Group C^⊠-algebras as inductive limits: a survey

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GROUP C^*-ALGEBRAS AS INDUCTIVE LIMITS: A SURVEY

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Abstract

This is a survey on our study for describing group C^*-algebras as inductive limits. It is shown by our results that the group C^*-algebras of certain type R Lie groups such as the Heisenberg Lie group, motion groups and more general CCR groups have the inductive limit structure by subhomogeneous C^*-algebras, and all non type R solvable Lie group C^*-algebras do not, and the group C^*-algebras of certain nilpotent discrete groups such as the generalized discrete Heisenberg groups have the decomposition into extensions by generalized ASH algebras. Reviews for Lie groups of type R and classification of inductive limit C^*-algebras are also included.

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Contents

Introduction

The main purpose of this paper is to show some results on our study for describing group C^*-algebras of Lie groups or discrete groups as inductive

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limits. Our mission for this project is still incomplete but we made some progress which would be crucial for further research on this topic ([20], [24], [25] and [26]), and the point of view should be interesting enough for us in a current situation that the classification theory for C^*-algebras has been of great interest ([6], [13] and [14]). We expect that our results obtained so far on this topic could be useful for some other topics such as the classification theory and K-theory for C^*-algebras. As for the K-theory ([2], [13] and [27]), using the structure of group C^*-algebras of (certain) connected or disconnected Lie groups (the structure means their decompositions into finite extensions, i.e., finite composition series) we estimated their stable rank (in terms of groups) (see [16], [17], [18], [19], [21], [22] and [23], and see M.A. Rieffel [12] for the stable rank of C^*-algebras).

In Preliminaries we review: Group C^*-algabras, Crossed products of C^*-algebras, and Solvable Lie groups. Actually we review the definitions of group C^*-algebras of locally compact groups and of crossed products of C^*-algebras and their basic properties, and further review the classes and examples of solvable Lie groups of type R or non type R. Refer to L. Auslander and C.C. Moore [1] for solvable Lie groups of type R.

In Section 1 we collect and review some results on the inductive limit structure of Lie group C^*-algebras obtained by us so far ([20], [24] and [25] from which we take some (part of) proofs). Our first discovery (or evidence) for the dichotomy for group C^*-algebras of (solvable) Lie groups was that (the unitizations of) the group C^*-algebras of simply connected solvable Lie groups of non type R are not ASH (approximately subhomogeneous).

In Section 2 we deal with decomposing the group C^*-algebra of the generalized discrete Heisenberg groups into extensions by generalized ASH algebras, and give its applications ([26]).

In Section 3 we review Lie groups of type R and their important properties. We refer to A.L. Onishchik and E.B. Vinberg [10] and L. Auslander and C.C Moore [1].

In Section 4 we collect and review some definitions and results on the classification of inductive limit C^*-algebras by K-theory. We refer to H. Lin [6], M. Rørdam, F. Larsen and N.J. Laustsen [13] and M. Rørdam and Størmer [14]. There are 8 subsections as follows: K-theory, Finite dimensional C^*-algebras, UHF algebras, AF algebras, AT algebras, AH algebras, ASH algebras and beyond them, and Purely infinite simple C^*-algebras. In particular, some important classification theorems are taken directly from the excellent (but unreadable) book of Lin [6] (and [7] and [14]) with some somewhat complete (or incomplete) proofs for the convenience to readers. For this reason this section has become very large in the end. However, we could not include all the details since the theory is vast and the time and space for publication are limited.

Preliminaries

Group C*-algebras (J. Dixmier [4] and G.K. Pedersen [11]) Let G be a locally compact group. The group C^*-algebra $C^*(G)$ of G is defined to be the C^*-completion of the Banach $*$-algebra $L^1(G)$ of integrable complex-valued measurable functions on G with convolution and involution:

$$f * g(t) = \int_G f(x)g(x^{-1}t)dx, \quad f^*(t) = \Delta(t)^{-1}\bar{f}(t^{-1})$$

for $t \in G$, $f, g \in L^1(G)$, where Δ is the modular function of G (a continuous homomorphism to $\mathbb{R}_+$), or to be the C^*-algebra generated by the image of $L^1(G)$ under the universal representation on the universal Hilbert space. The reduced group C^*-algebra $C_r^*(G)$ of G is defined to be the C^*-algebra generated by the image of $L^1(G)$ under the left regular representation λ on the Hilbert space $L^2(G)$ of square integrable complex-valued measurable functions on G:

$$\lambda(f)\xi(t) = \int_G f(x)\xi(x^{-1}t)dx \quad \text{for } t \in G,\ f \in L^1(G),\ \xi \in L^2(G).$$

By universality of $C^*(G)$ we always have the quotient: $C^*(G) \to C_r^*(G) \to 0$. Note that $C^*(G) \cong C_r^*(G)$ if and only if G is amenable [11, Theorem 7.3.9], that is, there exists a left invariant mean (or state) on the Banach $*$-algebra (or the von Neumann algebra) $L^\infty(G)$ of essentially bounded measurable functions on G with pointwise multiplication and supreme norm (or on $C^b(G)$ the C^*-algebra of bounded continuous functions on G). Solvable groups and compact groups are amenable. The class of amenable groups is closed under taking quotients, closed subgroups, extensions (by closed normal subgroups) and the unions of increasing nets. Free groups are not amenable.

The unitary dual of a locally compact group G consisting of equivalence classes of irreducible unitary representations of G is identified with the spectrum of $C^*(G)$ consisting of equivalence classes of irreducible representations of $C^*(G)$. In this sense G is usually identified with $C^*(G)$ in the level of representation theory. Moreover, G (or $C^*(G)$) is said to be CCR (= liminal) if for any irreducible representation π of $C^*(G)$, its image is isomorphic to the C^*-algebra $\mathbb{K}(H)$ of compact operators on a finite or infinite dimensional Hilbert space H, and G (or $C^*(G)$) is said to be of type I if for any irreducible representation π of $C^*(G)$, the image $\pi(C^*(G))$

contains $\mathbb{K}(H)$. We can define a C^*-algebra $\mathfrak{A}$ to be CCR (= liminal) or of type I by replacing $C^*(G)$ with $\mathfrak{A}$.

Crossed products of C*-algebras [11] Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system that consists of a C^*-algebra $\mathfrak{A}$, a locally compact group G, and an action α of G on $\mathfrak{A}$ by automorphisms. Let $L^1(G, \mathfrak{A})$ be the Banach $*$-algebra of all integrable $\mathfrak{A}$-valued measurable functions on G with convolution and involution:

$$f * g(t) = \int_G f(x)\alpha_x(g(x^{-1}t))dx, \quad f^*(t) = \Delta(t)^{-1}\alpha_t(f(t^{-1})^*)$$

for $f, g \in L^1(G, \mathfrak{A})$ and $t \in G$. The (full) crossed product of the C^*-dynamical system $(\mathfrak{A}, G, \alpha)$ (denoted by $\mathfrak{A} \rtimes_\alpha G$) is defined to be the norm closure of $\Phi(L^1(G, \mathfrak{A}))$ in $\mathbb{B}(H_\Phi)$, where Φ is the universal representation of $L^1(G, \mathfrak{A})$ defined by the direct sum representation of all non-degenerate representations of $L^1(G, \mathfrak{A})$, and H_Φ is its representation space and $\mathbb{B}(H_\Phi)$ is the C^*-algebra of all bounded operators on the Hilbert space H_Φ. Note that if $\mathfrak{A} = \mathbb{C}$, then $\mathbb{C} \rtimes_\alpha G \cong C^*(G)$ with α trivial. Let $H \rtimes_\alpha G$ be a semi-direct product of locally compact groups H, G for an action α of G on H. Then the group C^*-algabra $C^*(H \rtimes_\alpha G)$ of $H \rtimes_\alpha G$ is isomorphic to the crossed product $C^*(H) \rtimes_\alpha G$ with α on $C^*(H)$ induced from α on H.

The regular representaion Λ of $L^1(G, \mathfrak{A})$ (or $\mathfrak{A} \rtimes_\alpha G$) is defined by

$$(\Lambda(f)\xi)(t) = \int_G (\Phi(f(x))\lambda_x\xi)(t)dx = \int_G \Phi(\alpha_{t^{-1}}(f(x)))\xi(x^{-1}t)dx$$

for $t \in G$, $f \in L^1(G, \mathfrak{A})$, $\xi \in L^2(G, H_\Phi)$ (the Hilbert space of all square integrable H_Φ-valued measurable functions on G), where (Φ, λ) is the covariant representation defined by

$$(\Phi(a)\xi)(t) = \Phi(\alpha_{t^{-1}}(a))\xi(t), \quad (\lambda_x\xi)(t) = \xi(x^{-1}t)$$

for $a \in \mathfrak{A}$. Then the reduced crossed product of $L^1(G, \mathfrak{A})$ (or $\mathfrak{A} \rtimes_\alpha G$) (denoted by $\mathfrak{A} \rtimes_{\alpha,r} G$) is defined to be the norm closure of $\Lambda(L^1(G, \mathfrak{A}))$ in $\mathbb{B}(L^2(G, H_\Phi))$, or $\Lambda(\mathfrak{A} \rtimes_\alpha G)$. Note that if $\mathfrak{A} = \mathbb{C}$, then $\mathbb{C} \rtimes_{\alpha,r} G \cong C_r^*(G)$. If G is amenable, then $\mathfrak{A} \rtimes_{\alpha,r} G \cong \mathfrak{A} \rtimes_\alpha G$ for any C^*-algebra $\mathfrak{A}$.

Solvable Lie groups.

Recall that a simple connected solvable Lie group G can be written as a successive semi-direct by $\mathbb{R}$, that is,

$$G \cong \mathbb{R} \rtimes \mathbb{R} \rtimes \cdots \rtimes \mathbb{R}.$$

Thus, we have

$$C^*(G) \cong C^*(\mathbb{R}) \rtimes \mathbb{R} \rtimes \cdots \rtimes \mathbb{R},$$

where the right hand side is a successive crossed product by $\mathbb{R}$.

Now recall that a Lie group G is of type R (or rigid) if all the eigenvalues of the adjoint representation of G have absolute value one, and equivalently, if all the eigenvalues of the adjoint representation of the Lie algebra of G are purely imaginary or zero. Any connected solvable Lie group of CCR is of type R, and type I Lie groups of type R are CCR (see [1, Chapter V]). See Section 3 below for more details.

We also recall the following:

Table 1: Classes and examples of solvable Lie groups

Classes	Examples
Type R and type I	Commutative $\mathbb{R}^n$, $\mathbb{T}^s$, $\mathbb{R}^n \times \mathbb{T}^s$ Heisenberg Lie group H_3 Nilpotent Lie groups
Type R and non type I	Mautner group M_5 Dixmier group D_7
Non type R and type I	$ax + b$ group A_2 $A_2 \times G$ with G type I
Non type R and non type I	$A_2 \times M_5$, $A_2 \times D_7$, $A_2 \times G$ with G non type I

where H_3 is a simply connected nilpotent Lie group defined by

$$H_3 = \{(c, b, a) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R}\},$$

and $H_3 \cong \mathbb{R}^2 \rtimes \mathbb{R}$, and A_2 is a simply connected solvable Lie group of type I defined by

$$A_2 = \{(b, a) = \begin{pmatrix} e^a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{R}\},$$

and $A_2 \cong \mathbb{R} \rtimes \mathbb{R}$, and M_5 is a simply connected solvable Lie group of non type I defined by

$$M_5 = \{(z, w, t) = \begin{pmatrix} e^{it} & 0 & z \\ 0 & e^{i\theta t} & w \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \mathbb{R}, z, w \in \mathbb{C}\}$$

for θ an irrational number, and $M_5 \cong \mathbb{R}^4 \rtimes \mathbb{R}$, and D_7 is a simply connected solvable Lie group of non type I defined by $\mathbb{C}^2 \rtimes_\beta H_3$ the semi-direct product by an action β of H_3 on $\mathbb{C}^2$ defined by $\beta_g(z, w) = (e^{2\pi i a}z, e^{2\pi i b}w)$ for $z, w \in \mathbb{C}$ and $g = (c, b, a) \in H_3$, and $D_7 \cong \mathbb{R}^4 \rtimes \mathbb{R}^2 \rtimes \mathbb{R}$. Note that compact groups and CCR groups are type R and type I, but not necessarily solvable since (connected) semi-simple Lie groups are CCR.

1 Lie group C^*-algebras as inductive limits

Recall that a C^*-algebra is ASH if it is isomorphic to an inductive limit of subhomogeneous C^*-algebras. A subhomogeneous C^*-algebra is a C^*-subalgebra of a homogeneous C^*-algebra. A C^*-algebra is homogeneous if its spectrum consists of equivalence classes of finite dimensional irreducible representations with their dimension fixed.

Example 1.1 The C^*-algebra $C_0(X, M_n(\mathbb{C})) \cong C_0(X) \otimes M_n(\mathbb{C})$ of continuous matrix algebra $M_n(\mathbb{C})$-valued functions on a locally compact Hausdorff space X vanishing at infinity is homogeneous (or n-homogeneous). An n-homogeneous C^*-algebra can be written as the C^*-algebra of a continuous field on its spectrum X with fibers $M_n(\mathbb{C})$, but it is not necessarily isomorphic to $C_0(X, M_n(\mathbb{C}))$. Also, a n-subhomogeneous C^*-algebra can be written as the C^*-algebra of a continuous field on a locally compact Hausdorff space with fibers given by C^*-subalgebras of $M_n(\mathbb{C})$, and its spectrum is not necessarily Hausforff. For example, let $\mathfrak{B}$ be the C^*-algebra of a continuous field on the interval $[0, 1]$ with fibers given by $\mathfrak{A}_0 = \mathbb{C}^2$ in $M_2(\mathbb{C})$ and $\mathfrak{A}_t = M_2(\mathbb{C})$ for $0 < t \leq 1$. Then $\mathfrak{B}$ is subhomogeneous but two points corresponding to $\mathfrak{A}_0$ in its spectrum are not separated.

Our conjecture for the inductive limit structure of solvable Lie group C^*-algebras is:

Conjecture *We have the following dichotomy for solvable Lie group C^*-algebras*:

Table 2: Dichotomy for solvable Lie group C^*-algebras

Classes	As inductive limits
Type R solvable Lie group C^*-algebras	ASH
Non type R solvable Lie group C^*-algebras	Non ASH

Actually we have first obtained the following:

Theorem 1.2 [20] *Let G be a simply connected solvable Lie group of non type R. Then the unitization $C^*(G)^+$ is not ASH.*

Sketch of Proof. Suppose that $C^*(G)^+$ is ASH. Use the property that ASH algebras are closed under taking quotients. For the contradiction, we use Theorem 1.4 below and that G is of non type R (Remark below). $\square$

Remark. Note that a non type R simply connected solvable Lie group always has a quotient isomorphic to one of the following:

$$A_2\,(ax+b \text{ group}), \quad A_3 = \mathbb{R}^2 \rtimes_{\alpha^c} \mathbb{R}, \quad A_4 = \mathbb{R}^2 \rtimes_\beta \mathbb{R}^2,$$

where the actions α^c and β are defined by

$$\alpha_t^c = e^{ct}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad \beta_{(s,t)} = e^s \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

for $c \in \mathbb{R} \setminus \{0\}$, $t, s \in \mathbb{R}$ (see [1, Proposition 2.2 at p. 172]). It is shown that the unitizations $C^*(A_j)^+$ of the group C^*-algebras $C^*(A_j)$ of A_j $(2 \leq j \leq 4)$ have quotients that contain Fredholm operators with index nonzero.

Example 1.3 The semi-direct products $G = \mathbb{R}^n \rtimes_\alpha \mathbb{R}$ with certain hyperbolic orbits on $\mathbb{R}^n$ by actions α of $\mathbb{R}$ such as $\alpha_t(x_j) = (e^t x_j)$ for $t \in \mathbb{R}$ and $(x_j) \in \mathbb{R}^n$ are solvable Lie groups of non type R. One hyperbolic orbit corresponds to a quotient of G (or $C^*(G)$) that is isomorphic to the real $ax+b$ group A_2 (or $C^*(A_2)$).

Morevoer, we have shown that

Theorem 1.4 [20] *Let $\mathfrak{A}$ be a C^*-algebra in $\mathbb{B}(H)$ the C^*-algebra of bounded operators on a Hilbert space H. If $\mathfrak{A}$ contains a Fredholm operator on H with index nonzero, then $\mathfrak{A}$ is not ASH.*

On the other hand, we have

Theorem 1.5 [20] *Let $\mathfrak{A}$ be a liminal C^*-algebra in $\mathbb{B}(H)$. Then $\mathfrak{A}$ does not contain any Fredholm operator on H with index nonzero.*

Corollary 1.6 [20] *Let G be a connected nilpotent Lie group or a connected real semi-simple Lie group. Then $C^*(G)$ and its unitization do not contain any Fredholm operator on a Hilbert space with index nonzero.*

Remark. Note that connected nilpotent Lie groups and connected real semi-simple Lie groups are CCR.

We first showed in [24] that

Theorem 1.7 *Let H_{2n+1} be the real $(2n+1)$-dimensional generalized Heisenberg Lie group defined by*

$$H_{2n+1} = \{g = \begin{pmatrix} 1 & a & c \\ 0_n^t & 1_n & b^t \\ 0 & 0_n & 0 \end{pmatrix} \mid a, b \in \mathbb{R}^n, c \in \mathbb{R}\},$$

where 1_n is the $n \times n$ identity matrix, $0_n = 0 \in \mathbb{R}^n$, and b^t, 0_n^t are transposes of b, 0_n respectively. Then the group C^*-algebra $C^*(H_{2n+1})$ of H_{2n+1} is ASH.

Sketch of Proof. Since $H_{2n+1} \cong \mathbb{R}^{n+1} \rtimes \mathbb{R}^n$ (semi-direct product) via the identification $g = (c, b, a)$, we have $C^*(H_{2n+1}) \cong C_0(\mathbb{R}^{n+1}) \rtimes \mathbb{R}^n$. Furthermore, we can view this crossed product C^*-algebra as $\Gamma_0(\mathbb{R}, \{\mathfrak{A}_t\}_{t\in\mathbb{R}})$ the C^*-algebra of a continuous field on $\mathbb{R}$ with fibers $\mathfrak{A}_t$ given by $\mathfrak{A}_0 = C_0(\mathbb{R}^{2n})$ and $\mathfrak{A}_t = \mathbb{K}$ for $t \neq 0$. Then we define subhomogeneous C^*-subalgebras $\mathfrak{B}_n$ of $\Gamma_0(\mathbb{R}, \{\mathfrak{A}_t\}_{t\in\mathbb{R}})$ by continuous operator fields taking values in $M_n(\mathbb{C}) \subset \mathbb{K} = \mathfrak{A}_t$ for $t \neq 0$. □

Furthermore, recall that an AL (or ACCR) algebra is defined to be an inductive limit of liminal (or CCR) C^*-algebras. Then

Theorem 1.8 [24] *Let M_{2n+1} be the real $(2n+1)$-dimensional generalized Mautner group defined by*

$$M_{2n+1} = \{g = \begin{pmatrix} e^{-2\pi i\theta_1 t} & & 0 & z_1 \\ & \ddots & & \vdots \\ & & e^{-2\pi i\theta_n t} & z_n \\ 0 & & 0 & 1 \end{pmatrix} \mid z_1, \cdots, z_n \in \mathbb{C}, t \in \mathbb{R}\},$$

where $\theta_1, \cdots, \theta_n$ are rationally independent irrational numbers. Then the group C^*-algebra $C^*(M_{2n+1})$ of M_{2n+1} is AL.

Sketch of Proof. Since we have $M_{2n+1} \cong \mathbb{C}^n \rtimes \mathbb{R}$ via the identification $g = (z_1, \cdots, z_n, t)$, we obtain $C^*(M_{2n+1}) \cong C_0(\mathbb{C}^n) \rtimes \mathbb{R}$. Furthermore, we can view this crossed product C^*-algebra as $\Gamma_0(X_n, \{\mathfrak{A}_t\}_{t\in X_n})$ the C^*-algebra of a continuous field on $X_n = \{0\} \cup (\cup_{k=1}^n (\sqcup^{{}_nC_k}(\mathbb{R}_+)^k)$ (${}_nC_k$ the combination from n to k, and $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$) with fibers $\mathfrak{A}_t$ given by $\mathfrak{A}_0 = C_0(\mathbb{R})$, and $\mathfrak{A}_t = C(\mathbb{T}) \otimes \mathbb{K}$ for $t \in \sqcup^n \mathbb{R}_+$, and $\mathfrak{A}_t = (C(\mathbb{T}^{k-1}) \rtimes_\Theta \mathbb{Z}) \otimes \mathbb{K}$ for $t \in \sqcup^{{}_nC_k}(\mathbb{R}_+)^k$ $(2 \leq k \leq n)$, where $C(\mathbb{T}^{k-1}) \rtimes_\Theta \mathbb{Z}$ is a simple noncommutative torus associated with the multi-rotation action Θ on $(k-1)$ torus $\mathbb{T}^{k-1}$ by the multi-angle $2\pi\Theta = (2\pi\theta_{j_s})_{s=1}^{k-1}$ $(1 \leq j_1 < \cdots < j_{k-1} \leq n)$. For more details for this decomposition into finite extensions, see [17]. It is shown by [5] that the crossed product $C(\mathbb{T}^{k-1}) \rtimes_\Theta \mathbb{Z}$ is an AT-algebra, that is, an inductive limit of finite direct sums of matrix algebras over $C(\mathbb{T})$. Hence, $\mathfrak{A}_{t_k}$ for $t_k \in \sqcup^{nC_k}(\mathbb{R}_+)^k$ $(2 \leq k \leq n)$ is an AH-algebra, that is, $\mathfrak{A}_{t_k} = \varinjlim \mathfrak{A}_{j,t_k}$ for $\mathfrak{A}_{j,t_k}$ homogeneous C^*-algebras. Then we define liminal C^*-subalgebras $\mathfrak{B}_j$ of $\Gamma_0(X_n, \{\mathfrak{A}_t\}_{t\in X_t})$ by continuous operator fields taking values in $\mathfrak{A}_{j,t_k}$ for $t \in \sqcup^{nC_k}(\mathbb{R}_+)^k$ $(2 \leq k \leq n)$. □

Moreover,

Theorem 1.9 [24] *Let $E_n = \mathbb{R}^n \rtimes SO(n)$ $(n \geq 2)$ be the motion groups. Then the group C^*-algebra $C^*(E_n)$ of E_n is ASH.*

Furthermore, Let $E_n^\sim = \mathbb{R}^n \rtimes \mathrm{Spin}(n)$ $(n \geq 2)$ be the universal covering group of E_n. Then the group C^*-algebra $C^*(E_n^\sim)$ of $E_n^\sim$ is ASH.

Sketch of Proof. Since $E_n = \mathbb{R}^n \rtimes SO(n)$, we have $C^*(E_n) \cong C_0(\mathbb{R}^n) \rtimes SO(n)$, and this crossed product is isomorphic to $\Gamma_0(\{0\}\cup\mathbb{R}_+, \{\mathfrak{A}_t\}_{t\in\{0\}\cup\mathbb{R}_+})$ the C^*-algebra of a continuous field on $\{0\} \cup \mathbb{R}_+$ with fibers $\mathfrak{A}_t$ given by $\mathfrak{A}_0 = C^*(SO(n))$ and $\mathfrak{A}_t = C(S^{n-1}) \rtimes SO(n)$ for $t \in \mathbb{R}_+$, where S^{n-1} is the $(n-1)$-dimensional sphere. Furthermore, we have $C^*(SO(2)) = C^*(\mathbb{T}) \cong C_0(\mathbb{Z})$, and $C^*(SO(n)) \cong \oplus_{SO(n)^\wedge} M_{n_j}(\mathbb{C})$ for certain integers $n_j \geq 1$ $(n \geq 3)$. In addition, the imprimitivity theorem implies

$$\begin{aligned} C(S^{n-1}) \rtimes SO(n) &\cong C(SO(n)/SO(n-1)) \rtimes SO(n) \\ &\cong C^*(SO(n-1)) \otimes \mathbb{K}(L^2(S^{n-1})), \end{aligned}$$

where $K(L^2(S^{n-1}))$ is the C^*-algebra of compact operators on the Hilbert space $L^2(S^{n-1})$. Therefore, note that all the fibers $\mathfrak{A}_t$ are AF algebras. Since $\mathfrak{A}_t \cong \varinjlim \mathfrak{A}_{j,t}$ for $\mathfrak{A}_{j,t}$ certain finite dimensional C^*-algebras $(t \in \mathbb{R}_+)$, we define subhomogeneous C^*-subalgebras of $\Gamma_0(\{0\} \cup \mathbb{R}_+, \{\mathfrak{A}_t\}_{t\in\{0\}\cup\mathbb{R}_+})$ by continuous operator fields taking values in $\mathfrak{A}_{j,t}$ for $t \in \mathbb{R}_+$.

Similarly, we can follow the same argument for $C^*(E_n^\sim)$. In particular, when $n = 2$ the fibers $\mathfrak{A}_t$ are given by $\mathfrak{A}_0 = C^*(\mathbb{R}) \cong C_0(\mathbb{R})$ and

$$\mathfrak{A}_t = C(\mathbb{T}) \rtimes \mathbb{R} \cong C(\mathbb{R}/\mathbb{Z}) \rtimes \mathbb{R} \cong C^*(\mathbb{Z}) \otimes \mathbb{K}(L^2(\mathbb{T}))$$

for $t \in \mathbb{R}_+$, and when $n \geq 3$ we have $\mathfrak{A}_0 = C^*(\mathrm{Spin}(n))$ and $\mathfrak{A}_t = C(S^{n-1}) \rtimes \mathrm{Spin}(n)$ for $t \in \mathbb{R}_+$. □

More generally, we have obtained

Theorem 1.10 [25] *Let $\mathfrak{A}$ be a CCR C^*-algebra. Then $\mathfrak{A}$ is ASH.*

Remark. For the proof we use the structure theorem for type I C^*-algebras that says that type I C^*-algebras have composition series of closed ideals such that subquotients are of continuous trace (see [4] or [11]), and consider their interpretation as successive extensions by pull back C^*-algebras in the case of CCR C^*-algebras. For more details, see [25].

As our intentional application,

Corollary 1.11 [25] *Let G be a CCR locally compact group. Then $C^*(G)$ is ASH.*

In particular, an important application is:

Corollary 1.12 [25] *Let G be either a connected nilpotent Lie group, a solvable Lie group of type R and type I, or a connected semi-simple Lie group. Then $C^*(G)$ is ASH.*

Example 1.13 It follows from the corollary above that the full group C^*-algebra $C^*(SL_n(\mathbb{R}))$ and reduced group C^*-algebra $C_r^*(SL_n(\mathbb{R}))$ of $SL_n(\mathbb{R})$ $(n \geq 2)$ are ASH since they are CCR.

Furthermore,

Corollary 1.14 [25] *Let $\mathfrak{A}$ be an AL C^*-algebra. Then $\mathfrak{A}$ is an inductive limit of ASH algebras.*

Remark. It is not known whether or not the class of ASH algebras is closed under taking inductive limits. If the class is closed, then we have AL $=$ ASH. It is known that the class of AH algebras is not closed under taking inductive limits.

(Half wrong) conjecture *We have the following dichotomy:*

Table 3: Extended dichotomy for Lie group C^*-algebras

Classes	As inductive limits
Type R Lie group C^*-algebras	ASH
Non type R Lie group C^*-algebras	Non ASH

Remark. Theorems 1.7 and 1.9 support this conjecture. Note also that the group C^*-algebras of compact groups are ASH, which follows from Theorem 1.9 since they are CCR. In fact, the group C^*-algebra of a compact group can be written as a c_0-direct sum of matrix algebras over $\mathbb{C}$, which is also an inductive limit of finite direct sums of matrix algebras over $\mathbb{C}$. Thus, the first half of this conjecture should be true, but the latter half is wrong since $SL_n(\mathbb{R})$ $(n \geq 2)$ are of non type R (see Section 3) but their full and reduced group C^*-algebras are ASH as given above.

Beyond group C^*-algebras, we give

Definition 1.15 Let $\mathfrak{A}$ be a C^*-algebra. We say that $\mathfrak{A}$ is of type R if it is ASH.

Trivially, we have the following:

Table 4: Dichotomy for C^*-algebras

Classes	As inductive limits
Type R C^*-algebras	ASH
Non type R C^*-algebras	Non ASH

Remark. This definition seems to be quite suitable in the sense that it is shown in [25] that

- CCR C^*-algebras are ASH

and shown in [20] that

- ASH algebras are CCR if and only if they are of type I

and these results just correspond to the results by [1]:

- simply connected solvable Lie groups of CCR are of type R, and
- simply connected solvable Lie groups of type R are CCR if and only if they are of type I (see Section 3 below).

Remark. It should be an interesting question to find another notion for C^*-algebras to be of type R as that for type R Lie groups. Hopefully, the class of type R C^*-algebras is the same as that of ASH C^*-algebras. Also, the class of type R C^*-algebras should contain type R Lie group C^*-algebras.

Remark. Furthermore, It is known that C^*-algebras of the classes ASH or AL are all quasidiagonal. A (separable) C^*-algebra $\mathfrak{A}$ is quasidiagonal if it has a faithful representation π to $\mathbb{B}(H)$ the C^*-algebra of bounded operators on a Hilbert space H such that there exists an increasing sequence of finite rank projections p_n such that $\lim_{n\to\infty} \|ap_n - p_n a\| = 0$ for $a \in \pi(\mathfrak{A})$ and $p_n \to \mathrm{id}_H$ the identity operator on H as $n \to \infty$ in the strong operator topology. The quasidiagonal might be the right concept for C^*-algebras to be of type R. However, the class of quasidiagonal C^*-algebras are not closed under taking quotients while the class of ASH algebras are closed.

2 As extensions by generalized ASH algebras

Recall from [26] that a C^*-algebra $\mathfrak{A}$ is a generalized ASH algabra if it contains subhomogeneous C^*-subalgabras whose union is dense in $\mathfrak{A}$. This notion seems to be different from another notion for C^*-algebras to be locally homogeneous (cf. locally subhomogeneous (LSH) below later) but certainly close and might be the same.

Theorem 2.1 [26] *Let $H_{2n+1}^{\mathbb{Z}}$ be the generalized discrete Heisenberg group of rank $2n+1$ defined by*

$$H_{2n+1}^{\mathbb{Z}} = \{g = \begin{pmatrix} 1 & a & c \\ 0_n^t & 1_n & b^t \\ 0 & 0_n & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}^n, c \in \mathbb{Z}\}.$$

Then the group C^*-algebra $C^*(H_{2n+1}^{\mathbb{Z}})$ of $H_{2n+1}^{\mathbb{Z}}$ is an extension of $C(\mathbb{T}^{2n})$ by a generalized ASH algebra.

Part of Proof. Since $H_{2n+1}^{\mathbb{Z}} \cong \mathbb{Z}^{n+1} \rtimes \mathbb{Z}^n$ via the identification: $g = (c, b, a)$, we have

$$C^*(H_{2n+1}^{\mathbb{Z}}) \cong C^*(\mathbb{Z}^{n+1}) \rtimes \mathbb{Z}^n \cong C(\mathbb{T}^{n+1}) \rtimes \mathbb{Z}^n.$$

This crossed product is isomorphic to $\Gamma(\mathbb{T}, \{\mathfrak{A}_t\}_{t\in\mathbb{T}})$ the C^*-algebra of a continuous field on $\mathbb{T}$ with fibers $\mathfrak{A}_t$ given by the crossed products $C(\mathbb{T}^n) \rtimes_t \mathbb{Z}^n \cong \otimes^n C(\mathbb{T}) \rtimes_t \mathbb{Z}$, where $C(\mathbb{T}) \rtimes_t \mathbb{Z}$ is the rotation algebra correspoinding to $t \in \mathbb{T}$. Furthermore, we need to use the following exact sequence:

$$0 \to \Gamma_0(\mathbb{T} \setminus \{1\}, \{\mathfrak{A}_t\}_{t\in\mathbb{T}\setminus\{1\}}) \to \Gamma(\mathbb{T}, \{\mathfrak{A}_t\}_{t\in\mathbb{T}}) \to \mathfrak{A}_0 \to 0$$

and $\mathfrak{A}_0 \cong C(\mathbb{T}^{2n})$. Then show that the ideal above is generalized ASH. □

Remark. It might be possible to have $C^*(H_{2n+1}^{\mathbb{Z}})$ as an extension of $C(\mathbb{T}^{2n})$ by an ASH algebra. However, it seems to be hopeless to have $C^*(H_{2n+1}^{\mathbb{Z}})$ as an ASH algebra since the Rieffel projections involved in this inductive limit structure are not continuous as continuous operator fields on the torus. Therefore, as the inductive limit structure for (discrete) group C^*-algebras it would be necessary to consider the decompositions into such (generalized) ASH extensions.

As applications,

Theorem 2.2 [26] *Let D_{6n+1} be the real $(6n+1)$-dimensional generalized Diximier group defined by the semi-direct product $\mathbb{C}^{2n} \rtimes_\alpha H_{2n+1}$ with an action α such that $\alpha_g(z,w) = ((e^{ia_j}z_j),(e^{ib_j}w_j))$ for $z=(z_j), w=(w_j) \in \mathbb{C}^n$ and $g=(c,b,a) \in H_{2n+1}$. Then the group C^*-algebra $C^*(D_{6n+1})$ of D_{6n+1} has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^L$ such that*

$$\mathfrak{I}_L/\mathfrak{I}_{L-1} \cong C_0(\mathbb{R}^{2n}), \quad \textit{and}$$
$$\mathfrak{I}_j/\mathfrak{I}_{j-1}\ (1 \le j \le L-1) \textit{ are generalized ASH.}$$

Remark. See [23] for the finite composition series of $C^*(D_{6n+1})$ and its generalizations. Indeed, the suquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ $(1 \le j \le L-1)$ are given by either

$$C_0(X_j) \otimes \mathbb{K} \quad \text{or,}$$
$$C_0(X_j) \otimes \mathbb{K} \otimes \Gamma_0(\mathbb{R}\setminus\{0\}, \{(\otimes^{s_j}(C(\mathbb{T}) \rtimes_t \mathbb{T})) \otimes C(\mathbb{T}^{t_j}) \otimes \mathbb{K}\}_{t\in\mathbb{R}\setminus\{0\}})$$

for X_j the product spaces of some copies of $\mathbb{R}$ and $\mathbb{T}$, and $s_j \ge 1$ and $t_j \ge 0$, where $C(\mathbb{T}) \rtimes_t \mathbb{T}$ means the rotation algebra corresponding to $t \in \mathbb{R}\setminus\{0\} \approx \mathbb{T}\setminus\{1\}$ (homeomorphic identification).

Theorem 2.3 [26] *Let D_{4n}^d be the real $4n$-dimensional generalized disconnected Diximier group defined by the semi-direct product $\mathbb{C}^{2n} \rtimes_\alpha H_{2n+1}^{\mathbb{Z}}$ with an action α such that $\alpha_g(z,w) = ((e^{ia_j}z_j),(e^{ib_j}w_j))$ for $z=(z_j), w=(w_j) \in \mathbb{C}^n$ and $g=(c,b,a) \in H_{2n+1}^{\mathbb{Z}}$. Then the group C^*-algebra $C^*(D_{4n}^d)$ of D_{4n}^d has a finite composition series $\{\mathfrak{D}_j\}_{j=1}^K$ such that*

$$\mathfrak{D}_K/\mathfrak{D}_{K-1} \cong C(\mathbb{T}^{2n}), \quad \textit{and}$$
$$\mathfrak{D}_j/\mathfrak{D}_{j-1}\ (1 \le j \le K-1) \textit{ are generalized ASH.}$$

Remark. See [18] for the finite composition series of $C^*(D_{4n}^d)$ and its generalizations. Indeed, the subquotients of $C^*(D_{4n}^d)$ are given by continuous field C^*-algebras on the torus $\mathbb{T}$ with fibers involving tensor products of simple noncommutative 2, 3, or 4 tori so that the structure of $C^*(D_{4n}^d)$ are much more complicated than that of $C^*(D_{6n+1})$.

Conjecture *Any group C^*-algebra of a Lie or discrete group has a finite composition series such that its subquotients are either AH, ASH or generalized ASH.*

Remark. It might be possible to remove "ASH or generalized ASH" in this conjecture. Indeed, see the next.

Example 2.4 We have the following:

Table 5: Group C^*-algebras as (successive) extensions

Classes	Examples
ASH algebras	$C^*(H_{2n+1})$, $C^*(E_n)$
Extensions by AH algebras Successive extensions by AH algebras	$C^*(H_{2n+1})$, $C^*(E_n)$, $C^*(A_2)$ $C^*(\mathbb{R}^n \rtimes \mathbb{R})$, $C^*(M_{2n+1})$,
Extensions by g-ASH algebras Successive extensions by g-ASH algebras	$C^*(H_{2n+1}^{\mathbb{Z}})$ $C^*(D_{6n+1})$, $C^*(D_{4n}^d)$

where "g-ASH algebras" mean generalized ASH algebras. It could be the right way to view group C^*-algebras as (successive) extensions. Indeed, we have also obtained that the group C^*-algebras of the generalized connected or disconnected Mautner groups (that are the semi-direct products $\mathbb{C}^n \rtimes \mathbb{R}^m$, $\mathbb{C}^n \rtimes \mathbb{Z}^m$ with certain actions by multi-rotations) and the disconnected semi-direct products $\mathbb{C}^n \rtimes \mathbb{Z}$ are decomposed into successitve extenstions by AH-algebras (see [19], [21] and [22]).

3 Lie groups of type R: A review

Recall that a Lie algebra $\mathfrak{G}$ over the field $\mathbb{R}$ or $\mathbb{C}$ is of type R if for any $X \in \mathfrak{G}$, the eigenvalues of the adjoint operator $\mathrm{ad}(X)$ on $\mathfrak{G}$ are all purely imaginary or zero. A Lie group is of type R if its Lie algebra is of type R. A connected Lie group is of type R if for any $g \in G$, the absolute values of all eigenvalues of the adjoint operator $\mathrm{Ad}(g)$ on G are 1.

Example 3.1 [10] All nilpotent Lie groups are of type R.

Let N be a connected nilpotent Lie group and K a compact subgroup of the group $\mathrm{Aut}N$ of automorphisms of N. Then the semi-direct product $N \rtimes K$ is a Lie group of type R. In particular, the semi-direct product $N \rtimes \mathbb{T}^k$ is a solvable Lie group of type R. Furthermore, the motion groups $E_n = \mathbb{R}^n \rtimes SO(n)$ are of type R.

Remark. The class of Lie groups of type R is stable under taking Lie subgroups and quotient groups.

Theorem 3.2 [10] (Auslander, Green and Hahn) *Let G be a connected Lie group, and $G = RS$ its Levi decomposition by the radial S (solvable) and a Levi subgroup S (semi-simple). Then G is of type R if and only if R is of type R and S is compact.*

Remark. For the motion groups E_n, its radical of type R is $\mathbb{R}^n$, and its compact Levi subgroup is given by $SO(n)$. On the other hand, it follows that noncompact connected semi-simple Lie groups such as $SL_n(\mathbb{R})$ $(n \geq 2)$ are of non type R. Moreover, noncompact connected reductive Lie groups such as $GL_n(\mathbb{R})_0$ the connected component of $GL_n(\mathbb{R})$ $(n \geq 2)$ containing the unit are of non type R.

Moreover,

Theorem 3.3 [10] (Auslander, Green and Hahn) *Let G be a simply connected solvable Lie group of type R. Then G is isomorphic to a virtual Lie subgroup of the semi-direct product $N \rtimes \mathbb{T}^k$ for a simply connected nilpotent Lie group N.*

Corollary 3.4 [10] *Let G be a simply connected Lie group of type R. Then G is isomorphic to a virtual Lie subgroup of the semi-direct product $N \rtimes K$ for a simply connected nilpotent Lie group N and a compact subgroup K.*

Remark. Recall from [9] that a virtual Lie subgroup H of a Lie group G is a subgroup endowed with a Lie group structure ([9, Page 33]). Then the embedding from H to G is a Lie group homomorphism. On the other hand, a subgroup H of a Lie group G is a Lie subgroup if H is a submanifold of the manifold G. Thus, any Lie subgroup is virtual.

Example 3.5 [9] Let f be a Lie group homomorphism from $\mathbb{R}$ to $\mathbb{T}^n$ defined by $f(t) = (e^{i\theta_j t})_{j=1}^n \in \mathbb{T}^n$ for $\theta_j \in \mathbb{R}$ $(1 \leq j \leq n)$. Then the image $f(\mathbb{R})$ is a Lie subgroup of $\mathbb{T}^n$ if and only if θ_j are rationally dependent. Fortunately, $f(\mathbb{R})$ is always a virtual Lie subgroup of $\mathbb{T}^n$.

We now check

Example 3.6 The generalized Mautner groups M_{2n+1} defined in Section 2 are of type R. The definition of M_{2n+1} by the matrices given above implies that M_{2n+1} is a virtual Lie subgroup of the semi-direct product $\mathbb{C}^n \rtimes \mathbb{T}^n$ consisting of the following matrices:

$$\begin{pmatrix} w_1 & & 0 & z_1 \\ & \ddots & & \vdots \\ & & w_n & z_n \\ 0 & & 0 & 1 \end{pmatrix} \quad \text{for } w_j \in \mathbb{T}, z_j \in \mathbb{C} \ (1 \leq j \leq n).$$

We also check

Example 3.7 The generalized Dixmier groups $D_{6n+1} = \mathbb{C}^{2n} \rtimes_\alpha H_{2n+1}$ defined in Section 2 are of type R. Indeed, D_{6n+1} is a virtual Lie subgroup of the semi-direct product $(\mathbb{C}^{2n} \times H_{2n+1}) \rtimes \mathbb{T}^{2n}$ consisting of the following matrices:

$$\begin{pmatrix} g & & & & & & & 0^t_{n+2} \\ & a'_1 & & & & & & z_1 \\ & & \ddots & & & & & \vdots \\ & & & a'_n & & & & z_n \\ & & & & b'_1 & & & w_1 \\ & & & & & \ddots & & \vdots \\ & & & & & & b'_n & w_n \\ 0 & & & & & & & 1 \end{pmatrix}$$

for $z_j, w_j \in \mathbb{C}$ $(1 \le j \le n)$, $g = (c, b, a) \in H_{2n+1}$, and $a'_j = e^{ia_j}, b'_j = e^{ib_j} \in \mathbb{T}$ with $a = (a_j), b = (b_j)$.

Theorem 3.8 (L. Auslander and C.C. Moore) [1] *Let G be a simply connected solvable Lie group of CCR. Then G is of type R.*

Theorem 3.9 (L. Auslander and C.C. Moore) [1] *Let G be a simply connected solvable Lie group of type R. Then G is CCR if and only if G is of type I (or GCR).*

Furthermore, in fact we have

Theorem 3.10 [1] *Let G be a connected solvable Lie group of CCR. Then G is of type R.*

Sketch of Proof. Let $G^\sim$ be the universal covering group of G. Suppose that G is of non type R. Then $G^\sim$ is also of non type R, and there exists a normal subgroup N of $G^\sim$ such that the quotient group $G^\sim/N$ is isomorphic to either A_2, A_3 or A_4 (see Remark of Theorem 1.2). Let Γ be the discrete central subgroup of $G^\sim$ such that $G \cong G^\sim/\Gamma$.

Since A_2 and A_3 are centerless, if $G^\sim/N \cong A_2$ or A_3, then $\Gamma \subset N$ and $G/(N/\Gamma)$ is not CCR. Hence, G is not CCR.

If $G^\sim/N \cong A_4$, then the projection of Γ to A_4 is contained in the discrete center Z of A_4. It follows that $N\Gamma/\Gamma$ is closed in G and that $G/(N\Gamma/\Gamma)$ is a covering group of A_4/Z. Since A_4/Z is not CCR, neither is G. □

Moreover,

Theorem 3.11 [1] *Let G be a simply connected solvable Lie group. Then there exists a unique nilpotent Lie group N and a unique abelian semi-simple group S of automorphisms of N such that $G \subset NS$ and G, N generate NS, and for $p : NS \to N$ the projection map, the restriction of p to G is a homeomorphism of G onto N.*

Remark. If S is trivial on $N/[N, N]$, then S acts trivially on N. Let $H = NS$. Then $[H, H] \subset N$ and hence $[S, S] \subset N$.

We call NS the semi-simple splitting of G, and S is called the semi-simple part of G. Further, $G \cap N$ is called the unipotent part of G.

We say that G is semi-simply regular if its semi-simple part is regular.

Let N^* be the dual space of N as a vector space. Then S acts on N^*. For each $\varphi \in N^*$, let $F(\varphi)$ be the subgroup of S leaving φ invariant. Let $H(\varphi) = \psi^{-1}(F(\varphi))$, where $\psi : G \to S$. We say that S is unipotently regular if for each $\varphi \in N^*$, the unipotent part $U(H(\varphi))$ of $H(\varphi)$ is closed in N^*.

Theorem 3.12 [1] *Let G be a simply connected solvable Lie group of type R. If G is not semi-simply regular, then there exists a homomorphism from G onto a generalized Mautner group. All generalized Mautner groups are of non type I.*

Corollary 3.13 [1] *Let G be a simply connected solvable Lie group of type R. If G is of type I, then G is semi-simply regular.*

Example 3.14 An extended Mautner group M is defined to be a semi-direct product $\mathbb{R}^n \rtimes_\alpha \mathbb{R}^m$ for α an action of $\mathbb{R}^m$ on $\mathbb{R}^n$ by orthogonal matrices such that $\mathrm{Ad}(M) = \mathbb{R} \times \mathbb{T}^{m-1}$, and $\mathrm{Ad}(M/A)$ is compact for A any normal subgroup of M. Then M is of type R.

Theorem 3.15 [1] *Let G be a simply connected solvable Lie group of type R and N its nilradical. Then N is regularly embedded in G if and only if G is semi-simply and unipotently regular.*

Proposition 3.16 [1] *Let G be a simply connected solvable Lie group of type R that is semi-simply regular and N its nilradical. Then N is regularly embedded in G if and only if G is unipotently regular.*

Theorem 3.17 [1] *Let G be a simply connected solvable Lie group of type R and N its nilradical. If N is not regularly embedded in G, then G is of non type I.*

Remark. Let G be a locally compact group, N a closed normal subgroup of G of type I and λ a factor representation of G (under which the von Neumann algebra generated by G is a factor, i.e., its center is trivial). Then there exists a unique measure class $C(\mu)$ on the dual space $N^\wedge$ of N such that the restriction of λ to N can be written as a direct integral representation $\int_{N^\wedge} n(\pi)\pi d\mu(\pi)$ on the direct integral Hilbert space $H = \int_{N^\wedge} H_\pi d\mu(\pi)$, where $\pi \in N^\wedge$ and $n(\pi)$ is the multiplicity of π and H_π is the representation Hilbert space of π (in fact H is defined to be the space of (square integrable in norm) measurable functions f from the measure space $(N^\wedge, \mu)$ to the union of H_π such that $f(\pi) \in H_\pi$ for $\pi \in N^\wedge$). Morevoer, the class $C(\mu)$ is a quasi-orbit for the action of G on $N^\wedge$.

If the action of G on $N^\wedge$ is smooth, that is, all quasi-orbits are transitive, then we say that N is regularly embedded in G. In this case, one can compute the dual space $G^\wedge$ of G in terms of $N^\wedge$ and the representation theory of its little groups K_π/N, where $K_\pi = \{g \in G \,|\, g\cdot\pi = \pi\}$ (stabilizer). If N is of type I, then G is of type I on $C(\mu)$ if and only if its little group is of type I.

4 Classification of inductive limit C^*-algebras: A review

K-theory We now recall the K-theory for C^*-algebras briefly.

The K_0-group $K_0(\mathfrak{A})$ of a unital C^*-algebra $\mathfrak{A}$ is defined to be the Grothendieck group of the abelian semigroup of stable equivalence classes of projections of matrix algebras $M_n(\mathfrak{A})$ over $\mathfrak{A}$ $(n \geq 1)$, where for the the classes $[p]$, $[q]$ of projections p, q of $M_n(\mathfrak{A})$ (n large enough), we have $[p] = [q]$ if and only if the diagonal sums $p \oplus 1_m$, $q \oplus 1_m$ for certain $m \geq 0$ are equivalent in $M_n(\mathfrak{A})$ as projections, where 1_m is the $m \times m$ identity matrix. The standard picture is:

$$K_0(\mathfrak{A}) = \{[p] - [q] \,|\, p, q \in \cup_{n=1}^\infty M_n(\mathfrak{A}) : \text{projections}\}.$$

For a non-unital C^*-algebra $\mathfrak{A}$, we consider its unitization $\mathfrak{A}^+$ by $\mathbb{C}$. Then the K_0-group $K_0(\mathfrak{A})$ of $\mathfrak{A}$ is defined to be the kernel of the splitting onto homomorphism from $K_0(\mathfrak{A}^+)$ to $K_0(\mathbb{C})$. Namely,

$$0 \to K_0(\mathfrak{A}) \to K_0(\mathfrak{A}^+) \to K_0(\mathbb{C}) \to 0,$$

and $K_0(\mathfrak{A}^+) \cong K_0(\mathfrak{A}) \oplus K_0(\mathbb{C}) \cong K_0(\mathfrak{A}) \oplus \mathbb{Z}$.

The K_1-group $K_1(\mathfrak{A})$ of a C^*-algebra $\mathfrak{A}$ is defined to be the abelian group of homotopy equivalence classes of unitary (or invertible) elements of matrix algebras over $\mathfrak{A}$ (or its unitization). The standard picture is:

$$K_1(\mathfrak{A}) = \{[u] \mid u \in \cup_{n=1}^{\infty} U_n(\mathfrak{A})\} = \{[u] \mid u \in \cup_{n=1}^{\infty} GL_n(\mathfrak{A})\},$$

where $U_n(\mathfrak{A})$ is the group of $n \times n$ unitary matrices over $\mathfrak{A}$, and $GL_n(\mathfrak{A})$ is the group of $n \times n$ invertible matrices over $\mathfrak{A}$. Note that $[u][v] = [uv] = [u \oplus v]$ and $[u] = [u \oplus 1_m]$ for $[u], [v] \in K_1(\mathfrak{A})$ and $m \geq 1$. Furthermore,

$$K_1(\mathfrak{A}) \cong \varinjlim U_n(\mathfrak{A})/U_n(\mathfrak{A})_0 \cong \varinjlim GL_n(\mathfrak{A})/GL_n(\mathfrak{A})_0,$$

where $U_n(\mathfrak{A})_0, GL_n(\mathfrak{A})_0$ are the connected components of $U_n(\mathfrak{A}), GL_n(\mathfrak{A})$ containing the identity matrix respectively.

Finite dimensional C^*-algebras

A finite dimensional C^*-algebra is isomorphic to a finite direct sum of matrix algebras over $\mathbb{C}$.

Proposition 4.1 *Let $\mathfrak{A} = \oplus_{j=1}^{s} M_{n_j}(\mathbb{C})$, $\mathfrak{B} = \oplus_{j=1}^{t} M_{m_j}(\mathbb{C})$ be finite dimensional C^*-algebras. Then $\mathfrak{A} \cong \mathfrak{B}$ if and only if $s = t$ and $(n_j) = (m_j)$ up to permutations.*

In other words, $n_j = \mathrm{Tr}(1_j)$, $m_j = \mathrm{Tr}(1_j)$, which mean traces of the identity matrices 1_j of $M_{n_j}(\mathbb{C})$, $M_{m_j}(\mathbb{C})$ respectively.

K-theoretically, we have

$$\mathfrak{A} \cong \mathfrak{B} \Leftrightarrow (K_0(\mathfrak{A}), [1_{\mathfrak{A}}]) \cong (K_0(\mathfrak{B}), [1_{\mathfrak{B}}]),$$

where $[1_{\mathfrak{A}}]$, $[1_{\mathfrak{B}}]$ are the classes of the units $1_{\mathfrak{A}}$, $1_{\mathfrak{B}}$ of $\mathfrak{A}$, $\mathfrak{B}$ respectively, and

$$K_0(\mathfrak{A}) \cong \oplus_{j=1}^{s} \mathbb{Z}[p_j], \quad [1_{\mathfrak{A}}] = \oplus_{j=1}^{s} n_j [p_j]$$

where $[p_j]$ are the classes of rank 1 projections p_j of $M_{n_j}(\mathbb{C})$.

Remark. As for K_1-groups, we have

$$K_1(\oplus_{j=1}^{s} M_{n_j}(\mathbb{C})) \cong \oplus_{j=1}^{s} K_1(M_{n_j}(\mathbb{C})) \cong \oplus_{j=1}^{s} K_1(\mathbb{C}) \cong 0.$$

The following is called the uniqueness and existence theorem for homomorphisms from finite dimensional C^*-algebras:

Theorem 4.2 [6] *Let $\mathfrak{A}$ be a finite dimensional C^*-algebra and $\mathfrak{B}$ a C^*-algebra with stable rank one. Suppose that $\psi : K_0(\mathfrak{A}) \to K_0(\mathfrak{B})$ is a unital positive homomorphism. Then there exists a unital homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ such that $h_* = \psi$.*

If $h_1, h_2 : \mathfrak{A} \to \mathfrak{B}$ are unital homomorphisms, then $(h_1)_* = (h_2)_*$ if and only if $h_2 = \mathrm{Ad}(u) \circ h_1$ for some unitary $u \in \mathfrak{B}$.

Sketch of Proof. Write $\mathfrak{A} = \oplus_{j=1}^{k} M_{n_j}(\mathbb{C})$ and set $\mathfrak{A}_j = M_{n_j}(\mathbb{C})$. Let $\{e_{ij}^l\}$ be the canonical generators of $\mathfrak{A}_l$ and let $e_l = 1_{\mathfrak{A}_l}$. We use the fact that in a finite dimensional C^*-algebra, the stable equivalence and usual equivalence for projections are the same.

We have $\pi([e_l]) = [p_l]$ for some projection p_l in $\cup_{n=1}^{\infty} M_n(\mathfrak{B})$, since ψ is positive. We also have

$$[p_1 \oplus \cdots \oplus p_k] = \psi(\sum_{l=1}^{k}[e_l]) = \psi([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}].$$

Therefore, there exist mutually orthogonal projections $q_1, \cdots, q_k \in \mathfrak{B}$ such that $q_1 + \cdots + q_k = 1_{\mathfrak{B}}$ and $q_l \sim p_l$ for $1 \le l \le k$. Since $\psi([e_l]) = [q_l]$ and $q_l \mathfrak{B} q_l$ is a unital C^*-algebra with stable rank one, it suffices to find a homomorphism $h : \mathfrak{A}_l \to q_l \mathfrak{B} q_l$ such that $h_* = (\psi_*)|_{K_0(\mathfrak{A}_l)}$. Therefore, we may assume that $\mathfrak{A} = \mathfrak{A}_1 = M_n(\mathbb{C})$ and $\{e_{ij}\}$ is the system of matrix units of $\mathfrak{A}$.

Let p_{11} be a projection of $\mathfrak{B}$ such that $\psi([e_{11}]) = [p_{11}]$. So

$$n[p_{11}] = \psi(n[e_{11}]) = \psi([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}].$$

Hence, there are mutually equivalent and mutually orthogonal projections $p_{11}, p_{22}, \cdots, p_{nn}$ of $\mathfrak{B}$ such that $\sum_{i=1}^{n} p_{ii} = 1_{\mathfrak{B}}$. Therefore, there are partial isometries $p_{ij} \in \mathfrak{B}$ such that $p_{ij}^* p_{ij} = p_{jj}$ and $p_{ij} p_{ij}^* = p_{ii}$ for $1 \le i, j \le n$. Define a map $h : \mathfrak{A} \to \mathfrak{B}$ which maps e_{ij} to p_{ij}. It is routine to check that h is a homomorphism. Since $h(e_{11}) = p_{11}$ and $\psi([e_{11}]) = [p_{11}]$, $h_* = \psi$.

If u is a unitary of $\mathfrak{B}$, then $\mathrm{Ad}(u)_* = \mathrm{id}_{K_0(\mathfrak{B})}$, so $(\mathrm{Ad}(u) \circ h_1)_* = (h_1)_*$.

Conversely, suppose that $(h_1)_* = (h_2)_*$. Set $p_{ij}^l = h_1(e_{ij}^l)$ and $q_{ij}^l = h_2(e_{ij}^l)$ for $1 \le l \le k$ and $1 \le i, j \le n_l$. Then

$$[p_{ii}^l] = (h_1)_*([e_{ii}^l]) = (h_2)_*([e_{ii}^l]) = [q_{ii}^l].$$

Since $\mathfrak{B}$ has stable rank one, there exist partial isometries $v_l \in \mathfrak{B}$ such that $p_{11}^l = v_l^* v_l$ and $q_{11}^l = v_l v_l^*$. Set

$$u^* = \sum_{l=1}^{k} \sum_{i=1}^{n_l} q_{i1}^l v_l p_{1i}^l.$$

One checks directly that u is a unitary of $\mathfrak{B}$ and that $u p_{ij}^l = q_{ij}^l u$ for all i, j, l. Thus

$$h_2(e_{ij}^l) = u^* h_1(e_{ij}^l) u = \mathrm{Ad}(u) \circ h_1(e_{ij}^l)$$

for all i, j, l, so $h_2 = \mathrm{Ad}(u) \circ h_1$. □

UHF algebras

A UHF algebra (or uniformly hyper finite C^*-algebra) is an inductive limit of matrix algebras $\{M_{n_k}(\mathbb{C})\}_{k=1}^{\infty}$ for some n_k $(1 \leq k \leq \infty)$. Note that a unital $*$-homomorphism from $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$ exists if and only if n divides m, that is, $m = dn$ for some integer $d > 0$.

Theorem 4.3 *Let $\mathfrak{A} = \varinjlim M_{n_k}(\mathbb{C})$ be a UHF algebra for integers $\{n_k\}_{k=1}^{\infty}$ with each n_k dividing n_{k+1}. Then*

$$K_0(\varinjlim M_{n_k}(\mathbb{C})) \cong \cup_{j=1}^{\infty} \mathbb{Z}/n_k, \quad \textit{and } [1_{\mathfrak{A}}] = 1.$$

Moreover, let $\mathfrak{A}' = \varinjlim M_{n'_k}(\mathbb{C})$ be another UHF algebra. Then

$$\mathfrak{A} \cong \mathfrak{A}' \Leftrightarrow (K_0(\mathfrak{A}), [1_{\mathfrak{A}}]) \cong (K_0(\mathfrak{A}'), [1_{\mathfrak{A}'}]),$$

which is equivalent to $\{n_k\}_{k=1}^{\infty} = \{n'_k\}_{k=1}^{\infty}$.

Furthermore, for any subgroup G of $\mathbb{Q}$ (additive) containing 1 there exists a UHF algebra $\mathfrak{A}$ such that

$$(G, 1) \cong (K_0(\mathfrak{A}), [1_{\mathfrak{A}}]).$$

Sketch of Proof [13]. Let $\mathfrak{A}$ be a UHF algebra. If $g \in K_0(\mathfrak{A})$, then $g \in K_0(\mathfrak{A})_+$ (positive cone) (see below) if and only if there exist $k \in \mathbb{Z}_+$ and $l \in \mathbb{N}$ such that $lg = k[1_{\mathfrak{A}}]$. Indeed, $K_0(\mathfrak{A})$ is isomorphic to a subgroup of the abelian group $\mathbb{Q}$ of all rational numbers, and so $lg = k[1_{\mathfrak{A}}]$ for some l, k. Since $K_0(\mathfrak{A})$ is unperforated, we find that g is positive if $k \geq 0$, and that $-g$ is positive if $k \leq 0$.

It follows that if $\alpha : K_0(\mathfrak{A}) \to K_0(\mathfrak{A}')$ is an isomorphism satisfying $\alpha([1_{\mathfrak{A}}]) = [1_{\mathfrak{A}'}]$, then automatically $\alpha(K_0(\mathfrak{A})_+) = K_0(\mathfrak{A}')_+$.

For $\{n_j\}_{j=1}^{\infty}$, put

$$k_j = \Pi_{i=1}^{j} p_i^{\min\{j, n_i\}},$$

where $\{p_i\}$ is the set of all positive prime numbers with $p_i < p_{i+1}$. Then $k_j | k_{j+1}$ for any j. For each j there is a $*$-homomorphism $\varphi_j : M_{k_j}(\mathbb{C}) \to M_{k_{j+1}}(\mathbb{C})$. Let $\mathfrak{A}$ be the inductive limit of the sequence:

$$M_{k_1}(\mathbb{C}) \xrightarrow{\varphi_1} M_{k_2}(\mathbb{C}) \xrightarrow{\varphi_2} M_{k_3}(\mathbb{C}) \xrightarrow{\varphi_3} \cdots \longrightarrow \mathfrak{A}.$$

Then $(K_0(\mathfrak{A}), [1_{\mathfrak{A}}])$ is isomorphic to $(G, 1)$, where G is the subgroup of $\mathbb{Q}$ consisting of all fractions x/y for $x \in \mathbb{Z}$ and $y = \Pi_{j=1}^{\infty} p_j^{m_j}$ for some positive integers $m_j \leq n_j$, where $m_j > 0$ for finitely many j.

Indeed, let τ_j be the normalized trace on $M_{k_j}(\mathbb{C})$ that is given by $\tau_j(a) = k_j^{-1}\mathrm{Tr}(a)$ for $a \in M_{k_j}(\mathbb{C})$, where Tr is the standard trace. Then the map

$K_0(\tau_j)$ induced by τ_j on K_0 is an isomorphism from $K_0(M_{k_j}(\mathbb{C}))$ onto $k_j^{-1}\mathbb{Z}$. Since $\tau_{j+1} \circ \varphi_j = \tau_j$, we have $K_0(\tau_{j+1}) \circ K_0(\varphi_j) = K_0(\tau_j)$ for all j. This gives the homomorphism β in the commutative diagram:

$$
\begin{array}{ccccc}
 & & K_0(M_{k_j}(\mathbb{C})) & & \\
 & \swarrow K_0(\mu_j)) & \downarrow \beta_j & \searrow K_0(\tau_j)) & \\
K_0(\mathfrak{A}) & \xrightarrow[\gamma]{} & \varinjlim K_0(M_{k_j}(\mathbb{C})) & \xrightarrow[\beta]{} & G
\end{array}
$$

where $\mu_j : M_{k_j}(\mathbb{C}) \to \mathfrak{A}$ and β_j are the inductive limit morphisms, and γ is the isomorphism. Since $G \cong \cup_{j=1}^{\infty} k_j^{-1}\mathbb{Z}$, β is surjective and it is injective because each $K_0(\tau_j)$ is injective. It follows that $\beta \circ \gamma$ is an isomorphism, and $(\beta \circ \gamma)([1_{\mathfrak{A}}]) = K_0(\tau_1)([1_{M_{k_1}(\mathbb{C})}]) = 1$. $\square$

Remark. In particular, if $n_k = 2^k$, then $\varinjlim M_{2^k}(\mathbb{C}) = M_{2^\infty}$ is called CAR algebra and $K_0(M_{2^\infty}) \cong \mathbb{Z}[1/2]$ (a group ring over $\mathbb{Z}$). Note that a UHF algebra is always simple since matrix algebras over $\mathbb{C}$ are simple. In fact, it is known that inductive limit C^*-algebras of simple C^*-algebras are simple (see [8, Theorem 6.1.4]).

AF algebras

An AF algebra (or an approximately finite dimensional C^*-algebra) is an inductive limit of finite dimensional C^*-algebras. Recall that the positive cone $K_0(\mathfrak{A})_+$ of the K_0-group $K_0(\mathfrak{A})$ of a C^*-algebra $\mathfrak{A}$ is defined by

$$K_0(\mathfrak{A})_+ = \{[p] \,|\, p \in \cup_{n=1}^{\infty} M_n(\mathfrak{A}) : \text{ projections}\} \subset K_0(\mathfrak{A}).$$

Define an order $\leq$ on $K_0(\mathfrak{A})$ by $[p] \leq [q]$ if $[q] - [p] \in K_0(\mathfrak{A})_+$ for $[p], [q] \in K_0(\mathfrak{A})$.

If $\mathfrak{A}$ is a unital stably finite C^*-algebra, then $(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+)$ is an ordered abelian group, that is, (1) $K_0(\mathfrak{A})_+ + K_0(\mathfrak{A})_+ \subset K_0(\mathfrak{A})_+$,

(2) $K_0(\mathfrak{A})_+ \cap (-K_0(\mathfrak{A})_+) = \{0\}$, and (3) $K_0(\mathfrak{A})_+ - K_0(\mathfrak{A})_+ = K_0(\mathfrak{A})$,

where (2) follows from $\mathfrak{A}$ unital and (3) follows from $\mathfrak{A}$ stably finite.

For an ordered abelian group (G, G_+), an element u of G_+ is called an order unit if for any $g \in G$, there exists a positive integer n such that $-nu \leq g \leq nu$. For a unital C^*-algebra $\mathfrak{A}$, the class $[1_{\mathfrak{A}}]$ of the unit $1_{\mathfrak{A}}$ of $\mathfrak{A}$ in $K_0(\mathfrak{A})$ is an order unit.

Theorem 4.4 (G. Elliott) *Let* $\mathfrak{A} = \varinjlim \oplus_{j=1}^{s_k} M_{n_{j,k}}(\mathbb{C})$ *be an AF algebra. Then*

$$K_0(\varinjlim \oplus_{j=1}^{s_k} M_{n_{j,k}}(\mathbb{C})) \cong \varinjlim \oplus_{j=1}^{s_k} \mathbb{Z},$$

and its positive cone $K_0(\mathfrak{A})_+$ *is* $\varinjlim \oplus_{j=1}^{s_k} \mathbb{Z}_+$.

Moreover, let $\mathfrak{B} = \varinjlim \oplus_{j=1}^{s'_k} M_{n'_{j,k}}(\mathbb{C})$ *be another AF algebra. Then*

$$\mathfrak{A} \cong \mathfrak{B} \Leftrightarrow (K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, [1_{\mathfrak{A}}]) \cong (K_0(\mathfrak{B}), K_0(\mathfrak{B})_+, [1_{\mathfrak{B}}]),$$

where we assume that $\mathfrak{A}$, $\mathfrak{B}$ *are unital, and the right isomorphism means an order unit preserving order isomorphism as ordered abelian groups.*

Remark. In general, for an inductive limit $\varinjlim \mathfrak{A}_n$ of C^*-algebras $\mathfrak{A}_n$, we have $K_j(\varinjlim \mathfrak{A}_n) \cong \varinjlim K_j(\mathfrak{A}_n)$ for $j = 0, 1$. Moreover, for ordered abelian groups $(K_0(\mathfrak{A}_n), K_0(\mathfrak{A}_n)_+)$ for $n \geq 1$, we have $K_0(\varinjlim \mathfrak{A}_n) \cong \varinjlim K_0(\mathfrak{A}_n)$ as ordered abelian groups. Note that K_1-groups of AF algebras are all zero, from which one can tell that a C^*-algebra with its K_1 nontrivial is not AF.

Remark. A dimension group is an ordered abelian group that is isomorphic to an inductive limit of ordered abelian groups $(\mathbb{Z}^n, (\mathbb{Z}^n)_+)$ for $n \geq 1$, where $(\mathbb{Z}^n)_+ = \{(x_j) \in \mathbb{Z}^n \,|\, x_j \geq 0 \, (1 \leq j \leq n)\}$. The ordered K_0-group $(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+)$ of an AF algebra $\mathfrak{A}$ is a dimension group. Conversely, any dimension group is isomorphic to the ordered K_0-group of some AF algebra.

Remark. For an ordered abelian group (G, G_+), G is said to be unperforated if for any $x \in G$, $nx \in G_+$ for some $n \in \mathbb{N}$ implies $x \in G_+$, and G is said to have Riesz interpolation property if for any $x_1, x_2, y_1, y_2 \in G$ with $x_i \leq y_j$ for $i, j = 1, 2$, there exists $z \in G$ such that $x_i \leq z \leq y_j$ for $i, j = 1, 2$. It is known that a countable ordered abelian group is a dimension group if and only if it is unperforated and has Riesz interpolation property.

Remark. For non-unital AF-algebras $\mathfrak{A}$, to classify them one need to replace the positive cone of $K_0(\mathfrak{A})$ and the position of the unit of $K_0(\mathfrak{A})$ with the so-called dimension range:

$$D(\mathfrak{A}) = \{[p] : p \in P(\mathfrak{A})\} \subset K_0(\mathfrak{A})_+$$

where $P(\mathfrak{A})$ is the set of all projections of $\cup_{n=1}^{\infty} M_n(\mathfrak{A})$. Then for AF algebras $\mathfrak{A}$, $\mathfrak{B}$, we have

$$\mathfrak{A} \cong \mathfrak{B} \Leftrightarrow (K_0(\mathfrak{A}), D(\mathfrak{A})) \cong (K_0(\mathfrak{B}), D(\mathfrak{B})).$$

Sketch of Proof for Theorem 4.4 [6]. We may assume that $\mathfrak{A}$, $\mathfrak{B}$ are the closures of the unions $\cup_n \mathfrak{A}_n$, $\cup_n \mathfrak{B}_n$ of finite dimensional C^*-algebras $\mathfrak{A}_n$, $\mathfrak{B}_n$ with $\mathfrak{A}_n \subset \mathfrak{A}_{n+1}$ and $\mathfrak{B}_n \subset \mathfrak{B}_{n+1}$ respectively. We may also assume that they have the common units, i.e., $1_{\mathfrak{A}} = 1_{\mathfrak{A}_n}$ and $1_{\mathfrak{B}} = 1_{\mathfrak{B}_n}$ for all n. Let i_n be the embedding from $\mathfrak{A}_n$ to $\mathfrak{A}$ and $i_{n,m}$ the embedding from $\mathfrak{A}_n$ to $\mathfrak{A}_m$ for $m \geq n$, and let j_n be the embedding from $\mathfrak{B}_n$ to $\mathfrak{B}$ and $j_{n,m}$ the embedding from $\mathfrak{B}_n$ to $\mathfrak{B}_m$ for $m \geq n$. Let ψ be the order isomorphism from $K_0(\mathfrak{A})$ to $K_0(\mathfrak{B})$ and let φ be its inverse order isomorphism from $K_0(\mathfrak{B})$ to $K_0(\mathfrak{A})$. Note that $\varphi([1_{\mathfrak{B}}]) = [1_{\mathfrak{A}}]$.

We have $K_0(\mathfrak{A})_+ = \cup_n (i_n)_*(K_0(\mathfrak{A}_n)_+)$ and $K_0(\mathfrak{A}) = \cup_n (i_n)_*(K_0(\mathfrak{A}_n))$. Also, $K_0(\mathfrak{B})_+ = \cup_n (j_n)_*(K_0(\mathfrak{B}_n)_+)$ and $K_0(\mathfrak{B}) = \cup_n (j_n)_*(K_0(\mathfrak{B}_n))$.

Put $n_1 = 1$. Since $K_0(\mathfrak{A}_1)_+$ is finitely generated, so is $\psi \circ (i_1)_*(K_0(\mathfrak{A}_1)_+)$. Therefore, there exists an integer $m_1 > 0$ such that $(i_1)_*(K_0(\mathfrak{A}_1)_+) \subset (j_{m_1})_*(K_0(\mathfrak{B}_{m_1})_+)$. It follows that there exists a positive homomorphism $\psi_1 : K_0(\mathfrak{A}_1) \to K_0(\mathfrak{B}_{m_1})$ such that the diagram

$$\begin{array}{ccc} K_0(\mathfrak{A}_1) & \xrightarrow{(i_1)_*} & K_0(\mathfrak{A}) \\ {\scriptstyle\psi_1}\downarrow & & {\scriptstyle\psi}\downarrow \\ K_0(\mathfrak{B}_{m_1}) & \xrightarrow{(j_{m_1})_*} & K_0(\mathfrak{B}) \end{array}$$

commutes. Since $K_0(\mathfrak{B}_{m_1})_+$ is finitely generated, there exists an integer $n_2' > n_1$ such that $\varphi((j_{m_1})_*(K_0(\mathfrak{B}_{m_1})_+) \subset (i_{n_2'})_*(K_0(\mathfrak{A}_{n_2'})_+)$. Then we obtain a positive homomorphism $\varphi_1' : K_0(\mathfrak{B}_{m_1}) \to K_0(\mathfrak{A}_{n_2'})$ such that the diagram

$$\begin{array}{ccc} K_0(\mathfrak{A}_{n_2'}) & \xrightarrow{(i_{n_2'})_*} & K_0(\mathfrak{A}) \\ {\scriptstyle\varphi_1'}\uparrow & & {\scriptstyle\varphi}\uparrow \\ K_0(\mathfrak{B}_{m_1}) & \xrightarrow{(j_{m_1})_*} & K_0(\mathfrak{B}) \end{array}$$

commutes.

Let $g \in K_0(\mathfrak{A}_1)$. Then

$$(i_{n_2'})_* \circ \varphi_1' \circ \psi_1(g) = \varphi \circ (j_{m_1})_* \circ \psi_1(g) = \varphi \circ \psi \circ (i_{n_1})_*(g) = (i_{n_1})_*(g).$$

Since $K_0(\mathfrak{A}_1)$ is finitely generated, there exists $n_2 \geq n_2'$ such that

$$(i_{n_2})_* \circ (i_{n_2',n_2})_*(f_i) = (i_{n_2})_* \circ (i_{n_1,n_2})_*(g_i),$$

where $f_i = \varphi_1' \circ \psi_1(g_i)$ for $1 \leq i \leq l$, and g_i are generators of $K_0(\mathfrak{A}_1)$. Define $\varphi_1 = (i_{n_2',n_2})_* \circ \varphi_1' : K_0(\mathfrak{B}_{m_1}) \to K_0(\mathfrak{A}_{n_2})$. Then $\varphi_1 \circ \psi_1 = (i_{1,n_2})_*$. Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccccc} K_0(\mathfrak{A}_1) & \xrightarrow{(i_{1,n_2})_*} & K_0(\mathfrak{A}_{n_2}) & \xrightarrow{(i_{n_2})_*} & K_0(\mathfrak{A}) \\ \downarrow\psi_1 & \nearrow\varphi_1 & & & \updownarrow \\ K_0(\mathfrak{B}_{m_1}) & \longrightarrow & \longrightarrow & \xrightarrow{(j_{m_1})_*} & K_0(\mathfrak{B}). \end{array}$$

Similarly, there exists $m_2 > m_1$ and a positive homomorphism $\psi_2 : K_0(\mathfrak{A}_{n_2}) \to K_0(\mathfrak{B}_{m_2})$ such that the following diagram commutes:

$$\begin{array}{ccccc}
K_0(\mathfrak{A}_1) & \xrightarrow{(i_{1,n_2})_*} & K_0(\mathfrak{A}_{n_2}) & \xrightarrow{(i_{n_2})_*} & K_0(\mathfrak{A}) \\
\downarrow\psi_1 & \nearrow\varphi_1 & \downarrow\psi_2 & & \updownarrow \\
K_0(\mathfrak{B}_{m_1}) & \xrightarrow{(j_{m_1,m_2})_*} & K_0(\mathfrak{B}_{m_2}) & \xrightarrow{(j_{m_2})_*} & K_0(\mathfrak{B}).
\end{array}$$

Continuing the construction above, we construct inductively two increasing sequences of integers: $\{n_k\}$ and $\{m_k\}$, and positive homomorphisms $\psi_k : K_0(\mathfrak{A}_{n_k}) \to K_0(\mathfrak{B}_{m_k})$ and $\varphi_k : K_0(\mathfrak{B}_{m_k}) \to K_0(\mathfrak{A}_{n_{k+1}})$ such that the following diagram commutes:

$$\begin{array}{ccccc}
K_0(\mathfrak{A}_{n_k}) & \xrightarrow{(i_{n_k,n_{k+1}})_*} & K_0(\mathfrak{A}_{n_{k+1}}) & \xrightarrow{(i_{n_{k+1}})_*} & K_0(\mathfrak{A}) \\
\downarrow\psi_k & \nearrow\varphi_k & \downarrow\psi_{k+1} & & \updownarrow \\
K_0(\mathfrak{B}_{m_k}) & \xrightarrow{(j_{m_k,m_{k+1}})_*} & K_0(\mathfrak{B}_{m_{k+1}}) & \xrightarrow{(j_{m_{k+1}})_*} & K_0(\mathfrak{B}).
\end{array}$$

There exist unital homomorphisms $h_1 : \mathfrak{A}_1 \to \mathfrak{B}_{m_1}$ and $H_1' : \mathfrak{B}_{m_1} \to \mathfrak{A}_{n_2}$ such that $(h_1)_* = \psi_1$ and $(H_1')_* = \varphi_1$. Since $\varphi_1 \circ \psi_1 = (i_{1,n_2})_*$, there exists a unitary $u_1 \in \mathfrak{A}_{n_2}$ such that

$$\mathrm{Ad}(u_1) \circ H_1' \circ h_1 = i_{1,n_2}.$$

Define $H_1 = \mathrm{Ad}(u_1) \circ H_1'$. Then the following diagram:

$$\begin{array}{ccc}
\mathfrak{A}_1 & \xrightarrow{i_{1,n_2}} & \mathfrak{A}_{n_2} \\
\downarrow h_1 & \nearrow H_1 & \\
\mathfrak{B}_{m_1} & &
\end{array}$$

commutes. Also, we obtain a unital homomorphism $h_2' : \mathfrak{A}_{n_2} \to \mathfrak{B}_{m_2}$ such that $(h_2')_* = \psi_2$. Since $\psi_2 \circ \varphi_1 = (j_{m_1,m_2})_*$, there exists a unitary $v_1 \in \mathfrak{B}_{m_2}$ such that

$$\mathrm{Ad}(v_1) \circ h_2' \circ H_1 = j_{m_1,m_2}.$$

Define $h_2 = \mathrm{Ad}(v_1) \circ h_2'$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
\mathfrak{A}_1 & \xrightarrow{i_{1,n_2}} & \mathfrak{A}_{n_2} \\
\downarrow h_1 & \nearrow H_1 & \downarrow h_2 \\
\mathfrak{B}_{m_1} & \xrightarrow{j_{m_1,m_2}} & \mathfrak{B}_{m_2}.
\end{array}$$

Continuing this process, we construct inductively unital homomorphisms $h_k : \mathfrak{A}_{n_k} \to \mathfrak{B}_{m_k}$ and $H_k : \mathfrak{B}_{m_k} \to \mathfrak{A}_{n_{k+1}}$ such that the following diagram:

$$\begin{array}{ccccc}
K_0(\mathfrak{A}_{n_k}) & \overset{(i_{n_k,n_{k+1}})_*}{\longrightarrow} & K_0(\mathfrak{A}_{n_{k+1}}) & \overset{(i_{n_{k+1}})_*}{\longrightarrow} & K_0(\mathfrak{A}) \\
\downarrow h_k & \nearrow H_k & \downarrow h_{k+1} & & \updownarrow \\
K_0(\mathfrak{B}_{m_k}) & \overset{(j_{m_k,m_{k+1}})_*}{\longrightarrow} & K_0(\mathfrak{B}_{m_{k+1}}) & \overset{(j_{m_{k+1}})_*}{\longrightarrow} & K_0(\mathfrak{B})
\end{array}$$

commutes. $\square$

AT algebras

An AT algebra (or an approximately circle C^*-algebra) is an inductive limit of finite direct sums of matrix algebras over $C(\mathbb{T})$.

Let $\mathfrak{A}$, $\mathfrak{B}$ be C^*-algebras and $L : \mathfrak{A} \to \mathfrak{B}$ a contractive completely positive linear map. For a finite subset F of $\mathfrak{A}$ and $\varepsilon > 0$, L is said to be (F, ε) multiplicative if

$$\|L(xy) - L(x)L(y)\| < \varepsilon$$

for all $x, y \in F$. Let $L_1, L_2 : \mathfrak{A} \to \mathfrak{B}$ be linear maps. We write

$$L_1 \approx_\varepsilon L_2$$

on F if $\|L_1(x) - L_2(x)\| < \varepsilon$ for $x \in F$. For a projection p of $\mathfrak{A}$, if L is (F, ε) multiplicative for F sufficiently large and ε sufficiently small, then $L(p)$ is close to a projection. Thus, we write $[L]([p]) = [L(p)]$ for $[p] \in K_0(\mathfrak{A})$.

Lemma 4.5 [6] *Let $\mathfrak{A}$ be a finite dimensional C^*-algebra. For any $\varepsilon > 0$ and finite subset F of $\mathfrak{A}$, there exists $\delta > 0$, finite subsets G of $\mathfrak{A}$, P of $K_0(\mathfrak{A})$ satisfying the following: if $L_i : \mathfrak{A} \to \mathfrak{B}$ $(i = 1, 2)$ is (G, δ) multiplicative contractive completely positive linear maps, where $\mathfrak{B}$ is a unital C^*-algebra with stable rank one such that*

$$[L_1]|_{G(P)} = [L_2]|_{G(P)},$$

where $G(P)$ is the subgroup generated by P, then there is a unitary w of $\mathfrak{B}$ such that

$$\mathrm{Ad}(w) \circ L_1 \approx_\varepsilon L_2 \quad \textit{on } F.$$

Sketch of Proof. Since $K_0(\mathfrak{A})$ is finitely generated, we may assume that $G(P) = K_0(\mathfrak{A})$. It follows that for any $\delta > 0$ and finite subset G of $\mathfrak{B}$ with sufficiently small δ and sufficiently large G, there are homomorphisms $h_i : \mathfrak{A} \to \mathfrak{B}$ such that

$$L_i \approx_\delta h_i$$

on G for $i = 1, 2$. Therefore, $(h_1)_* = (h_2)_*$ on $K_0(\mathfrak{A})$. $\square$

Lemma 4.6 [6] *Let $\mathfrak{A}$ be a unital simple C^*-algebra with tracial rank zero and u a unitary of $\mathfrak{A}$. For any $\varepsilon > 0$ and finite subset F of $\mathfrak{A}$, there exists $\delta > 0$, finite subsets G of $\mathfrak{A}$ and P of $K_0(\mathfrak{A})$ satisfying the following: if $L_i : \mathfrak{A} \to \mathfrak{B}$ $(i = 1, 2)$ is (G, δ) multiplicative contractive completely positive linear maps, where $\mathfrak{B}$ is a unital C^*-algebra with stable rank one and real rank zero, such that*

$$[L_1]|_P = [L_2]|_P, \quad \textit{and} \quad [L_1]([u]) = [L_2]([u]),$$

then there is a unitary w of $\mathfrak{B}$ such that

$$\mathrm{Ad}(w) \circ L_1(u) \approx_\varepsilon L_2(u).$$

Sketch of Proof. We first consider the case where the spectrum of u is not $\mathbb{T}$. Then there are nonzero mutually orthogonal projections $d_1, \cdots, d_l \in \mathfrak{A}$ and $\lambda_1, \cdots, \lambda_l \in \mathbb{T}$ such that

$$\|u - v \oplus \sum_{j=1}^{l} \lambda_j d_j\| < \varepsilon/4$$

where $v \in q\mathfrak{A}q$ and $q = 1 - \sum_{j=1}^{l} d_j$. Then choose P which contains $[d_1], \cdots, [d_l]$. With sufficiently large G and sufficiently small δ, we obtain mutually orthogonal projections $d_1', \cdots, d_l'$ and $d_1'', \cdots, d_l''$ such that

$$\|L_1(d_i) - d_i'\| < \varepsilon/(4l+1), \quad \|L_2(d_i) - d_i''\| < \varepsilon/(4l+1)$$

and $[d_i'] = [d_i'']$ for $1 \le i \le l$. Since $\mathfrak{B}$ has stable rank one, we obtain a unitary $w \in \mathfrak{B}$ such that $w^* d_i' w = d_i''$ for $1 \le i \le l$. Thus, $\mathrm{Ad}(w) \circ L_1(v) \approx_{\varepsilon/4} L_2(v)$. Therefore,

$$\mathrm{Ad}(w) \circ L_1(u) \approx_{\varepsilon/4} \mathrm{Ad}(w) \circ L_1(v) \approx_{\varepsilon/4} L_2(v) \approx \mathrm{Ad}(w) \circ L_2(u).$$

Now we consider the case where the spectrum of u is $\mathbb{T}$. Fix $\varepsilon > 0$ and $m_0 \ge 1$. Let n be the integer associated with $\varepsilon/2$ and m_0. Fix an $\eta > 0$. Then we obtain a nonzero projection $p \in \mathfrak{A}$, a unitary $v_1 \in p\mathfrak{A}p$ with $[(1-p) + v_1] = [u]$, z and v_2 satisfy the conditions associated with η, m_0 and n. Note that $z = \sum_{i=1}^{N} \lambda_i q_i$ with $m_0[q_i] \ge [p]$ and $v_2 = \sum_{j=1}^{l} \alpha_j g_j$, where $\alpha_1, \cdots, \alpha_l \in \mathbb{T}$ and $g_1, \cdots, g_l$ are mutually orthogonal projections.

Let P be a finite subset of $K_0(\mathfrak{A})$ which contains

$$\{\mathrm{diag}(q_i, 0, \cdots, 0), \mathrm{diag}(0, q_i, \cdots, 0), \cdots, \mathrm{diag}(0, \cdots, 0, q_i) \,|\, 1 \le i \le N\},$$

$[p]$, $[1-p]$, $\{g_1, \cdots, g_l\}$, $Q = \sum_{j=1}^{l} g_j$, $[P+Q]$ and $[1-p-Q]$, where diag means the diagonal sum. Let F_1 be a finite subset of $\mathfrak{A}$ which contains

$$\mathrm{diag}(z, 0, \cdots, 0), \cdots, \mathrm{diag}(0, \cdots, 0, z), \mathrm{diag}(q_i, 0, \cdots, 0), \cdots, \mathrm{diag}(0, \cdots, 0, q_i)$$

for $1 \le i \le N$ and u, v_1, v_2, $g_1, \cdots, g_l$.

Suppose that L_1, L_2 are (F_1, η) multipicative contractive completely positive linear maps from $\mathfrak{A}$ to $\mathfrak{B}$ such that $[L_1]|_P = [L_2]|_P$. Without loss of generality, we may assume that L_1, L_2 are unital and that there are projections P', P'', Q', $Q'' \in \mathfrak{B}$ such that

$$\|P' - L_1(p)\| < \eta, \quad \|Q' - L_1(v_2^* v_2)\| < \eta, \quad \|Q'' - L_2(v_2^* v_2)\| < \eta$$

and $\|P'' - L_2(p)\| < \eta$. Then $[P'] = [P'']$ and $[1''_Q] = [1 - Q''] \in K_0(\mathfrak{B})$. Since $\mathfrak{A}$ has stable rank one, by replacing L_1 with $\mathrm{Ad}(W') \circ L_1$ for some suitable W', without loss of generality, we may assume that $P' = P''$, $P'Q' = 0$ and $Q' = Q''$.

With a sufficiently small η, there are unitaries v_1', v_1'' of $P'\mathfrak{B}P'$, unitaries v_2', v_2'' of $Q'\mathfrak{B}Q'$ and unitaries z', z'' of $(1 - P' - Q')\mathfrak{B}(1 - P' - Q')$ such that

$$\begin{aligned}
&v_1' \approx_{\varepsilon/16} L_1(v_1), \quad v_1'' \approx_{\varepsilon/16} L_2(v_1), \quad v_2' \approx_{\varepsilon/16} L_1(v_2),\\
&v_2'' \approx_{\varepsilon/16} L_2(v_2), \quad z' \approx_{\varepsilon/16} L_1(z), \quad z'' \approx_{\varepsilon/16} L_2(z),\\
&\mathrm{diag}(z', \cdots, z') \approx_{\varepsilon/16} L_1(\mathrm{diag}(z, \cdots, z)),
\end{aligned}$$

and $\mathrm{diag}(z'', \cdots, z'') \approx_{\varepsilon/16} L_2(\mathrm{diag}(z, \cdots, z))$ and $z' = \sum_{i=1}^{N} \lambda_i q_i'$, $z'' = \sum_{i=1}^{N} \lambda_i q_i''$, where $\{q_i' : 1 \le i \le l\}$ are mutually orthogonal projections, $\{q_i'' : 1 \le i \le l\}$ are mutually orthogonal projections, $[q_i'] = [q_i'']$, and $v_2' = \sum_{j=1}^{l} \alpha_j g_j'$ and $v_2'' = \sum_{j=1} -l\alpha_j g_j''$, where $\{g_j' : 1 \le j \le l\}$ are mutually orthogonal projections, $\{g_j'' : 1 \le j \le l\}$ are mutually orthogonal projections and $[g_i'] = [g_i'']$ in $K_0(\mathfrak{B})$. Since $\mathfrak{B}$ has stable rank one, we obtain a unitary W_1 of $(1 - P')\mathfrak{B}(1 - P')$ such that

$$W_1^* \mathrm{diag}(z', \cdots, z') W_1 = \mathrm{diag}(z'', \cdots, z'')$$

and $W_1^* v_2' W_1 = v_2''$. Set $W_2 = P' \oplus W_1$. Then

$$W_2^* L_1(u) W_2 \approx_{3\varepsilon/16} v_1' \oplus \mathrm{diag}(z'', \cdots, z'') \oplus v_2''.$$

We also have

$$v_2' \oplus \mathrm{diag}(z'', \cdots, z'') \oplus v_2'' \approx_{3\varepsilon/16} L_2(u).$$

Note that with sufficiently small η, $[v_1'] = [v_1''] = [L_1(u)] = [L_2(u)]$. We also have $m_0[q_i'] \geq [P']$. By the choice of n we obtain

$$v_1' \oplus \mathrm{diag}(z'', \cdots, z'') \oplus v_2'' \approx_{\varepsilon/2} v_2' \oplus \mathrm{diag}(z'', \cdots, z'') \oplus v_2''.$$

Therefore, we obtain $L_1(u) \approx_\varepsilon L_2(u)$. □

Lemma 4.7 [6] *Let $\mathfrak{A}$ be a unital simple AT algebra. For any $\varepsilon > 0$ and finite subset F of $\mathfrak{A}$, there exists $\delta > 0$, finite subsets G of $\mathfrak{A}$, P of $K_0(\mathfrak{A})$ and U of $K_1(\mathfrak{A})$ satisfying the following: if $L_i : \mathfrak{A} \to \mathfrak{B}$ $(i = 1, 2)$ is (G, δ) multiplicative contractive completely positive linear maps, where $\mathfrak{B}$ is a unital C^*-algebra with stable rank one and real rank zero, such that*

$$[L_1]|_P = [L_2]|_P, \quad \text{and} \quad [L_1]|_U = [L_2]|_U,$$

then there is a unitary w of $\mathfrak{B}$ such that

$$\mathrm{Ad}(w) \circ L_1 \approx_\varepsilon L_2 \quad \text{on } F.$$

Sketch of Proof. Without loss of generality, we may assume that $F \subset C = \oplus_{j=1}^k M_{n_j}(C(X_j))$, where X_j is a connected compact subset of $\mathbb{T}$.

We first consider the case $C = M_n(C(X_1))$. Let $\{e_{ij}\}$ be the system of matrix units of $M_n(\mathbb{C})$ and let $u \in e_{11}Ce_{11}$ be the unitary generator given by the function $z \mapsto z \in \mathbb{T}$. So, we may assume that $F = \{u\} \cup \{e_{ij} : 1 \leq i, j \leq n\}$.

Suppose that P contains $\{e_{11}, \cdots, e_{nn}\}$. So $[L_1](e_{11}) = [L_2](e_{11})$. Therefore, since $\mathfrak{B}$ has stable rank one, by replacing L_1 with $\mathrm{Ad}(w_0) \circ L_1$ for some unitary w_0, we may assume that $L_1(e_{11}) = L_2(e_{11}) = q$ is a projection of $\mathfrak{B}$.

Then there is $\delta_1 > 0$, a finite subset P_1 of $K_0(\mathfrak{A})$ and a finite subset G_1 of $e_{11}\mathfrak{A}e_{11}$ such that if $L_i' : e_{11}\mathfrak{A}e_{11} \to q\mathfrak{B}q$ is a (G_1, δ_1) multiplicative contractive completely positive linear map such that

$$[L_1']|_{P_1} = [L_2']|_{P_1}, \quad \text{and} \quad [L_1'](u) = [L_2'](u),$$

then there is a unitary w_1 of $\mathfrak{B}$ such that

$$\mathrm{Ad}(w_1) \circ L_1'(u) \approx_{\varepsilon/2} L_2'(u).$$

Thus, choosing a small $0 < \delta < \delta_1/n^2$ and a finite subset $G \supset G_1$, we obtain a unitary u_2 of $\mathfrak{B}$ such that

$$\mathrm{Ad}(w_2) \circ L_1 \approx_{\varepsilon/n^2} L_2 \quad \text{on } F_1,$$

where $F_1 = \{e_{ij} : 1 \le i, j \le n\}$. Then we may assume that $\mathrm{Ad}(w_2) \circ L_1(e_{11}) = L_2(e_{11})$ is a projection q of $\mathfrak{B}$. Then we consider $L_1' = \mathrm{Ad}(w_2) \circ L_1|_{e_{11}\mathfrak{A}e_{11}}$ and $L_2' = L_2|_{e_{11}\mathfrak{A}e_{11}}$. Then there is a unitary W of $\mathfrak{B}$ such that

$$\mathrm{Ad}(W) \circ \mathrm{Ad}(w_2) \circ L_1 \approx_\varepsilon L_2 \quad \text{on } F.$$

For the general case $C = \oplus_{j=1}^k M_{n_j}(C(X_j))$, we let $q_1, \cdots, q_k$ be the identities of each summand. We may assume that $L_j(q_i)$ are projections $(j = 1, 2)$. We may reduce the general case to the case in which $L_1(q_i) = L_2(q_i)$. By considering each summand, we further reduce the general case to the case in which C has only one summand. □

Lemma 4.8 [6] *Let $\mathfrak{A} = \oplus_{j=1}^s M_{n_j}(C(X_j))$ for X_j connected compact subsets of $\mathbb{T}$ and $\mathfrak{B}$ a unital C^*-algebra with stable rank one. Let $\alpha_i ; K_i(\mathfrak{A}) \to K_i(\mathfrak{B})$ $(i = 0, 1)$ be homomorphisms and α_0 positive. Then there exists a homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ such that $h_{*i} = \alpha_i$ for $i = 0, 1$.*

Sketch of Proof. Let $\{u_l, e_{ij}^l : i, j = 1, \cdots, n_l, 1 \le l \le s\}$ be the system of standard generators of $\mathfrak{A}$. Let $F = \oplus_{j=1}^s M_{n_j}(\mathbb{C})$. We identify $\{e_{ij}^l : i, j = 1, \cdots, n_l, 1 \le l \le s\}$ with the set of standard generators of F. Note that $(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+) = (K_0(F), K_0(F)_+)$. Thus, there exists a homomorphism $\varphi : F \to \mathfrak{B}$ such that $\varphi_{*0} = \alpha_0$. Set $q_{ij}^l = \varphi(e_{ij}^l)$ for $i, j = 1, \cdots, n_l, 1 \le l \le s$.

It follows that there are unitaries $v_1, \cdots, v_s \in q_{11}^l \mathfrak{B} q_{11}^l$ with $[v_l + (1 - q_{11}^l] = \alpha_1([u_l + (1_{\mathfrak{A}} - e_{11}^l])$ for $1 \le l \le s$. Note that if the spectrum $\mathrm{sp}(u_l) \ne \mathbb{T}$, then $\alpha_1([u_l + (1_{\mathfrak{A}} - e_{11}^l]) = [1_{\mathfrak{B}}]$. Therefore, we may assume that $\mathrm{sp}(v_l + (1 - q_{11}^l)) = \mathrm{sp}(u_l + (1 - q_{11}^l))$. By sending u_l to v_l and e_{ij}^l to q_{ij}^l respectively for $i, j = 1, \cdots, n_l, 1 \le l \le s$, we obtain a homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ such that $h_{*i} = \alpha_i$ for $i = 0, 1$ by the construction. □

Lemma 4.9 [6] *Let $\mathfrak{A}$ be a unital AT algebra and $\mathfrak{B}$ a unital C^*-algebra with stable rank one. Let $\alpha_i ; K_i(\mathfrak{A}) \to K_i(\mathfrak{B})$ $(i = 0, 1)$ be homomorphisms and α_0 positive. Then for any $\varepsilon > 0$, finite subset F of $\mathfrak{A}$, and any finitely generated subgroup P_i of $K_i(\mathfrak{A})$ $(i = 0, 1)$, there exists an (F, ε) multiplicative contractive completely positive linear map $L : \mathfrak{A} \to \mathfrak{B}$ such that*

$$[L]|_{P_0} = (\alpha_0)|_{P_0}, \quad \text{and} \quad [L]|_{P_1} = (\alpha_1)|_{P_1}.$$

Sketch of Proof. Fix F, P_0, P_1 and $\varepsilon > 0$. Since $\mathfrak{A}$ is an AT algebra, we may assume that $\mathfrak{A}$ is the closure of the union of an increasing sequence $\{\mathfrak{A}_n\}$ of $\mathfrak{A}_n$ finite direct sums of circle algebras. Let $j_n : \mathfrak{A}_n \to \mathfrak{A}$ be the embeddings. We may assume that for some $N_0 > 0$, $(j_n)_{*,i}(K_i(\mathfrak{A}_n)) \supset P_i$ for all $n \geq N_0$ $(i = 0, 1)$. Thus we may also assume that $x \in_{\varepsilon/4} \mathfrak{A}_n$ for all $x \in F$.

We may assume that there exist $\mathfrak{B}_n \subset \mathfrak{A}_n$, where $\mathfrak{B}_n$ is a finite direct sum of C^*-algebras of the form $M_k(C(X))$ for X a compact connected subset of $\mathbb{T}$ such that $(j_n)_{*i} \circ (j'_n)_{*i}(K_i(\mathfrak{B}_n)) \supset P_i$ and $x \in_{\varepsilon/2} \mathfrak{B}_n$ for all $x \in F$, where $j'_n : \mathfrak{B}_n \to \mathfrak{A}_n$ is the embeddings $(i = 0, 1)$. Let $i_n = j_n \circ j'_n$ and Q_i be a finitely generated subgroup of $K_i(\mathfrak{B}_n)$ such that $(i_n)_{*i}(Q_i) = P_i$ $(i = 0, 1)$. Then there exists a homomorphism $h : \mathfrak{B}_n \to \mathfrak{B}$ such that

$$h_{*i}|_{Q_i} = \alpha_i \circ (i_n)_{*i}|_{Q_i}$$

for $i = 0, 1$. Since $\mathfrak{A}$ is amenable, for any $\delta > 0$ and finite subset G of $\mathfrak{A}$, there exists a contractive completely positive linear map $L : \mathfrak{A} \to \mathfrak{B}$ such that

$$L \approx_{\delta/2} h$$

on G. We choose G large enough so that $x \in_{\delta/2} G$. Since h is a homomorphism, L is (F, ε) multiplicative if G is large enough and δ is small enough. It is also clear that we can assume that $[L]|_{P_i}$ is well defined and

$$[L]|_{P_i} = \alpha_i|_{P_i}$$

for $i = 0, 1$ (where $[L]([p]) = [L(p)]$ and $[L]([u]) = [L(u)]$ for some projections p and unitaries u), provided that G is large enough and δ is small enough. □

Using the lemmas above, we obtain

Theorem 4.10 (G. Elliott) *Let $\mathfrak{A}$, $\mathfrak{B}$ be unital simple AT algebras with real rank zero. If there exists a positive isomorphism:*

$$\alpha : (K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, [1_{\mathfrak{A}}], K_1(\mathfrak{A})) \to (K_0(\mathfrak{B}), K_0(\mathfrak{B})_+, [1_{\mathfrak{B}}], K_1(\mathfrak{B})),$$

that is, $\alpha(K_0(\mathfrak{A})_+) \subset K_0(\mathfrak{B})_+$. Then there exists an isomorphism $h : \mathfrak{A} \to \mathfrak{B}$ such that $\alpha = (h_*)_0 \oplus (h_*)_1$ the induced map on K-groups from h.

Sketch of Proof [6]. Define $\beta = \alpha^{-1}$ the inverse of α. Let $\{a_n\}$, $\{b_n\}$ be dense sequences of elements of $\mathfrak{A}$, $\mathfrak{B}$ respectively. Let $\{r_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} r_n < \infty$.

Let F_1 be a finite subset of $\mathfrak{A}$ containing a_1. Let $\delta_1 = \delta(r_1/2, F_1) > 0$ a positive number, $G_1 = G(r_1/2, F_1)$ a finite subset of $\mathfrak{A}$, and $P_1 = P(r_1/2, F_1)$ be a finitely generated subgroup of $K_0(\mathfrak{A})$, and $U_1 = U(r_1/2, F_1)$ a finite subset of $K_1(\mathfrak{A})$ corresponding to $r_1/2$ and F_1. We may assume that $F_1 \subset G_1$ and $\delta_1 < r_1/2$. Then it follows that there exists a $(G_1, \delta_1/2)$ multiplicative contractive completely positive linear map $L_1 : \mathfrak{A} \to \mathfrak{B}$ such that

$$[L_1]|_{P_1} = \alpha_0|_{P_1}, \quad \text{and} \quad [L_1]|_{U_1} = \alpha|_{U_1}.$$

Let S_1 be a finite subset of $\mathfrak{B}$ containing b_1 and $L_1(G_1)$. Set $t_1 = \min\{r_1/4, \delta_1/2\}$. Let $\eta_1 = \eta(t_1/2, S_1) > 0$ a positive number, $H_1 = H(t_1, S_1)$ be a finite subset of $\mathfrak{A}$, and $Q_1 = Q(t_1, S_1)$ be a finitely generated subgroup of $K_0(\mathfrak{B})$ and $V_1 = V(t_1, S_1)$ a finitely generated subgroup of $K_1(\mathfrak{B})$ corresponding to t_1 and S_1. We may assume that $\eta_1 < t_1/2$, $H_1 \supset S_1$, $[L_1](P_1) \subset Q_1$ and $[L_1](U_1) \subset V_1$. Then it follows that there exists a (H_1, η_1) multiplicative contractive completely positive linear map $\Delta_1' : \mathfrak{B} \to \mathfrak{A}$ such that

$$[\Delta_1']|_{Q_1} = \beta_0|_{Q_1}, \quad \text{and} \quad [\Delta_1']|_{V_1} = \beta_1|_{V_1}.$$

Therefore, $\Delta_1' \circ L_1$ is (G_1, δ_1) multiplicative and

$$[\Delta_1' \circ L_1]|_{P_1} = [\mathrm{id}_{\mathfrak{A}}]|_{P_1}, \quad \text{and} \quad [\Delta_1' \circ L_1]|_{U_1} = [\mathrm{id}_{\mathfrak{A}}]|_{U_1}.$$

Then it follows from the choice of δ_1, G_1, P_1 and U_1 that there exists a unitary $u_1 \in \mathfrak{A}$ such that

$$\mathrm{Ad}(u_1) \circ \Delta_1' \circ L_1 \approx_{r_2/2} \mathrm{id}_{\mathfrak{A}}$$

on F_1. Set $\Delta_1 = \mathrm{Ad}(u_1) \circ \Delta_1'$.

Let F_2 be a finite subset of $\mathfrak{A}$ containing a_2, F_2 and $\Delta_1(H_1)$. Set $s_2 = \min\{r_2/2, \eta_1/2\}$. Let $\delta_2 = \delta(s_2/4, F_2) > 0$ a positive number, $G_2 = G(s_2/4, F_2)$ be a finite subset of $\mathfrak{A}$, and $P_2 = P(s_2/4, F_2)$ be a finitely generated subgroup of $K_0(\mathfrak{A})$ and $U_2 = U(s_2/4, F_2)$ a finitely generated subgroup of $K_1(\mathfrak{A})$ corresponding to $s_2/4$ and F_2. We may assume that $\delta_2 < s_2/2$, $G_2 \supset F_2 \cup H_1$, $[\Delta_1](Q_1) \subset P_2$ and $[\Delta_1](V_1) \subset U_2$. Then it follows that there exists a $(G_2, \delta_2/2)$ multiplicative contractive completely positive linear map $L_2' : \mathfrak{A} \to \mathfrak{B}$ such that

$$[L_2']|_{P_2} = \alpha_0|_{P_2}, \quad \text{and} \quad [L_2']|_{U_2} = \alpha_1|_{U_2}.$$

Therefore, $L_2' \circ \Delta_1$ is (H_1, η_1) multiplicative and

$$[L_2' \circ \Delta_1]|_{Q_1} = [\mathrm{id}_{\mathfrak{B}}]|_{Q_1}, \quad \text{and} \quad [L_2' \circ \Delta_1]|_{V_1} = [\mathrm{id}_{\mathfrak{B}}]|_{V_1}.$$

Then it follows that there exists a unitary $w_1 \in \mathfrak{B}$ such that

$$\mathrm{Ad}(w_1) \circ L_2' \approx_{s_2/2} \mathrm{id}_{\mathfrak{B}}$$

on H_1. Set $L_2 = \mathrm{Ad}(w_1) \circ L_2'$. Then the following (not necessarily commutative) diagram:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\mathrm{id}_{\mathfrak{A}}} & \mathfrak{A} \\ \downarrow L_1 & \nearrow \Delta_1 & \downarrow L_2 \\ \mathfrak{B} & \xrightarrow{\mathrm{id}_{\mathfrak{B}}} & \mathfrak{B} \end{array}$$

is approximately commutative on F_1 within r_1 and on S_1 within s_1.

Let S_2 be a finite subset of $\mathfrak{B}$ containing b_3, S_1 and $L_2(G_2)$. Set $t_2 = \min\{r_2/2, \delta_2/2\}$. Let $\eta_2 = \eta(t_2, S_2) > 0$ a positive number, $H_2 = H(t_2, S_2)$ be a finite subset of $\mathfrak{A}$, and $Q_2 = Q(t_2, S_2)$ be a finitely generated subgroup of $K_0(\mathfrak{B})$ and $V_2 = V(t_2, S_2)$ a finitely generated subgroup of $K_1(\mathfrak{B})$ corresponding to t_2 and S_2. We may assume that $\eta_2 < t_2/2$, $H_2 \supset S_2$, $[L_2](P_2) \subset Q_2$ and $[L_2](U_2) \subset V_2$. Then it follows that there exists a $(H_2, \eta_2/2)$ multiplicative contractive completely positive linear map $\Delta_2' : \mathfrak{B} \to \mathfrak{A}$ such that

$$[\Delta_2']|_{Q_2} = \beta_0|_{Q_2}, \quad \text{and} \quad [\Delta_2']|_{V_2} = \beta_1|_{V_2}.$$

Therefore, $\Delta_2' \circ L_2$ is (G_2, δ_2) multiplicative and

$$[\Delta_2' \circ L_2]|_{P_2} = [\mathrm{id}_{\mathfrak{A}}]|_{P_2}, \quad \text{and} \quad [\Delta_2' \circ L_2]|_{U_2} = [\mathrm{id}_{\mathfrak{A}}]|_{U_2}.$$

Then it follows from the choice of δ_2, G_2, P_2 and U_2 that there exists a unitary $u_2 \in \mathfrak{A}$ such that

$$\mathrm{Ad}(u_2) \circ \Delta_2' \circ L_2 \approx_{t_2} \mathrm{id}_{\mathfrak{A}}$$

on F_2. Set $\Delta_2 = \mathrm{Ad}(u_2) \circ \Delta_2'$.

Repeating this process we obtain a sequence of (F_n, r_n) multiplicative contractive completely positive linear maps $L_n : \mathfrak{A} \to \mathfrak{B}$ and a sequence of (S_n, s_n) multiplicative contractive completely positive linear maps $\Delta_n : \mathfrak{B} \to \mathfrak{A}$ such that the following (not necessarily commutative) diagram:

$$\begin{array}{ccccccccc} \mathfrak{A} & \xrightarrow{\mathrm{id}_{\mathfrak{A}}} & \mathfrak{A} & \xrightarrow{\mathrm{id}_{\mathfrak{A}}} & \mathfrak{A} & \xrightarrow{\mathrm{id}_{\mathfrak{A}}} & \cdots & \mathfrak{A} \\ \downarrow L_1 & \nearrow \Delta_1 & \downarrow L_2 & \nearrow \Delta_2 & \downarrow L_3 & & & \\ \mathfrak{B} & \xrightarrow{\mathrm{id}_{\mathfrak{B}}} & \mathfrak{B} & \xrightarrow{\mathrm{id}_{\mathfrak{B}}} & \mathfrak{B} & \xrightarrow{\mathrm{id}_{\mathfrak{B}}} & \cdots & \mathfrak{B} \end{array}$$

is two-sided approximately intertwining. Hence, it follows that there exists an isomorphism $h : \mathfrak{A} \to \mathfrak{B}$ (and $h^{-1} : \mathfrak{B} \to \mathfrak{A}$) such that

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\mathrm{id}_{\mathfrak{A}}} & \mathfrak{A} \\ \downarrow L_n & \nearrow \Delta_n & \downarrow h \\ \mathfrak{B} & \xrightarrow{\mathrm{id}_{\mathfrak{B}}} & \mathfrak{B} \end{array}$$

is approximately commutative on F_n within r_n and on S_n within s_n. Therefore, $(h_*)_0 = (\alpha_*)_i$ for $i = 0, 1$. $\square$

Theorem 4.11 [6] *Let $G = \lim_{n\to\infty}(G_n, \varphi_n)$, where G_n is a finitely generated abelian free group and each partial map of φ_n is positive and has multiplicity at leanst 2, and let H be a countable torsion free abelian group. Then there exists a unital simple AT algebra $\mathfrak{A}$ with real rank zero such that*

$$(G, G_+) \cong (K_0(\mathfrak{A}), K_0(\mathfrak{A})_+), \quad \textit{and} \quad H \cong K_1(\mathfrak{A}).$$

Sketch of Proof. Suppose that $H = \lim_{k\to\infty}(H_k, \alpha_k)$, where H_k is the sum of n_k copies of $\mathbb{Z}$. Let $\alpha_{k,i,j}$ be the multiplicity of the partial map from i-th copy of $\mathbb{Z}$ to j-th copy of $\mathbb{Z}$. Set $\beta_k = \sum_{i,j}\alpha_{k,i,j}$, $\gamma_k = \Pi_{i=1}^k \beta_i$ and $s(k) = k\gamma_k$. We may assume that each G_n is the sum of m_k copies of $\mathbb{Z}$ with $m_k \geq n_k$. By passing to a subsequence if necessary, we may assume that each partial map of φ_k has positive multiplicity at least $s(k) + 1$.

If follows that there is a unital AF algebra $\mathfrak{B} = \lim_{k\to\infty}(\mathfrak{B}_k, \psi_k)$, where $\mathfrak{B}_k = \oplus_{j=1}^{m_k} M_{l(k,j)}(\mathbb{C})$, $(K_0(\mathfrak{B}_k), K_0(\mathfrak{B}_k)_+) = (G_k, (G_k)_+)$ and $(\psi_k)_{*0} = \alpha_k$ for $k \geq 1$. Denote by $\psi_k^{(i,j)}$ the partial map of ψ_k from the i-th summand of $\mathfrak{B}_k$ to the j-th summand of $\mathfrak{B}_{k+1}$. Define

$$\mathfrak{A}_k = \oplus_{j=1}^{n_k} M_{l(k,j)}(C(\mathbb{T})) \oplus M_{l(k,n_k+1)}(\mathbb{C}) \oplus \cdots \oplus M_{l(k,m_k)}(\mathbb{C}).$$

Define $\pi_{k,i} : \mathfrak{A}_k \to M_{l(k,j)}(C(\mathbb{T}))$ for $j < n_k$ and $\pi_{k,i} : \mathfrak{A}_k \to M_{l(k,i)}(\mathbb{C})$ for $i \geq n_k$ be projections. Let $\{\xi_{k,j}\}$ be a sequence of points of $\mathbb{T}$ so that $\{\xi_{k,1}, \cdots, \xi_{k,s(k)}\}$ is $2\pi/s(k)$ dense in $\mathbb{T}$. Define $h_{k,i,j} : M_{l(k,i)}(C(\mathbb{T})) \to M_{l(k+1,j)}(C(\mathbb{T}))$ by

$$h_{k,i,j}(f) = \mathrm{diag}(f \circ t_{k,i,j}, f(\xi_{k,1}), \cdots, f(\xi_{k,\tau(k,i,j)}))$$

for $i \leq n_k$ and $j \leq n_{k+1}$, where $t_{k,i,j} : \mathbb{T} \to \mathbb{T}$ is defined so that $t_{k,i,j} = z^{\alpha_{k,i,j}}$ for $z = e^{i\theta} \in \mathbb{T}$ and $\tau(k, i, j) + 1$ is the multiplicity of the partial map of $\varphi_k^{(i,j)}$. If $i > n_k$ and $j \leq n_{k+1}$, define $h_{k,i,j}$ to be the composition of $M_{l(k+1,j)}(\mathbb{C}) \to M_{l(k+1,j)}(C(\mathbb{T}))$ and $\psi_k^{(i,j)}$. If $i > n_k$ and $j > n_{k+1}$, define $h_{k,i,j} = \psi_k^{(i,j)}$. Set $h_k = \oplus_{i,j} h_{k,i,j} : \mathfrak{A}_k \to \mathfrak{A}_{k+1}$, $\mathfrak{A} = \lim_{n\to\infty}(\mathfrak{A}_k, h_k)$ and $h_{k,n} = h_{n-1} \circ \cdots \circ h_k$ $(n > k)$. Therefore,

$$(K_0(\mathfrak{A}_k), K_0(\mathfrak{A}_k)_+) = (G_k, (G_k)_+), \quad K_1(\mathfrak{A}_k) = F_k$$

and $(h_k)_{*0} = \varphi_k$ and $(h_k)_{*1} = \alpha_k$. So $\mathfrak{A}$ is a unital AT algebra and

$$(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+) = (G, G_+), \quad K_1(\mathfrak{A}) = F.$$

To see that $\mathfrak{A}$ is simple, it suffices to show that for any closed ideal $\mathfrak{I}$ of $\mathfrak{A}$ with $\mathfrak{I} \neq \mathfrak{A}$, $h_{k,\infty}(\mathfrak{A}_k) \cap \mathfrak{I} = \{0\}$ for all k. Let $f \in \mathfrak{A}_k$ be a nonzero element. Without loss of generality, we may assume that $f \geq 0$.

Case 1: $\pi_{k,i}(f) \neq 0$ for some $i \geq n_k$. It is clear from the construction that the closed ideal generated by $h_k(\pi_{k,i}(f))$ is just $\mathfrak{A}_{k+1}$.

Case 2: $\pi_{k,i}(f) \neq 0$ for some $i < n_k$. There exists $n > k$ such that for any $2\pi/n$-dense subset X of $\mathbb{T}$, $\pi_{k,i}(f)(\xi) \neq 0$ for some $\xi \in X$. Note that $\{\xi^s_{n,1}, \xi^s_{n,2}, \cdots, \xi^s_{n,r(k,i,j)}\}$ is $2\pi/n$-dense in $\mathbb{T}$ for any $s \leq \gamma_n$. Therefore, for each j, we have $\pi_{n,j} \circ h_{k,n}(\pi_{k,i}(f)) \geq f(\zeta(i,j))e(i,j)$, where $\zeta_{i,j} \in \mathbb{T}$ with $f(\zeta(i,j)) > 0$ and $e(i,j)$ is a nonzero projection in $\pi_{n,j}(\mathfrak{A}_n)$. Therefore, the closed ideal generated by $f(\zeta(i,j))e(i,j)$ in $\pi_{n,j}(\mathfrak{A}_n)$ is $\pi_{n,j}(\mathfrak{A}_n)$ itself. Since this holds for each j, the closed ideal generated by $h_{k,n}(f)$ in $\mathfrak{A}_n$ is $\mathfrak{A}_n$ itself. Thus, in both cases, if $h_{k,\infty}(f) \in \mathfrak{I}$, then $\mathfrak{I} \supset h_{n,\infty}(\mathfrak{A}_n) \supset h_{k,\infty}(\mathfrak{A}_k)$. This implies that $\mathfrak{I} = \mathfrak{A}$. Thus $\mathfrak{I} \cap h_{k,\infty}(\mathfrak{A}_k) = \{0\}$ for all k. It follows that $\mathfrak{I} = \{0\}$. Hence $\mathfrak{A}$ is simple.

It remains to show that $\mathfrak{A}$ has real rank zero. We show that $\mathfrak{A}$ has tracial rank zero. Since $\mathfrak{A}$ is an AT algebra, $\mathfrak{A}$ has tracial rank ≤ 1. It follows that $\mathfrak{A}$ satisfies the property (SP), that is, any hereditary C^*-subalgebra of $\mathfrak{A}$ has a nonzero projection. Thus, for any nonzero positive element $a \in \mathfrak{A}$, there is a nonzero projection $e \in a\mathfrak{A}a$. So $[e] \leq [a]$. Therefore, to show that $\mathrm{TR}(\mathfrak{A}) = 0$, it suffices to show the following: for any $\varepsilon > 0$, finite subset F of $\mathfrak{A}$ and nonzero projection $e \in \mathfrak{A}$, there is a finite dimensional C^*-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ with $1_{\mathfrak{B}} = p$ such that $\|px - xp\| < \varepsilon$, $pxp \in_\varepsilon \mathfrak{B}$ for all $x \in F$ and $\tau(1-p) \leq \tau(e)$ for all tracial states $\tau \in T(\mathfrak{A})$ on $\mathfrak{A}$.

Without loss of generality, we may assume that $F \subset h_{1,\infty}(\mathfrak{A}_1)$. For any $k > 1$, there is a projection $p_k \in \mathfrak{A}_k$ such that

$$h_{1,k}(x) = (1-p_k)h_{1,k}(x)(1-p_k) \oplus p_k h_{1,k}(x) p_k$$

for all $x \in \mathfrak{A}_1$, $p_k(h_{1,k}(\mathfrak{A}_1))p_k$ is a finite dimensional C^*-algebra and $\tau(1-p_k) < 1/k$ for all $\tau \in T(\mathfrak{A}_k)$. Since for any trace $\tau \in T(\mathfrak{A})$, $\tau \circ h_{k,\infty}$ is a normalized trace on $\mathfrak{A}_k$, $\tau(h_{k,\infty}(p_k)) < 1/k$. On the other hand, since $\mathfrak{A}$ is simple,

$$\inf\{\tau(e) \mid \tau \in T(\mathfrak{A})\} > 0.$$

Therefore, for large k, $\tau(h_{k,\infty}(p_k)) < (1/2)\tau(e)$ for all $\tau \in T(\mathfrak{A})$. Choose $p = h_{k,\infty}(p_k)$. □

Remark. A C^*-algebra $\mathfrak{A}$ has real rank zero (denoted by $\mathrm{RR}(\mathfrak{A}) = 0$) if the set of all invertible self-adjoint elements of $\mathfrak{A}$ is dense in the set of all self-adjoint elements of $\mathfrak{A}$. This is equivelent to that any self-adjoint element of $\mathfrak{A}$ is approximated by self-adjoint elements of $\mathfrak{A}$ with finite spectrums

(FS property). This is also equivalent to that any non-zero hereditary C^*-subalgebra of $\mathfrak{A}$ has an approximate identity of projections (HP property). See [3] for the proofs. Furthermore, that is also equivalent to that any unitary of $\mathfrak{A}$ (unital) connecting with the identify in its unitary group is approximated by unitaries connecting with the identify in its unitary group and with finite spectrums (weak FU property).

A C^*-algebra $\mathfrak{A}$ has stable rank one (denoted by $\mathrm{sr}(\mathfrak{A}) = 1$) if the set of all invertible elements of $\mathfrak{A}$ is dense in $\mathfrak{A}$. If a separable C^*-algebra has stable rank one, then its hereditary C^*-subalgebra also has stable rank one, which follows from that a σ-unital hereditary C^*-subalgebra $\mathfrak{B}$ of a σ-unital C^*-algebra is stably isomorphic to the closed ideal generated by $\mathfrak{B}$, and that a C^*-algebra $\mathfrak{C}$ has stable rank one if and only if $\mathfrak{C} \otimes \mathbb{K}$ does.

A unital simple C^*-algebra $\mathfrak{A}$ has tracial rank $\leq k$ (denoted by $\mathrm{TR}(\mathfrak{A}) \leq k$) if for any $\varepsilon > 0$, any finite subset F of $\mathfrak{A}$ and any nonzero positive element a of $\mathfrak{A}$, there exists a nonzero projection p of $\mathfrak{A}$ and a C^*-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ that is isomorphic to $\otimes_{j=1}^{s} M_{n_j}(\mathbb{C}) \otimes C(X)$ for X a k-dimensional finite CW complex, and $1_{\mathfrak{B}} = p$ such that $\|px - xp\| < \varepsilon$ for $x \in F$, and for any $x \in F$, there exists $b \in \mathfrak{B}$ with $\|pxp - b\| < \varepsilon$, and $1 - p$ is equivalent to a projection of the hereditary C^*-subalgebra $a\mathfrak{A}a$. In particular, a unital simple C^*-algebra $\mathfrak{A}$ has tracial rank zero ($\mathrm{TR}(\mathfrak{A}) = 0$) if we can take $k = 0$ in the sense above. The standard picture is:

$$F \subset_{\varepsilon} \begin{pmatrix} (1-p)\mathfrak{A}(1-p) & \\ & \mathfrak{B} \end{pmatrix} \quad \text{with } 1 - p \text{ sufficiently small,}$$

where $\subset_{\varepsilon}$ means that F is contained in a ε-neighborhood of the right hand side.

For $\mathfrak{A}$ an inductive limit of C^*-algebras $\mathfrak{A}_n$, if $\mathrm{RR}(\mathfrak{A}_n) \leq k$, $\mathrm{sr}(\mathfrak{A}_n) \leq k$, and $\mathrm{TR}(\mathfrak{A}_n) \leq k$ for all n, then $\mathrm{RR}(\mathfrak{A}) \leq k$, $\mathrm{sr}(\mathfrak{A}) \leq k$, and $\mathrm{TR}(\mathfrak{A}) \leq k$ respectively (see Brown-Pedersen [3] and Rieffel [12] for the definitions of higher real and stable ranks for C^*-algebras). In particular, all AF algebras have real rank zero, stable rank one and tracial rank zero since all finite dimensional C^*-algebras do have.

Theorem 4.12 [6] *A unital simple AT algebra $\mathfrak{A}$ has real rank zero if and only if its projections separate its trace, i.e., for any different normalized traces τ_1, τ_2 of $\mathfrak{A}$, there exists a projection p of $\mathfrak{A}$ such that $\tau_1(p) \neq \tau_2(p)$.*

Also, this is equivalent to that $\mathfrak{A}$ has tracial rank zero.

Sketch of Proof. If $\mathfrak{A}$ has real rank zero, any self-adjoint element of $\mathfrak{A}$ is in the closed real linear span of projections of $\mathfrak{A}$. Thus, the span of projections of $\mathfrak{A}$ is dense in $\mathfrak{A}$. Hence the projections of $\mathfrak{A}$ separate traces of $\mathfrak{A}$.

Write $\mathfrak{A} = \lim_{n\to\infty}(\mathfrak{A}_n, \varphi_n)$, where $\mathfrak{A}_n = \oplus_{j=1}^{r_n} C(X_{n,j}, M_{n(j)}(\mathbb{C}))$, where $X_{n,j}$ is a compact connected subset of $\mathbb{T}$.

Claim: if $x \in (\mathfrak{A}_n)_+$ and $\varepsilon > 0$, there is an integer $m_0 > n$ such that

$$\sup_{s,t\in X_{n,j}} (1/n(j))|\mathrm{Tr}(\varphi_{n,m}^{(j)}(x)(s)) - \mathrm{Tr}(\varphi_{n,m}^{(j)}(x)(t)| < \varepsilon$$

for all j and $m \geq m_0$.

Suppose, to the contrary, that there is $x \in (\mathfrak{A}_n)_+$ and $\varepsilon_0 > 0$ so that for every $m > n$, there is a summand $C(X_{m,j}, M_{m(j)}(\mathbb{C}))$ of $\mathfrak{A}_m$ and points s_m and t_m in $X_{m,j}$ such that

$$(1/m(j))|\mathrm{Tr}(\varphi_{n,m}^{(j)}(x)(s_m)) - \mathrm{Tr}(\varphi_{n,m}^{(j)}(x)(t_m)| \geq \varepsilon_0.$$

Let σ_m be any state of $\mathfrak{A}$ extending the trace $\sigma_m(a) = (1/m(j))\mathrm{Tr}(a(s_m))$ on $\mathfrak{A}_m$, and let τ_m be a state extending the trace $\tau_m(a) = (1/m(j))\mathrm{Tr}(a(t_m))$. Choose a subset S of $\mathbb{N}$ such that $\sigma = \lim_S \sigma_m$ and $\tau = \lim_S \tau_m$ exist as weak-$*$ limits. These are easily seen to be traces on $\mathfrak{A}$ such that $|\sigma(x) - \tau(x)| > \varepsilon_0$. Hence $\sigma \neq \tau$.

Now suppose that $p \in \mathfrak{A}$ is a projection. Then p is the limit of projections $\varphi_{n,\infty}(p_n) \in \mathfrak{A}_n$. For any $m > n$, $\varphi_{n,m}(p_n)$ is a projection in $\mathfrak{A}_m$, and thus has a constant rank on each summand. As the trace of projections in $M_k(\mathbb{C})$ depends only on rank, it follows that $\sigma_m(\varphi_{n,m}(p_n)) = \tau_m(\varphi_{n,m}(p_n))$. Therefore, σ and τ agree on each $\varphi_{n,\infty}(p_n)$, and hence on every projections in $\mathfrak{A}$. This contradicts the hypothesis that projections separate traces.

Let $a_i \in (\mathfrak{A}_n)_+$ $(i = 1, \cdots, 6)$ such that $a_i a_{i+1} = a_i$ for $1 \leq i \leq 5$. If $\delta \in (0,1)$ is not in the spectrum $\mathrm{sp}(a_4)$ of a_4, then $p = \chi_{[\delta,\infty)}(a_4)$ is a projection of $\mathfrak{A}_n$ such that $a_1 p = a_1$ and $a_5 p = p$, where $\chi_{[\delta,\infty)}$ is the characteristic function on $[\delta, \infty)$. If $(0,1) \cap \mathrm{sp}(a_4) \neq \emptyset$ and $f(t) = \max\{t(1-t), 0\}$, $b = f(a_4)$ is nonzero, then $ba_2 = a_2 b = 0$ and $a_2(a_5 - b) = a_2$. Let $\delta > 0$ and $m_0 \geq n$ for $b \in (\mathfrak{A}_n)_+$. Thus, if $m_0 \geq n$, then $(1/m(j))\mathrm{Tr}(\varphi_{n,m}^{(j)}(b)(s)) \geq \delta$ for all $s \in X_{n,j}$, j and $m \geq m_0$. By applying the claim above and by choosing m_0 larger if necessary, we may assume that

$$\sup_{s,t\in X_{m,j}} (1/m(j))|\mathrm{Tr}(\varphi_{n,m}^{(j)}(a_5 - b)(t)) - \mathrm{Tr}(\varphi_{n,m}^{(j)}(a_5 - b)(s)| < \delta/2$$

for all j and $m \geq m_0$. Hence, for any $s, t \in X_{m,j}$,

$$\begin{aligned}
\mathrm{rank}(\varphi_{n,m}^{(j)}(a_2)(s)) &\leq \mathrm{Tr}(\varphi_{n,m}^{(j)}(a_5 - b)(s)) \\
&\leq \mathrm{Tr}(\varphi_{n,m}^{(j)}(a_5 - b)(t)) + m(j)\delta/2 \\
&\leq \mathrm{Tr}(\varphi_{n,m}^{(j)}(a_5 - b)(t)) + \mathrm{Tr}(\varphi_{n,m}^{(j)}(b)(t)) \\
&\leq \mathrm{Tr}(\varphi_{n,m}^{(j)}(a_5)(t)) \leq \mathrm{rank}(\varphi_{n,m}^{(j)}(a_5)(t)).
\end{aligned}$$

Let $k' = \max_s\{\mathrm{rank}(\varphi^{(j)}_{n,m}(a_2)(s))\}$ and $k = \max_s\{\mathrm{rank}(\varphi^{(j)}_{n,m}(a_5)(s))\}$. Then $k - k' \geq 1$. It follows that there is a projection $p \in \mathfrak{A}_m$ such that $a_3 p = a_3$ and $pa_6 = p$. It follows that $\mathfrak{A}$ has real rank zero.

Now we may assume that $\mathfrak{A}$ is the closure of the union of $\mathfrak{A}_n$, where $\mathfrak{A}_n \subset \mathfrak{A}_{n+1}$, $\mathfrak{A}_n \cong \oplus_{j=1}^{k(n)} \mathfrak{B}_{j,n}$, where $\mathfrak{B}_{j,n} = M_{s(j,n)}(C(X_{j,n}))$ and $X_{j,n}$ is a connected compact subset of $\mathbb{T}$. Let $e \in \mathfrak{A}$ be a nonzero projection and $\varepsilon > 0$.

Fix a finite subset F of $\mathfrak{A}$. Without loss of generality, we may assume that $F \subset \mathfrak{A}_n$ for some n. Then there are $k(n)$ mutually orthogonal projections $e_1, \cdots, e_{k(n)} \in e_0 \mathfrak{A} e_0$, where $e_0 \leq e$ and $r(n)[e_0] \leq [e]$ for $r(n) = \max\{s(j,n) : 1 \leq j \leq k(n)\}$.

Let $e^{j,n} \in M_{s(j,n)}(C(X_{j,n}))$ be a constant rank one projection. We identify $e^{j,n}\mathfrak{B}_{j,n}e^{j,n}$ with $C(X_{j,n})$. Let $z_{j,n}$ be the identity function in $C(X_{j,n})$ which is the standard unitary generator of $C(X_{j,n})$. We assume that $z_{j,n}$ is a unitary of $e^{j,n}\mathfrak{A}e^{j,n}$. Let $\delta > 0$. Then we obtain a projection $p_j \leq e^{j,n}$ such that $[p_j] \leq [e_1]$ and a unitary $v_{j,n}$ of $p_j \mathfrak{A} p_j$ and a unitary $u_{j,n}$ of $(e^{j,n} - p_j)\mathfrak{A}(e^{j,n} - p_j)$ in which it is connecting with the identity such that $z_{j,n} \approx_{\delta/2} v_{j,n} + u_{j,n}$. It follows that there is a finite dimensional C^*-subalgebra $C_{j,n}$ of $(e^{j,n} - p_j)\mathfrak{A}(e^{j,n} - p_j)$ such that $u_{j,n} \in_{\delta/2} C_{j,n}$. We also have

$$\|p_j z_{j,n} - z_{j,n} p_j\| < \delta, \quad \text{and} \quad p_j z_{j,n} p_j \in_\delta C_{j,n}.$$

Set $C'_{j,n} = M_{s(j,n)}(C_{j,n})$, $p'_j = \mathrm{diag}(p_j, \cdots, p_j)$ (p_j repeats n times) and we view p'_j as a projection of $M_{s(j,n)}(e^{j,n}\mathfrak{A}e^{j,n}) \subset 1_{\mathfrak{B}_{j,n}}\mathfrak{A}1_{\mathfrak{B}_{j,n}}$. With $z_{j,n}$ and with a set of matrix units generate $\mathfrak{B}_{j,n}$, with sufficiently small δ, we have $\|p'_j b - b p'_j\| < \varepsilon$ and $p'_j b p'_j \in_\varepsilon C'_{j,n}$ for all $b \in 1_{\mathfrak{B}_{j,n}} F 1_{\mathfrak{B}_{j,n}}$ and $[p'_j] \leq r(n)[e_j]$. We may assume that $1_{\mathfrak{B}_{j,n}} b = b 1_{\mathfrak{B}_{j,n}}$. Finally, set $p = \sum_{j=1}^{k(n)} p'_j$ and $C = \sum_{j=1}^{k(n)} C'_{j,n}$. Then $\|pb - bp\| < \varepsilon$ and $pbp \in_\varepsilon C$ for all $b \in F$ and $[p] \leq r(n)[e_0] \leq [e]$. Therefore, $\mathfrak{A}$ has tracial rank zero. □

Remark. Let $\mathfrak{A} = \varinjlim \mathfrak{A}_n$ be an inductive limit of C^*-algebras $\mathfrak{A}_n$. If for any n and $a, b, c, d, e, f \in (\mathfrak{A}_n)_+$ with $ab = a$, $bc = b$, $dc = c$, $ed = d$ and $fe = e$, there is a projection $p \in \mathfrak{A}$ such that $ap = a$ and $cp = p$, then $\mathfrak{A}$ has real rank zero.

Corollary 4.13 *A unital simple AT algebra with the unique normalized trace has real rank zero and tracial rank zero.*

AH algebras

An AH algebra (or an approximately homogeneous C^*-algebra) is an inductive limit of finite direct sums of matrix algebras over commutative C^*-algebras (or an inductive limit of homogeneous C^*-algebras).

Let I^n be the n-dimensional unit cube. Fix $k > 0$. A $1/k$-frame Ω is defined to be a closed subset of I^n which is a union of hyperplanes intersecting with I^n such that

$$\Omega = (X_1 \times I^{n-1}) \cup (I \times X_2 \times I^{n-2}) \cup \cdots \cup (I^{n-1} \times X_n)$$

where $X_i = \{t_j^i\}_{j=0}^{k+2}$ with $t_0^i = 0$ and $t_{k+2}^i = 1$ such that $1/2k < |t_j^i - t_{j+1}^i| < 1/k$ for $0 \leq j \leq k+1$. Let $d > 0$. Define

$$\Omega^d = \{x \in I^n \mid \mathrm{dist}(x, \Omega) \leq d\}.$$

A $1/k$-frame is called standard if each $X_i = \{0, 1/(k+1), 2/(k+1), \cdots, 1\}$.

Lemma 4.14 [6] *Fix $k > 0$, $1/8k > d > 0$ and a $1/k$-frame Ω of I^n. Let $N > \max\{(8k)^2, 8/d\}$ be an integer. There exist $\eta > 0$ and finitely many $(1/k)$-frames $\{\Omega_j\}_{j=1}^L$ such that $\Omega_j^{4\eta} \subset \Omega^{d/2}$ for $1 \leq j \leq L$, and for any $1/k$-frame Ω_0 with $\Omega_0^{4/N} \subset \Omega^{d/2}$, there exists Ω_j $(1 \leq j \leq L)$ such that $\Omega_j^\eta \subset \Omega_0^{1/N}$.*

Lemma 4.15 [6] *Fix $k > 0$, $1/8k > d > 0$ and a $1/k$-frame Ω of I^n. For any $\sigma > 0$, there exists an integer $N > \max\{(8k)^2, 8/d\}$ such that for any normalized Borel measure μ on I^n, there exists a $(1/k)$-frame Ω_0 such that $\Omega_0^{2/N} \subset \Omega^{d/2}$ and $\mu(\Omega_0^{1/N}) < \sigma/2$.*

Using the lemmas above we obtain

Theorem 4.16 [6] *Let $\mathfrak{A}$ be a simple AH algebra $\mathfrak{A}$ with real rank zero and with the property that if $\tau(p) < \tau(q)$ for any projections p, q of $M_n(\mathfrak{A})$ (for any n) and for any trace τ of $M_n(\mathfrak{A})$, then p is equivalent to a subprojection of q. Then $\mathfrak{A}$ has tracial rank zero.*

Sketch of Proof. Let $\mathfrak{A} = \lim_n(\mathfrak{A}_n, \varphi_n)$ as an AH algebra with φ_n connecting maps. Fix a finite subset F of $\mathfrak{A}$, $\varepsilon > 0$ and a nonzero positive element $a \in \mathfrak{A}$. Without loss of generality, we may assume that there exists a finite subset G of $\mathfrak{A}_1$ such that $\varphi_{1,\infty}(G) = F$. We may further assume that F, G are in the unit balls of $\mathfrak{A}$, $\mathfrak{A}_1$ respectively. We show that there exists a finite dimensional C^*-subalgebra C of $\mathfrak{A}$ with $1_C = p$ such that (1): $\|px - xp\| < \varepsilon$ and (2): $pxp \in_\varepsilon C$ for all $x \in F$ and (3): $1 - p$ is equivalent to a projection of $a\mathfrak{A}a$.

Since $\mathfrak{A}$ has real rank zero, there exists a nonzero projection $q \in a\mathfrak{A}a$. Thus, we may replace a with the projection q, and we may replace the condition (3) with: for any trace τ of $\mathfrak{A}$, $\tau(1 - p) < \sigma$ for $\sigma > 0$ given.

Let $\{p_i\}_{j=1}^m$ be central projections of $\mathfrak{A}_1$. Considering $\varphi_{1,\infty}(p_i)\mathfrak{A}\varphi_{1,\infty}(p_i)$ we can reduce the general case to the case where $\mathfrak{A}_1 = PM_l(C(X))P$ where X is a connected finite CW complex and P is a projection.

We first consider the case $\mathfrak{A}_1 = M_l(C(X))$. If the conditions (1), (2) and (3) can be established for the case $\mathfrak{A}_1 = C(X)$, then they can be established for the case $\mathfrak{A}_1 = M_l(C(X))$. Thus, we consider the case $\mathfrak{A}_1 = C(X)$. Then there exists an integer n such that $X \subset I^n$ the n-dimensional unit cube. Then there exists a surjective map $\psi : C(I^n) \to C(X)$. Thus we may assume that $G \subset C(I^n)$.

Now choose $\delta > 0$ for $\varepsilon/2$ and G with $X = I^n$.

We apply the lemmas above. Fix $k > 0$ so that $1/k < \delta/4$, $d = 1/(8k+1)$ and a standard $1/k$-frame Ω of I^n. Choose N for k, d and σ. Let $\eta > 0$ and $\{\Omega_j\}_{j=1}^L$ be finitely many $1/k$-frames on I^n.

Let $f_i \in C(I^n)$ such that $0 \leq f_i \leq 1$, $f_i(t) = 1$ on $\Omega_i^{\eta/2}$ and $f_i(t) = 0$ on $X \setminus \Omega_i^\eta$ for $1 \leq i \leq L$. Since $\mathfrak{A}$ has real rank zero, there are mutually orthogonal projections $\{p_{i,1}, \cdots, p_{i,l(i)}\} \subset \mathfrak{A}$ such that

$$\|\varphi_{1,\infty}(f_i) - \sum_{j=1}^{l(i)} \lambda(i,j)p_{i,j}\| < \sigma/16$$

for $1 \leq i \leq L$, where $0 \leq \lambda(i,j) \leq 1$ are positive mumbers. Choosing a large number m, for each i there are mutually orthogonal projections $\{q_{i,1}, \cdots, q_{i,l(i)}\} \subset \mathfrak{A}_m$ such that

$$\|\varphi_{1,\infty}(f_i) - \sum_{j=1}^{l(i)} \lambda(i,j)\varphi_{m,\infty}(q_{i,j})\| < \sigma/8$$

for $1 \leq i \leq L$. It follows that for m large, we may assume that

$$\|\varphi_{1,\infty}(f_i) - \sum_{j=1}^{l(i)} \lambda(i,j)q_{i,j}\| < \sigma/4$$

for $1 \leq i \leq L$.

We may assume that $\mathfrak{A}_m = PM_K(C(Y))P$, where P is a projection of $M_K(C(Y))$ and Y is a disjoint union of $\{Y_j\}_{j=1}^J$ for Y_j a connected finite CW complex $(1 \leq j \leq L)$. Let $y_j \in Y_j$ and $\tau_j = \mathrm{tr} \circ p_{y_j} \circ \varphi_{1,m}$ for $1 \leq j \leq J$, where tr is the normalized trace on $M_{s(j)}(\mathbb{C})$ for $s(j)$ the rank of $P_j = P|_{Y_j}$, and p_{y_j} is the point-evaluation at y_j. Let μ_j be the normalized measure induced by τ_j.

We now fix j. Let $\mathfrak{B}_j = \varphi_{m,\infty}(P_j)\mathfrak{A}\varphi_{m,\infty}(P_j)$. Then there exists a $1/k$-frame $\Omega_{0,j}$ such that $\mu_j(\Omega_{0,j}^{1/N}) < \sigma/2$ and $\Omega_{0,j}^{2/N} \subset \Omega^{d/2}$. Therefore, $\Omega_{i(j)}^{\eta} \subset \Omega_{0,j}^{1/N}$ for some $i(j)$. It follows that $\tau_j(f_{i(j)}) < \sigma/2$. Let $\tau_\xi = \mathrm{tr} \circ p_\xi \circ \varphi_{1,m}$, where ξ is any other point of Y_j. Since Y_j is connected, $\tau_\xi(q_{i,j}) = \tau_j(q_{i,j})$. This implies that $\tau_\xi(f_{i(j)}) < \sigma$ for $\xi \in Y_j$. Since for any tracial state $\tau_{\mathfrak{B}_j}$ on $\mathfrak{B}_j$, there is a normalized measure μ on Y_j such that $\tau_{\mathfrak{B}_j}(f_{i(j)}) = \int_{Y_j} \tau_\xi(f_{i(j)})d\mu$, we conclude that $\tau_{\mathfrak{B}_j}(f_{i(j)}) < \sigma$.

Let $I^n \setminus \Omega^{\eta/16} = \cup_{i=1}^K U_i$, where U_i are mutually disjoint open subsets of I^n with diameter $< \delta$. Let $h_i, h_i' \in C_0(U_i) \subset C(I^n)$ such that $0 \leq h_i, h_i' \leq 1$, $h_i(t) = 1$ on $U_i \cap (I^n \setminus \Omega^{\eta/2})$ and $h_i(t) = 0$ on $\Omega^{\eta/4}$, and $h_i'(t) = 1$ on $U_i \cap (I^n \setminus \Omega^{\eta/4})$ and $h_i'(t) = 0$ on $\Omega^{\eta/8}$. Note that $h_i h_i' = h_i$. Then it follows that since $\mathfrak{B}_i$ has real rank zero, there is a projection $e_i \in \varphi_{1,\infty}(h_i')\mathfrak{B}_i\varphi_{1,\infty}(h_i')$ such that $e_i\varphi_{i,\infty}(h_i) = \varphi_{i,\infty}(h_i)$. Let $Q_j = \sum_{i=1}^K e_i$. Then $1_{\mathfrak{B}_j} - Q_j \leq \varphi_{1,\infty}(f_{i(j)})$.

It follows that $\|Q_j x - xQ_j\| < \varepsilon$, $Q_j x Q_j \in_\varepsilon C_j$ for $x \in \varphi_{1,\infty}(G)$, where C_j is the finite dimensional C^*-algebra genrated by $e_1, \cdots, e_K$, and $\tau(1_{\mathfrak{B}_j} - Q_j) \leq \sigma\tau(1_{\mathfrak{B}_j})$ for all tracial states τ on $\mathfrak{A}$.

Applying this to each j, we obtain a finite dimensional C^*-subalgebra C of $\mathfrak{A}$ with $1_C = p$ such that $\|pz - zp\| < \varepsilon$, $pzp \in_\varepsilon C$ for all $z \in \varphi_{1,\infty}(G)$, and $\tau(1 - p) < \sigma$ for all tracial states τ on $\mathfrak{A}$.

Now we consider the case $\mathfrak{A}_1 = PM_l(C(X))P$. Then there is a positive integer K and a projection $Q \in M_K(PM_l(C(X))P)$ such that

$$QM_K(PM_l(C(X))P)Q \cong M_L(C(X))$$

for some L. Let $e = 1_{\mathfrak{A}_1}$ be identified with a projection of $M_L(C(X))$. Let $F_1 = \{e\} \cup F$. If the three conditions can be established for the case $\mathfrak{A}_1 = M_L(C(X))$, then they can be established for $\varphi_{1,\infty}(Q)M_K(\mathfrak{A})\varphi_{1,\infty}(Q)$ for F_1 and $\varepsilon/32 < 1/64$. In particular, $\|pe - ep\| < \varepsilon/32$ and $pep \in_{\varepsilon/32} C$. Thus there is a projection p' of $e\varphi_{1,\infty}(Q)M_K(\mathfrak{A})\varphi_{1,\infty}(Q)e = \mathfrak{A}$ such that $\|p' - pep\| < \varepsilon/16$. There is a projection $q \in C$ such that $\|q - p'\| < \varepsilon/8$. There is a unitary $u \in \mathfrak{A}$ such that $\|u - 1\| < \varepsilon/4$ with $u^*qu = p'$. Set $C_1 = u^*(qCq)u$. Then C_1 is a finite dimensional C^*-subalgebra and $1_{C_1} = p'$. Moreover, since $\|(e - p') - (e - pep)\| < \varepsilon/16$, $e - p'$ is equivalent to a subprojection of $1 - p$. Now we have that $\|p'x - xp'\| < \varepsilon/4$, $p'xp' \in_{\varepsilon/2} C_1$, and $\tau(e - p') < \sigma$ for all tracial states τ on $\mathfrak{A}$. □

Corollary 4.17 [6] *A unital simple AH algebra with real rank zero and slow dimension growth has tracial rank zero.*

Theorem 4.18 [6] *Let $\mathfrak{A}$ be a separable unital simple C^*-algebra. For any $\varepsilon > 0$ and finite subset F of $\mathfrak{A}$, there exist $\delta > 0$, a finite subset G of $\mathfrak{A}$,* *a finite subset P of $P(\mathfrak{A})$ and an integer $n > 0$ satisfying the following: for any simple unital C^*-algebra $\mathfrak{B}$ with real rank zero and stable rank one and weakly unperforated $K_0(\mathfrak{B})$, if $\varphi, \psi, \sigma : \mathfrak{A} \to \mathfrak{B}$ are (G, δ) multiplicative contractive completely positive linear maps such that $[\varphi]|_P = [\psi]|_P$ and σ is unital, then there is a unitary u of $M_{n+1}(\mathfrak{B})$ such that*

$$\|u^* \mathrm{diag}(\varphi(a), \sigma(a), \cdots, \sigma(a))u - \mathrm{diag}(\psi(a), \sigma(a), \cdots, \sigma(a))\| < \varepsilon$$

for all $a \in F$, where $\sigma(a)$ is repeated n times.

Theorem 4.19 [6] *Let $\mathfrak{A}$ be a separable unital amenable simple C^*-algebra with tracial rank zero and with the UCT. For any $\varepsilon > 0$ and finite subset F of $\mathfrak{A}$, there exist $\delta > 0$, a finite subset G of $\mathfrak{A}$, a finite subset P of $P(\mathfrak{A})$ and an integer $n > 0$ satisfying the following: for any unital simple C^*-algebra $\mathfrak{B}$ with real rank zero and stable rank one and weakly unperforated $K_0(\mathfrak{B})$, if $L_1, L_2 : \mathfrak{A} \to \mathfrak{B}$ are (G, δ) multiplicative contractive completely positive linear maps such that $[L_1]|_P = [L_2]|_P$, then there is a unitary u of $\mathfrak{B}$ such that*

$$\mathrm{Ad}(u) \circ L_1 \approx_\varepsilon L_2 \quad \text{on } F.$$

Theorem 4.20 [6] *Let $\mathfrak{A}$, $\mathfrak{B}$ be unital separable simple C^*-algebras with $\mathrm{TR}(\mathfrak{A}) = 0$ and $\mathrm{TR}(\mathfrak{B}) = 0$ and with the UCT such that*

$$(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, [1_\mathfrak{A}], K_1(\mathfrak{A})) \cong (K_0(\mathfrak{B}), K_0(\mathfrak{B})_+, [1_\mathfrak{B}], K_1(\mathfrak{B})).$$

Let $\alpha \in KL(\mathfrak{A}, \mathfrak{B})$ and $\beta \in KL(\mathfrak{B}, \mathfrak{A})$ be elements which corresponds to the isomorphism above. If for any finite subsets $P \subset P(\mathfrak{A})$ and $G \subset P(\mathfrak{B})$, there exist sequences of contractive completely positive linear maps $L_n : \mathfrak{A} \to \mathfrak{B}$ and $\Lambda_n : \mathfrak{B} \to \mathfrak{A}$ such that

$$\begin{aligned}
&\|L_n(ab) - L_n(a)L_n(b)\| \to 0, \\
&\|\Lambda_n(cd) - \Lambda_n(c)\Lambda_n(d)\| \to 0, \\
&[L_n]|_P = \alpha|_P, \quad \text{and} \quad [\Lambda_n]|_P = \beta|_P
\end{aligned}$$

for $a, b \in \mathfrak{A}$ and $c, d \in \mathfrak{B}$, then $\mathfrak{A} \cong \mathfrak{B}$.

Sketch of Proof. Since $\mathfrak{A}$, $\mathfrak{B}$ satisfy the UCT, there are $\alpha \in KL(\mathfrak{A}, \mathfrak{B})$ and $\beta \in KL(\mathfrak{B}, \mathfrak{A})$ such that $\alpha \circ \beta = [\mathrm{id}_\mathfrak{B}]$ and $\beta \circ \alpha = [\mathrm{id}_\mathfrak{A}]$. Let $\{F_n\}$, $\{F'_n\}$ be increasing sequences of finite subsets of the unit ball of $\mathfrak{A}$, $\mathfrak{B}$ respectively. Let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers such that $\sum_{n=1}^\infty \varepsilon_n < \infty$. Let $\delta_1 = \delta(\varepsilon_1, F_1)$, $G_1 = G(\varepsilon_1, F_1)$ and $P_1 = P(\varepsilon_1, F_1)$. We may assume that $\delta_1 < \varepsilon_1/2$ and $G_1 \supset F_1$. By assumption, there exists a (G_1, δ_1) multiplicative contractive completely positive linear map $L_1 : \mathfrak{A} \to \mathfrak{B}$ such that $[L_1]|_{P_1} = \alpha|_{P_1}$. Let $\eta_1 = \eta(\delta_1/2, F_1' \cup L_1(G_1))$, $G_1' = G(\delta_1/2, F_1' \cup L_1(G_1))$ and $P_1' = P(\delta_1/2, F_1' \cup L_1(G_1))$. By assumption, we obtain a (G_1', η_1) multiplicative contractive completely positive linear map $\Lambda_n' : \mathfrak{B} \to \mathfrak{A}$ such that $[\Lambda_n']|_{P_2 \cup [L_1](P_1)} = \beta|_{P_2 \cup [L_1](P_1)}$. Therefore,

$$[\Lambda_1' \circ L_1]|_{P_2 \cup [L_1](P_1)} = [\mathrm{id}_{\mathfrak{A}}]|_{P_2 \cup [L_1](P_1)}.$$

It follows that there exists a unitary $u_1 \in \mathfrak{A}$ such that

$$\mathrm{id}_{\mathfrak{A}} \approx_{\varepsilon_1} \mathrm{Ad}(u_1) \circ \Lambda_1' \circ L_1$$

on F_1. Define $\Lambda_n = \mathrm{Ad}(u_1) \circ \Lambda_n'$. We may assume that $\eta_1 < \varepsilon/2$. We may also assume without loss of generality that $F_2 \supset \Lambda_n(G_2')$ and $\varepsilon_2 < \min\{\varepsilon/2, \eta_1/2\}$. Let $\delta_2 = \delta(\varepsilon_2/2, F_2)$, $G_2 = G(\varepsilon_2/2, F_2)$ and $P_2 = P(\varepsilon_2/2, F_2)$. We may further assume that $[\Lambda_1](P_1') \subset P_2$. By assumption, there exists a (G_2, δ_2) multiplicative contractive completely positive linear map $L_2' : \mathfrak{A} \to \mathfrak{B}$ such that $[L_2']|_{P_2} = \alpha|_{P_2}$. Thus,

$$[L_2' \circ \Lambda_1]|_{P_2} = [\mathrm{id}_{\mathfrak{B}}]|_{P_2}.$$

It follows that there exists a unitary $v_2 \in \mathfrak{B}$ such that

$$\mathrm{id}_{\mathfrak{B}} \approx_{\varepsilon_2} \mathrm{Ad}(v_2) \circ \Lambda_2' \circ \Lambda_1$$

on F_1'. Therefore, we obtain the following (not necessarily commutative) diagram:

$$\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{\mathrm{id}_{\mathfrak{A}}} & \mathfrak{A} \\
\downarrow L_1 & \nearrow \Lambda_1 & \downarrow L_2 \\
\mathfrak{B} & \xrightarrow{\mathrm{id}_{\mathfrak{B}}} & \mathfrak{B}
\end{array}$$

so that the upper triangle is (F_1, ε_1) commutative and the lower triangle is (F_1', ε_1) commutative. Repeating this process we obtain the following approximately intertwining diagram:

$$\begin{array}{cccccccc}
\mathfrak{A} & \xrightarrow{\mathrm{id}_{\mathfrak{A}}} & \mathfrak{A} & \xrightarrow{\mathrm{id}_{\mathfrak{A}}} & \mathfrak{A} & \xrightarrow{\mathrm{id}_{\mathfrak{A}}} & \cdots & \mathfrak{A} \\
\downarrow L_1 & \nearrow \Lambda_1 & \downarrow L_2 & \nearrow \Lambda_2 & \downarrow L_3 & & & \\
\mathfrak{B} & \xrightarrow{\mathrm{id}_{\mathfrak{B}}} & \mathfrak{B} & \xrightarrow{\mathrm{id}_{\mathfrak{B}}} & \mathfrak{B} & \xrightarrow{\mathrm{id}_{\mathfrak{B}}} & \cdots & \mathfrak{B}.
\end{array}$$

Since L_n and Λ_n are (F_n, ε_n) multiplicative and (F_n', ε_n) multiplicative respectively, it follows that there exists an isomorphism $h_1 : \mathfrak{A} \to \mathfrak{B}$ and $h_2 : \mathfrak{B} \to \mathfrak{A}$ such that $h_1 = h_2^{-1}$. □

Using the theorem above, we obtain

Theorem 4.21 (G. Elliott and G. Gong) *Let $\mathfrak{A}$, $\mathfrak{B}$ be unital simple AH algebras with real rank zero and with slow dimension growth. Then*

$$(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, [1_{\mathfrak{A}}], K_1(\mathfrak{A})) \cong (K_0(\mathfrak{B}), K_0(\mathfrak{B})_+, [1_{\mathfrak{B}}], K_1(\mathfrak{B})).$$

if and only if $\mathfrak{A} \cong \mathfrak{B}$.

Sketch of Proof [6]. It follows that $\mathrm{TR}(\mathfrak{A}) = 0 = \mathrm{TR}(\mathfrak{B})$. □

Remark. Precisely, AH algebras in the theorem above are inductive limits of finite direct sums $\oplus_{j=1}^{s_k} p_{k,j} M_{n_j,k}(\mathbb{C}) \otimes C(X_{j,k}) p_{k,j}$ for $p_{k,j}$ projections of $M_{n_j,k}(\mathbb{C}) \otimes C(X_{j,k})$ for $X_{j,k}$ connected finite CW complexes, and they have slow dimension growth if

$$\lim_{k\to\infty} \max_j \{\dim X_{j,k}/\mathrm{rank}(p_{k,j})\} = 0.$$

A simple AH algebra with slow dimension growth has real rank zero if and only if its projections separate its traces.

Recall that an ordered abelian group (G, G_+) is said to be simple if any nonzero element of G_+ is an order unit. A simple ordered abelian group (G, G_+) is said to be weakly unperforated if for any $g \in G$ and $n \in \mathbb{N}$, $ng \in G_+ \setminus \{0\}$ implies $g \in G_+$.

Remark. For a unital simple AH algebra $\mathfrak{A}$ with real rank zero and with slow dimension growth, $(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+)$ is a simple weakly unpereforated ordered abelian group with Riesz interpolation property. Conversely, any simple weakly unpereforated ordered abelian group with Riesz interpolation property can be realized as $(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+)$ for $\mathfrak{A}$ as above. Furthermore, any countable abelian group is realized as $K_1(\mathfrak{A})$ for $\mathfrak{A}$ as above.

Theorem 4.22 (H. Lin) *Let $\mathfrak{A}$, $\mathfrak{B}$ be unital separable simple nuclear C^*-algebras with tracial rank zero and with UCT. Then*

$$(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, [1_{\mathfrak{A}}], K_1(\mathfrak{A})) \cong (K_0(\mathfrak{B}), K_0(\mathfrak{B})_+, [1_{\mathfrak{B}}], K_1(\mathfrak{B})).$$

if and only if $\mathfrak{A} \cong \mathfrak{B}$.

Remark [6]. A unital simple C^*-algebra with tracial rank zero has real rank zero and stable rank one and its K_0-group weakly unperforated and with Riesz interpolation property, and is quasidiagonal, and has fundamental comparison property, that is, if p, q are projections of $\mathfrak{A}$ such that $\tau(p) < \tau(q)$ for any tracial state τ of $\mathfrak{A}$, then p is equivalent to a subprojection of q.

Theorem 4.23 (G. Elliott and G. Gong) [6] *Let (G, G_+, g) be a countable weakly unperforated simple ordered group with the Riesz property and with an order unit g, and let H be a countable abelian group. Then there exists a simple AH algebra $\mathfrak{A} = \varinjlim \mathfrak{A}_n$, where $\mathfrak{A}_n = P_n M_{k(n)}(C(X_n))P_n$ for X_n a finite CW complex with dimension ≤ 3 such that $\mathrm{TR}(\mathfrak{A}) = 0$ and*

$$(G, G_+, g, H) \cong (K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, [1_{\mathfrak{A}}], K_1(\mathfrak{A})).$$

For C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$, the Kasparov's $KK(\mathfrak{A}, \mathfrak{B})$ in Cuntz's picture is defined to be the abelian group of homotopy equivalence classes of quasi-homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$, where a quasi-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is a pair of $*$-homomorphisms $\varphi_\pm \colon \mathfrak{A} \to M(\mathfrak{B} \otimes \mathbb{K})$ the multiplier algebra of $\mathfrak{B} \otimes \mathbb{K}$ such that $\varphi_+(a) - \varphi_-(a) \in \mathfrak{B} \otimes \mathbb{K}$ for any $a \in \mathfrak{A}$. Addition of $KK(\mathfrak{A}, \mathfrak{B})$ is defined by

$$[\varphi_\pm] + [\psi_\pm] = [\lambda_\pm]$$

where $\lambda_\pm = s_1\varphi_\pm(a)s_1^* + s_2\psi_\pm(a)s_2^*$ for $a \in \mathfrak{A}$, $[\varphi_\pm]$, $[\psi_\pm] \in KK(\mathfrak{A}, \mathfrak{B})$, where s_1, s_2 are isometries of $M(\mathfrak{B} \otimes \mathbb{K})$ such that $s_1 s_1^* + s_2 s_2^* = 1$.

Remark [14]. UCT in the theorems above means the universal coefficient theorem that states that there exists the following short exact sequence:

$$0 \to \mathrm{Ext}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) \to KK(\mathfrak{A}, \mathfrak{B}) \xrightarrow{\gamma} \oplus_{j=0,1} \mathrm{Hom}(K_j(\mathfrak{A}), K_j(\mathfrak{B})) \to 0$$

for C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$ in the class $\mathfrak{N}$ of C^*-algebras that are KK-equivalent to commutative C^*-algebras, where $K_* = K_0 \oplus K_1$, and the quotient map γ is defined by the sum of the group homomorphisms: $KK(\mathfrak{A}, \mathfrak{B}) \to \mathrm{Hom}(K_j(\mathfrak{A}), K_j(\mathfrak{B}))$ for $j = 0, 1$ that are induced from the following Kasparov products:

$$KK(\mathbb{C}, \mathfrak{A}) \times KK(\mathfrak{A}, \mathfrak{B}) \to KK(\mathbb{C}, \mathfrak{B})$$

with $KK(\mathbb{C}, \mathfrak{A}) = K_0(\mathfrak{A})$ and $KK(\mathbb{C}, \mathfrak{B}) = K_0(\mathfrak{B})$, and

$$KK(C_0(\mathbb{R}), \mathfrak{A}) \times KK(\mathfrak{A}, \mathfrak{B}) \to KK(C_0(\mathbb{R}), \mathfrak{B})$$

with $KK(C_0(\mathbb{R}), \mathfrak{A}) = K_1(\mathfrak{A})$ and $KK(C_0(\mathbb{R}), \mathfrak{B}) = K_1(\mathfrak{B})$. On the other hand, $\mathrm{Ext}(K_*(\mathfrak{A}), K_*(\mathfrak{B}))$ corresponds to the extension groups:

$$0 \to K_j(\mathfrak{B}) \to K_j(\mathfrak{E}) \to K_j(\mathfrak{A}) \to 0$$

($j = 0, 1$) that are induced from the extension: $0 \to \mathfrak{B} \otimes \mathbb{K} \to \mathfrak{E} \to \mathfrak{A} \to 0$ that corresponds to an element of $KK_1(\mathfrak{A}, \mathfrak{B})$ (that consists of stable equivalent classes of such extensions) and its six term exact sequence of K-groups:

$$\begin{array}{ccccc} K_0(\mathfrak{B}) & \longrightarrow & K_0(\mathfrak{E}) & \longrightarrow & K_0(\mathfrak{A}) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\partial} \\ K_1(\mathfrak{A}) & \longleftarrow & K_1(\mathfrak{E}) & \longleftarrow & K_1(\mathfrak{B}) \end{array}$$

where the index maps ∂: $K_j(\mathfrak{A}) \to K_{j+1}(\mathfrak{B})$ are zero (see [15]).

C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$ are KK-equivalent if $KK(\mathfrak{A}, \mathfrak{B})$ contains an invertible element in the sense that there exist $x \in KK(\mathfrak{A}, \mathfrak{B})$ and $y \in KK(\mathfrak{B}, \mathfrak{A})$ such that $x \cdot y = \mathrm{id}_{\mathfrak{A}}$ the identity map of $\mathfrak{A}$ and $y \cdot x = \mathrm{id}_{\mathfrak{B}}$ the identity map of $\mathfrak{B}$ via the following Kasparov products:

$$KK(\mathfrak{A}, \mathfrak{B}) \times KK(\mathfrak{B}, \mathfrak{A}) \to KK(\mathfrak{A}, \mathfrak{A}),$$
$$KK(\mathfrak{B}, \mathfrak{A}) \times KK(\mathfrak{A}, \mathfrak{B}) \to KK(\mathfrak{B}, \mathfrak{B}).$$

C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$ in the class $\mathfrak{N}$ are KK-equivalent if and only if $K_j(\mathfrak{A}) \cong K_j(\mathfrak{B})$ for $j = 0, 1$.

What's more, the class $\mathfrak{N}_n$ of nuclear C^*-algebras of $\mathfrak{N}$ defined above is in fact the smallest class N (the bootstrap category) of separable nuclear C^*-algebras such that (1): N contains $\mathbb{C}$, (2): N is closed under countable inductive limits, (3): N is closed under extensions and reductions to ideals and quotients by C^*-algebras of N, (4): N is closed under KK-equivalence (see [2, Section 22]). Also, $\mathfrak{N}_n$ is closed under taking crossed products of C^*-algebras of $\mathfrak{N}_n$ by $\mathbb{Z}$ or $\mathbb{R}$. It is not known whether $\mathfrak{N} = \mathfrak{N}_n$ or not.

Remark[14]. A subgroup G' of an abelian group G is pure if $nG' = nG \cap G'$ for all $n \in \mathbb{N}$. An extension: $0 \to G' \to G \to G'' \to 0$ of abelian groups is pure if G' is pure, and equivalently, if any torsion element of G'' lifts to a torsion element (with the same order) of G. Let $\mathrm{pExt}(G'', G')$ be the set of elements of $\mathrm{Ext}(G'', G')$ that are represented by pure extensions of G'' by G'. Set

$$[\mathrm{Ext}](G'', G') = \mathrm{Ext}(G'', G')/\mathrm{pExt}(G'', G').$$

Now assume that $\mathfrak{A}$ is a C^*-algebra in the class N. Then the map

$$\varepsilon : \mathrm{Ext}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) \to KK(\mathfrak{A}, \mathfrak{B})$$

is defined for any C^*-algebra $\mathfrak{B}$. Define

$$KL(\mathfrak{A}, \mathfrak{B}) = KK(\mathfrak{A}, \mathfrak{B})/\varepsilon(\mathrm{pExt}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})).$$

Then we obtain the following exact sequence:

$$0 \to [\mathrm{Ext}](K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) \xrightarrow{\varepsilon} KL(\mathfrak{A}, \mathfrak{B}) \xrightarrow{\gamma} \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to 0.$$

Moreover, it is shown that if φ, ψ: $\mathfrak{A} \to \mathfrak{B}$ are approximately unitarily quivalent $*$-homomorphisms, then $KL(\varphi) = KL(\psi)$. If $K_0(\mathfrak{A})$ and $K_1(\mathfrak{A})$ are finitely generated, then $KL(\mathfrak{A}, \mathfrak{B}) = KK(\mathfrak{A}, \mathfrak{B})$ since $\mathrm{pExt}(G'', G') = 0$ if G'' is finitely generated.

Let $\mathfrak{A}$, $\mathfrak{B}$ be C^*-algebras and φ, ψ: $\mathfrak{A} \to \mathfrak{B}$ $*$-homomorphisms. Then

$$\begin{array}{ccc} \varphi \approx_{uh} \psi & \Longrightarrow & KK(\varphi) = KK(\psi) \\ \Downarrow & & \Downarrow \\ \varphi \approx_u \psi & \Longrightarrow & K_*(\varphi) = K_*(\psi), \end{array}$$

where we say that φ and ψ are approximately unitarily equivalent (denoted by $\varphi \approx_u \psi$) if for any $\varepsilon > 0$ and finite subset F of $\mathfrak{A}$, there exists a unitary u of the multiplier algebra $M(\mathfrak{B})$ of $\mathfrak{B}$ such that $\|u\varphi(a)u^* - \psi(a)\| < \varepsilon$ for $a \in F$, and they are asymptotically unitarily equivalent (denoted by $\varphi \approx_{uh} \psi$) if there exists a norm continuous path $\{u_t\}_{t\in[0,\infty)}$ of unitaries of $M(\mathfrak{B})$ such that $\mathrm{Ad}(u_t) \circ \varphi \to \psi$ as $t \to \infty$. If $\mathfrak{A}$ is separable and if $\varphi \approx_u \psi$, then there is a sequence $\{u_n\}_{n=1}^{\infty}$ of unitaries of $M(\mathfrak{B})$ such that $\mathrm{Ad}(u_n) \circ \varphi \to \psi$ pointwise as $n \to \infty$.

ASH algebras and beyond them

We have the following inclusions:

$$\mathrm{UHF} \subset \mathrm{AF} \subset \mathrm{AT} \subset \mathrm{AH} \subset \mathrm{ASH} \subset \mathrm{AL} \subset \mathrm{AI}$$

where AI means the class of inductive limits of type I C^*-algebras.

Remark. All C^*-algebras in these classes are nuclear, but there exist nuclear C^*-algebras not contained in AI such as the Cuntz algebras O_n ($n \geq 2$) generated by n orthogonal isometries s_j with $\sum_{j=1}^{n} s_j s_j^* = 1$.

As we have mensioned in Section 1, the classes ASH and AL might be the same. The class ASH is strictly contained in the class AI. For example, non type R and type I group C^*-algebras are not ASH as given in Section 1. The vast class AI seems to be untouchable to clasify at this moment. Indeed, a complete classification theorem even for the class ASH seems to be not known (without the real rank zero condition). In fact, it is known that there exists an infinite dimensional, simple unital ASH algebra called Jiang-Su algebra that has the same K-theory as $\mathbb{C}$ (see [14]). However, with the tracial rank zero condition, we have some classification theorems by H. Lin below.

Remark. Recall that a C^*-algebra $\mathfrak{A}$ is said to be amenable (or nuclear) if the identity map on $\mathfrak{A}$ is approximated pointwise in norm by contractive completely positive linear maps through finite dimensional C^*-algebras, i.e., for any finite subset F of $\mathfrak{A}$ and $\varepsilon > 0$, there exists a finite dimensional C^*-algebra $\mathfrak{B}$, a contractive completely positive linear map $\varphi : \mathfrak{A} \to \mathfrak{B}$ and a contractive completely positive linear map $L : \mathfrak{B} \to \mathfrak{A}$ such that

$$\|a - L \circ \varphi(a)\| < \varepsilon$$

for all $a \in F$, that is,

$$\mathrm{id}_{\mathfrak{A}} \approx_\varepsilon L \circ \varphi \quad \text{on } F.$$

Commutative C^*-algebras and finite dimensional C^*-algebras are amenable. The class of amenable C^*-algebras is closed under taking hereditary C^*-subalgebras, extensions and inductive limits.

Remark. Let $\mathfrak{A}, \mathfrak{B}$ be C^*-algebras and $L : \mathfrak{A} \to \mathfrak{B}$ a linear map. If $L^{(n)} = L \otimes \mathrm{id}_{M_n(\mathbb{C})} : M_n(\mathfrak{A}) \to M_n(\mathfrak{B})$ is positive for any n, then L is said to be completely positive. Define

$$\|L\|_{cb} = \sup_n \|L^{(n)}\|.$$

If $\|L\|_{cb}$ is finite, L is completely bounded, and if $\|L\|_{cb} \leq 1$, then L is completely contractive. A homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is a contractive completely positive linear map. If $\mathfrak{B}$ is commutative, any positive linear map from a C^*-algebra $\mathfrak{A}$ to $\mathfrak{B}$ is completely positive. Also, if $\mathfrak{A}$ is commutative, any positive linear map from $\mathfrak{A}$ to a C^*-algebra $\mathfrak{B}$ is completely positive.

A C^*-algebra $\mathfrak{A}$ is said to be locally subhomogeneous (LSH) if for any ε and finite subset F of $\mathfrak{A}$, there exists a subhomogeneous C^*-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathrm{dist}(x, \mathfrak{B}) < \varepsilon$ for all $x \in F$.

Let C_n be a commutative C^*-algebra with $K_0(C_n) \cong \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$ and $K_1(C_n) \cong 0$. Define $K_j(\mathfrak{A}, \mathbb{Z}_k) = K_j(\mathfrak{A} \otimes C_k)$ for a C^*-algebra $\mathfrak{A}$. Let $P(\mathfrak{A})$ be the set of all projections of $\cup_{n=1}^\infty M_n(\mathfrak{A})$, $\cup_{n=1}^\infty M_n(\mathfrak{A} \otimes C(\mathbb{T}))$, $\cup_{n=1}^\infty M_n((\mathfrak{A} \otimes C_k)^+)$ and $\cup_{n=1}^\infty M_n((\mathfrak{A} \otimes C(\mathbb{T}) \otimes C_m)^+)$. We have the following commutative diagram:

$$\begin{array}{ccccc} K_0(\mathfrak{A}) & \longrightarrow & K_0(\mathfrak{A}, \mathbb{Z}_k) & \longrightarrow & K_1(\mathfrak{A}) \\ \uparrow & & & & \downarrow \\ K_0(\mathfrak{A}) & \longleftarrow & K_1(\mathfrak{A}, \mathbb{Z}_k) & \longleftarrow & K_1(\mathfrak{A}). \end{array}$$

Define

$$\underline{K}(\mathfrak{A}) = \oplus_{j=0,1,n\in\mathbb{Z}_+} K_j(\mathfrak{A}, \mathbb{Z}_n).$$

Let $\mathrm{Hom}_\Lambda(\underline{K}(\mathfrak{A}), \underline{K}(\mathfrak{B}))$ denote the set of all homomorphisms from $\underline{K}(\mathfrak{A})$ to $\underline{K}(\mathfrak{B})$, which respect the direct sum decomposition and the so-called

Bockstein operations. If $\mathfrak{A}$ satisfies the UCT, then $\mathrm{Hom}_\Lambda(\underline{K}(\mathfrak{A}), \underline{K}(\mathfrak{B})) = KL(\mathfrak{A}, \mathfrak{B})$.

Let G be a countable unperforated simple ordered group and T the state space of G. Let $\rho_G : G \to \mathrm{Aff}(T)$ be the map defined by $\rho_G(g)(t) = t(g)$ for $g \in G$ and $t \in T$. It is known that

$$G_+ = \{g \in G \mid \rho_G(g)(t) > 0 \quad \text{for all } t \in T\} \cup \{0\}.$$

Lemma 4.24 [7] *Let $\mathfrak{A}$ be a unital simple LSH C^*-algebra that has stable rank one and weakly unperforated $K_0(\mathfrak{A})$ and let $\mathfrak{B}$ be a unital simple AH algebra of finite direct sums of unital hereditary C^*-subalgebras of $M_n(C(X))$ (for various n) with real rank zero and with X connected finite CW complexes of dimension ≤ 3. Suppose that $\alpha \in \mathrm{Hom}_\Lambda(\underline{K}(\mathfrak{A}), \underline{K}(\mathfrak{B}))$ which gives an order isomorphism:*

$$(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, [1_{\mathfrak{A}}], K_1(\mathfrak{A})) \to (K_0(\mathfrak{B}), K_0(\mathfrak{B})_+, [1_{\mathfrak{B}}], K_1(\mathfrak{B})).$$

Then there exists a sequence of contractive completely positive linear maps $L_n : \mathfrak{A} \to \mathfrak{B}$ such that for any finite subset P of $P(\mathfrak{A})$, $[L_n]|_P = \alpha|_P$ for n sufficiently large and

$$\|L_n(ab) - L_n(a)L_n(b)\| \to 0$$

as $n \to \infty$ for all $a, b \in \mathfrak{A}$.

Sketch of Proof. Fix a finite subset F of $\mathfrak{A}$ and a finite subset P of $P(\mathfrak{A})$. Since $\mathfrak{A}$ is LSH, we may assume that there exists a subhomogeneous C^*-subalgebra $\mathfrak{A}_m$ of $\mathfrak{A}$ such that $F \subset \mathfrak{A}_m$ and $P \subset [j](G)$, where $G \subset P(\mathfrak{A}_m)$ is a finite subset and $j : \mathfrak{A}_m \to \mathfrak{A}$ is the embedding. Set $\alpha = \beta \circ [j] \in \mathrm{Hom}_\Lambda(\underline{K}(\mathfrak{A}), \underline{K}(\mathfrak{B}))_+$. Since both $\mathfrak{A}$ and $\mathfrak{B}$ are simple, and $\alpha|_{K_0(\mathfrak{A})}$ is an order isomorphism, for any $g \in K_0(\mathfrak{A}) \setminus \{0\}$, $\rho_{\mathfrak{B}} \circ \beta(g) \neq 0$, where $\rho_{\mathfrak{B}} : K_0(\mathfrak{B}) \to \mathrm{Aff}(T(\mathfrak{B}))$ is the homomorphism given by the traces of $\mathfrak{B}$, and every finitely generated subgroup of $K_0(\mathfrak{B})$ can be order-embedded into $\mathbb{Z}^k$ under $\rho_{\mathfrak{B}}$ for some k. Note that $\mathfrak{A}_m$ satisfies the UCT. It follows that there exists a sequence of contractive completely positive linear maps $\psi_n : \mathfrak{A}_m \to \mathfrak{B} \otimes \mathbb{K}$ and a homomorphism $H_n : \mathfrak{A}_m \to \mathfrak{B} \otimes \mathbb{K}$ with finite dimensional range such that

$$[\psi_n]|_G = \beta|_G + [H_n]|_G.$$

Let $G' = K_0(\mathfrak{A}_m) \cap G$, where G by the same symbol is the finitely generated subgroup generated by G. We may assume that there are projections $p_1, \cdots, p_l \in M_k(\mathfrak{A}_m)$ for some k such that G' is generated by $[p_1], \cdots, [p_l]$.

Let $G_0 = \rho_{\mathfrak{A}_m}(G')$. It follows that $G_0 \subset K_0(\pi(\mathfrak{A}_m)) \subset \mathbb{Z}^k$ for some integer $k > 0$, where $\pi : \mathfrak{A}_m \to C$ is a surjective homomorphism from $\mathfrak{A}_m$ to a finite dimensional C^*-algebra C. Let K be the integer associated with G_0.

Let K_1 be the integer such that $G \cap K_0(\mathfrak{A}, \mathbb{Z}_k) = \emptyset$ for all $k > K_1$.

Let $\Psi_n = \psi_n \oplus H_n \oplus \cdots \oplus H_n$, the direct sum of $K(K_1)! - 1$ copies of H_n. Thus,

$$[\Psi_n]|_G = \alpha|_G + K(K_1)![H_n]|_G.$$

If F is a finite dimensional C^*-algebra, then one has the following commutative diagram:

$$\begin{tikzcd}
K_0(F) \arrow[r] & K_0(F, \mathbb{Z}_k) \arrow[r] & K_1(F) \arrow[d, "k"] \\
K_0(F) \arrow[u, "k"] & K_1(F, \mathbb{Z}_k) \arrow[l] & K_1(F) \arrow[l]
\end{tikzcd}$$

where $K_0(F, \mathbb{Z}_k) \cong K_0(F)/kK_0(F)$, $K_1(F) \cong 0$, $K_1(F, \mathbb{Z}_k) \cong 0$. Since H_n factors through a finite dimensional C^*-algebra, it is easy to check that $[H_n]|_{K_1(\mathfrak{A}) \cap G} = 0$, $[H_n]|_{K_1(\mathfrak{A}, \mathbb{Z}_k) \cap G} = 0$ and $[H_n]|_{\ker \rho_{\mathfrak{A}}(K_0(\mathfrak{A}) \cap G} = 0$. Moreover, $(K_1)![H_n]|_{K_0(\mathfrak{A}, \mathbb{Z}_k) \cap G} = 0$ for $k \le K_1$. Therefore,

$$\begin{aligned}
[\Psi_n]|_{K_1(\mathfrak{A}) \cap G} &= \alpha|_{K_1(\mathfrak{A}) \cap G}, \\
[\Psi_n]|_{K_1(\mathfrak{A}, \mathbb{Z}_k) \cap G} &= \alpha|_{K_1(\mathfrak{A}, \mathbb{Z}_k) \cap G}, \\
[\Psi_n]|_{\ker \rho_{\mathfrak{A}}(K_0(\mathfrak{A}) \cap G} &= \alpha|_{\ker \rho_{\mathfrak{A}}(K_0(\mathfrak{A}) \cap G}, \\
[\Psi_n]|_{K_0(\mathfrak{A}, \mathbb{Z}_k) \cap G} &= \alpha|_{K_0(\mathfrak{A}, \mathbb{Z}_k) \cap G}.
\end{aligned}$$

Choose $r > 0$ such that $r < 1/(JK(K_1)! + 1)$, where J is an integer so that $[H_n(1_{\mathfrak{A}})] \le [1_{M_J(\mathfrak{B})}]$. Set $G_1' = [\Psi_n](G)$. Let e_i be projections such that $[\bar{e}_i] = [\Psi_n]([p_i]) - [\bar{q}_i](= \alpha([p_i])$, where $[\bar{q}_i] \in K(K_1)![H_n]([p_i])$ for $1 \le i \le l$. Set $G_1 = [\Psi_n](G) \cup \{[\bar{e}_i], \alpha([p_i]), [\bar{q}_i], i = 1, \cdots, l\}$.

Let $L : \mathfrak{A} \to \mathfrak{B}$ be associated with G_1 (with $\varepsilon > 0$ and F to be determined later). Set $\Phi_n = L \circ \Psi_n$. Then

$$\begin{aligned}
[\Phi_n]|_{K_1(\mathfrak{A}_m) \cap G} &= \alpha|_{K_1(\mathfrak{A}_m) \cap G}, \\
[\Phi_n]|_{K_1(\mathfrak{A}_m, \mathbb{Z}_k) \cap G} &= \alpha|_{K_1(\mathfrak{A}_m, \mathbb{Z}_k) \cap G}, \\
[\Phi_n]|_{\ker \rho_{\mathfrak{A}}(K_0(\mathfrak{A}_m) \cap G} &= \alpha|_{\ker \rho_{\mathfrak{A}_m}(K_0(\mathfrak{A}_m) \cap G}, \\
[\Phi_n]|_{K_0(\mathfrak{A}_m, \mathbb{Z}_k) \cap G} &= \alpha|_{K_0(\mathfrak{A}_m, \mathbb{Z}_k) \cap G}.
\end{aligned}$$

We also have $\rho \circ [\Psi_n](g) \le r\rho_{\mathfrak{B}} \circ \alpha(g)$ for $g \in K_0(\mathfrak{A}) \cup G$ and

$$\alpha([p_i]) - [\Phi_n]([p_i]) = K(K_1)![f_i]$$

for $1 \leq i \leq l$, where $f_i \in K_0(\mathfrak{B})$. Note that $(\alpha - [\Psi_n])(g) > 0$ for all $g \in G_0 \setminus \{0\}$ since $r < 1/(KJ(K_1)! + 1)$. Note also that with the order embedding $G_0 \subset K_0(\pi(\mathfrak{A})) \subset \mathbb{Z}^k$ and the choice of K, there exists a positive homomorphism $\Psi : K_0(\pi(\mathfrak{A}_m)) \to K_0(\mathfrak{B})$ such that

$$\Psi|_{G_0} = (\alpha - [\Phi_n])|_{G_0}.$$

Since $\mathfrak{B}$ has real rank zero and stable rank one, we obtain a homomorphism $h_n : \pi(\mathfrak{A}_m) \to (1-p)\mathfrak{B}(1-p)$ such that $[h_n] = \Psi$, where $p = 1_{\mathfrak{B}} - \Phi_n(1_{\mathfrak{A}})$ (we may assume that $\Psi_n(1_{\mathfrak{A}}) \leq 1_{\mathfrak{B}}$, since $r < 1/2(KJ(K_1)! + 1)$). We set $L_n = \Phi_n \oplus h_n \circ \pi$ with sufficiently small ε (depends on n) and the finite subset F_1 (which is larger than $\Phi_n(F)$). Now $[L_n]|_P = \alpha|_P$. $\square$

Theorem 4.25 [7] *Let $\mathfrak{A}$, $\mathfrak{B}$ be unital LSH algebras with* $\mathrm{TR}(\mathfrak{A}) = 0 = \mathrm{TR}(\mathfrak{B})$ *and with the UCT. Suppose that there exists an order isomorphism*

$$\alpha : (K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, [1_{\mathfrak{A}}], K_1(\mathfrak{A})) \to (K_0(\mathfrak{B}), K_0(\mathfrak{B})_+, [1_{\mathfrak{B}}], K_1(\mathfrak{B})).$$

Then there is an isomorphism $h : \mathfrak{A} \to \mathfrak{B}$ such that $h_* = \alpha$.

Corollary 4.26 [7] *Let $\mathfrak{A}$, $\mathfrak{B}$ be unital ASH algebras with* $\mathrm{TR}(\mathfrak{A}) = 0 = \mathrm{TR}(\mathfrak{B})$. *Suppose that there exists an order isomorphism*

$$\alpha : (K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, [1_{\mathfrak{A}}], K_1(\mathfrak{A})) \to (K_0(\mathfrak{B}), K_0(\mathfrak{B})_+, [1_{\mathfrak{B}}], K_1(\mathfrak{B})).$$

Then there is an isomorphism $h : \mathfrak{A} \to \mathfrak{B}$ such that $h_* = \alpha$.

Theorem 4.27 [7] *Let $\mathfrak{A}$ be a unital LSH C^*-algebra with unique normalized trace. If $\mathfrak{A}$ has real rank zero, stable rank one and weakly unperforated $K_0(\mathfrak{A})$, then* $\mathrm{TR}(\mathfrak{A}) = 0$.

Theorem 4.28 (Q. Lin and N.C. Phillips) *Let $\mathfrak{A} = C(X) \rtimes_\alpha \mathbb{Z}$ be the crossed product of $C(X)$ by an action α of $\mathbb{Z}$, where X is a compact manifold and α is induced by a minimal diffeomorphism on X. Then $\mathfrak{A}$ is simple and LSH. In fact, $\mathfrak{A}$ is ASH. Furthermore, $\mathfrak{A}$ has stable rank one and $K_0(\mathfrak{A})$ is weakly unperforated.*

Theorem 4.29 [7] *Let $\mathfrak{A}_j = C(X_j) \rtimes_\alpha \mathbb{Z}$ $(j = 1, 2)$ be the crossed products of $C(X_j)$ by an action α_j of $\mathbb{Z}$, where X_j are compact manifolds and α_j are induced by minimal diffeomorphisms on X_j respectively. If* $\mathrm{TR}(\mathfrak{A}_j) = 0$ *for $j = 1, 2$, or if X_j have unique invariant measures associated with α_j and the range of $K_0(\mathfrak{A}_j)$ under the trace is dense in $\mathbb{R}$, then $\mathfrak{A}_1 \cong \mathfrak{A}_2$ if and only if*

$$(K_0(\mathfrak{A}_1), K_0(\mathfrak{A}_1)_+, [1_{\mathfrak{A}_1}], K_1(\mathfrak{A}_1)) \cong (K_0(\mathfrak{A}_2), K_0(\mathfrak{A}_2)_+, [1_{\mathfrak{A}_2}], K_1(\mathfrak{A}_2)).$$

Remark. It follows from the second assumption that $\mathfrak{A}_j$ have real rank zero.

Theorem 4.30 [7] *Let* $\mathfrak{A}_j = C(\mathbb{T}^k) \rtimes_{\alpha_j} \mathbb{Z}$ $(j = 1, 2)$ *be simple noncommutative* $k+1$*-tori generated by* $k+1$ *unitaries* U_j $(1 \le j \le k+1)$ *such that* $U_j U_{k+1} = e^{2\pi i\theta_j} U_{k+1} U_j$ *for* $1 \le j \le k$ *and* θ_j *rationally independent irrational real numbers. Then* $\mathfrak{A}_1 \cong \mathfrak{A}_2$ *if and only if* $[\alpha_1] = [\alpha_2]$ *on* $K_i(C(\mathbb{T}^k))$ $(i = 0, 1)$. *Furthermore,* $\mathfrak{A}_j$ *are AT algebras.*

Proof. We have $K_i(\mathfrak{A}_1) \cong K_i(\mathfrak{A}_2)$ for $i = 0, 1$ by Pimsner-Voiculescu six term exact sequence of K-groups of crossed products by $\mathbb{Z}$. Then it follows that $\mathfrak{A}_1 \cong \mathfrak{A}_2$, and $\mathfrak{A}_j$ are isomorphic to certain unital simple AH algebras. Since $K_i(\mathfrak{A}_j)$ are torsion free, $\mathfrak{A}_j$ are in fact isomorphic to certain AT algebras. $\square$

Purely infinite simple C^*-algebras

The classification theorem for purely infinite, simple, separable nuclear C^*-algebras such as O_n has been completed (see [14]):

Theorem 4.31 (E. Kirchberg and N.C. Phillips) *Let* $\mathfrak{A}$, $\mathfrak{B}$ *be purely infinite, simple, separable nuclear* C^**-algebras.*

If they are stable, then $\mathfrak{A} \cong \mathfrak{B}$ *if and only if they are KK-equivalent. Moreover, for any invertible element* x *of* $KK(\mathfrak{A}, \mathfrak{B})$ *there exists an isomorphism* $\varphi : \mathfrak{A} \to \mathfrak{B}$ *with* $KK(\varphi) = x$.

If they are stable and of $\mathfrak{N}$, *then*

$$\mathfrak{A} \cong \mathfrak{B} \Leftrightarrow (K_0(\mathfrak{A}), K_1(\mathfrak{A})) \cong (K_0(\mathfrak{B}), K_1(\mathfrak{B})).$$

Moreover, for any pair of isomorphisms $\alpha_j : K_j(\mathfrak{A}) \to K_j(\mathfrak{B})$ *for* $j = 0, 1$, *there exists an isomorphism* $\varphi : \mathfrak{A} \to \mathfrak{B}$ *with* $K_j(\varphi) = \alpha_j$ *for* $j = 0, 1$.

If they are unital, then $\mathfrak{A} \cong \mathfrak{B}$ *if and only if they are KK-equivalent by preserving the* K_0*-classes of their units* $1_{\mathfrak{A}}$, $1_{\mathfrak{B}}$, *i.e., there exists an invertible element* x *of* $KK(\mathfrak{A}, \mathfrak{B})$ *with* $\gamma_0(x)([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}]$. *For any such* x *there is an isomorphism* $\varphi : \mathfrak{A} \to \mathfrak{B}$ *with* $KK(\varphi) = x$.

If they are unital and of $\mathfrak{N}$, *then*

$$\mathfrak{A} \cong \mathfrak{B} \Leftrightarrow (K_0(\mathfrak{A}), K_1(\mathfrak{A}), [1_{\mathfrak{A}}]) \cong (K_0(\mathfrak{B}), K_1(\mathfrak{B}), [1_{\mathfrak{B}}]).$$

For any such isomorphism there is an isomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ *with* $K_j(\varphi) = \alpha_j$ *for* $j = 0, 1$.

Sketch of Proof [14]. If $\mathfrak{A}$, $\mathfrak{B}$ are isomorphic, then they are KK-equivalent. Conversely, suppose that $x \in KK(\mathfrak{A}, \mathfrak{B})$ and $y \in KK(\mathfrak{B}, \mathfrak{A})$ such that

$x \cdot y = \mathrm{id}_{\mathfrak{A}}$ and $y \cdot x = \mathrm{id}_{\mathfrak{B}}$. Then we obtain nonzero $*$-homomorphisms $\varphi_0 : \mathfrak{A} \to \mathfrak{B}$ and $\psi_0 : \mathfrak{B} \to \mathfrak{A}$ satisfying $KK(\varphi_0) = x$ and $KK(\psi_0) = y$. Then

$$KK(\psi_0 \circ \varphi_0) = KK(\varphi_0) \cdot KK(\psi_0) = x \cdot y = \mathrm{id}_{\mathfrak{A}} = KK(\mathrm{id}_{\mathfrak{A}}).$$

Then we have $\psi_0 \circ \varphi_0 \approx_u \mathrm{id}_{\mathfrak{A}}$. It follows that $\varphi_0 \circ \psi_0 \approx_u \mathrm{id}_{\mathfrak{B}}$. We can apply the approximate intertwining to obtain that $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ and that there is an isomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ with $\varphi \approx_u \varphi_0$.

Let $\alpha_j : K_j(\mathfrak{A}) \to K_j(\mathfrak{B})$ for $j = 0, 1$ be isomorphisms given. Since $\mathfrak{A}$, $\mathfrak{B}$ belong to the class N we can apply the UCT to find an invertible element x of $KK(\mathfrak{A}, \mathfrak{B})$ with $\gamma_j(x) = \alpha_j$ for $j = 0, 1$. Then $\mathfrak{A}, \mathfrak{B}$ are isomorphic and there is an isomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ with $KK(\varphi) = x$. Then $K_j(\varphi) = \gamma_j(KK(\varphi)) = \gamma_j(x) = \alpha_j$ for $j = 0, 1$. □

Remark. In the statement, if $\mathfrak{A}$ is unital such that $KK(\mathfrak{A}, \mathfrak{A}) \cong 0$, then $\mathfrak{A} \cong O_2$ because $KK(O_2, O_2) \cong 0$. Any separable nuclear C^*-algebra is KK-equivalent to some $\mathfrak{A}$ (p-i-s-s-n).

Proposition 4.32 [14] *Let G_0, G_1 be countable abelian groups and let $g \in G_0$. Then there exists a unital purely infinite, simple nuclear separable C^*-algebra $\mathfrak{A}$ with the UCT and that is an inductive limit of $\mathfrak{B}_j \otimes C(\mathbb{T})$, where $\mathfrak{B}_j = \oplus_{l=1}^{r_j} M_{n_l}(O_{k_l})$ for some $r_j, k_l \in \mathbb{N}$ and $2 \leq n_l \leq \infty$ such that*

$$(G_0, g, G_1) \cong (K_0(\mathfrak{A}), [1_{\mathfrak{A}}], K_1(\mathfrak{A})).$$

Remark. A C^*-algebra $\mathfrak{A}$ is stable if $\mathfrak{A} \cong \mathfrak{A} \otimes \mathbb{K}$, where $\mathbb{K}$ is the C^*-algebra of compact operators.

A simple C^*-algebra $\mathfrak{A}$ (not $\mathbb{C}$) is purely infinite if it satisfies one of the following equivalent 6 conditions:

1. For nonzero positive elements $a, b \in \mathfrak{A}$, there is $x \in \mathfrak{A}$ with $b = x^*ax$.

2. For nonzero elements $a, b \in \mathfrak{A}$, there are $x, y \in \mathfrak{A}$ with $b = xay$.

3. $\mathfrak{A}$ has real rank zero, and any nonzero projection of $\mathfrak{A}$ is properly infinite.

4. $\mathfrak{A}$ has no nonzero abelian quotients, and for positive elements $a, b \in \mathfrak{A}$ such that b is contained in the closed two-sided ideal $\mathfrak{A}a\mathfrak{A}$ generated by a, there is a sequence $\{x_n\}_{n=1}^{\infty}$ of $\mathfrak{A}$ with $\lim x_n^* a x_n = b$.

5. Any nonzero hereditary C^*-subalgebra of $\mathfrak{A}$ contains an infinite projection.

6. Any nonzero hereditary C^*-subalgebra of $\mathfrak{A}$ contains a stable C^*-subalgebra.

Remark. Note that projections p, q of a C^*-algebra $\mathfrak{A}$ are (Murray-von Neumann) equivalent ($p \sim q$) if $p = v^*v$ and $q = vv^*$ for some partial isometry v in $\mathfrak{A}$, and write $p \precsim q$ if p is equivalent to a subprojection of q. A projection p of a C^*-algebra $\mathfrak{A}$ is infinite if p is equivalent to a proper subprojection of itself, and otherwise p is called finite, and p is properly infinite if there are mutually orthogoanl projections p_1, p_2 of $\mathfrak{A}$ such that $p_1 + p_2 \leq p$ and $p \sim p_1 \sim p_2$.

A C^*-algebra is infinite if it contains an infinite projection. A C^*-algebra $\mathfrak{A}$ is finite if all projections of $\mathfrak{A}$ are finite and $\mathfrak{A}$ contains an approximate unit of projections, or if for any $u \in \mathfrak{A}$ with $u^*u = 1$, we have $uu^* = 1$. A C^*-algebra $\mathfrak{A}$ is stably finite if $\mathfrak{A} \otimes \mathbb{K}$ is finite, or if for any $u \in M_n(\mathfrak{A})$ ($n \geq 1$) with $u^*u = 1$, we have $uu^* = 1$.

(Zhang's dichotomy): Any separable purely infinite simple C^*-algebra is either stable or unital.

The class of purely infinite simple C^*-algebras is closed under stable isomorphism, taking inductive limits, and taking minimal tensor products.

Finally, we have the following:

Table 6: Classes and examples of C^*-algebras by projections

Classes	Examples
Stably finite	Commutative C^*-algebras, $M_n(\mathbb{C})$ Homogeneous and subhomogeneous C^*-algebras $\mathbb{K}$, AF algebras, AT algebras C^*-algebras with stable rank one
Infinite	The usual Toeplitz algebra (with a isometry) The C^*-algebra $\mathbb{B}$ of bounded operators
Purely infinite	Cuntz algebras, Calkin algebra $\mathbb{B}/\mathbb{K}$ (non separable)

Remark. There exists a finite C^*-algebra that is not stably finite. For example, the Toeplitz algebra T_{S^3} on the unit ball of $\mathbb{C}^2$, that is an extension of $C(S^3)$ by $\mathbb{K}$, is finite but $M_2(T_{S^3})$ not finite. C^*-algebras with stable rank one have the cancellation, which implies that they are stably finite.

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