K-theory for amalgams and multi-ones of C^⊠-algebras

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K-THEORY FOR AMALGAMS AND MULTI-ONES OF C^*-ALGEBRAS

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Dedicated to Professor Toshihiko Nishishiraho on his sixtieth birthday

Abstract

We obtain a six-term exact sequence of K-theory for (full) amalgams of C^*-algebras. We also obtain a six-term exact sequence of K-theory for the full crossed product of a C^*-algebra by an amalgam of discrete groups, as well. Furthermore, it follows from this result and K-amenability for discrete (or locally compact) groups that the same is true for its reduced crossed product.

We also study K-theory of (full) multi-amalgams of the (full) group C^*-algebras of discrete groups. As an important application we compute K-groups of the full group C^*-algebras of $SL_n(\mathbb{Z})$ ($n \geq 3$).

In addition, an appendix on K-theory and beyond and an appendix on finite subgroups of $GL_3(\mathbb{Z})$ and $SL_3(\mathbb{Z})$ are presented.

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Preface

This paper consists of two sections as contributions and two sections as appendixes. This is a continuation (from [16]) of studying K-theory for amalgams (or amalgamted free products) and multi-ones of C^*-algebras, especially, those of the (full) group C^*-algebras of discrete groups. The first and second sections include their own introductions explained in more details. As in the abstract, in the first section we obtain the six-term exact sequence of K-theory groups for (full) amalgams of C^*-algebras, which implies that the K-theory conjecture for the full crossed products by amalgams of discrete groups is solved. By assuming K-amenability for groups, the similar conjecture for their reduced crossed products is solved. In the

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second section, similarly we consider K-theory of (full) multi-amalgams of the (full) group C^*-algebras of discrete groups, which implies the application to K-theory of the full group C^*-algebra of $SL_n(\mathbb{Z})$. As an appendix, in the third section we review K-theory for the reduced crossed products by free groups from Pimsner-Voiculescu [12], K-theory for the reduced C^*-algebras of certain discrete free product groups from Lance [7] (and Natsume [8]), and K-theory for the reduced C^*-algebras of HNN groups from Anderson and Paschke [1], and K-theory for the reduced C^*-algebras of one-relator groups from Béguin, Bettaieb, and Valette [2]. Furthermore, KK-theory and E-theory for amalgams of C^*-algebras from Thomsen [18] and E-theory basics from Blackadar [3] are reviewed. Further added are the lists of finite subgroups of $GL_3(\mathbb{Z})$ and $SL_3(\mathbb{Z})$ from Tahara [17].

Contents

1 K-theory of amalgams of C^*-algebras

1.0 Introduction to the first section

K-theory for group C^*-algebras of discrete groups and crossed products of C^*-algebras by discrete groups has been studied well to some extent. It was Pimsner-Voiculesce [12] who first obtained K-theory for reduced crossed products of C^*-algebras by free groups. Cuntz [4] studied K-theory for (full) free products of C^*-algebras (cf. Blackadar [3, 10.11.11]). After [12], Lance [7] studied K-theory for reduced group C^*-algebras of some free product groups such as free groups. Furthermore, Natsume [8] extended Lance's result to reduced crossed products of C^*-algebras by certain amalgams of discrete groups such as $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. On the other hand, K-amenability for discrete groups was invented and studied by Cuntz [5]. Furthermore, K-amenability for locally compact groups was introduced and studied by Julg and Valette [6].

The problem that we would like to consider is whether the six-term exact sequence of K-groups:

$$\begin{array}{ccccc} K_0(\mathfrak{C}) & \longrightarrow & K_0(\mathfrak{A}) \oplus K_0(\mathfrak{B}) & \longrightarrow & K_0(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) & \longleftarrow & K_1(\mathfrak{A}) \oplus K_1(\mathfrak{B}) & \longleftarrow & K_1(\mathfrak{C}) \end{array}$$

associated to an amalgam (or amalgamated free product) $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ of two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ over a common C^*-subalgebra $\mathfrak{C}$ holds or not in general. A rather restrictive partial positive answer was obtained by Cuntz [4] (see also [3, 10.11.11]). In this section we give a positive answer to certainly full generality. It should be noticed that Thomsen [18] has already obtained the same six-term exact sequence of K-groups and the corresponding ones of KK-theory and E-theory, provided that $\mathfrak{C}$ is finite dimensional or nuclear respectively. However, our method on K-theory only is quite different from Thomsen's one used in KK-theory and E-theory settings.

A conjecture ([3, Conjecture 10.8.3]) that we would like to next consider is whether the six-term exact sequence of K-groups:

$$\begin{array}{ccccc} K_0(\mathfrak{A} \rtimes K) & \longrightarrow & K_0(\mathfrak{A} \rtimes G) \oplus K_0(\mathfrak{A} \rtimes H) & \longrightarrow & K_0(\mathfrak{A} \rtimes \Gamma) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{A} \rtimes \Gamma) & \longleftarrow & K_1(\mathfrak{A} \rtimes G) \oplus K_1(\mathfrak{A} \rtimes H) & \longleftarrow & K_1(\mathfrak{A} \rtimes K), \end{array}$$

associated to the full crossed products of a C^*-algebra $\mathfrak{A}$ by two discrete groups G and H and their amalgam $\Gamma = G *_K H$ over a common subgroup K holds or not in general. As mentioned above, a rather restrictive partial positive answer in the case of reduced crossed products was obtained by Natsume. In this section we obtain a positive answer for that conjecture to certainly full generality.

For those problem and conjecture, since we deal with full amalgams and full crossed products, we can use the universality argument. (Another attempt for proofing them has been made by using a stable isomorphism between a C^*-algebra and its full hereditary C^*-subalgebra and a deformation argument, but it was not successful completely.)

Furthermore, using both our result for that conjecture for full crossed products and a result of Julg and Valette [6, Corollary 3.6] for K-group isomorphisms between full and reduced crossed products by K-amenable groups, we have the same conjecture for the reduced crossed products by K-amenable discrete groups solved. Refer to [9] for crossed products of C^*-algebras.

Also, using the universal coefficient theorem (UCT) or the Künneth formula (KF) for KK-theory (see [3]) we can extend those results obtained to ones of the six-term exact sequences of KK-groups.

1.1 Amalgams of C^*-algebras

Let $K_0(\mathfrak{A})$ be the K_0-group of a C^*-algebra $\mathfrak{A}$ and $K_1(\mathfrak{A})$ the K_1-group of $\mathfrak{A}$. For a unital C^*-algebra $\mathfrak{A}$, by definition, $K_0(\mathfrak{A})$ is defined to be the abelian group of formal differences ($-$: minus) of stable equivalence classes of projections of matrix algebras $M_n(\mathfrak{A})$ over $\mathfrak{A}$ ($n \geq 1$) and $K_1(\mathfrak{A})$ is defined to the abelian group of (homotopy) equivalence classes of unitaries (or invertible elements) of $M_n(\mathfrak{A})$ ($n \geq 1$). Namely,

$$\begin{aligned} K_0(\mathfrak{A}) &= \cup_{n\geq 1}\mathrm{Proj}_n(\mathfrak{A})/\sim_s - \cup_{n\geq 1}\mathrm{Proj}_n(\mathfrak{A})/\sim_s, \\ K_1(\mathfrak{A}) &= \cup_{n\geq 1}U_n(\mathfrak{A})/\sim_h = \cup_{n\geq 1}GL_n(\mathfrak{A})/\sim_h \end{aligned}$$

where $\mathrm{Proj}_n(\mathfrak{A})$ is the set of projections of $M_n(\mathfrak{A})$, $U_n(\mathfrak{A})$ is the group of unitaries of $M_n(\mathfrak{A})$, and $GL_n(\mathfrak{A})$ is the group of invertible elements of $M_n(\mathfrak{A})$, and $p \sim_s q$ for $p, q \in \mathrm{Proj}_n(\mathfrak{A})$ if there exists a unitary u such that $p \oplus 1_k = u(q \oplus 1_k)u^*$ for some $k \geq 0$, where 1_k is the $k \times k$ identity matrix, and $\oplus$ means the diagonal sum. The addition of $K_0(\mathfrak{A})$ is given by $[p] + [q] = [p \oplus q]$. Also,

$$K_1(\mathfrak{A}) = \varinjlim U_n(\mathfrak{A})/U_n(\mathfrak{A})_0 = \varinjlim GL_n(\mathfrak{A})/GL_n(\mathfrak{A})_0,$$

as inductive limits of quotient groups, where $U_n(\mathfrak{A})_0$, $GL_n(\mathfrak{A})_0$ are connected components of $U_n(\mathfrak{A})$, $GL_n(\mathfrak{A})$ containing the identity matrix, respectively. The multiplication of $K_1(\mathfrak{A})$ is given by $[u][v] = [uv] = [u \oplus v] = [vu]$.

Refer to the text books [3] or [19] for K-theory of C^*-algebras.

Theorem 1.1.1 (Cuntz [4] and Blackadar [3, 10.11.11]) *Let $\mathfrak{A}$, $\mathfrak{B}$ be C^*-algebras and $\mathfrak{A} * \mathfrak{B}$ their full free product C^*-algebra. Then for $j = 0, 1$,*

$$K_j(\mathfrak{A} * \mathfrak{B}) \cong K_j(\mathfrak{A}) \oplus K_j(\mathfrak{B}).$$

If $\mathfrak{A}$, $\mathfrak{B}$ are unital C^*-algebras, let $\mathfrak{A} *_{\mathbb{C}} \mathfrak{B}$ be their unital full free product C^*-algebra. Then we have

$$K_j(\mathfrak{A} *_{\mathbb{C}} \mathfrak{B}) \cong (K_j(\mathfrak{A}) \oplus K_j(\mathfrak{B}))/K_j(\mathbb{C}) \cong \begin{cases} (K_0(\mathfrak{A}) \oplus K_0(\mathfrak{B}))/\mathbb{Z} & j = 0, \\ K_1(\mathfrak{A}) \oplus K_1(\mathfrak{B}) & j = 1, \end{cases}$$

where $\mathbb{Z} \cong \{n([1_{\mathfrak{A}}], -[1_{\mathfrak{B}}]) \mid n \in \mathbb{Z}\}$ for $[1_{\mathfrak{A}}] \in K_0(\mathfrak{A})$, $[1_{\mathfrak{B}}] \in K_0(\mathfrak{B})$, where $1_{\mathfrak{A}}$, $1_{\mathfrak{B}}$ are the units of $\mathfrak{A}$, $\mathfrak{B}$ respectively.

Remark. Note that $K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K_1(\mathbb{C}) \cong 0$.

Lemma 1.1.2 *Let $\mathfrak{A} * \mathfrak{B}$ be the full free product of C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$. Suppose that there exists a common C^*-subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ and $\mathfrak{B}$. Then the following is exact:*

$$0 \to K_*(\mathfrak{C}) \to K_*(\mathfrak{A}) \oplus K_*(\mathfrak{B}) \to (K_*(\mathfrak{A}) \oplus K_*(\mathfrak{B}))/K_*(\mathfrak{C}) \to 0$$

for $* = 0, 1$, where a K-theory class $[p]$ of $K_0(\mathfrak{C})$ is mapped to $([p], -[p]) \in K_0(\mathfrak{A}) \oplus K_0(\mathfrak{B})$ and a K-theory class $[u]$ of $K_1(\mathfrak{C})$ is mapped to $([u], [u^{-1}]) \in K_1(\mathfrak{A}) \oplus K_1(\mathfrak{B})$.

Proof. Note that $K_j(\mathfrak{A} * \mathfrak{B}) \cong K_j(\mathfrak{A}) \oplus K_j(\mathfrak{B})$ ($j = 0, 1$) (see [3, 10.11.11]). Note that there is a canonical inclusion from the $n \times n$ matrix algebra $M_n(\mathfrak{C})$ over $\mathfrak{C}$ to $M_n(\mathfrak{A}) \subset M_n(\mathfrak{A} * \mathfrak{B})$ (or $M_n(\mathfrak{B}) \subset M_n(\mathfrak{A} * \mathfrak{B})$), where we identify those images. It follows from freeness in $M_n(\mathfrak{A} * \mathfrak{B})$ that the inclusion induces the (diagonal) injections from $K_j(\mathfrak{C})$ to $K_j(\mathfrak{A} * \mathfrak{B}) \cong K_j(\mathfrak{A}) \oplus K_j(\mathfrak{B})$ ($j = 0, 1$) as in the statement. Onto-ness of the exact sequence is clear. $\square$

Proposition 1.1.3 *Let $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ be an amalgam of C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ over a common C^*-subalgebra $\mathfrak{C}$. Then*

$$K_*(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) \cong (K_*(\mathfrak{A}) \oplus K_*(\mathfrak{B}))/K_*(\mathfrak{C}).$$

Proof. By universality, there exists a canonical onto $*$-homomorphism φ from $\mathfrak{A} * \mathfrak{B}$ to $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ (cf. [10] or [11]). Thus, there exists a group homomorphism from $K_*(\mathfrak{A} * \mathfrak{B})$ to $K_*(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B})$. Furthermore, any element of $M_n(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B})$ can be viewed as that of $M_n(\mathfrak{A} * \mathfrak{B})$, so that the group homomorphism is onto. Moreover, the quotients $(K_*(\mathfrak{A}) \oplus K_*(\mathfrak{B})/K_*(\mathfrak{C})$ can be viewed as subgroups of $K_*(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B})$. Therefore, we obtain the isomorphisms as desired. □

Combining the arguments above, we obtain

Theorem 1.1.4 *Let $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ be an amalgam of C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ over a common C^*-subalgebra $\mathfrak{C}$. Then*

$$\begin{array}{ccccc} K_0(\mathfrak{C}) & \longrightarrow & K_0(\mathfrak{A}) \oplus K_0(\mathfrak{B}) & \longrightarrow & K_0(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) & \longleftarrow & K_1(\mathfrak{A}) \oplus K_1(\mathfrak{B}) & \longleftarrow & K_1(\mathfrak{C}) \end{array}$$

1.2 Crossed products by amalgams of groups

Let $\mathfrak{A}$ be a C^*-algebra and Γ a discrete (or locally compact) group. We denote by $\mathfrak{A} \rtimes \Gamma$ the full crossed product of $\mathfrak{A}$ by an action of Γ (see [9]).

Theorem 1.2.1 *Let $\mathfrak{A}$ be a C^*-algebra and $G *_K H \equiv \Gamma$ an amalgam of discrete groups G and H over a common subgroup K. Then*

$$\begin{array}{ccccc} K_0(\mathfrak{A} \rtimes K) & \longrightarrow & K_0(\mathfrak{A} \rtimes G) \oplus K_0(\mathfrak{A} \rtimes H) & \longrightarrow & K_0(\mathfrak{A} \rtimes \Gamma) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{A} \rtimes \Gamma) & \longleftarrow & K_1(\mathfrak{A} \rtimes G) \oplus K_1(\mathfrak{A} \rtimes H) & \longleftarrow & K_1(\mathfrak{A} \rtimes K). \end{array}$$

Proof. Note that $\mathfrak{A} \rtimes \Gamma \cong (\mathfrak{A} \rtimes G) *_{\mathfrak{A} \rtimes K} (\mathfrak{A} \rtimes H)$. □

Corollary 1.2.2 *Let $G_1 *_H G_2$ be an amalgam of discrete groups G_1 and G_2 over a common subgroup H, and $C^*(G_1 *_H G_2)$ its full group C^*-algebra. Then*

$$\begin{array}{ccccc} K_0(C^*(H)) & \longrightarrow & \oplus_{j=1}^2 K_0(C^*(G_j)) & \longrightarrow & K_0(C^*(G_1 *_H G_2)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(G_1 *_H G_2)) & \longleftarrow & \oplus_{j=1}^2 K_1(C^*(G_j)) & \longleftarrow & K_1(C^*(H)), \end{array}$$

where $C^*(G_j)$ and $C^*(H)$ are the full group C^*-algebras of G_j and H respectively.

Remark. Note that $\mathbb{C} \rtimes \Gamma \cong C^*(\Gamma)$ for a discrete (or locally compact) group Γ. Note also that $C^*(G *_K H) \cong C^*(G) *_{C^*(K)} C^*(H)$ (see [3, 10.11.11]). Also, in the theorem above, the groups G, H, and K may not be discrete if $\mathfrak{A} \rtimes K$ can be embedded into $\mathfrak{A} \rtimes (G *_K H)$.

Furthermore, for a C^*-algebra $\mathfrak{A}$ and a discrete (or locally compact) group Γ, we denote by $\mathfrak{A} \rtimes_r \Gamma$ the reduced crossed product of $\mathfrak{A} \rtimes \Gamma$ ([10]). Then

Theorem 1.2.3 *Let $\mathfrak{A}$ be a C^*-algebra and $G *_K H \equiv \Gamma$ a K-amenable amalgam of countable discrete groups G and H over a common subgroup K. Then*

$$\begin{array}{ccccc} K_0(\mathfrak{A} \rtimes_r K) & \longrightarrow & K_0(\mathfrak{A} \rtimes_r G) \oplus K_0(\mathfrak{A} \rtimes_r H) & \longrightarrow & K_0(\mathfrak{A} \rtimes_r \Gamma) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{A} \rtimes_r \Gamma) & \longleftarrow & K_1(\mathfrak{A} \rtimes_r G) \oplus K_1(\mathfrak{A} \rtimes_r H) & \longleftarrow & K_1(\mathfrak{A} \rtimes_r K). \end{array}$$

Proof. It is shown by [6, Corollary 3.6] that if Γ is a K-amenable, locally compact (in particular, discrete) group, then

$$K_j(\mathfrak{A} \rtimes \Gamma) \cong K_j(\mathfrak{A} \rtimes_r \Gamma) \quad (j = 0, 1)$$

for any full crossed product $\mathfrak{A} \rtimes \Gamma$ of a C^*-algebra $\mathfrak{A}$ by Γ and its reduced crossed product $\mathfrak{A} \rtimes_r \Gamma$. It is also shown by [5] that all subgroups of a K-amenable countable discrete group are K-amenable. Thus, the statement follows from the theorem obtained above immediately. $\square$

Corollary 1.2.4 *Let $G_1 *_H G_2$ be a K-amenable amalgam of countable discrete groups G_1 and G_2 over a common subgroup H, and $C_r^*(G_1 *_H G_2)$ its reduced group C^*-algebra. Then*

$$\begin{array}{ccccc} K_0(C_r^*(H)) & \longrightarrow & \oplus_{j=1}^2 K_0(C_r^*(G_j)) & \longrightarrow & K_0(C_r^*(G_1 *_H G_2)) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(G_1 *_H G_2)) & \longleftarrow & \oplus_{j=1}^2 K_1(C_r^*(G_j)) & \longleftarrow & K_1(C_r^*(H)), \end{array}$$

where $C_r^*(G_j)$ and $C_r^*(H)$ are the reduced group C^*-algebras of G_j and H respectively.

Remark. Note that $\mathbb{C} \rtimes_r \Gamma \cong C_r^*(\Gamma)$ the reduced group C^*-algebra for a discrete (or locally compact) group Γ. The class of countable K-amenable discrete groups is closed under taking subgroups, their direct products and free products, and extensions of them by amenable normal subgroups (see [5] for more details). The class of K-amenable locally compact groups is closed under taking closed subgroups, their direct products, and extensions of them by amenable closed normal subgroups. An amalgam of amenable discrete groups is K-amenable. A non-compact locally compact group with property T is not K-amenable. See [6] for more details.

1.3 Application to KK-theory

Recall that the universal coefficient theorem (UCT) implies that if $\mathfrak{A}$, $\mathfrak{B}$ are separable C^*-algebras and $\mathfrak{A}$ is in the so called bootstrap category or UCT class, and if $K_j(\mathfrak{A})$ $(j = 0, 1)$ are free or $K_j(\mathfrak{B})$ $(j = 0, 1)$ are divisible, then

$$\begin{aligned} KK^0(\mathfrak{A}, \mathfrak{B}) &\cong \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{B})) \oplus \mathrm{Hom}(K_1(\mathfrak{A}), K_1(\mathfrak{B})), \\ KK^1(\mathfrak{A}, \mathfrak{B}) &\cong \mathrm{Hom}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \oplus \mathrm{Hom}(K_1(\mathfrak{A}), K_0(\mathfrak{B})), \end{aligned}$$

where $KK^j(\mathfrak{A}, \mathfrak{B})$ are KK-groups of $\mathfrak{A}$ and $\mathfrak{B}$ (see [3]).

Proposition 1.3.1 *For the six-term exact sequence of K-groups:*

$$\begin{tikzcd} K_0(\mathfrak{C}) \arrow[r] & K_0(\mathfrak{D}) \arrow[r] & K_0(\mathfrak{B}) \arrow[d] \\ K_1(\mathfrak{B}) \arrow[u] & K_1(\mathfrak{D}) \arrow[l] & K_1(\mathfrak{C}) \arrow[l] \end{tikzcd}$$

of separable C^*-algebras $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$, if $\mathfrak{A}$ is a separable C^*-algebra in the UCT class, and if $K_j(\mathfrak{A})$ are free, or $K_j(\mathfrak{B})$, $K_j(\mathfrak{C})$, and $K_j(\mathfrak{D})$ are divisible, then we obtain

$$\begin{tikzcd} KK^0(\mathfrak{A}, \mathfrak{C}) \arrow[r] & KK^0(\mathfrak{A}, \mathfrak{D}) \arrow[r] & KK^0(\mathfrak{A}, \mathfrak{B}) \arrow[d] \\ KK^1(\mathfrak{A}, \mathfrak{B}) \arrow[u] & KK^1(\mathfrak{A}, \mathfrak{D}) \arrow[l] & KK^1(\mathfrak{A}, \mathfrak{C}). \arrow[l] \end{tikzcd}$$

Furthermore, we can exchange $\mathfrak{A}$ with $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$ in each $KK_j(\cdot, \cdot)$ with those conditions on K-groups, respectively.

Corollary 1.3.2 *The six-term exact sequences of K-groups obtained in subsections* 1.1 *and* 1.2 *imply the six-term exact sequences of KK-groups under the assumptions as above.*

Also, the Künneth formula (KF) implies that if $\mathfrak{A}$, $\mathfrak{B}$ are separable C^*-algebras and $\mathfrak{A}$ is in the so called bootstrap category or UCT class, and if

$K^j(\mathfrak{A})$ $(j=0,1)$ or $K_j(\mathfrak{B})$ $(j=0,1)$ are finitely generated and if $K^j(\mathfrak{A})$ $(j=0,1)$ or $K_j(\mathfrak{B})$ $(j=0,1)$ are torsion-free, then

$$\begin{aligned} KK^0(\mathfrak{A},\mathfrak{B}) &\cong (K^0(\mathfrak{A})\otimes K_0(\mathfrak{B}))\oplus(K^1(\mathfrak{A})\otimes K_1(\mathfrak{B})), \\ KK^1(\mathfrak{A},\mathfrak{B}) &\cong (K^0(\mathfrak{A})\otimes K_1(\mathfrak{B}))\oplus(K^1(\mathfrak{A})\otimes K_0(\mathfrak{B})), \end{aligned}$$

where $K^j(\mathfrak{A})$ are K-homology groups of $\mathfrak{A}$ (see [3]).

Proposition 1.3.3 *For the six-term exact sequence of K-groups:*

$$\begin{tikzcd} K_0(\mathfrak{C}) \arrow[r] & K_0(\mathfrak{D}) \arrow[r] & K_0(\mathfrak{B}) \arrow[d] \\ K_1(\mathfrak{B}) \arrow[u] & K_1(\mathfrak{D}) \arrow[l] & K_1(\mathfrak{C}) \arrow[l] \end{tikzcd}$$

of separable C^*-algebras $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$, if $\mathfrak{A}$ is a separable C^**-algebra in the UCT class, and if $K_j(\mathfrak{A})$, or $K_j(\mathfrak{B})$, $K_j(\mathfrak{C})$, and $K_j(\mathfrak{D})$ are finitely generated, and if $K_j(\mathfrak{A})$, or $K_j(\mathfrak{B})$, $K_j(\mathfrak{C})$, and $K_j(\mathfrak{D})$ are torsion-free, then we obtain

$$\begin{tikzcd} KK^0(\mathfrak{A},\mathfrak{C}) \arrow[r] & KK^0(\mathfrak{A},\mathfrak{D}) \arrow[r] & KK^0(\mathfrak{A},\mathfrak{B}) \arrow[d] \\ KK^1(\mathfrak{A},\mathfrak{B}) \arrow[u] & KK^1(\mathfrak{A},\mathfrak{D}) \arrow[l] & KK^1(\mathfrak{A},\mathfrak{C}). \arrow[l] \end{tikzcd}$$

Corollary 1.3.4 *The six-term exact sequences of K-groups obtained in subsections* 1.1 *and* 1.2 *imply the six-term exact sequences of KK-groups under the assumptions as above.*

1.4 Amalgams of the full C^*-algebras of discrete groups

Now recall that the group $SL_2(\mathbb{Z})$ is isomorphic to the amalgam $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ of finite cyclic groups $\mathbb{Z}_4$, $\mathbb{Z}_6$ over $\mathbb{Z}_2$, where its generators are given by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$$

which are identified with the generators of $\mathbb{Z}_2$, $\mathbb{Z}_4$, and $\mathbb{Z}_6$ respectively. Let $C^*(SL_2(\mathbb{Z}))$ be the full group C^*-algebra of $SL_2(\mathbb{Z})$. Then $C^*(SL_2(\mathbb{Z}))$ is isomorphic to the amalgam $C^*(\mathbb{Z}_4) *_{C^*(\mathbb{Z}_2)} C^*(\mathbb{Z}_6)$ of the group C^*-algebras $C^*(\mathbb{Z}_4)$ and $C^*(\mathbb{Z}_6)$ over $C^*(\mathbb{Z}_2)$, where its unitary generators are identified with the generators of $SL_2(\mathbb{Z})$ under the universal representation of $C^*(SL_2(\mathbb{Z}))$ (or $SL_2(\mathbb{Z})$).

Theorem 1.4.1 *Let $C^*(SL_2(\mathbb{Z}))$ be the full group C^*-algebra of $SL_2(\mathbb{Z})$. Then*

$$K_0(C^*(SL_2(\mathbb{Z}))) \cong (\mathbb{Z}^4 \oplus \mathbb{Z}^6)/\mathbb{Z}^2, \quad K_1(C^*(SL_2(\mathbb{Z}))) \cong 0.$$

Moreover, let $C_r^*(SL_2(\mathbb{Z}))$ be the reduced group C^*-algebra of $SL_2(\mathbb{Z})$. Then

$$K_0(C_r^*(SL_2(\mathbb{Z}))) \cong (\mathbb{Z}^4 \oplus \mathbb{Z}^6)/\mathbb{Z}^2, \quad K_1(C_r^*(SL_2(\mathbb{Z}))) \cong 0.$$

Proof. Since $C^*(SL_2(\mathbb{Z})) = C^*(\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6)$, it follows that $C^*(SL_2(\mathbb{Z})) \cong C^*(\mathbb{Z}_4) *_{C^*(\mathbb{Z}_2)} C^*(\mathbb{Z}_6)$ so that $K_j(C^*(SL_2(\mathbb{Z}))) \cong K_j(C^*(\mathbb{Z}_4) *_{C^*(\mathbb{Z}_2)} C^*(\mathbb{Z}_6))$ for $j = 0, 1$.

By the Fourier transform, we have $C^*(\mathbb{Z}_j) \cong C((\mathbb{Z}_j)^\wedge) \cong C(\mathbb{Z}_j) \cong \mathbb{C}^j$ for $j = 2, 4, 6$, where $(\mathbb{Z}_j)^\wedge$ is the dual group of $\mathbb{Z}_j$ and $(\mathbb{Z}_j)^\wedge \cong \mathbb{Z}_j$ and $C(\mathbb{Z}_j)$ is the C^*-algebra of all continuous functions on $\mathbb{Z}_j$. Therefore, we have $K_0(C^*(\mathbb{Z}_j)) \cong K_0(\mathbb{C}^j) \cong \mathbb{Z}^j$ and $K_1(C^*(\mathbb{Z}_j)) \cong K_1(\mathbb{C}^j) \cong 0$.

It is shown by Cuntz [5] that the group $SL_2(\mathbb{Z})$ is K-amenable, i.e, we have $K_*(C^*(SL_2(\mathbb{Z}))) \cong K_*(C_r^*(SL_2(\mathbb{Z})))$ for $* = 0, 1$. □

Remark. The result above was first obtained by Natsume [8] (and it is shown that $(\mathbb{Z}^4 \oplus \mathbb{Z}^6)/\mathbb{Z}^2 \cong \mathbb{Z}^8$), but his method to deduce it is completely different from ours. The same is just using K-amenablity of Cuntz.

Let $\mathbb{Z}_n$, $\mathbb{Z}_m$, and $\mathbb{Z}_l$ be finite cyclic groups and $C^*(\mathbb{Z}_n)$, $C^*(\mathbb{Z}_m)$, and $C^*(\mathbb{Z}_l)$ their group C^*-algebras, respectively. Let $\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m$ be the amalgam of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ over $\mathbb{Z}_l$. Let $C^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)$ be the full group C^*-algebra of $\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m$. Then $C^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)$ is isomorphic to the amalgam $C^*(\mathbb{Z}_n) *_{C^*(\mathbb{Z}_l)} C^*(\mathbb{Z}_m)$.

Theorem 1.4.2 *Let $C^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)$ be the full group C^*-algebra of an amalgam $\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m$. Then we have*

$$K_0(C^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)) \cong (\mathbb{Z}^n \oplus \mathbb{Z}^m)/\mathbb{Z}^l, \quad K_1(C^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)) \cong 0.$$

Moreover, let $C_r^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)$ be the reduced group C^*-algebra of $\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m$. Then

$$K_0(C_r^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)) \cong (\mathbb{Z}^n \oplus \mathbb{Z}^m)/\mathbb{Z}^l, \quad K_1(C_r^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m)) \cong 0.$$

Proof. We have $C^*(\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m) \cong C^*(\mathbb{Z}_n) *_{C^*(\mathbb{Z}_l)} C^*(\mathbb{Z}_m)$. By the Fourier transform, we have $C^*(\mathbb{Z}_j) \cong C((\mathbb{Z}_j)^\wedge) \cong C(\mathbb{Z}_j) \cong \mathbb{C}^j$ for $j = l, n, m$, where $(\mathbb{Z}_j)^\wedge$ is the dual group of $\mathbb{Z}_j$ and $(\mathbb{Z}_j)^\wedge \cong \mathbb{Z}_j$ and $C(\mathbb{Z}_j)$ is the C^*-algebra of continuous functions on $\mathbb{Z}_j$. Therefore, we have $K_0(C^*(\mathbb{Z}_j)) \cong K_0(\mathbb{C}^j) \cong \mathbb{Z}^j$ and $K_1(C^*(\mathbb{Z}_j)) \cong K_1(\mathbb{C}^j) \cong 0$.

It is shown by Cuntz [5, Theorem 2.4] that the amalgam $\mathbb{Z}_n *_{\mathbb{Z}_l} \mathbb{Z}_m$ is also K-amenable. $\square$

Remark. The result in the corollary above was first obtained by Natsume [8] (and it is shown that $(\mathbb{Z}^n \oplus \mathbb{Z}^m)/\mathbb{Z}^l \cong \mathbb{Z}^{n+m-l}$), but his method to deduce it is completely different from ours. The same is just using K-amenablity of Cuntz.

Furthermore,

Theorem 1.4.3 *Let G_1 and G_2 be finitely generated abelian discrete groups with a common subgroup G_3 such that*

$$G_j \cong \mathbb{Z}^{g_j} \times \mathbb{Z}_{n_{j1}} \times \cdots \times \mathbb{Z}_{n_{jl_j}}$$

for some $g_j \geq 1$ and $n_{j1} \geq 0, \cdots, n_{jl_j} \geq 0$ for some $l_j \geq 1$ $(j = 1, 2, 3)$. Let $G_1 *_{G_3} G_2$ be the amalgam of G_1 and G_2 over G_3, and $C^*(G_1 *_{G_3} G_2)$ its full group C^*-algebra. Then

$$\begin{aligned}&K_*(C^*(G_1 *_{G_3} G_2)) \cong \\ &\mathbb{Z}^{(n_{11}\times\cdots\times n_{1l_1}\times 2^{g_1-1})+(n_{21}\times\cdots\times n_{2l_2}\times 2^{g_2-1})}/\mathbb{Z}^{(n_{31}\times\cdots\times n_{3l_3}\times 2^{g_3-1})}\end{aligned}$$

for $* = 0, 1$.

If each $g_j = 0$, then

$$\begin{aligned}&K_0(C^*(G_1 *_{G_3} G_2)) \cong \mathbb{Z}^{(n_{11}\times\cdots\times n_{1l_1})+(n_{21}\times\cdots\times n_{2l_2})}/\mathbb{Z}^{(n_{31}\times\cdots\times n_{3l_3})}, \\ &K_1(C^*(G_1 *_{G_3} G_2)) \cong 0.\end{aligned}$$

Proof. By the Fourier transform,

$$C^*(G_j) \cong C(\mathbb{T}^{g_j}) \otimes C(\mathbb{Z}_{n_{j1}}^{\wedge}) \otimes \cdots \otimes C(\mathbb{Z}_{n_{jl_j}}^{\wedge}),$$

where each dual space $\mathbb{Z}_{n_{jk}}^{\wedge}$ of $\mathbb{Z}_{n_{jk}}$ is homeomorphic to $\mathbb{Z}_{n_{jk}}$. Therefore,

$$\begin{aligned}K_*(C^*(G_j)) &\cong \oplus^{n_{j1}\times\cdots\times n_{jl_j}} K_*(C(\mathbb{T}^{g_j}) \\ &\cong \oplus^{n_{j1}\times\cdots\times n_{jl_j}} \mathbb{Z}^{2^{g_j-1}} \cong \mathbb{Z}^{n_{j1}\times\cdots\times n_{jl_j}\times 2^{g_j-1}}.\end{aligned}$$

$\square$

Moreover, it is obtained that

Theorem 1.4.4 *Let G_1 and G_2 be finite groups such that $|G_1| = n$, $|G_2| = m$, and there is a common subgroup G_3 with $|G_3| = l$. Let $G_1 *_{G_3} G_2$ be the amalgam of G_1 and G_2 over G_3, and $C^*(G_1 *_{G_3} G_2)$ its full group C^*-algebra. Then*

$$K_0(C^*(G_1 *_{G_3} G_2)) \cong \mathbb{Z}^{n+m}/\mathbb{Z}^l, \quad K_1(C^*(G_1 *_{G_3} G_2)) \cong 0.$$

Proof. We may assume that there exist the following isomorphisms to the direct products:

$$G_j \cong G_{j1} \times G_{j2} \times \cdots \times G_{jp_j}, \quad (j = 1, 2, 3)$$

for some $1 \leq p_1$, $1 \leq p_2$, and $1 \leq p_3$, where each finite subgroup G_{ij} for $1 \leq i \leq 3$, $1 \leq j \leq p_i$ has no nontrivial direct factor group, i.e., indecomposable. Then we have the following isomorphisms to the tensor products of the group C^*-algebras:

$$C^*(G_j) \cong C^*(G_{j1}) \otimes C^*(G_{j2}) \otimes \cdots \otimes C^*(G_{jp_j}), \quad (j = 1, 2, 3).$$

Furthermore,

$$C^*(G_{js}) \cong \oplus_{\chi \in G_{js}^\wedge} M_{t(\chi)}(\mathbb{C})$$

where $G_{js}^\wedge$ is the unitary dual of G_{js}, and $G_{js}^\wedge \cong G_{js}$ (as a space) since G_{js} is finite (so that it is compact), and the positive integers $t(\chi)$ (≥ 1) correspond to irreducible unitary representations $\chi \in G_{js}^\wedge$, i.e., those numbers are dimensions of their representation spaces.

Thus, it follows that

$$\begin{aligned} K_0(C^*(G_j)) &\cong \otimes_{s=1}^{p_j} K_0(\oplus_{\chi \in G_{js}^\wedge} M_{t(\chi)_s}(\mathbb{C})) \\ &\cong \otimes_{s=1}^{p_j} (\oplus^{n_{js}} K_0(M_{t(\chi)_s}(\mathbb{C}))) \cong \otimes_{s=1}^{p_j} \mathbb{Z}^{n_{js}} \cong \mathbb{Z}^{n_{j1} \times \cdots \times n_{jp_j}}, \end{aligned}$$

since $K_0(M_n(\mathbb{C})) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ for any integer $n \geq 1$, where $|G_{js}^\wedge| = |G_{js}| = n_{js}$. Note that $|G_j| = n_{j1} \times \cdots \times n_{jp_j}$.

Similarly, it follows that $K_1(C^*(G_j)) \cong 0$ since $K_1(M_n(\mathbb{C})) \cong K_1(\mathbb{C}) \cong 0$ for any integer $n \geq 1$. □

2 K-theory of multi-amalgams of C^*-algebras

2.0 Introduction to the second section

K-theory for C^*-algebras has been well developed (for instance, see Blackadar [3] and Wegge-Olsen [19]). Especially, K-theory for (full and reduced) group C^*-algebras of discrete groups has several interesting and important applications to geometry, topology, and analysis, via the Baum-Connes conjecture (see [3]). It has been quite important to compute and determine K-theory of those C^*-algebras. See Pimsner-Voiculescu [12] for the case of the reduced group C^*-algebras of free groups and Lance [7] for the reduced group C^*-algebras of certain free products and Natsume [8] for the reduced and full group C^*-algebras of $SL_2(\mathbb{Z})$, where note that $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ (an amalgam of groups) is not amenable but K-amenable (see Cuntz [5]). See Cuntz [4] for certain free products of C^*-algebras including the full group C^*-algebras of free groups.

On the other hand, recall that infinite discrete groups with property T such as $SL_n(\mathbb{Z})$ $(n \geq 3)$ are not amalgams, not amenable, and not K-amenable (see Serre [14, Section 6.6] and [5]). However, it is obtained by Soulé [15] that $SL_n(\mathbb{Z})$ $(n \geq 3)$ can be written as multi-amalgams of certain finite subgroups over their intersections.

Now, this section is organized as follows. In the first subsection, from those backgrouds we study K-theory of (full) multi-amalgams of the group C^*-algebras of discrete groups, and it should be of independent interest. In the second subsection, we give an important application of the results obtained in the subsection 2.1. Namely, we compute K-groups of the full group C^*-algebras of $SL_n(\mathbb{Z})$ $(n \geq 3)$ by our theory developed here and by using Soulé [15, Theorem 9] mentioned above. This goal (in a direction) should have been achieved (almost).

Refer to Pedersen [10] (or [11]) for (full) amalgams of C^*-algebras.

2.1 Multi-amalgams of the full C^*-algebras of discrete groups

Let G_j be discrete groups for $j \in J = \{1, 2, \cdots, |J|\}$ a finite set and $C^*(G_j)$ their group C^*-algebras. Let $G_{ij} = G_i \cap G_j$ be intersections of G_i, G_j for $i, j \in J$, where G_{ij} may be the trivial subgroup of G_i and G_j. Then we have the inclusions $C^*(G_i) \supset C^*(G_{ij}) \subset C^*(G_j)$ as a C^*-subalgebra.

Let $G_i *_{G_{ij}} G_j$ be the amalgam of G_i, G_j over G_{ij}. Let $*_{i,j\in J} G_i *_{G_{ij}} G_j$ be the multi-amalgam of G_j $(j \in J)$ over G_{ij}, which is generated by G_j $(j \in J)$ as a multi-free product group and is amalgamated over G_{ij}. Note that $*_{i,j\in J} G_i *_{G_{ij}} G_j$ symbolically looks like the multi-free product of $G_i *_{G_{ij}} G_j$ (which are disjoint for $i, j \in J$), but we do not mean it by this symbol. If we use this sense, we need to identify any G_i for $i \in J$ with that of any $G_i *_{G_{ij}} G_j$ for $i, j \in J$ in $*_{i,j\in J} G_i *_{G_{ij}} G_j$ (or we assume this condition).

Let $C^*(*_{i,j\in J} G_i *_{G_{ij}} G_j)$ be the full group C^*-algebra of the multi-amalgam $*_{i,j\in J} G_i *_{G_{ij}} G_j$. Then $C^*(*_{i,j\in J} G_i *_{G_{ij}} G_j)$ is isomorphic to the unital (full) multi-amalgam $*_{\mathbb{C},i,j\in J} C^*(G_i) *_{C^*(G_{ij})} C^*(G_j)$, which is generated by $C^*(G_j)$ $(j \in J)$ as a multi-free product C^*-algebra and is amalgamated over $C^*(G_{ij})$, where we identify its unitary generators with those of $*_{i,j\in J} G_i *_{G_{ij}} G_j$ under the universal representation.

Let $*_{j\in J} G_j$ be the free product group of G_j $(j \in J)$ and $C^*(*_{j\in J} G_j)$ its full group C^*-algebra. Then $C^*(*_{j\in J} G_j)$ is isomorphic to the unital (full) multi-free product C^*-algebra $*_{\mathbb{C},j\in J} C^*(G_j)$ of $C^*(G_j)$ $(j \in J)$.

Lemma 2.1.1 *Let $*_{j\in J}G_j$ be the free product of discrete groups G_j. Suppose that G_{ij} is a common subgroups of G_i and G_j. Then we have the following exact sequence:*

$$\begin{aligned}0 \to K_*(C^*(G_{ij})) &\to K_*(C^*(*_{j\in J}G_j)) \\ &\to K_*(C^*(*_{j\in J}G_j))/K_*(C^*(G_{ij})) \to 0\end{aligned}$$

for $* = 0, 1$.

Proof. Since $C^*(*_{j\in J}G_j) \cong *_{j\in J}C^*(G_j)$, for $* = 0, 1$ we have

$$K_*(*_{j\in J}C^*(G_j)) \cong \oplus_{j\in J}K_*(C^*(G_j))$$

(see [3, 10.11.11]). There is a canonical inclusion from the $n \times n$ matrix algebra $M_n(C^*(G_{ij}))$ over $C^*(G_{ij})$ to $M_n(C^*(G_j)) \subset M_n(*_{j\in J}C^*(G_j))$ (or $M_n(C^*(G_i)) \subset M_n(*_{j\in J}C^*(G_j))$), where we identity those images. It follows from freeness in $M_n(*_{j\in J}C^*(G_j))$ that the inclusion induces the (diagonal) injections from $K_*(C^*(G_{ij}))$ to $K_*(*_{j\in J}C^*(G_j)) \cong \oplus_{j\in J}K_*(C^*(G_j))$ $(* = 0, 1)$. Therefore, the exact sequence as in the statement is obtained. □

Proposition 2.1.2 *Let $H = *_{k\in J\setminus\{i,j\}}G_k * (G_i *_{G_{ij}} G_j)$ be an amalgam of G_j over a G_{ij}. Then*

$$K_*(H) \cong K_*(C^*(*_{j\in J}G_j))/K_*(C^*(G_{ij})).$$

Proof. By universality, there exists a $*$-homomorphism from $*_{j\in J}C^*(G_j)$ onto $C^*(H)$. This implies a group homomorphism from $K_*(*_{j\in J}C^*(G_j))$ to $K_*(C^*(H))$. Note that any element (in particular, projection) of $M_n(C^*(H))$ can be viewed as that of $M_n(*_{j\in J}C^*(G_j))$, so that the group homomorphism is onto. Furthermore, the quotient group obtained in the lemma above can be viewed as a subgroup of $K_*(C^*(H))$. Therefore, we obtain the isomorphism as in the statement. □

Theorem 2.1.3 *Let $C^*(*_{i,j\in J}G_i *_{G_{ij}} G_j)$ be the full group C^*-algebra of a multi-amalgam $*_{i,j\in J}G_i *_{G_{ij}} G_j$ of discrete groups G_i, G_j over their intersections G_{ij}. Then*

$$K_*(C^*(*_{i,j\in J}G_i *_{G_{ij}} G_j)) \cong K_*(C^*(*_{j\in J}G_j))/\langle K_*(C^*(G_{ij}))\rangle_{ij},$$

where $\langle K_*(C^*(G_{ij}))\rangle_{ij}$ means the subgroup generated by $K_*(C^*(G_{ij}))$ in $K_*(C^*(*_{j\in J}G_j))$.

Proof. This can be proved using the above proposition repeatedly. □

Corollary 2.1.4 *If* $\langle K_*(C^*(G_{ij}))\rangle_{ij} \cong \oplus_{ij} K_*(C^*(G_{ij}))$*, then*

$$K_*(C^*(*_{i,j\in J} G_i *_{G_{ij}} G_j)) \cong \oplus_j K_*(C^*(G_j))/\oplus_{ij} K_*(C^*(G_{ij})).$$

This is the case where $*_{i,j\in J} G_i *_{G_{ij}} G_j$ *is a chain (or loop) amalgam in the sense that each* G_j *is viewed as a vertex and each* G_{ij} *is viewed as an edge between* G_i *and* G_j*.*

2.2 Application to the full C^*-algebra of $SL_n(\mathbb{Z})$

We now briefly review about multi-amalgams of (discrete) groups, and about $SL_n(\mathbb{Z})$ ($n \geq 3$) as multi-amalgams.

Let G be a (discrete) group and $(G_j)_j \in J$ a (finite) family of subgroups of G. Recall that G is a multi-amalgam of G_j over their intersection subgroups, i.e., $G = *_{i,j\in J} G_i *_{G_{ij}} G_j$ in our notation above, if the following universal property holds: For H a group with homomorphisms $\varphi_j : G_j \to H$ such that $\varphi_i = \varphi_j$ on the intersection subgroup $G_i \cap G_j$, there exists a unique homomorphism φ from G to H such that $\varphi = \varphi_j$ on G_j (see [15]). It is not difficult to see that the multi-amalgam $*_{i,j\in J} G_i *_{G_{ij}} G_j$ has this universal property.

It is shown by Soulé [15, Theorem 9] that the group $SL_3(\mathbb{Z})$ is isomorphic to a multi-amalgam of four copies of the symmetric group S_4, one of the dyhedral group D_6 and two of the dyhedral group D_4 over their intersection subgroups.

Moreover, it is shown by Soulé [15, Theorem 9] that the group $SL_n(\mathbb{Z})$ ($n \geq 4$) is a multi-amalgam of $8 \cdot {}_n\mathrm{C}_3$ copies of S_4, $9 \cdot {}_n\mathrm{C}_3$ copies of D_4, $3 \cdot {}_n\mathrm{C}_3$ copies of D_6, and ${}_n\mathrm{C}_4$ copies of the following direct products of finite cyclic groups: $\mathbb{Z}_4 \times \mathbb{Z}_3$, $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ over their intersections.

It is known (by Serre [13] or [15]) that S_4 has two generators a, b such that $a^4 = b^2 = (ab)^3 = 1$, and D_4 has two generators a, b such that $a^4 = b^2 = (ab)^2 = 1$, and D_6 has two generators a, b such that $a^6 = b^2 = (ab)^2 = 1$.

It is known (by Serre [13]) that the symmetric group S_4 has two 1-dimensional irreducible representations and one 2-dimensional irreducible representation. It follows from this fact that the group C^*-algebra $C^*(S_4)$ of S_4 is isomorphic to the direct sum: $\mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) = \mathbb{C}^2 \oplus M_2(\mathbb{C})$.

It is also known ([13]) that the dyhedral group D_n for n even has four 1-dimensional irreducible representations and two 2-dimensional irreducible representations. It follows from this fact that the group C^*-algebra $C^*(D_n)$

of D_n is isomorphic to the direct sum:

$$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) = \mathbb{C}^4 \oplus M_2(\mathbb{C})^2.$$

The group C^*-algebra $C^*(\mathbb{Z}_s \times \mathbb{Z}_t)$ of the product $\mathbb{Z}_s \times \mathbb{Z}_t$ of cyclic groups $\mathbb{Z}_s, \mathbb{Z}_t$ is isomorphic to the tensor product: $C^*(\mathbb{Z}_s) \otimes C^*(\mathbb{Z}_t) \cong \mathbb{C}^s \otimes \mathbb{C}^t \cong \mathbb{C}^{st}$.

It is shown by Cuntz [5] that $SL_2(\mathbb{Z})$ is K-amenable so that K-theory of its full and reduced group C^*-algebras is the same. However, the groups with Kazhdan's property T such as $SL_n(\mathbb{Z})$ $(n \geq 3)$ are not K-amenable (see [3, Section 20.9]).

It follows from the facts above (especially by virtue of Soulé [15, Theorem 9]) and our Theorem 2.1.3 that

Theorem 2.2.1 *Let $C^*(SL_3(\mathbb{Z}))$ be the full group C^*-algebra of $SL_3(\mathbb{Z})$. Then*

$$K_0(C^*(SL_3(\mathbb{Z}))) \cong K_0(C^*(*_{1 \leq j \leq 7} G_j))/\langle K_0(C^*(G_{ij}))\rangle_{ij}$$
$$K_1(C^*(SL_3(\mathbb{Z}))) \cong 0 \cong K_1(C^*(*_{1 \leq j \leq 7} G_j)),$$

where $G_j = S_4$ for $1 \leq j \leq 4$, $G_j = D_4$ for $j = 5, 6$, and $G_7 = D_6$, and $G_{ij} = G_i \cap G_j$ in $SL_3(\mathbb{Z})$, and $C^*(*_{1 \leq j \leq 7} G_j) \cong *_{\mathbb{C}, 1 \leq j \leq 7} C^*(G_j)$ the unital full free product of $C^*(G_j)$, for the free product $*_{1 \leq j \leq 7} G_j$.

Corollary 2.2.2 *We have*

$$K_0(C^*(SL_3(\mathbb{Z}))) \cong (\mathbb{Z}^3 \oplus \mathbb{Z}^3 \oplus \mathbb{Z}^3 \oplus \mathbb{Z}^3 \oplus \mathbb{Z}^6 \oplus \mathbb{Z}^6 \oplus \mathbb{Z}^6)/\mathbb{Z}^L \cong \mathbb{Z}^{30}/\mathbb{Z}^L,$$
$$K_1(C^*(SL_3(\mathbb{Z}))) \cong 0,$$

where L is the number of generators of $\langle K_0(C^*(G_{ij}))\rangle_{ij}$.

Proof. Let $\mathfrak{A}_j = C^*(G_j)$ for $1 \leq j \leq 7$. Note that we have

$$K_0(\mathfrak{A}_j) \cong \oplus^3 K_0(\mathbb{C}) \cong \mathbb{Z}^3, \quad K_1(\mathfrak{A}_j) \cong \oplus^3 K_1(\mathbb{C}) \cong 0$$

for $\mathfrak{A}_j = C^*(S_4) \cong \mathbb{C}^2 \oplus M_2(\mathbb{C})$ for $1 \leq j \leq 4$ since $K_*(M_2(\mathbb{C})) \cong K_*(\mathbb{C})$ $(* = 0, 1)$, and

$$K_0(\mathfrak{A}_j) \cong \oplus^6 K_0(\mathbb{C}) \cong \mathbb{Z}^6, \quad K_1(\mathfrak{A}_j) \cong \oplus^6 K_1(\mathbb{C}) \cong 0$$

for $\mathfrak{A}_j = C^*(D_4) \cong \mathbb{C}^4 \oplus M_2(\mathbb{C})^2$ for $j = 5, 6$, and $\mathfrak{A}_7 = C^*(D_6) \cong \mathbb{C}^4 \oplus M_2(\mathbb{C})^2$. □

Remark. The number L could be computed from [15]. But this task that seems to involve direct computation to take a large space would be carried out elsewhere (or it might be not difficult for specialists on this subject).

Moreover, by Theorem 2.1.3 and the facts above we obtain

Theorem 2.2.3 *Let $C^*(SL_n(\mathbb{Z}))$ be the full group C^*-algebra of $SL_n(\mathbb{Z})$ $(n \geq 4)$. Then*

$$K_0(C^*(SL_n(\mathbb{Z}))) \cong K_0(C^*(*_{1\leq j\leq N_6} G_j))/\langle K_0(C^*(G_{ij}))\rangle_{ij}$$
$$K_1(C^*(SL_n(\mathbb{Z}))) \cong 0 \cong K_1(C^*(*_{1\leq j\leq N_6} G_j)),$$

where $G_j = S_4$ for $1 \leq j \leq 8 \cdot {}_n\mathrm{C}_3 = N_1$, and $G_j = D_4$ for $N_1 + 1 \leq j \leq N_1 + 9 \cdot {}_n\mathrm{C}_3 = N_2$, and $G_j = D_6$ for $N_2 + 1 \leq j \leq N_2 + 3 \cdot {}_n\mathrm{C}_3 = N_3$, and $G_j = \mathbb{Z}_3 \times \mathbb{Z}_3$ for $N_3 + 1 \leq j \leq N_3 + {}_n\mathrm{C}_4 = N_4$, and $G_j = \mathbb{Z}_4 \times \mathbb{Z}_3$ for $N_4+1 \leq j \leq N_4+{}_n\mathrm{C}_4 = N_5$, and $G_j = \mathbb{Z}_4\times\mathbb{Z}_4$ for $N_5+1 \leq j \leq N_5+{}_n\mathrm{C}_4 = N_6$, and $G_{ij} = G_i \cap G_j$ in $SL_n(\mathbb{Z})$, and $C^*(*_{1\leq j\leq N_6} G_j) \cong *_{\mathbb{C},1\leq j\leq N_6} C^*(G_j)$ the unital full free product of $C^*(G_j)$, for the free product $*_{1\leq j\leq N_6} G_j$.

Corollary 2.2.4 *For $n \geq 4$, we have*

$$\begin{aligned}
K_0(C^*(SL_n(\mathbb{Z}))) &\cong [(\oplus^{N_1}\mathbb{Z}^3) \oplus (\oplus^{N_2-N_1}\mathbb{Z}^6) \oplus (\oplus^{N_3-N_2}\mathbb{Z}^6) \\
&\quad \oplus (\oplus^{N_4-N_3}\mathbb{Z}^9) \oplus (\oplus^{N_5-N_4}\mathbb{Z}^{12}) \oplus (\oplus^{N_6-N_5}\mathbb{Z}^{16})]/\mathbb{Z}^L \\
&\cong \mathbb{Z}^{-3N_1-3N_3-3N_4-4N_5+16N_6}/\mathbb{Z}^L, \\
&= \mathbb{Z}^{(6\cdot 20-24)_n C_3+(16\cdot 3-11)_n C_4+1}/\mathbb{Z}^L \\
&= \mathbb{Z}^{96_n C_3+37_n C_4+1}/\mathbb{Z}^L, \\
K_1(C^*(SL_n(\mathbb{Z}))) &\cong 0,
\end{aligned}$$

where L is the number of generators of $\langle K_0(C^*(G_{ij}))\rangle_{ij}$.

Proof. Note that we have

$$K_0(\mathfrak{A}_j) \cong \oplus^3 K_0(\mathbb{C}) \cong \mathbb{Z}^3, \quad K_1(\mathfrak{A}_j) \cong \oplus^3 K_1(\mathbb{C}) \cong 0$$

for $\mathfrak{A}_j = C^*(S_4) \cong \mathbb{C}^2 \oplus M_2(\mathbb{C})$ for $1 \leq j \leq N_1$, and

$$K_0(\mathfrak{A}_j) \cong \oplus^6 K_0(\mathbb{C}) \cong \mathbb{Z}^6, \quad K_1(\mathfrak{A}_j) \cong \oplus^6 K_1(\mathbb{C}) \cong 0$$

for $\mathfrak{A}_j = C^*(D_4) \cong \mathbb{C}^4 \oplus M_2(\mathbb{C})^2$ for $N_1 + 1 \leq j \leq N_2$, and

$$K_0(\mathfrak{A}_j) \cong \oplus^6 K_0(\mathbb{C}) \cong \mathbb{Z}^6, \quad K_1(\mathfrak{A}_j) \cong \oplus^6 K_1(\mathbb{C}) \cong 0$$

for $\mathfrak{A}_j = C^*(D_6) \cong \mathbb{C}^4 \oplus M_2(\mathbb{C})^2$ for $N_2 + 1 \leq j \leq N_3$, and

$$K_0(\mathfrak{A}_j) \cong \oplus^9 K_0(\mathbb{C}) \cong \mathbb{Z}^9, \quad K_1(\mathfrak{A}_j) \cong \oplus^9 K_1(\mathbb{C}) \cong 0$$

for $\mathfrak{A}_j = C^*(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong C^*(\mathbb{Z}_3) \otimes C^*(\mathbb{Z}_3) \cong \mathbb{C}^9$ for $N_3 + 1 \leq j \leq N_4$, and

$$K_0(\mathfrak{A}_j) \cong \oplus^{12} K_0(\mathbb{C}) \cong \mathbb{Z}^{12}, \quad K_1(\mathfrak{A}_j) \cong \oplus^{12} K_1(\mathbb{C}) \cong 0$$

for $\mathfrak{A}_j = C^*(\mathbb{Z}_4 \times \mathbb{Z}_3) \cong C^*(\mathbb{Z}_4) \otimes C^*(\mathbb{Z}_3) \cong \mathbb{C}^{12}$ for $N_4 + 1 \leq j \leq N_5$, and

$$K_0(\mathfrak{A}_j) \cong \oplus^{16} K_0(\mathbb{C}) \cong \mathbb{Z}^{16}, \quad K_1(\mathfrak{A}_j) \cong \oplus^{16} K_1(\mathbb{C}) \cong 0$$

for $\mathfrak{A}_j = C^*(\mathbb{Z}_4 \times \mathbb{Z}_4) \cong C^*(\mathbb{Z}_4) \otimes C^*(\mathbb{Z}_4) \cong \mathbb{C}^{16}$ for $N_5 + 1 \leq j \leq N_6$.

By definition, note that $N_1 = 8{}_nC_3$, and

$$N_4 = 8{}_nC_3 + 9{}_nC_3 + 3{}_nC_3 + {}_nC_4,$$

and $N_5 = N_4 + {}_nC_4$, and $N_6 = N_5 + {}_nC_4$. By direct computation we can show the equalities in the statement. □

Remark. The number L could be computed from [15] (but its computation would be complicated), and we could say the similar thing as in the remark for Corollary 2.2.2.

Moreover, in general

Theorem 2.2.5 *Let $G = *_{i,j\in J} G_i *_{G_{ij}} G_j$ be a multi-amalgam of finite groups G_j over their intersections $G_{ij} = G_i \cap G_j$, where $J = \mathbb{N}$ is the set of all natural numbers. Then*

$$\begin{aligned} K_0(C^*(G)) &\cong \lim_{t\to\infty} K_0(C^*(H_t)) \\ &\cong \lim_{t\to\infty} K_0(C^*(*_{j\in J_t} G_j))/\langle K_0(C^*(G_{ij}))\rangle_{i,j\in J_t} \\ &\cong \lim_{t\to\infty} (\oplus_{j\in J_t} \mathbb{Z}^{n_j})/\mathbb{Z}^{L_t} \\ &\cong \lim_{t\to\infty} \mathbb{Z}^{n_1+\cdots+n_t}/\mathbb{Z}^{L_t}, \\ K_1(C^*(G)) &\cong \lim_{t\to\infty} K_1(C^*(H_t)) \cong 0, \end{aligned}$$

where $\lim_{t\to\infty}$ means inductive limit of groups, $J_t = \{1, 2, \cdots, t\}$ are finite subsets of J, and $H_t = *_{i,j\in J_t} G_i *_{G_{ij}} G_j$, and $|G_j| = n_j$ for $j \in J$, and L_t is the number of generators of $\langle K_0(C^*(G_{ij}))\rangle_{i,j\in J_t}$.

Proof. It is clear that G is an inductive limit of H_t. Since G and H_t are discrete, $C^*(G)$ is also an inductive limit of $C^*(H_t)$. Therefore,

$$K_*(C^*(G)) \cong K_*(\lim_{t\to\infty} C^*(H_t)) \cong \lim_{t\to\infty} K_*(C^*(H_t))$$

for $* = 0, 1$ by continuity of K-groups. Since J_t are finite, we use Theorem 2.1.3 to deduce the conclusions. Note that $C^*(G_j) \cong \oplus_{\chi\in G_j^\wedge} M_{n_\chi}(\mathbb{C})$ for some $n_\chi \geq 1$, where $G_j^\wedge$ means the space of equivalence classes of irreducible representations χ of G_j. Since $|G_j| = |G_j^\wedge| = n_j$, we have $K_0(C^*(G_j)) \cong \mathbb{Z}^{n_j}$ and $K_1(C^*(G_j)) \cong 0$. □

Remark. The numbers L_t could be computed if G_j are given concretely. Moreover, we could apply our arguments for more general C^*-algebras beyond multi-amalgams of group C^*-algebras of finite groups.

2.3 Generalization to multi-amalgams of C^*-algebras

Let $*_{j\in J}\mathfrak{A}_j$ be a full free product C^*-algebra of C^*-algebras $\mathfrak{A}_j$ for $j \in J$ a countable set, where if J is infinite, then $*_{j\in J}\mathfrak{A}_j$ is defined to be an inductive limit of the full free product C^*-algebras corresponding to finite subsets of J.

Lemma 2.3.1 *Let $*_{j\in J}\mathfrak{A}_j$ be the full free product of C^*-algebras $\mathfrak{A}_j$. Suppose that $\mathfrak{B}_{ij}$ is a common C^*-subalgebra of $\mathfrak{A}_i$ and $\mathfrak{A}_j$. Then we have the following exact sequence:*

$$0 \to K_*(\mathfrak{B}_{ij}) \to K_*(*_{j\in J}\mathfrak{A}_j) \to K_*(*_{j\in J}\mathfrak{A}_j)/K_*(\mathfrak{B}_{ij}) \to 0$$

for $* = 0, 1$.

Proof. Note that $K_*(*_{j\in J}\mathfrak{A}_j) \cong \oplus_{j\in J}K_*(\mathfrak{A}_j)$ (see [3, 10.11.11]), where if J is infinite, we use continuity of K-theory groups with respect to inductive limits of C^*-algebras. There is a canonical inclusion from the $n \times n$ matrix algebra $M_n(B_{ij})$ over $\mathfrak{B}_{ij}$ to $M_n(\mathfrak{A}_j) \subset M_n(*_{j\in J}\mathfrak{A}_j)$ (or $M_n(\mathfrak{A}_i) \subset M_n(*_{j\in J}\mathfrak{A}_j)$), where we identify those images. It follows from freeness in $M_n(*_{j\in J}\mathfrak{A}_j)$ that the inclusion induces the (diagonal) injections from $K_*(\mathfrak{B}_{ij})$ to $K_*(*_{j\in J}\mathfrak{A}_j) \cong \oplus_{j\in J}K_*(\mathfrak{A}_j)$ $(* = 0,1)$. Therefore, the exact sequence as in the statement is obtained. □

Proposition 2.3.2 *Let $\mathfrak{C} = *_{k\in J\setminus\{i,j\}}\mathfrak{A}_k * (\mathfrak{A}_i *_{\mathfrak{B}_{ij}} \mathfrak{A}_j)$ be an amalgam of C^*-algebras $\mathfrak{A}_j$ over a $\mathfrak{B}_{ij}$. Then*

$$K_*(\mathfrak{C}) \cong K_*(*_{j\in J}\mathfrak{A}_j)/K_*(\mathfrak{B}_{ij}).$$

Proof. By universality, there exists a $*$-homomorphism from $*_{j\in J}\mathfrak{A}_j$ onto $\mathfrak{C}$. This implies a group homomorphism from $K_*(*_{j\in J}\mathfrak{A}_j)$ to $K_*(\mathfrak{C})$. Note that any element (in particular, projection) of $M_n(\mathfrak{C})$ can be viewed as that of $M_n(*_{j\in J}\mathfrak{A}_j)$, so that the group homomorphism is onto. Furthermore, the quotient group obtained in the lemma above can be viewed as a subgroup of $K_*(\mathfrak{C})$. Therefore, we obtain the isomorphism as in the statement. □

Let $*_{i,j\in J}(\mathfrak{A}_i *_{\mathfrak{B}_{ij}} \mathfrak{A}_j)$ be the full multi-amalgam of C^*-algebras $\mathfrak{A}_i$, $\mathfrak{A}_j$ over their common C^*-subalgebras $\mathfrak{B}_{ij}$ for J a countable set, where we assume the notational convention as used before, i.e., each $\mathfrak{A}_i$ (or $\mathfrak{A}_j$) in $\mathfrak{A}_i *_{\mathfrak{B}_{ij}} \mathfrak{A}_j$ is identified in its copy in $*_{i,j\in J}(\mathfrak{A}_i *_{\mathfrak{B}_{ij}} \mathfrak{A}_j)$ (such as $\mathfrak{A}_k *_{\mathfrak{B}_{ki}} \mathfrak{A}_i$).

Theorem 2.3.3 *Let $*_{i,j\in J}(\mathfrak{A}_i *_{\mathfrak{B}_{ij}} \mathfrak{A}_j)$ be the full multi-amalgam of C^*-algebras $\mathfrak{A}_i$, $\mathfrak{A}_j$ over their common C^*-subalgebras $\mathfrak{B}_{ij}$. Then*

$$K_*(*_{i,j\in J}(\mathfrak{A}_i *_{\mathfrak{B}_{ij}} \mathfrak{A}_j)) \cong K_*(*_{j\in J}\mathfrak{A}_j)/\langle K_*(\mathfrak{B}_{ij})\rangle_{ij},$$

where $\langle K_*(\mathfrak{B}_{ij})\rangle_{ij}$ means the subgroup generated by $K_*(\mathfrak{B}_{ij})$ in $K_*(*_{j\in J}\mathfrak{A}_j)$.

Proof. This can be proved using the above proposition repeatedly. □

3 Appendix on K-theory and beyond

3.1 The reduced crossed products by free groups

Taken from Pimsner-Voiculescu [12] are those as in what follows.

Let F_n be the free group of n generators g_j $(1 \le j \le n)$. Let Γ_k denote the subset of F_n consisting of elements $g_{i_1}^{m_1} \cdots g_{i_s}^{m_s}$ $(s \ge 0,\ i_1 \ne i_2,\ \cdots,\ i_{s-1} \ne i_s,\ m_1 \ne 0,\ \cdots,\ m_s \ne 0)$ such that $m_s > 0$ if $i_s = k$. Note that the identity element e of F_n is in Γ_k and $g_j\Gamma_k = \Gamma_k$ if $j \ne k$ and $g_k\Gamma_k = \Gamma_k \setminus \{e\}$.

Let $\mathfrak{A}$ be a unital C^*-algebra with an action $\alpha : F_n \to \mathrm{Aut}(\mathfrak{A})$. Assume that $\mathfrak{A}$ acts on a Hilbert space H_0 and there exists a unitary representation v of F_n on H_0 such that $v_g a v_g^* = \alpha(g)a$ for $a \in \mathfrak{A}$ and $g \in F_n$.

Denote by $l^2(X)$ the Hilbert space of all square summable functions on a discrete space X in what follows. The reduced crossed product $\mathfrak{A} \rtimes_{\alpha,r} F_n$ of $\mathfrak{A}$ by α is defined to be the C^*-algebra generated by $\pi(a) = 1 \otimes a$ and u_g for $a \in \mathfrak{A}$ and $g \in F_n$ on the Hilbert space $l^2(F_n, H_0) \cong l^2(F_n) \otimes H_0$, where $(u_g f)(h) = v_g f(g^{-1}h)$ for $h \in F_n$ and $f \in l^2(F_n, H_0)$. Set $u_j = u_{g_j}$ and $v_j = v_{g_j}$.

Let ρ, $U_j (1 \le j \le n-1)$, and S denote the restrictions of π, $u_j (1 \le j \le n-1)$, and u_n respectively. The Toeplitz algebra $\mathfrak{T}_n$ is defined to be the C^*-algebra generated by $\rho(\mathfrak{A})$, U_j $(1 \le j \le n-1)$, and S_n.

We have $1 - S_n S_n^* = P_e$, where P_e is the orthogonal projection of $l^2(\Gamma_n, H_0)$ onto its subspace $l^2(\{e\}, H_0)$. Note that U_j $(1 \le j \le n-1)$ are unitaries and $U_j\rho(a)U_j^* = \rho(\alpha(g_j)a)$ for $1 \le j \le n-1$, and $S_n^*\rho(a)S_n = \rho(\alpha(g_n^{-1})a)$.

Denote by $\mathbb{K}(H) = \mathbb{K}$ the C^*-algebra of all compact operators on a Hilbert space H in what follows. The closed ideal $\mathfrak{J}$ in $\mathfrak{T}_n$ generated by P_e is isomorphic to $\mathfrak{A} \otimes \mathbb{K}(l^2(\Gamma_n))$. In fact, denote by $e(g', g)$ the natural matrix unit for $\mathbb{K}(l^2(\Gamma_n))$. The isomorphism is given by the correspondence: $e(g', g) \mapsto w(g')P_e w(g^{-1})\rho(\alpha(g)a)$, where $w(g)$ for $g = g_{i_1}^{k_1} \cdots g_{i_m}^{k_m}$ is given by replacing g_j with U_j for $1 \le j \le n-1$, g_n^k with S_n^k for $k > 0$ and g_n^{-k} with $(S_n^*)^k$ for $k > 0$.

Lemma 3.1.1 *We have the following exact sequence:*

$$0 \longrightarrow \mathfrak{A} \otimes \mathbb{K} \xrightarrow{\psi} \mathfrak{T}_n \xrightarrow{p} \mathfrak{A} \rtimes_{\alpha,r} F_n \longrightarrow 0,$$

where the map p is defined by $p(\rho(a)) = \pi(a)$, $p(U_j) = u_j$ $(1 \le j \le n-1)$, and $p(S_n) = u_n$, and the kernel of p is generated by P_e and is isomorphic to $\mathfrak{A} \otimes \mathbb{K}$.

It is easy to see that $\rho(\mathfrak{A})$ and U_j $(1 \le j \le n-1)$ determine an injective $*$-homomorphism $d : \mathfrak{A} \rtimes_{\alpha,r} F_{n-1} \to \mathfrak{T}_n$. Let $i : \mathfrak{A} \to \mathfrak{A} \rtimes_{\alpha,r} F_{n-1}$ be the canonical inclusion, so that $d \circ i = \rho$. Let $\alpha_n : \mathbb{Z} \to \mathrm{Aut}(\mathfrak{A})$ be the action given by $\alpha_n(k) = \alpha(g_n^k)$ for $k \in \mathbb{Z}$. Let $\mathfrak{T}(\mathfrak{A}, \alpha_n, \mathbb{Z})$ the Toeplitz algebra corresponding to the action α_n (or the crossed product of $\mathfrak{A}$ by α_n of $\mathbb{Z}$). Since $\rho(\mathfrak{A}) \cup \{S_n\} \subset \mathfrak{T}_n$, we have an injective $*$-homomorphism $t : \mathfrak{T}(\mathfrak{A}, \alpha_n, \mathbb{Z}) \to \mathfrak{T}_n$.

Lemma 3.1.2 *The following diagram commutes:*

$$\begin{tikzcd} K_j(\mathfrak{A} \otimes \mathbb{K}) \arrow[r, "\psi_*"] \arrow[d, equal] & K_j(\mathfrak{T}_n) \\ K_j(\mathfrak{A}) \arrow[r, "\mathrm{id}_* - (\alpha(g_n^{-1}))_*"] & K_j(\mathfrak{A}). \arrow[u, "\rho_*"'] \end{tikzcd}$$

Sketch of Proof. Use the following factorization $\mathfrak{A} \to \mathfrak{T}(\mathfrak{A}, \alpha_n, \mathbb{Z}) \to \mathfrak{T}_n$. □

Theorem 3.1.3 *Let $\mathfrak{A}$ be a C^*-algebra. Then there exists the following six-term exact sequence:*

$$\begin{tikzcd} K_0(\mathfrak{A}) \arrow[r, "\iota_*"] & K_0(\mathfrak{A} \rtimes_{\alpha,r} F_{n-1}) \arrow[r, "k_*"] & K_0(\mathfrak{A} \rtimes_{\alpha,r} F_n) \arrow[d] \\ K_1(\mathfrak{A} \rtimes_{\alpha,r} F_n) \arrow[u] & K_1(\mathfrak{A} \rtimes_{\alpha,r} F_{n-1}) \arrow[l, "k_*"'] & K_1(\mathfrak{A}) \arrow[l, "\iota_*"'] \end{tikzcd}$$

where $\iota_* = i_* \circ (\mathrm{id}_* - \alpha(g_n^{-1})_*)$ and the map k is the natural inclusion from $\mathfrak{A} \rtimes_{\alpha,r} F_{n-1}$ to $\mathfrak{A} \rtimes_{\alpha,r} F_n$ and the vertical arrows correspond to the connecing homomorphisms in the exact sequence for the Toeplitz extension.

Sketch of Proof. Consider the follwoing diagram for the Toeplitz extension:

$$\begin{tikzcd} K_0(\mathfrak{A} \otimes \mathbb{K}) \arrow[r, "\psi_*"] & K_0(\mathfrak{T}_n) \arrow[r, "p_*"] & K_0(\mathfrak{A} \rtimes_{\alpha,r} F_n) \arrow[d] \\ K_1(\mathfrak{A} \rtimes_{\alpha,r} F_n) \arrow[u] & K_1(\mathfrak{T}_n) \arrow[l, "p_*"'] & K_1(\mathfrak{A} \otimes \mathbb{K}). \arrow[l, "\psi_*"'] \end{tikzcd}$$

We have the commutative diagram

$$\begin{array}{ccc} K_j(\mathfrak{A}\otimes\mathbb{K}) & \xrightarrow{\psi_*} & K_j(\mathfrak{T}_n) \\ \| & & \uparrow{\scriptstyle\rho_*} \\ K_j(\mathfrak{A}) & \xrightarrow{\mathrm{id}_*-\alpha(g_n^{-1})_*} & K_j(\mathfrak{A}) \end{array}$$

for $j = 0, 1$. Since $\rho = d \circ i$, the diagram

$$\begin{array}{ccc} K_j(\mathfrak{A}\otimes\mathbb{K}) & \xrightarrow{\psi_*} & K_j(\mathfrak{T}_n) \\ \| & & \uparrow{\scriptstyle d_*} \\ K_j(\mathfrak{A}) & \xrightarrow{i_*\circ(\mathrm{id}_*-\alpha(g_n^{-1})_*)} & K_j(\mathfrak{A}\rtimes_{\alpha,r}F_{n-1}) \end{array}$$

is also commutative. Now we have that d_* is an isomorphism so that $\psi_* : K_j(\mathfrak{A}\otimes\mathbb{K}) \to K_j(\mathfrak{T}_n)$ can be replaced with $i_* \circ (\mathrm{id}_* - \alpha(g_n^{-1})_*) : K_j(\mathfrak{A}) \to K_j(\mathfrak{A}\rtimes_{\alpha,r}F_{n-1})$. □

Corollary 3.1.4 *We have $K_0(C_r^*(F_n)) \cong \mathbb{Z}$ whose generator is $[1]$, and $K_1(C_r^*(F_n)) \cong \mathbb{Z}^n$ whose generators are $[u_j]$ $(1 \le j \le n)$.*

Sketch of Proof. Let $\mathfrak{A} = \mathbb{C}$ and α trivial in Theorem 3.1.3. Then we have $K_0(C_r^*(F_{n-1}) \cong K_0(C_r^*(F_n))$ for $n \ge 1$, where $C_r^*(F_0) = \mathbb{C}$. This shows that $K_0(C_r^*(F_n)) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$.

For K_1, we obtain the following:

$$0 \longrightarrow K_1(C_r^*(F_{n-1})) \xrightarrow{k_*} K_1(C_r^*(F_n)) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the map $K_1(C_r^*(F_n)) \to \mathbb{Z}$ is given by the index corresponding to the extension: $0 \to \mathbb{K} \to \mathfrak{T}_n \to C_r^*(F_n) \to 0$. Since $u_n \in C_r^*(F_n)$ has index 1, it follows that $K_1(C_r^*(F_n))$ is isomorphic to the sum $k_*(K_1(C_r^*(F_{n-1}))) \oplus \mathbb{Z}[u_n]$, where k_* is injective. Repeating this argument we obtain the conclusion. □

Corollary 3.1.5 *There exists no non-trivial projection of $C_r^*(F_n)$.*

Sketch of Proof. There exists a faithful tracial state τ on $C_r^*(F_n)$. The range of the homomorphism from $K_0(C_r^*(F_n))$ to $\mathbb{R}$ by τ is $\mathbb{Z}$ by the corollary above. Hence the trace of a projection of $C_r^*(F_n)$ can be only 0 or 1. Thus, the only projections of $C_r^*(F_n)$ are 0 and 1. $\square$

Denote by $\mathfrak{T}_{n,k}$ the Toeplitz algebra arising from Γ_k as $\mathfrak{A} \rtimes_{\alpha,r} F_n$, and by $\partial_{n,k}$ the corresponding connecting homomorphism $K_j(\mathfrak{A} \rtimes_{\alpha,r} F_n) \to K_{j+1}(\mathfrak{A} \otimes \mathbb{K})$. Define

$$\mathfrak{B}_n = \{x_1 \oplus \cdots \oplus x_n \in \oplus_{k=1}^n \mathfrak{T}_{n,k} \mid p_{n,1}(x_1) = \cdots = p_{n,n}(x_n)\}$$

as a fibered product of $\mathfrak{T}_{n,k}$ over $\mathfrak{A} \rtimes_{\alpha,r} F_n$, where $p_{n,k} : \mathfrak{T}_{n,k} \to \mathfrak{A} \rtimes_{\alpha,r} F_n$ $(1 \leq k \leq n)$ are surjections. Thus, there exists the following:

$$0 \to \oplus^n (\mathfrak{A} \otimes \mathbb{K}) \to \mathfrak{B}_n \to \mathfrak{A} \rtimes_{\alpha,r} F_n \to 0.$$

Consider $\rho_{n,k} : \mathfrak{A} \to \mathfrak{T}_{n,k}$ as ρ for $k = n$ and consider

$$\rho_n : \mathfrak{A} \to \mathfrak{B}_n, \quad \rho_n(a) = \oplus_{j=1}^n \rho_{n,j}(a).$$

Then $\mathfrak{T}_{n,k}$ is generated by $\rho_{n,k}(\mathfrak{A})$, $S_{n,k}$, and $U_{n,k,j}$ for $1 \leq j \leq n$ and $j \neq k$. Also, $\mathfrak{B}_n$ is generated by the isometries:

$$\sigma_{n,j} = U_{n,1,j} \oplus \cdots \oplus U_{n,j-1,j} \oplus S_{n,j} \oplus U_{n,j+1,j} \oplus \cdots \oplus U_{n,n,j}$$

for $1 \leq j \leq n$ and $\rho_n(\mathfrak{A})$. There exist unital $*$-homomorphisms $d_n : \mathfrak{B}_{n-1} \to \mathfrak{B}_n$ such that $d_n(\rho_{n-1}(a)) = \rho_n(a)$ and $d_n(\sigma_{n-1,j}) = \sigma_{n,j}$ for $1 \leq j \leq n-1$.

Theorem 3.1.6 *There exists the following six-term exact sequence:*

$$\begin{array}{ccccc}
\oplus^n K_0(\mathfrak{A}) & \xrightarrow{\beta} & K_0(\mathfrak{A}) & \xrightarrow{\pi_*} & K_0(\mathfrak{A} \rtimes_{\alpha,r} F_n) \\
\delta \uparrow & & & & \downarrow \delta \\
K_1(\mathfrak{A} \rtimes_{\alpha,r} F_n) & \xleftarrow{\pi_*} & K_1(\mathfrak{A}) & \xleftarrow{\beta} & \oplus^n K_1(\mathfrak{A}),
\end{array}$$

where $\beta(\oplus_{j=1}^n \gamma_j) = \sum_{j=1}^n (\gamma_j - (\alpha(g_j^{-1}))_* \gamma_j)$ *and* $\delta x = \oplus_{j=1}^n \partial_{n,j} x$.

Sketch of Proof. The extension

$$0 \to \oplus^n (\mathfrak{A} \otimes \mathbb{K}) \to \mathfrak{B}_n \to \mathfrak{A} \rtimes_{\alpha,r} F_n \to 0$$

gives rise to the following diagram:

$$\begin{array}{ccccc}
\oplus^n K_0(\mathfrak{A}) & \longrightarrow & K_0(\mathfrak{B}_n) & \longrightarrow & K_0(\mathfrak{A} \rtimes_{\alpha,r} F_n) \\
\uparrow & & & & \downarrow \\
K_1(\mathfrak{A} \rtimes_{\alpha,r} F_n) & \longleftarrow & K_1(\mathfrak{B}_n) & \longleftarrow & \oplus^n K_1(\mathfrak{A}),
\end{array}$$

where the vertical arrows coincide with δ. We have the isomorphisms $(\rho_n)_* : K_j(\mathfrak{A}) \to K_j(\mathfrak{B}_n)$, so that what we need to prove is the commutativity of

$$\begin{array}{ccc} \oplus^n K_j(\mathfrak{A} \otimes \mathbb{K}) & \longrightarrow & K_j(\mathfrak{B}_n) \\ \| & & \uparrow (\rho_n)_* \\ \oplus^n K_j(\mathfrak{A}) & \xrightarrow{\beta} & K_j(\mathfrak{A}) \end{array}$$

since the composition of $\rho_n : \mathfrak{A} \to B_n$ with $\mathfrak{B}_n \to \mathfrak{A} \rtimes_{\alpha,r} F_n$ is π. To prove the commutativity, consider the C^*-algebra generated by $\rho_n(\mathfrak{A})$ and $\sigma_{n,j}$, which is isomorphic to the Toeplitz algebra for the action $\alpha_j : \mathbb{Z} \to \mathrm{Aut}(\mathfrak{A})$ defined by $\alpha_j(m) = \alpha(g_j^m)$ for $m \in \mathbb{Z}$. □

3.2 The reduced C^*-algebras of certain free product groups

Taken from Lance [7] are those in what follows.

Denote by λ the left regular representation of a discrete group G defined by $\lambda(g)\delta_h = \delta_{gh}$ for $g \in G$, where δ_h is the function taking value 1 at h and 0 elsewhere, and the family $\{\delta_h : h \in G\}$ forms the canonical orthonormal basis of $l^2(G)$. The reduced group C^*-algebra $C_r^*(G)$ of G is generated by $\{\lambda(g) : g \in G\}$.

We say that a representation λ_1 of G on $l^2(G)$ has a fixed point if, for some unit vector ξ in $l^2(G)$, we have $\lambda_1(g)\xi = \xi$ for all $g \in G$. Let $\mathbb{K}$ be the closed ideal of all compact operators in the C^*-algebra $\mathbb{B}(l^2(G))$ of all bounded operators on $l^2(G)$.

The group G has property Λ if λ considered as a representation of the full group C^*-algebra $C^*(G)$ is $\mathbb{K}$-homotopic to a representation λ_1 which has a fixed point.

Theorem 3.2.1 *Any countable amenable discrete group has property Λ.*

Sketch of Proof. To say that λ_1 has a fixed point is the same as to say that λ_1 contains the trivial representation τ of G as a subrepresentation. If G is finite, then λ contains τ. □

For $G = \mathbb{Z}$, constructed is a $\mathbb{K}$-homotopy as follows. Given a 2×2 unitary matrix u, we can associate with u a unitary operator $u^\wedge$ on $l^2(\mathbb{Z})$ by assuming that u acts on the subspace genetated by δ_0 and δ_1 by multiplication and on the orthogonal complement trivially. Let λ be the left regular representation of $\mathbb{Z}$, which is just the shift on $l^2(\mathbb{Z})$. Let $(u_t)_{t\in[0,1]}$ be a continuous path of 2×2 unitary matrices between the identity matix and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{where, for instance} \quad u_t = \begin{pmatrix} c_t & s_t \\ s_t & c_t \end{pmatrix}$$

$c_t = (1+e^{\pi it})/2$ and $s_t = (1-e^{\pi it})/2$, and define $\lambda_t(1) = u_t^{\wedge}\lambda(1)$. This gives a $\mathbb{K}$-homotopy λ_t of $\mathbb{Z}$ between λ and a representation that fixes δ_0.

Proposition 3.2.2 *The free group F_n of n generators has property Λ.*

Sketch of Proof. Let g_j $(1 \le j \le n)$ be generators of F_n and G_j the subgroup generated by one generator g_j. Let λ be the left regular representation of F_n. Define unitary operators $\lambda_t(g_j)$ $(0 \le t \le 1)$ as follows. On the subspace $l^2(G_j) \cong l^2(\mathbb{Z})$, $\lambda_t(g_j)$ is a copy of the operator $\lambda_t(1)$, and on the orthogonal subspace $l^2(F_n \backslash G_j)$ the operator $\lambda_t(g_j)$ is equal to $\lambda(g_j)$. Then λ_t extends in a unique way to a unitary representation of F_n, which gives a $\mathbb{K}$-homotopy from λ to a representation which fixes δ_e. □

Proposition 3.2.3 *A nonamenable discrete group with property T cannot have property Λ.*

Sketch of Proof. Suppose that a nonamenable discrete group G has property T. There exist a finite subset F of G and $\varepsilon > 0$ such that if ψ is a representation of G on a Hilbert space H and ξ is a unit vector of H such that $\|\psi(g)\xi - \xi\| < \varepsilon$ for $g \in F$, then ψ contains the trivial representation τ. It follows that if ψ contains τ weakly, then ψ contains τ.

Suppose that G also has property Λ. Let (λ_t) be a $\mathbb{K}$-homotopy between the left regular representation of G and a representation which contains τ. Let $X = \{t \in [0,1] : \lambda_t \text{ contains } \tau\}$. Since G has property T, the set X is open. On the other hand, it is easily seen that the set of all t for which λ_t weakly contains τ is closed, so that X is closed. Since $X \ni 1$, it also contains 0, and so G is amenable. □

Theorem 3.2.4 *Let $\Gamma = \Gamma_1 * \Gamma_2$, where Γ_1 has property Λ. Then*

$$0 \to K_*(\mathbb{C}) \xrightarrow{\chi_{1*}-\chi_{2*}} K_*(C_r^*(\Gamma_1) \oplus C_r^*(\Gamma_2)) \xrightarrow{\varepsilon_{1*}+\varepsilon_{2*}} K_*(C_r^*(\Gamma)) \to 0,$$

where χ_j $(j=1,2)$ are the natural embeddings of $\mathbb{C}$ to the scalar multiples of the identity of $C_r^*(\Gamma_j)$, and ε_j $(j=1,2)$ are the natural embeddings of $C_r^*(\Gamma_j)$ into $C_r^*(\Gamma)$.

Sketch of Proof. From the short exact sequence:

$$0 \to \mathbb{K} \to E \to C_r^*(\Gamma) \to 0$$

we obtain the following diagram:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] & K_0(E) \arrow[r] & K_0(C_r^*(\Gamma)) \arrow[d] \\
K_1(C_r^*(\Gamma)) \arrow[u] & K_1(E) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

and we have an isomorphism from $K_*(C_r^*(\Gamma_1) \oplus C_r^*(\Gamma_2))$ to $K_*(E)$. It follows that the map d_0 from $K_0(\mathbb{K})$ to $K_0(C_r^*(\Gamma_1) \oplus C_r^*(\Gamma_2))$ takes the generator of $\mathbb{Z} = K_0(\mathbb{K})$ to the class $[p \otimes 1_1] - [p \otimes 1_2]$, where 1_j are the units of $C_r^*(\Gamma_j)$ and p is a compact operator, from which one has $d_* = \chi_{1*} - \chi_{2*}$. Since $C_r^*(\Gamma_1)$ is finite and has a canonical trace, $[p \otimes 1_1]$ can not be zero of $K_0(C_r^*(\Gamma_1))$. Therefore, d_0 is not zero. It follows that the index map from $K_1(C_r^*(\Gamma))$ to $\mathbb{Z}$ is zero. Thus, the above diagram splits into two short exact sequences, as desired. □

Corollary 3.2.5 *Let G_j $(1 \leq j \leq n)$ be countable amenable discrete groups and Γ their free product. Then*

$$K_0(C_r^*(\Gamma)) \cong (\oplus_{j=1}^n K_0(C_r^*(G_j)))/\mathbb{Z}^{n-1},$$
$$K_1(C_r^*(\Gamma)) \cong \oplus_{j=1}^n K_1(C_r^*(G_j)).$$

Sketch of Proof. Use induction for $(*_{j=1}^k G_j)$ and G_{k+1} for $1 \leq k \leq n-1$. □

Taken from Natsume [8] are those in what follows.

Let G be a countable discrete group and H a subgroup of G. Let λ^- be the unitary representation of G on $l^2(G/H)$ induced from the left multiplication.

Definition 3.2.6 The pair (G, H) has (relative) property Λ if there exists a family (λ_t) of unitary representations of G on $l^2(G/H)$ such that $\lambda_0 = \lambda^-$, $\lambda_1(g)\delta_{e^-} = \delta_{e^-}$ for any $g \in G$, (λ_t) is a $\mathbb{K}$-homotopy, i.e., for each $x \in C^*(G)$, the map $G \ni t \mapsto \lambda_t(x)$ is a continuous path in $\mathbb{B}(l^2(G/H))$, and $\lambda_t(x) - \lambda^-(x) \in \mathbb{K}(l^2(G/H))$, and $\lambda_t(h) = \lambda^-(h)$ for any $h \in H$.

In particular, G has property Λ in the sense of Lance if the pair $(G, \{e\})$ has (relative) property Λ.

Theorem 3.2.7 *Let Γ be an amalgam of countable discrete groups G and H over a common subgroup K. Assume that (G, H) has property Λ. Then the following six-term exact sequence holds:*

$$\begin{tikzcd}
K_0(\mathfrak{A} \rtimes_r K) \arrow[r, "\mu_*^1 - \mu_*^2"] & K_0(\mathfrak{A} \rtimes_r G) \oplus K_0(\mathfrak{A} \rtimes_r H) \arrow[r, "\nu_*^1 + \nu_*^2"] & K_0(\mathfrak{A} \rtimes_r \Gamma) \arrow[d] \\
K_1(\mathfrak{A} \rtimes_r \Gamma) \arrow[u] & K_1(\mathfrak{A} \rtimes_r G) \oplus K_1(\mathfrak{A} \rtimes_r H) \arrow[l, "\nu_*^1 + \nu_*^2"'] & K_1(\mathfrak{A} \rtimes_r K) \arrow[l, "\mu_*^1 - \mu_*^2"']
\end{tikzcd}$$

for the reduced crossed products of a C^*-algebra $\mathfrak{A}$ by K, G, H, and Γ, where μ^1 and μ^2 are the inclusions from $\mathfrak{A} \rtimes_r K$ to $\mathfrak{A} \rtimes_r G$ and $\mathfrak{A} \rtimes_r H$

respectively, and ν^1 and ν^2 are the inclusions from $\mathfrak{A} \rtimes_r G$ and $\mathfrak{A} \rtimes_r H$ to $\mathfrak{A} \rtimes_r \Gamma$ respectively.

Remark. If K is trivial, this theorem coincides with Lance's result.

If H is a normal subgroup of G, and the quotient group G/H has proptery Λ, then (G, H) has property Λ. Any countable amenable group has property Λ, so that if G/H is amenable, then (G, H) has property Λ.

Example 3.2.8 Let $G = F_n$ be the free group with n generators and H its subgroup generated by k ones of n generators. Then the pair (G, H) has property Λ.

Since $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$, it follows that $K_0(C_r^*(SL_2(\mathbb{Z}))) \cong \mathbb{Z}^8$ and $K_1(C_r^*(SL_2(\mathbb{Z}))) \cong 0$.

Furthermore, we have $K_0(C_r^*(\mathbb{Z}_m *_{\mathbb{Z}_k} \mathbb{Z}_n)) \cong \mathbb{Z}^{m+n-k}$ and $K_1(C_r^*(\mathbb{Z}_m *_{\mathbb{Z}_k} \mathbb{Z}_n)) \cong 0$.

3.3 The reduced C^*-algebras of HNN groups

Taken from Anderson and Paschke [1] are those in what follows.

Proposition 3.3.1 *Let $\Gamma = G *_K H$ be an amalgam of countable groups G and H over a common subgroup K and $\mathfrak{T}$ the associated Toeplitz extension. Assume that G is a subgroup of a group G'. Let $\Gamma' = G' *_K H$ and $\mathfrak{T}'$ the associated Toeplitz extension. Then there is a canonical $*$-monomorphism φ from $\mathfrak{T}$ to $\mathfrak{T}'$ such that the following diagram commutes:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{J}' & \longrightarrow & \mathfrak{T}' & \longrightarrow & C_r^*(\Gamma') & \longrightarrow & 0 \\
\| & & \varphi\uparrow & & \varphi\uparrow & & \uparrow & & \| \\
0 & \longrightarrow & \mathfrak{J} & \longrightarrow & \mathfrak{T} & \longrightarrow & C_r^*(\Gamma) & \longrightarrow & 0
\end{array}$$

where the first and second horizontal sequences are the Toeplitz extensions.

Let G be a group and A its subgroup with $\theta : A \to G$ a monomorphism. Let $\Gamma = G *_A$ be the corresponding HNN group by the Higman-Neumann-Neumann construction, that is, Γ is generated by G and an element $s \in \Gamma$ such that $sas^{-1} = \theta(a)$ for $a \in A$, with the universal property. The group Γ can be viewed as a semi-direct product $H \rtimes \mathbb{Z}$, where

$$H = \cdots *_A G *_A G *_A G *_A \cdots$$

is the amalgam of copies of G over copies of A injected left and right by θ and the inclusion $i : A \subset G$, and $\mathbb{Z}$ acts on H by the shift.

Let $G_1, \cdots, G_n$ be countable groups and A_k a proper subgroup of G_k and G_{k+1} for $1 \leq k \leq n-1$. Let $\Gamma_n = G_1 *_{A_1} G_2 *_{A_2} \cdots *_{A_{n-1}} G_n$. Then

$$\Gamma_n = (G_1 *_{A_1} G_2 *_{A_2} \cdots *_{A_{k-1}} G_k) *_{A_k} (G_{k+1} *_{A_{k+1}} \cdots *_{A_{n-1}} G_n)$$

for each k. Let

$$0 \to \mathfrak{J}_k \cong C_r^*(A_k) \otimes \mathbb{K} \to \mathfrak{T}_k \xrightarrow{\pi_k} C_r^*(\Gamma_n) \to 0$$

be the corresponding Toeplitz extension. Define the pullback $\mathfrak{T}$ as

$$\mathfrak{T} = \{(t_k) \in \oplus_{k=1}^{n-1} \mathfrak{T}_{n-1} \mid \pi_1(t_1) = \pi_2(t_2) = \cdots = \pi_{n-1}(t_{n-1})\}$$

with the quotient map $\pi : \mathfrak{T} \to C_r^*(\Gamma_n)$ whose kernel is $\oplus_{k=1}^{n-1}\mathfrak{J}_k$. We have the following commutative diagram:

$$\begin{array}{ccc}
\oplus_{k=1}^{n-1} K_*(\mathfrak{J}_k) & \longrightarrow & K_*(\mathfrak{T}) \\
\uparrow & & \downarrow \\
\oplus_{k=1}^{n-1} K_*(C_r^*(A_k)) & \longrightarrow & \oplus_{j=1}^{n} K_*(C_r^*(G_j)).
\end{array}$$

Theorem 3.3.2 *Suppose that the pairs $(G_1, A_1), \cdots, (G_{n-1}, A_{n-1})$ all have property Λ of Natsume. Then $\oplus_{j=1}^{n} K_*(C_r^*(G_j))$ is isomorphic to $K_*(\mathfrak{T})$.*

Corollary 3.3.3 *In the same situation as above, it follows that*

$$\begin{array}{ccccc}
\oplus_{k=1}^{n-1} K_0(C_r^*(A_k)) & \longrightarrow & \oplus_{j=1}^{n} K_0(C_r^*(G_j)) & \longrightarrow & K_0(C_r^*(\Gamma_n)) \\
\uparrow & & & & \downarrow \\
K_1(C_r^*(\Gamma_n)) & \longleftarrow & \oplus_{j=1}^{n} K_1(C_r^*(G_j)) & \longleftarrow & \oplus_{k=1}^{n-1} K_1(C_r^*(A_k)).
\end{array}$$

Theorem 3.3.4 *Let Γ be an inductive limit of Γ_n as above (and Γ_{-n} defined similarly). Suppose that each pair (G_k, A_k) has property Λ of Natsume. Then*

$$\begin{array}{ccccc}
\oplus_{k=-\infty}^{\infty} K_0(C_r^*(A_k)) & \longrightarrow & \oplus_{j=-\infty}^{\infty} K_0(C_r^*(G_j)) & \longrightarrow & K_0(C_r^*(\Gamma)) \\
\uparrow & & & & \downarrow \\
K_1(C_r^*(\Gamma)) & \longleftarrow & \oplus_{j=-\infty}^{\infty} K_1(C_r^*(G_j)) & \longleftarrow & \oplus_{k=-\infty}^{\infty} K_1(C_r^*(A_k)).
\end{array}$$

Now let Γ_1 be the set of reduced words in $\Gamma = G*_A$ ending in sA. Note that $l^2(\Gamma_1)$ reduces $\lambda(G)$ and is invariant for $\lambda(s)$, where λ is the left regular representation of Γ on $l^2(\Gamma)$. Let S and $\beta(g)$ for $g \in G$ denote respectively the restrictions of $\lambda(s)$ and $\lambda(g)$ to $l^2(\Gamma_1)$. Let $\mathfrak{D}$ be the C^*-algebra generated by S and $\beta(G)$.

Let R be the unitary on $l^2(\Gamma)$ corresponding the right multiplication by s. For any $x \in \Gamma$, we have $xs^n \in \Gamma_1$ for sufficiently large n. Note that $R^{-n}SR^n \to \lambda(s)$ and $R^{-n}\beta(g)R^n \to \lambda(g)$ for $g \in G$ in the strong operator topology. Thus we obtain a $*$-homomorphism $\pi : \mathfrak{D} \to C_r^*(\Gamma)$ such that $\pi(S) = \lambda(s)$ and $\pi \circ \beta = \lambda|_G$. Let $q = 1 - SS^*$ the projection from $l^2(\Gamma_1)$ onto $l^2(sA)$. Then the kernel of π is the closed ideal of $\mathfrak{D}$ generated by q. Denote by Q the ideal. We have $Q \cong C_r^*(A) \otimes \mathbb{K}$.

For this, let $\{g_0, g_1, \cdots\}$ be coset representatives for G/A with $g_0 = e$ the identity element. Let $\{h_0, h_1, \cdots\}$ be those for $G/\theta(A)$ with $h_0 = e$. Each element of Γ_1 can be written uniquely in the forms

$$h_{i_1}sh_{i_2}s \cdots h_{i_n}sh_{i_{n+1}}sa,$$

where $n \geq 0, a \in A$, and each $h_{i_k}s$ $(1 \leq k \leq n)$ may be replaced with $g_{j_k}s^{-1}$, and g_0 is not allowed to lie between s and s^{-1}, and h_0 is not done between s^{-1} and s. Denote by Ω_0 the set of products of these forms in which $a = e$. Let $\Omega = \Omega_0 s^{-s}$, which is the union of G, words ending in sG, and words ending in $s^{-1}(G \setminus \theta(A))$, and is also the difference set of Γ from words ending in $s^{-1}\theta(A)$. Also, Γ_1 is the union of the elements wsA for $w \in \Omega$. For $w \in \Omega$, let V_w be the corresponding product in $\mathfrak{D}$. The isomorphism of $C_r^*(A) \otimes \mathbb{K}(l^2(\Omega))$ with Q is obtained by identifying $l^2(A) \otimes l^2(\Omega)$ with $l^2(\Gamma_1)$ via the bijection $A \times \Omega \ni (a, w) \mapsto wsa \in \Gamma_1$, from which $\lambda_A(a) \otimes E_{w',w}$ is mapped to $V_{w'}\beta(\theta(a))qV_w^*$, where $a \in A$ and λ_A is the left regular representation of A, and $E_{w',w}$ is the rank one projection associated with w' and w. Note that $\beta(\theta(a))$ commutes with q.

There is a homomorphism $\varphi : \Gamma = G*_A \to \mathbb{Z}$ annihilating G with $\varphi(s) = 1$. For $z \in \mathbb{T}$, define a unitary operator U_z on $l^2(\Gamma_1)$ multiplying the basis element corresponding to $\gamma \in \Gamma_1$ by $z^{\varphi(\gamma)}$. Conjugation by U_z gives an action α of $\mathbb{T}$ on $\mathfrak{D}$ fixing $\beta(G)$ and spinning S. Let $\mathfrak{D}_0$ be the subalgebra of $\mathfrak{D}$ fixed by α. Define $\sigma_0 : \mathfrak{D}_0 \to (1-q)\mathfrak{D}_0(1-q) \subset \mathfrak{D}_0$ by $\sigma_0(x) = SxS^*$.

Proposition 3.3.5 *The crossed product of $\mathfrak{D}_0$ by the endomorphism σ_0 is $\mathfrak{D}$.*

Proof. Let $\mathfrak{D}^\sim$ denote the crossed product of $\mathfrak{D}_0$ by σ_0, which is the universal C^*-algebra generated by $\mathfrak{D}_0$ and an isometry $S^\sim$ such that $S^\sim x(S^\sim)^* = \sigma_0(x)$ for $x \in \mathfrak{D}_0$. By universality, there is an action $\alpha^\sim$ of $\mathbb{T}$ on $\mathfrak{D}^\sim$ fixing $\mathfrak{D}_0$ and spinning $S^\sim$, and there is a map $\psi : \mathfrak{D}^\sim \to \mathfrak{D}$ sending $S^\sim$ to S, whose restriction to $\mathfrak{D}_0$ is the identity map. For each $z \in \mathbb{T}$,

$\psi \circ \alpha_z^\sim = \alpha_z \circ \psi$. Integrating $\alpha^\sim$ and α yields faithful conditional expectations $E^\sim : \mathfrak{D}^\sim \to \mathfrak{D}_0$ and $E : \mathfrak{D} \to \mathfrak{D}_0$. We have $\psi \circ E^\sim = E \circ \psi$. If $y \in \mathfrak{D}^\sim$ and $\psi(y) = 0$, then $\psi(E^\sim(y^*y)) = E(\psi(y^*y)) = 0$. Since $E^\sim(y^*y) \in \mathfrak{D}_0$, so $E^\sim(y^*y) = 0$. Hence $y = 0$. Thus ψ is an isomorphism. □

Let $\Omega_k = \Omega \cap \varphi^{-1}(k)$ for $k \in \mathbb{Z}$. Define Q_k to be the closed subalgebra of $\mathfrak{D}_0$ generated by the elements $V_{w'}\beta(\theta(a))qV_w^*$ for $w, w' \in \Omega_k$ and $a \in A$. This is a closed ideal of $\mathfrak{D}_0$ and isomorphic to $C_r^*(A) \otimes \mathbb{K}(l^2(\Omega_k))$. We have $Q \cap \mathfrak{D}_0 \cong \oplus_{-\infty}^{\infty} Q_k$. Thus,

$$\cdots \to K_1(C_r^*(H)) \to \oplus_{-\infty}^{\infty} K_0(C_r^*(Q_k)) \to K_0(\mathfrak{D}_0) \to K_0(C_r^*(H)) \to \cdots$$

Note that

$$H \cong \cdots *_{A_{-1}} G_{-1} *_{A_0} G_0 *_{A_1} G_1 *_{A_2} \cdots$$

where $G_k = s^k G s^{-k}$ and $A_k = s^k A s^{-k} = s^{k-1}\theta(A)s^{1-k}$. If we assume that $(G, \theta(A))$ has property Λ, then the following is exact:

$$\begin{aligned} \cdots \to K_1(C_r^*(H)) &\to \oplus_{-\infty}^{\infty} K_0(C_r^*(A_k)) \\ &\to \oplus_{-\infty}^{\infty} K_0(C_r^*(G_k)) \to K_0(C_r^*(H)) \to \cdots. \end{aligned}$$

Assume that $A \neq G \neq \theta(A)$. For $k \geq 0$, define $\Phi_k : C_r^*(G) \to \mathfrak{D}_0$ by $\Phi_k(g) = S^k\beta(g)(S^*)^k$. Since S is an isometry, Φ_k is a $*$-monomorphism.

Lemma 3.3.6 *Given $j \geq 1$, there exist $x_j, y_j \in \mathfrak{D}_0$ such that $x_jS^j(S^*)^jx_j^* + y_jS^j(S^*)^jy_j^* = 1$.*

With $j = 1$, it follows that $(1-q)\mathfrak{D}_0(1-q)$ is a full corner of $\mathfrak{D}_0$.

Proof. Note that $1 - S^j(S^*)^j$ is the projection of $l^2(\Gamma_1)$ onto $l^2(sA \cup s^2A \cup \cdots \cup s^jA)$. Take $g \in G \backslash \theta(G)$. Then $1 - \beta(g)S^j(S^*)^j\beta(g^{-1})$ is the projection onto $l^2(gsA \cup gs^2A \cup \cdots \cup gs^jA)$, and is orthogonal to $1 - S^j(S^*)^j$. Therefore, $S^j(S^*)^j + \beta(g)S^j(S^*)^j\beta(g^{-1}) = 2 - p$ for p a projection of $\mathfrak{D}_0$. Take $x_j = (2-p)^{-1/2}$ and $y_j = x_j\beta(g)$. □

For $j \geq 1$, define $\Phi_{-j} : C_r^*(G) \to \mathfrak{D}_0 \otimes M_2(\mathbb{C})$ by

$$\Phi_{-j}(g) = R_j^* \begin{pmatrix} \beta(g) & 0 \\ 0 & 0 \end{pmatrix}, \quad R_j = \begin{pmatrix} x_jS^j & y_jS^j \\ 0 & 0 \end{pmatrix} \in \mathfrak{D}_0 \otimes M_2(\mathbb{C})$$

for $g \in C_r^*(G)$. Note that $R_jR_j^* = 1 \oplus 0$ (the diagonal sum). Then Φ_{-j} is a $*$-monomorphism.

Lemma 3.3.7 *For all $k \in \mathbb{Z}$, we have*

$$(\sigma_0)_*(\Phi_k)_* = (\Phi_{k+1})_* : K_*(C_r^*(G)) \to K_*(\mathfrak{D}_0)).$$

Lemma 3.3.8 *The following diagram for $k \in \mathbb{Z}$ commutes:*

$$\begin{array}{ccc} K_*(Q_k) & \longrightarrow & K_*(\mathfrak{D}_0) \\ \uparrow & & \uparrow{\scriptstyle (\Phi_k \circ \theta)_* - (\Phi_{k+1} \circ i)_*} \\ K_*(C_r^*(A)) & = & K_*(C_r^*(A)). \end{array}$$

Define $\Phi : \oplus_{-\infty}^{\infty} K_*(C_r^*(G_k)) \to K_*(\mathfrak{D}_0)$ to be the map obtained by summing the maps $(\Phi_k)_*$, where each G_k is identified with G.

Lemma 3.3.9 *Suppose that $A \neq G \neq \theta(A)$ and $(G, \theta(A))$ has property Λ. Then the map $\Phi : \oplus_{-\infty}^{\infty} K_*(C_r^*(G)) \to K_*(\mathfrak{D}_0)$ is an isomorphism intertwining the forward shift on the direct sum with $(\sigma_0)_*$ on $K_*(\mathfrak{D}_0)$.*

Sketch of Proof. Consider the following diagram:

$$\begin{array}{ccccccc} \to K_1(C_r^*(H)) & \longrightarrow & L_{-\infty}^{\infty} & \longrightarrow & K_0(\mathfrak{D}_0) & \longrightarrow & K_0(C_r^*(H)) \\ \| & & \uparrow & & {\scriptstyle \Phi}\uparrow & & \| \\ \to K_1(C_r^*(H)) & \longrightarrow & L_{-\infty}^{\infty} & \longrightarrow & \oplus_{-\infty}^{\infty} K_0(C_r^*(G)) & \longrightarrow & K_0(C_r^*(H)) \end{array}$$

where $L_{-\infty}^{\infty} = \oplus_{-\infty}^{\infty} K_0(C_r^*(A))$. The top sequence is exact, which comes from $\mathfrak{D}_0 \to C_r^*(H)$ after identifying the kernel of this map with $\oplus_{-\infty}^{\infty} C_r^*(A) \otimes \mathbb{K}$. The bottom sequence is also exact. The vertical map between $L_{-\infty}^{\infty}$ is the backward shift. Those squares are commutative. The desired isomorphism follows from the five-lemma. $\square$

Let $\mathfrak{B}$ be a unital C^*-algebra and ρ a unital $*$-monomorphism of $\mathfrak{B}$ into itself. An isometry V is a $(\mathfrak{B}, \rho)$-isometry if V acts on a Hilbert space on which $\mathfrak{B}$ is faithfully represented and satisfies $bV = V\rho(b)$ for all $b \in \mathfrak{B}$. This condition forces VV^* to commute with $\mathfrak{B}$. We say that V is faithful if $b \mapsto b(1 - VV^*)$ is injective on $\mathfrak{B}$.

Lemma 3.3.10 *Let V be a faithful $(\mathfrak{B}, \rho)$-isometry. For a $(\mathfrak{B}, \rho)$-isometry W, there is a $*$-homomorphism φ from the C^*-algebra generated by $\mathfrak{B}$ and V to that by $\mathfrak{B}$ and W such that $\varphi(b) = b$ for $b \in \mathfrak{B}$ and $\varphi(V) = W$.*

Theorem 3.3.11 *If V is a faithful $(\mathfrak{B}, \rho)$-isometry, then the inclusion from $\mathfrak{B}$ to the C^*-algebra generated by $\mathfrak{B}$ and V induces an isomorphism on K-theory.*

Corollary 3.3.12 *Whenever $G = \theta(A)$, the map $\beta : C_r^*(G) \to \mathfrak{D}$ induces an isomorphism on K-theory.*

Proof. The isometry S is a $(C_r^*(G), \rho)$-isometry, where $\rho = \theta^{-1} : C_r^*(G) \to C_r^*(A) \subset C_r^*(G)$ comes from $\theta^{-1} : G \to A \subset G$. The defect space of S is $l^2(sA) = l^2(Gs)$, on which $\beta(C_r^*(G))$ acts faithfully. □

Theorem 3.3.13 *Let G be a countable group and A its subgroup with $\theta : A \to G$ a monomorphism. Let $\Gamma = G*_A$ be the corresponding HNN group. If either (G, A) or $(G, \theta(A))$ has (relative) property Λ, there is the following exact sequence:*

$$\begin{array}{ccccc} K_0(C_r^*(A)) & \xrightarrow{\theta_* - i_*} & K_0(C_r^*(G)) & \xrightarrow{j_*} & K_0(C_r^*(\Gamma)) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(\Gamma)) & \xleftarrow{j_*} & K_1(C_r^*(G)) & \xleftarrow{\theta_* - i_*} & K_1(C_r^*(A)) \end{array}$$

where i_*, j_*, and θ_* are induced from the inclusions $i : A \to G$, $j : G \to \Gamma$, and $\theta : A \to G$.

Sketch of Proof. Consider the case where $A \neq G \neq \theta(A)$ and $(G, \theta(A))$ has property Λ. The Toeplitz algebra $\mathfrak{D}$ is the crossed product of $\mathfrak{D}_0$ by the endomorphism $\sigma_0 : \mathfrak{D}_0 \to (1-q)\mathfrak{D}_0(1-q)$, whose range is a full corner of $\mathfrak{D}_0$, so that we have the following exact sequence:

$$\cdots \to K_1(\mathfrak{D}) \to K_0(\mathfrak{D}_0) \xrightarrow{(\sigma_0)_* - \mathrm{id}} K_0(\mathfrak{D}_0) \to K_0(\mathfrak{D}) \to \cdots .$$

When we identify $K_*(\mathfrak{D}_0)$ with $\oplus_{-\infty}^{\infty} K_*(C_r^*(G))$, the map $(\sigma_0)_*$ becomes the forward shift. Hence $(\sigma_0)_* - \mathrm{id}$ has trivial kernel, and the image of $K_*(C_r^*(G))$ in $K_*(\mathfrak{D}_0)$ under $(\Phi_0)_*$ is mapped isomorphically onto $K_*(\mathfrak{D})$. Hence $\beta_* : K_*(C_r^*(G)) \to K_*(\mathfrak{D})$ is an isomorphism. The exact sequence in the statement is now seen to be the exact sequence of K-groups associated with the extension:

$$0 \to Q \to \mathfrak{D} \to C_r^*(\Gamma) \to 0.$$

If $G = \theta(A)$, in which case the property Λ is superfluous, then β is an isomorphism. The remaining cases are obtained by interchanging the roles of A and $\theta(A)$. □

Theorem 3.3.14 *Let $G_1, \cdots, G_n$ be countable groups and $A_1, \cdots, A_n$ subgroups with imbeddings $i_j : A_j \to G_j$ and $\theta_j : A_j \to G_{j-1}$ $(1 \leq j \leq n)$, i.e., a loop of groups with length n, with fundamental group Δ. Assume that all the pairs (G_j, A_j) have property Λ. Then*

$$\begin{array}{ccccc} \oplus_{j=1}^n K_0(C_r^*(A_j)) & \longrightarrow & \oplus_{j=1}^n K_0(C_r^*(G_j)) & \longrightarrow & K_0(C_r^*(\Delta)) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(\Delta)) & \longleftarrow & \oplus_{j=1}^n K_1(C_r^*(G_j)) & \longleftarrow & \oplus_{j=1}^n K_1(C_r^*(A_j)). \end{array}$$

The map from $\oplus_{j=1}^{n} K_*(C_r^*(A_j))$ to $\oplus_{j=1}^{n} K_*(C_r^*(G_j))$ sends the summand $K_*(C_r^*(A_j))$ to $K_*(C_r^*(G_{j-1})) \oplus K_*(C_r^*(G_j))$ via $(-(\theta_j)_*, (i_j)_*)$. The map from $\oplus_{j=1}^{n} K_*(C_r^*(G_j))$ to $K_*(C_r^*(\Delta))$ sums the maps induced on K-theory by the natural injections $C_r^*(G_j) \to C_r^*(\Delta)$.

Remark. The fundamental group Δ for the loop of groups is defined as follows. For each j, break the loop at A_j and let

$$\Gamma_j = G_j *_{A_{j+1}} G_{j+1} *_{A_{j+2}} \cdots *_{A_{j-1}} G_{j-1}.$$

Then A_j is a subgroup of Γ_j via $i_j : A_j \to G_j$, and we have $\theta_j : A_j \to \Gamma_j$ via $\theta_j : A_j \to G_{j-1}$. Let $\Delta_j = \Gamma_j *_{A_j}$ in this sense. The groups Δ_j are all isomorphic. Define Δ to be a single group identified with Δ_j.

Sketch of Proof. We identity Δ with $\Delta_j = \Gamma_j *_{A_j}$ via θ_j $(1 \le j \le n)$ and form $\pi_j : \mathfrak{D}_j \to C_r^*(\Delta)$ as before. Let $(\pi, \mathfrak{D})$ be the pullback of $(\pi_j, \mathfrak{D}_j)$ as before. The strategy is to obtain the exact sequence in the theorem as the exact sequence of K-groups induced by $\pi : \mathfrak{D} \to C_r^*(\Delta)$. The main thing is to exhibit an isomorphism of $K_*(\mathfrak{D})$ with $\oplus_{j=1}^{n} K_*(C_r^*(G_j))$. □

3.4 The reduced C^*-algebras of one-relator groups

Taken from Béguin, Bettaieb, and Valette [2] are those in what follows.

A one-relator group Γ has a presentation $\Gamma = \langle X | r \rangle$, where X is a countable set of generators and the relator r is a cyclically reduced word in the free group $F(X)$ on X. The class of one-relator groups contains fundamental groups of closed surfaces, groups of torus knots, and Baumslag-Solitar groups. Write $r = s^n$, where $n \ge 1$ and s is not a power in $F(X)$. It is known that Γ is torsion-free if and only if $n = 1$. Denote by $\bar{r}$ the image of r in the abelianized group $F(X)^{ab}$.

Theorem 3.4.1 *Let $\Gamma = \langle X | r \rangle$ be a one-relator group, with $r = s^n$ as above. Then*

$$K_0(C_r^*(\Gamma)) \cong \begin{cases} \mathbb{Z}^{n+1} & \text{if } \bar{r} \text{ is trivial in } F(X)^{ab}, \\ \mathbb{Z}^n & \text{if not.} \end{cases}$$

Moreover, in the second case, it is generated by the n spectral projections of s in $C_r^*(\Gamma)$. In particular, if Γ is torsion-free and $\bar{r} \ne 0$, then $K_0(C_r^*(\Gamma))$ is generated by the class of the unit.

Any discrete group Γ is embedded into the unitary group of $C_r^*(\Gamma)$, which is mapped to $K_1(C_r^*(\Gamma))$ so that by composition, a homomorphism from Γ to $K_1(C_r^*(\Gamma))$ is obtained. As K_1 is an abelian group, there is a homomorphism k_Γ from the abelianized group Γ^{ab} of Γ to $K_1(C_r^*(\Gamma))$. It is known in general that k_Γ is rationally injective.

Theorem 3.4.2 *Let $\Gamma = \langle X|s^n\rangle$ be a one-relator group. Let $\langle \bar{s}\rangle$ be the finite cyclic group generated by $\bar{s} \in \Gamma^{ab}$. Then*

$$0 \to \langle \bar{s}\rangle \to \Gamma^{ab} \xrightarrow{k_\Gamma} K_1(C_r^*(\Gamma)) \to 0.$$

In particular, if Γ is torsion-free, then k_Γ is an isomorphism.

For any group Γ, there is a homomorphism $\beta : \Gamma^{ab} \to K_1^\Gamma(\underline{E\Gamma})$ the Γ-equivariant K_1-homology for $\underline{E\Gamma}$ the universal space for Γ-proper actions such that $\mu_1^\Gamma \circ \beta = k_\Gamma$.

To prove the theorems above,

Lemma 3.4.3 *Let Γ be a group acting properly on an oriented tree X, then $\mu_0^\Gamma : K_0^\Gamma(\underline{E\Gamma}) \to K_0(C_r^*(\Gamma))$ is an isomorphism.*

Let $\Gamma = \mathbb{Z}_n * F_k$ and s the generator of $\mathbb{Z}_n$. Then μ_0^Γ is an isomorphism, and $K_0(C_r^*(\Gamma)) \cong \mathbb{Z}^n$ is generated by the n spectral projections of s. Furthermore, the following diagram:

$$\begin{tikzcd}
K_1^\Gamma(\underline{E\Gamma}) \arrow[r, "\mu_1^\Gamma"] & K_1(C_r^*(\Gamma)) \\
\Gamma^{ab} = \mathbb{Z}_n \oplus \mathbb{Z}^k \arrow[u, "\beta"] \arrow[r, equal] & \Gamma^{ab} \arrow[u, "k_\Gamma"']
\end{tikzcd}$$

commutes, where both of the kernels of β and k_Γ are $\mathbb{Z}_n$, and μ_1^Γ is an isomorphism.

Sketch of Proof. The first part follows from the idea of Baum and Connes. Denote by X^0 and X^1 the sets of vertices and edges of X respectively. Let $|X|$ be the geometric realization of X. Identify every geometric edge of $|X|$ with $[0,1]$. There is the following Γ-equivariant exact sequence:

$$0 \to C_0(X^1 \times (0,1)) \to C_0(|X|) \xrightarrow{i} C_0(X^0) \to 0,$$

which implies the six-term exact sequence in Γ-equivariant K-homology:

$$\begin{tikzcd}
K_0^\Gamma(X^0) \arrow[r, "i_*"] & K_0^\Gamma(|X|) \arrow[r] & K_1^\Gamma(X^1) \arrow[d] \\
K_0^\Gamma(X^1) \arrow[u] & K_1^\Gamma(|X|) \arrow[l] & K_1^\Gamma(X^0) \arrow[l, "i_*"']
\end{tikzcd}$$

where $K_j^\Gamma(X^1 \times (0,1)) \cong K_{j+1}^\Gamma(X^1)$ by Bott periodicity, and the vertical maps are induced by the coboundary operator $d : X^1 \to X^0$. Let Σ be the fundamental domain for the action of Γ on X. Then

$$K_j^\Gamma(X^k) = \oplus_{x\in\Sigma^k} K_j^\Gamma(\Gamma\cdot x) = \oplus_{x\in\Sigma^k} K_j^{\Gamma_x}(\mathrm{point})$$

for $k = 0,1$ and $j = 0,1$. Since the stabilizers are finite, it follows that

$$0 \to K_1^\Gamma(|X|) \to \oplus_{y\in\Sigma^1} K_0^{\Gamma_y}(\mathrm{pt}) \xrightarrow{d_*} \oplus_{x\in\Sigma^0} K_0^{\Gamma_x}(\mathrm{pt}) \to K_0^\Gamma(|X|) \to 0.$$

On the other hand, $K_1(C_r^*(\Gamma_x)) \cong 0$ for $x \in \Sigma^k$ $(k = 0,1)$ since the stabilizers are finite, Therefore, it follows from the six-term exact sequence that

$$\begin{aligned} 0 \to K_1(C_r^*(\Gamma)) &\to \oplus_{y\in\Sigma^1} K_0(C_r^*(\Gamma_y)) \\ &\xrightarrow{d_*} \oplus_{x\in\Sigma^0} K_0(C_r^*(\Gamma_x)) \to K_0(C_r^*(\Gamma)) \to 0. \end{aligned}$$

For a finite group F, we can take $\underline{EF} = \mathrm{point}$. Thus, the two above exact sequences are linked by assembly maps as follows:

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] \\
K_1^\Gamma(|X|) \arrow[r, "\mu_1^\Gamma"] \arrow[d] & K_1(C_r^*(\Gamma)) \arrow[d] \\
\oplus_{y\in\Sigma^1} K_0^{\Gamma_y}(\mathrm{pt}) \arrow[r, "\oplus\mu_0^{\Gamma_y}"] \arrow[d, "d_*"'] & \oplus_{y\in\Sigma^1} K_0(C_r^*(\Gamma_y)) \arrow[d, "d_*"'] \\
\oplus_{x\in\Sigma^0} K_0^{\Sigma_x}(\mathrm{pt}) \arrow[r, "\oplus\mu_0^{\Gamma_x}"] \arrow[d, "i_*"'] & \oplus_{x\in\Sigma^0} K_0(C_r^*(\Gamma_x)) \arrow[d, "i_*"'] \\
K_0^\Gamma(|X|) \arrow[r, "\mu_0^\Gamma"] \arrow[d] & K_0(C_r^*(\Gamma)) \arrow[d] \\
0 \arrow[r, equal] & 0.
\end{tikzcd}$$

Since the maps d_* and i_* come from group inclusions, and the assembly map is natural with respect to group monomorphisms, the thrid and fourth squares (from the top) in the middle are commutative. The Baum-Connes conjecture is true for finite groups, so that the five-lemma implies that μ_0^Γ is an isomorphism. (Note that it is not always clear that the second square is commutative, so that the five-lemma does not immediately apply to conclude that μ_1^Γ is an isomorphism.)

For the second part of the statement, let Y be the Cayley graph of Γ with respect to $\{s^{\pm 1}, a_1^{\pm 1}, \cdots, a_k^{\pm 1}\}$, where s and a_j $(1 \le j \le k)$ are generators of $\mathbb{Z}_n$ and F_k respectively. The graph Y is tree-like: it is the free product of the n-gon and the $2k$-regular tree. In Y, collapse every n-gon to a point, to get the $2kn$-regular tree X. In this way, a proper action of Γ on X is obtained, which is transitive on vertices. It follows from the first part of this proof that μ_0^Γ is an isomorphism. Also, $K_0^\Gamma(|X|) \cong K_0^{\mathbb{Z}_n}(\mathrm{pt}) \cong \mathbb{Z}^n$. Furthermore, the last claim follows from the computation of K-theory for $C_r^*(\mathbb{Z}_n * F_k)$. □

Denote by $\Gamma^\sim$ the torsion-free one-relator group defined by $\Gamma^\sim = \langle X | s \rangle$. It is a fact that all torsion elements in Γ are conjugates of powers of s, so that the map k_Γ factors through an homomorphism $k_\Gamma^\sim : (\Gamma^\sim)^{ab} \to K_1(C_r^*(\Gamma))$ which leads to the commutative diagram:

$$\begin{array}{ccccccc}
0 \to \langle \bar{s} \rangle & \longrightarrow & \Gamma^{ab} & \xrightarrow{k_\Gamma} & K_1(C_r^*(\Gamma)) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \| \\
0 & \longleftarrow & (\Gamma^\sim)^{ab} & \xrightarrow{k_\Gamma^\sim} & K_1(C_r^*(\Gamma)) & \longrightarrow & 0.
\end{array}$$

Therefore, to prove the second theorem above is equivalent to prove that $k_\Gamma^\sim$ is an isomorphism.

We need to use the principle of induction on the length of s. Note that if only one generator is left in s, then Γ is a free product of a free group with a cyclic group, and this is the first step of the induction. For the general case, we may assume that $X = \{t, b, c, d, \cdots\}$ with at least two distinct generators t and b occuring in s. For $x \in X$, denote by $\sigma_x(s)$ the sum of exponents of occurrences of x in s.

Case 1. There exists $t \in X$ such that $\sigma_t(s) = 0$. Set $b_j = t^j b t^{-j}$, $c_j = t^j c t^{-j}, \cdots$, $(j \in \mathbb{Z})$. Rewrite s as a cyclically reduced word s' on b_j, $c_j, \cdots$. Then the length of s' (with respect to $X' = \{t, b_j, c_j, \cdots\}$) is less than the length of s with respect to X. Let ν and μ be the minimum and maximum subscripts on b respectively, that occurs in s'. It turns out that Γ is an HNN extension $H*_A$, where $H = \langle b_\nu, \cdots, b_m, c_j, d_j, \cdots (j \in \mathbb{Z}) | (s')^n \rangle$, a free group $A = \langle b_\nu, \cdots, b_{m-1}, c_j, d_j, \cdots (j \in \mathbb{Z}) \rangle$ if $\nu < m$, and $= \langle c_j, d_j, \cdots (j \in \mathbb{Z}) \rangle$ if $\nu = m$, and the inclusion $\theta : A \to H$ given by $\theta(b_j) = b_{j+1}$, $\theta(c_j) = c_{j+1}, \cdots$. Similarly, $\Gamma^\sim$ is an HNN extension $H^\sim *_A$, where only $(s')^n$ is replaced with s' to define $H^\sim$.

Case 2. Suppose that $\sigma_z(s) \neq 0$ for every $z \in X$. Set $\alpha = \sigma_t(s)$ and $\beta = \sigma_b(s)$. Consider the group $G = \langle y, x, c, d, \cdots | s^n(yx^{-\beta}, x^\alpha, c, d, \cdots)\rangle$ and the map $\psi : \Gamma \to G$ defined by $\psi(t) = yx^{-\beta}$, $\psi(b) = x^\alpha$, $\psi(c) = c, \cdots$. Then ψ is injective, and G is an amalgam $\Gamma *_{\langle b = x^\alpha\rangle} \langle x\rangle$. Let s_1 be the word obtained by cyclically reducing $s(yx^{-\beta}, x^\alpha, c, d, \cdots)$. Then $\sigma_x(s_1) = 0$ while $\sigma_y(s_1) = \sigma_t(s) \neq 0$. So $G = \langle y, x, c, d, \cdots | s_1^n\rangle$ is a one-relator group for which the case 1 is applied. Namely, G is an HNN extension of a one-relator group $H = \langle y_j, c_j, d_j, \cdots | (s')^n\rangle$ with $|s'| < |s|$. It is not true in general that $|s_1| < |s|$. Let $s_1 = w^k$ for some $k \geq 2$ in what follows.

Lemma 3.4.4 *When the reduction terminates with one generator left in s, then Γ is isomorphic to $\mathbb{Z}_n * F_k$, where n is the integer given in $\langle X|s^n\rangle$ and $0 \leq k \leq \infty$.*

Sketch of Proof for the first and second theorems above.

Case 1. Suppose $\sigma_t(s) = 0$, so that $\Gamma = H*_A$ and $\Gamma^\sim = H^\sim *_A$ via $\theta : A \to H$. Denote by i and $i^\sim$ the inclusions of H and $H^\sim$ in Γ and $\Gamma^\sim$ respectively. We have the following six-term exact sequence:

$$\begin{array}{ccccc}
K_0(C_r^*(A)) & \xrightarrow{\mathrm{id}-\theta_*} & K_0(C_r^*(H)) & \xrightarrow{i_*} & K_0(C_r^*(\Gamma)) \\
{\scriptstyle\partial_1}\uparrow & & & & \downarrow{\scriptstyle\partial_0} \\
K_1(C_r^*(\Gamma)) & \xleftarrow{i_*} & K_1(C_r^*(H)) & \xleftarrow{\mathrm{id}-\theta_*} & K_1(C_r^*(A))
\end{array}$$

(due to Anderson and Paschke). Since A is free we have $K_0(C_r^*(A)) = \mathbb{Z}[1]$. Since θ_* is induced by a unital $*$-homomorphism, we have $\theta_*[1] = [1]$, so that the map $\mathrm{id} - \theta_*$ on the K_0-group is zero. For any $g \in \Gamma$, we denote by δ_g the corresponding unitary in $C_r^*(\Gamma)$, and by $[\delta_g]$ its class in $K_1(C_r^*(\Gamma))$. Then we have $\partial_1[\delta_t] = -[1]$. This follows as follows. The six-term exact sequence comes from the following Toeplitz extension: $0 \to C_r^*(A) \otimes \mathbb{K} \to \mathfrak{D} \to C_r^*(\Gamma) \to 0$, where $\mathfrak{D}$ is the C^*-algebra acting on $l^2(\Gamma)$, generated by δ_h for $h \in H$ and an isometry S such that $1 - SS^* = 1 \otimes p$ with p a rank one projection in $\mathbb{K}$. Since $\delta_t \in C_r^*(\Gamma)$ lifts to $S \in \mathfrak{D}$, it follows that

$$\partial_1[\delta_t] = [1 - S^*S] - [1 - SS^*] = -[1 \otimes p] \in K_0(C_r^*(A) \otimes \mathbb{K}).$$

Under the identification between $K_0(C_r^*(A) \otimes \mathbb{K})$ and $K_0(C_r^*(A))$, $[1 \otimes p]$ corresponds to $[1]$.

Now observe that there is a group homomorphism $p^\sim : \Gamma^\sim \to \mathbb{Z}$ by mapping t to -1 and all other generators to 0. Denote by θ^{ab}, $(i^\sim)^{ab}$, $(p^\sim)^{ab}$ the homomorphisms on the abelianized groups by θ, $i^\sim$, $p^\sim$ respectively.

For $g \in \Gamma^{\sim}$, denote by $\bar{g}$ the image of g in $(\Gamma^{\sim})^{ab}$. Then the following is exact:

$$A^{ab} \xrightarrow{\mathrm{id}-\theta^{ab}} (H^{\sim})^{ab} \xrightarrow{(i^{\sim})^{ab}} (\Gamma^{\sim})^{ab} \xrightarrow{(p^{\sim})^{ab}} \mathbb{Z} \to 0.$$

Consider the following commutative diagram:

$$\begin{tikzcd}
A^{ab} \arrow[r, "k_A"] \arrow[d, "\mathrm{id}-\theta^{ab}"'] & K_1(C_r^*(A)) \arrow[d, "\mathrm{id}-\theta_*"] \\
(H^{\sim})^{ab} \arrow[r, "k_H^{\sim}"] \arrow[d, "(i^{\sim})^{ab}"'] & K_1(C_r^*(H)) \arrow[d, "i_*"] \\
(\Gamma^{\sim})^{ab} \arrow[r, "k_\Gamma^{\sim}"] \arrow[d, "(p^{\sim})^{ab}"'] & K_1(C_r^*(\Gamma)) \arrow[d, "\partial_1"] \\
\mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[d] \\
0 \arrow[r, equal] & 0.
\end{tikzcd}$$

By induction, k_A and $k_H^{\sim}$ are isomorphisms. It follows from the five-lemma that $k_\Gamma^{\sim}$ is an isomorphism. We also get

$$\begin{tikzcd}
0 \arrow[r] & K_0(C_r^*(H)) \arrow[r, "i_*"] & K_0(C_r^*(\Gamma)) \arrow[r, "\partial_0"] & A^{ab} \arrow[d, "\mathrm{id}-\theta^{ab}"] \\
0 & \mathbb{Z} \arrow[l] & (\Gamma^{\sim})^{ab} \arrow[l, "(p^{\sim})^{ab}"'] & (H^{\sim})^{ab}. \arrow[l, "(i^{\sim})^{ab}"']
\end{tikzcd}$$

By induction, the proof for the first case is completed.

Case 2. Suppose that $\sigma_z(s) \neq 0$ for any $z \in X$. Then the first case is applied to $G = \Gamma *_{\langle b = x^\alpha \rangle} \langle x \rangle$ with $\alpha = \sigma_t(s)$. By the first case, $K_0(C_r^*(G)) = \mathbb{Z}^n$ is generated by $n^{-1} \sum_{j=0}^{n-1} w^{jk} \delta_{s_1^j}$ $(k = 0, \cdots, n-1)$ and $k_G^{\sim} : (G^{\sim})^{ab} \to K_1(C_r^*(G))$ is an isomorphism, where $G = \langle y, x, c, d, \cdots | s_1^n \rangle$ with $\sigma_y(s_1) = \alpha \neq 0$ and $\bar{s}_1 \neq 0$, as before.

We have the following six-term exact sequence:

$$\begin{tikzcd}
K_0(C_r^*(\mathbb{Z})) \arrow[r, "(i_{1*}, -i_{2*})"] & K_0(C_r^*(\Gamma)) \oplus K_0(C_r^*(\mathbb{Z})) \arrow[r, "j_{1*}+j_{2*}"] & K_0(C_r^*(G)) \arrow[d, "\partial_0"] \\
K_1(C_r^*(G)) \arrow[u, "\partial_1"] & K_1(C_r^*(\Gamma)) \oplus K_1(C_r^*(\mathbb{Z})) \arrow[l, "j_{1*}+j_{2*}"'] & K_1(C_r^*(\mathbb{Z})) \arrow[l, "(i_{1*}.-i_{2*})"']
\end{tikzcd}$$

where $i_1(1) = b \in \Gamma$, $i_2(1) = \alpha \in \mathbb{Z}$ for $1 \in \mathbb{Z}$, and j_1 and j_2 are the inclusions from Γ and $\mathbb{Z} = \langle x \rangle$ to G respectively. The maps $(i_{1*}, -i_{2*})$ on the K-groups are injective. Indeed,

$$(i_{1*}, -i_{2*})[1] = ([1], -[1]), \quad (i_{1*}, -i_{2*})[\delta_1] = ([\delta_b], -\alpha[\delta_1]).$$

It follows that $j_{1*} : K_0(C_r^*(\Gamma)) \to K_0(C_r^*(G))$ is an isomorphism. Since $j_1(s) = s_1$ and $K_0(C_r^*(G)) \cong \mathbb{Z}^n$ is generated by $n^{-1}\sum_{j=0}^{n-1} w^{jk}\delta_{s_1^j}$ $(k = 0, \cdots, n-1)$, $K_0(C_r^*(\Gamma)) \cong \mathbb{Z}^n$ is generated by $n^{-1}\sum_{j=0}^{n-1} w^{jk}\delta_{s^j}$ $(k = 0, \cdots, n-1)$.

The following is exact:

$$0 \to \mathbb{Z} \xrightarrow{((i_1^\sim)^{ab}, -(i_2^\sim)^{ab})} (\Gamma^\sim)^{ab} \oplus \mathbb{Z} \xrightarrow{j_1^{ab} + j_2^{ab}} (G^\sim)^{ab} \to 0$$

where note that $j_1^{ab}(\bar{t}) = \bar{y} - \beta\bar{x}$, $j_1^{ab}(\bar{b}) = \alpha\bar{x}$, $j_1^{ab}(\bar{c}) = \bar{c}$, and $j_2^{ab}(1) = \bar{x}$. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] \\
\mathbb{Z} \arrow[r, "k_{\mathbb{Z}}"] \arrow[d, "((i_1^\sim)^{ab}, -(i_2^\sim)^{ab})"'] & K_1(C_r^*(\mathbb{Z})) \arrow[d, "(i_{1*}, -i_{2*})"] \\
(\Gamma^\sim)^{ab} \oplus \mathbb{Z} \arrow[r, "k_\Gamma^\sim \oplus k_{\mathbb{Z}}"] \arrow[d, "(j_1^\sim)^{ab} + (j_2^\sim)^{ab}"'] & K_1(C_r^*(\Gamma)) \oplus K_1(C_r^*(\mathbb{Z})) \arrow[d, "j_{1*} + j_{2*}"] \\
(G^\sim)^{ab} \arrow[r, "k_G^\sim"] \arrow[d] & K_1(C_r^*(\Gamma)) \arrow[d] \\
0 \arrow[r, equal] & 0.
\end{tikzcd}$$

Since $k_G^\sim$ and $k_{\mathbb{Z}}$ are isomorphisms, it follows from the five-lemma that $k_\Gamma^\sim$ is an isomorphism. □

Corollary 3.4.5 *Let $\Gamma = \langle X | s^n \rangle$ be a one-relator group with s a not proper power in $F(X)$, and $\bar{s} \neq 0$. Then $\tau_*(K_0(C_r^*(\Gamma))) = (1/n)\mathbb{Z}$. In particular, if $n = 1$, then $C_r^*(\Gamma)$ has no non-trivial projections, where τ is the canonical trace on $C_r^*(\Gamma)$.*

A one-relator group $\Gamma = \langle X | r \rangle$ with $|X| \geq 3$ is not amenable since Γ contains the free group of two generators (by Magnus).

Theorem 3.4.6 *Any one-relator group is K-amenable.*

Sketch of Proof. Use the same induction principle as before. If only one generator is left in the relator r, then Γ a one-relator group is either a free group or a free product of a free group with a finite cyclic group, so that Γ is K-amenable (by Cuntz). For general $|r|$, consider two cases as follows.

Case 1. Suppose that $\sigma_t(r) = 0$. Then Γ is an HNN extension $H*_A$, with H a one-relator group with a relation shorter than r. It follows from the Serre's theory for HNN extensions that Γ admits an action on a certain tree, transitive both on vertices and edges, with vertex stabilizers conjugate to H and edge stabilizers conjugate to A. By induction, H and A are K-amenable. For Γ, we can apply a Pimsner's result that a group acting on a tree with K-amenable stabilizers is K-amenable.

Case 2. Suppose that $\sigma_x(r) \neq 0$ for every $x \in X$. Then Γ is embedded as a subgroup in a one-relator group G to which the first case is applied. Thus, G is K-amenable, and K-amenability is stable under taking subgroups. □

Let Γ be a discrete group. The Baum-Connes geometric groups for Γ are the Γ-equivariant K-homology groups $K_j^\Gamma(\underline{E\Gamma})$ $(j = 0, 1)$ (with compact supports), where $\underline{E\Gamma}$ is the universal space for Γ-proper actions. Let μ_j^Γ denote the assembly map from $K_j^\Gamma(\underline{E\Gamma})$ to $K_j(C_r^*(\Gamma))$. The Baum-Connes conjecture is that μ_j^Γ are isomorphisms. Let ch_Γ denote the Chern character from $K_j^\Gamma(\underline{E\Gamma})$ to $\oplus_{k\in\mathbb{N}} H_{j+2k}(\Gamma, F\Gamma)$, which is an isomorphism after tensoring $\mathbb{C}$, where $F\Gamma$ is the Γ-module of $\mathbb{C}$-valued functions with finite support on the set of torsion elements of Γ, with Γ acting by conjugation. Note that if Γ is torsion-free, $K_j^\Gamma(\underline{E\Gamma}) = K_j^\Gamma(B\Gamma)$ the K-homology (with compact supports) of the classifying space $B\Gamma$, and $H_{j+2k}(\Gamma, F\Gamma) = H_{j+2k}(\Gamma, \mathbb{C})$.

Now assume that $\Gamma = \langle X | s^n \rangle$ is a one-relator group. Then Γ has homological dimension ≤ 2 over $\mathbb{C}$ (by Lyndon), so that

$$
\begin{aligned}
K_0^\Gamma(\underline{E\Gamma}) \otimes_{\mathbb{Z}} \mathbb{C} &\cong H_0(\Gamma, F\Gamma) \oplus H_2(\Gamma, F\Gamma), \\
K_1^\Gamma(\underline{E\Gamma}) \otimes_{\mathbb{Z}} \mathbb{C} &\cong H_1(\Gamma, F\Gamma).
\end{aligned}
$$

A set of representatives for the conjugacy classes of torsion elements in Γ is given by $\{1, s, \cdots, s^{n-1}\}$. By Shapiro's lemma,

$$H_j(\Gamma, F\Gamma) \cong \oplus_{k=0}^{n-1} H_j(Z_\Gamma(s^k), \mathbb{C}),$$

where $Z_\Gamma(s^k)$ is the centralizer of s^k, which gives $H_0(\Gamma, F\Gamma) \cong \mathbb{C}^n$. The centralizer $Z_\Gamma(s^k)$ for $k \geq 1$ is the cyclic subgroup generated by s, so that

$$
\begin{aligned}
H_1(\Gamma, F\Gamma) &\cong H_1(\Gamma, \mathbb{C}) = \Gamma^{ab} \otimes_{\mathbb{Z}} \mathbb{C}, \\
H_2(\Gamma, F\Gamma) &\cong H_2(\Gamma, \mathbb{C}).
\end{aligned}
$$

It follows that

$$H_2(\Gamma, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } \bar{s} \in [F(X), F(X)], \\ 0 & \text{if } \bar{s} \notin [F(X), F(X)]. \end{cases}$$

It follows from the above theorems on the K-theory groups that

$$K_j(C_r^*(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{C} \cong K_j^{\Gamma}(\underline{E\Gamma}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Theorem 3.4.7 *Let Γ be a one-relator group. Then the Baum-Connes conjecture holds for Γ, i.e., the map $\mu_*^{\Gamma} : K_*^{\Gamma}(\underline{E\Gamma}) \to K_*(C_r^*(\Gamma))$ is an isomorphism.*

Remark. It is known in general that injectivity of the assembly map implies the Novikov conjecture on the homotopy invariance of higher signatures, and its surjectivity in the torsion-free case implies the Kadison-Kaplansky conjecture on projection non-existence for reduced group C^*-algebras.

Corollary 3.4.8 *The reduced C^*-algebra of a torsion-free, one-relator group has no non-trivial projections.*

This corollary and the theorem above follows from a more general theorem given below.

Let Γ be a discrete group. Suppose that Γ acts on a C^*-algebra $\mathfrak{A}$ by automorphisms. The Baum-Connes conjecture for Γ with coefficients in $\mathfrak{A}$ is that the assembly map μ_j^{Γ} from the group $K_j^{\Gamma}(\underline{E\Gamma}, \Gamma)$ of Baum, Connes, and Higson to $K_j(\mathfrak{A} \rtimes_r \Gamma)$ $(j = 0, 1)$ is an isomorphism, where $\mathfrak{A} \rtimes_r \Gamma$ is the reduced crossed product of $\mathfrak{A}$ by Γ.

Theorem 3.4.9 *The Baum-Connes conjecture with coefficients holds for one-relator groups.*

To prove the theorem above,

Lemma 3.4.10 *Let Γ be a group acting properly on an oriented tree X. Then the Baum-Connes conjecture with coefficients holds for Γ.*

Sketch of Proof. Let $|X|$ be the geometric realization of Γ. It follows that we may assume $\underline{E\Gamma} = |X|$. Let $\mathfrak{A}_X$ be the C^*-algebra accociated with the simplicial complex $|X|$ (defined by Julg and Kasparov-Skandalis). There is a Dirac element $D \in KK^{\Gamma}(\mathfrak{A}_X, \mathbb{C})$ and a dual-Dirac element $\eta \in KK^{\Gamma}(\mathbb{C}, \mathfrak{A}_X)$ such that $D \otimes_{\mathbb{C}} \eta = 1_{\mathfrak{A}_X} \in KK^{\Gamma}(\mathfrak{A}_X, \mathfrak{A}_X)$, and $\eta \otimes_{\mathfrak{A}_X} \mathfrak{D} = 1_{\mathbb{C}} \in KK^{\Gamma}(\mathbb{C}, \mathbb{C})$. Then it follows that μ_j^{Γ} is an isomorphism (by Tu). $\square$

The above theorem is proved by using the lemma above and the induction principle as before.

Remarks. (1). Let $\Gamma = \langle X|r\rangle$ be a torsion-free one-relator group with $\bar{r} = 0$. Write r as a product of g commutators in $F(X)$. Consider the associated homomorphism $f : \Gamma_g \to \Gamma$, where Γ_g is the surface group with genus g. Then the induced map $f_* : K_0(C^*(\Gamma_g)) \to K_0(C^*(\Gamma))$ between the K_0-groups of the full group C^*-algebras is an isomorphism. Indeed, let $\bar{\mu}_0^\Gamma : K_0(B\Gamma) \to K_0(C^*(\Gamma))$ be the Baum-Connes assemply map defined at the level of the full C^*-algebras, which is natural with respect to group homomorphisms. Then the following is commutative:

$$\begin{array}{ccc} K_0(B\Gamma_g) & \xrightarrow{f_*} & K_0(B\Gamma) \\ \bar{\mu}_0^{\Gamma_g}\Big\downarrow & & \Big\downarrow\bar{\mu}_0^{\Gamma} \\ K_0(C^*(\Gamma_g)) & \xrightarrow{f_*} & K_0(C^*(\Gamma)). \end{array}$$

The upper horizontal map is an isomorphism and the two vertical maps are isomorphisms. Hence the lower map is an isomorphism. The Baum-Connes conjecture for Γ_g has been obtained by Kasparov in a completely different way of the equivariant KK-theory, and note that Γ_g for $g \geq 2$ is a co-compact lattice in $PSL_2(\mathbb{R})$.

(2). Let $\Gamma = \langle X|r\rangle$ be a torsion-free one-relator group. There is a surjective $*$-homomorphism $\pi : C^*(F(X)) \to C^*(\Gamma)$. Denote by $\mathfrak{J}$ the kernel of π. Then we have the following:

$$\begin{array}{ccccc} K_0(\mathfrak{J}) & \longrightarrow & K_0(C^*(F(X))) & \xrightarrow{\pi_*} & K_0(C^*(\Gamma)) \\ \Big\uparrow & & & & \Big\downarrow \\ K_1(C^*(\Gamma)) & \xleftarrow{\pi_*} & K_1(C^*(F(X))) & \longleftarrow & K_1(\mathfrak{J}). \end{array}$$

Then π_* on K_0 is injective and π_* on K_1 is surjective. It follows that $K_0(\mathfrak{J}) = 0$. In the case $\bar{r} \neq 0$, π_* on K_0 is an isomorphism. Since $K_1(C^*(F(X))) \cong F(X)^{ab}$ and $K_1(C^*(\Gamma)) \cong \Gamma^{ab} \cong F(X)^{ab}/\langle\bar{r}\rangle$, we have $K_1(\mathfrak{J}) \cong \mathbb{Z}$, whose generator corresponds to the relator r as a unitary in $C^*(F(X))$. In the case $\bar{r} = 0$, π_* is an isomorphism on K_1, so that we have

$$0 \to K_0(C^*(F(X)) = \mathbb{Z}[1] \to K_0(C^*(\Gamma)) = \mathbb{Z}^2 \to K_1(\mathfrak{J}) \cong \mathbb{Z} \to 0.$$

(3). Let B_n be the braid group on n strings. It is conjectured by Baum and Connes that $K_0(C_r^*(B_n)) \cong \mathbb{Z} \cong K_1(C_r^*(B_n))$. This is true if $n = 2$, since $B_2 = \mathbb{Z}$. It follows that this is true if $n = 3$, since $B_3 = \langle a, b|abab^{-1}a^{-1}b^{-1}\rangle$.

(4). Let $\mathbb{C}\Gamma$ be the group algebra of a torsion-free one-relator group Γ. It is implied that $\mathbb{C}\Gamma$ has no idempotents. However, it has shown by Lewin(s) that $\mathbb{C}\Gamma$ has no zero divisor.

Now let M be a compact, connected, oriented 3-manifold. We say that M is irreducible if every embedded 2-sphere in M bounds an embedded 3-disk. A properly embedded, orientable, connected compact surface in M is incompressible if it is not a 2-sphere, and its inclusion induces a monomorphism on fundamental groups. An irreducible 3-manifold is called Haken if it contains an incompressible surface. It is known that if the boundary ∂M contains a surface other than a 2-sphere, then $H_1(M, \mathbb{Z})$ is infinite, which implies that M is Haken. Knot spaces that are complements of small tubular neighbourhoods of knots in S^3 are Haken manifolds.

Theorem 3.4.11 *The Baum-Connes conjecture with coefficients holds for fundamental groups of Haken 3-manifolds. Furthermore, the groups are K-amenable.*

Sketch of Proof. Let M be a Haken 3-manifold. The hierarchy for M is a finite sequence of pairs $(M_1, F_1), \cdots, (M_n, F_n)$, where $M_1 = M$, each F_j is an incompressible surface in M_j which is not boundary parallel, each M_{j+1} is the manifold obtained from M_j by cutting it open along F_j (or $M_{j+1} = M_j \setminus V_j$, where V_j is an open tubular neighbourhood of F_j in M_j), and each component of M_n is the 3-disk. Then the fundamental group $\pi_1(M)$ is obtained as a sequence of amalgamated products and HNN extensions. Indeed, if F_j separates M_j, we have M_{j+1} is a disjoint union $M'_{j+1} \sqcup M''_{j+1}$ and $\pi_1(M_j) \cong \pi_1(M'_{j+1}) *_{\pi_1(F_j)} \pi_1(M''_{j+1})$ an amalgam. If F_j does not separate M_j, then two embeddings of F_j into ∂M_{j+1} induce two monomorphisms $\iota, \theta : \pi_1(F_j) \to \pi_1(M_{j+1})$ and $\pi_1(M_j) \cong \pi_1(M_{j+1}) *_{\pi_1(F_j)}$ an HNN extension via ι, θ. By induction on the length of the hierarchy, it follows by Tu that the BC conjecture holds for $\pi_1(M)$. □

Remark. Note that a Haken manifold M is a $K(\pi, 1)$-space. This implies that $\pi_1(M) = \pi$ is torsion-free. Since the assemply map $\mu_j^\pi : K_j(M) \to K_j(C_r^*(\pi))$ is an isomorphism, so that $C_r^*(\pi)$ has no nontrivial projections.

3.5 KK- and E-theories of amalgams of C^*-algebras

Taken from Thomsen [18] are those in what follows.

KK-theory.

Let A and $\mathfrak{D}$ be separable C^*-algebras, $\mathfrak{D}$ stable. Let $M(\mathfrak{D})$ be the multiplier algebra of $\mathfrak{D}$. Then there are isometries $V_1, V_2 \in M(\mathfrak{D})$ such that $V_1V_1^* + V_2V_2^* = 1$ and $V_1^*V_2 = 0$. Define the orthogonal sum $a \oplus b$ of $a, b \in M(\mathfrak{D})$ to be $V_1aV_1^* + V_2bV_2^*$. Similarly, define the orthogonal sum of maps $\varphi, \psi : A \to M(\mathfrak{D})$ by $(\varphi \oplus \psi)(a) = V_1\varphi(a)V_1^* + V_2\psi(a)V_2^*$. A $*$-homomorphism $\varphi : A \to M(\mathfrak{D})$ is absorbing if for a $*$-homomorphism $\pi : A \to M(\mathfrak{D})$, there is a sequence of unitaries $\{U_n\} \subset M(\mathfrak{D})$ such that $\lim_{n\to\infty} U_n(\varphi\oplus\pi)(a)U_n^* = \varphi(a)$ for $a \in A$. There always exists an absorbing $*$-homomorphism.

Lemma 3.5.1 *Let $\mathfrak{B}$ be a C^*-subalgebra of a separable C^*-algebra A. Assume that there is a sequence of completely positive contractions $R_n : A \to \mathfrak{B}$ such that $\lim_n R_n(b) = b$ for $b \in \mathfrak{B}$. If $\pi : A \to M(\mathfrak{D})$ is an absorbing $*$-homomorphism, then the restriction $\pi|_{\mathfrak{B}}$ to $\mathfrak{B}$ is absorbing.*

Sketch of Proof. We need to show that the unitization $(\pi|_{\mathfrak{B}})^+ : \mathfrak{B}^+ \to M(\mathfrak{D})$ is unitarily absorbing. Consider a completely positive contraction $\varphi : \mathfrak{B}^+ \to \mathfrak{D}$. Then $\varphi \circ R_k^+ : A^+ \to \mathfrak{D}$ is also a completely positive contraction. Since $\pi^+ : A^+ \to M(\mathfrak{D})$ is unitarily absorbing, there is a sequence $\{W_n^k\} \subset M(\mathfrak{D})$ such that $\lim_{n\to\infty} \|\varphi\circ R_k^+(a) - W_n^{k*}\pi^+(a)W_n^k\| = 0$ for $a \in A^+$ and $\lim_{n\to\infty} \|W_n^{k*}d\| = 0$ for $d \in \mathfrak{D}$. Since $\lim_{k\to\infty} R_k^+(b) = b$ for $b \in \mathfrak{B}^+$, it follows that $(\pi|_{\mathfrak{B}})^+ = \pi^+|_{\mathfrak{B}^+}$ satisfies the desired condition. □

Lemma 3.5.2 *Let $\mathfrak{B}$ be a C^*-subalgebra of a separable C^*-algebra A. Assume that $\mathfrak{B}$ is nuclear. If $\pi : A \to M(\mathfrak{D})$ is an absorbing $*$-homomorphism, then the restriction $\pi|_{\mathfrak{B}}$ to $\mathfrak{B}$ is absorbing.*

Proof. Since $\mathfrak{B}$ is nuclear, there are sequences of completely positive contractions $S_n : \mathfrak{B} \to F_n$ and $T_n : F_n \to \mathfrak{B}$ $(n \in \mathbb{N})$, where F_n are finite dimensional C^*-algebras, such that $\lim_{n\to\infty} T_n \circ S_n(b) = b$ for $b \in \mathfrak{B}$. By Arveson's extension theorem, there is for each n, a completely positive contraction $V_n : A \to F_n$ extending S_n. Set $R_n = T_n \circ V_n$ and apply the above lemma. □

Assume now that $\mathfrak{A}_1$, $\mathfrak{A}_2$, and $\mathfrak{B}$ are separable C^*-algebras with embeddings $i_k : \mathfrak{B} \to \mathfrak{A}_k$ for $k = 1, 2$. Let $j_k : \mathfrak{A}_k \to \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2 = \mathfrak{A}$ be the canonical maps. Assume that $\mathfrak{B}$ is finite dimensional. Then there is an absorbing $*$-homomorphism $\alpha : \mathfrak{A} \to M(\mathfrak{D})$ such that $\alpha \circ j_k$ and $\alpha \circ j_k \circ i_k$ $(k = 1, 2)$ are absorbing. Indeed, since $\mathfrak{B}$ is finite dimensional, there are surjective conditional expectations $P_k : \mathfrak{A}_k \to \mathfrak{B}$ $(k = 1, 2)$, and it follows that P_1 and P_2 give rise to the conditional expectations $\mathrm{id}_{\mathfrak{A}_1} *_{\mathfrak{B}} P_2 : \mathfrak{A} \to \mathfrak{A}_1$ and $P_1 *_{\mathfrak{B}} \mathrm{id}_{\mathfrak{A}_2} : \mathfrak{A} \to \mathfrak{A}_2$. Any absorbing $*$-homomorphism $\alpha : \mathfrak{A} \to M(\mathfrak{D})$

has the desired property and α does exist by Thomsen. Set $\alpha_k = \alpha \circ j_k$ $(k = 1, 2)$. Set

$$\mathcal{A}_k = \{x \in M(\mathfrak{D}) : x\alpha_k(a) - \alpha_k(a)x \in \mathfrak{D}, a \in \mathfrak{A}_k\},$$
$$\mathcal{A} = \{x \in M(\mathfrak{D}) : x(\alpha_1 \circ i_1)(b) - (\alpha_1 \circ i_1)(b)x \in \mathfrak{D}, b \in \mathfrak{B}\}.$$

It follows that $KK^0(\mathfrak{A}_k, \mathfrak{D}) \cong K_1(\mathcal{A}_k)$ $(k = 1, 2)$ and $KK^0(\mathfrak{B}, \mathfrak{D}) \cong K_1(\mathcal{A})$. Set

$$\mathcal{A}_0 = \{x \in M(\mathfrak{D}) : x(\alpha_1 \circ i_1)(b) = (\alpha_1 \circ i_1)(b)x, b \in \mathfrak{B}\}.$$

Proof. Let $x \in \mathcal{A}$. Note that $\alpha_1 \circ i_1(\mathfrak{B}) \cong \oplus_{j=1}^N M_{n_j}(\mathbb{C})$ for some n_j. Choose a set of matix units $\{e_{ij}^d : i, j = 1, \cdots, n_d, 1 \leq d \leq N\}$ for the direct sum. Let $q = \sum_{d=1}^N \sum_{i=1}^{n_d} e_{ii}^d$ the unit of $\alpha_1 \circ i_1(\mathfrak{B})$. Define $R : M(\mathfrak{D}) \to M(\mathfrak{D}) \cap (\alpha_1 \circ i_1(\mathfrak{B}))'$ by

$$R(x) = (1 - q)x(1 - q) + \sum_{d=1}^N \sum_{j=1}^{n_d} d_{i1}^d x e_{1i}^d.$$

Then $R(x) \in x + \mathfrak{D}$ because x commetes with the matrix units modulo $\mathfrak{D}$. Since $R(x) \in \mathcal{A}_0$, this shows that $\mathcal{A} \subset \mathcal{A}_0 + \mathfrak{D}$. Its converse inclusion is trivial. □

Given a $*$-homomorphism $\varphi : E \to F$ between C^*-algebras, denote by $1_n \otimes \varphi$ the $*$-homomorphism from E to $M_n(F)$ given by $(1_n \otimes \varphi)(e) = \varphi(e) \oplus \cdots \oplus \varphi(e)$ the diagonal sum in $M_n(F)$. Let $q_{\mathfrak{D}} : M(\mathfrak{D}) \to Q(\mathfrak{D}) = M(\mathfrak{D})/\mathfrak{D}$ be the quotient map. Let u be a unitary of $M_n(\mathcal{A})$. Then $u((1_n \otimes \alpha_1) \circ i_1)(b)u^* - ((1_n \otimes \alpha_2) \circ i_2)(b) \in M_n(\mathfrak{D})$ for $b \in \mathfrak{B}$, so define $\rho : KK^0(\mathfrak{B}, \mathfrak{D}) \to \mathrm{Ext}^{-1}(\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2, \mathfrak{D})$ by

$$\rho(u) = (q_{M_n(\mathfrak{D})} \circ \mathrm{Ad}(u) \circ (1_n \otimes \alpha_1)) *_{\mathfrak{B}} (q_{M_n(\mathfrak{D})} \circ (1_n \otimes \alpha_2))$$

where $\rho(u) : \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2 \to Q(M_n(\mathfrak{D})) \cong Q(\mathfrak{D})$.

Lemma 3.5.3 *The extension $\rho(u)$ is invertible. In fact, the diagonal sum $\rho(u) \oplus \rho(u^*)$ is a split extension.*

Proof. Note that the diagonal sum $u \oplus u^*$ is in the connected component of the unit in the unitary group of $M_{2n}(\mathcal{A})$. If follows from the above lemma that there is a unitary $w \in M_{2n}(\mathcal{A}_0)$ such that $u \oplus u^* = w$ modulo $M_n(\mathfrak{D})$. Then

$$\begin{aligned}\rho(u) \oplus \rho(u^*) &= (q_{M_{2n}(\mathfrak{D})} \circ \mathrm{Ad}(w) \circ (1_{2n} \otimes \alpha_1)) *_{\mathfrak{B}} (q_{M_{2n}(\mathfrak{D})} \circ (1_{2n} \otimes \alpha_2)) \\ &= q_{m_{2n}(\mathfrak{D})} \circ \varphi, \quad \varphi = (\mathrm{Ad}(w) \circ (1_{2n} \otimes \alpha_1)) *_{\mathfrak{B}} (1_{2n} \otimes \alpha_2).\end{aligned}$$

□

Lemma 3.5.4 *The following sequence is exact:*

$$\begin{array}{ccccc} \oplus_{j=1}^{2} KK^0(\mathfrak{A}_j,\mathfrak{D}) & \xrightarrow{i_1^*-i_2^*} & KK^0(\mathfrak{B},\mathfrak{D}) & \xrightarrow{\rho} & \mathrm{Ext}^{-1}(\mathfrak{A},\mathfrak{D}) \\ & & & & \big\downarrow{\scriptstyle (j_1^*,j_2^*)} \\ & & \mathrm{Ext}^{-1}(\mathfrak{B},\mathfrak{D}) & \xleftarrow{i_1^*-i_2^*} & \oplus_{k=1}^{2}\mathrm{Ext}^{-1}(\mathfrak{A}_k,\mathfrak{D}). \end{array}$$

Proof. To show exactness at $KK^0(\mathfrak{B},\mathfrak{D})$, consider unitaries $v_k \in M_n(\mathcal{A}_k)$. Then $(q_{M_n(\mathfrak{D})} \circ \mathrm{Ad}(v_2^* v_1) \circ (1_n \otimes \alpha_1)) *_{\mathfrak{B}} (q_{M_n(\mathfrak{D})} \circ (1_n \otimes \alpha_2))$ is unitarily equivalent to

$$\begin{aligned} &(q_{M_n(\mathfrak{D})} \circ \mathrm{Ad}(v_1) \circ (1_n \otimes \alpha_1)) *_{\mathfrak{B}} (q_{M_n(\mathfrak{D})} \circ \mathrm{Ad}(v_2) \circ (1_n \otimes \alpha_2)) \\ &= (q_{M_n(\mathfrak{D})} \circ (1_n \otimes \alpha_1)) *_{\mathfrak{B}} (q_{M_n(\mathfrak{D})} \circ (1_n \otimes \alpha_2)) = q_{M_n(\mathfrak{D})} \circ (1_n \otimes \alpha), \end{aligned}$$

which is a split extension. Thus, $\rho \circ (i_1^* - i_2^*) = 0$. Consider a unitary $u \in M_n(\mathcal{A})$ and assume that $[\rho(u)] = 0$ in $\mathrm{Ext}^{-1}(\mathfrak{A},\mathfrak{D})$. Since α is absorbing, this implies

$$\mathrm{Ad}(q_{M_{n+1}(\mathfrak{D})}(W)) \circ (\rho(u) \oplus (q_{\mathfrak{D}} \circ \alpha)) = (q_{M_n(\mathfrak{D})} \circ (1_n \otimes \alpha)) \oplus (q_{\mathfrak{D}} \circ \alpha)$$

for some unitary $W \in M_{n+1}(M(\mathfrak{D}))$. Alternatively,

$$\begin{aligned} &\mathrm{Ad}(q_{M_{n+1}(\mathfrak{D})}(W)) \circ ((q_{M_n(\mathfrak{D})} \circ \mathrm{Ad}(u) \circ (1_n \otimes \alpha_1)) \oplus (q_{\mathfrak{D}} \circ \alpha_1)) \\ &= (q_{M_n(\mathfrak{D})} \circ (1_n \otimes \alpha_1)) \oplus (q_{\mathfrak{D}} \circ \alpha_1), \\ &\mathrm{Ad}(q_{M_{n+1}(\mathfrak{D})}(W)) \circ ((q_{M_n(\mathfrak{D})} \circ (1_n \otimes \alpha_2)) \oplus (q_{\mathfrak{D}} \circ \alpha_2)) \\ &= (q_{M_n(\mathfrak{D})} \circ (1_n \otimes \alpha_2)) \oplus (q_{\mathfrak{D}} \circ \alpha_2). \end{aligned}$$

Hence, W^* and $W(u \oplus 1)$ are unitaries of $M_{n+1}(\mathcal{A}_2)$ and $M_{n+1}(\mathcal{A}_1)$ respectively. Since the product of their images in $M_{n+1}(\mathcal{A})$ is $u \oplus 1$, it follows that the class $[u]$ is in the range of $i_1^* - i_2^*$.

To show exactness at $\mathrm{Ext}^{-1}(\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2, \mathfrak{D})$, note that the composition $(j_1^*, j_2^*) \circ \rho$ is zero, so that we consider an extension $\varphi : \mathfrak{A} \to Q(\mathfrak{D})$ such that $\varphi \circ j_k$ $(k = 1, 2)$ are split. Since α_k is absorbing, there are unitaries $S_k \in M_2(M(\mathfrak{D}))$ such that

$$\mathrm{Ad}_{q_{M_2(\mathfrak{D})}}(S_k) \circ ((\varphi \circ j_k) \oplus (q_{\mathfrak{D}} \circ \alpha_k)) = (q_{\mathfrak{D}} \circ \alpha_k) \oplus (q_{\mathfrak{D}} \circ \alpha_k)$$

$(k = 1, 2)$. It follows that $S_2 S_1^* \in M_2(\mathcal{A})$ and that $\varphi \oplus (q_{\mathfrak{D}} \circ \alpha)$ is unitarily equivalent to $(\mathrm{Ad}_{q_{M_2(\mathfrak{D})})}(S_2 S_1^*) \circ (1_2 \otimes \alpha_1)) *_{\mathfrak{B}} (1_2 \otimes \alpha_2)$ which is in the range of ρ. In particular, φ is invertible.

To show exactness at $\oplus_{j=1}^{2}\mathrm{Ext}^{-1}(\mathfrak{A}_j, \mathfrak{D})$, note that $(i_1^* - i_2^*) \circ (j_1^*, j_2^*) = 0$, so that we consider a pair of invertible extensions $\varphi_k : \mathfrak{A}_k \to Q(\mathfrak{D})$ $(k = 1, 2)$ such that $i_1^*[\varphi_1] = i_2^*[\varphi_2]$. Then there is a unitary $S \in M_2(M(\mathfrak{D}))$ such that

$$\mathrm{Ad}_{q_{M_2(\mathfrak{D})}}(S) \circ ((\varphi_1 \circ i_1) \oplus (q_{\mathfrak{D}} \circ \alpha_1 \circ i_1)) = ((\varphi_2 \circ i_2) \oplus (q_{\mathfrak{D}} \circ \alpha_2 \circ i_2)).$$

Adding $q_{\mathfrak{D}} \circ \alpha_k$ to φ_k we may assume that $\varphi_1 \circ i_1 = \varphi_2 \circ i_2$. Similarly, if $\psi_k : \mathfrak{A}_k \to Q(\mathfrak{D})$ represents the inverse of φ_k in $\mathrm{Ext}^{-1}(\mathfrak{A}_k, \mathfrak{D})$, we may assume that $\psi_1 \circ i_1 = \psi_2 \circ i_2$. Then we consider the extensions $\varphi_1 *_{\mathfrak{B}} \varphi_2$ and $\psi_1 *_{\mathfrak{B}} \psi_2 : \mathfrak{A} \to Q(\mathfrak{D})$ whose sum μ satisfies that $\mu \circ j_k : \mathfrak{A}_k \to Q(\mathfrak{D})$ $(k = 1, 2)$ are split. It follows that μ and $\varphi_1 *_{\mathfrak{B}} \varphi_2$ are invertible. Since $\varphi_k = (\varphi_1 *_{\mathfrak{B}} \varphi_2) \circ j_k$, we are done. □

Theorem 3.5.5 *Let $\mathfrak{A} = \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2$ be an amalgam of separable C^*-algebras. Assume that $\mathfrak{B}$ is finite dimensional. For any separable C^*-algebra $\mathfrak{D}$,*

$$\begin{tikzcd}
KK^0(\mathfrak{D},\mathfrak{B}) \arrow[r, "(i_{1*},i_{2*})"] & KK^0(\mathfrak{D},\mathfrak{A}_1) \oplus KK^0(\mathfrak{D},\mathfrak{A}_2) \arrow[r, "j_{1*}-j_{2*}"] & KK^0(\mathfrak{D},\mathfrak{A}) \arrow[d] \\
KK^1(\mathfrak{D},\mathfrak{A}) \arrow[u] & KK^1(\mathfrak{D},\mathfrak{A}_1) \oplus KK^1(\mathfrak{D},\mathfrak{A}_2) \arrow[l, "j_{1*}-j_{2*}"'] & KK^1(\mathfrak{D},\mathfrak{B}) \arrow[l, "(i_{1*},i_{2*})"']
\end{tikzcd}$$

where $i_k : \mathfrak{B} \to \mathfrak{A}_k$ are embeddings, and $j_k : \mathfrak{A}_k \to \mathfrak{A}$ are the canonical maps. Furthermore,

$$\begin{tikzcd}
KK^0(\mathfrak{B},\mathfrak{D}) \arrow[d] & KK^0(\mathfrak{A}_1,\mathfrak{D}) \oplus KK^0(\mathfrak{A}_2,\mathfrak{D}) \arrow[l, "i_1^*-i_2^*"'] & KK^0(\mathfrak{A},\mathfrak{D}) \arrow[l, "(j_1^*,j_2^*)"'] \\
KK^1(\mathfrak{A},\mathfrak{D}) \arrow[r, "(j_1^*,j_2^*)"] & KK^1(\mathfrak{A}_2,\mathfrak{D}) \oplus KK^1(\mathfrak{A}_2,\mathfrak{D}) \arrow[r, "i_1^*-i_2^*"] & KK^1(\mathfrak{B},\mathfrak{D}). \arrow[u]
\end{tikzcd}$$

Sketch of Proof. The mapping cone for the embedding defined by $\mathfrak{B} \ni b \mapsto (i_1(b), i_2(b)) \in \mathfrak{A}_1 \oplus \mathfrak{A}_2$ is given by

$$\begin{aligned}C = \{&(b, g_1, g_2) \in \mathfrak{B} \oplus (C_0((0,1]) \otimes \mathfrak{A}_1) \oplus (C_0((0,1]) \otimes \mathfrak{A}_2) : \\ &g_1(1) = i_1(b), g_2(1) = i_2(b)\}.\end{aligned}$$

The $*$-homomorphism of Germain $G : C \to S\mathfrak{A} = C_0(0,1) \otimes \mathfrak{A}$ is defined by

$$G(b, g_1, g_2)(t) = \begin{cases} j_1(g_1(2t)) & t \in (0, 1/2], \\ j_2(g_2(2-2t)) & t \in [1/2, 1). \end{cases}$$

Using the above lemma and the Puppe exact sequence of Cuntz and Skandalis, we obtain the following diagram:

$$\begin{tikzcd}
KK^0(\mathfrak{A}_1,\mathfrak{D})\oplus KK^0(\mathfrak{A}_2,\mathfrak{D}) \arrow[r, equal] \arrow[d, "i_1^*-i_2^*"'] & KK^0(\mathfrak{A}_1,\mathfrak{D})\oplus KK^0(\mathfrak{A}_2,\mathfrak{D}) \arrow[d, "i_1^*-i_2^*"'] \\
KK^0(\mathfrak{B},\mathfrak{D}) \arrow[r, equal] \arrow[d, "\rho"'] & KK^0(\mathfrak{B},\mathfrak{D}) \arrow[d, "p"'] \\
\mathrm{Ext}^{-1}(\mathfrak{A},\mathfrak{D}) \arrow[r, "G^*"] \arrow[d, "(j_1^*,-j_2^*)"'] & KK^0(\mathfrak{C},\mathfrak{D}) \arrow[d, "i^*"'] \\
\mathrm{Ext}^{-1}(\mathfrak{A}_1,\mathfrak{D})\oplus \mathrm{Ext}^{-1}(\mathfrak{A}_2,\mathfrak{D}) \arrow[r, equal] \arrow[d, "i_1^*+i_2^*"'] & \mathrm{Ext}^{-1}(\mathfrak{A}_1,\mathfrak{D})\oplus \mathrm{Ext}^{-1}(\mathfrak{A}_2,\mathfrak{D}) \arrow[d, "i_1^*+i_2^*"'] \\
\mathrm{Ext}^{-1}(\mathfrak{B},\mathfrak{D}) \arrow[r, equal] & \mathrm{Ext}^{-1}(\mathfrak{B},\mathfrak{D})
\end{tikzcd}$$

where $i : S\mathfrak{A}_1 \oplus S\mathfrak{A}_2 \to C$ is given by $i(g_1, g_2) = (0, g_1, g_2)$, and $p : C \to \mathfrak{B}$ by $p(b, g_1, g_2) = b$, and we have the natural identification $KK^1(\cdot, \mathfrak{D})$ with $\mathrm{Ext}^{-1}(\cdot, \mathfrak{D})$. This diagram is shown to be commutative. Indeed, note that $G \circ i$ is homotopic to $((Sj_1), (-Sj_2))$. Note also the following square:

$$\begin{tikzcd}
KK^0(\mathfrak{B},\mathfrak{D}) \arrow[r, "\mu"] \arrow[d, "S\circ\rho"'] & \mathrm{Ext}^{-1}(\mathfrak{B},S\mathfrak{D}) \arrow[d, "p^*"] \\
\mathrm{Ext}^{-1}(S\mathfrak{A},S\mathfrak{D}) \arrow[r, "G^*"] & \mathrm{Ext}^{-1}(\mathfrak{C},S\mathfrak{D}),
\end{tikzcd}$$

where $\mu : KK^0(\mathfrak{B}, \mathfrak{D}) \to \mathrm{Ext}^{-1}(\mathfrak{B}, S\mathfrak{D})$ is defined as follows. Realize an element of $KK^0(\mathfrak{B}, \mathfrak{D})$ as a pair (u, α), where u is a unitary of $M(\mathfrak{D})$ such that $u(\alpha_1 \circ i_1)(b) - (\alpha_1 \circ i_1)(b)u \in \mathfrak{D}$ for all $b \in \mathfrak{B}$. Define $\mu^\sim : \mathfrak{B} \to M(S\mathfrak{D})$ by

$$\mu^\sim(b)f(t) = [t(\alpha_1 \circ i_1)(b) + (1 - t)u(\alpha_1 \circ i_1)(b)u^*]f(t)$$

for $f \in S\mathfrak{D} = C_0((0,1)) \otimes \mathfrak{D}$. Then $\mu[(u, \alpha)] = [q_{S\mathfrak{D}} \circ \mu^\sim]$, where $q_{S\mathfrak{D}} : M(S\mathfrak{D}) \to Q(S\mathfrak{D})$ is the quotient map. We need to show that $q_{S\mathfrak{D}} \circ \mu^\sim \circ p : \mathfrak{C} \to Q(S\mathfrak{D})$ is the same as $G^* \circ S \circ \rho[(u, \alpha)]$ in $\mathrm{Ext}^{-1}(\mathfrak{C}, S\mathfrak{D})$. For $\lambda \in (0, 1/2]$, G is homotopic to the $*$-homomorphism $G_\lambda : \mathfrak{C} \to \mathfrak{A}$ defined as $G_\lambda(b.g_1, g_2)(t) = j_1(g_1(t/\lambda))$ for $t \in (0, \lambda)$, it $= j_1 \circ i_1(b)$ for $t \in [\lambda, 1-\lambda]$, and it $= j_2(g_2((1-t)/\lambda))$ for $t \in [1-\lambda, 1)$. It follows that $G^* \circ S \circ \rho[(u, \alpha)]$ is represented by $q_{S\mathfrak{D}} \circ \Phi_\lambda$ for $\lambda \in (0, 1/2]$, where $\Phi_\lambda : \mathfrak{C} \to M(S\mathfrak{D})$ is defined as $\Phi_\lambda(b, g_1, g_2)f(t) = u\alpha_1(g_1(t/\lambda))u^*f(t)$ for $t \in (0, \lambda]$, and it

$$= [(t - \lambda)(1 - 2\lambda)^{-1}u(\alpha_1 \circ i_1)(b)u^* + (1 - \lambda - t)(1 - 2\lambda)^{-1}(\alpha_1 \circ i_1)(b)]f(t)$$

for $t \in [\lambda, 1-\lambda]$, and it $= \alpha_2(g_2((1-t)/\lambda))f(t)$ for $t \in [1-\lambda, 1)$, where $f \in S\mathfrak{D}$. Note that $\lim_{\lambda\to 0} \Phi_\lambda(b, g_1, g_2) = \mu^\sim(b)$ in the strict topology and that

$$\lim_{\lambda\to 0} \Phi_\lambda(b, g_1, g_2)\Phi_\lambda(b', g_1', g_2') - \Phi_\lambda(bb', g_1g_1', g_2g_2') = \mu^\sim(b)\mu^\sim(b') - \mu^\sim(bb')$$

in the norm topology of $S\mathfrak{D}$ for (b, g_1, g_2), $(b', g_1', g_2') \in \mathfrak{C}$. It follows that $G^* \circ S \circ \rho[(u, \alpha)]$ is homotopic to $q_{S\mathfrak{D}} \circ \mu^\sim \circ p$.

It follows immediately from the above diagram and the five-lemma that $G^* : KK^1(\mathfrak{A}, \mathfrak{D}) \to KK^0(\mathfrak{C}, \mathfrak{D})$ is an isomorphism. It follows from the standard KK-theory argument that G is invertible in KK-theory. This completes the proof. □

Proposition 3.5.6 *Assume the same settings as above, with* $\mathfrak{D}$ *stable. If* $\mathrm{Ext}(\mathfrak{A}_k, \mathfrak{D})$ $(k = 1, 2)$ *are groups, then* $\mathrm{Ext}(\mathfrak{A}, \mathfrak{D})$ *is a group.*

Proof. It follows that every extension of $\mathfrak{A} = \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2$ by $\mathfrak{D}$ is invertible. □

Remark. L. Brown has shown that $\mathrm{Ext}(\mathfrak{A})$ is a group when $\mathrm{Ext}(\mathfrak{A}_k)$ $(k = 1, 2)$ are groups.

E-theory.

The suspension and the cone of a C^*-algebra $\mathfrak{A}$ are denoted by $S\mathfrak{A}$ and $C\mathfrak{A}$ respectively, where $S\mathfrak{A} = C_0((0,1)) \otimes \mathfrak{A}$ and $C\mathfrak{A} = C_0((0,1]) \otimes \mathfrak{A}$.

Let $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{B}$, $\mathfrak{D}$ be separable C^*-algebras, with $\mathfrak{D}$ stable, and $i_k : \mathfrak{B} \to \mathfrak{A}_k$ $(k = 1, 2)$ embeddings and $j_k : \mathfrak{A}_k \to \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2$ the canonical maps.

Lemma 3.5.7 *Assume that* $\mathfrak{B}$ *is nuclear or that there are surjective conditional expectations* $P_k : \mathfrak{A}_k \to i_k(\mathfrak{B})$ $(k = 1, 2)$. *Let* $\alpha : C(\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2) \to M(\mathfrak{D})$ *be an absorbing* $*$*-homomorphism. It follows that* $\alpha \circ j_1 \circ i_1 : C\mathfrak{B} \to M(\mathfrak{D})$ *is absorbing.*

Proof. When $\mathfrak{B}$ is nuclear, it is shown before. When the conditional expectations are given, there is a surjective conditional expectation $P_1 *_{\mathfrak{B}} P_2 : \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2 \to j_1 \circ i_1(\mathfrak{B})$, and thus also a surjective conditional expectation from $C(\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2)$ to $C(j_1 \circ i_1(\mathfrak{B}))$. Note that $\mathfrak{A}_1^+ *_{\mathfrak{B}^+} \mathfrak{A}_2^+ = (\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2)^+$. It follows as shown before. □

Lemma 3.5.8 *Assume the same assumption as above. For any pair of asymptotic* $*$*-homomorphisms* $\varphi = (\varphi_t)_{t\in[1,\infty)}$, $\psi = (\psi_t)_{t\in[1,\infty)} : C\mathfrak{B} \to \mathfrak{D}$, *there is an asymptotic $*$-homomorphism $\mu : C(\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2) \to \mathfrak{D}$ and a norm-continuous path of unitaries $\{W_t\}_{t\in[1,\infty)}$ in $M_2(\mathfrak{D})^+$ such that*

$$\lim_{t\to\infty}\{W_t[\varphi_t(b) \oplus (\mu_t \circ j_1 \circ i_1(b))]W_t^* - [\psi_t(b) \oplus (\mu_t \circ j_1 \circ i_1(b))]\} = 0$$

for $b \in C\mathfrak{B}$.

Let $\mathfrak{A}$, $\mathfrak{D}$ be separable C^*-algebras, with $\mathfrak{D}$ stable. An E-pair for $(\mathfrak{A}, \mathfrak{D})$ is a pair (W, φ), where $\varphi : C\mathfrak{A} \to \mathfrak{D}$ is an asymptotic $*$-homomorphism and $W = \{W_t\}_{t\in[1,\infty)}$ is a strictly continuous path of unitaries in $M(\mathfrak{D})$ such that $\lim_{t\to\infty} \|W_t\varphi_t(a) - \varphi_t(a)W_t\| = 0$ for $a \in S\mathfrak{A}$. The pair (W, φ) is degenerate if the equation holds for $a \in C\mathfrak{A}$. Let $X_0(\mathfrak{A}, \mathfrak{D})$ denote the set of homotopy classes of E-pairs, where a homotopy is given by an E-pair for $(\mathfrak{A}, C([0,1]) \otimes \mathfrak{D})$. The direct sum of E-pairs is defined by using any pair V_1, V_2 of isometries of $M(\mathfrak{D})$ such that $V_1^*V_2 = 0$ and $V_1V_1^* + V_2V_2^* = 1$, to make $X_0(\mathfrak{A}, \mathfrak{D})$ into an abelian semi-group. Denote by $X_{00}(\mathfrak{A}, \mathfrak{D})$ the subsemigroup of $X_0(\mathfrak{A}, \mathfrak{D})$ consisting of the classes represented by degenerate E-pairs. Then the quotient $X(\mathfrak{A}, \mathfrak{D}) = X_0(\mathfrak{A}, \mathfrak{D})/X_{00}(\mathfrak{A}, \mathfrak{D})$ becomes an abelian group, where it is shown that $(W, \varphi) \oplus (W^*, \varphi)$ is homotopic to the degenerate E-pair $(1, \varphi) \oplus (1, \varphi)$.

Given an E-pair (W, φ), we define an asymptotic $*$-homomorphism $W \otimes \varphi : C(\mathbb{T}) \otimes S\mathfrak{A} \to \mathfrak{D}$ such that $\lim_{t\to\infty}\{(W\otimes\varphi)_t(g\otimes f) - g(W_t)\varphi_t(f)\} = 0$ for $g \in C(\mathbb{T})$ and $f \in S\mathfrak{A}$. Then the class $[W \otimes \varphi]$ of $W \otimes \varphi$ in $[[C(\mathbb{T}) \otimes S\mathfrak{A}, \mathfrak{D}]]$ only depends on the class $[W, \varphi]$ of (W, φ) in $X(\mathfrak{A}, \mathfrak{D})$. Thus, define $\kappa : X(\mathfrak{A}, \mathfrak{D}) \to [[S^2\mathfrak{A}, \mathfrak{D}]]$ by $\kappa[W, \varphi] = i^*[W \otimes \varphi]$, where $i : S^2\mathfrak{A} \to C(\mathbb{T}) \otimes S\mathfrak{A}$ is the canonical embedding.

Theorem 3.5.9 *The map κ is an isomorphism from $X(\mathfrak{A}, \mathfrak{D})$ to $[[S^2\mathfrak{A}, \mathfrak{D}]]$.*

Given an E-pair (W, φ), there are asymptotic $*$-homomorphisms μ^k, $\nu^k : C\mathfrak{A}_k \to \mathfrak{D}$ $(k = 1, 2)$ such that $\mu^1 \circ i_1 = \mu^2 \circ i_2$, $\nu^1 \circ i_1 = \nu^2 \circ i_2$, and $\varphi \oplus (\mu^1 \circ i_1) \sim \nu^1 \circ i_1$ on $C\mathfrak{B}$. Then $\lim_{t\to\infty}\{\mathrm{Ad}(W_t \oplus 1) \circ \nu_t^1 \circ i_1(b) - \nu_t^2 \circ i_2(b)\} = 0$ for $b \in S\mathfrak{B}$, i.e., $\mathrm{Ad}(W \oplus 1) \circ \nu^1 \circ i_1 \sim \nu^2 \circ i_2$ on $S\mathfrak{B}$. Therefore we can consider $(\mathrm{Ad}(W \oplus 1) \circ \nu^1) *_{S\mathfrak{B}} \nu^2 : C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2 \to \mathfrak{D}$. Define $\rho : X(\mathfrak{B}, \mathfrak{D}) \to [[C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2, \mathfrak{D}]]$ by $\rho[W, \varphi] = [(\mathrm{Ad}(W \oplus 1) \circ \nu^1) *_{S\mathfrak{B}} \nu^2]$.

Lemma 3.5.10 *The map ρ is an isomorphism from $X(\mathfrak{B}, \mathfrak{D})$ to $[[C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2, \mathfrak{D}]]$.*

Theorem 3.5.11 *Assume that $\mathfrak{B}$ is nuclear or that there are surjective conditional expectations $P_k : \mathfrak{A}_k \to i_k(\mathfrak{B})$ $(k = 1, 2)$. Then there is an asymptotic $*$-homomorphism $\Phi : C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2 \to \mathfrak{B} \otimes \mathbb{K}$ which is invertible in E-theory.*

Define $*$-homomorphisms $\mathrm{ev}_k : C\mathfrak{A}_k \to \mathfrak{A}_k$ $(k = 1, 2)$ by evaluation at 1, which annihilate the image of $S\mathfrak{B}$. Thus we get a $*$-homomorphism $p : C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2 \to \mathfrak{A}_1 * \mathfrak{A}_2$ that is surjective since ev_k are so. There is a $*$-homomorphism $i : S\mathfrak{A}_1 *_{S\mathfrak{B}} S\mathfrak{A}_2 \to C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2$ which is injective by Pedersen. Given embeddings $i_k : \mathfrak{B} \to \mathfrak{A}_k$ $(k = 1, 2)$ that proper in the sense that the closure of $i_k(\mathfrak{B})\mathfrak{A}_k$ is $\mathfrak{A}_k$, the natural map from $S\mathfrak{A}_1 *_{S\mathfrak{B}} S\mathfrak{A}_2$ to $S(\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2)$ is an isomorphism by Pedersen. Furthermore, $S\mathfrak{A}_1 *_{S\mathfrak{B}} S\mathfrak{A}_2$ is viewed as a closed ideal of $C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2$ in this case.

Lemma 3.5.12 *Assume that the closure of $i_k(\mathfrak{B})\mathfrak{A}_k$ is $\mathfrak{A}_k$ $(k = 1, 2)$. Then*

$$0 \to S\mathfrak{A}_1 *_{S\mathfrak{B}} S\mathfrak{A}_2 \xrightarrow{i} C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2 \xrightarrow{p} \mathfrak{A}_1 * \mathfrak{A}_2 \to 0$$

i.e., the image of i is the kernel of p.

Proof. Since the image of i is contained in the kernel of p, there is a surjection $p^{\sim} : C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2/i(S\mathfrak{A}_1 *_{S\mathfrak{B}} S\mathfrak{A}_2) \to \mathfrak{A}_1 * \mathfrak{A}_2$ induced by p. Let $q : C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2 \to C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2/i(S\mathfrak{A}_1 *_{S\mathfrak{B}} S\mathfrak{A}_2)$ be the quotient map. Define $s_k : \mathfrak{A}_k \to C\mathfrak{A}_k$ by $s_k(a)(t) = ta$. Then $q \circ j_k \circ s_k : \mathfrak{A}_k \to C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2/i(S\mathfrak{A}_1 *_{S\mathfrak{B}} S\mathfrak{A}_2)$ $(k = 1, 2)$ are $*$-homomorphisms such that $((q \circ j_1 \circ s_1) * (q \circ j_2 \circ s_2)) \circ p^{\sim} = \mathrm{id}$. □

Using the identification $S\mathfrak{A}_1 *_{S\mathfrak{B}} S\mathfrak{A}_2 = S(\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2)$ we get the following extension:

$$0 \to S(\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2) \to C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2 \to \mathfrak{A}_1 * \mathfrak{A}_2 \to 0.$$

Applying the functor $E(\mathfrak{D}, \cdot)$ to the above exact sequence we get the following six-term exact sequence:

$$\begin{tikzcd}
E(\mathfrak{D}, C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2) \arrow[r, "p_*"] & E(\mathfrak{D}, \mathfrak{A}_1 * \mathfrak{A}_1) \arrow[r, "\partial"] & E(\mathfrak{D}, \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2) \arrow[d, "i_*"] \\
E(S\mathfrak{D}, \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2) \arrow[u, "i_*"] & E(S\mathfrak{D}, \mathfrak{A}_1 * \mathfrak{A}_2) \arrow[l, "\partial"'] & E(S\mathfrak{D}, C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2). \arrow[l, "p_*"']
\end{tikzcd}$$

The KK-equivalence between $\mathfrak{A}_1 * \mathfrak{A}_2$ and $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ is induced by the $*$-homomorphism $J : \mathfrak{A}_1 \oplus \mathfrak{A}_2 \to M_2(\mathfrak{A}_1 * \mathfrak{A}_2)$ given by $J(a_1, a_2) = j'_1(a_1) \oplus j'_2(a_2)$ the diagonal sum, where $j'_k : \mathfrak{A}_k \to \mathfrak{A}_1 * \mathfrak{A}_2$ are the canonical $*$-homomorphisms.

Lemma 3.5.13 *The following diagram commutes:*

$$\begin{tikzcd}
E(\mathfrak{D}, \mathfrak{A}_1 * \mathfrak{A}_2) \arrow[r, "\partial"] & E(\mathfrak{D}, \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2) \\
E(\mathfrak{D}, \mathfrak{A}_1 \oplus \mathfrak{A}_2) \arrow[u, "J_*"] \arrow[r, equal] & E(\mathfrak{D}, \mathfrak{A}_1 \oplus \mathfrak{A}_2). \arrow[u, "-j_{1*}-j_{2*}"']
\end{tikzcd}$$

Proof. Set

$$C_p = \{(f,x) \in C(\mathfrak{A}_1 * \mathfrak{A}_2) \oplus (C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2) : f(1) = p(x)\}$$

and let $I : S(\mathfrak{A}_1 * \mathfrak{A}_2) \to C_p$ and $E : S(\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2) \to C_p$ be the $*$-homomorphisms given by $I(f) = (f,0)$ and $E(g) = (0,g)$ respectively. Then $\partial = E_*^{-1} \circ I_*$. Since $J_* = j'_{1*} + j'_{2*}$, it suffices to show that $I_* \circ j'_{k*} = -E_* \circ j_{k*}$. For $\lambda \in [0,1]$, consider the functions ψ_λ and $\varphi_\lambda : [0,1] \to [0,1]$ defined by $\psi_\lambda(t) = t$ for $t \leq \lambda$ and it $= \lambda$ for $t \geq \lambda$, and $\varphi_\lambda(t) = 1-t$ for $t \leq 1-\lambda$ and it $= \lambda$ for $t \geq 1-\lambda$. Then $\Phi_\lambda(f) = (j'_k(f \circ \psi_\lambda), j_k(f \circ \varphi_\lambda))$ a homotopy of $*$-homomorphisms $S\mathfrak{A}_k \to C_p$ such that $\Phi_{1*} = I_* \circ j'_{k*}$ and $\Phi_{0*} = -E_* \circ j_{k*}$. □

There is an asymptotic $*$-homomorphism $\Phi : C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2 \to \mathfrak{B} \otimes \mathbb{K}$ which is invertible in E-theory. What we need to show is that

$$\pm[(\mathrm{ev}_1 *_{S\mathfrak{B}} 0) \oplus (0 *_{S\mathfrak{B}} \mathrm{ev}_2)] = (i_{1*}, -i_{2*})[\Phi]$$

in $E(C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2.\mathfrak{A}_1 \oplus \mathfrak{A}_2)$. This is proved by reductions as follows. Let $\iota : \mathfrak{B} \to \mathfrak{A}_1 \oplus \mathfrak{A}_2$ be the embedding by $\iota(b) = (i_1(b), i_2(b))$. Consider the corresponding amalgam $C(\mathfrak{A}_1 \oplus \mathfrak{A}_2) *_{S\mathfrak{B}} C(\mathfrak{A}_1 \oplus \mathfrak{A}_2)$. The projections $p_k : \mathfrak{A}_1 \oplus \mathfrak{A}_2 \to \mathfrak{A}_k$ induce a $*$-homomorphism $j = (j_1 \circ p_1) *_{S\mathfrak{B}} (j_2 \circ p_2)$ from $C(\mathfrak{A}_1 \oplus \mathfrak{A}_2) *_{S\mathfrak{B}} C(\mathfrak{A}_1 \oplus \mathfrak{A}_2)$ to $C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2$. There is also an asymptotic $*$-homomorphism $\Phi' : C(\mathfrak{A}_1 \oplus \mathfrak{A}_2) *_{S\mathfrak{B}} C(\mathfrak{A}_1 \oplus \mathfrak{A}_2)$ to $\mathfrak{B} \otimes \mathbb{K}$ which is invertible in E-theory. It follows that $\Phi \circ j = \Phi'$ in $E(C(\mathfrak{A}_1 \oplus \mathfrak{A}_2) *_{S\mathfrak{B}} C(\mathfrak{A}_1 \oplus \mathfrak{A}_2), \mathfrak{B} \otimes \mathbb{K})$.

Now assume that $(\iota_*, -\iota_*)[\Phi'] = \pm[(\mathrm{ev} *_{S\mathfrak{B}} 0) \oplus (0 *_{S\mathfrak{B}} \mathrm{ev})]$, where $\mathrm{ev} : C(\mathfrak{A}_1 \oplus \mathfrak{A}_2) \to \mathfrak{A}_1 \oplus \mathfrak{A}_2$ is the evaluation at 1. If $r : \oplus^2(\mathfrak{A}_1 \oplus \mathfrak{A}_2) \to \mathfrak{A}_1$ is the projection to the first coordinate, it follows that

$$i_{1*}[\Phi \circ j] = r_* \circ (\iota_*, -\iota_*)[\Phi'] = \pm r_*[(\mathrm{ev} *_{S\mathfrak{B}} 0) \oplus (0 *_{S\mathfrak{B}} \mathrm{ev})] = \pm[(\mathrm{ev}_1 *_{S\mathfrak{B}} 0) \circ j].$$

Similarly, $-i_{2*}[\Phi \circ j] = \pm[(0 *_{S\mathfrak{B}} \mathrm{id}_{\mathfrak{A}_1}) \circ j]$. Since Φ and Φ' are invertible in E-theory, so is j. We now may assume that $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{A}$ and $i_1 = i_2 = \iota$.

For the next reduction, consider the surjection $P : C\mathfrak{A} *_{S\mathfrak{B}} C\mathfrak{A} \to C\mathfrak{A} *_{S\mathfrak{A}} C\mathfrak{A}$. For the asymptotic $*$-homomorphism $\Phi^\# : C\mathfrak{A} *_{S\mathfrak{A}} C\mathfrak{A} \to \mathfrak{A} \otimes \mathbb{K}$ with $\mathfrak{B} = \mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{A}$, we have $i_*[\Phi] = [\Phi^\# \circ P]$ in $E(C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2, \mathfrak{A} \otimes \mathbb{K})$. Indeed, the following diagram:

$$\begin{array}{ccc}
X(\mathfrak{A}, \mathfrak{A} \otimes \mathbb{K}) & \xrightarrow{\rho} & [[C\mathfrak{A} *_{S\mathfrak{A}} C\mathfrak{A}, \mathfrak{A} \otimes \mathbb{K}]] \\
{\scriptstyle \iota^*}\downarrow & & \downarrow{\scriptstyle P^*} \\
X(\mathfrak{B}, \mathfrak{A} \otimes \mathbb{K}) & \xrightarrow{\rho} & [[C\mathfrak{A} *_{S\mathfrak{B}} C\mathfrak{A}, \mathfrak{A} \otimes \mathbb{K}]]
\end{array}$$

commutes. Therefore,

$$\begin{aligned}\iota_*[\Phi] &= \iota_*(\rho \circ \kappa^{-1}[\mathrm{id}_{\mathfrak{B}} \otimes \varphi_0]) = \rho \circ \kappa^{-1} \circ \iota_*[\mathrm{id}_{\mathfrak{B}} \otimes \varphi_0] \\ &= \rho \circ \kappa^{-1}[\iota \otimes \varphi_0] = \rho \circ \kappa^{-1}(\iota^*[\mathrm{id}_{\mathfrak{A}} \otimes \varphi_0]) \\ &= \rho \circ i^* \circ \kappa^{-1}[\mathrm{id}_{\mathfrak{A}} \circ \varphi_0] = P^* \circ \rho \circ \kappa^{-1}[\mathrm{id}_{\mathfrak{A}} \otimes \varphi_0].\end{aligned}$$

Also note that $((\mathrm{ev} *_{S\mathfrak{A}} 0) \oplus (0 *_{S\mathfrak{A}} \mathrm{ev})) \circ P = (\mathrm{ev} *_{S\mathfrak{B}} 0) \oplus (0 *_{S\mathfrak{B}} \mathrm{ev})$.

For the final reduction, let $C = C_0((0,1])$ and $S = C_0((0,1))$. Let $\Phi_0 : C *_S C \to \mathbb{K}$ and $\Phi : C\mathfrak{A} *_{S\mathfrak{A}} C\mathfrak{A} \to \mathfrak{A} \otimes \mathbb{K}$ be the asymptotic $*$-homomorphisms obtained from the cases $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{B} = \mathbb{C}$ and $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{B} = \mathfrak{A}$ respectively. Since $\Phi = \rho \circ \kappa^{-1}[\mathrm{id}_{\mathfrak{A}} \otimes \varphi_0]$, it follows that $\Phi = (\mathrm{id}_{\mathfrak{A}} \otimes \Phi_0) \circ Q$, where $Q : C\mathfrak{A} *_{S\mathfrak{A}} C\mathfrak{A} \to \mathfrak{A} \otimes (C *_S C)$ is the surjection induced by the maps $\mathrm{id}_{\mathfrak{A}} \otimes \iota_1$, $\mathrm{id}_{\mathfrak{A}} \otimes \iota_2 : C\mathfrak{A} \to \mathfrak{A} \otimes (C *_S C)$, where $\iota_k : C \to C *_S C$ are the canonical embeddings. Applying the functor $E(\mathbb{C}, \cdot)$ to the extension $0 \to S \to C *_S C \to \mathbb{C} * \mathbb{C} \to 0$, we get the following:

$$0 \to E(\mathbb{C}, C *_S C) \xrightarrow{p_*} E(\mathbb{C}, \mathbb{C} * \mathbb{C}) \xrightarrow{\partial} E(\mathbb{C}, \mathbb{C}).$$

Note that $(\mathrm{id}_{\mathbb{C}} * 0) \oplus (0 * \mathrm{id}_{\mathbb{C}}) : \mathbb{C} * \mathbb{C} \to \mathbb{C}^2$ is a KK-equivalence, with the inverse given by $J : \mathbb{C} \oplus \mathbb{C} \to M_2(\mathbb{C} * \mathbb{C})$. Since $((\mathrm{id}_{\mathbb{C}} * 0) \oplus (0 * \mathrm{id}_{\mathbb{C}})) \circ p = (\mathrm{ev} *_S 0) \oplus (0 *_S \mathrm{ev})$, it follows that

$$0 \to E(\mathbb{C}, \mathbb{C}) \xrightarrow{X} E(\mathbb{C}, \mathbb{C} \oplus \mathbb{C}) \xrightarrow{\partial'} E(\mathbb{C}, \mathbb{C}),$$

where $X = (\mathrm{ev} *_S 0) \oplus (0 *_S \mathrm{ev})_* \circ \Phi_{0*}^{-1}$ and $\partial' = \partial \circ ((\mathrm{id}_{\mathbb{C}} * 0) \oplus (0 * \mathrm{id}_{\mathbb{C}}))_*^{-1} = \partial \circ J_* = -j_{1*} - j_{2*}$. Since $E(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$, the above sequence can be exact if $X = \pm(\iota_*, -\iota_*)$.

Theorem 3.5.14 *Let $\mathfrak{A} = \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2$ be an amalgam of separable C^*-algebras, with $i_k : \mathfrak{B} \to \mathfrak{A}_k$ embeddings. Assume that $\mathfrak{B}$ is nuclear or that there are surjective conditional expectations $P_k : \mathfrak{A}_k \to i_k(\mathfrak{B})$ $(k = 1, 2)$. For any separable C^*-algebra $\mathfrak{D}$,*

$$\begin{array}{ccccc} E(\mathfrak{D}, \mathfrak{B}) & \xrightarrow{(i_{1*}, i_{2*})} & E(\mathfrak{D}, \mathfrak{A}_1) \oplus E(\mathfrak{D}, \mathfrak{A}_2) & \xrightarrow{j_{1*} - j_{2*}} & E(\mathfrak{D}, \mathfrak{A}) \\ \uparrow & & & & \downarrow \\ E(S\mathfrak{D}, \mathfrak{A}) & \xleftarrow{j_{1*} - j_{2*}} & E(S\mathfrak{D}, \mathfrak{A}_1) \oplus E(S\mathfrak{D}, \mathfrak{A}_2) & \xleftarrow{(i_{1*}, i_{2*})} & E(S\mathfrak{D}, \mathfrak{B}) \end{array}$$

where $j_k : \mathfrak{A}_k \to \mathfrak{A}$ are the canonical maps. Furthermore,

$$\begin{array}{ccccc} E(\mathfrak{B}, \mathfrak{D}) & \xleftarrow{i_1^* - i_2^*} & E(\mathfrak{A}_1, \mathfrak{D}) \oplus E(\mathfrak{A}_2, \mathfrak{D}) & \xleftarrow{(j_1^*, j_2^*)} & E(\mathfrak{A}, \mathfrak{D}) \\ \downarrow & & & & \uparrow \\ E(\mathfrak{A}, S\mathfrak{D}) & \xrightarrow{(j_1^*, j_2^*)} & E(\mathfrak{A}_2, S\mathfrak{D}) \oplus E(\mathfrak{A}_2, S\mathfrak{D}) & \xrightarrow{i_1^* - i_2^*} & E(\mathfrak{B}, S\mathfrak{D}). \end{array}$$

Sketch of Proof. Consider the case where $\mathfrak{A}_1$, $\mathfrak{A}_2$, and $\mathfrak{B}$ are unital and $i_k(1) = 1$ $(k = 1, 2)$. Using the E-theory equivalence between $C\mathfrak{A}_1 *_{S\mathfrak{B}} C\mathfrak{A}_2$ and $\mathfrak{B}$ and the KK-theory equivalence between $\mathfrak{A}_1 * \mathfrak{A}_2$ and $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ we get the following exact sequence:

$$\begin{array}{ccccc} E(\mathfrak{D},\mathfrak{B}) & \xrightarrow{J_*^{-1} \circ p_* \Phi_*^{-1}} & E(\mathfrak{D},\mathfrak{A}_1 \oplus \mathfrak{A}_2) & \xrightarrow{j_{1*}+j_{2*}} & E(\mathfrak{D},\mathfrak{A}) \\ \uparrow & & & & \downarrow \\ E(S\mathfrak{D},\mathfrak{A}) & \xleftarrow{j_{1*}+j_{2*}} & E(S\mathfrak{D},\mathfrak{A}_1 \oplus \mathfrak{A}_2) & \xleftarrow{J_*^{-1} \circ p_* \circ \Phi_*^{-1}} & E(S\mathfrak{D},\mathfrak{B}). \end{array}$$

Since the inverse of J in KK-theory is represented by the $*$-homomorphism $(\mathrm{id}_{\mathfrak{A}_1} * 0) \oplus (0 * \mathrm{id}_{\mathfrak{A}_2})$, and since

$$[(\mathrm{id}_{\mathfrak{A}_1} * 0) \oplus (0 * \mathrm{id}_{\mathfrak{A}_2})] \circ p = (\mathrm{ev}_1 *_{S\mathfrak{B}} 0) \oplus (0 *_{S\mathfrak{B}} \mathrm{ev}_2),$$

we see that $J_*^{-1} \circ p_* \circ \Phi_*^{-1} = \pm(i_{1*}, -i_{2*})$. This gives us the first six-term exact sequence. The second one is obtained in a similar way. The non-unital case is deduced from the unital case by considering the unitization.
$\square$

Taking $\mathfrak{D}$ as $\mathbb{C}$,

Corollary 3.5.15 *Under the same assumptions as above,*

$$\begin{array}{ccccc} K_0(\mathfrak{B}) & \xrightarrow{(i_{1*},i_{2*})} & K_0(\mathfrak{A}_1) \oplus K_0(\mathfrak{A}_2) & \xrightarrow{j_{1*}-j_{2*}} & K_0(\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2) & \xleftarrow{j_{1*}-j_{2*}} & K_1(\mathfrak{A}_1) \oplus K_1(\mathfrak{A}_2) & \xleftarrow{(i_{1*},i_{2*})} & K_1(\mathfrak{B}). \end{array}$$

3.6 E-theory basics

Taken from Blackadar [3] are those in what follows.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. An asymptotic $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is a family $\{\varphi_t\}_{t\in[1,\infty)} (= \{\varphi_t\})$ of maps $\varphi_t : \mathfrak{A} \to \mathfrak{B}$ such that the map $[1,\infty) \ni t \mapsto \varphi_t(a)$ is continuous for $a \in \mathfrak{A}$, and $\{\varphi_t\}$ is asymptotically $*$-linear and multiplicative in the sense that

$$\lim_{t\to\infty} \|\varphi_t(a+b) - (\varphi_t(a) + \varphi_t(b))\| = 0,$$
$$\lim_{t\to\infty} \|\varphi_t(a^*) - \varphi_t(a)^*\| = 0, \quad \lim_{t\to\infty} \|\varphi_t(ab) - \varphi_t(a)\varphi_t(b)\| = 0$$

for $a, b \in \mathfrak{A}$. Two asymptotic $*$-homomorphisms $\{\varphi_t\}$ and $\{\psi_t\}$ are equivalent if $\lim_{t\to\infty} \|\varphi_t(a) - \psi_t(a)\| = 0$ for $a \in \mathfrak{A}$. A homotopy between $\{\varphi_t\}$ and

$\{\psi_t\}$ is an asymptotic $*$-homomorphism $\{\lambda_t\}$ from $\mathfrak{A}$ to $C([0,1],\mathfrak{B})$ such that $(\lambda_t(a))(0) = \varphi_t(a)$ and $(\lambda_t(a))(1) = \psi_t(a)$ for $a \in \mathfrak{A}$ and $t \in [1,\infty)$.

We denote by $[[\mathfrak{A},\mathfrak{B}]]$ the set of homotopy classes of asymptotic $*$-homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$.

Example 3.6.1 A $*$-homomorphism φ from $\mathfrak{A}$ to $\mathfrak{B}$ becomes an asymptotic $*$-homomorphism by setting $\varphi_t = \varphi$ for $t \in [1,\infty)$. A homotopy of $*$-homomorphisms gives a homotopy of the corresponding asymptotic $*$-homomorphisms, so that there is a natural map $[\mathfrak{A},\mathfrak{B}] \to [[\mathfrak{A},\mathfrak{B}]]$, where $[\mathfrak{A},\mathfrak{B}]$ is the set of homotopy classes of $*$-homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$.

A continuous deformation from $\mathfrak{A}$ to $\mathfrak{B}$ is a continuous field of C^*-algebras over the interval $[0,1]$ such that the fiber at 0 is isomorphic to $\mathfrak{A}$ and the fibers over $(0,1]$ are isomorphic to $\mathfrak{B}$. Every continuous deformation from $\mathfrak{A}$ to $\mathfrak{B}$ gives an asymptotic $*$-homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$, where $t \in [1,\infty)$ is replaced by $t \in (0,1]$.

If $\mathfrak{A}$ and $\mathfrak{B}$ are unital C^*-algebras, an asymptotic $*$-homomorphism $\{\varphi_t\} : \mathfrak{A} \to \mathfrak{B}$ is said to be unital if each φ_t is unital. Any asymptotic $*$-homomorphism $\{\varphi_t\} : \mathfrak{A} \to \mathfrak{B}$ is extended canonically to a unital asymptotic $*$-homomorphism $\{\varphi_t^+\} : \mathfrak{A}^+ \to \mathfrak{B}^+$, where $\varphi_t^+(a + \lambda 1) = \varphi_t(a) + \lambda 1$ for $a + \lambda 1 \in \mathfrak{A}^+$ the unitization of $\mathfrak{A}$ by $\mathbb{C}$.

An asymptotic $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ gives an asymptotic $*$-homomorphism $M_n(\mathfrak{A})$ to $M_n(\mathfrak{B})$ coordinatewise.

Equivalent asymptotic $*$-homomorphisms $\{\varphi_t\}$ and $\{\psi_t\}$ are homotopic by the straight line homotopy: $\lambda_t(a)(s) = s\varphi_t(a) + (1-s)\psi_t(a)$ for $s \in [0,1]$ and $a \in \mathfrak{A}$.

Proposition 3.6.2 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and $\{\varphi_t\}_{t\in[1,\infty)}$ an asymptotic $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Then the function $t \mapsto \varphi_t(a)$ for each $a \in \mathfrak{A}$ is bounded. Indeed, $\limsup_t \|\varphi_t(a)\| \le \|a\|$.*

Remark. It follows that an asymptotic $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ defines a $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}_\infty$, where $\mathfrak{B}_\infty$ is the quotient C^*-algebra $C^b([1,\infty),\mathfrak{B})/C_0([1,\infty),\mathfrak{B})$, where $C^b([1,\infty),\mathfrak{B})$ and $C_0([1,\infty),\mathfrak{B})$ are the C^*-algebras of all $\mathfrak{B}$-valued, bounded continuous functions and continuous functions vanishing at infinity on $[1,\infty)$ respectively. Two asymptotic $*$-homomorphisms define the same $*$-homomorphism if and only if they are equivalent.

From this, we may view an asymptotic $*$-homomorphism as a generalized mapping cone. An asymptotic $*$-homomorphism φ indexed by $(0,1]$ with $\varphi_1 = 0$ may be viewed as an extension: $0 \to S\mathfrak{B} \to E \to \mathfrak{A} \to 0$. If φ satisfies $\lim_{t\to 0} \varphi_t(a) = \psi(a)$ for $a \in \mathfrak{A}$, where ψ is a $*$-homomorphism, then E is just the mapping cone of ψ. Conversely, if $\mathfrak{B}$ is unital, then any extension of $\mathfrak{A}$ by $S\mathfrak{B}$ which is concentrated at 0 gives an asymptotic $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ via the Busby invariant since the multiplier algebra $M(S\mathfrak{B})$ is $C^b((0,1),\mathfrak{B})$.

By elementary linear algebra, a $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}_\infty$ has a $*$-linear (not necessarily bounded) lifting to $C^b([1,\infty),\mathfrak{B})$. Thus, every asymptotic $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is equivalent to that of $*$-linear maps. By using a selection theorem of Bartle and Graves, a $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}_\infty$ has a continuous (not necessarily linear) lifting to $C^b([1,\infty),\mathfrak{B})$. Thus, every asymptotic $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is equivalent to that of continuous maps φ_t that become asymptotically $*$-linear and multiplicative on compact sets as $t \to \infty$, i.e., for every $\varepsilon > 0$ and compact subset K of $\mathfrak{A}$, there is t_0 such that for $t \geq t_0$,

$$\begin{aligned}
&\|\varphi_t(x) + \lambda\varphi_t(y) - \varphi_t(x+\lambda y)\| < \varepsilon, \quad \|\varphi_t(x)\varphi_t(y) - \varphi_t(xy)\| < \varepsilon, \\
&\|\varphi_t(x)^* - \varphi_t(x^*)\| < \varepsilon, \quad \text{and } \|\varphi_t(x)\| < \|x\| + \varepsilon
\end{aligned}$$

for $x, y \in K$ and $\lambda\mathbb{C}$ with $|\lambda| \leq 1$. In such a case, the asymptotic $*$-homomorphism φ is called uniform. If $\mathfrak{A}$ is nuclear, then we can take such a lifting as a completely positive linear contraction, and the similar equivalence holds.

Let $\mathfrak{A}$, $\mathfrak{B}$ be unital C^*-algebras. A unital asymptotic $*$-homomorphism φ from $\mathfrak{A}$ to $\mathfrak{B}$ induces a homomorphism $\varphi_* : K_1(\mathfrak{A}) \to K_1(\mathfrak{B})$. By suspension a map $\varphi_* : K_0(\mathfrak{A}) \cong K_1(S\mathfrak{A}) \to K_1(S\mathfrak{B}) \cong K_0(\mathfrak{B})$ is obtained. For the non-unital case, we can consider the unital extension of φ.

Proposition 3.6.3 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras, with $\mathfrak{A}$ semiprojective. Then every asymptotic $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is homotopic to a $*$-homomorphism. Thus, there is a bijection between* $[\mathfrak{A},\mathfrak{B}]$ *and* $[[\mathfrak{A},\mathfrak{B}]]$.

Remark. A separable C^*-algebra $\mathfrak{A}$ is said to be semiprojective if for any increasing sequence of closed ideals $\mathfrak{I}_j$ of a C^*-algebra $\mathfrak{B}$, and any $*$-homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}/\mathfrak{I}$, where $\mathfrak{I}$ is the norm closure of the union of $\mathfrak{I}_j$, there is a $*$-homomorphism $\psi : \mathfrak{A} \to \mathfrak{B}/\mathfrak{I}_n$ for some n such that $\varphi = \pi \circ \psi$, where $\pi : \mathfrak{B}/\mathfrak{I}_n \to \mathfrak{B}/\mathfrak{I}$ is the natural quotient map. A separable C^*-algebra $\mathfrak{A}$ is said to be projective if for any $\varphi : \mathfrak{A} \to \mathfrak{B}/\mathfrak{I}$, there is a $*$-homomorphism $\psi : \mathfrak{A} \to \mathfrak{B}$ such that $\varphi = \pi \circ \psi$, where $\pi : \mathfrak{B} \to \mathfrak{B}/\mathfrak{I}$.

The universal C^*-algebra generated by a positive element of norm 1, that is $C_0((0,1])$, is projective. It is known that $M_n(\mathbb{C})$, $C(\mathbb{T})$, $S = C_0(\mathbb{R})$, the Toeplitz algebra, the Cuntz algebras O_n and O_∞, and the Cuntz-Krieger algebras are all semiprojective but not projective.

Corollary 3.6.4 *For any C^*-algebra $\mathfrak{B}$,*

$$[[S\mathbb{C}, \mathfrak{B} \otimes \mathbb{K}]] \cong K_1(\mathfrak{B}), \quad [[S\mathbb{C}, S\mathfrak{B} \otimes \mathbb{K}]] \cong K_0(\mathfrak{B}).$$

Suppose that $\{\varphi_t\}$ and $\{\psi_t\}$ are asymptotic $*$-homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ and from $\mathfrak{C}$ to $\mathfrak{D}$ respectively. Then $\{\varphi_t \otimes 1\}$ and $\{1 \otimes \psi_t\}$ define asymptotic $*$-homomorphisms from $\mathfrak{A}$ and $\mathfrak{B}$ to $\mathfrak{C}^+ \otimes_{\max} \mathfrak{D}^+$, and hence to $(\mathfrak{C}^+ \otimes_{\max} \mathfrak{D}^+)_\infty$, where $\otimes_{\max}$ is the maximal C^*-tensor product. Since the images commute, there is an induced $*$-homomorphism from $\mathfrak{A} \otimes_{\max} \mathfrak{B}$ to $(\mathfrak{C}^+ \otimes_{\max} \mathfrak{D}^+)_\infty$. Since the image is contained in $(\mathfrak{C} \otimes_{\max} \mathfrak{D})_\infty$, there is an asymptotic $*$-homomorphism $\varphi \otimes \psi$ from $\mathfrak{A} \otimes_{\max} \mathfrak{B}$ to $\mathfrak{C} \otimes_{\max} \mathfrak{D}$. The class of $\varphi \otimes \psi$ in $[[\mathfrak{A} \otimes_{\max} \mathfrak{B}, \mathfrak{C} \otimes_{\max} \mathfrak{D}]]$ depends only on the classes of φ and ψ in $[[\mathfrak{A}, \mathfrak{C}]]$ and $[[\mathfrak{B}, \mathfrak{D}]]$. Similarly, one can replace $\otimes_{\max}$ for $\mathfrak{C} \otimes_{\max} \mathfrak{D}$ with the minimal C^*-tensor product, but it is not obvious in general that that for $\mathfrak{A} \otimes_{\max} \mathfrak{B}$ is done with. If $\mathfrak{A}$ or $\mathfrak{B}$ is nuclear, then it is done. In particular, there is a suspension map S from $[[\mathfrak{A}, \mathfrak{B}]]$ to $[[S\mathfrak{A}, S\mathfrak{B}]]$ defined by $S\varphi = \mathrm{id}_S \otimes \varphi$.

Theorem 3.6.5 *Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ be separable C^*-algebras, and $\{\varphi_t\}$: $\mathfrak{A} \to \mathfrak{B}$, $\{\psi_t\} : \mathfrak{B} \to \mathfrak{C}$ be uniform asymptotic $*$-homomorphisms. Then $\{\psi_{r(t)} \circ \varphi_t\}$ is an asymptotic $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{C}$, where a real valued function $r(\geq 1)$ on $[1, \infty)$ increases sufficiently quickly.*

The resulting asymptotic $*$-homomorphism (up to homotopy) depends only on the homotopy classes of $\{\varphi_t\}$ and $\{\psi_t\}$, so that a composition from $[[\mathfrak{A}, \mathfrak{B}]] \times [[\mathfrak{B}, \mathfrak{C}]]$ to $[[\mathfrak{A}, \mathfrak{C}]]$ is defined. This composition is associative, commutes with tensor products, and agrees with composition for $*$-homomorphisms.

Let φ and ψ be asymptotic $*$-homomorphisms from $\mathfrak{A}$ to $S\mathfrak{B}$. Move them homotopically to asymptotic $*$-homomorphisms φ', ψ' supported on $(0, 1/2)$ and $(1/2, 1)$ respectively. Then $\{\varphi'_t + \psi'_t\}$ is an asymptotic $*$-homomorphism from $\mathfrak{A}$ to $S\mathfrak{B}$ whose homotopy class depends only on $[\varphi]$ and $[\psi]$ in $[[\mathfrak{A}, S\mathfrak{B}]]$. Denote this class by $[\varphi] + [\psi]$.

Proposition 3.6.6 *Under the addition above, $[[\mathfrak{A}, S\mathfrak{B}]]$ is a group, where the inverse of $[\varphi]$ is given by $\varphi \circ (\rho \otimes \mathrm{id})$, where $\rho : S\mathbb{C} \to S\mathbb{C}$ is defined by $\rho(f)(s) = f(1-s)$.*

Set $E(\mathfrak{A}, \mathfrak{B}) = [[S\mathfrak{A}, S\mathfrak{B} \otimes \mathbb{K}]]$ for $\mathfrak{A}$, $\mathfrak{B}$ (separable) C^*-algebras.

Proposition 3.6.7 *Let $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{B} \to 0$ be an exact sequence of C^*-algebras. Suppose that $\mathfrak{J}$ has a continuous approximate unit $\{u_t\}$ which* *is quasicentral for* $\mathfrak{A}$ (*this holds if* $\mathfrak{A}$ *is separable*). *Choose a cross section* σ *for the quotient map* $q : \mathfrak{A} \to \mathfrak{B}$. *Then an asymptotic* $*$*-homomorphism* $\{\varphi_t\}$ *from* $S\mathfrak{B}$ *to* $\mathfrak{J}$ *is obtained as* $\varphi_t(f \otimes b) = f(u_t)\sigma(b)$ *for* $f \in C_0((0,1))$ *and* $b \in \mathfrak{B}$.

Its class in $[[S\mathfrak{B}, \mathfrak{J}]]$ *denoted by* ε_q *is independent of the choice of* σ *and* $\{u_t\}$.

Remark. This connecting asymptotic $*$-homomorphism is defined even for extensions which are not semi-split. This is the crucial difference between E-theory and KK-theory because semi-splitness is required in KK.

Example 3.6.8 Let φ be an asymptotic $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. As before, we have an extension $0 \to S\mathfrak{B} \to E \to \mathfrak{A} \to 0$ associated with φ, with E contained in $\mathfrak{A} \oplus C\mathfrak{B}$. Let $\{h_t\}$ $(t \geq 3)$ be an approximate unit for $C_0((0,1))$ such that each h_t takes 0 on $[0, 1/(t+1)]$ and $[1 - 1/(t+1), 1]$, 1 on $[1/t, 1 - 1/t]$ and linear on $[1/(t+1), t/1]$ and $[1 - 1/t, 1 - 1/(t+1)]$. If $f \in C_0((0,1))$, then $f(h_t)$ consists of two copies of f transfered to $[1/(t+1), 1/t]$ and $[1 - 1/t, 1 - 1/(t+1)]$. Choose a cross section for the quotient map $E \to \mathfrak{A}$ which vanishes on $[1/2, 1]$. Then the copy of f on $[1 - 1/t, 1 - 1/(t+1)]$ has no effect on the asymptotic $*$-homomorphism ψ from $S\mathfrak{A}$ to $S\mathfrak{B}$. Expanding the interval $[1/(t+1), 1/t]$ to $[0, 1/2]$ via a homotopy converts ψ to an asymptotic $*$-homomorphism homotopic to the suspension $S\varphi$.

Let $0 \to S\mathbb{C} \to C\mathbb{C} \to \mathbb{C} \to 0$ be the standard extension of $\mathbb{C}$ by $S\mathbb{C} \cong C_0((0,1)) = S$. The asymptotic $*$-homomorphism associated with this is homotopic to the identity map on S.

For a split exact sequence: $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{B} \to 0$, there is an associated exact sequence: $0 \to S\mathfrak{J} \to E \to \mathfrak{A} \to 0$, where E is the C^*-subalgebra of $C([0,1], \mathfrak{A})$ generated by $S\mathfrak{J}$ and $\{\tau(a) : a \in \mathfrak{A}\}$, where $\tau(a)(s) = (1-s)a + s(\sigma \circ q)(a)$, with $q : \mathfrak{A} \to \mathfrak{B}$ the quotient map and $\sigma : \mathfrak{B} \to \mathfrak{A}$ a splitting. The class of the corresponding asymptotic $*$-homomorphism from $S\mathfrak{A}$ to $S\mathfrak{J}$ is the splitting $*$-homomorphism of the exact sequence, denoted by η_q. This is the KK-element defined by the extension under the identification between $\mathrm{Ext}(\mathfrak{A}, S\mathfrak{J})^{-1}$ and $KK(\mathfrak{A}, \mathfrak{J})$.

Proposition 3.6.9 *Let* $0 \to \mathfrak{J} \to \mathfrak{A} \xrightarrow{q} \mathfrak{B} \to 0$ *be an exact sequence of separable* C^**-algebras. For* $0 \to S\mathfrak{J} \to S\mathfrak{A} \to S\mathfrak{B} \to 0$ *the suspended exact sequence,* $\varepsilon_{Sq} = S\varepsilon_q$ *in* $[[S^2\mathfrak{B}, S\mathfrak{J}]]$, *where* $Sq : S\mathfrak{A} \to S\mathfrak{B}$ *is the suspended quotient map.*

If the first exact sequence splits, then $\varepsilon_q = [0]$ *in* $[[S\mathfrak{B}, \mathfrak{J}]]$.

Proposition 3.6.10 *For a split exact sequence $0 \to \mathfrak{J} \xrightarrow{i} \mathfrak{A} \xrightarrow{q} \mathfrak{B} \to 0$ of separable C^*-algebras, it follows that $\eta_q \circ [Si] = [\mathrm{id}_{S\mathfrak{J}}]$ in $[[S\mathfrak{J}, S\mathfrak{J}]]$, and $[Si] \circ \eta_q = [\mathrm{id}_{S\mathfrak{A}}] - [S(\sigma \circ q)]$ in $[[S\mathfrak{A}, S\mathfrak{A}]]$, where $Si : S\mathfrak{J} \to S\mathfrak{A}$ is the suspended inclusion and σ is a cross section for q.*

Furthermore, $\eta_q \oplus [Sq]$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{J} \oplus \mathfrak{B}$ in $[[S(\cdot), S(\cdot)]]$, with inverse $[Si] + [S\sigma]$.

Corollary 3.6.11 *There is a functor that sends $KK(\mathfrak{A}, \mathfrak{B})$ to $E(\mathfrak{A}, \mathfrak{B})$ for every separable C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, which respects addition and tensor products.*

Corollary 3.6.12 *If φ is the homomorphism from S to $M_2(S^2S)$ corresponding to the generator of $K_1(S^2S)$, then $[\varphi] \in E(\mathbb{C}, S^2)$ is an isomorphism in E-theory, whose inverse is $S\varepsilon_q$ for the Toeplitz extension of S by $\mathbb{K}$.*

It follows that the Bott periodicity for E-theory holds:

$$E(\mathfrak{A}, \mathfrak{B}) \cong E(S^2\mathfrak{A}, \mathfrak{B}) \cong E(\mathfrak{A}, S^2\mathfrak{B}) \cong E(S\mathfrak{A}, S\mathfrak{B})$$

for any C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$.

Proof. The isomorphism from $E(\mathfrak{A}, \mathfrak{B})$ to $E(S^2\mathfrak{A}, \mathfrak{B})$ is given by tensoring with ε_q, or composing on the left with $\varepsilon_q \otimes [\mathrm{id}_{S\mathfrak{A}}]$. Similarly, the isomorphism $E(\mathfrak{A}, \mathfrak{B}) \cong E(\mathfrak{A}, S^2\mathfrak{B})$ is given by tensoring with $[\varphi]$. The isomorphism from $E(\mathfrak{A}, \mathfrak{B})$ to $E(S\mathfrak{A}, S\mathfrak{B})$ is given by tensoring with id_S. □

Set $E^0(\mathfrak{A}, \mathfrak{B}) = E(\mathfrak{A}, \mathfrak{B})$ and $E^1(\mathfrak{A}, \mathfrak{B}) = E(\mathfrak{A}, S\mathfrak{B}) \cong E(S\mathfrak{A}, \mathfrak{B})$. Then $E^1(S\mathfrak{A}, \mathfrak{B}) \cong E^1(\mathfrak{A}, S\mathfrak{B}) \cong E^0(\mathfrak{A}, \mathfrak{B})$.

Theorem 3.6.13 *For any extension $0 \to \mathfrak{J} \xrightarrow{i} \mathfrak{A} \xrightarrow{q} \mathfrak{B} \to 0$ of separable C^*-algebras and any separable C^*-algebra $\mathfrak{D}$,*

$$E(\mathfrak{D}, \mathfrak{J}) \xrightarrow{i_*} E(\mathfrak{D}, \mathfrak{A}) \xrightarrow{q_*} E(\mathfrak{D}, \mathfrak{B}),$$
$$E(\mathfrak{J}, \mathfrak{D}) \xleftarrow{i^*} E(\mathfrak{A}, \mathfrak{D}) \xleftarrow{q^*} E(\mathfrak{B}, \mathfrak{D})$$

are exact in the middle, where the induced maps on E-theory are given by compositions. Namely, E-theory is half-exact.

Sketch of Proof. It follows that the following are exact in the middle:

$$[[S\mathfrak{D}, S\mathfrak{J}]] \xrightarrow{Si_*} [[S\mathfrak{D}, S\mathfrak{A}]] \xrightarrow{Sq_*} [[S\mathfrak{D}, S\mathfrak{B}]],$$
$$[[S^2\mathfrak{B}, S^2\mathfrak{D}]] \xrightarrow{S^2q^*} [[S^2\mathfrak{A}, S^2\mathfrak{D}]] \xrightarrow{S^2i^*} [[S^2\mathfrak{J}, S^2\mathfrak{D}]]$$

where the image of Si_* is contained in the kernel of Sq_* by functoriality. Combining these with the Bott periodicity gives the desired exact sequences for E-theory. $\square$

Corollary 3.6.14 (The six-term exact sequences) *Under the same assumption as above,*

$$\begin{tikzcd}
E^0(\mathfrak{D},\mathfrak{I}) \arrow[r, "i_*"] & E^0(\mathfrak{D},\mathfrak{A}) \arrow[r, "q_*"] & E^0(\mathfrak{D},\mathfrak{B}) \arrow[d, "\varepsilon_{q*}"] \\
E^1(\mathfrak{D},\mathfrak{B}) \arrow[u, "\varepsilon_{q*}"] & E^1(\mathfrak{D},\mathfrak{A}) \arrow[l, "q_*"'] & E^1(\mathfrak{D},\mathfrak{I}), \arrow[l, "i_*"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
E^0(\mathfrak{I},\mathfrak{D}) \arrow[d, "\varepsilon_{q*}"'] & E^0(\mathfrak{A},\mathfrak{D}) \arrow[l, "i_*"'] & E^0(\mathfrak{B},\mathfrak{D}) \arrow[l, "q_*"'] \\
E^1(\mathfrak{B},\mathfrak{D}) \arrow[r, "q_*"] & E^1(\mathfrak{A},\mathfrak{D}) \arrow[r, "i_*"] & E^1(\mathfrak{I},\mathfrak{D}). \arrow[u, "\varepsilon_{q*}"']
\end{tikzcd}$$

Theorem 3.6.15 *If F is a homotopy invariant, stable, half-exact covariant (or contravariant) functor from the category of all separable C^*-algebras and $*$-homomorphisms to an additive category, then there is a pairing*

$$F(\mathfrak{A}) \times E(\mathfrak{A},\mathfrak{B}) \to F(\mathfrak{B}) \quad (\textit{or } E(\mathfrak{A},\mathfrak{B}) \times F(\mathfrak{B}) \to F(\mathfrak{A}))$$

which is compatible with respect to composition in E-theory, and agrees with usual functoriality for homomorphisms (respectively).

Theorem 3.6.16 *Suppose that $KK(\mathfrak{A},\cdot)$ is half exact for a separable C^*-algebra $\mathfrak{A}$. Then $E(\mathfrak{A},\mathfrak{B}) \cong KK(\mathfrak{A},\mathfrak{B})$ for any separable C^*-algebra $\mathfrak{B}$.*

In particular, if $\mathfrak{A}$ is separable nuclear (or K-nuclear), then the same is true.

Sketch of Proof. There is a pairing from $KK(\mathfrak{A},\mathfrak{D})\times E(\mathfrak{D},\mathfrak{B})$ to $KK(\mathfrak{A},\mathfrak{B})$ for every $\mathfrak{D}$ and $\mathfrak{B}$. Setting $\mathfrak{D}=\mathfrak{A}$ and applying for $1_{\mathfrak{A}}$, this pairing gives a homomorphism from $E(\mathfrak{A},\mathfrak{B})$ to $KK(\mathfrak{A},\mathfrak{B})$. This is an inverse to the canonical homomorphism from $KK(\mathfrak{A},\mathfrak{B})$ to $E(\mathfrak{A},\mathfrak{B})$. $\square$

4 Appendix added

4.1 Finite subgroups of $GL_3(\mathbb{Z})$ and $SL_3(\mathbb{Z})$

Taken from Tahara [17] are those in what follows.

Let c and d be invertible $n \times n$ matrices over $\mathbb{Z}$, i.e., $c, d \in GL_n(\mathbb{Z})$. We say that c is conjugate to d in $GL_n(\mathbb{Z})$ if there is a matrix $v \in GL_n(\mathbb{Z})$ such that $c = v^{-1}dv$, and denote by $c \sim d$. Let H, K be subgroups of $GL_n(\mathbb{Z})$. We say that H is conjugate to K if there is a matrix $v \in GL_n(\mathbb{Z})$ such that $H = v^{-1}Kv$, and denote by $H \sim K$. In these definitions, $GL_n(\mathbb{Z})$ may be replaced with $SL_n(\mathbb{Z})$. Note that if n is odd, then matrices or subgroups of $GL_n(\mathbb{Z})$ are conjugate in $GL_n(\mathbb{Z})$ if and only if they are conjugate in $SL_n(\mathbb{Z})$.

Note that groups with order ≤ 5 are all abelian. Indeed, they are the trivial group, $\mathbb{Z}_2$ the cyclic group of degree 2, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ the product group of $\mathbb{Z}_2$, and $\mathbb{Z}_5$, up to isomorphism. Namely, there exist 6 non-isomorphic groups of order ≤ 5.

Denote by $|G|$ the order of a finite group G. It is known that if G is a non-abelian group with $|G| \leq 24$, then each element of G has order either 1, 2, 3, 4, or 6. The following in the next page is the list of non-abelian (and abelian) groups G with $|G| \leq 24$, up to isomorphism, due to Coxeter-Moser: *Remarks.* In the list, denote by $\mathbb{Z}_k$ the cyclic group of order k, and by $\mathfrak{D}_k$ the dihedral group of degree k. Denote by S_k the symmetric group of degree k, and by A_k the alternating group of degree k. Denote by $\mathcal{Q}$ the quaternion group, and by $\langle 2, 2, 3 \rangle$ the ZS-metacyclic group.

Recall from some references about some definitions and elementary facts as follows.

The dihedral group $\mathfrak{D}_n$ is generated by the rotation r of angel $2\pi/n$ and a symmetry s such that $r^n = 1$, $s^2 = 1$, and $srs = r^{-1}$. Since $sr^ks = r^{-k}$, we have $(sr^k)^2 = 1$. Note that $|\mathfrak{D}_n| = 2n$ and $|\mathfrak{D}_n \times \mathbb{Z}_2| = 4n$. The subgroup generated by r is isomorphic to $\mathbb{Z}_n$. The symmetric group S_n is generated by permutations on n different elements. Note that $|S_n| = n!$. The alternating group A_n is generated by even permutations on n different elements. There is a short exact sequence: $1 \to A_n \to S_n \to \mathbb{Z}_2 \to 1$ induced by signature. Hence $|S_n| = [S_n : A_n] \cdot |A_n| = 2 \cdot |A_n|$.

Let G be a (finite) group. The commutator subgroup $[G, G]$ of G is generated by commutators $a^{-1}b^{-1}ab = [a, b]$ for elements $a, b \in G$. The quotient group $G/[G, G]$ is an abelian group. If $[G, G]$ is an abelian group, then G is said to be a meta-abelian group. In particual, if $[G, G]$ is a cyclic group, then G is said to be a meta-cyclic group.

Any p-group G for a prime number p, with $|G| = p^k$ for some $k \geq 1$ is nilpotent. A finite group G is nilpotent if and only if it is a direct product of Sylow subgroups, so that $|G| = p^k \cdot q^l$ for some prime numbers p and q and $k, l \geq 1$. Any group with order either a prime number p or p^2 is abelian.

Table 1: The list of non-abelian (and abelian) groups with order ≤ 24

Order	Number	Group	Relation
$6 = 2 \cdot 3$	1	S_3	$s^3 = t^2 = (st)^2 = 1$
	(1)	$(\mathbb{Z}_6)$	$(s^6 = 1)$
$8 = 2^3$	2	$\mathcal{Q}$	$i^2 = j^2 = k^2 = ijk = -1$
		$\mathfrak{D}_4$	$s^4 = t^2 = (st)^2 = 1$
	(3)	$(\mathbb{Z}_8;\ \mathbb{Z}_4 \times \mathbb{Z}_2;$	$(s^8 = 1;\ s^4 = s^2 = 1,$
		$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$	$st = ts;\ \cdots\cdots)$
$12 = 2^2 \cdot 3$	3	$\mathfrak{D}_6 \cong S_3 \times \mathbb{Z}_2$	$s^6 = t^2 = (st)^2 = 1$
		A_4	$s^3 = t^2 = (st)^3 = 1$
		$\langle 2, 2, 3 \rangle$	$s^3 = t^2 = (st)^2$
	(3)	$(\mathbb{Z}_{12};\ \mathbb{Z}_6 \times \mathbb{Z}_2;$	$(s^{12} = 1;\ s^6 = t^2 = 1,$
		$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$	$st = ts;\ \cdots\cdots)$
$16 = 2^4$	5	$\mathfrak{D}_4 \times \mathbb{Z}_2$	Omitted
		$\mathcal{Q} \times \mathbb{Z}_2$	Omitted
		$\langle 2, 2 \mid 4, 2 \rangle$	$s^4 = t^4 = 1,$
			$t^{-1}st = s^3$
		$\langle 4, 4 \mid 2, 2 \rangle$	$s^4 = t^4 = 1,$
			$(st)^2 = (s^{-1}t)^2 = 1$
		$\langle r, s, t \rangle$	$r^2 = s^2 = t^2 = 1,$
			$rst = str = trs$
	(5)	$(\mathbb{Z}_{16};\ \mathbb{Z}_8 \times \mathbb{Z}_2;$	(Omitted)
		$(\mathbb{Z}_4)^2;\ (\mathbb{Z}_2)^4;$	
		$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4)$	
$24 = 2^3 \cdot 3$	6	$A_4 \times \mathbb{Z}_2$	Omitted
		$\mathfrak{D}_6 \times \mathbb{Z}_2$	Omitted
		$\langle 2, 2, 3 \rangle \times \mathbb{Z}_2$	Omitted
		S_4	$s^4 = t^2 = (st)^3 = 1$
		$\langle 2, 3, 3 \rangle$	$s^3 = t^3 = (st)^2$
		$\langle 4, 6 \mid 2, 2 \rangle$	$s^4 = t^6 = 1,$
			$(st)^2 = (s^{-1}t)^2 = 1$
	(4)	$(\mathbb{Z}_{24};\ \mathbb{Z}_{12} \times \mathbb{Z}_2;$	(Omitted)
		$\mathbb{Z}_6 \times \mathbb{Z}_4;$	
		$\mathbb{Z}_6 \times (\mathbb{Z}_2)^2)$	
Total	17 (16)	Non-abel. (Abel.)	Generators: $s, t, \cdots$

Remarks. In Table 1 above, each s, t, and i, j, k, and R are generators of corresponding groups. Note that if $n = n_1 \times n_2$, and n_1 and n_2 are relatively prime, then $\mathbb{Z}_n \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$, so that $\mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_2$, $\mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$, and $\mathbb{Z}_{24} \cong \mathbb{Z}_8 \times \mathbb{Z}_3$. Also, $(\mathbb{Z}_n)^k$ means the k-fold direct product of $\mathbb{Z}_n$.

It is known by Vaidyanathaswamy that any element of $GL_3(\mathbb{Z})$ has order either 1, 2, 3, 4, 6, or ∞. Therefore, the order of any finite cyclic subgroup of $GL_3(\mathbb{Z})$ is either 1, 2, 3, 4, or 6.

Reiner determined all non-conjugate cyclic subgroups of order 2 or 3 in $GL_3(\mathbb{Z})$. The case for order 4 or 6 was considered by Tahara. Furthermore, Tahara determined all non-conjugate non-cyclic subgroups of $GL_3(\mathbb{Z})$.

Corollary 4.1.1 *There exist* 17 *non-isomorphic non-abelian groups of oder* ≤ 24, *and there exist* 16 *non-isomorphic abelian groups of order* ≤ 24 *and* ≥ 6; *and there do* $16 + 6 = 22$ *those of order* ≤ 24.

The following table is well known and due to Voskresenskiĭ:

Table 2: Non-conjugate cyclic subgroups in $GL_2(\mathbb{Z})$

Order	Group	Generator
2	$H_{2,1} \cong \mathbb{Z}_2$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \equiv -1_2$
	$H_{2,2} \cong \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv 1 \oplus -1$
	$H_{2,3} \cong \mathbb{Z}_2$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
3	$H_3 \cong \mathbb{Z}_3$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$
4	$H_4 \cong \mathbb{Z}_4$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
6	$H_6 \cong \mathbb{Z}_6$	$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

Corollary 4.1.2 *It follows from Table* 2 *that non-conjugate cyclic subgroups in* $SL_2(\mathbb{Z})$ *are* $H_{2,1} \cong \mathbb{Z}_2$, $H_3 \cong \mathbb{Z}_3$, $H_4 \cong \mathbb{Z}_4$, *and* $H_6 \cong \mathbb{Z}_6$.

Remark. Note that if two subgroups in a group G is non-conjugate and if they are subgroups of a subgroup of G, then they are non-conjugate in the subgroup. This principle is used in what follows.

Table 3: Non-conjugate subgroups of order **2** in $GL_3(\mathbb{Z})$

Order	Group	Generator
2	$C_{2,1} \cong \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \equiv 1 \oplus -1_2$
	$C_{2,2} \cong \mathbb{Z}_2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv -1 \oplus 1_2$
	$C_{2,3} \cong \mathbb{Z}_2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
	$C_{2,4} \cong \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
	$C_{2,5} \cong \mathbb{Z}_2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \equiv -1_3$

Remark. Table 3 follows from a result of Reiner.

Corollary 4.1.3 *It follows from Table* 3 *that non-conjugate subgroups of order* 2 *in* $SL_3(\mathbb{Z})$ *are* $C_{2,1} \cong \mathbb{Z}_2$ *and* $C_{2,3} \cong \mathbb{Z}_2$.

Without it, a large space had been created below so that we made *Part of Proof for Table* 2. Assume that

$$gs \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv tg$$

in $GL_2(\mathbb{Z})$. It follows that $c = -b$, $d = -a$, and $a^2 + b^2 \neq 0$. Hence $s \sim t$.

Suppose that $s^3 = 1$ and $t^2 = 1$. Then $s \not\sim t$. Indeed, if $s \sim t$, then $s = g^{-1}tg$ for some $g \in GL_3(\mathbb{Z})$. Then $s^2 = (g^{-1}tg)^2 = g^{-1}t^2g = 1$, which is a contradiction.

Assume that $x^2 \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv 1_2$. It follows that if $b = c = 0$, then $a = \pm 1$ and $b = \pm 1$, and if $b = c = 1$ (or -1), then $a = d = 0$.

Assume that $x^3 = 1_2$. It follows that if $a = 0$, then $d = -1$ and either $b = 1$ and $c = -1$, or $b = -1$ and $c = 1$. □

Table 4: Non-conjugate subgroups of order **3** in $GL_3(\mathbb{Z})$

Order	Group	Generator
3	$C_{3,1} \cong \mathbb{Z}_3$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$
	$C_{3,2} \cong \mathbb{Z}_3$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Corollary 4.1.4 *The groups $C_{3,1} \cong \mathbb{Z}_3$ and $C_{3,2} \cong \mathbb{Z}_3$ are non-conjutate subgroups in $SL_3(\mathbb{Z})$.*

Without it, a large space had been created below so that we made *Part of Proof for Table* 3. Assume that

$$gs \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv tg$$

in $GL_3(\mathbb{Z})$. It follows that $a = 0$, $d = 0$, and $bc \neq 0$. Hence $s \sim t$. Using this observation, we have the equivalences $-1 \oplus 1 \oplus 1 \sim 1 \oplus -1 \oplus 1 \sim 1 \oplus 1 \oplus -1$ among these diagonal matrices. Similarly, $-1 \oplus -1 \oplus 1 \sim -1 \oplus 1 \oplus -1 \sim 1 \oplus -1 \oplus -1$.

Assume that

$$g \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} g$$

in $GL_3(\mathbb{Z})$, where $g = (a_{ij})_{i,j=1}^3$. It follows that $a_{11} = 0$, $a_{12} = a_{13}$, $a_{21} = a_{31}$, $a_{23} + a_{32} = 0$, and $a_{22} + a_{33} = 0$, and we have the rank of the matrix g two. Hence we must have those matrices non-conjugate. □

Part of Proof for Table 4. Assume that

$$g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} g$$

in $GL_3(\mathbb{Z})$, where $g = (a_{ij})_{i,j=1}^3$. It follows that $a_{11} = a_{21} = a_{33} \equiv k$, $a_{13} = a_{22} \equiv l$, $a_{23} = a_{32} \equiv m$, $a_{12} = a_{33} \equiv n$, and $l + m + n = 0$, with $ln - m^2 = -(l^2 + n^2 + ln) = \pm 1$. It forces that $l = \pm 1$, $n = \mp 1$, and $m = 0$. But in each case the matrix g is not in $GL_3(\mathbb{Z})$. Hence we have those matrices non-conjugate. □

Table 5: Non-conjugate subgroups of order **4** in $GL_3(\mathbb{Z})$: I

Group	Generator
$C_{4,1} \cong \mathbb{Z}_4$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$
$C_{4,2} \cong \mathbb{Z}_4$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$
$C_{4,3} \cong \mathbb{Z}_4$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$
$C_{4,4} \cong \mathbb{Z}_4$	$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$
$C_{4,5}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$C_{4,6}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$C_{4,7}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$C_{4,8}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
$C_{4,9}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$C_{4,10}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$C_{4,11}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$C_{4,12}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$

Table 6: Non-conjugate subgroups of order 4 in $GL_3(\mathbb{Z})$: II

Group	Generator
$C_{4,13}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$
$C_{4,14}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
$C_{4,15}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Note also that $C_{4,5}^2 \cong C_{2,1} \times C_{2,5}$ and $C_{4,11}^2 \cong C_{2,3} \times C_{2,5}$.

Corollary 4.1.5 *Non-conjugate cyclic subgroups of order* 4 *in* $SL_3(\mathbb{Z})$ *are* $C_{4,1} \cong \mathbb{Z}_4$ *and* $C_{4,3} \cong \mathbb{Z}_4$. *Non-conjugate non-cyclic subgroups of order* 4 *in* $SL_3(\mathbb{Z})$ *are* $C_{4,6}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $C_{4,8}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $C_{4,12}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, *and* $C_{4,14}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Several lemmas are put in the following:

Sketch of Proof for Tables 5 *and* 6. First of all, find all non-conjugate cyclic subgroups of order 4 in $GL_3(\mathbb{Z})$. Let $y \in GL_3(\mathbb{Z})$ be of order 4. It follows from Table 3 for non-conjugate subgroups of order 2 in $GL_3(\mathbb{Z})$ that

$$y^2 \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \equiv m_1, \quad \text{or} \quad y^2 \sim \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \equiv m_2.$$

Case 1. Assume that $y^2 = g^{-1} m_1 g$ for some $g \in GL_3(\mathbb{Z})$.

Lemma 4.1.6 *Let* $x \in GL_3(\mathbb{Z})$. *If* x^2 *is equal to* m_1, $-1 \oplus 1 \oplus -1$, *or* $-1 \oplus -1 \oplus 1$ *(diagonal matrices), then* x *is of the form:*

$$\pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & -a \end{pmatrix}, \quad \pm \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & -a \end{pmatrix} \quad \text{or} \quad \pm \begin{pmatrix} a & b & 0 \\ c & -a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

respectively, where $a^2 + bc + 1 = 0$.

Therefore,

$$gyg^{-1} = \pm\{1 \oplus \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\}, \quad a^2 + bc + 1 = 0.$$

It follows from Table 2 that

$$1 \oplus \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \sim 1 \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so that the group generated by y is conjugate to $C_{4,1}$ or $C_{4,2}$.

Case 2. Assume that $y^2 = g^{-1}m_2 g$ for some $g \in GL_3(\mathbb{Z})$.

Lemma 4.1.7 *Let $x \in GL_3(\mathbb{Z})$. If x^2 is equal to m_2 or $-1 \oplus \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, then x is of the form:*

$$\pm \begin{pmatrix} a & b & -b \\ \frac{-1-a^2}{2b} & \frac{1-a}{2} & \frac{1+a}{2} \\ \frac{1+a^2}{2b} & \frac{1+a}{2} & \frac{1-a}{2} \end{pmatrix} \equiv \pm n, \quad or \quad \pm \begin{pmatrix} a & b & b \\ \frac{-1-a^2}{2b} & \frac{1-a}{2} & \frac{-1-a}{2} \\ \frac{-1-a^2}{2b} & \frac{-1-a}{2} & \frac{1-a}{2} \end{pmatrix}$$

respectively, where $b \neq 0$, a, and $(1+a^2)/2b$ are all odd integers.

Therefore, $gyg^{-1} = \pm n$. We then claim that

$$y \sim \pm \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \equiv \pm m_3$$

It is enough to show that $n \sim m_3$. This holds if there is a matrix $z \in GL_3(\mathbb{Z})$ defined by

$$\begin{pmatrix} z_{11} & z_{12} & z_{13} \\ (-1-a)z_{11} + \frac{1+a^2}{2b}k & -bz_{11} + \frac{a-1}{2}k & bz_{11} + \frac{1-a}{2}k \\ (a-1)z_{11} - \frac{1+a^2}{2b}k & bz_{11} - \frac{1+a}{2}k & -bz_{11} + \frac{1+a}{2}k \end{pmatrix}$$

with $k = z_{12} - z_{13}$, and

$$\det(z) = -(z_{12} + z_{13})(2bz_{11}^2 - 2az_{11}k + (1+a^2)(2b)^{-1}k^2) = \pm 1,$$

i.e., each factor is ± 1. Hence $n \sim m_3$ if $s \equiv z_{11}$ and $t \equiv 2z_{12} \mp 1 = k$ with $k \pm 1 = 2z_{12}$ are integers satisfying the diophantine equation: $(2|b|s + at)^2 + t^2 = 2|b|$. This equation has integral solutions by LeVeque.

Next find all non-conjugate non-cyclic subgroups in $GL_3(\mathbb{Z})$ that are isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$. Let s, t be generators of $\mathbb{Z}_2\times\mathbb{Z}_2$ such that $s^2 = t^2 = 1$ and $st = ts$. By Table 3, we need to consider three cases $3, 4$, and 5 as follows.

Case 3. Suppose that $s = \pm g^{-1}(1 \oplus -1_2)g$ for some $g \in GL_3(\mathbb{Z})$, where 1_2 means the 2×2 identity matrix. Since $st = ts$, we have $gtg^{-1}(1\oplus -1_2) = (1 \oplus -1_2)gtg^{-1}$.

Lemma 4.1.8 *Let $x \in GL_3(\mathbb{Z})$. If x commutes with $1 \oplus (-1_2)$, then x is of the form:* $\pm 1 \oplus \begin{pmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{pmatrix}$, *where* $x_{22}x_{33} - x_{23}x_{32} = 1$.

Therefore, $t = \pm g^{-1}(1 \oplus t_1)g$, where $t_1 = \begin{pmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{pmatrix}$ with $x_{22}x_{33} - x_{23}x_{32} = 1$. Since t_1 has order 2, it follows from Table 2 that t_1 is conjugate to either $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It follows that the group $\langle s, t\rangle$ generated by s and t is conjugate to either $C^2_{4,5}$, $C^2_{4,6}, \cdots$, or $C^2_{4,10}$, which are not conjugate to each other.

Case 4. Suppose that $s = \pm g^{-1}\{-1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}g$ for some $g \in GL_3(\mathbb{Z})$. The relation $st = ts$ implies

$$gtg^{-1}\{-1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\} = \{-1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}gtg^{-1}.$$

Lemma 4.1.9 *Let $x \in GL_3(\mathbb{Z})$. If x commutes with* $\{-1\oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$, *then x is of the form:*

$$\begin{pmatrix} x_{11} & x_{12} & -x_{12} \\ x_{21} & x_{22} & x_{23} \\ -x_{21} & x_{23} & x_{22} \end{pmatrix},$$

where $(x_{22} + x_{23})\{x_{11}(x_{22} - x_{23}) - 2x_{12}x_{21}\} = 1$.

Furthermore, if x has order 2, then x is determined among matrices of 9 forms. Also, there is no such matrix x of order 3.

Using this lemma, it is shown case by case that the group $\langle s, t\rangle$ generated by s and t is conjugate to either $C^2_{4,8}$, $C^2_{4,8}$, $C^2_{4,9}$, $C^2_{4,10}$, $C^2_{4,11}$, $C^2_{4,12}$, $C^2_{4,13}$, $C^2_{4,14}$, or $C^2_{4,15}$. □

Table 7: Non-conjugate subgroups of order **6** in $GL_3(\mathbb{Z})$

Group	Generator
$C_{6,1} \cong \mathbb{Z}_6$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$
$C_{6,2} \cong \mathbb{Z}_6$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$
$C_{6,3} \cong \mathbb{Z}_6$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$
$C_{6,4} \cong \mathbb{Z}_6$	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$
$S_{6,5} \cong S_3$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
$S_{6,6} \cong S_3$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$S_{6,7} \cong S_3$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$S_{6,8} \cong S_3$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
$S_{6,9} \cong S_3$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$
$S_{6,10} \cong S_3$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Corollary 4.1.10 *Only one cyclic subgroup of order* 6 *up to conjugacy in* $SL_3(\mathbb{Z})$ *is* $C_{6,1} \cong \mathbb{Z}_6$. *Non-conjugate non-abelian subgroups of order* 6 *in* $SL_3(\mathbb{Z})$ *are* $S_{6,5} \cong S_3$, $S_{6,7} \cong S_3$, $S_{6,7} \cong S_3$, *and* $S_{6,9} \cong S_3$.

Sketch of Proof for Table 7. The case for cyclic groups of order 6 has been considered by Matuljauskas. Determine all non-conjugate subgroups that are isomorphic to S_3. Let s, t be generators of such a group satisfying $s^3 = t^2 = (st)^2 = 1$. By Table 4, it follows that

$$(1): \quad s \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \quad \text{or} \quad (2): \quad s \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \equiv u.$$

Case 1. Assume that $s = g^{-1}\{1 \oplus \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\}g$ for some $g \in GL_3(\mathbb{Z})$. Since $ts = s^2t$, we get $gtg^{-1}\{1 \oplus \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\} = \{1 \oplus \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\}gtg^{-1}$.

Lemma 4.1.11 *Let* $x \in GL_3(\mathbb{Z})$. *Assume that* $x\{1 \oplus \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\} = \{1 \oplus \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\}x$. *Then* x *is determined among matrices of* 6 *types, all of which have order* 2.

If x *commutes with either* $\{1 \oplus \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\}$ *or* $\{1 \oplus \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\}$, *then* x *is determined among matrices of* 6 *types.*

Using this lemma, it is shown case by case that the group $\langle s, t\rangle$ generated by s and t is conjugate to either $S_{6,5}$, $S_{6,6}$, $S_{6,7}$, or $S_{6,8}$.

Case 2. Assume that $s = g^{-1}ug$ for some $g \in GL_3(\mathbb{Z})$. Since $ts = s^2t$, we have $gtg^{-1}u = vgtg^{-1}$, where $v = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Lemma 4.1.12 *Let* $x \in GL_3(\mathbb{Z})$. *Assume that* $xu = vx$. *Then* x *is determined among matrices of* 3 *types, all of which have order* 2.

If x *commutes with* u *or* v, *then* x *is determined among matrices of* 3 *types.*

It follows that $\langle s, t\rangle \sim S_{6,9}$ or $\sim S_{6,10}$. □

Table 8: Non-conjugate subgroups of order **8** in $GL_3(\mathbb{Z})$: I

Group	Generator
$C^2_{8,1} \cong \mathbb{Z}_4 \times \mathbb{Z}_2$	$\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$
$C^2_{8,2} \cong \mathbb{Z}_4 \times \mathbb{Z}_2$	$\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$
$C^3_{8,3} \cong (\mathbb{Z}_2)^3$	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$
$C^3_{8,4} \cong (\mathbb{Z}_2)^3$	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$
$C^3_{8,5} \cong (\mathbb{Z}_2)^3$	$\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\1&0&-1\\-1&-1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$
$C^3_{8,6} \cong (\mathbb{Z}_2)^3$	$\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}-1&1&-1\\0&0&-1\\0&-1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$
$\mathfrak{D}_{8,7} \cong \mathfrak{D}_4$	$\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}$
$\mathfrak{D}_{8,8} \cong \mathfrak{D}_4$	$\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$
$\mathfrak{D}_{8,9} \cong \mathfrak{D}_4$	$\begin{pmatrix}-1&0&0\\0&0&1\\0&-1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}$
$\mathfrak{D}_{8,10} \cong \mathfrak{D}_4$	$\begin{pmatrix}-1&0&0\\0&0&1\\0&-1&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$
$\mathfrak{D}_{8,11} \cong \mathfrak{D}_4$	$\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$
$\mathfrak{D}_{8,12} \cong \mathfrak{D}_4$	$\begin{pmatrix}1&0&1\\0&0&-1\\0&1&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$

Table 9: Non-conjugate subgroups of order **8** in $GL_3(\mathbb{Z})$: II

Group	Generator
$\mathfrak{D}_{8,13} \cong \mathfrak{D}_4$	$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
$\mathfrak{D}_{8,14} \cong \mathfrak{D}_4$	$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Remark. In Table 8, $(\mathbb{Z}_2)^3$ means the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. It is shown by Vaidyanathaswamy that there is no cyclic subgroup of order 8 in $GL_3(\mathbb{Z})$. Also, there is no subgroup in $GL_3(\mathbb{Z})$ that is isomorphic to the quaternion group. It follows that any subgroup of order 8 in $GL_3(\mathbb{Z})$ is isomorphic to either $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathfrak{D}_4$.

Corollary 4.1.13 *There is no abelian subgroup of order* 8 *in* $SL_3(\mathbb{Z})$. *Non-conjugate non-abelian sugroups of order* 8 *in* $SL_3(\mathbb{Z})$ *are* $\mathfrak{D}_{8,7} \cong \mathfrak{D}_4$ *and* $\mathfrak{D}_{8,11} \cong \mathfrak{D}_4$.

To show Tables 8 and 9 above, we use in part the following lemmas:

Lemma 4.1.14 *Let* $x \in GL_3(\mathbb{Z})$. *If* x *commutes with either* $\{1 \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\} \equiv u$ *or* $\{1 \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\} \equiv v$, *then* x *is equal to either* $\pm u$, $\pm v$, $\pm(1 \oplus -1_2)$, *or* $\pm 1_3$.

Assume that $xu = vx$. *then* x *is equal to either* $\pm\{-1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$, $\pm\{-1 \oplus \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\}$, $\pm(-1 \oplus 1 \oplus -1)$, *or* $\pm(-1_2 \oplus 1)$, *all of which have order* 2.

Lemma 4.1.15 *Let* $x \in GL_3(\mathbb{Z})$. *If* x *commutes with either* $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = u$ *or* $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \equiv v$, *then* x *is determined among matrices of* 4 *types.*

Assume that $xu = vx$. *Then* x *is determined among matrices of* 4 *types, all of which have order* 2.

Table 10: Non-conjugate subgroups of order **12** in $GL_3(\mathbb{Z})$

Group	Generator
$C^2_{12,1} \cong \mathbb{Z}_6 \times \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$\mathfrak{D}_{12,2} \cong \mathfrak{D}_6$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$\mathfrak{D}_{12,3} \cong \mathfrak{D}_6$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
$\mathfrak{D}_{12,4} \cong \mathfrak{D}_6$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$\mathfrak{D}_{12,5} \cong \mathfrak{D}_6$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
$\mathfrak{D}_{12,6} \cong \mathfrak{D}_6$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
$\mathfrak{D}_{12,7} \cong \mathfrak{D}_6$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$\mathfrak{D}_{12,8} \cong \mathfrak{D}_6$	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$
$A_{12,9} \cong A_4$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$A_{12,10} \cong A_4$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$
$A_{12,11} \cong A_4$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Remark. There exists no subgroup of order 9 in $GL_3(\mathbb{Z})$. Hence the order of any finite subgroup of $GL_3(\mathbb{Z})$ is of the form $2^i \times 3^j$, where $j = 0$ or 1.

Any group of order 12, whose elements have order either 1, 2, 3, 4, or 6, is isomorphic to either $\mathbb{Z}_6 \times \mathbb{Z}_2$, $\mathfrak{D}_6 \cong S_3 \times \mathbb{Z}_2$, A_4, or the group $\langle 2, 2, 3\rangle$.

Corollary 4.1.16 *There is no abelian subgroup of order* 12 *in* $SL_3(\mathbb{Z})$. *Non-conjugate subgroups of order* 12 *in* $SL_3(\mathbb{Z})$ *are* $\mathfrak{D}_{12,2} \cong \mathfrak{D}_6$, $A_{12,9} \cong A_4$, $A_{12,10} \cong A_4$, *and* $A_{12,11} \cong A_4$.

To show Table 10 above, we use in part the following lemma:

Lemma 4.1.17 *Let* $x \in GL_3(\mathbb{Z})$. *If* x *commutes with* $\{1 \oplus \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\} \equiv u$ *or* $\{1 \oplus \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\} \equiv v$, *then* x *is determined among matrices of* 6 *types.*

Assume that $xu = vx$. *Then* x *is determined among matrices of* 6 *types, all of which have order* 2.

Table 11: Non-conjugate subgroups of order **16** in $GL_3(\mathbb{Z})$

Group	Generator
$H_1 \cong \mathfrak{D}_4 \times \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$H_2 \cong \mathfrak{D}_4 \times \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Remark. There is no abelian subgroup of order 16 in $GL_3(\mathbb{Z})$. It is shown that there exists no non-abelian subgroup of order 16 in $SL_3(\mathbb{Z})$.

Any non-abelian group of order 16, whose elements have order either 1, 2, 3, 4, or 6, is isomorphic to either $\mathfrak{D}_4 \times \mathbb{Z}_2$, $\mathcal{Q} \times \mathbb{Z}_2$, $\langle 2, 2 \mid 4, 2\rangle$, $\langle 4, 4 \mid 2, 2\rangle$, or $\langle r, s, t\rangle$.

Corollary 4.1.18 *There is no subgroup of order* 16 *in* $SL_3(\mathbb{Z})$.

Remark. If follows from this corollary that the order of any finite subgroup of $GL_3(\mathbb{Z})$ is of the form $2^i \cdot 3^j$ for $i \leq 4$ and $j \leq 1$ (and that of $SL_3(\mathbb{Z})$ is $2^i \cdot 3^j$ for $i \leq 3$ and $j \leq 1$).

Sketch of Proof for Table 11. Let H be a subgroup of the type $\mathfrak{D}_4 \times \mathbb{Z}_2$. Then $\mathfrak{D}_4$ is conjugate to one of $\mathfrak{D}_{8,j}$ $(7 \le j \le 14)$ in Tables 8 and 9. Let t be a generator of $\mathbb{Z}_2$.

Suppose that $\mathfrak{D}_4 = g^{-1}\mathfrak{D}_{8,j}g$ for some $7 \le j \le 10$ and $g \in GL_3(\mathbb{Z})$. Then gTg^{-1} commutes with $\{1 \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}$ and $\{-1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$. It follows that H is conjugate to H_1 in Table 11.

Suppose that $\mathfrak{D}_4 = g^{-1}\mathfrak{D}_{8,j}g$ for some $11 \le j \le 14$ and $g \in GL_3(\mathbb{Z})$. Similarly, it follows that H is conjugate to H_2 in Table 11.

Since there is no subgroup of order 8 in $GL_3(\mathbb{Z})$ that is isomorphic to the quaternion group $\mathcal{Q}$, there exists no subgroup of the type $\mathcal{Q} \times \mathbb{Z}_2$.

Let $\langle s, t\rangle$ be the subgroup of the type $\langle 2, 2 \,|\, 4, 2\rangle$ such that $s^4 = t^4 = 1$ and $t^{-1}st = s^3$. First assume that $s = \pm g^{-1}\{1 \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}g$ for some $g \in GL_3(\mathbb{Z})$. The relation $t^{-1}st = s^3$ implies that $gt^{-1}g^{-1}\{1 \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\} = \{1 \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}gt^{-1}g^{-1}$. Thus, $gt^{-1}g^{-1}$ must have order 2, which contradicts to $t^4 = 1$. Secondly assume that $s = \pm g^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} g$ for some $g \in GL_3(\mathbb{Z})$. Similarly, it follows that there is no such matrix t. Hence there exists no subgroup of the type $\langle 2, 2 \,|\, 4, 2\rangle$ in $GL_3(\mathbb{Z})$.

Let $\langle s, t\rangle$ be the group of the type $\langle 4, 4 \,|\, 2, 2\rangle$ such that $s^4 = t^4 = 1$ and $(st)^2 = (s^{-1}t)^2 = 1$. Assume that for some $g \in GL_3(\mathbb{Z})$, either

$$t = \pm g^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} g, \quad \text{or} \quad gtg^{-1} = \pm \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \equiv v.$$

The relation $s^2t^3 = t^3s^2$ implies a commutative relation with respect to gs^2g^{-1} in each case. It follows that s is conjugate to $\{1 \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$ or $1 \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}$ in the first case, and to v in the second case. Therefore, $\langle s, t\rangle$ does not have order 16. Hence there exists no subgroup of the type $\langle 4, 4 \,|\, 2, 2\rangle$ in $GL_3(\mathbb{Z})$.

Furthermore, it is shown that there is no subgroup in $GL_3(\mathbb{Z})$ that is isomorphic to the group $\langle r, s, t\rangle$ generated by r, s, t such that $r^2 = s^2 = t^2 = 1$ and $rst = str = trs$. □

Table 12: Non-conjugate subgroups of order **24** in $GL_3(\mathbb{Z})$

Group	Generator
$A_4 \times \mathbb{Z}_2$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$A_4 \times \mathbb{Z}_2$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$A_4 \times \mathbb{Z}_2$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$\mathfrak{D}_6 \times \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$\mathfrak{D}_6 \times \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$S_{24,6} \cong S_4$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
$S_{24,7} \cong S_4$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$S_{24,8} \cong S_4$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$S_{24,9} \cong S_4$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$S_{24,10} \cong S_4$	$\begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$S_{24,11} \cong S_4$	$\begin{pmatrix} -1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$

Remark. There is no abelian subgroup of order 24 in $GL_3(\mathbb{Z})$. Any non-abelian group of order 24, whose elements have order either 1, 2, 3, 4, or 6, is isomorphic to either $A_4 \times \mathbb{Z}_2$, $\langle 2,2,3\rangle \times \mathbb{Z}_2$, $\mathfrak{D}_6 \times \mathbb{Z}_2$, S_4, $\langle 2,3,3\rangle$, or the group $\langle 4,6 \,|\, 2,2\rangle$.

Corollary 4.1.19 *Non-conjugate subgroups of order* 24 *in* $SL_3(\mathbb{Z})$ *are* $S_{24,6} \cong S_4$, $S_{24,8} \cong S_4$, *and* $S_{24,10} \cong S_4$.

Table 13: Non-conjugate subgroups of order **48** in $GL_3(\mathbb{Z})$

Group	Generator
$S_{48,1} \cong S_4 \times \mathbb{Z}_2$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$S_{48,2} \cong S_4 \times \mathbb{Z}_2$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$S_{48,3} \cong S_4 \times \mathbb{Z}_2$	$\begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Remark. There is no subgroup of order 48 in $SL_3(\mathbb{Z})$. Hence any subgroup of order 48 in $GL_3(\mathbb{Z})$ is generated by a subgroup of order 24 in $SL_3(\mathbb{Z})$ and a matrix with determinant -1. Indeed, $S_{48,1} = \langle S_{24,6}, -1_3\rangle$, $S_{48,2} = \langle S_{24,8}, -1_3\rangle$, and $S_{48,3} = \langle S_{24,10}, -1_3\rangle$, where each $\langle S_{24,j}, -1_3\rangle$ is the group generated by $S_{24,j}$ and -1_3, and is isomorphic to $S_4 \times \mathbb{Z}_2$, and 1_3 is the unit of $GL_3(\mathbb{Z})$.

Proposition 4.1.20 *There is no subgroup of order* > 48 *in* $GL_3(\mathbb{Z})$.

Table 13 and the proposition above imply

Corollary 4.1.21 *There is no subgroup of order* ≥ 48 *in* $SL_3(\mathbb{Z})$.

To show Table 13 and the proposition above, we use in part the following lemmas:

Lemma 4.1.22 *Let* $x \in GL_3(\mathbb{Z})$. *If* x *commutes with* $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \equiv u$, *then* x *is determined among matrices of* 4 *types.*

Assume that either $xu = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} x$, $xu = \{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus 1\}x$, $xu = \{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus 1\}x$, $xu = \{1 \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}x$, *or* $xu = \{1 \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$. *Then in each case x is determined among matrices of 4 types.*

Lemma 4.1.23 *Let $x \in GL_3(\mathbb{Z})$. If x commutes with* $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \equiv u$, *then x is determined among matrices of 4 types. Assume that either*

$$xu = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} x, \quad xu = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} x,$$

$$xu = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} x, \quad xu = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} x,$$

$$\text{or} \quad xu = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} x.$$

Then in each case x is determined among matrices of 4 types.

Lemma 4.1.24 *Let $x \in GL_3(\mathbb{Z})$. If x commutes with* $\begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \equiv u$, *then x is determined among matrices of 4 types. Assume that either*

$$xu = \begin{pmatrix} -1 & -1 & -1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x, \quad xu = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & -1 & -1 \end{pmatrix} x,$$

$$xu = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} x, \quad xu = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} x,$$

$$\text{or} \quad xu = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} x.$$

Then in the first, third, and fifth cases x is determined among matrices of 4 types, and in the second and fourth cases, x is done of 2 types.

Table 14: Non-conjugate subgroups in $GL_3(\mathbb{Z})$

Order	Number	Abelian	Non-abelian
1	1	$* \langle 1_3 \rangle$	None
2	5	$* C_{2,j} \cong \mathbb{Z}_2$ $(1 \leq j \leq 5)$	None
3	2	$* C_{3,j} \cong \mathbb{Z}_3$ $(1 \leq j \leq 2)$	None
4	4 + 11	$* C_{4,j} \cong \mathbb{Z}_4$ $(1 \leq j \leq 4)$ $C^2_{4,j} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ $(5 \leq j \leq 15)$	None
6	4 + 6	$* C_{6,j} \cong \mathbb{Z}_6$ $(1 \leq j \leq 4)$	$S_{6,j} \cong S_3$ $(5 \leq j \leq 10)$
8	6 + 8	$C^2_{8,j} \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ $(j = 1, 2)$ $C^3_{8,j} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $(3 \leq j \leq 6)$	$\mathfrak{D}_{8,j} \cong \mathfrak{D}_4$ $(7 \leq j \leq 14)$
12	1 +10	$C^2_{12,1} \cong \mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathfrak{D}_{12,j} \cong \mathfrak{D}_6$ $(2 \leq j \leq 8)$ $A_{12,j} \cong A_4$ $(9 \leq j \leq 11)$
16	2	None	$\mathfrak{D}_4 \times \mathbb{Z}_2$ (2 copies)
24	11	None	$A_4 \times \mathbb{Z}_2$ (3 copies) $\mathfrak{D}_6 \times \mathbb{Z}_2$ (2 copies) $S_{24,j} \cong S_4$ $(6 \leq j \leq 11)$
48	3	None	$\langle S_{24,j}, -1_3 \rangle$ $\cong S_4 \times \mathbb{Z}_2$ $(j = 6, 8, 10)$
Any ≥ 49	0	None	None
Total	74	$*$ Cyclic 16; Non-cyclic 18	Non-abelian 40

Table 15: Non-conjugate subgroups in $SL_3(\mathbb{Z})$

Order	Number	Abelian	Non-abelian
1	1	$*\ \langle 1_3 \rangle$	None
2	2	$*\ C_{2,j} \cong \mathbb{Z}_2\ (j = 1, 3)$	None
3	2	$*\ C_{3,j} \cong \mathbb{Z}_3\ (j = 1, 2)$	None
4	$2 + 4$	$*\ C_{4,j} \cong \mathbb{Z}_4\ (j = 1, 3)$ $C_{4,j}^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ $(j = 6, 8, 12, 14)$	None
6	$1 + 3$	$*\ C_{6,1} \cong \mathbb{Z}_6$	$S_{6,j} \cong S_3$ $(j = 5, 7, 9)$
8	2	None	$\mathfrak{D}_{8,j} \cong \mathfrak{D}_4$ $(j = 7, 11)$
12	4	None	$\mathfrak{D}_{12,2} \cong \mathfrak{D}_6$ $A_{12,j} \cong A_4$ $(9 \leq j \leq 11)$
16	0	None	None
24	3	None	$S_{24,j} \cong S_4$ $(j = 6, 8, 10)$
48	0	None	None
Any ≥ 49	0	None	None
Total	24	$*$ Cyclic 8; Non-cyclic 4	Non-abelian 12

Corollary 4.1.25 *Any abelian subgroup of $GL_3(\mathbb{Z})$ is isomorphic to either*

$$\{1\}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, (\mathbb{Z}_2)^2, \mathbb{Z}_6, \mathbb{Z}_4 \times \mathbb{Z}_2, (\mathbb{Z}_2)^3, \textit{or } \mathbb{Z}_6 \times \mathbb{Z}_2$$

of 9 types, and any non-abelian subgroup of $GL_3(\mathbb{Z})$ is isomorphic to either

$$S_3, \mathfrak{D}_4, \mathfrak{D}_6, A_4, \mathfrak{D}_4 \times \mathbb{Z}_2, A_4 \times \mathbb{Z}_2, \mathfrak{D}_6 \times \mathbb{Z}_2, S_4, \textit{or } S_4 \times \mathbb{Z}_2$$

of 9 types, where $GL_3(\mathbb{Z})$ itself is excluded.

Corollary 4.1.26 *Any abelian subgroup of $SL_3(\mathbb{Z})$ is isomorphic to either*

$$\{1\}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, (\mathbb{Z}_2)^2, \textit{or } \mathbb{Z}_6 \textit{ of } 6 \textit{ types.}$$

Any non-abelian subgroup of $SL_3(\mathbb{Z})$ is isomorphic to either

$$S_3, \mathfrak{D}_4, \mathfrak{D}_6, A_4, \textit{or } S_4 \textit{ of } 5 \textit{ types.}$$

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