorname{csr}(C^*(D_4)) \le 2$, $\operatorname{gsr}(C^*(D_4)) = 1$.

Remark. Applying Theorem 2.1 to $C^*(D_4)\otimes C^*(D_4)$ with the structure as the C^*-algebra of a continuous field as given above, we deduce

$$\begin{aligned}
&\operatorname{sr}(C^*(D_4)\otimes C^*(D_4)) \le 3 < 4 = 2\operatorname{sr}(C^*(D_4)),\\
&\operatorname{csr}(C^*(D_4)\otimes C^*(D_4)) \le 3.
\end{aligned}$$

However, if we use another strategy,

Theorem 3.2. *We have*

$$\begin{cases} \operatorname{sr}(C^*(D_4)\otimes C^*(D_4)) = 2, \quad \operatorname{csr}(C^*(D_4)\otimes C^*(D_4)) \le 2, \\ \operatorname{gsr}(C^*(D_4)\otimes C^*(D_4)) = 1. \end{cases}$$

Proof. First note that

$$C^*(D_4)\otimes C^*(D_4) \cong \Gamma_0(\mathbb{R}, \{(\mathfrak{A}_t\rtimes_{\hat\alpha}\mathbb{R})\otimes C^*(D_4)\}_{t\in\mathbb{R}}),$$

and we have the following exact sequence:

$$\begin{aligned}
0 \to \mathfrak{I} = \Gamma_0(\mathbb{R}\setminus\{0\}, \{(\mathfrak{A}_t\rtimes_{\hat\alpha}\mathbb{R})^+\otimes C^*(D_4)^+\}_{t\in\mathbb{R}\setminus\{0\}})\\
\to \mathfrak{C} \to (\mathfrak{A}_0\rtimes_{\hat\alpha}\mathbb{R})^+\otimes C^*(D_4)^+ \to 0
\end{aligned}$$

where $\mathfrak{C} = \Gamma_0(\mathbb{R}, \{\mathfrak{B}_t \otimes C^*(D_4)^+\}_{t\in\mathbb{R}})$ with $\mathfrak{B}_t = (\mathfrak{A}_t \rtimes_{\hat{\alpha}} \mathbb{R})^+$ for $t \in \mathbb{R}$, which contains $C^*(D_4) \otimes C^*(D_4)$ as a closed ideal, and we have $M(\mathfrak{J}) \cong \Gamma^b(\mathbb{R} \setminus \{0\}, \{\mathfrak{B}_t \otimes C^*(D_4)^+\}_{t\in\mathbb{R}\setminus\{0\}})$, which is the C^*-algebra of a bounded continuous field on $\mathbb{R} \setminus \{0\}$ (cf. [APT]). By (F3), Theorem 2.5, and [NOP, Proposition 1.6] (for stable rank),

$$\begin{cases} \mathrm{sr}(C^*(D_4) \otimes C^*(D_4)) \le \\ \quad \mathrm{sr}(\mathfrak{C}) \le \mathrm{sr}(M(\mathfrak{J})) \vee \mathrm{sr}(\mathfrak{B}_0 \otimes C^*(D_4)^+), \\ \mathrm{sr}(M(\mathfrak{J})) \le \sup_{t\in\mathbb{R}\setminus\{0\}} \mathrm{sr}(C^b(\mathbb{R} \setminus \{0\}) \otimes \mathfrak{B}_t \otimes C^*(D_4)^+) \le 2, \end{cases}$$

and by (F3) and (F) in Section 2 we get

$$\begin{aligned} &\mathrm{csr}(C^*(D_4) \otimes C^*(D_4)) \\ &\le \sup_{t\in\mathbb{R}\setminus\{0\}} [\mathrm{sr}(C_0(\mathbb{R} \setminus \{0\}) \otimes (\mathfrak{A}_t \rtimes_{\hat{\alpha}} \mathbb{R}) \otimes C^*(D_4)) \\ &\vee \mathrm{csr}(C_0(\mathbb{R} \setminus \{0\}) \otimes (\mathfrak{A}_t \rtimes_{\hat{\alpha}} \mathbb{R}) \otimes C^*(D_4))] \\ &\vee \mathrm{csr}((\mathfrak{A}_0 \rtimes_{\hat{\alpha}} \mathbb{R}) \otimes C^*(D_4)) \le 2 \vee \mathrm{csr}((\mathfrak{A}_0 \rtimes_{\hat{\alpha}} \mathbb{R}) \otimes C^*(D_4)). \end{aligned}$$

Similarly, we compute the ranks for the following exact sequence:

$$0 \to \Gamma_0(\mathbb{R} \setminus \{0\}, \{\mathfrak{B}_0 \otimes \mathfrak{B}_s\}_{s\in\mathbb{R}\setminus\{0\}}) \to \mathfrak{D} \to \mathfrak{B}_0 \otimes \mathfrak{B}_0 \to 0$$

where $\mathfrak{D} = \Gamma_0(\mathbb{R}, \{\mathfrak{B}_0 \otimes \mathfrak{B}_s\}_{s\in\mathbb{R}})$. Moreover, $\mathfrak{A}_0 \rtimes_{\hat{\alpha}} \mathbb{R}$ has the following structure:

$$\begin{cases} 0 \to \oplus^4(C_0(\mathbb{R}) \otimes \mathbb{K}) \to \mathfrak{A}_0 \rtimes_{\hat{\alpha}} \mathbb{R} \to \mathcal{E} \to 0, \quad \text{and} \\ 0 \to \oplus^4(C_0(\mathbb{R}_+) \rtimes \mathbb{R}) \to \mathcal{E} \to C_0(\mathbb{R}) \to 0, \end{cases}$$

where $\mathcal{E} = C_0(\mathbb{R}^2 \setminus (\cup^4\mathbb{R}^2_+)) \rtimes \mathbb{R}$ and $C_0(\mathbb{R}_+) \rtimes \mathbb{R} \cong \mathbb{K}$. Finally, we compute $\mathrm{sr}(\mathfrak{B}_0 \otimes \mathfrak{B}_0)$ and $\mathrm{csr}((\mathfrak{A}_0 \rtimes_{\hat{\alpha}} \mathbb{R}) \otimes (\mathfrak{A}_0 \rtimes_{\hat{\alpha}} \mathbb{R}))$ as above from the bottom exact sequence of the structure of $\mathfrak{B}_0 \otimes \mathfrak{B}_0$ or $(\mathfrak{A}_0 \rtimes_{\hat{\alpha}} \mathbb{R}) \otimes (\mathfrak{A}_0 \rtimes_{\hat{\alpha}} \mathbb{R})$ inductively. □

The disconnected case. We next define the disconnected diamond Lie group D_3^d to be the semi-direct product $H_3 \rtimes_\alpha \mathbb{Z}$, where α is the restriction of α to $\mathbb{Z}$ in definition of D_4. Then the group C^*-algebra $C^*(D_3^d)$ of D_3^d is isomorphic to $\Gamma_0(\mathbb{R}, \{\mathfrak{A}_t \rtimes_{\hat{\alpha}} \mathbb{Z}\}_{t\in\mathbb{R}})$ with $\mathfrak{A}_t \rtimes_{\hat{\alpha}} \mathbb{Z} \cong \mathbb{K} \otimes C(\mathbb{T})$ for $t \in \mathbb{R} \setminus \{0\}$. For $t = 0$, we have

$$0 \to C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathbb{R} \to C_0(\mathbb{R}^2) \rtimes_{\hat{\alpha}} \mathbb{Z} \to C(\mathbb{T}) \to 0,$$

$$0 \to \oplus^4(C_0(\mathbb{R} \times \mathbb{T}) \otimes \mathbb{K}) \to C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathbb{Z} \to \oplus^4(C(\mathbb{T}) \otimes \mathbb{K}) \to 0.$$

By [Eh, Theorem 2.2], we have $\mathrm{csr}(C^*(D_3^d)) \ge 2$ since $C^*(D_3^d) \cong C^*(H_3) \rtimes_\alpha \mathbb{Z}$. Similarly as before Theorem 3.1, we get

Theorem 3.3. *Let $D_3^d = H_3 \rtimes_\alpha \mathbb{Z}$ be the disconnected diamond Lie group. Then* $\mathrm{sr}(C^*(D_3^d)) = 2$, $\mathrm{csr}(C^*(D_3^d)) = 2$, $\mathrm{gsr}(C^*(D_3^d)) = 1$.

THE HIGHER DIMENSIONAL CASE

The real $(2n+1)$-dimensional Heisenberg Lie group H_{2n+1} is

$$\left\{ (c, b, a) = \begin{pmatrix} 1 & a & c \\ & 1_n & b^t \\ 0 & & 1 \end{pmatrix} \mid c \in \mathbb{R}, b, a \in \mathbb{R}^n \right\},$$

where 1_n is the $n \times n$ identity matrix and b^t is the transpose of b. We define the real $(3n+1)$-dimensional, generalized diamond Lie group D_{3n+1} to be the semi-direct product $H_{2n+1} \rtimes_\alpha \mathbb{R}^n$, where

$$\alpha_x(c, b, a) = (c, e^{x_1} b_1, \cdots, e^{x_n} b_n, e^{-x_1} a_1, \cdots, e^{-x_n} a_n)$$

with $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Note that since $H_{2n+1} \cong \mathbb{R}^{n+1} \rtimes \mathbb{R}^n$ as the identification of the above matrices,

$$\begin{aligned} C^*(H_{2n+1}) &\cong C_0(\mathbb{R}^{n+1}) \rtimes_\beta \mathbb{R}^n \\ &\cong \Gamma_0(\mathbb{R}, \{C_0(\mathbb{R}^n) \rtimes_{\beta^t} \mathbb{R}^n\}_{t \in \mathbb{R}}) \cong \Gamma_0(\mathbb{R}, \{\otimes^n \mathfrak{A}_t\}_{t \in \mathbb{R}}), \end{aligned}$$

where the action β is defined by $\beta_a(t, (m_j)_{j=1}^n) = (t, (m_j + a_j t)_{j=1}^n$ for $(t, (m_j)_{j=1}^n) \in \mathbb{R} \times \mathbb{R}^n$, and β^t means the restriction of β to $\{t\} \times \mathbb{R}^n$, and furthermore, $C_0(\mathbb{R}^n) \rtimes_{\beta^t} \mathbb{R}^n \cong \otimes^n C_0(\mathbb{R}) \rtimes_{\beta^t} \mathbb{R}$ (an n-fold C^*-tensor product) and $C_0(\mathbb{R}) \rtimes_{\beta^t} \mathbb{R} \cong \mathbb{K}$ for $t \in \mathbb{R} \setminus \{0\}$, and $C_0(\mathbb{R}) \rtimes_{\beta^0} \mathbb{R} \cong C_0(\mathbb{R}^2)$. Then

$$\begin{aligned} C^*(D_{3n+1}) &\cong C^*(H_{2n+1}) \rtimes_\alpha \mathbb{R}^n \cong (C_0(\mathbb{R}^{n+1}) \rtimes \mathbb{R}^n) \rtimes_\alpha \mathbb{R}^n \\ &\cong \Gamma_0(\mathbb{R}, \{\otimes^n (\mathfrak{A}_t \rtimes_{\hat{\alpha}} \mathbb{R})\}_{t \in \mathbb{R}}) \end{aligned}$$

with $\otimes^n (\mathfrak{A}_t \rtimes_{\hat{\alpha}} \mathbb{R}) \cong \mathbb{K} \otimes C_0(\mathbb{R}^n)$ for $t \in \mathbb{R} \setminus \{0\}$, and $\otimes^n (C_0(\mathbb{R}^2) \rtimes_{\hat{\alpha}} \mathbb{R})$ for $t = 0$. Then $\otimes^n (C_0(\mathbb{R}^2) \rtimes_{\hat{\alpha}} \mathbb{R})$ has a finite composition series such that its subquotients are given by

$$\begin{aligned} &\mathfrak{D}_{j_1} \otimes \cdots \otimes \mathfrak{D}_{j_n}, \quad \text{for } 1 \le j_s \le 3, \text{ and} \\ &\mathfrak{D}_1 = \oplus^4 C_0(\mathbb{R}) \otimes \mathbb{K}, \quad \mathfrak{D}_2 = \oplus^4 \mathbb{K}, \quad \mathfrak{D}_3 = C_0(\mathbb{R}). \end{aligned}$$

Note that $C^*(D_{3n+1})$ is of type I from the above analysis. Using the same methods with Theorems 3.1 and 3.2, we get

Theorem 3.4. *Let $D_{3n+1} = H_{2n+1} \rtimes_\alpha \mathbb{R}^n$ be the real $(3n+1)$-dimensional, generalized diamond Lie group. Then $C^*(D_{3n+1})$ is isomorphic to $\Gamma_0(\mathbb{R}, \{\mathfrak{B}_t\}_{t\in\mathbb{R}})$ the C^*-algebra of a continuous field on $\mathbb{R}$ with fibers $\mathfrak{B}_t$ given by $\mathfrak{B}_t = \mathbb{K} \otimes C_0(\mathbb{R}^n)$ for $t \in \mathbb{R} \setminus \{0\}$ and $\mathfrak{B}_0 = \otimes^n C_0(\mathbb{R}^2) \rtimes \mathbb{R}$ which has a composition series $\{\mathfrak{I}_j\}_{j=1}^{3^n}$ such that its subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ are given by*

$$\mathfrak{D}_{j_1} \otimes \cdots \otimes \mathfrak{D}_{j_n}, \quad \text{for } 1 \le j_s \le 3, \text{ and}$$
$$\mathfrak{D}_1 = \oplus^4 C_0(\mathbb{R}) \otimes \mathbb{K}, \quad \mathfrak{D}_2 = \oplus^4 \mathbb{K}, \quad \mathfrak{D}_3 = C_0(\mathbb{R}),$$

where $\mathfrak{I}_1 = \mathfrak{D}_1 \otimes \cdots \otimes \mathfrak{D}_1$ and $\mathfrak{I}_{3^n}/\mathfrak{I}_{3^n-1} = \mathfrak{D}_3 \otimes \cdots \otimes \mathfrak{D}_3$. Moreover,

$$\begin{cases} \mathrm{sr}(C^*(D_{3n+1})) = 2 \vee ([n/2]+1) = 2 \vee \dim_{\mathbb{C}}(D_{3n+1})^\wedge_1, \\ \mathrm{csr}(C^*(D_{3n+1})) \le [(n+1)/2]+1. \end{cases}$$

Proof. Note that $\mathrm{csr}(C_0(\mathbb{R})) = 2$, $\mathrm{csr}(C_0(\mathbb{R}^2)) = 1$, $\mathrm{csr}(C_0(\mathbb{R}^n)) = [(n+1)/2]+1$ for $n \ge 3$ [Sh, p.381] (cf. the proof of Theorem 3.1). □

Remark. If n is even, we get $\mathrm{csr}(C^*(D_{3n+1})) \ge 2$ by [Eh, Corollary 1.6] since the K_1-group of $C^*(D_{3n+1})$ is $\mathbb{Z}$ by Connes' Thom isomorphism of K-groups (cf. [Bl]). Applying Theorem 2.1 to the structure of $C^*(D_{3n+1})$ directly, we have that for $n \ge 2$,

$$\begin{cases} \mathrm{sr}(C^*(D_{3n+1}) \otimes C^*(D_{3n+1})) \le n+2 \ge 2\,\mathrm{sr}(C^*(D_{3n+1})), \\ \mathrm{csr}(C^*(D_{3n+1}) \otimes C^*(D_{3n+1})) \le n+2, \end{cases}$$

which suggests that the stable rank estimate of Theorem 2.1 is not sharper in general than the product formula of the stable rank: $\mathrm{sr}(\mathfrak{A} \otimes \mathfrak{B}) \le \mathrm{sr}(\mathfrak{A}) + \mathrm{sr}(\mathfrak{B})$ for C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$, which has not been proved in general yet. However, by the same way with Theorem 3.2,

$$\begin{cases} \mathrm{sr}(C^*(D_{3n+1}) \otimes C^*(D_{3n+1})) = n+1, \\ \mathrm{csr}(C^*(D_{3n+1}) \otimes C^*(D_{3n+1})) \le n+1. \end{cases}$$

We now define the real $(2n+1)$-dimensional, disconnected, generalized diamond Lie group D^d_{2n+1} to be the semi-direct product $H_{2n+1} \rtimes_\alpha \mathbb{Z}^n$, where α is the restriction of α to $\mathbb{Z}^n$ in definition of D_{3n+1}. By the same analysis as that for $C^*(D_{3n+1})$, we have

Theorem 3.5. *Let $D^d_{2n+1} = H_{2n+1} \rtimes_\alpha \mathbb{Z}^n$ be the real $(2n+1)$-dimensional, generalized, disconnected diamond Lie group. Then*

$$\begin{cases} \mathrm{sr}(C^*(D^d_{2n+1})) = 2 \vee ([n/2]+1) = 2 \vee \dim_{\mathbb{C}}(D^d_{2n+1})^\wedge_1, \\ 2 \leq \mathrm{csr}(C^*(D^d_{2n+1})) \leq [(n+1)/2]+1. \end{cases}$$

Remark. The estimates of the above remark hold by replacing D_{3n+1} with D^d_{2n+1}. The group C^*-algebras $C^*(D_{3n+1})$ and $C^*(D^d_{2n+1})$ have no nontrivial projections, which is deduced from that they are C^*-algebras of continuous fields on $\mathbb{R}$ (connected and non-compact). Under the situations of Theorems 3.4 and 3.5, we obtain

$$\begin{cases} \mathrm{sr}(M(C^*(D_{3n+1}))) = \infty, & \mathrm{sr}(M(C^*(D^d_{2n+1}))) = \infty, \\ \mathrm{csr}(M(C^*(D_{3n+1}))) = \infty, & \mathrm{csr}(M(C^*(D^d_{2n+1}))) = \infty, \\ \mathrm{gsr}(M(C^*(D_{3n+1}))) = \infty, & \mathrm{gsr}(M(C^*(D^d_{2n+1}))) = \infty, \end{cases}$$

which follows from [Sd5, Theorem 1.4] and that $C^*(D_{3n+1})$ and $C^*(D^d_{2n+1})$ have stable quotients from their continuous field structures as given above.

THE DISCRETE DIAMOND CROSSED PRODUCTS

Let $H^{\mathbb{Z}}_3$ be the discrete Heisenberg group of rank 3. The group C^*-algebra $C^*(H^{\mathbb{Z}}_3)$ of $H^{\mathbb{Z}}_3$ is generated by three unitaries U, V, W which correspond to the following generators of $H^{\mathbb{Z}}_3$ respectively:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that $C^*(H^{\mathbb{Z}}_3) = C^*(W, V, U) \cong C(\mathbb{T}^2) \rtimes_\beta \mathbb{Z} \cong \Gamma(\mathbb{T}, \{\mathfrak{A}_w\}_{w\in\mathbb{T}})$, where the action β is defined by $\beta_a(w, z) = (w, w^a z)$ for $(w, z) \in \mathbb{T}^2$ and $a \in \mathbb{Z}$, and $\mathfrak{A}_w = C(\{w\} \times \mathbb{T}) \rtimes \mathbb{Z}$ is the rotation algebra with the multiplication action by w. We define the discrete diamond crossed product to be $C^*(H^{\mathbb{Z}}_3) \rtimes_\alpha \mathbb{Z}$, where $\alpha_t(U) = e^{2\pi i \theta_1 t} U$, $\alpha_t(V) = e^{2\pi i\theta_2 t} V$ and $\alpha_t(W) = W$ for $t \in \mathbb{Z}$, and $\theta_1, \theta_2 \in \mathbb{R}$ which are irrational and rationally independent. Note that $H^{\mathbb{Z}}_3 \rtimes_\alpha \mathbb{Z}$ is not defined as D^d_3 since $e^t \notin \mathbb{Z}$ for $t \in \mathbb{Z} \setminus \{0\}$. Then we have

$$\begin{aligned} C^*(H^{\mathbb{Z}}_3) \rtimes_\alpha \mathbb{Z} &\cong (C(\mathbb{T}^2) \rtimes \mathbb{Z}) \rtimes_\alpha \mathbb{Z} \\ &\cong \Gamma(\mathbb{T}, \{\mathfrak{A}_w\}_{w\in\mathbb{T}}) \rtimes_\alpha \mathbb{Z} \cong \Gamma(\mathbb{T}, \{\mathfrak{A}_w \rtimes_{\hat\alpha} \mathbb{Z}\}_{w\in\mathbb{T}}). \end{aligned}$$

Then for any $w \in \mathbb{T}$, the crossed product $\mathfrak{A}_w \rtimes_{\hat\alpha} \mathbb{Z}$ is a simple non-commutative 3-torus (cf. [Ln]). Therefore, we obtain

Theorem 3.6. *Let $\mathfrak{B} = C^*(H_3^{\mathbb{Z}}) \rtimes_\alpha \mathbb{Z}$ be the discrete diamond crossed product. Then $\mathfrak{B}$ is isomorphic to $\Gamma(\mathbb{T}, \{\mathfrak{A}_w \rtimes_{\hat{\alpha}} \mathbb{Z}\}_{w\in\mathbb{T}})$ the C^*-algebra of a continuous field on $\mathbb{T}$ with fibers $\mathfrak{A}_w \rtimes_{\hat{\alpha}} \mathbb{Z}$ simple noncommutative 3-tori, and* $\mathrm{sr}(\mathfrak{B}) = 2$, $\mathrm{csr}(\mathfrak{B}) = 2$, $\mathrm{gsr}(\mathfrak{B}) = 1$.

Proof. For the latter part, we use the decomposition of simple noncommutative 3-tori into A$\mathbb{T}$-algebras, i.e. inductive limits of matrix algebras over $C(\mathbb{T})$ [Ln]. By [Rf1, Theorem 5.1] and its modification for the connected stable rank, we get

$$\mathrm{sr}(C(\mathbb{T}) \otimes (\mathfrak{A}_w \rtimes_{\hat{\alpha}} \mathbb{Z})) = 2, \quad \mathrm{csr}(C(\mathbb{T}) \otimes (\mathfrak{A}_w \rtimes_{\hat{\alpha}} \mathbb{Z})) = 2.$$

Use (F) of Section 2 to have $\mathrm{sr}(\mathfrak{B}) \leq 2$, $\mathrm{csr}(\mathfrak{B}) \leq 2$. So $\mathrm{gsr}(\mathfrak{B}) = 1$.

Consider the following exact sequence: for $z_j \in \mathbb{T}$ $(1 \leq j \leq n)$,

$$0 \to \mathfrak{I}_n \to \mathfrak{B} \to \oplus_{j=1}^n (\mathfrak{A}_{z_j} \rtimes_{\hat{\alpha}} \mathbb{Z}) \to 0$$

where $\mathfrak{I}_n = \Gamma_0(\mathbb{T}\setminus\{z_j\}_{j=1}^n, \{\mathfrak{A}_w \rtimes_{\hat{\alpha}} \mathbb{Z}\}_{w\in\mathbb{T}\setminus\{z_j\}_{j=1}^n})$. Now suppose that $\mathrm{sr}(\mathfrak{B}) = 1$. Then the index map of K-groups from $K_1(\oplus_{j=1}^n(\mathfrak{A}_{z_j} \rtimes_{\hat{\alpha}} \mathbb{Z}))$ to $K_0(\mathfrak{I}_n)$ must be zero by [Ny] (cf. [Ns2]). Hence, using the 6-term exact sequence of K-groups (cf. [Bl], [Wo]), we have the onto map from $K_1(\mathfrak{B})$ to $K_1(\oplus_{j=1}^n(\mathfrak{A}_{z_j} \rtimes_{\hat{\alpha}} \mathbb{Z}))$. However, the Pimsner-Voiculescu exact sequence of K-groups implies that this is impossible. Indeed, we have

$$0 \to K_1(C^*(H_3^{\mathbb{Z}})) \cong \mathbb{Z}^3 \to K_1(\mathfrak{B}) \to K_0(C^*(H_3^{\mathbb{Z}})) \cong \mathbb{Z}^3 \to 0,$$
$$K_1(\oplus_{j=1}^n(\mathfrak{A}_{z_j} \rtimes_{\hat{\alpha}} \mathbb{Z})) \cong \oplus_{j=1}^n K_1(\mathfrak{A}_{z_j} \rtimes_{\hat{\alpha}} \mathbb{Z}) \cong \oplus_{j=1}^n \mathbb{Z}^4 \cong \mathbb{Z}^{4n}.$$

See [AP] for the K-groups of $C^*(H_3^{\mathbb{Z}})$. By [Eh, Theorem 2.2], we have $\mathrm{csr}(C^*(H_3^{\mathbb{Z}}) \rtimes_\alpha \mathbb{Z}) \geq 2$. $\square$

Remark. In the definition of the action α, if one of θ_1, θ_2 is rational, or θ_1, θ_2 are rationally dependent, then we have the same rank estimates as in the statement.

By the same way as above, we define the generalized, discrete diamond crossed product to be $C^*(H_{2n+1}^{\mathbb{Z}}) \rtimes_\alpha \mathbb{Z}^n$, where $\alpha_t(U_j) = e^{2\pi i\theta_{1j}t_j}U_j$, $\alpha_t(V_j) = e^{2\pi i\theta_{2j}t_j}V_j$ and $\alpha_t(W) = W$ for $t = (t_j) \in \mathbb{Z}^n$, and $\theta_{1j}, \theta_{2j} \in \mathbb{R}$ are irrational and rationally independent, where

U_j, V_j, W are the generators of $C^*(H^{\mathbb{Z}}_{2n+1})$ corresponding to those of $H^{\mathbb{Z}}_{2n+1}$ as in the case of $n = 1$. Then we have

$$\begin{aligned} C^*(H^{\mathbb{Z}}_{2n+1}) \rtimes_\alpha \mathbb{Z}^n &= C^*(W, V_1, \cdots, V_n, U_1, \cdots, U_n) \rtimes_\alpha \mathbb{Z}^n \\ \cong (C(\mathbb{T}^{n+1}) \rtimes_\beta \mathbb{Z}^n) \rtimes_\alpha \mathbb{Z} &\cong \Gamma(\mathbb{T}, \{\otimes^n \mathfrak{A}_w\}_{w\in\mathbb{T}}) \rtimes_\alpha \mathbb{Z} \\ &\cong \Gamma(\mathbb{T}, \{\otimes^n (\mathfrak{A}_w \rtimes_{\hat{\alpha}} \mathbb{Z})\}_{w\in\mathbb{T}}), \end{aligned}$$

where the action β is defined by $\beta_a(w, z) = (w, (w^{a_j} z_j)_{j=1}^n)$ for $w \in \mathbb{T}$, $z = (z_j) \in \mathbb{T}^n$ and $a = (a_j) \in \mathbb{Z}^n$, and $C(\{w\} \times \mathbb{T}^n) \rtimes_\beta \mathbb{Z}^n \cong \otimes^n C(\{w\} \times \mathbb{T}) \rtimes_\beta \mathbb{Z} = \otimes^n \mathfrak{A}_w$. Using the same methods with Theorem 3.6, we get

Theorem 3.7. *Let $\mathfrak{B}_n = C^*(H^{\mathbb{Z}}_{2n+1}) \rtimes_\alpha \mathbb{Z}^n$ be the generalized, discrete diamond crossed product. Then $\mathfrak{B}_n$ is isomorphic to the C^*-algebra $\Gamma(\mathbb{T}, \{\otimes^n \mathfrak{A}_w \rtimes_{\hat{\alpha}} \mathbb{Z}\}_{w\in\mathbb{T}})$ of a continuous field on $\mathbb{T}$ with fibers $\otimes^n \mathfrak{A}_w \rtimes_{\hat{\alpha}} \mathbb{Z}$ n-fold tensor products of simple noncommutative 3-tori. Moreover, $\mathrm{sr}(\mathfrak{B}_n) = 2$, $\mathrm{csr}(\mathfrak{B}_n) = 2$ and $\mathrm{gsr}(\mathfrak{B}_n) = 1$.*

Remark. The same equalities hold by replacing $\mathfrak{B}_n$ with its finite tensor products.

Some improvements

To make the estimates of the connected stable rank of Theorems 3.1 and 3.2 more exact, we shall show the following, which might have been proved elsewhere:

Proposition 3.8. *Let $\mathfrak{A}$ be a nonunital C^*-algebra. If the K_1-group $K_1(\mathfrak{A})$ is trivial, then the group $GL(\mathfrak{A}^+)$ of all invertible elements of $\mathfrak{A}^+$ is connected.*

Proof. First note that

$$0 = K_1(\mathfrak{A}) \cong K_1(\mathfrak{A} \otimes \mathbb{K}) \cong GL((\mathfrak{A} \otimes \mathbb{K})^+)/GL((\mathfrak{A} \otimes \mathbb{K})^+)_0,$$

where $GL(\mathfrak{B})_0$ is the connected component of $GL(\mathfrak{B})$ containing the unit of a C^*-algebra $\mathfrak{B}$. Then for any $(a, \lambda), (b, \lambda') \in GL(\mathfrak{A}^+)$ identified with $(a \oplus 0_\infty, \lambda), (b \oplus 0_\infty, \lambda') \in GL((\mathfrak{A} \otimes \mathbb{K})^+)$ for $a, b \in \mathfrak{A}$, $\lambda, \lambda' \in \mathbb{C}$, $\oplus$ the diagonal sum and 0_∞ the infinite zero matrix, there exists $((c_{ij}), \mu) \in GL((\mathfrak{A} \otimes \mathbb{K})^+)_0$ such that

$$((c_{ij}), \mu)(a \oplus 0_\infty, \lambda) = (b \oplus 0_\infty, \lambda'),$$

which implies that

$$\left(\begin{pmatrix} c_{11}a & 0 & \cdots \\ c_{21}a & 0 & \cdots \\ \vdots & \vdots & 0_\infty \end{pmatrix} + \lambda(c_{ij}) + \mu(a \oplus 0_\infty), \mu\lambda\right) = (b \oplus 0_\infty, \lambda').$$

Hence we obtain that

$$((c_{ij}), \mu) = \left(\begin{pmatrix} c_{11} & 0 & \cdots \\ c_{21} & 0 & \cdots \\ \vdots & \vdots & 0_\infty \end{pmatrix}, \mu\right)$$

and $(c_{11}, \mu)(a, \lambda) = (b, \lambda')$ with $(c_{11}, \mu) \in GL(\mathfrak{A}^+)_0$. Thus $GL(\mathfrak{A}^+)$ is connected. $\square$

Corollary 3.9. *In Theorems* 3.1 *and* 3.2, *we have* $\operatorname{csr}(C^*(D_4)) = 1$ *and* $\operatorname{csr}(C^*(D_4) \otimes C^*(D_4)) = 1$.

Proof. By Connes' Thom isomorphism, we have $K_1(C^*(D_4)) = 0$ and $K_0(C^*(D_4)) = \mathbb{Z}$. By the Künneth formula (cf.[Wo]),

$$K_1(C^*(D_4) \otimes C^*(D_4)) \cong \oplus^2[K_0(C^*(D_4)) \otimes K_1(C^*(D_4))] \cong 0. \quad \square$$

Remark. By Proposition 3.8, the groups $GL(C^*(D_{3n+1})^+)$ for n odd and $GL(C^*(D_{3n+1} \times D_{3n+1})^+)$ for $n \geq 2$ are connected, but it does not imply in general that $\operatorname{csr}(C^*(D_{3n+1})) = 1$ for n odd and $\operatorname{csr}(C^*(D_{3n+1} \times D_{3n+1})) = 1$.

Corollary 3.10. *Let $\mathfrak{A}$ be a nonunital C^*-algebra such that* $\operatorname{csr}(\mathfrak{A}) \leq 2$ *and $K_1(\mathfrak{A})$ is trivial. Then* $\operatorname{csr}(\mathfrak{A}) = 1$.

Combining Corollary 3.10 with (F2) and [Eh, Corollary 1.6],

Proposition 3.11. *For any simply connected, solvable Lie group G,*

$$\operatorname{csr}(C^*(G) \otimes \mathbb{K}) = \begin{cases} 1 & \textit{if } \dim G \textit{ even}, \\ 2 & \textit{if } \dim G \textit{ odd}, \end{cases}$$

while $\operatorname{gsr}(C^*(G) \otimes \mathbb{K}) = 1$.

Proof. Since G is obtained by successive semi-direct products by $\mathbb{R}$, by Connes' Thom isomorphism for K-groups we have $K_1(C^*(G)) \cong 0$

if $\dim G$ even, and $K_1(C^*(G)) \cong \mathbb{Z}$ if $\dim G$ odd. Since $(C^*(G)\otimes\mathbb{K})^+$ is finite, we have $\mathrm{gsr}(C^*(G) \otimes \mathbb{K}) = 1$. $\square$

Remark. This is a generalization of Sheu's result for $G = \mathbb{R}^n$ [Sh, p.386]. On the other hand, we have by [ST, Lemma 3.7] that for G any simply connected, solvable Lie group,

$$\mathrm{sr}(C^*(G) \otimes \mathbb{K}) = \begin{cases} 1 & \text{if } \dim G = 1, \\ 2 & \text{if } \dim G \geq 2. \end{cases}$$

Finally, we give an example slightly different from D_4 (cf. [Dp]).

Example 3.12. Let $E_4 = H_3 \rtimes_\gamma \mathbb{R}$ with the action γ defined by

$$\gamma_t(c, b, a) = (c, \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix}), \quad t \in \mathbb{R}.$$

Using the setting before Theorem 3.1 we have

$$C^*(E_4) \cong C^*(H_3) \rtimes_\gamma \mathbb{R} \cong \Gamma_0(\mathbb{R}, \{\mathfrak{A}_t\}_{t\in\mathbb{R}}) \rtimes_\gamma \mathbb{R} \cong \Gamma_0(\mathbb{R}, \{\mathfrak{A}_t \rtimes_{\hat{\gamma}} \mathbb{R}\}_{t\in\mathbb{R}})$$

with $\mathfrak{A}_t \rtimes_{\hat{\gamma}} \mathbb{R} \cong \mathbb{K} \otimes C_0(\mathbb{R})$ for $t \in \mathbb{R} \setminus \{0\}$, and

$$0 \to C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathbb{R} \to \mathfrak{A}_0 \rtimes_{\hat{\gamma}} \mathbb{R} \to C_0(\mathbb{R}) \to 0$$

with $C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathbb{R} \cong C_0(\mathbb{R}_+) \otimes (C(\mathbb{T}) \rtimes \mathbb{R}) \cong C_0(\mathbb{R} \times \mathbb{T}) \otimes \mathbb{K}$. By the similar analysis as for $C^*(D_4)$, we get the rank estimates of $C^*(E_4)$, its disconnected version $C^*(E_3^d)$, and their higher dimensional cases $C^*(E_{3n+1})$ and $C^*(E_{2n+1}^d)$ as in the above theorems for the versions of $C^*(D_4)$. However, we can not define a crossed product as $C^*(H_3^{\mathbb{Z}}) \rtimes_\alpha \mathbb{Z}$ since the action γ of $\mathbb{Z}$ such that $\gamma_t(U) = U\cos t + V \sin t$, $\gamma_t(V) = -U\sin t + V\cos t$ for $t \in \mathbb{Z}$ is not an automorphism.

4. The properties for groups and group C^*-algebras

This is an appendix for convenience to readers. It consists of 5 subsections (A to E) as in the following (cf. [Mg], [dH2], [BL], [DyH], and [Jl]):

A: Groups with Properties (T) or (R).

For a locally compact (second countable) group G, we denote by $G^\wedge$ the space of equivalence classes of irreducible unitary representations of G, and $G^\sim$ the set of equivalence classes of (continuous) unitary representations of G. Set $H_\rho^G = \{\xi \in H_\rho \mid \rho(G)\xi = \xi\}$, where H_ρ is the representation space of $\rho \in G^\sim$. Write $\rho \geq 1_G$ if H_ρ^G is non-empty, that is, ρ contains 1_G, where 1_G is the trivial representation of G. The representation ρ is close to 1_G if for any $\varepsilon > 0$ and any compact $K \subset G$, there exists $\xi \in H_\rho$ such that $\|\rho(h)\xi - \xi\| < \varepsilon\|\xi\|$ for all $h \in K$.

Definitions. A locally compact group G has the property (T) (denoted by $G \in (T)$) if the trivial representation 1_G of G is isolated in $G^\wedge$, or if $\rho \geq 1_G$ for any $\rho \in G^\sim$ close to 1_G. A locally compact group G has the property (R) (denoted by $G \in (R)$) if one of the following is satisfied:

(1) The regular representation λ_G of G is close to 1_G.
(2) G is amenable, that is, the space $C^b(G)$ of continuous bounded functions on G has a left invariant mean, that is, a linear functional m on $C^b(G)$ such that $m(f) \geq 0$ for $f \geq 0$, $m(1) = 1$ and $m(\lambda_g f) = m(f)$ for $g \in G$ and $f \in C^b(G)$.
(3) The trivial representation 1_G of G is weakly contained in λ_G, that is, the kernel of 1_G contains that of λ_G (cf.[Dx, 3]).
(4) Every element of $G^\wedge$ is weakly contained in λ_G.

Remark. Every compact group has the property (T). If G is non-compact, then the regular representation λ_G of G dose not contain 1_G. Since λ_G is close to 1_G if and only if G is amenable, an amenable group with (T) is compact. In particular, a discrete amenable group with (T) is finite. A compact group G has (R) by $m(f) = \int_G f(g) d\mu_G(g)$.

A1. *A commutative group has (T) if and only if it is compact.*

Proof. The dual group of a commutative group G is discrete if and only if G is compact. $\square$

A2. *If H is a closed normal subgroup of $G \in (T)$, then $G/H \in (T)$.*

Proof. Note that $\pi \in (G/H)^\sim$ is close to (or contains) $1_{G/H}$ if and only if $\pi \circ q$ is close to (or contains) 1_H, where $q : G \to G/H \to 0$. $\square$

A3. *If $G \in (T)$, then $G/[G,G]^-$ (closure) is compact. This implies that $G \in (T)$ is unimodular.*

Proof. If $G \in (T)$, then $G/[G,G]^- \in (T)$ and commutative. Hence $G/[G,G]^-$ should be compact. Let $\Delta_G : G \to \mathbb{R}^+$ be the modular function of G. Since Δ_G is trivial on $[G,G]^-$, $\Delta_G(G)$ is a compact subgroup of $\mathbb{R}^+$. Hence $\Delta_G(g) = 1$ for any $g \in G$. $\square$

A4. *If $G \in (T)$, then G is compactly generated. If in addition G is discrete, then G is finitely generated.*

A5. *If $(T) \ni G/H, H$ (a closed normal subgroup), then $G \in (T)$.*

A6. *If H is a closed subgroup of G such that G/H has a finite G-invariant Borel measure (in particular, G/H is compact), then $G \in (T)$ if and only if $H \in (T)$.*

A7. *If $G \in (T)$, there exists no unbounded continuous negative definite function f on G, that is, $\sum_{i,j=1}^{n} f(g_i g_j^{-1}) z_i \bar{z}_j \leq 0$ for $n \in \mathbb{N}$, $g_i, g_j \in G$, $z_i, z_j \in \mathbb{C}$ with $\sum_{j=1}^{n} z_j = 0$, and $f(g^{-1}) = \overline{f(g)}$ for $g \in G$.*

A8. *If $G \in (T)$, then G is not an amalgam, where an amalgam is $H = H_1 *_K H_2$ with H_1, H_2 open subgroups of H and $K = H_1 \cap H_2$.*

Sketch of proof. Suppose the contrary, that is, G is an amalgam. Then construct an unbounded function on G. But the function must be bounded by property (T). $\square$

A9. (1) *If H is a closed subgroup of $G \in (R)$, then $H \in (R)$.* (2) *If H is a closed normal subgroup of G and if $H, G/H \in (R)$, then $G \in (R)$.* (3) *If G is solvable, then $G \in (R)$. Moreover, if there is a closed solvable normal subgroup H of G with G/H compact, then $G \in (R)$.*

A10. *The free groups do not have (R). The group $SL_2(\mathfrak{K})$ for $\mathfrak{K}$ a local field does not have (R).*

Part of proof. The following matrices for a suitable $\lambda \in \mathfrak{K}$:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} A \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$$

generate a free discrete subgroup in $SL_2(\mathfrak{K})$. $\square$

A11. *Let $\mathfrak{K}$ be a local field. Then the group $SL_3(\mathfrak{K})$ has (T), and the group $Sp_2(\mathfrak{K})$ has (T), where*

$$Sp_2(\mathfrak{K}) = \{g \in GL_4(\mathfrak{K}) \mid g^t J g = J\}, \quad J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$$

Part of proof. Define the following two subgroups of $SL_3(\mathfrak{K})$:

$$H = \left\{ \begin{pmatrix} a & b & c \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix} \in SL_3(\mathfrak{K}) \right\}, \quad N = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y \in \mathfrak{K} \right\}$$

to show that if $\rho \in H^{\sim}$ is close to 1_H, then $\rho|_N \geq 1_N$, and to deduce from this that if $\rho \in G^{\sim}$ is close to 1_G, then $\rho \geq 1_G$. $\square$

Remark. A local field is a commutative non-discrete locally compact field. A local field is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, a finite extension of $\mathbb{Q}_p$, or a field of formal power series in one variable over a finite field. A local field is archimedean if it is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.

A12. *Let G be a connected semisimple group over a local field and H the simply connected covering group of G. Then $G \in (T)$ if and only if $H \in (T)$.*

Sketch. We take the central k-isogeny π from H to G. Then $\pi(H)$ is closed in G, and $H/\ker\pi \cong \pi(H)$ as a topological group. Since $\ker\pi$ is finite, it has (T). Thus, H has (T) if and only so does $\pi(H)$. Also note that $\pi(H)$ is normal in G and $G/\pi(H)$ is compact. $\square$

A13. *Let $\mathfrak{K}$ be a local field, $G(\mathfrak{K})$ a connected non-commutative almost $\mathfrak{K}$-simple $\mathfrak{K}$-group and* $\mathrm{rank}_{\mathfrak{K}} G(\mathfrak{K}) \neq 1$. *Then $G(\mathfrak{K})$ has (T).*

Sketch. If $\mathrm{rank}_{\mathfrak{K}} G(\mathfrak{K}) = 0$, then $G(\mathfrak{K})$ is compact and hence has (T). If $\mathrm{rank}_{\mathfrak{K}} G(\mathfrak{K}) \geq 2$, then $G(\mathfrak{K})$ (simply connected) contains a semisimple $\mathfrak{K}$-subgroup of $\mathfrak{K}$-rank 2 the simply connected covering of which is isomorphic over $\mathfrak{K}$ either to $SL_3(\mathfrak{K})$ or $Sp_4(\mathfrak{K})$. $\square$

Remark. If $\mathfrak{K}$ is non-archimedean and $\mathrm{rank}_{\mathfrak{K}} G(\mathfrak{K}) = 1$, then $G(\mathfrak{K}) \notin (T)$. If $\mathfrak{K} = \mathbb{C}$ and $G(\mathbb{C})$ is locally isomorphic to $SL_2(\mathbb{C})$, then $G(\mathbb{C}) \notin (T)$. If $\mathfrak{K} = \mathbb{R}$ and $G(\mathbb{R})$ is locally isomorphic to either $SO(n,1)$ or $SU(n,1)$, then $G(\mathbb{R}) \notin (T)$.

B: Residually finite groups.

Definition. A group Γ is residually finite if for any $\gamma \in \Gamma \setminus 1_\Gamma$, there exists a finite group F and a homomorphism φ from Γ to F such that $\varphi(\gamma) \neq 1_F$.

B1. *Any subgroup of a residually finite group is residually finite. For $n \geq 1$, $SL_n(\mathbb{Z})$ is residually finite. The free groups are residually finite. Any free product of residually finite groups is residually finite.*

B2. *A group Γ is residually finite if and only if there is a Hausdorff space X and a faithful action of Γ by homeomorphisms on X which is chaotic, that is, the union of all finite orbits is dense in X, and there exists $\gamma \in \Gamma$ such that $\gamma(U) \cap V \neq \emptyset$ for any non-empty open subsets U, V of X (the action is topologically transitive). $SL_n(\mathbb{Z})$ has such a chaotic action on the n-torus.*

B3. *If a group has a residually finite subgroup of finite index, then it is residually finite. A semi-direct product by residually finite groups is residually finite.*

Remark. There exists a non-residually finite group with a finite central subgroup with its quotient residually finite.

B4. *If Γ is a residually finite, finitely generated group, then the group of automorphisms of Γ is residually finite.*

Example. The group $GL_n(\mathbb{Z})$ for $n \geq 1$ is finitely generated. The additive group $\mathbb{Q}$ is not finitely generated. The group $SL_n(\mathbb{Q})$ for $n \geq 2$ is not finitely generated. The amalgamated free product $\Gamma_1 *_K \Gamma_2$ of Γ_1, Γ_2 over K is finitely generated as soon as Γ_1, Γ_2 are.

Example. The Baumslag-Solitar group $\Gamma_{p,q}$ for integers p, q with $(p, q) \neq (1, 1)$ is defined by $\Gamma_{p,q} = \langle s, t \mid st^p s^{-1} = t^q \rangle$. If $p = 1$, $q = 1$ or $p = q$, then $\Gamma_{p,q}$ is residually finite. In other cases, $\Gamma_{p,q}$ is not residually finite. A group Γ by the HNN-extension of a group Γ' is defined by $\Gamma = \Gamma' *_A = \langle \Gamma', t \mid t^{-1}At = B \rangle$ with A, B subgroups of Γ'

C: The factorization property.

Definition. A discrete group Γ has the factorization property (Fc) if the representation β of $\Gamma \times \Gamma$ on $\mathbb{B}(l^2(\Gamma))$ defined by $\beta_{(\gamma,\gamma')} f(x) = f(\gamma^{-1} x \gamma')$ for $\gamma, \gamma', x \in \Gamma$ factorizes through the spatial tensor product $C^*(\Gamma) \otimes C^*(\Gamma)$, that is, β from $C^*(\Gamma) \otimes_{\max} C^*(\Gamma)$ to $\mathbb{B}(l^2(\Gamma))$ is equal to the composition from $C^*(\Gamma) \otimes_{\max} C^*(\Gamma)$ to $C^*(\Gamma) \otimes C^*(\Gamma)$ to $\mathbb{B}(l^2(\Gamma))$.

C1. *Let Γ be a discrete group with the factorization property. Then the following are equivalent:* (1) $C^*(\Gamma)$ *is nuclear;* (2) $C^*(\Gamma)$ *is exact;* (3) Γ *is amenable.*

Sketch of proof. If Γ is amenable, then $C^*(\Gamma)$ is nuclear. A nuclear C^*-algebra is exact. If $C^*(\Gamma)$ is exact, the following sequence:

$$0 \to \ker\lambda_\Gamma \otimes C^*(\Gamma) \to C^*(\Gamma) \otimes C^*(\Gamma) \to C^*_r(\Gamma) \otimes C^*(\Gamma) \to 0$$

is exact. Thus, $\ker(\lambda_\Gamma \otimes \mathrm{id}) = \ker\lambda_\Gamma \otimes C^*(\Gamma)$, where λ_Γ and id are the regular and identity representations of $C^*(\Gamma)$ respectively. By (Fc), $\ker\lambda_\Gamma \otimes C^*(\Gamma) \subset \ker\beta$. Hence β is weakly contained in $\lambda_\Gamma \otimes \mathrm{id}$. Let Δ be the diagonal subgroup of $\Gamma \times \Gamma$. Then the restriction $\beta|_\Delta$ is weakly contained in $(\lambda_\Gamma \otimes \Phi_G)|_\Delta$, where Φ_G is the universal representation of Γ. Thus, 1_Γ is weakly contained in the inner tensor product $\lambda_\Gamma \otimes \Phi_G$. Since $\lambda_\Gamma \otimes \Phi_G \cong \infty\lambda_\Gamma$, 1_Γ is weakly contained in λ_Γ. $\square$

C2. *Let Γ be a residually finite group. Then Γ has (Fc).*

Sketch. Note that for distinct $\gamma_j \in \Gamma$ for $1 \leq j \leq n$,

$$\langle \beta(\gamma_i, \gamma_j)\delta_e, \delta_e \rangle = \langle \beta_{\Gamma/N}(\sigma(\gamma_i), \sigma(\gamma_j))\delta_e, \delta_e \rangle,$$

where N is a normal subgroup of Γ with finite index such that $\sigma : \Gamma \to \Gamma/N$ and $\sigma(\gamma_i) \neq \sigma(\gamma_j)$ for $i \neq j$ $\square$

C3. *Let Γ be a finitely generated linear group, that is, a finitely generated subgroup of some $GL_n(\mathbb{C})$. Then Γ has (Fc).*

Proof. It is Mal'cev's theorem that a finitely generated linear group is residually finite. $\square$

C4. *A group with (T) and (Fc) is residually finite.*

Sketch. We need to show that β is weakly contained in the set of all finite dimensional representations of $\Gamma \times \Gamma$. Then, the restriction of β to $\Gamma \times \{e\} \cong \Gamma$, which is a multiple of the regular representation of Γ, is weakly contained in the set of finite dimensional unitary representations of Γ. This implies that Γ has a separating family of finite dimensional unitary representations. $\square$

D: Residually finite dimensional C^*-algebras.

Definition. A group is a MAP-group (maximally almost periodic) if is has a separating family of finite dimensional unitary representations. A C^*-algebra is residually finite dimensional (RFD) if it has a separating family of finite dimensional representations.

D1. *A group* Γ *is an amenable MAP-group if and only if* $C_r^*(\Gamma)$ *is residually finite dimensional* (*RFD*).

Sketch. If $C_r^*(\Gamma)$ has a finite dimensional representation π, then Γ is amenable. In fact, $\pi \otimes \pi^*$ contains the trivial representation 1_Γ. Note that $\pi \otimes \pi^*(\gamma)T = \pi_\gamma T \pi_\gamma^{-1}$ for $T \in M_n(\mathbb{C}) = \pi(\Gamma)$, $\gamma \in \Gamma$. Hence 1_Γ is weakly contained in $\lambda_\Gamma \otimes \lambda_\Gamma \cong \infty\lambda_\Gamma$ as π is weakly contained in the regular representation λ_Γ. If $C^*(\Gamma)$ is RFD, then Γ is a MAP-group (maximally almost periodic), that is, finite dimensional representations of Γ separate points of Γ. For the converse, it suffices to show that $\langle \lambda_\Gamma(\gamma)\delta_e, \delta_e \rangle$ for $\gamma \in \Gamma$ is approximated by positive definite functions associated to finite dimensional representations of Γ, pointwise on Γ. $\square$

D2. *If* $\Gamma = F_n$ *the free group of* n *generators, or if* $\Gamma = SL_2(\mathbb{Z})$, *then* $C^*(\Gamma)$ *is residually finite dimensional.*

Sketch. Let $\Gamma = F_2$ and Φ be the universal representation from Γ to $C^*(\Gamma)$ in $\mathbb{B}(H)$ the C^*-algebra of bounded operators on a Hilbert space H. Let (P_n) be a sequence of increasing projections of $\mathbb{B}(H)$ with $\dim P_n = n$ and their strong limit the identity operator. Define the desired representation by $\pi = \oplus_n^\infty \pi_n$ where

$$\pi_n(\Phi(g)) = \begin{pmatrix} G_n & (P_n - G_n G_n^*)^{1/2} \\ (P_n - G_n^* G_n)^{1/2} & -G_n^* \end{pmatrix} \in M_{2n}(\mathbb{C})$$

for $G_n = P_n \Phi(g) P_n$ and $g \in \Gamma$ (two generators).

Furthermore, the property (RFD) is inherited by subalgebras, $C^*(F_n)$ is embeddable to $C^*(F_2)$, and the subgroup generated by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$$

is isomorphic to F_2 and of index 12 in $SL_2(\mathbb{Z})$. $\square$

D3. *If Λ is a subgroup of a discrete group Γ with finite index, and if $C^*(\Lambda)$ is RFD, then $C^*(\Gamma)$ is RFD.*

Sketch. Take a unitary representation π of Γ. The restriction $\pi|_\Lambda$ to Λ is weakly contained in the set of all finite dimensional unitary representations of Λ. Hence the induced representation of $\pi|_\Lambda$ to Γ is weakly contained in the set of all induced representations from finite dimensional unitary representations of Λ, which are finite dimensional since Λ has finite index in Γ. Moreover, π is equivalent to a subrepresentation of the induced representation of $\pi|_\Lambda$. $\square$

D4. *Let G be a simply connected, simple algebraic group over a number field $\mathcal{K}$. Let $\mathcal{O}$ be the ring of integers of $\mathcal{K}$, and $\Gamma = G(\mathcal{O})$ the group of all integral points in G. Suppose that* $\mathrm{rank}_{\mathcal{K}} G \geq 2$ *or* $\mathrm{rank}_{\mathcal{K}} G = 1$ *with $\mathcal{K} \neq \mathbb{Q}$ and $\mathcal{K} \neq \mathbb{Q}(\sqrt{d})$ for $d < 0$. Then $C^*(\Gamma)$ is not RFD.*

Sketch. In the case where $K = \mathbb{Q}(\sqrt{2})$, $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$, $\Gamma = SL_2(\mathbb{Z}[\sqrt{2}])$. Then Γ is embeddable to a lattice in $SL_2(\mathbb{R})^2$ by $\Gamma \ni \gamma \mapsto (\gamma, \gamma^\sigma)$, where σ is the non-trivial automorphism of $\mathbb{Q}(\sqrt{2})$ with $\sigma(\sqrt{2}) = -\sqrt{2}$. Since $SL_2(\mathbb{R})^2$ does not have (T), so does Γ. Also, Γ has the congruence subgroup property, that is, every normal subgroup N of Γ with index finite contains the kernel $\Gamma(S)$ of the map from $SL_2(\mathbb{Z}[\sqrt{2}])$ to $SL_2(\mathbb{Z}[\sqrt{2}])/S$, where S is an ideal of $\mathbb{Z}[\sqrt{2}]$. By Selberg's inequality, the trivial representation 1_Γ is isolated in the set $\Gamma_f^\wedge$ of all irreducible, finite dimensional representations of Γ. On the other hand, if $C^*(\Gamma)$ were RFD, then $\Gamma_f^\wedge$ is dense in $\Gamma^\wedge$. $\square$

Remark. As Γ we may take $SL_n(\mathbb{Z})$ for $n \geq 3$ and $Sp_n(\mathbb{Z})$ for $n \geq 2$.

D5. *An RFD C^*-algebra $\mathfrak{A}$ has a finite faithful trace.*

Sketch. There is an injection i from $\mathfrak{A}$ to a direct sum C^*-algebra $\oplus_{j=1}^\infty M_{n_j}(\mathbb{C})$ of certain matrix algebras $M_{n_j}(\mathbb{C})$. Let τ_j be the finite faithful normalized trace on $M_{n_j}(\mathbb{C})$. Then define the finite faithful trace τ on $\oplus_{j=1}^\infty M_{n_j}(\mathbb{C})$ by $\tau = \sum_{j=1}^\infty 2^{-j}\tau_j$. Then $\tau \circ i$ is the desired trace on $\mathfrak{A}$. $\square$

E: Some properties for group C^*-algebras.

Let $\mathfrak{A}$ be a C^*-algebra and τ a faithful tracial state on $\mathfrak{A}$, that is, $\tau(ab) = \tau(ba)$ for $a, b \in \mathfrak{A}$, and if $\tau(a^*a) > 0$ for any nonzero $a \in \mathfrak{A}$.

The spectral radius and l^2-spectral radius of $a \in \mathfrak{A}$ are defined by

$$r(a) = \lim_{n\to\infty} \sqrt[n]{\|a^n\|}, \quad r_2(a) = \limsup_{n\to\infty} \sqrt[n]{\|a^n\|_2},$$

where $\|a\|_2 = \sqrt{\tau(a^*a)}$. As $\|a\|_2 \leq \|a\|$, we have $0 \leq r_2(a) \leq r(a)$. For any group Γ, the canonical faithful trace τ on the reduced C^*-algebra $C_r^*(\Gamma)$ of Γ is defined by $\tau(a) = \langle a\delta_e, \delta_e\rangle$ for $a \in C_r^*(\Gamma)$.

Definition. A finite set F of a group Γ is semi-free if the semigroup generated by F is free over F, that is, if $x_j, y_k \in F$ for $1 \leq j \leq n$, $1 \leq k \leq m$ and $x_1x_2\cdots x_n = y_1y_2\cdots y_m$, then $n = m$ and $x_j = y_j$ for $1 \leq j \leq n$. Also, F has the l^2-spectral radius (spr) property if for any $a \in \mathrm{span}F$ in $\mathbb{C}\Gamma$, $r(a) = r_2(a)$. Furthermore, Γ has the free semigroup property if for any finite subset F, there is $\gamma \in \Gamma$ such that $\gamma F = \{\gamma x \mid x \in F\}$ is semifree.

E1. *Let Γ be a discrete group and suppose that for any finite subset F of Γ, there is $\gamma \in \Gamma$ such that γF is semi-free and has the l^2-spectral radius property. Then $C_r^*(\Gamma)$ has stable rank* 1.

Sketch of proof. For any element a of a unital C^*-algebra $\mathfrak{A}$, we have $d(a, GL(\mathfrak{A})) \leq r(a)$, where $d(x, y) = \|x - y\|$. Indeed, for $\varepsilon > 0$, $b = a - (r(a) + \varepsilon) \in GL(\mathfrak{A})$ and $\|a - b\| = r(a) + \varepsilon$. Hence $d(a, GL(\mathfrak{A})) \leq r(a) + \varepsilon$.

For $c = \sum_{x\in X} c_x\lambda_x \in \mathbb{C}\Gamma$, where X is a semifree subset of Γ, one has $r_2(c) = \|c\|_2$. Indeed, for any integer $n \geq 1$, one has $c^n = \sum_{y\in X^n} c_y\lambda_y$ with $c_y = c_{x_1}c_{x_2}\cdots c_{x_n}$ whenever $y = x_1x_2\cdots x_n \in X^n$. Thus, $\|c^n\|_2 = \|c\|_2^n$.

Suppose for contradiction that $GL(\mathfrak{A})$ for $\mathfrak{A} = C_r^*(\Gamma)$ is not dense in $\mathfrak{A}$. Then there is $a \in \mathfrak{A}$ such that $\|a\| = 1$ and $d(a, GL(\mathfrak{A})) = 1$. If $\|a\|_2 = 1$, that is, $\tau(a^*a) = 1$, then $a^*a = 1$ by faithfulness of τ. Thus, one must have $\varepsilon = 1 - \|a\|_2 > 0$. Let $b = \sum_{x\in X} b_x\lambda_x \in \mathbb{C}\Gamma$ with $\|b - a\| < \varepsilon/3$, where $X \subset \Gamma$ is the support of b. Then $d(b, GL(\mathfrak{A})) \geq d(a, GL(\mathfrak{A})) - (\varepsilon/3)1 - (\varepsilon/3)$ and

$$\begin{aligned}\|b\|_2 &\leq \|a\|_2 + \|b - a\| \\ &\leq 1 - \varepsilon + (\varepsilon/3) < 1 - (\varepsilon/3) \leq d(b, GL(\mathfrak{A})).\end{aligned}$$

By assumption, there exists $\gamma \in \Gamma$ such that $Y = \gamma X$ is semifree and has the l^2-spectral radius property. If $c = \lambda_\gamma b \in \mathbb{C}\Gamma$, then $\|c\|_2 =$

$\|b\|_2$, $d(c, GL(\mathfrak{A})) = d(b, GL(\mathfrak{A}))$, and $r_2(c) = \|c\|_2$. Furthermore, $r(c) = r_2(c)$. Consequently, the contradiction follows from

$$\|b\|_2 < d(b, GL(\mathfrak{A})) = d(c, GL(\mathfrak{A})) \leq r(c) = \|c\|_2 = \|b\|_2. \qquad \square$$

Remark. It follows from $E1$ that if Γ is a group wit both the l^2-(spr) property and the free semigroup property, then $C_r^*(\Gamma)$ has stable rank one. The following is obtained from this and the observations below.

E2. *Let Γ be a hyperbolic discrete group and suppose that Γ is either torsion free and non-elementary or a cocompact lattice in a real noncompact simple connected Lie group with real rank* 1 *and center trivial. Then $C_r^*(\Gamma)$ has stable rank* 1.

E3. *Let $\Gamma = G_1 *_H G_2$ be an amalgamated free product of discrete groups with H finite. Suppose that there is $\gamma \in \Gamma$ such that $\gamma^{-1}H\gamma \cap H = \{1\}$. Then $C_r^*(\Gamma)$ has stable rank* 1.

Sketch. If H is trivial, the statement has already been known to be true. Assume that H is nontrivial. If G_1, G_2 are finite, then they have property (RD) (see below). Hence Γ has (RD). Then Γ has the l^2-(spr) property. $\square$

Definition. A length function on a group Γ is a map $L : \Gamma \to \mathbb{R}_+$ such that (1) $L(gh) \leq L(g) + L(h)$ for $g, h \in \Gamma$; (2) $L(g) = L(g^{-1})$ for $g \in \Gamma$; (3) $L(1) = 0$. The Sobolev space $H_L^s(\Gamma)$ of order $s \in \mathbb{R}$ with respect to a length function L on a group Γ is the set of all functions ξ on Γ such that $\xi(1+L)^s \in l^2(\Gamma)$. The space of rapidly decreasing functions on Γ with respect to L is $H_L^\infty(\Gamma) = \cap_{s\in\mathbb{R}} H_L^s(\Gamma)$.

Remark. The space $H_L^s(\Gamma)$ is a Hilbert space with the inner product:

$$\langle \xi | \xi \rangle_{2,s,L} = \sum_{g\in\Gamma} |\xi(g)|^2 (1 + L(g))^{2s}, \quad \xi \in H_L^s(\Gamma),$$

and $H_L^\infty(\Gamma)$ is a Fréchet space with the projective limit topology.

Definition. A discrete group Γ has the property (RD) if there is a length function L on Γ such that $H_L^\infty(\Gamma)$ is contained in $C_r^*(\Gamma)$, or if for some length function L on Γ, some positive constants c, s, and any $a = \sum_{\gamma \in \Gamma} a_\gamma \lambda_\gamma \in \mathbb{C}\Gamma$, we have $\|a\| \leq c\|a\|_{2,s,L}$, where

$$\|a\|_{2,s,L} = \left(\sum_{\gamma \in \Gamma} |a_\gamma|^2 (1 + L(\gamma))^{2s} \right)^{1/2}.$$

Example. (1) If Γ is the infinite cyclic group, then Γ has (RD) since $\|a\| \leq (\pi/\sqrt{3})\|a\|_{2,1,L}$ for $a \in \mathbb{C}\Gamma$. (2) If F_n is the free group of rank ≥ 2, then F_n has (RD) since $\|f\| \leq 2\|f\|_{2,2,L}$ for f any function with finite support on F_n.

E4. *A subgroup of a group with (RD) has (RD). If E is an extension of Γ with finite index, then $E \in (RD)$ if and only if $\Gamma \in (RD)$. If Γ has a subgroup of finite index with (RD), then $\Gamma \in (RD)$.*

E5. *A group Γ with (RD) has the l^2-spectral radius property.*

Sketch of Proof. Let L be a length function on Γ such that $\Gamma \in (RD)$ with respect to Γ. Then for $a \in \mathbb{C}\Gamma$, we have $\|a\| \leq c(1+L(a))^s\|a\|_2$, where $L(a) = \max\{L(g) \mid g \in \mathrm{supp}(a)\}$. Now $\|a^n\|_2 \leq \|a^n\|$ and $L(a^n) \leq nL(a)$ for every $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} r_2(a) \leq r(a) &\leq \liminf_{n \to \infty} c^{1/n}(1 + nL(a))^{s/n}\|a^n\|_2^{1/n} \\ &= \liminf_{n \to \infty} \|a^n\|_2^{1/n} \leq r_2(a). \qquad \square \end{aligned}$$

Definition. A (finitely generated) group Γ is of polynomial growth (with respect to L) if there exist $c, r \in \mathbb{R}_+$ such that $|B_{k,L}| \leq c(1+k)^r$ for every $k \geq 0$, where $B_{k,L} = \{g \in \Gamma \mid L(g) \leq k\}$ (the ball of radius k), and $|B_{k,L}|$ is the cardinal of $B_{k,L}$. A (finitely generated) group Γ is of exponential growth (with respect to L) if there exist $u > 0$ and $v > 1$ such that $|B_{k,L}| \geq uv^k$ for every $k \geq 0$.

Example. If Γ is the abelian free group of rank $N \geq 1$, then Γ is of polynomial growth with respect to L by the canonical set of generators of Γ since there exist $c_1, c_2 > 0$ such that $c_1 k^N \leq |B_{k,L}| \leq c_2 k^N$ for every $k \geq 1$, and $|B_{k,L}| = \sum_{l=0}^{N} 2^l \binom{N}{l}\binom{K}{l}$. On the other hand, $\mathbb{Z}$ is of exponential growth with respect to L defined by $L(n) =$

$\log(1+|n|)$. If a finitely generated group Γ is of polynomial growth if and only if it is almost nilpotent, that is, Γ contains a nilpotent subgroup of finite index, and also if and only if $H_L^\infty(\Gamma)$ is contained in $l^1(\Gamma)$. If Γ is amenable, then Γ is of polynomial growth if and only if Γ has (RD).

Examples. As more groups with property (RD),

(1) Gromov's hyperbolic groups ([dH1]).
(2) Free products of groups with (RD), and amalgamated free products of groups with (RD) over finite groups.

A group Γ is δ-hyperpolic if (Γ, L_S) (a metric space) for the word metric L_S with respect to a generator system S of Γ is δ-hyperpolic, that is, for any fixed $t \in \Gamma$, $(x|y)_t \geq \min\{(x|z)_t, (y|z)_t\} - \delta$ for every $x, y, z \in \Gamma$, where $(x|y)_t = 2^{-1}(L_S(x,t) + L_S(y,t) - L_S(x,y))$ (Gromov product). A group Γ is hyperbolic if it is δ-hyperpolic for $\delta \geq 0$. The fundamental group of a closed manifold with negative curvature is hyperbolic. This follows from that a complete simply connected Riemann manifold with negative sectional curvature is hyperbolic, and that a group Γ acting on a proper metric space X isometrically and properly discontinuously with X/Γ compact is hyperbolic if and only if X is hyperbolic.

Examples. As more groups with the l^2-spectral radius property,

(1) Finitely generated groups with subexponential growth.
(2) Inductive limits of groups with the l^2-(spr) property.

A group Γ has subexponential growth if $\lim_{k\to\infty} |F^k|^{1/k} = 1$ for F a finite symmetric generating subset of Γ with $e \in F$ ([HRV]).

Definition. An action of a group Γ on a Hausdorff space X by homeomorphisms is minimal if every orbit of the action is dense in X, is strongly faithful if for any finite subset F of $\Gamma \setminus \{1\}$, there is $x \in X$ such that $fx \neq x$ for $f \in F$, and is strongly hyperbolic if there are $g, h \in \Gamma$ that act hyperbolically on X and are transverse.

E6. *Let Γ be a discrete group acting on a Hausdorff space. If the action is minimal, strongly faithful and strongly hyperbolic, then Γ has free semigroup property.*

Examples. As groups with free semigroup property,

(1) Torsion-free non-elementary hyperbolic groups.

(2) Lattices in simple non-compact connected real Lie groups with real rank 1 and center trivial.
(3) Free products of discrete groups with more than one or two elements respectively. Amalgamated free products Γ with H a common subgroup such that for any finite F of $\Gamma \setminus \{1\}$, there is $\gamma \in \Gamma$ with $\gamma^{-1} F \gamma \cap H = \emptyset$.

E7. *Let* $\Gamma = G_1 *_H G_2$ *be an amalgamated free product of discrete groups. Suppose that there is* $\gamma \in \Gamma$ *such that* $\gamma^{-1} H \gamma \cap H = \{1\}$. *Then for any finite subset* F *of* $\Gamma \setminus \{1\}$, *there is* $g \in \Gamma$ *such that* $g^{-1} F g \cap H = \emptyset$. *Hence* Γ *has the free semigroup property.*

E8. *Let* Γ *be a torsion free group that is either amenable or a discrete subgroup in a connected Lie group whose semi-simple part is locally isomorphic to a product of either compact groups, Lorentz groups* $SO(n, 1)$ *or* $SU(n, 1)$ $(n \geq 2)$. *Then* $C_r^*(\Gamma)$ *has no nontrivial idempotents. Hence* $\mathrm{RR}(C_r^*(\Gamma)) \geq 1$, *that is, the real rank of* $C_r^*(\Gamma)$ *is nonzero.*

Remark. This is the case where the Kaplansky-Kadison conjecture is true, and this is also a consequence from that the Baum-Connes conjecture has been proved for such groups (see [DyH]).

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