Noncommutative continuous deformation theory by soft C^⊠-algebras : a review

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NONCOMMUTATIVE CONTINUOUS DEFORMATION THEORY BY SOFT C^*-ALGEBRAS — A REVIEW

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Dedicated to Professor Kichi-Suke Saito on his sixtieth birthday

Abstract

We review and study noncommutative continuous deformation theory for soft C^*-algebras such as the soft torus of Exel and other variations by several others. This theory includes structure theory, K-theory, and continuous field theory for those C^*-algebras. In addition, their (finite dimensional) representation theory is reviewed and considered. Furthermore, noncommutative shape theory for C^*-algebras is also done.

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Introduction

This paper is devoted to reviewing and studying the noncommutative continuous deformation theory for (certain) C^*-algebras. Especially, we first focus on the soft tori of Exel, that are parameterized on a closed interval and are viewed as a softly noncommutative continuous deformation from (or to) the commutative C^*-algebra of all continuous functions on the usual 2-torus. In particular, the structure, K-theory, and continuous fields of the soft tori are explicitly given and computed. Some proofs for these things become more detailed and should be more (easily) readable by some efforts.

As mentioned in the abstract, this paper is organized as follows.

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Notes about original sources and some references are put in the last part of each section with several subsections.

1 Soft torus

1.1 The soft torus as a crossed product

The C^*-algebra $C(\mathbb{T}^2)$ of all continuous complex valued functions on the two-torus $\mathbb{T}^2$ is the universal C^*-algebra generated by two commuting unitaries, and which is also the C^*-tensor product $C(\mathbb{T}) \otimes C(\mathbb{T})$ of $C(\mathbb{T})$ for the torus $\mathbb{T}$.

The soft two-torus $C(\mathbb{T}^2)_\varepsilon$ of Exel for $\varepsilon \in [0,2]$ is defined to be the universal C^*-algebra generated by two ε-almost commuting unitaries u_ε and v_ε in the sense that

$$\|u_\varepsilon v_\varepsilon - v_\varepsilon u_\varepsilon\| \leq \varepsilon.$$

Note that $C(\mathbb{T}^2)_0 = C(\mathbb{T}^2)$, and $C(\mathbb{T}^2)_2$ is isomorphic to the full group C^*-algebra of the free group $\mathbb{F}_2$ of two generators, and which is also the universal C^*-algebra of generated by two unitaries which have no relations, that is, the unital free product C^*-algebra $C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T})$. Note that the above inequality always holds when $\varepsilon = 2$. Therefore, the soft tori $C(\mathbb{T}^2)_\varepsilon$ defined on the closed interval $[0,2]$ contain two such commutative and the most noncommutative unital cases at the boundary points. That is of crucial interest. However, it will be shown below in this subsection that highly or lowly noncommutative $C(\mathbb{T}^2)_\varepsilon$ for $\varepsilon \in (0,2)$ have rather different structure from two extreme cases for $\varepsilon = 0$, 2, but their K-theory is the same as that of $C(\mathbb{T}^2)$ and not the same as that of $C(\mathbb{T}^2)_2$, while their stable rank is the same infinity as that of $C(\mathbb{T}^2)_2$ and not the same as that of $C(\mathbb{T}^2)$ (in the subsection 4.1). Furthermore, the soft tori will be viewed as fibers of a continuous field of C^*-algebras over $[0,2]$ (in the section 2).

There is a $*$-homomorphism from $C(\mathbb{T}^2)_\varepsilon$ onto $C(\mathbb{T}^2)$ by corresponding their generators. This follows from the universality of $C(\mathbb{T}^2)_\varepsilon$ since the relation of $C(\mathbb{T}^2)$ is stronger than that of $C(\mathbb{T}^2)_\varepsilon$.

Lemma 1.1.1. *Let u, v be unitaries of a C^*-algebra $\mathfrak{A}$ with $\|u-v\| = \varepsilon < 2$. Then there is a continuous path $u(t)$ of unitaries in $\mathfrak{A}$ such that $u(0) = u$ and $u(1) = v$ and $\|u(t) - u(s)\| \leq \varepsilon$ for all $t, s \in [0,1]$.*

Proof. Note that $\|1 - u^{-1}v\| = \varepsilon < 2$. Indeed, using the C^*-norm property,

$$\begin{aligned} \|1 - u^{-1}v\|^2 &= \|u^{-1}(u-v)\|^2 \\ &= \|(u^* - v^*)(u^{-1})^* u^{-1}(u-v)\| = \|(u^* - v^*)(u-v)\| \\ &= \|u - v\|^2. \end{aligned}$$

Therefore, -1 is not in the spectrum of $u^{-1}v$. Put $h = i^{-1}\log(u^{-1}v)$. Then

$$h^* = -i^{-1}\log(v^*(u^{-1})^*) = -i^{-1}\log(v^{-1}u) = -i^{-1}\log((u^{-1}v)^{-1}) = h.$$

Also, the spectral theorem implies that

$$\|h\| = \|\log(u^{-1}v)\| \leq \|\log(e^{i\theta})\|_\infty = \|i\theta\|_\infty \leq \pi,$$

where $e^{i\theta}$ is in the spectrum of $u^{-1}v$, and $\|\log(e^{i\theta})\|_\infty$ means the supremum norm of continuous functions on the spectrum of $u^{-1}v$ (cf. [14]). Thus, $\|h\| \leq \pi$. Note that

$$\begin{aligned}|1-e^{ix}|^2 &= |1-(\cos x + i\sin x)|^2 \\ &= (1-\cos x)^2 + \sin^2 x \\ &= 2 - 2\cos x = 2(1-\cos x) \\ &= 2^2 \sin^2(\frac{x}{2}) = |2\sin(\frac{x}{2})|^2.\end{aligned}$$

It follows from the spectral theorem that

$$\begin{aligned}\varepsilon = \|1-u^{-1}v\| &= \|1-e^{\log(u^{-1}v)}\| \\ &= \|1-e^{ih}\| = \|2\sin(\frac{h}{2})\| = 2\sin(\frac{\|h\|}{2}).\end{aligned}$$

Hence, one has $\|h\| = 2\sin^{-1}(\frac{\varepsilon}{2})$. Let $u(t) = ue^{ith}$. Then

$$\begin{aligned}\|u(t)-u(s)\| &= \|u(t)(1-e^{i(s-t)h})\| = \|1-e^{i(s-t)h}\| \\ &= 2\sin\frac{\|(s-t)h\|}{2} \leq 2\sin\frac{\|h\|}{2} = \varepsilon.\end{aligned}$$

□

Define $\mathfrak{B}_\varepsilon$ to be the universal C^*-algebra generated by unitaries u_n for $n \in \mathbb{Z}$ such that $\|u_{n+1} - u_n\| \leq \varepsilon$ for all n. Let z be the canonical generator of the C^*-algebra $C(\mathbb{T})$ of all continuous functions on the 1-torus $\mathbb{T}$. Let ψ_ε be the $*$-homomorphism from $\mathfrak{B}_\varepsilon$ onto $C(\mathbb{T})$ by universality such that $\psi_\varepsilon(u_n) = z$ for all n.

Theorem 1.1.2. *The map ψ_ε for $\varepsilon < 2$ is a homotopy equivalence between $\mathfrak{B}_\varepsilon$ and $C(\mathbb{T})$.*

Proof. Let $\sigma : C(\mathbb{T}) \to \mathfrak{B}_\varepsilon$ be given by $\sigma(z) = u_0$. Then $\psi_\varepsilon \circ \sigma$ is the identity map of $C(\mathbb{T})$.

For any interger $p \geq 0$, let $u_p^+(t)$ and $u_p^-(t)$ be continuous paths of unitaries in $\mathfrak{B}_\varepsilon$ such that $u_p^\pm(0) = u_{\pm p}$ and $u_p^\pm(1) = u_{\pm(p+1)}$. In particular, $\|u_p^\pm(t) - u_\pm\| \leq \varepsilon$ for all $t \in [0,1]$.

Concatenating these paths and by reparametrization we obtain continuous paths $\gamma^+(t)$ and $\gamma^-(t)$ for $0 \leq t < 1$ such that $\gamma^\pm(p/(p+1)) = u_{\pm p}$, and furthermore, if $p/(p+1) \leq t \leq (p+1)/(p+2)$, then $\|\gamma^\pm(t) - u_{\pm p}\| \leq \varepsilon$.

Define a continuous path v_n in $\mathfrak{B}_\varepsilon$ by

$$v_n(t) = \begin{cases} \gamma^{\mathrm{sgn}(n)}(t) & 0 \le t < \frac{|n|}{|n|+1}, \\ u_n & \frac{|n|}{|n|+1} \le t \le 1 \end{cases}$$

where $\mathrm{sgn}(n) = 1$ if $n > 0$, and $\mathrm{sgn}(n) = -1$ if $n < 0$, and $v_0(t) = u_0$. It follows that $\|v_{n+1}(t) - v_n(t)\| \le \varepsilon$ for all t and n. Let ρ_t be the endomorphism of $\mathfrak{B}_\varepsilon$ such that $\rho_t(u_n) = v_n(t)$. This yields a homotopy from $\sigma \circ \psi_\varepsilon$ to the identity map of $\mathfrak{B}_\varepsilon$. □

Define α to be the automorphism of $\mathfrak{B}_\varepsilon$ defined by $\alpha(u_n) = u_{n+1}$. Let $\mathfrak{B}_\varepsilon \rtimes_\alpha \mathbb{Z}$ be the crossed product C^*-algebra corresponding to the C^*-dynamical system $(\mathfrak{B}_\varepsilon, \alpha, \mathbb{Z})$.

Proposition 1.1.3. *The soft torus $C(\mathbb{T}^2)_\varepsilon$ is isomorphic to the crossed product $\mathfrak{B}_\varepsilon \rtimes_\alpha \mathbb{Z}$.*

Proof. Let w be the unitary of $\mathfrak{B}_\varepsilon \rtimes_\alpha \mathbb{Z}$ corresponding to the action α such that $wu_nw^* = u_{n+1}$. Then

$$\begin{aligned} \|u_0w - wu_0\| &= \|(u_0w - wu_0)w^*\| \\ &= \|u_0 - wu_0w^*\| = \|u_0 - u_1\| \le \varepsilon. \end{aligned}$$

Therefore, there is a $*$-homomorphism $\varphi : C(\mathbb{T}^2)_\varepsilon \to \mathfrak{B}_\varepsilon \rtimes_\alpha \mathbb{Z}$ such that $\varphi(u_\varepsilon) = u_0$ and $\varphi(v_\varepsilon) = w$.

On the other hand,

$$\begin{aligned} \|v_\varepsilon^{n+1}u_\varepsilon v_\varepsilon^{-(n+1)} - v_\varepsilon^n u_\varepsilon v_\varepsilon^{-n}\| &= \|v_\varepsilon^n(v_\varepsilon u_\varepsilon v_\varepsilon^{-1} - u_\varepsilon)v_\varepsilon^{-n}\| \\ &= \|v_\varepsilon u_\varepsilon v_\varepsilon^{-1} - u_\varepsilon\| \le \varepsilon. \end{aligned}$$

Thus, there is a $*$-homomorphism $\psi : \mathfrak{B}_\varepsilon \to C(\mathbb{T}^2)_\varepsilon$ such that $\psi(u_n) = v_\varepsilon^n u_\varepsilon v_\varepsilon^{-n}$. Since $v_\varepsilon \psi(u_n) v_\varepsilon^* = \psi(u_{n+1})$ one can extend ψ to $\mathfrak{B}_\varepsilon \rtimes_\alpha \mathbb{Z}$ by setting $\psi(w) = v_\varepsilon$.

By construction, φ and ψ are inverses each other. □

Let $\varphi_\varepsilon : C(\mathbb{T}^2)_\varepsilon \to C(\mathbb{T}^2)$ be the $*$-homomorphism defined by $\varphi_\varepsilon(u_\varepsilon) = z_1$ and $\varphi_\varepsilon(v_\varepsilon) = z_2$, where z_1 and z_2 are the canonical generators of $C(\mathbb{T}^2)$.

Theorem 1.1.4. *We have K-theory group isomorphisms from $K_j(C(\mathbb{T}^2)_\varepsilon)$ to $K_j(C(\mathbb{T}^2))$ induced by φ_ε, where $j = 0, 1$ and $\varepsilon < 2$.*

Proof. Regard $C(\mathbb{T}^2)$ as $C(\mathbb{T}) \rtimes_{\mathrm{id}} \mathbb{Z}$ the crossed product with the identity action id. Note that there is a $*$-homomorphism from $\mathfrak{B}_\varepsilon \rtimes_\alpha \mathbb{Z}$ to $C(\mathbb{T}) \rtimes_{\mathrm{id}} \mathbb{Z}$ by extending $\psi_\varepsilon : \mathfrak{B}_\varepsilon \to C(\mathbb{T})$. Identify $\mathfrak{B}_\varepsilon \rtimes_\alpha \mathbb{Z}$ with $C(\mathbb{T}^2)_\varepsilon$.

The Pimsner-Voiculescu exact sequence of K-groups for crossed products by $\mathbb{Z}$ implies the following diagram:

$$\begin{array}{ccccc} K_0(\mathfrak{B}_\varepsilon) & \xrightarrow{\mathrm{id}-\alpha_*} & K_0(\mathfrak{B}_\varepsilon) & \longrightarrow & K_0(C(\mathbb{T}^2)_\varepsilon) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1(C(\mathbb{T}^2)_\varepsilon) & \longleftarrow & K_1(\mathfrak{B}_\varepsilon) & \xleftarrow{\mathrm{id}-\alpha_*} & K_1(\mathfrak{B}_\varepsilon) \end{array}$$

and the similar diagram for $C(\mathbb{T}) \rtimes_{\mathrm{id}} \mathbb{Z}$. Since the maps $\mathrm{id} - \alpha_*$ both vanish, the above diagram splits into the following short exact sequences to make another diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_j(\mathfrak{B}_\varepsilon) & \longrightarrow & K_j(C(\mathbb{T}^2)_\varepsilon) & \xrightarrow{\partial} & K_{j+1}(\mathfrak{B}_\varepsilon) & \longrightarrow & 0 \\ \| & & \downarrow\psi_{\varepsilon,*} & & \downarrow\varphi_{\varepsilon,*} & & \downarrow\psi_{\varepsilon,*} & & \| \\ 0 & \longrightarrow & K_j(C(\mathbb{T})) & \longrightarrow & K_j(C(\mathbb{T}^2)) & \xrightarrow{\partial} & K_{j+1}(C(\mathbb{T})) & \longrightarrow & 0 \end{array}$$

where $j+1$ mod 2. Homotopy invariance of K-groups and the five lemma completes the proof. □

Corollary 1.1.5. *We have $K_j(C(\mathbb{T}^2)_\varepsilon) \cong \mathbb{Z}^2$ where $j = 0, 1$ and $\varepsilon < 2$.*

1.2 Invariants for almost commuting unitaries

Let $\mathfrak{A}$ be a C^*-algebra. Suppose that u and v are unitaries of $\mathfrak{A}$ such that $\|uv - vu\| \le \varepsilon < 2$. There is a $*$-homomorphism $\rho : C(\mathbb{T}^2)_\varepsilon \to \mathfrak{A}$ such that $\rho(u_\varepsilon) = u$ and $\rho(v_\varepsilon) = v$.

The K-theory invariant of such a pair (u, v) is defined to be the element of $K_0(\mathfrak{A})$ defined by $k(u, v) = \rho_*(b_\varepsilon)$, where $b_\varepsilon \in K_0(C(\mathbb{T}^2)_\varepsilon)$ is defined by $b_\varepsilon = \varphi_{\varepsilon,*}^{-1}(b)$, where $b \in K_0(C(\mathbb{T}^2))$ is the class of the Bott projection of $M_2(C(\mathbb{T}^2))$. Note that $k(u, v)$ does not depend on ε if $\varepsilon \ge \|uv - vu\|$.

Let $\mathfrak{A} = M_n(\mathbb{C})$ be the C^*-algebra of all $n \times n$ complex matrices. Since $K_0(\mathfrak{A}) \cong \mathbb{Z}$ induced by the standard trace Tr on $M_n(\mathbb{C})$, we identify $k(u, v)$ with its image under Tr_*.

Let u and v be unitaries of $M_n(\mathbb{C})$. Define $w(u, v)$ the winding invariant of the pair (u, v) to be the winding number of the following closed complex path:

$$\gamma(t) = \det(tvu + (1-t)uv)$$

around zero.

Lemma 1.2.1. *Let u and v be unitaries of $M_n(\mathbb{C})$ with $\|uv - vu\| = \varepsilon < 2$. Then*

$$w(u,v) = \frac{1}{2\pi i}\mathrm{Tr}(\log(vuv^*u^*)),$$

where Tr *denotes the standard trace.*

Proof. Since $\|1 - vuv^*u^*\| = \varepsilon < 2$, the spectrum of vuv^*u^* does not contain -1. Therefore, let $h = i^{-1}\log(vuv^*u^*)$ by spectral theory. Then $\|h\| = 2\arcsin(\frac{\varepsilon}{2})$. Indeed,

$$\begin{aligned}\varepsilon = \|1 - vuv^*u^*\| &= \|1 - e^{\log(vuv^*u^*)}\| \\ &= \|1 - e^{ih}\| = \|2\sin(\frac{h}{2})\| = 2\sin(\frac{\|h\|}{2})\end{aligned}$$

since $\|h\| = \|\log(vuv^*u^*)\| \leq \pi$.

Claim that the following continuous paths:

$$t \mapsto e^{ith}uv \quad \text{and} \quad t \mapsto tvu + (1-t)uv$$

are homotopic in $GL_n(\mathbb{C})$. Indeed,

$$\begin{aligned}&\|tvu + (1-t)uv - e^{ith}uv\| \\ &= \|tvuv^*u^* + (1-t) - e^{ith}\| \\ &= \|te^{ih} + (1-t) - e^{ith}\| \leq \|te^{i\theta} + (1-t) - e^{it\theta}\|_\infty\end{aligned}$$

by spectral theory, where $e^{i\theta}$ is in the spectrum of e^{ih}. Furthermore,

$$\begin{aligned}|te^{i\theta} + (1-t) - e^{it\theta}| &\leq 1 - \sqrt{1 - (\frac{\varepsilon}{2})^2} \\ &= 1 - \sqrt{1 - \sin^2(\frac{\|h\|}{2})} = 1 - \cos\frac{\|h\|}{2} < 1\end{aligned}$$

for all $t \in [0,1]$ and $e^{i\theta}$. In fact, we view $te^{i\theta} + (1-t)$ as the point divided internally on the line segment between $e^{i\theta}$ and 1 with ratio $1-t : t$, so that we need to estimate the distance between that point and $(e^{i\theta})^t$ as the above inequality. This task should be done by using elementary geometry (but it seems to involve a somewhat complicate estimate to complete it analytically, and see also the remark below). It follows from this estimate that $tvu+(1-t)uv$ are invertible since $e^{ith}uv$ are unitary. Also, the estimate obtained implies that the continuous paths are homotopic in $GL_n(\mathbb{C})$.

Therefore, the winding number $w(u,v)$ of $\gamma(t)$ is equal to the winding number of the path: $t \mapsto \det(e^{ith}uv)$. Since $\det(e^{ith}uv) = \det(e^{ith})\det(uv)$, the winding number is the same as the winding number of the path: $t \mapsto e^{ith}$. Since h is hermitian, there is a unitary p such that p^*hp is diagonal, so let

$$p^*hp = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

It follows that

$$\begin{aligned} \det(e^{ith}) &= \det(p^* e^{ith} p) \\ &= \det(e^{itp^*hp}) \\ &= \det \begin{pmatrix} e^{it\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{it\lambda_n} \end{pmatrix} \\ &= e^{it(\lambda_1 + \cdots + \lambda_n)} \\ &= e^{it\mathrm{Tr}(h)} \end{aligned}$$

which implies that the winding number of the path: $t \mapsto e^{ith}$ is equal to the following

$$\frac{\mathrm{Tr}(h)}{2\pi} = \frac{1}{2\pi i}\mathrm{Tr}(\log(vuv^*u^*))$$

as desired. □

Remark. The estimate:

$$|te^{i\theta} + (1-t) - e^{it\theta}| \leq 1 - \cos\frac{\|h\|}{2}$$

is attained if $t = 1/2$ and $\theta = \|h\|$. Indeed,

$$\begin{aligned} |\frac{1}{2}e^{i\|h\|} + \frac{1}{2} - e^{i\|h\|/2}| &= |\frac{1}{2}(1 + \cos\|h\| + i\sin\|h\|) - (\cos\frac{\|h\|}{2} + i\sin\frac{\|h\|}{2})| \\ &= |\frac{1+\cos\|h\|}{2} - \cos\frac{\|h\|}{2} + i(\frac{\sin\|h\|}{2} - \sin\frac{\|h\|}{2})| \\ &= |\cos\frac{\|h\|}{2}(\cos\frac{\|h\|}{2} - 1) + i\sin\frac{\|h\|}{2}(\cos\frac{\|h\|}{2} - 1)| \\ &= |\cos\frac{\|h\|}{2} + i\sin\frac{\|h\|}{2}||\cos\frac{\|h\|}{2} - 1| \\ &= |\cos\frac{\|h\|}{2} - 1| = 1 - \cos\frac{\|h\|}{2}. \end{aligned}$$

Lemma 1.2.2. *Let α be an automorphism of a C^*-algebra. Let τ^1 and τ^2 be traces on $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ such that $\tau^1 = \tau^2$ on $\mathfrak{A}$. Then $\tau^1_* = \tau^2_*$ on $K_0(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$.*

Proof. Without loss of generality we may assume that $\mathfrak{A}$ is unital. Let p be a projection of $M_k(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$ for some k. It suffices to prove that $\tau^1(p) = \tau^2(p)$.

Without loss of generality we may assume that $k = 1$. Let $\alpha^\wedge$ be the dual action of α , i.e., an action of $\mathbb{T}$ on $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ defined by $\alpha^\wedge_z(a) = a$ for $a \in \mathfrak{A}$ and $\alpha^\wedge_z(U) = zU$ for any $z \in \mathbb{T}$, where U is the unitary corresponding to the action α. Observe that $\tau^j(\alpha^\wedge_z(p))$ does not depend on $z \in \mathbb{T}$ via trace property since close projections are (unitarily) equivalent. Thus,

$$\tau^j(p) = \int_{\mathbb{T}} \tau^j(\alpha^\wedge_z(p))d\mu(z) = \tau^j(E(p)),$$

where E is the canonical conditional expectation from $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ to $\mathfrak{A}$ defined by $E(f) = \int_{\mathbb{T}} \alpha^\wedge_z(f)d\mu(z)$ for $f \in \mathfrak{A} \rtimes_\alpha \mathbb{Z}$, where $d\mu(z) = (2\pi i)^{-1}z^{-1}dz$ is the normalized Lebesgue measure on $\mathbb{T}$. Hence $\tau^1(p) = \tau^2(p)$. □

Remark. The conditional expectation E defined above is a positive, unital, idempotent map and satisfies the following:

$$E(af) = aE(f), \quad E(fb) = E(f)b$$

for $a, b \in \mathfrak{A}$ and $f \in \mathfrak{A} \rtimes_\alpha \mathbb{Z}$. Indeed,

$$\begin{aligned} E(afb) &= \int_{\mathbb{T}} \alpha^\wedge_z(afb)d\mu(z) \\ &= a(\int_{\mathbb{T}} \alpha^\wedge_z(f)d\mu(z))b = aE(f)b. \end{aligned}$$

Also, for every $a \in \mathfrak{A}$,

$$\begin{aligned} E(a) &= \int_{\mathbb{T}} \alpha^\wedge_z(a)d\mu(z) = a\frac{1}{2\pi i}\int_{\mathbb{T}} z^{-1}dz \\ &= a\frac{1}{2\pi i}\int_0^1 e^{-2\pi i t}2\pi i e^{2\pi i t}dt = a[t]_0^1 = a. \end{aligned}$$

Hence $E^2 = E$, i.e., an idempotent map. Note that $\mathfrak{A}$ is just the fixed point algebra under the dual action $\alpha^\wedge$. Furthermore, if $k \neq 0$, then

$$\begin{aligned} E(U^k) &= \int_{\mathbb{T}} \alpha^\wedge_z(U^k)d\mu(z) = U^k\frac{1}{2\pi i}\int_{\mathbb{T}} z^{k-1}dz \\ &= U^k\frac{1}{2\pi i}\int_0^1 e^{2\pi i k t}2\pi i dt = U^k[\frac{e^{2\pi i k t}}{2\pi i k}]_0^1 = 0. \end{aligned}$$

It follows that for a finite sum $\sum_k a_k U^k \in \mathfrak{A} \rtimes_\alpha \mathbb{Z}$ with $a_k \in \mathfrak{A}$,

$$E(\sum_k a_k U^k) = \sum_k a_k E(U^k) = a_0.$$

Theorem 1.2.3. *Let u and v be unitaries of $M_n(\mathbb{C})$ with $\|uv - vu\| = \varepsilon < 2$. Then the K-theory invariant $k(u,v)$ is equal to the winding number $w(u,v)$.*

Proof. Let $\rho : C(\mathbb{T}^2)_\varepsilon \to M_n(\mathbb{C})$ be a $*$-homomorphism such that $\rho(u_\varepsilon) = u$ and $\rho(v_\varepsilon) = v$. Define a unital trace τ on $C(\mathbb{T}^2)_\varepsilon$ by $\tau = n^{-1}\mathrm{Tr} \circ \rho$.

Identify $\mathfrak{B}_\varepsilon$ with a subalgebra of $C(\mathbb{T}^2)_\varepsilon$. By restriction, τ is an α-invariant trace on $\mathfrak{B}_\varepsilon$. Note that τ is an integral trace on $\mathfrak{B}_\varepsilon$.

Let $\tau^\sim$ be the canonical extension of τ on $\mathfrak{B}_\varepsilon$ to $C(\mathbb{T}^2)_\varepsilon$. Then $\tau^\sim$ is often different from τ on $C(\mathbb{T}^2)_\varepsilon$. As shown above, $\tau_* = \tau_*^\sim$ on $K_0(C(\mathbb{T}^2)_\varepsilon)$.

We have

$$\tau_*^\sim(b_\varepsilon) = \tau_*(b_\varepsilon) = \frac{1}{n}\mathrm{Tr}_*\rho_*(b_\varepsilon) = \frac{1}{n}k(u,v),$$

$b_\varepsilon \in K_0(C(\mathbb{T}^2)_\varepsilon)$ that corresponds to the class of $K_0(C(\mathbb{T}^2))$ for the Bott projection.

The commutative diagram of Exel for an integral unital C^*-algebra with a trace-preserving automorphism α is:

$$\begin{array}{ccc} K_0(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) & \xrightarrow{\ \tau\ } & \mathbb{R} \\ \partial \big\downarrow & & \big\downarrow e^{2\pi i(\cdot)} \\ K_1(\mathfrak{A})^\alpha & \xrightarrow{\ \rho_\alpha^\tau\ } & \mathbb{T} \end{array}$$

where the connecting map ∂ is of the Pimsner-Voiculescu sequence, and for $[p] \in K_0(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$ with $\partial([p]) = [u]$, we have

$$\exp(2\pi i \tau_*([p])) = \rho_\alpha^\tau([u]) = \det_\tau(\alpha(u^*)u)$$

where the second equality is the definition of the rotation number map ρ_α^τ for $[u] \in K_1(\mathfrak{A})^\alpha$ the subgroup of fixed points under α_*. i.e., $\alpha_*([u]) = [u] \in K_1(\mathfrak{A})$, where $\det_\tau$ is a group homomorphism from the group of unitaries of $\mathfrak{A}$ to $\mathbb{T}$ satisfying $\det(e^{ih}) = e^{i\tau(h)}$, where h is a self-adjoint element of $\mathfrak{A}$.

Applying that diagram to the those equations above we obtain

$$\exp(\frac{2\pi i}{n}k(u,v)) = \det_\tau(\alpha(u_0)u_0^*),$$

where we identify $C(\mathbb{T}^2)_\varepsilon$ with $\mathfrak{B}_\varepsilon \rtimes_\alpha \mathbb{Z}$, $K_0(C(\mathbb{T}^2)_\varepsilon)$ with $K_0(C(\mathbb{T}^2))$, and $K_0(\mathfrak{B}_\varepsilon)$ with $K_0(C(\mathbb{T}))$, and the Bott element of $K_0(C(\mathbb{T}^2))$ is mappped to the class of $z^* \in C(\mathbb{T})$ under the connecting map ∂ from $K_0(C(\mathbb{T}^2))$ to $K_1(C(\mathbb{T}))$, so that $\partial(b_\varepsilon) = [u_0^*] \in K_1(\mathfrak{B}_\varepsilon)$.

On the other hand,

$$\alpha(u_0)u_0^* = u_1u_0^* = \exp(\log(u_1u_0^*)) = \exp(ii^{-1}\log(u_1u_0^*)).$$

Therefore,

$$\det_\tau(\alpha(u_0)u_0^*) = \exp(i\tau(i^{-1}\log(u_1u_0^*))) = \exp(\tau(\log(u_1u_0^*))).$$

Furthermore,

$$\begin{aligned}\tau(\log(u_1u_0^*)) &= \frac{1}{n}(\mathrm{Tr}\circ\rho)(\log(u_1u_0^*)) = \frac{1}{n}\mathrm{Tr}(\log(\rho(u_1u_0^*))) \\ &= \frac{1}{n}\mathrm{Tr}(\log(\rho((v_\varepsilon u_\varepsilon v_\varepsilon^*)u_\varepsilon^*))) \\ &= \frac{1}{n}\mathrm{Tr}(\log(vuv^*u^*)) = \frac{2\pi i}{n}w(u,v).\end{aligned}$$

Hence we obtain

$$\exp(\frac{2\pi i}{n}k(u,v)) = \exp(\frac{2\pi i}{n}w(u,v))$$

so that $(k(u,v) - w(u,v))/n \in \mathbb{Z}$.

Replace u and v by $u\oplus 1_m$ and $v\oplus 1_m$ respectively, where 1_m is the $m\times m$ identity matrix. Then note that $k(u,v) = k(u\oplus 1_m, v\oplus 1_m)$ and $w(u,v) = w(u\oplus 1_m, v\oplus 1_m)$. Thus, it follows that $(k(u,v) - w(u,v))/(n+m) \in \mathbb{Z}$ for all m. Hence $k(u,v) = w(u,v)$.

Theorem 1.2.4. *The unital traces of $C(\mathbb{T}^2)_\varepsilon$ form a separating family of maps for its K_0-group.*

Proof. Recall that Voiculescu's unitaries S_n and Ω_n for $n \geq 2$ are defined by

$$S_n = \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}, \quad \Omega_n = \begin{pmatrix} \omega_n & & & 0 \\ & \omega_n^2 & & \\ & & \ddots & \\ 0 & & & \omega_n^n \end{pmatrix}$$

where $\omega_n = e^{2\pi i/n}$. Compute:

$$S_n\Omega_n - \Omega_n S_n = \begin{pmatrix} 0 & & & \omega_n^n \\ \omega_n & 0 & & \\ & \ddots & \ddots & \\ 0 & & \omega_n^{n-1} & 0 \end{pmatrix} - \begin{pmatrix} 0 & & & \omega_n \\ \omega_n^2 & 0 & & \\ & \ddots & \ddots & \\ 0 & & \omega_n^n & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & & \omega_n(\omega_n^{n-1} - 1) \\ \omega_n(1-\omega_n) & 0 & & \\ & \ddots & \ddots & \\ 0 & & \omega_n^{n-1}(1-\omega_n) & 0 \end{pmatrix}$$

It follows that $\|S_n\Omega_n - \Omega_n S_n\|$ tends to zero as n tends to infinity.

Given $0 < \varepsilon < 2$, let n_0 be such that $\|S_n\Omega_n - \Omega_n S_n\| \leq \varepsilon$ whenever $n \geq n_0$. For each $n \geq n_0$, let $\rho_n : C(\mathbb{T}^2)_\varepsilon \to M_n(\mathbb{C})$ be a $*$-homomorphism defined by $\rho_n(u_\varepsilon) = S_n$ and $\rho_n(v_\varepsilon) = \Omega_n$. Let τ_n be the unital trace on $C(\mathbb{T}^2)_\varepsilon$ given by $\tau_n = n^{-1}\mathrm{Tr} \circ \rho_n$.

Claim that the set $\{\tau_{n,*} : n \geq n_0\}$ is a separating family for $K_0(C(\mathbb{T}^2)_\varepsilon)$. In fact, note that $K_0(C(\mathbb{T}^2)_\varepsilon)$ is generated by b_ε and $[1]$. We have $\tau_{n,*}([1]) = 1$, while

$$\tau_{n,*}(b_\varepsilon) = \frac{1}{n}k(S_n, \Omega_n) = \frac{1}{n}w(S_n, \Omega_n) = \frac{1}{n}.$$

Indeed, the last equality holds as follows:

$$\Omega_n S_n \Omega_n^* S_n^*$$

$$= \begin{pmatrix} 0 & & & \omega_n \\ \omega_n^2 & 0 & & \\ & \ddots & \ddots & \\ 0 & & \omega_n^n & 0 \end{pmatrix} \begin{pmatrix} \bar{\omega}_n & & & 0 \\ & \bar{\omega}_n^2 & & \\ & & \ddots & \\ 0 & & & \bar{\omega}_n^n \end{pmatrix} \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & & \omega_n \\ \omega_n^2 & 0 & & \\ & \ddots & \ddots & \\ 0 & & \omega_n^n & 0 \end{pmatrix} \begin{pmatrix} 0 & \bar{\omega}_n & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \bar{\omega}_n^{n-1} \\ \bar{\omega}_n^n & & 0 & \end{pmatrix}$$

$$= \begin{pmatrix} \omega_n & & & 0 \\ & \omega_n & & \\ & & \ddots & \\ 0 & & & \omega_n \end{pmatrix} \equiv \oplus^n \omega_n$$

which implies that $\log(\Omega_n S_n \Omega_n^* S_n^*) = \oplus^n \log \omega_n$, from which it follows that

$$\begin{aligned} w(S_n, \Omega_n) &= \frac{1}{2\pi i}\mathrm{Tr}(\log(\Omega_n S_n \Omega_n^* S_n^*)) \\ &= \frac{1}{2\pi i}\mathrm{Tr}(\oplus^n \log \omega_n) \\ &= \frac{1}{2\pi i}\sum_n \log e^{2\pi i/n} \\ &= \frac{1}{2\pi i}\sum_n \frac{2\pi i}{n} = 1. \end{aligned}$$

The conclusion now follows from observing:

$$\tau_{n,*}(s[1] + tb_\varepsilon) = s + \frac{t}{n}$$

for $s, t \in \mathbb{Z}$. Indeed, $s + \frac{t}{n} = s' + \frac{t'}{n}$ if and only if $r - r' = \frac{1}{n}(s' - s)$. However, if n is large enough, always $r - r' \neq \frac{1}{n}(s' - s)$. □

Notes. This section of two subsections is based on the paper [9] of Exel. In [22] of the author, a version of the soft torus by replacing almost commuting unitaries with almost commuting isometries has been considered. Also, almost commuting unitary operators as well as almost commuting unitary matrices and their invariants have been of interest. Its interesting history is omitted. Another topic on almost commuting self-adjoint operators is also not contained in this review.

2 Continuous fields of Soft tori

2.1 Soft tori as continuous fields

For $\varepsilon \in [0, 2]$, let $C(\mathbb{T}^2)_\varepsilon$ be the soft torus that is the universal C^*-algebra generated by unitary elements u_ε and v_ε such that $\|u_\varepsilon v_\varepsilon - v_\varepsilon u_\varepsilon\| \leq \varepsilon$.

By universality, if $\varepsilon_1 \leq \varepsilon_2$, there is a $*$-homomorphism $\phi_{\varepsilon_2,\varepsilon_1}$ from $C(\mathbb{T}^2)_{\varepsilon_2}$ to $C(\mathbb{T}^2)_{\varepsilon_1}$ sending the generators of $C(\mathbb{T}^2)_{\varepsilon_2}$ to those of $C(\mathbb{T}^2)_{\varepsilon_1}$. In particular, in the case where $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = 2$, denote by $\phi_\varepsilon : C(\mathbb{T}^2)_2 \to C(\mathbb{T}^2)_\varepsilon$ such a map. Let $\mathfrak{I}_\varepsilon = \ker(\phi_\varepsilon)$ be the kernel of ϕ_ε, which is a closed two-sided ideal of $C(\mathbb{T}^2)_\varepsilon$. Since $C(\mathbb{T}^2)_2/\mathfrak{I}_\varepsilon \cong C(\mathbb{T}^2)_\varepsilon$ we have

$$\|\phi_\varepsilon(a)\| = \inf_{b\in\mathfrak{I}_\varepsilon} \|a - b\| = \mathrm{dist}(a, \mathfrak{I}_\varepsilon)$$

for $a \in C(\mathbb{T}^2)_2$, where the first equality is the definition of the quotient norm and the second is also the definition of the distance between a and $\mathfrak{I}_\varepsilon$.

If $\varepsilon < \varepsilon'$, then $\mathfrak{I}_\varepsilon \supset \mathfrak{I}_{\varepsilon'}$ since we have the following commutative diagram:

$$\begin{array}{ccc} C(\mathbb{T}^2)_2 & \xrightarrow{\phi_{\varepsilon'}} & C(\mathbb{T}^2)_{\varepsilon'} \\ \| & & \downarrow \phi_{\varepsilon',\varepsilon} \\ C(\mathbb{T}^2)_2 & \xrightarrow{\phi_\varepsilon} & C(\mathbb{T}^2)_\varepsilon \end{array}$$

by uniqueness of those maps, so that $\|\phi_\varepsilon(a)\| \leq \|\phi_{\varepsilon'}(a)\|$.

Denote by $\mathfrak{I}_\varepsilon^+$ the closure of the union $\cup_{\varepsilon<\varepsilon'}\mathfrak{I}_{\varepsilon'}$ of $\mathfrak{I}_{\varepsilon'}$ for $\varepsilon' > \varepsilon$, and by $\mathfrak{I}_\varepsilon^-$ the intersection $\cap_{\varepsilon'<\varepsilon}\mathfrak{I}_{\varepsilon'}$ of $\mathfrak{I}_{\varepsilon'}$ for $\varepsilon' < \varepsilon$. Note that

$$\mathfrak{I}_{\varepsilon'} \supset \mathfrak{I}_\varepsilon^- \supset \mathfrak{I}_\varepsilon \supset \mathfrak{I}_\varepsilon^+ \supset \mathfrak{I}_{\varepsilon''}$$

where $\varepsilon' < \varepsilon < \varepsilon''$.

Proposition 2.1.1. *Let $\varepsilon \in [0,2)$. If $\mathfrak{I}_\varepsilon = \mathfrak{I}_\varepsilon^+$, then the function defined by $f_a(\varepsilon) = \|\phi_\varepsilon(a)\|$ for $a \in C(\mathbb{T}^2)_2$ is right continuous at ε.*

Let $\varepsilon \in (0,2]$. If $\mathfrak{I}_\varepsilon = \mathfrak{I}_\varepsilon^-$, then the same function is left continuous at ε.

Proof. Note that for $a \in C(\mathbb{T}^2)_2$, we have

$$\mathrm{dist}(a, \mathfrak{I}_\varepsilon^+) = \inf_{\varepsilon<\varepsilon''} \mathrm{dist}(a, \mathfrak{I}_{\varepsilon''}), \quad \mathrm{dist}(a, \mathfrak{I}_\varepsilon^-) = \sup_{\varepsilon'<\varepsilon} \mathrm{dist}(a, \mathfrak{I}_{\varepsilon'}).$$

Indeed, since $\mathfrak{I}_\varepsilon^+ \supset \mathfrak{I}_{\varepsilon''}$, we have $\mathrm{dist}(a, \mathfrak{I}_\varepsilon^+) \leq \mathrm{dist}(a, \mathfrak{I}_{\varepsilon''})$. Thus,

$$\mathrm{dist}(a, \mathfrak{I}_\varepsilon^+) \leq \inf_{\varepsilon<\varepsilon''} \mathrm{dist}(a, \mathfrak{I}_{\varepsilon''}).$$

Conversely, let $b \in \mathfrak{I}_\varepsilon^+$. Then for any $k > 0$ there exists $c \in \mathfrak{I}_{\varepsilon''}$ with $\varepsilon < \varepsilon''$ such that $\|b - c\| < k^{-1}$. Then

$$\inf_{\varepsilon<\varepsilon''} \mathrm{dist}(a, \mathfrak{I}_{\varepsilon''}) \leq \mathrm{dist}(a, \mathfrak{I}_{\varepsilon''}) \leq \|a - c\| \leq \|a - b\| + \frac{1}{k}.$$

Hence $\inf_{\varepsilon<\varepsilon''} \mathrm{dist}(a, \mathfrak{I}_{\varepsilon''}) \leq \mathrm{dist}(a, \mathfrak{I}_\varepsilon^+)$ as $k \to \infty$.

To prove the second, let $\phi : C(\mathbb{T}^2)_2 \to \Pi_{\varepsilon'<\varepsilon} C(\mathbb{T}^2)_{\varepsilon'}$ be given by $\phi(a) = (\phi_{\varepsilon'}(a))_{\varepsilon'<\varepsilon}$. Observe that $\mathfrak{I}_\varepsilon^- = \ker(\phi)$ the kernel of ϕ. In fact, note that $\mathfrak{I}_{\varepsilon'} = \ker(\phi_{\varepsilon'})$, and $\phi(a) = 0$ if and only if $\phi_{\varepsilon'}(a) = 0$ for every $\varepsilon' < \varepsilon$. Thus,

$$\mathrm{dist}(a, \mathfrak{I}_\varepsilon^-) = \|\phi(a)\| = \sup_{\varepsilon'<\varepsilon} \|\phi_{\varepsilon'}(a)\| = \sup_{\varepsilon'<\varepsilon} \mathrm{dist}(a, \mathfrak{I}_{\varepsilon'}).$$

The conclusions follow from those equalities and the assumptions respectively. □

Remark. Now let $E = C([0,1])$. For $f \in E$, set

$$\|f\| = |f(1)| + \sup_{t\in[0,1]} |f(t)|.$$

Let $E_\varepsilon = \{f \in E \mid f([0,1-\varepsilon]) = 0\}$ for $0 \le \varepsilon \le 1$. If $\varepsilon' < \varepsilon$, then $E_{\varepsilon'} \supset E_\varepsilon$, so that E_ε are decreasing closed ideals as ε increases. Let $g = 1$ be the constant function. Then $\mathrm{dist}(g, E_\varepsilon) = 1$ for all $0 \le \varepsilon < 1$. Indeed, define $g_\varepsilon \in E_\varepsilon$ by $g_\varepsilon(t) = 0$ for $t \in [0, 1-\varepsilon]$, $g_\varepsilon(t) = 1$ for $t \in [1-(\varepsilon/2), 1]$, and $g_\varepsilon(t) = 2\varepsilon^{-1}(t - 1 + \varepsilon)$ for $t \in [1-\varepsilon, 1-(\varepsilon/2)]$. Then $\|g - g_\varepsilon\| = 1$. Also, for any $f \in E_\varepsilon$, we have $\|g - f\| \ge |g(0) - f(0)| = 1$.

On the other hand, note that $E_\varepsilon^- = \cap_{\varepsilon'<\varepsilon} E_{\varepsilon'}$ is equal to E_ε for any $\varepsilon \in (0,1]$, but we have $\mathrm{dist}(g, \cap_{\varepsilon'<1} E_{\varepsilon'}) = \|g\| = 2$ since $\cap_{\varepsilon'<1} E_{\varepsilon'} = \{0\}$. It follows that

$$\mathrm{dist}(g, E_1^-) = \|g\| = 2 \ne 1 = \sup_{\varepsilon'<1} \mathrm{dist}(g, E_{\varepsilon'}).$$

Hence the above statements can not be generalized to Banach spaces in general.

Proposition 2.1.2. *One has that $\mathfrak{I}_\varepsilon^+$ the closure of the union $\cup_{\varepsilon<\varepsilon'} \mathfrak{I}_{\varepsilon'}$ is equal to $\mathfrak{I}_\varepsilon$ for every $\varepsilon \in [0,2)$.*

Proof. Denote by u_ε^+ and v_ε^+ the images of u_2 and v_2 in $C(\mathbb{T}^2)_2/\mathfrak{I}_\varepsilon^+$. Then

$$\|u_\varepsilon^+ v_\varepsilon^+ - v_\varepsilon^+ u_\varepsilon^+\| \le \varepsilon.$$

Indeed, we have

$$\begin{aligned}\|u_\varepsilon^+ v_\varepsilon^+ - v_\varepsilon^+ u_\varepsilon^+\| &= \mathrm{dist}(u_2 v_2 - v_2 u_2, \mathfrak{I}_\varepsilon^+) \\ &\le \mathrm{dist}(u_2 v_2 - v_2 u_2, \mathfrak{I}_{\varepsilon'}) = \|u_{\varepsilon'} v_{\varepsilon'} - v_{\varepsilon'} u_{\varepsilon'}\| \le \varepsilon'\end{aligned}$$

for every $\varepsilon' > \varepsilon$.

Universality of $C(\mathbb{T}^2)_\varepsilon$ implies that there is a $*$-homomorphism from $C(\mathbb{T}^2)_\varepsilon \cong C(\mathbb{T}^2)/\mathfrak{I}_\varepsilon$ onto $C(\mathbb{T}^2)_2/\mathfrak{I}_\varepsilon^+$ sending u_ε and v_ε to u_ε^+ and v_ε^+ respectively. Therefore, $\mathfrak{I}_\varepsilon \subset \mathfrak{I}_\varepsilon^+$. Also, $\mathfrak{I}_\varepsilon$ contains $\mathfrak{I}_\varepsilon^+$ by definition. □

Now assume that $0 < \varepsilon < 2$ in the following:

Lemma 2.1.3. *Suppose that u_j $(0 \le j \le n)$ are unitaries of a C^*-algebra $\mathfrak{B}$ such that $\|u_{j-1} - u_j\| \le \varepsilon$ for $1 \le j \le n$. Then for every $\delta > 0$, there are unitaries v_j $(0 \le j \le n)$ of $\mathfrak{B}$ such that*

$$\|v_j - u_j\| \le \delta \quad (0 \le j \le n), \quad \|v_{j-1} - v_j\| < \varepsilon \quad (1 \le j \le n).$$

Proof. Since $\|u_{j-1} - u_j\| \le \varepsilon < 2$, it follows that $\|u_j u_{j-1}^* - 1\| < 2$, so that -1 is not in the spectrum of $u_j u_{j-1}^*$. Therefore, define $h_j = \log(u_j u_{j-1}^*)$, which is a skew-adjoint element of $\mathfrak{B}$, that is, $h_j^* = -h_j$, and we have $u_j = e^{h_j} u_{j-1}$ for $1 \le j \le n$.

Choose v_j $(0 \le j \le n)$ as follows: $v_0 = u_0$ and $v_j = e^{t_j h_j} u_{j-1}$ for $1 \le j \le n$, where each t_j is a suitably chosen positive real number approaching 1 from below.

Define
$$d(x) = |1 - e^{ix}| = 2|\sin(\frac{x}{2})|$$
for $x \in \mathbb{R}$, where the second equality is shown above. Also, if h is skew-adjoint and $\|h\| \le \pi$, then

$$\|1 - e^h\| = \|1 - e^{i(i^{-1}h)}\| = d(\|i^{-1}h\|) = d(\|h\|)$$

by the spectral theorem. Moreover, for $1 \le j \le n$,

$$d(\|h_j\|) = \|1 - e^{h_j}\| = \|u_{j-1} - u_j\| \le \varepsilon$$

which implies that
$$\|h_k\| \le \theta \equiv d^{-1}(\varepsilon) < \pi$$
because $\varepsilon < 2$, where d^{-1} is the inverse of d since d is increasing on $[0, \pi]$.

Observe that

$$\|v_0 - v_1\| = \|u_0 - e^{t_1 h_1} u_0\| = d(t_1 \|h_1\|) \le d(t_1 \theta)$$

and for $j \ge 1$ we have

$$\begin{aligned}
\|v_j - v_{j+1}\| &\le \|v_j - u_j\| + \|u_j - v_{j+1}\| \\
&= \|e^{t_k h_k} u_{k-1} - e^{h_k} u_{k-1}\| + \|u_k - e^{t_{k+1} h_{k+1}} u_k\| \\
&= \|e^{(t_k - 1) h_k} - 1\| + \|1 - e^{t_{k+1} h_{k+1}}\| \\
&= d((1 - t_k)\|h_k\|) + d(t_{k+1}\|h_{k+1}\|) \\
&\le d((1 - t_k)\theta) + d(t_{k+1}\theta).
\end{aligned}$$

Note that the derivative $d'(x) = \cos(x/2)$ for $x \in [0, \theta]$, and $1 \geq d'(x) \geq m \equiv \cos(\theta/2) > 0$. By the mean value theorem, for $t, (t >) s \in [0, \theta]$ we have $|d(t) - d(s)| = d'(s + \theta(t - s))|t - s|$ for some $0 < \theta < 1$, so that

$$m|t - s| \leq | d(t) - d(s)| \leq | t - s|.$$

Using this we obtain

$$\begin{aligned}\|v_0 - v_1\| &\leq d(t_1\theta) = \varepsilon - (d(\theta) - d(t_1\theta)) \\ &\leq \varepsilon - m(\theta - t_1\theta) = \varepsilon - m\theta(1 - t_1),\end{aligned}$$

while for $j \geq 1$,

$$\begin{aligned}\|v_j - v_{j+1}\| &\leq d((1 - t_j)\theta) + d(t_{j+1}\theta) \\ &\leq (1 - t_j)\theta + \varepsilon - (d(\theta) - d(t_{j+1}\theta)) \\ &\leq (1 - t_j)\theta + \varepsilon - m(\theta - t_{j+1}\theta) \\ &= \varepsilon + (1 - t_j - m(1 - t_{j+1}))\theta.\end{aligned}$$

Therefore, the condition that $\|v_{j-1} - v_j\| < \varepsilon$ $(1 \leq j \leq n)$ holds whenever $1 - t_1 > 0$ and $1 - t_{j+1} > (1 - t_j)/m$.

If we thus put $t_j = 1 - (2^j\sigma/m^j)$ for $\sigma < (m/2)^n (\leq (m/2)^j)$, then each $t_j \in (0, 1)$ and

$$1 - t_{j+1} = \frac{2^{j+1}\sigma}{m^{j+1}} = 2 \cdot \frac{1}{m} \cdot \frac{2^j\sigma}{m^j} > \frac{1 - t_j}{m}$$

As σ tends to zero,

$$\begin{aligned}\|v_j - u_j\| &= \|e^{t_j h_j} u_{j-1} - e^{h_j} u_{j-1}\| = \|e^{t_j h_j} - e^{h_j}\| \\ &= \|e^{(t_j - 1)h_j} - 1\| = \sup_x |e^{i(t_j - 1)x} - 1| \to 0\end{aligned}$$

where the last equality is by the spectral theorem and the supremum is taken over x in the spectrum of $i^{-1}h_j$, which implies $\|v_j - u_j\| \leq \delta$. □

As shown before, $C(\mathbb{T}^2)_\varepsilon \cong \mathfrak{B}_\varepsilon \rtimes_\alpha \mathbb{Z}$, where $\mathfrak{B}_\varepsilon$ is the universal C^*-algebra generated by unitaries $u_{\varepsilon,n}$ for $n \in \mathbb{Z}$ such that $\|u_{\varepsilon,n} - u_{\varepsilon,n+1}\| \leq \varepsilon$ for all n, and the action α is defined by $\alpha(u_{\varepsilon,n}) = u_{\varepsilon,n+1}$ for $n \in \mathbb{Z}$.

Proposition 2.1.4. *There exist endomorphisms ψ_n $(n \in \mathbb{N})$ of $\mathfrak{B}_\varepsilon$ converging pointwise to the identity map, such that*

$$\sup_{k \in \mathbb{Z}} \|\psi_n(u_{\varepsilon,k}) - \psi_n(u_{\varepsilon,k+1})\| < \varepsilon.$$

Proof. By the previous lemma, there exist unitaries $v_{n,j}$ $(-n \leq j \leq n)$ of $\mathfrak{B}_\varepsilon$ such that

$$\|v_{n,j} - u_{\varepsilon,j}\| \leq 1/n \quad (-n \leq j \leq n), \quad \|v_{n,j-1} - v_{n,j}\| < \varepsilon \quad (-n+1 \leq j \leq n).$$

Define $\psi_n : \mathfrak{B}_\varepsilon \to \mathfrak{B}_\varepsilon$ by $\psi_n(u_{\varepsilon,k}) = v_{n,-n}$ for $k < -n$, and $= v_{n,k}$ for $-n \leq k \leq n$, and $= v_{n,n}$ for $k > n$. It follows that

$$\lim_{n\to\infty} \psi_n(u_{\varepsilon,k}) = u_{\varepsilon,k}$$

for all k, which implies that ψ_n converges pointwise to the identity.

Since, by definition,

$$\|\psi_n(u_{\varepsilon,k}) - \psi_n(u_{\varepsilon,k+1})\| = \|v_{n,k} - v_{n,k+1}\| < \varepsilon$$

for $-n \leq k \leq n-1$, and the norm zero otherwise, so that the supremum in the statement is $< \varepsilon$. □

Let $\mathfrak{C}_\varepsilon$ be the closed ideal of $\mathfrak{B}_2$ given by the kernel of the $*$-homomorphism $\phi_\varepsilon : \mathfrak{B}_2 \to \mathfrak{B}_\varepsilon$.

Theorem 2.1.5. *One has* $\cap_{\varepsilon'<\varepsilon}\mathfrak{C}_{\varepsilon'} = \mathfrak{C}_\varepsilon$.

Proof. Let $x \in \cap_{\varepsilon'<\varepsilon}\mathfrak{C}_{\varepsilon'} \subset \mathfrak{B}_2$. Put $y = \phi_\varepsilon(x) \in \mathfrak{B}_\varepsilon$. We have

$$y = \lim_{n\to\infty} \psi_n(y) = \lim_{n\to\infty} \psi_n(\phi_\varepsilon(x))$$

because ψ_n converge pointwise to the identity map. Observe that

$$\begin{aligned} \varepsilon_n' &\equiv \sup_k \|\psi_n(\phi_\varepsilon(u_{2,k})) - \psi_n(\phi_\varepsilon(u_{2,k+1}))\| \\ &= \sup_k \|\psi_n((u_{\varepsilon,k}) - \psi_n(u_{\varepsilon,k+1}))\| < \varepsilon. \end{aligned}$$

Hence $\psi_n \circ \phi_\varepsilon$ factors through $\mathfrak{B}_{\varepsilon_n'}$. That is, we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{B}_2 & \xrightarrow{\phi_\varepsilon} & \mathfrak{B}_\varepsilon \\ {\scriptstyle \phi_{\varepsilon_n'}}\downarrow & & \downarrow{\scriptstyle \psi_n} \\ \mathfrak{B}_{\varepsilon_n'} & \longrightarrow & \mathfrak{B}_\varepsilon \end{array}$$

where the bottom map to the image of $\psi_n \circ \phi_\varepsilon$ comes from universality of $\mathfrak{B}_{\varepsilon_n'}$. Since $x \in \mathfrak{C}_{\varepsilon_n'}$ the kernel of $\phi_{\varepsilon_n'}$, we have $\psi_n(\phi_\varepsilon(x)) = 0$ by the diagram. Thus, $y = 0$, which implies that $x \in \mathfrak{C}_\varepsilon$.

On the other hand, the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{B}_2 & \xrightarrow{\phi_\varepsilon} & \mathfrak{B}_\varepsilon \\ \| & & \downarrow \\ \mathfrak{B}_2 & \xrightarrow{\phi_{\varepsilon'}} & \mathfrak{B}_{\varepsilon'} \end{array}$$

for $\varepsilon' < \varepsilon$, so that $\mathfrak{C}_\varepsilon$ is always contained in $\mathfrak{C}_{\varepsilon'}$, which implies that $\mathfrak{C}_\varepsilon \subset \bigcap_{\varepsilon'<\varepsilon}\mathfrak{C}_{\varepsilon'}$. □

Lemma 2.1.6. *Let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a C^*-algebra homomorphism. Suppose that φ is equivariant with respect to automorphisms α and β of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Let J and $J^\sim$ be the kernels of φ and $\varphi^\sim$ respectively, where $\varphi^\sim$ is the canonical extension of φ from the crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ to $\mathfrak{B} \rtimes_\beta \mathbb{Z}$. Then*

$$J^\sim = \{f \in \mathfrak{A} \rtimes_\alpha \mathbb{Z} \,|\, E_{\mathfrak{A}}(fu^{-n}) \in J, n \in \mathbb{Z}\},$$

where $E_{\mathfrak{A}} : \mathfrak{A} \rtimes_\alpha \mathbb{Z} \to \mathfrak{A}$ is the associated conditional expectation, and u is the unitary implementing α.

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{A} \rtimes_\alpha \mathbb{Z} & \xrightarrow{E_{\mathfrak{A}}} & \mathfrak{A} \\ \varphi^\sim \downarrow & & \downarrow \varphi \\ \mathfrak{B} \rtimes_\beta \mathbb{Z} & \xrightarrow{E_{\mathfrak{B}}} & \mathfrak{B} \end{array}$$

where $E_{\mathfrak{B}}$ is the same as for $E_{\mathfrak{A}}$. Indeed, for a finite sum $f = \sum_j a_j u^j \in \mathfrak{A} \rtimes_\alpha \mathbb{Z}$ for $a_j \in \mathfrak{A}$,

$$\begin{aligned} (\varphi \circ E_{\mathfrak{A}})(f) &= \varphi(\int_{\mathbb{T}} \alpha_z^\wedge(f) d\mu(z)) \\ &= \varphi(\int_{\mathbb{T}} (\sum_j a_j \alpha_z^\wedge(u^j)) d\mu(z)) \\ &= \int_{\mathbb{T}} (\sum_j \varphi(a_j) \beta_z^\wedge(v^j)) d\mu(z) \cdot \\ &= \int_{\mathbb{T}} \beta_z^\wedge(\sum_j \varphi(a_j) v^j) d\mu(z) \\ &= (E_{\mathfrak{B}} \circ \varphi^\sim)(f) \end{aligned}$$

where v is the unitary implementing the action β.

Given $f \in \mathfrak{A} \rtimes_\alpha \mathbb{Z}$, we have that f is in $J^\sim$ if and only if $\varphi^\sim(f) = 0$, which is equivalent to that $E_{\mathfrak{B}}(\varphi^\sim(f)v^{-n}) = 0$ for all $n \in \mathbb{Z}$, which is equivalent to that $\varphi(E_{\mathfrak{A}}(xu^{-n})) = 0$ for all n by the diagram. This says that the equality of the statement holds. □

Theorem 2.1.7. *One has $\mathfrak{J}_\varepsilon^- = \mathfrak{J}_\varepsilon$ for every $\varepsilon \in (0, 2)$.*

Proof. Let $E : C(\mathbb{T}^2)_2 \to \mathfrak{B}_2$ be the conditional expectation induced by the isomorphism $C(\mathbb{T}^2)_2 \cong \mathfrak{B}_2 \rtimes_\alpha \mathbb{Z}$. Given f in $\mathfrak{J}_\varepsilon^-$ the intersection $\cap_{\varepsilon' < \varepsilon} \mathfrak{J}_{\varepsilon'}$, we have that $E(fu^{-n})$ is in $\mathfrak{C}_{\varepsilon'}$ for all $n \in \mathbb{Z}$ and $\varepsilon' < \varepsilon$. Note that the extension $\mathfrak{C}_{\varepsilon'}^\sim$ of $\mathfrak{C}_{\varepsilon'}$ as in the lemma above is $\mathfrak{J}_{\varepsilon'}$. Hence $E(fu^{-n}) \in \cap_{\varepsilon' < \varepsilon} \mathfrak{C}_{\varepsilon'} = \mathfrak{C}_\varepsilon$ for all $n \in \mathbb{Z}$, which shows that $f \in \mathfrak{J}_\varepsilon$ the extension of $\mathfrak{C}_\varepsilon$. Thus, $\mathfrak{J}_\varepsilon^- \subset \mathfrak{J}_\varepsilon$. The converse inclusion is clear. □

Now let us consider the case where $\varepsilon = 2$ to prove that $\mathfrak{J}_2^- = \mathfrak{J}_2$, that is, $\cap_{\varepsilon < 2} \mathfrak{J}_\varepsilon = \{0\}$.

Lemma 2.1.8. *Let w_1 and w_2 be $n \times n$ unitary matrices. Then $\|w_1 - w_2\| = 2$ if and only if $\det(w_1 + w_2) = 0$.*

Proof. Note that $\|w_1 - w_2\| = 2$ if and only if $\|w_2 w_2^* - 1\| = 2$ which is equivalent to that -1 is in the spectrum of $w_1 w_2^*$, which is to say that $\det(w_1 w_2^{-1} + 1) = 0$ which is equivalent to that $\det(w_1 + w_2) = 0$. □

Proposition 2.1.9. *Given $n \times n$ unitary matrices u and v such that $\|uv - vu\| = 2$, there is, for every $\delta > 0$, a unitary u' such that $\|u' - u\| < \delta$ and $\|u'v - vu'\| < 2$.*

Proof. Write $u = e^h$ for some skew adjoint h. Let $u(t) = ue^{-th}$ for t real. Put $f(t) = \det(u(t)v + vu(t))$. Observe that

$$f(1) = \det(2v) \neq 0, \quad f(0) = \det(uv + vu) = 0.$$

Therefore, f is not a constant function. Since f is analytic, its zeros are isolated. Thus, there are arbitrarily small values of t for which $f(t) \neq 0$, which is to say that $\|u(t)v - vu(t)\| < 2$. Taking t sufficiently small implies that $\|u(t) - u\| < \delta$. □

Remark. As for f, in fact, observe that the matrix-valued function $u(t)v + vu(t)$ is analytic, so that its determinant $f(t)$ is also analytic. Hence $f(t) = \sum_{n=0}^\infty a_n t^n$ Taylor expansion around 0. Moreover, since $f(0) = 0$ we have $f(t) = t^k g(t)$ for some $k \geq 1$ and a holomorphic function $g(t)$ with $g(0) \neq 0$ (This is the case where $a_0 = 0, \cdots, a_{k-1} = 0$ and $a_k \neq 0$). Therefore, $f(t) \neq 0$ for t in a neighbourhood of 0.

Theorem 2.1.10. *One has* $\mathfrak{I}_2^- = \mathfrak{I}_2$.

Proof. Assume by way of contradiction that there is a non-zero $a \in \mathfrak{I}_2^-$. Note that $C(\mathbb{T}^2)_2$ is isomorphic to the full group C^*-algebra of the free group of two generators. It is shown by Choi [5] that the group C^*-algebra has a separating family of finite dimensional representations. Therefore, there is a $*$-homomorphism $\pi : C(\mathbb{T}^2)_2 \to M_n(\mathbb{C})$ such that $\pi(a) \neq 0$.

Put $u = \pi(u_2)$ and $v = \pi(v_2)$. Write $u = \lim_{j\to\infty} u'_j$, where $\|u'_j v - v u'_j\| < 2$. For each j, let π_j be a representation of $C(\mathbb{T}^2)_2$ such that $\pi_j(u_2) = u'_j$ and $\pi_j(v_2) = v$. Set $\|u'_j v - v u'_j\| \equiv \varepsilon_j$. Then π_j vanishes on $\mathfrak{I}_{\varepsilon_j}$. Indeed, use the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{I}_{\varepsilon_j} & \longrightarrow & C(\mathbb{T}^2)_2 & \longrightarrow & C(\mathbb{T}^2)_{\varepsilon_j} & \longrightarrow & 0 \\
 & & & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}(\pi) & \longrightarrow & C(\mathbb{T}^2)_2 & \xrightarrow{\pi} & M_n(\mathbb{C}) & \longrightarrow & 0
\end{array}$$

It follows that π_j vanishes on $\mathfrak{I}_2^-$, so that $\pi_j(a) = 0$. Furthermore, π_j converges pointwise to π, so that $\pi(a) = 0$, which is a contradiction. □

Remark. In the proof above, it seems there might be a small gap because

$$\|uv - vu\| \leq \|u_2 v_2 - v_2 u_2\| = 2.$$

But we could modify π to have $\|uv - vu\| = 2$ and $\pi(a) \neq 0$ since we always have a representation of $C(\mathbb{T}^2)_2$ by sending u_2 and v_2 respectively to any unitaries u' and v' such that $\|u'v' - v'u'\| = 2$.

Theorem 2.1.11. *There exists a continuous field of C^*-algebras over the interval $[0, 2]$ such that $C(\mathbb{T}^2)_\varepsilon$ is the fiber at $\varepsilon \in [0, 2]$ and for every $a \in C(\mathbb{T}^2)_2$, the section f_a defined by $f_a(\varepsilon) = \phi_\varepsilon(a) \in C(\mathbb{T}^2)_\varepsilon$ is continuous.*

Proof. Note that the set of all the sections f_a for $a \in C(\mathbb{T}^2)_2$ becomes a $*$-algebra under the point-wise operations such as: for $a, b, a^* \in C(\mathbb{T}^2)_2$,

$$\begin{aligned}
(f_a + f_b)(\varepsilon) &= \phi_\varepsilon(a) + \phi_\varepsilon(b) = \phi_\varepsilon(a + b) = f_{a+b}(\varepsilon), \\
f_{ab}(\varepsilon) &= \phi_\varepsilon(ab) = \phi_\varepsilon(a)\phi_\varepsilon(b) = f_a(\varepsilon) f_b(\varepsilon), \\
f_{a^*}(\varepsilon) &= \phi_\varepsilon(a^*) = \phi_\varepsilon(a)^* = f_a(\varepsilon)^*.
\end{aligned}$$

Since it has been shown above that the norm $\|\phi_\varepsilon(a)\|$ is continuous over $[0, 2]$, the existence of the continuous field in the statement is obtained by continuous field theory (for C^*-algebras). □

Now let us consider the following related problem:

Problem. Given unitary operators u and v with $uv \neq vu$, do there exist unitaries u' and v' perturbing u and v respectively, such that $\|u'v' - v'u'\| < \|uv - vu\|$?

In other words, this is a characterization of pairs of unitary operators which are not local minimum points for the commutator norm. The last proposition above is a partial answer to this problem and it says that a pair (u, v) of unitary matrices with $\|uv - vu\| = 2$ is never such a local minimum point.

Theorem 2.1.12. *Let u and v be non-commuting unitary operators on a Hilbert space H. Then there are nets (u_j) and (v_j) of unitary operators on $\oplus^\infty H$ the direct sum of infinitely many copies of H such that $\|u_j v_j - v_j u_j\| < \|uv - vu\|$ and the compressions pu_jp and pv_jp of u_j and v_j to H converge $*$-strongly to u and v respectively, where p means the projection from $\oplus^\infty H$ to H.*

Proof. Let $\varepsilon = \|uv - vu\|$. Consider the set N of states on $C(\mathbb{T}^2)_2$ which vanish on some $\mathfrak{J}_{\varepsilon'}$ for $\varepsilon' < \varepsilon$. Since $\bigcap_{\varepsilon' < \varepsilon} \mathfrak{J}_{\varepsilon'} = \mathfrak{J}_\varepsilon$, it follows that N is weakly dense in the set M of states of $C(\mathbb{T}^2)_2$ that vanish on $\mathfrak{J}_\varepsilon$. In fact, a point is to treat the case where the kernel of a state f of M is just $\mathfrak{J}_\varepsilon$. If $f(a) \neq 0$, then note that $a \notin \mathfrak{J}_{\varepsilon'}$ for some $\varepsilon' < \varepsilon$. Then there exists $g \in N$ such that $g(a) = f(a)$ and the kernel of g is just $\mathfrak{J}_{\varepsilon'}$. Inductively, it can be proved.

Let π be a representation of $C(\mathbb{T}^2)_2$ on H such that $\pi(u_2) = u$ and $\pi(v_2) = v$. Assume without loss of generality that π is cyclic with a cyclic vector $\xi \in H$. Put $f(a) = \langle \pi(a)\xi, \xi \rangle$ the inner product for $a \in C(\mathbb{T}^2)_2$. Since π factors through $C(\mathbb{T}^2)_\varepsilon$ we have that f vanishes on $\mathfrak{J}_\varepsilon$, so there exists a net (f_j) in N converging weakly to f. For every j, let π_j be the GNS representation of $C(\mathbb{T}^2)_2$ corresponding to f_j. Since each f_j vanishes on some $\mathfrak{J}_{\varepsilon'_j}$ with ε'_j, the same is true for π_j. Hence $\|\pi_j(u_2)\pi_j(v_2) - \pi_j(v_2)\pi_j(u_2)\| \leq \varepsilon'_j < \varepsilon$.

One may assume that the space H_j on which π_j acts is a subspace of $\oplus^\infty H$, and the conclusion holds with

$$u_j = \pi_j(u_2) + 1 - p_j, \quad v_j = \pi_j(v_2) + 1 - p_j,$$

where p_j is the projection onto H_j. Indeed, note that

$$(\pi_j(u_2) + (1 - p_j))(\pi_j(v_2) + (1 - p_j)) = \pi_j(u_2)\pi_j(v_2) + (1 - p_j).$$

Also,

$$\langle u\xi, \xi\rangle = \langle \pi(u_2)\xi, \xi\rangle = \lim_j \langle \pi_j(u_2)\xi_j, \xi_j\rangle$$
$$= \lim_j \langle \pi_j(u_2)p_j(\oplus_k \eta_k), p_j(\oplus_k \eta_k)\rangle = \lim_j \langle p_j u_j p_j(\oplus_k \eta_k), \oplus_k \eta_k\rangle,$$

where $p_j(\oplus_k \eta_k) = \xi_j$ and we may identified p_j with p. The same is true for v. □

Theorem 2.1.13. *For $n \geq 3$, there exists a neighbourhood O of the pair (Ω_n, S_n) of Voiculescu $n \times n$ unitary matrices such that*

$$\|uv - vu\| \geq \|\Omega_n S_n - S_n \Omega_n\|$$

for all (u, v) of O in $U(n) \times U(n)$.

Proof. Note that $\Omega_n S_n \Omega_n^* S_n^* = \omega_n I_n$ with $\omega_n = e^{2\pi i/n}$ and I_n the $n \times n$ identity matrix, as computed before. If (u, v) is close enough to (Ω_n, S_n), then the spectrum of uvu^*v^* is in a small neighbourhood of ω_n in $\mathbb{C}$.

On the other hand, note that $\det(uvu^*v^*) = 1$, so that if the spectrum of uvu^*v^* a unitary is the set $\{e^{i\theta_j}\}_{j=1}^n$ with $-\pi < \theta_j < \pi$, one has that $\sum_{j=1}^n \theta_j$ is in $2\pi\mathbb{Z}$. In fact, set $uvu^*v^* \equiv w$. There is a unitary matrix q such that $q^*wq = \oplus_{j=1}^n e^{i\theta_j}$ a diagonal matrix, so that

$$1 = \det(w) = \det(q^*wq) = \det(\oplus_{j=1}^n e^{i\theta_j}) = e^{i\sum_{j=1}^n \theta_j}.$$

Since each θ_j is near $2\pi/n$, it follows that $\sum_{j=1}^n \theta_j = 2\pi$. Therefore, for some k_0 we must have $\theta_{k_0} \geq 2\pi/n$ (by Pigeon Hole Principle). It follows that

$$\|uv - vu\| \geq |e^{i\theta_{k_0}} - 1| \geq |\omega_n - 1|$$
$$= \|\Omega_n S_n \Omega_n^* S_n^* - 1\| = \|\Omega_n S_n - S_n \Omega_n\|.$$

□

Remark. The same is also true for any pair of unitary matrices whose multiplicative commutator is a scalar multiple of the identity matrix, but not equal to $-I_n$.

We say that a pair of unitary operators is irreducible if there is no proper invariant subspace for both operators of the pair.

Denote by γ the map $\gamma : U(n) \times U(n) \to SU(n)$ defined by $\gamma(u, v) = uvu^*v^*$.

Lemma 2.1.14. *A point $(u, v) \in U(n) \times U(n)$ is regular for γ in the sense that γ is a submersion at (u, v) if and only if (u, v) is an irreducible pair.*

Theorem 2.1.15. *If (u, v) is an irreducible pair in $U(n) \times U(n)$ and is a local minimum for the commutator norm, then uvu^*v^* is a scalar.*

Let us consider a reducible pair $(u, v) = (\oplus_j u_j, \oplus_j v_j)$ of unitary matrices, where each (u_j, v_j) is an irreducible pair of unitary matrices u_j and v_j. Note that $\|uv - vu\| = \max_j \|u_j v_j - v_j u_j\|$.

Theorem 2.1.16. *Let $(u, v) = (\oplus_j u_j, \oplus_j v_j)$ be a reducible pair as above. Suppose that (u, v) is a local minimum for the commutator norm. Then $u_j v_j u_j^* v_j^*$ is a scalar for j such that $\|u_j v_j - v_j u_j\| = \|uv - vu\|$.*

Notes. This section of one subsection is based on the paper [10] of Exel. In [22] of the author, it is shown that the versions of the soft tori by replacing almost commuting unitaries with almost commuting isometries also have continuous field structure.

3 Softening the 2-sphere

As before, let $C(\mathbb{T}^2)_\varepsilon$ be the soft 2-torus generated by two unitaries u and v subject to the relation $\|uv - vu\| \leq \varepsilon$. Consider the flip (or symmetry) σ on $C(\mathbb{T}^2)_\varepsilon$ defined by $\sigma(u) = u^*$ and $\sigma(v) = v^*$, that is, an automorphism of $C(\mathbb{T}^2)_\varepsilon$ with σ^2 the identity map on $C(\mathbb{T}^2)_\varepsilon$. The connection with the 2-sphere S^2 is that

$$C(\mathbb{T}^2)^\sigma \cong C(S^2), \quad C(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}_2 \subset C(S^2, M_2(\mathbb{C})),$$

where $C(\mathbb{T}^2)^\sigma$ is the fixed point algebra under σ on $C(\mathbb{T}^2)_0 = C(\mathbb{T}^2)$, and $C(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}_2$ is the crossed product of $C(\mathbb{T}^2)_0$ by σ of the 2-cyclic group $\mathbb{Z}_2$, and $C(S^2, M_2(\mathbb{C}))$ is the C^*-algebra of all continuous $M_2(\mathbb{C})$-valued functions on S^2 (see [8] and [3]). Therefore, as a reasonable replacement for $C(S^2)$, we may accept $C(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}_2$. There are two ways to soften this crossed product. The first one is to consider the crossed product $C(\mathbb{T}^2)_\varepsilon \rtimes_\sigma \mathbb{Z}_2$ defined similarly as above and we call it the soft sphere. The second is to consider the soft flip. That is, rather than adjoining an order-two unitary w implementing σ to $C(\mathbb{T}^2)_\varepsilon$ we require that

$$\|wuw^* - u^*\| \leq \varepsilon, \quad \|wvw^* - v^*\| \leq \varepsilon.$$

For $0 \leq \theta \leq 1$, let $\mathbb{T}^2_\theta$ the rotation algebra defined to be the crossed product $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ by the rotation action θ of $\mathbb{Z}$ on $\mathbb{T}$ by $\theta(z) = e^{2\pi i\theta}z$ for $z \in \mathbb{T}$. The noncommutative sphere $\mathbb{T}^2_\theta \rtimes_\sigma \mathbb{Z}_2$ that is the crossed product by the flip is regarded as a quantized sphere and not a softened sphere, and is shown to be an AF algebra.

A truely soft torus would be a unital C^*-algebra T_ε generated by two elements a and b subject to the relations: $\|ab - ba\| \leq \varepsilon$, $\|a^*a - 1\| \leq \varepsilon$, $\|aa^* - 1\| \leq \varepsilon$, $\|b^*b - 1\| \leq \varepsilon$, and $\|bb^* - 1\| \leq \varepsilon$. The question whether the natural surjection from T_ε to $C(\mathbb{T}^2)$ induces an isomorphism on K-theory might be still open.

3.1 The soft sphere

Recall that $C(\mathbb{T}^2)_\varepsilon \cong \mathfrak{B}_\varepsilon \rtimes_\alpha \mathbb{Z}$, where $\mathfrak{B}_\varepsilon$ is generated by unitaries u_n such that $\|u_{n+1} - u_n\| \leq \varepsilon$ for $n \in \mathbb{Z}$, and $\alpha(u_n) = u_{n+1}$.

Proposition 3.1.1. *For all $\varepsilon \in [0, 2]$, the soft sphere $C(\mathbb{T}^2)_\varepsilon \rtimes_\sigma \mathbb{Z}_2$ is isomorphic to the crossed product $\mathfrak{B}_\varepsilon \rtimes_{\beta*\gamma} (\mathbb{Z}_2 * \mathbb{Z}_2)$, where β and γ are automorphisms of $\mathfrak{B}_\varepsilon$ defined by $\beta(u_n) = u^*_{-n}$ and $\gamma(u_n) = u^*_{1-n}$ for $n \in \mathbb{Z}$, and $\beta * \gamma$ is the action on $\mathfrak{B}_\varepsilon$ extended to the free product $\mathbb{Z}_2 * \mathbb{Z}_2$.*

Proof. Note that $C(\mathbb{T}^2)_\varepsilon \rtimes_\sigma \mathbb{Z}_2$ is the universal unital C^*-algebra generated by unitaries u, v, s such that $\|uv - vu\| \leq \varepsilon$, $sus^* = u^*$, and $svs^* = v^*$, and $s^2 = 1$. On the other hand, $\mathfrak{B}_\varepsilon \rtimes_{\beta*\gamma} (\mathbb{Z}_2 * \mathbb{Z}_2)$ is the universal unital C^*-algebra generated by unitaries z_1, z_2, and u_n for $n \in \mathbb{Z}$ such that $z_1^2 = 1 = z_2^2$, $\|u_n - u_{n+1}\| \leq \varepsilon$, $z_1 u_n z_1^* = u^*_{-n}$, and $z_2 u_n z_2^* = u^*_{1-n}$ for $n \in \mathbb{Z}$.

Consider the $*$-homomorphisms:

$$\varphi : C(\mathbb{T}^2)_\varepsilon \rtimes_\sigma \mathbb{Z}_2 \to \mathfrak{B}_\varepsilon \rtimes_{\beta*\gamma} (\mathbb{Z}_2 * \mathbb{Z}_2), \quad \psi : \mathfrak{B}_\varepsilon \rtimes_{\beta*\gamma} (\mathbb{Z}_2 * \mathbb{Z}_2) \to C(\mathbb{T}^2)_\varepsilon \rtimes_\sigma \mathbb{Z}_2$$

given by $\varphi(u) = u_0$, $\varphi(v) = z_1 z_2$, and $\varphi(s) = z_1$, while $\psi(u_n) = v^n u (v^*)^n$, $\psi(z_1) = s$, and $\psi(z_2) = vs$, respectively. They are each other's inverse. Indeed,

$$C(\mathbb{T}^2)_\varepsilon \rtimes_\sigma \mathbb{Z}_2 \cong (\mathfrak{B}_\varepsilon \rtimes_\alpha \mathbb{Z}) \rtimes_\sigma \mathbb{Z}_2$$

by sending u, v, and s to u_0, w, and s respectively. Note that $\mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$

by sending w and s to z_1z_2 and z_2 respectively. In details, check that

$$\begin{aligned}
\varphi(sus^*) &= z_1u_0z_1^* = u_0^* = \varphi(u^*),\\
\varphi(svs^*) &= z_1(z_1z_2)z_1^* = z_1^2z_2z_1 = (z_1z_2)^* = \varphi(v^*),\\
\psi(z_1u_nz_1^*) &= sv^nu(v^*)^ns = (v^*)^nsus(v)^n\\
&= (v^*)^nu^*v^n = (u_{-n})^*,\\
\psi((u_{-n})^*) &= v^{-n}u^*(v^*)^{-n} = (v^*)^nu^*v^n = (u_{-n})^*,\\
\psi(z_2u_nz_2) &= vsv^nu(v^*)^nvs = v(v^*)^nsus(v)^{n-1}\\
&= (v^*)^{n-1}u^*v^{n-1} = (u_{1-n})^*,\\
\psi(u_{1-n}^*) &= v^{1-n}u^*(v^*)^{1-n} = (v^*)^{n-1}u^*v^{n-1} = (u_{1-n})^*.
\end{aligned}$$

□

Definition 3.1.2. We say that two C^*-dynamical systems $(\mathfrak{A}, \alpha, G)$ and $(\mathfrak{B}, \beta, H)$ are homotopically equivalent if there are $*$-homomorphisms $\varphi : \mathfrak{A} \to \mathfrak{B}$ and $\psi : \mathfrak{B} \to \mathfrak{A}$ such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are both homotopic to the respective identity maps on $\mathfrak{A}$ and $\mathfrak{B}$ in such a way that the homotopies involved commute with the group actions α and β of G and H respectively.

Homotopically equivalent C^*-dynamical systems give rise to homotopically equivalent crossed product C^*-algebras.

Consider the map $\rho : \mathfrak{B}_\varepsilon \to C(S^1)$ given by $\rho(u_n) = z$ the standard unitary generator of $C(S^1)$ for all $n \in \mathbb{Z}$.

Proposition 3.1.3. *For $\varepsilon < 2$, the map ρ is a homotopy equivalence from both $(\mathfrak{B}_\varepsilon, \beta, \mathbb{Z}_2)$ and $(\mathfrak{B}_\varepsilon, \gamma, \mathbb{Z}_2)$ to $(C(S^1), r, \mathbb{Z}_2)$, where $r(z) = z^*$.*

Sketch of Proof. Let us consider the case for γ. Put $m = (u_0 + u_1)/2$. Since $\varepsilon < 2$, one can check that m is an invertible element of $\mathfrak{B}_\varepsilon$. Indeed,

$$\begin{aligned}
2 > \varepsilon &\geq \|u_0 - u_1\|\\
&= \|2u_0 - (u_0 + u_1)\| = 2\|u_0 - (\frac{u_0 + u_1}{2})\|\\
&= 2\|1 - u_0^*(\frac{u_0 + u_1}{2})\|
\end{aligned}$$

which implies that $u_0^*(u_0 + u_1)/2$ is invertible in $\mathfrak{B}_\varepsilon$.

Define a $*$-homomorphism $\varphi : C(S^1) \to \mathfrak{B}_\varepsilon$ by $\varphi(z) = u_m$, where u_m stands for the unitary part of the polar decomposition of m, i.e., $u_m =$

$m|m|^{-1} = m(m^*m)^{-1/2}$. We have that

$$\gamma(m) = z_2(u_0 + u_1)2^{-1}z_2^* = 2^{-1}(u_1^* + u_0^*) = m^*,$$
$$\gamma(\varphi(z)) = \gamma(m(m^*m)^{-1/2}) = m^*(mm^*)^{-1/2}$$
$$= (m^*m)^{-1/2}m^* = (u_m)^* = \varphi(r(z)),$$

which says that φ is equivariant with respect to γ and r. Indeed, we check (only) a non-trivial $m^*(mm^*)^{-1/2} = (m^*m)^{-1/2}m^*$ as follows:

$$\begin{aligned} m^*(mm^*)^{-1/2} &= (m^*m)^{-1}m^*mm^*(mm^*)^{-1/2} \\ &= (m^*m)^{-1/2}(m^*m)^{-1/2}m^*mm^*(mm^*)^{-1/2} \\ &= (m^*m)^{-1/2}m^*m(m^*m)^{-1/2}m^*(mm^*)^{-1/2}. \end{aligned}$$

Moreover, $m(m^*m)^{-1/2}m^* = (mm^*)^{1/2}$. Indeed,

$$(m(m^*m)^{-1/2}m^*)(m(m^*m)^{-1/2}m^*) = mm^*.$$

Clearly, the composition $\rho \circ \varphi$ is the identity map on $C(S^1)$. Indeed, $\rho(m) = (z + z)/2 = z$, so that $\rho(u_m) = z(z^*z)^{-1/2} = z$.

We need to check that $\varphi \circ \rho$ is equivariantly homotopic to the identity map on $\mathfrak{B}_\varepsilon$. We claim that $\varphi \circ \rho$ is equivariantly homotopic to the map $\psi : \mathfrak{B}_\varepsilon \to \mathfrak{B}_\varepsilon$ given by $\psi(u_n) = u_0$ if $n \le 0$ and $= u_1$ if $n \ge 1$. Let $a_t = (1 - t)u_0 + tu_1$ for all $t \in [0, 1]$. Note that each a_t is invertible in $\mathfrak{B}_\varepsilon$. Indeed, check that

$$\begin{aligned} \|1 - u_0^*((1 - t)u_0 + tu_1)\| &= \|1 - (1 - t)1 - tu_0^*u_1\| \\ &= \|t(1 - u_0^*u_1)\| = t\|u_0 - u_1\| \le t\varepsilon < 2t \end{aligned}$$

so that if $t \le 1/2$, then $u_0^*((1-t)u_0+tu_1)$ is invertible, hence, $(1-t)u_0+tu_1$ is invertible. Also.

$$\begin{aligned} &\|1 - u_1^*((1 - t)u_0 + tu_1)\| = \|1 - (1 - t)u_1^*u_0 - t1\| \\ &= \|(1 - t)(1 - u_1^*u_0)\| = (1 - t)\|u_1 - u_0\| \le (1 - t)\varepsilon < 2(1 - t) \end{aligned}$$

so that if $1-t \le 1/2$, i.e., $t \ge 1/2$, then $u_1^*((1-t)u_0+tu_1)$ is invertible, hence, $(1 - t)u_0 + tu_1$ is invertible. Let u_{a_t} denote the unitary part of the polar decomposition of a_t. For $t \in [0, 1/2]$, define $\psi_t : \mathfrak{B}_\varepsilon \to \mathfrak{B}_\varepsilon$ by $\psi_t(u_n) = u_{a_t}$ if $n \ge 0$ and $= u_{a_{1-t}}$ if $n \ge 1$.

In order to verify that each ψ_t is a well-defined endomorphism of $\mathfrak{B}_\varepsilon$ one needs to check that $\|\psi_t(u_{n+1}) - \psi_t(u_n)\| \le \varepsilon$ for all n. For this we prove that $\|u_{a_t} - u_{a_s}\| \le \varepsilon$ for $t, s \in [0, 1]$. We have

$$\|u_{a_t} - u_{a_s}\| = \|a_t|a_t|^{-1} - a_s|a_s|^{-1}\| = \|u_0^*a_t|u_0^*a_t|^{-1} - u_0^*a_s|u_0^*a_s|^{-1}\|,$$

where note that $(u_0^* a_t)^*(u_0^* a_t) = a_t^* a_t$ so that $|a_t| = |u_0^* a_t|$ for all t. Let $b_t = u_0^* a_t = 1 - t + t u_0^* u_1$. Then

$$\|u_{a_t} - u_{a_s}\| = \|u_{b_t} - u_{b_s}\|.$$

Note that b_t is in the commutative C^*-algebra $C^*(u_0^* u_1)$ generated by $u_0^* u_1$. Therefore,

$$\|u_{b_t} - u_{b_s}\| = \sup_{\chi} |\chi(u_{b_t}) - \chi(u_{b_s})|$$

where the supremum is taken over all characters $\chi : C^*(u_0^* u_1) \to \mathbb{C}$. Since $\chi(b_t) = 1 - t + t\chi(u_0^* u_1)$, the path $\{\chi(b_t) : 0 \leq t \leq 1\}$ is just the line segment joining $\chi(b_0) = 1$ and $\chi(b_1) = \chi(u_0^* u_1)$ which are points in the unit circle within ε of each other. Now, $\chi(u_{b_t})$ is the radial projection of $\chi(b_t)$ onto the unit circle and lies in the arc from $\chi(b_0)$ to $\chi(b_1)$. It is verified that any two points in the arc are within ε of each other. This shows ψ_t to be well defined for all t.

To show that ψ_t is equivariant for γ, we check that if $n \leq 0$,

$$\begin{aligned}
\gamma(\psi_t(u_n)) &= \gamma(u_{a_t}) = \gamma(a_t |a_t|^{-1}) \\
&= |a_{1-t}^*|^{-1} a_{1-t}^* = u_{a_{1-t}}^* \\
&= \psi_t(u_{1-n}^*) = \psi_t(\gamma(u_n))
\end{aligned}$$

since $\gamma(a_t) = (1-t)u_1^* + t u_0^* = a_{1-t}^*$, and also, if $n \geq 1$,

$$\begin{aligned}
\gamma(\psi_t(u_n)) &= \gamma(u_{a_{1-t}}) = \gamma(a_{1-t}|a_{1-t}|^{-1}) \\
&= |a_t^*|^{-1} a_t^* = u_{a_t}^* \\
&= \psi_t(u_{1-n}^*) = \psi_t(\gamma(u_n)).
\end{aligned}$$

The assertion is thus proved since $\psi_0 = \psi$ and $\psi_{1/2} = \varphi \circ \rho$.

It is shown similarly as in the second section that ψ is equivariantly homotopic to the identity map on $\mathfrak{B}_\varepsilon$.

The proof for β is essentially contained in the argument in the second section. Indeed, one just needs to observe that the homotopy given there is equivariant for β and r. □

Theorem 3.1.4. *If $\varepsilon < 2$, the natural $*$-homomorphism from $C(\mathbb{T}^2)_\varepsilon \rtimes_\sigma \mathbb{Z}_2$ to $C(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}_2$ induces isomorphisms at the level of K-theory groups. It follows that*

$$K_0(C(\mathbb{T}^2)_\varepsilon \rtimes_\sigma \mathbb{Z}_2) \cong \mathbb{Z}^6, \quad K_1(C(\mathbb{T}^2)_\varepsilon \rtimes_\sigma \mathbb{Z}_2) \cong 0.$$

Proof. Using the isomorphisms obtained above, it is enough to prove the corresponding result for the natural map:

$$\varphi : \mathfrak{B}_\varepsilon \rtimes_{\beta*\gamma} (\mathbb{Z}_2 * \mathbb{Z}_2) \equiv \mathfrak{D}_\varepsilon \to \mathfrak{B}_0 \rtimes_{\beta*\gamma} (\mathbb{Z}_2 * \mathbb{Z}_2).$$

Now set $\alpha^1 = \beta$ and $\alpha^2 = \gamma$. There is the following six-term exact sequence:

$$\begin{array}{ccccc} K_0(\mathfrak{B}_\varepsilon) & \xrightarrow{(\iota_*^1, -\iota_*^2)} & \oplus_{j=1,2} K_0(\mathfrak{B}_\varepsilon \rtimes_{\alpha^j} \mathbb{Z}_2) & \xrightarrow{\kappa_*^1 + \kappa_*^2} & K_0(\mathfrak{D}_\varepsilon) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{D}_\varepsilon) & \xleftarrow{\kappa_*^1 + \kappa_*^2} & \oplus_{j=1,2} K_1(\mathfrak{B}_\varepsilon \rtimes_{\alpha^j} (\mathbb{Z}_2 * \mathbb{Z}_2)) & \xleftarrow{(\iota_*^1, -\iota_*^2)} & K_1(\mathfrak{B}_\varepsilon) \end{array}$$

where ι^j is the natural inclusion of $\mathfrak{B}_\varepsilon$ into $\mathfrak{B}_\varepsilon \rtimes_{\alpha^j} \mathbb{Z}_2$ ($j = 1, 2$), and κ^j is the natural inclusion of $\mathfrak{B}_\varepsilon \rtimes_{\alpha^j} \mathbb{Z}_2$ into $\mathfrak{D}_\varepsilon$ (see [15] and [17] also). Since $K_*(\mathfrak{B}_\varepsilon \rtimes_{\alpha^j} \mathbb{Z}_2) \cong K_*(C(S^1) \rtimes_r \mathbb{Z}_2)$ for $* = 0, 1$ by the homotopy equivalence shown above, applying the five lemma for the half splitting short exact sequences of K_0 and K_1 in the above six-term diagrams for $\varepsilon \neq 0$ and $\varepsilon = 0$ we obtain the isomorphisms at the level of K-theory groups for $\mathfrak{D}_\varepsilon$ and $\mathfrak{D}_0$.

In fact, the six-term exact sequence becomes as follows:

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{(\iota_*^1, -\iota_*^2)} & \mathbb{Z}^3 \oplus \mathbb{Z}^3 & \xrightarrow{\kappa_*^1 + \kappa_*^2} & K_0(\mathfrak{C}_\varepsilon) \\ {\scriptstyle 0}\uparrow & & & & \downarrow \\ K_1(\mathfrak{C}_\varepsilon) & \xleftarrow{\kappa_*^1 + \kappa_*^2} & 0 \oplus 0 & \xleftarrow{(\iota_*^1, -\iota_*^2)} & \mathbb{Z} \end{array}$$

(see [23]). Therefore, $K_0(\mathfrak{C}_\varepsilon) \cong \mathbb{Z}^6$ and $K_1(\mathfrak{C}_\varepsilon) \cong 0$. □

Lemma 3.1.5. *Let Γ be a discrete amenable group and α_ε an action of Γ on $\mathfrak{B}_\varepsilon$ (for each $\varepsilon \in [0, 2]$) such that the canonical map $\phi_\varepsilon : \mathfrak{B}_2 \to \mathfrak{B}_\varepsilon$ is Γ-equivariant. Then there exists a continuous field of C^*-algebras over the interval $[0, 2]$ such that $\mathfrak{B}_\varepsilon \rtimes_{\alpha_\varepsilon} \Gamma$ is the fiber at ε, and the map f_a for every $a \in \mathfrak{B}_2 \rtimes_{\alpha_2} \Gamma$ defined by $f_a(\varepsilon) = \phi_\varepsilon^\sim(a) \in \mathfrak{B}_\varepsilon \rtimes_{\alpha_\varepsilon} \Gamma$ is a continuous section, where $\phi_\varepsilon^\sim : \mathfrak{B}_2 \rtimes_{\alpha_2} \Gamma \to \mathfrak{B}_\varepsilon \rtimes_{\alpha_\varepsilon} \Gamma$ is the natural extension of ϕ_ε.*

Sketch of Proof. Let L_ε be the kernel of $\varphi_\varepsilon^\sim$. We claim that

$$L_\varepsilon = \overline{\cup_{\varepsilon < \varepsilon'} L_{\varepsilon'}} \quad \text{for } \varepsilon \in [0, 2), \text{ and}$$
$$L_\varepsilon = \cap_{\varepsilon' < \varepsilon} L_{\varepsilon'} \quad \text{for } \varepsilon \in (0, 2].$$

The first assertion can be proved by the universal properties of both $\mathfrak{B}_\varepsilon$ and the full crossed products $\mathfrak{B}_\varepsilon \rtimes_{\alpha_\varepsilon} \Gamma$ as it was done in the previous section.

As for the second claim, recall that if $\mathfrak{C}_\varepsilon$ is the kernel of φ_ε, then for $\varepsilon \in (0, 2)$, we have

$$\cap_{\varepsilon' < \varepsilon} \mathfrak{C}_{\varepsilon'} = \mathfrak{C}_\varepsilon.$$

The same also holds for $\varepsilon = 2$ since we have $\mathfrak{C}_\varepsilon = \mathfrak{J}_\varepsilon \cap \mathfrak{B}_2$ so that

$$\cap_{\varepsilon' < 2} \mathfrak{C}_{\varepsilon'} \subset \cap_{\varepsilon' < 2} \mathfrak{J}_{\varepsilon'} = \{0\} = \mathfrak{C}_2,$$

where $\mathfrak{J}_\varepsilon$ is the kernel of the map from $\mathfrak{B}_2 \rtimes_\alpha \mathbb{Z}$ to $\mathfrak{B}_\varepsilon \rtimes_\alpha \mathbb{Z}$.

We next need to check that the lemma in the previous section for a C^*-algebra homomorphism and its crossed products by $\mathbb{Z}$ extends to crossed products by a discrete amenable group Γ. The key point for this is the fact that an element x in $\mathfrak{B}_\varepsilon \rtimes_{\alpha_\varepsilon} \Gamma$ is zero if and only if $E_{\mathfrak{B}_\varepsilon}(x\lambda_{t^{-1}}) = 0$ for all $t \in \Gamma$, where $E_{\mathfrak{B}_\varepsilon} : \mathfrak{B}_\varepsilon \rtimes_{\alpha_\varepsilon} \Gamma \to \mathfrak{B}_\varepsilon$ is the canonical conditional expectation and λ is the regular representation of Γ. This is a consequence of Γ being amenable.

In fact, note that it is known (by [18, Theorem 7.3.9]) that a locally compact group G is amenable if and only if the regular representation of the full group C^*-algebra $C^*(G)$ of G is faithful, so that $C^*(G)$ is isomorphic to the reduced group C^*-algebra of G.

The second assertion now follows from the same consideration as for $\mathbb{Z}$, extended to the case of Γ.

The proof is concluded as the same way as for $\mathbb{Z}$. □

Theorem 3.1.6. *There exists a continuous field of C^*-algebras over the interval $[0, 2]$ such that the soft sphere $C(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}_2$ is the fiber at ε, and the function f_a for $a \in C(\mathbb{T}^2)_2 \rtimes_\sigma \mathbb{Z}_2$ defined by $f_a(\varepsilon) = \varphi_\varepsilon(a) \in C(\mathbb{T}^2)_\varepsilon \rtimes_\sigma \mathbb{Z}_2$ is a continuous section, where $\varphi_\varepsilon : C(\mathbb{T}^2)_2 \rtimes_\sigma \mathbb{Z}_2 \to C(\mathbb{T}^2)_\varepsilon \rtimes_\sigma \mathbb{Z}_2$ is the natural map.*

Proof. Recall that

$$C(\mathbb{T}^2)_\varepsilon \rtimes_\sigma \mathbb{Z}_2 \cong \mathfrak{B}_\varepsilon \rtimes_{\beta * \gamma} (\mathbb{Z}_2 * \mathbb{Z}_2).$$

The statement follows from the previous lemma since $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2$ is amenable. □

3.2 Softening crossed products

Let $(\mathfrak{A}, \alpha, \Gamma)$ be a C^*-dynamical system, where Γ is a discrete group and $\mathfrak{A}$ is a unital C^*-algebra. Assume that $\mathfrak{A}$ is generated by a set $\{a_i\}_{i \in I}$ as a C^*-algebra and Γ is generated by a set $\{g_j\}_{j \in J}$ as a group.

Definition 3.2.1. For every $\varepsilon \geq 0$, the soft crossed product associated to the C^*-dynamical system $(\mathfrak{A}, \alpha, \Gamma)$ with $\{a_i\}_{i\in I}$ and $\{g_j\}_{j\in J}$ generating sets of $\mathfrak{A}$ and Γ respectively is defined to be the universal unital C^*-algebra generated by $\mathfrak{A}$ and unitaries u_g for $g \in \Gamma$ subject to the relations:

$$\|u_{g_j} a_i u_{g_j}^* - \alpha_{g_j}(a_i)\| \leq \varepsilon \quad \text{and} \quad u_g u_h = u_{gh}$$

for $i \in I$, $j \in J$, and $g, h \in \Gamma$. Denote it by $\mathfrak{A} \rtimes_{\alpha,\varepsilon} \Gamma$.

Remark. When $\varepsilon = 0$ we recover the usual crossed product $\mathfrak{A} \rtimes_\alpha \Gamma$. The soft torus $C(\mathbb{T}^2)_\varepsilon$ is viewed as the soft crossed product $C(\mathbb{T}) \rtimes_{\mathrm{id},\varepsilon} \mathbb{Z}$, where id is the trivial action. On the other hand,

$$C(\mathbb{T}^2)_\varepsilon \rtimes_\sigma \mathbb{Z}_2 \cong (C(\mathbb{T}) \rtimes_{\mathrm{id},\varepsilon} \mathbb{Z}) \rtimes_\sigma \mathbb{Z}_2$$

can be considered as a semi-soft crossed product.

Let $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ on $\mathbb{Z}$ is given by the involution: $n \mapsto -n \in \mathbb{Z}$. The semi-direct product Γ admits the presentation:

$$\Gamma = \langle m, s : sms^{-1} = m^{-1}, s^2 = 1\rangle.$$

Consider the action ρ of Γ on $C(S^1)$ given by

$$\rho_m(z) = z, \quad \rho_s(z) = z^{-1}.$$

Note that $\Gamma \cong \mathbb{Z}_2 * \mathbb{Z}_2$ via the identitication of ms with w, where $w^2 = 1$.

Proposition 3.2.2. *One has that the (truly) soft crossed product $C(S^1) \rtimes_{\rho,\varepsilon} \Gamma$ is isomorphic to the (usual) crossed product $\mathfrak{B}_\varepsilon \rtimes_{\gamma*\delta} (\mathbb{Z}_2 * \mathbb{Z}_2)$, where δ is the involution of $\mathfrak{B}_\varepsilon$ defined by $\delta(u_n) = u_{3-n}^*$.*

Proof. In order to simplify the notation, describe $\mathfrak{B}_\varepsilon$ as the universal unital C^*-algebra generated by unitaries a_n and b_n for $n \in \mathbb{Z}$ subject to the relations:

$$\|a_n - b_n\| \leq \varepsilon, \quad \|b_n - a_{n+1}\| \leq \varepsilon, \quad n \in \mathbb{Z}.$$

In other words, it is relabeling by $a_n = u_{2n}$ and $b_n = u_{2n+1}$. The automorphisms γ and δ are given by

$$\begin{aligned}
\gamma(a_n) &= \gamma(u_{2n}) = u_{1-2n}^* = u_{2(-n)+1}^* = b_{-n}^*, \\
\gamma(b_n) &= \gamma(u_{2n+1}) = u_{-2n}^* = a_{-n}^*, \\
\delta(a_n) &= \delta(u_{2n}) = u_{3-2n}^* = u_{2(1-n)+1}^* = b_{1-n}^*, \\
\delta(b_n) &= \delta(u_{2n+1}) = u_{2-2n}^* = u_{2(1-n)}^* = a_{1-n}^*.
\end{aligned}$$

Therefore, $\mathfrak{B}_\varepsilon \rtimes_{\gamma*\delta} (\mathbb{Z}_2 * \mathbb{Z}_2)$ is described as the universal unital C^*-algebra generated by unitaries a_n, b_n $(n \in \mathbb{Z})$ and s and t subject to the relations:

$$\begin{aligned}
&\|a_n - b_n\| \le \varepsilon, \quad \|b_n - a_{n+1}\| \le \varepsilon, \\
&s a_n s^* = b_{-n}^*, \quad s b_n s^* = a_{-n}^*, \\
&t a_n t^* = b_{1-n}^*, \quad t b_n t^* = a_{1-n}^*, \\
&s^2 = 1, \quad t^2 = 1,
\end{aligned}$$

where we are denoting implement unitaries corresponding to the actions by the same symbols s and t.

On the other hand, $C(S^1) \rtimes_{\rho,\varepsilon} \Gamma$ is the universal unital C^*-algebra generated by unitaries z, w, and s such that

$$\|wzw^* - z^{-1}\| \le \varepsilon, \quad \|szs^* - z^{-1}\| \le \varepsilon,$$

and $w^2 = 1$ and $s^2 = 1$, where we are using the same symbol as s above.

The map φ from $\mathfrak{B}_\varepsilon \rtimes_{\gamma*\delta} (\mathbb{Z}_2 * \mathbb{Z}_2)$ to $C(S^1) \rtimes_{\rho,\varepsilon} \Gamma$ (extended to a $*$-homomorphism) is given by

$$\begin{aligned}
&\varphi(b_n) = (ws)^n z (ws)^{-n}, \quad \varphi(a_n) = s(ws)^{-n} z^* (ws)^n s, \\
&\varphi(t) = w, \quad \varphi(s) = s,
\end{aligned}$$

and the map ψ from $C(S^1) \rtimes_{\rho,\varepsilon} \Gamma$ to $\mathfrak{B}_\varepsilon \rtimes_{\gamma*\delta} (\mathbb{Z}_2 * \mathbb{Z}_2)$ is given by

$$\psi(w) = t, \quad \psi(s) = s, \quad \psi(z) = b_0,$$

so that φ and ψ are inverses each other. Indeed, check that

$$\begin{aligned}
\|\varphi(a_n) - \varphi(b_n)\| &= \|s(ws)^{-n} z^* (ws)^n s - (ws)^n z (ws)^{-n}\| \\
&= \|s(sw)^n z^* (ws)^n s - (ws)^n z (sw)^n\| \\
&= \|(ws)^{n-1} w z^* w (sw)^{n-1} - (ws)^n z (sw)^n\| \\
&= \|wz^*w - wszsw\| = \|z^* - szs\| \le \varepsilon,
\end{aligned}$$

and also

$$\begin{aligned}
\|\varphi(a_{n+1}) - \varphi(b_n)\| &= \|s(ws)^{-(n+1)} z^* (ws)^{n+1} s - (ws)^n z (ws)^{-n}\| \\
&= \|s(sw)^{n+1} z^* (ws)^{n+1} s - (ws)^n z (sw)^n\| \\
&= \|(ws)^n w z^* w (sw)^n - (ws)^n z (sw)^n\| \\
&= \|wz^*w - z\| \le \varepsilon,
\end{aligned}$$

which implies the existence of φ by universality, and on the other hand,

$$\|\psi(w)\psi(z)\psi(w^*) - \psi(z^{-1})\| = \|tb_0t^* - b_0^*\| = \|a_1^* - b_0^*\| \le \varepsilon,$$
$$\|\psi(s)\psi(z)\psi(s^*) - \psi(z^{-1})\| = \|sb_0s^* - b_0^*\| = \|a_0^* - b_0^*\| \le \varepsilon,$$

which implies the existence of ψ by universality, and moreover,

$$\begin{aligned}\psi\circ\varphi(b_n) &= (ts)^n b_0 (ts)^{-n} = (ts)^{n-1} tsb_0s^*t^*(ts)^{-(n-1)} \\ &= (ts)^{n-1} ta_0^*t^*(ts)^{-(n-1)} = (ts)^{n-1} b_1 (ts)^{-(n-1)} \\ &= \cdots = b_n, \\ \psi\circ\varphi(a_n) &= s(ts)^{-n} b_0^*(ts)^n s = s(ts)^{-(n-1)} s^*t^*b_0^*ts(ts)^{n-1}s \\ &= s(ts)^{-(n-1)} s^*a_1 s(ts)^{n-1}s = s(ts)^{-(n-1)} b_{-1}^*(ts)^{n-1}s \\ &= \cdots = sb_{-n}^*s = a_n\end{aligned}$$

and also $\varphi\circ\psi(z) = \varphi(b_0) = z$. □

Theorem 3.2.3. *The canonical map from the soft $C(S^1) \rtimes_{\rho,\varepsilon} \Gamma$ for $\varepsilon < 2$ to the usual $C(S^1) \rtimes_{\rho,0} \Gamma$ induces an isomorphism at the level of K-theory groups. It follows that their K_0-group is $\mathbb{Z}^6$ and K_1 is zero.*

Proof. This is done by using the isomorphism obtained above and the six-term exact sequence for the crossed product by $\Gamma \cong \mathbb{Z}_2 * \mathbb{Z}_2$ used above and the five lemma. □

Theorem 3.2.4. *There exists a continuous field of C^*-algebras over the interval $[0, 2]$ such that $C(S^1) \rtimes_{\rho,\varepsilon} \Gamma$ is the fiber at ε, and the function f_a for $a \in C(S^1) \rtimes_{\rho,2} \Gamma$ defined by $f_a(\varepsilon) = \varphi_\varepsilon(a) \in C(S^1) \rtimes_{\rho,\varepsilon} \Gamma$ is a continuous section.*

Remark. Without using the identification of $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}_2$ with $\mathbb{Z}_2 * \mathbb{Z}_2$, the soft $C(S^1) \rtimes_{\rho,\varepsilon} \Gamma$ can be isomorphic to the hard crossed product of Γ by the C^*-algebra defined to be the universal unital C^*-algebra generated by unitaries $u_{n,m}$ for $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_2$ such that

$$\|u_{n,0} - u_{n,1}\| \le \varepsilon, \quad \|u_{n,m} - u_{n+1,m}\| \le \varepsilon$$

for $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_2$. However, its homotopy class has not been determined yet (probably).

Notes. This section of two subsections is based on the paper [8] of Elliott-Exel-Loring. In [23] of the author, the flip crossed products of the isometric versions of the soft tori are also considered.

4 More Soft C^*-algebras

4.1 Softening C^*-algebras

Definition 4.1.1. Define a soft C^*-algebra by unitaries to be the universal C^*-algebra generated by unitaries $u_1, \cdots u_k$ such that

$$\|r_p(u_1, \cdots, u_k) - 1\| \leq \varepsilon$$

$(1 \leq p \leq l)$ for a fixed $\varepsilon \in [0, 2]$, where each r_p is a monomial of k variables. Denote it by F_ε.

Definition 4.1.2. Let Γ be a finitely generated, finitely presented group with generators $g_1, g_2, \cdots, g_k$ and relations $r_p(g_1, \ldots, g_k) = 1$ $(1 \leq p \leq l)$, where τ_p are monomials in g_j and their inverses. Define a soft group C^*-algebra to be the universal C^*-algebra generated by unitaries $u_1, \cdots, u_k$ such that $\|r_p(u_1, \cdots, u_k) - \rho_p\| \leq \varepsilon_p$ for some $\rho_p \in \mathbb{T}$ (1-torus) and $\varepsilon_p \in [0, 2]$. Denote it by $C_\varepsilon^*(\Gamma)$.

Definition 4.1.3. Let $\mathfrak{A}$ be a unital C^*-algebra generated by a finite set $\{a_i\}$ and Γ a discrete group finitely generated by $\{g_j\}$. Define a soft crossed product C^*-algebra to be the universal C^*-algebra generated by $\mathfrak{A}$ and unitaries u_g $(g \in \Gamma)$ such that for each i, j,

$$\|u_{g_j} a_i u_{g_j}^* - \rho_{i,j} \alpha_{g_j}(a_i)\| \leq \varepsilon_{i,j}$$

for some $\rho_{i,j} \in \mathbb{T}$ and $\varepsilon_{i,j} \in [0, 2]$, where α means an action of generators (only) of Γ on $\mathfrak{A}$, and u is a unitary representaion of Γ. Denote it by $\mathfrak{A} \rtimes_{\alpha,\varepsilon} \Gamma$. Denote by $\mathfrak{A} \rtimes_\alpha \Gamma$ the ordinary crossed product C^*-algebra of $\mathfrak{A}$ by α of Γ.

Example 4.1.4. Define the soft group C^*-algebra $C_\varepsilon^*(\mathbb{Z}^3)$ of $\mathbb{Z}^3$ to be the universal C^*-algebra generated by three unitaries u_1, u_2, u_3 such that for each i, j,

$$\|u_i u_j - u_j u_i\| \leq \varepsilon_{i,j}$$

for some $\varepsilon_{i,j} > 0$.

Proposition 4.1.5. *The C^*-algebra $C_\varepsilon^*(\mathbb{Z}^3)$ is isomorphic to $W_\varepsilon \rtimes_\tau \mathbb{Z}$, where W_ε is the universal C^*-algebra generated by unitaries U_l, V_l $(l \in \mathbb{Z})$ such that $\|U_l V_l - V_l U_l\| \leq \varepsilon_{1,2}$, $\|U_{l+1} - U_l\| \leq \varepsilon_{1,3}$, and $\|V_{l+1} - V_l\| \leq \varepsilon_{2,3}$ for $l \in \mathbb{Z}$, and the action τ on W_ε is defined by $\tau(U_l) = U_{l+1}$ and $\tau(V_l) = V_{l+1}$ for $l \in \mathbb{Z}$.*

Proof. Set $U'_l = u_3^l u_1 u_3^{-l}$ and $V'_l = u_3^l u_2 u_3^{-l}$ for $l \in \mathbb{Z}$. Then

$$\|U'_l V'_l - V'_l U'_l\| \le \varepsilon_{1,2}, \quad \|U'_{l+1} - U'_l\| \le \varepsilon_{1,3}, \quad \|V'_{l+1} - V'_l\| \le \varepsilon_{2,3}.$$

Therefore, there is a $*$-homomorphism π from W_ε to $C^*_\varepsilon(\mathbb{Z}^3)$ such that $\pi(U_l) = U'_l$ and $\pi(V_l) = V'_l$, which can be extended to $W_\varepsilon \rtimes_\tau \mathbb{Z}$ by setting $\pi(w) = u_3$, where w is the unitary implementing the action τ of $\mathbb{Z}$. Conversely, the unitaries U_0, V_0, and w satisfy the same relations as u_1, u_2, u_3 in $C^*_\varepsilon(\mathbb{Z}^3)$. Hence there is a $*$-homomorphism ρ from $C^*_\varepsilon(\mathbb{Z}^3)$ to $W_\varepsilon \rtimes_\tau \mathbb{Z}$ such that $\rho(u_1) = U_0$, $\rho(u_2) = V_0$, and $\rho(u_3) = w$. By definition, π and ρ are inverses each other. □

More generally,

Example 4.1.6. Define the soft group C^*-algebra $C^*_\varepsilon(\mathbb{Z}^{n+1})$ of $\mathbb{Z}^{n+1}$ to be the universal C^*-algebra generated by unitaries u_j $(1 \le j \le n+1)$ such that for each i, j,

$$\|u_i u_j - u_j u_i\| \le \varepsilon_{i,j}$$

for some $\varepsilon_{i,j} > 0$.

Proposition 4.1.7. *The C^*-algebra $C^*_\varepsilon(\mathbb{Z}^{n+1})$ is isomorphic to $W_{\varepsilon,n} \rtimes_\tau \mathbb{Z}$, where $W_{\varepsilon,n}$ is the universal C^*-algebra generated by unitaries $U_{j,l}$ $(1 \le j \le n, l \in \mathbb{Z})$ such that $\|U_{j,l}U_{k,l} - U_{k,l}U_{j,l}\| \le \varepsilon_{j,k}$, $\|U_{j,l+1} - U_{j,l}\| \le \varepsilon_{n+1,j}$, and and the action τ on $W_{\varepsilon,n}$ is defined by $\tau(U_{j,l}) = U_{j,l+1}$ for $l \in \mathbb{Z}$.*

Proof. Set $U'_{j,l} = u_{n+1}^l u_j u_{n+1}^{-l}$ for $1 \le j \le n$, $l \in \mathbb{Z}$. Then

$$\|U'_{j,l}U'_{k,l} - U'_{k,l}U'_{j,l}\| \le \varepsilon_{j,k}, \quad \|U'_{j,l+1} - U'_{j,l}\| \le \varepsilon_{n+1,j}.$$

Therefore, there is a $*$-homomorphism π from $W_{\varepsilon,n}$ to $C^*_\varepsilon(\mathbb{Z}^{n+1})$ such that $\pi(U_{j,l}) = U'_{j,l}$, which can be extended to $W_{\varepsilon,n} \rtimes_\tau \mathbb{Z}$ by setting $\pi(w) = u_{n+1}$, where w is the unitary implementing the action τ of $\mathbb{Z}$. Conversely, the unitaries $U_{j,0}$ $(1 \le j \le n)$, and w satisfy the same relations as u_j $(1 \le j \le n+1)$ in $C^*_\varepsilon(\mathbb{Z}^{n+1})$. Hence there is a $*$-homomorphism ρ from $C^*_\varepsilon(\mathbb{Z}^{n+1})$ to $W_{\varepsilon,n} \rtimes_\tau \mathbb{Z}$ such that $\rho(u_j) = U_{j,0}$ $(1 \le j \le n)$, and $\rho(u_{n+1}) = w$. By definition, π and ρ are inverses each other. □

Theorem 4.1.8. *Let $\mathfrak{A}$ be a finitely generated and finitely polynomially presented C^*-algebra. Let Γ be a finitely generated and finitely presented group. Suppose that there is an action α of generators (only) of Γ on $\mathfrak{B}$ by monomial automorphisms. Then there is an action β of Γ on a C^*-algebra $\mathfrak{B}$ such that*

$$\mathfrak{A} \rtimes_{\alpha,\varepsilon} \Gamma \cong \mathfrak{B} \rtimes_\beta \Gamma.$$

Proof. Suppose that $\mathfrak{A}$ is generated by unitaries a_j $(1 \leq j \leq m)$ subject to the polynomial relations r_k, $k \in K$ a finite set. Let g_j $(1 \leq j \leq n)$ be generators of Γ such that $z_p(g_1, \cdots, g_n) = 1$ for $p \in P$ a finite set. Assume that $\alpha_{g_l}(a_j) = p_{j,l}(a_1, \cdots, a_m)$. Note that $r_k(\alpha^s_{g_l}(a_1), \cdots, \alpha^s_{g_l}(a_m)) = 0$ for $s \in \mathbb{Z}$.

Define $\mathfrak{B}$ to be the universal C^*-algebra generated by unitaries $b_{j,g}$ and $b_{j,g,l}$ for $1 \leq j \leq m$, $1 \leq l \leq n$, $g \in \Gamma$ subject to the relations

$$r_k(b_{1.g}, \cdots, b_{m,g}) = 0, \quad r_k(b_{1,g,l}, \cdots, b_{m,g,l}) = 0,$$

and $\|b_{j,gg_l} - b_{j,g,l}\| \leq \varepsilon_{j,l}$ and $b_{j,g,l} = p_{j,l}(b_{1,g}, \cdots, b_{m,g})$.

Note that $\mathfrak{A} \rtimes_{\alpha,\varepsilon} \Gamma$ is the universal C^*-algebra generated by $\mathfrak{A}$ and unitaries $u_1, \cdots, u_n$ such that $z_p(u_1, \cdots, u_n) = 1$, $\|u_l a_j u_l^* = \alpha_{g_l}(a_j)\| \leq \varepsilon_{j,l}$. Set

$$a_{j,g} = u_g a_j u_g^*, \quad a_{j,g,l} = u_g \alpha_{g_l}(b_j) u_g^*.$$

Then

$$\begin{aligned} \|a_{j,gg_l} - a_{j,g,l}\| &= \|u_{gg_l} a_j u_{gg_l}^* - u_g \alpha_{g_l}(a_j) u_g^*\| \\ &\leq \|u_l a_j u_l^* - \alpha_{g_l}(a_j)\| \leq \varepsilon_{j,l}. \end{aligned}$$

Moreover, it follows that

$$r_k(a_{1,g}, \cdots, a_{m,g}) = 0, \quad r_k(a_{1,g,l}, \cdots, a_{m,g,l}) = 0$$

Note also that $a_{j,g,l} = p_{j,l}(a_{1,g}, \cdots, a_{m,g})$. Hence there is a $*$-homomorphism φ from $\mathfrak{B}$ to $\mathfrak{A} \rtimes_{\alpha,\varepsilon} \Gamma$ such that $\varphi(b_{j,g}) = a_{j,g}$ and $\varphi(b_{j,g,l}) = a_{j,g,l}$. Define the automorphisms β^t of $\mathfrak{B}$ $(1 \leq t \leq n)$ by $\beta^t(b_{j,g}) = b_{j,g_tg}$ and $\beta^t(a_{j,g,l}) = b_{j,g_tg,l}$. They satisfy $z_p(\beta^1, \cdots, \beta^n) = 1$. Hence they determine an action β of Γ on $\mathfrak{B}$. Since $a_{j,g_tg} = u_t a_{j,g} u_t^*$ and $a_{j,g_tg,l} = u_t a_{j,g,l} u_t^*$, we can extend φ to $\mathfrak{B} \rtimes_\beta \Gamma$ by setting $\varphi(U_t) = u_t$, where we denote by U_t the unitary implementing β^t.

On the other hand, the elements $b_{j,1}$, $b_{j,1,l}$, and U_l satisfy the following relations:

$$\begin{aligned} &\|U_l b_{j,1} U_l^* - b_{j,1,l}\| \leq \varepsilon_{j,l}, \quad r_k(b_{1,1}, \cdots, b_{m,1}) = 0, \\ &r_k(b_{1,1,l}, \cdots, b_{m,1,l}) = 0, \quad a_{j,1,l} = p_{j,l}(b_{1,1}, \cdots, b_{m,1}). \end{aligned}$$

Hence there is a $*$-homomorphism ψ from $\mathfrak{A} \rtimes_{\alpha,\varepsilon} \Gamma$ to $\mathfrak{B} \rtimes_\beta \Gamma$ such that $\psi(a_j) = b_{j,1}$, $\psi(\alpha_{g_l}(a_j)) = b_{j,1,l}$, and $\psi(u_l) = U_l$. Clearly, φ and ψ are inverses each other. □

Proposition 4.1.9. *The soft C^*-algebras $\{F_\varepsilon\}_{\varepsilon\in[0,2]}$ for a set of monomials $\{r_p\}_{p=1}^k$ and $\varepsilon \in [0,2]$ form a right continuous field of C^*-algebras over $[0,2]$.*

Proof. Assume that each F_ε is finitely generated by l unitaries. Let $C^*(\mathbb{F}_l)$ be the full group C^*-algebra of the free group F_l of l generators. There is the canonical $*$-homomorphism φ_ε from $C^*(\mathbb{F}_l)$ to F_ε by universality. Let J_ε be the kernel of φ_ε. To have right continuity, we show that $J_\varepsilon = J_\varepsilon^+$ for $\varepsilon \in [0,2)$, where J_ε^+ is defined to the closure of the union $\cup_{\delta>\varepsilon} J_\delta$. There is a $*$-homomorphism from $C^*(\mathbb{F}_l)/J_\varepsilon$ to $C^*(\mathbb{F}_l)/J_\varepsilon^+$ by universality. Therefore, $J_\varepsilon \subset J_\varepsilon^+$. Its inverse inclusion also holds. □

Proposition 4.1.10. *The soft crossed product C^*-algebras $\{C^*(\mathbb{F}_n) \rtimes_{\mathrm{id},\varepsilon} \mathbb{Z}\}_{\varepsilon\in[0,2]}$ for the identity representation* id *of $\mathbb{Z}$ on $C^*(\mathbb{F}_n)$ form a continuous field of C^*-algebras over $[0,2]$.*

Proof. It is shown above that $C^*(\mathbb{F}_n) \rtimes_{\mathrm{id},\varepsilon} \mathbb{Z}$ is isomorphic to the crossed product $\mathfrak{B} \rtimes_\beta \mathbb{Z}$, where $\mathfrak{B}$ is the universal C^*-algebra generated by unitaries $U_{i,j}$ for $1 \leq i \leq n$, $j \in \mathbb{Z}$ subject to the relations $\|U_{i,j} - U_{i,j+1}\| \leq \varepsilon$, and the implementing unitary V for β satisfies $VU_{i,j}V^* = U_{i,j+1}$. The rest of the proof is the same as above. □

Theorem 4.1.11. *The soft group C^*-algebra $C_\varepsilon^*(\mathbb{Z}^2)$ $(0 < \varepsilon < 2)$ has stable rank equal to ∞.*

Sketch of Proof. There is a unital $*$-homomorphism ψ from $C^*(\mathbb{F}_2)$ onto the tensor product $C([0,1]^{n^2}) \otimes M_{n+1}(\mathbb{C})$ for any $n \in \mathbb{N}$. It follows that the stable rank of $C^*(\mathbb{F}_2)$ is ∞. It is shown that ψ factors through $C_\varepsilon^*(\mathbb{Z}^2)$. □

Proposition 4.1.12. *Let $C_\varepsilon^*(\mathbb{Z}^2) \cong W_{\varepsilon,1} \rtimes_\tau \mathbb{Z}$ as above. Then $W_{\varepsilon,1}$ has stable rank equal to ∞.*

Proof. Use the formula of Rieffel [21]: $\mathrm{sr}(\mathfrak{B} \rtimes_\beta \mathbb{Z}) \leq \mathrm{sr}(\mathfrak{B}) + 1$ for the crossed product $\mathfrak{B} \rtimes_\beta \mathbb{Z}$ of a C^*-algebra $\mathfrak{B}$ by an action β of $\mathbb{Z}$, where $\mathrm{sr}(\cdot)$ means the stable rank (see [21]). □

Similarly,

Proposition 4.1.13. *The soft group C^*-algebras $C_\varepsilon^*(\mathbb{Z}^{n+1})$ $(n \geq 1)$ and $W_{\varepsilon,n}$ both have stable rank ∞.*

4.2 Soft torus extended

Lemma 4.2.1. *If u and v are unitary elements in a C^*-algebra $\mathfrak{A}$, then $\|uv - vu\| \le \varepsilon < 2$ if and only if there exists a self-adjoint element $h \in \mathfrak{A}$ such that $uvu^*v^* = e^{ih}$ and $\|h\| \le \alpha = 2\arcsin(\varepsilon/2)$.*

Proof. Note that

$$2 > \varepsilon \ge \|uv - vu\| = \|uvu^*v^* - 1\|.$$

Thus, let $h = i^{-1}\log(uvu^*v^*)$. Then

$$h^* = -i^{-1}\log(vuv^*u^*) = -i^{-1}\log((uvu^*v^*)^{-1}) = h$$

and $e^{ih} = uvu^*v^*$ and

$$\|h\| = \|\log(uvu^*v^*)\| \le \|\log(e^{i\theta})\|_\infty = \|i\theta\|_\infty$$

where $e^{i\theta}$ is in the spectrum of uvu^*v^*. Since

$$2\sin(\frac{\theta}{2}) = |1 - e^{i\theta}| \le \| 1 - uvu^*v^*\| \le \varepsilon$$

so that $\|h\| \le \sup_\theta |\theta| \le 2\arcsin(\varepsilon/2)$.

Conversely,

$$\|uv - vu\| = \|1 - uvu^*v^*\| = \|1 - e^{ih}\|.$$

Moreover,

$$\|1 - e^{ih}\| = \sup_t |1 - e^{it}| = \sup_t |2\sin(t/2)| \le 2\sin(\alpha/2) = \varepsilon$$

where t is in the spectrum of h. □

Now, for $s \in [0, \infty)$, define $\mathfrak{C}_s$ to be the universal C^*-algebra generated by unitaries u_s and v_s, and a self-adjoint h_s such that $u_s v_s u_s^* v_s^* = e^{ih_s}$ and $\|h_s\| \le s$.

For $\varepsilon \in [0, 2)$, the soft torus $C(\mathbb{T}^2)_\varepsilon$ is isomorphic to $\mathfrak{C}_s$ with $s = 2\arcsin(\varepsilon/2)$ by universality. Therefore, we may call $\mathfrak{C}_s$ the extended soft torus (of Cerri). However, only $C(\mathbb{T}^2)_2$ that is isomorphic to the full group C^*-algebra of the free group F_2 never happen in $\mathfrak{C}_s$.

Let $\mathfrak{D}_s$ be the universal C^*-algebra generated by unitaries u_n and self-adjoint h_n for $n \in \mathbb{Z}$ such that $u_n u_{n+1}^* = e^{ih_n}$ and $\|h_n\| \le s$ for all n. Define an automorphism β of $\mathfrak{D}_s$ by $\beta(u_n) = u_{n+1}$ and $\beta(h_n) = h_{n+1}$ for all n.

Proposition 4.2.2. *The extended soft torus $\mathfrak{C}_s$ of Cerri is isomorphic to the crossed product $\mathfrak{D}_s \rtimes_\beta \mathbb{Z}$.*

Proof. Note that $\mathfrak{D}_s \rtimes_\beta \mathbb{Z}$ is the universal C^*-algebra generated by $\mathfrak{D}_s$ and an unitary element w implementing the action β, so that $\beta(x) = wxw^*$ for $x \in \mathfrak{D}_s$, and

$$e^{ih_0} = u_0 u_1^* = u_0 \beta(u_0)^* = u_0 w u_0^* w^*.$$

By the universal property of $\mathfrak{C}_s$, there exists a $*$-homomorphism $\varphi : \mathfrak{C}_s \to \mathfrak{D}_s \rtimes_\beta \mathbb{Z}$ such that $\varphi(u_s) = u_0$, $\varphi(v_s) = w$, and $\varphi(h_s) = h_0$.

In order to define the inverse of φ, observe that

$$\begin{aligned}
&(v_s^n u_s v_s^{-n})(v_s^{n+1} u_s v_s^{-n-1})^* = v_s^n u_s v_s u_s^* v_s^* v_s^{-n} \\
&= v_s^n e^{ih_s} v_s^{-n} = e^{iv_s^n h_s v_s^{-n}}, \quad \|v_s^n h_s v_s^{-n}\| \le s, \\
&v_s(v_s^n u_s v_s^{-n})v_s^{-1} = v_s^{n+1} u_s v_s^{-n-1}, \quad \text{and} \\
&v_s(v_s^n h_s v_s^{-n})v_s^{-1} = v_s^{n+1} h_s v_s^{-n-1}.
\end{aligned}$$

Thus, by the universal property of $\mathfrak{D}_s \rtimes_\beta \mathbb{Z}$ there is a $*$-homomorphism $\psi : \mathfrak{D}_s \rtimes_\beta \mathbb{Z} \to \mathfrak{C}_s$ such that $\psi(u_n) = v_s^n u_s v_s^{-n}$, $\psi(h_n) = v_s^n h_s v_s^{-n}$, and $\psi(w) = v_s$.

Also compute that

$$\begin{aligned}
&\psi \circ \varphi(u_s) = \psi(u_0) = u_s, \quad \psi \circ \varphi(v_s) = \psi(w) = v_s, \\
&\psi \circ \varphi(h_s) = \psi(h_0) = h_s, \quad \text{and} \\
&\varphi \circ \psi(u_n) = \varphi(v_s^n u_s v_s^{-n}) = w^n u_0 w^{-n} = u_n, \\
&\varphi \circ \psi(h_n) = \varphi(v_s^n h_s v_s^{-n}) = w^n h_0 w^{-n} = h_n
\end{aligned}$$

as required. □

By the universal property of $\mathfrak{D}_s$, there exists a $*$-homomorphism $\psi_s : \mathfrak{D}_s \to C(S^1)$ such that $\psi_s(u_n) = z$ and $\psi(h_n) = 0$ for all $n \in \mathbb{Z}$, where z is the canonical unitary generator of $C(S^1)$.

Proposition 4.2.3. *The homomorphism ψ_s is a homotopy equivalence between $\mathfrak{D}_s$ and $C(S^1)$, so that the induced maps on their K-theory groups are isomorphisms, so that $K_j(\mathfrak{D}_s) \cong \mathbb{Z}$ for $j = 0, 1$.*

Theorem 4.2.4. *The K-theory group $K_j(\mathfrak{C}_s)$ is isomorphic to $\mathbb{Z}^2$, for $j =$ $0, 1$.*

Theorem 4.2.5. *There exists a continuous field of C^*-algebras over the interval $[0, t]$ for any $t \ge 0$ such that the fiber at $s \in [0, t]$ is $\mathfrak{C}_s$ and continuous operator fields are given as in the case of the soft torus.*

Recall that the rotation algebra denoted by $\mathbb{T}^2_\theta$ for $\theta \in \mathbb{R}$ is defined to be the universal C^*-algebra generated by two unitaries u_θ and v_θ such that $u_\theta v_\theta u_\theta^* v_\theta^* = e^{2\pi i\theta}$, and is isomorphic to the crossed product $C(S^1) \rtimes_\theta \mathbb{Z}$, where the action θ is given by $\theta(f)(z) = f(e^{-2\pi i\theta}z)$ for $f \in C(S^1)$ and $z \in S^1$ (cf. Rieffel [20]).

Suppose that $s \geq 2\pi|\theta|$. Then there exists a $*$-homomorphism $\varphi : \mathfrak{C}_s \to \mathbb{T}^2_\theta$ such that $\varphi(u_s) = u_\theta$, $\varphi(v_s) = v_\theta$, and $\varphi(h_s) = 2\pi\theta 1$.

Theorem 4.2.6. *For $s \geq 2\pi|\theta|$, the induced map $\varphi_* : K_j(\mathfrak{C}_s) \to K_j(\mathbb{T}^2_\theta)$ is an isomorphism, for $j = 0, 1$.*

Proof. Note that $v_\theta u_\theta v_\theta^{-1} = e^{-2\pi i\theta}u_\theta$. Also, $(e^{-2\pi in\theta}z)(e^{-2\pi i(n+1)\theta}z)^{-1} = e^{2\pi i\theta}1$. Thus, there is a $*$-homomorphism $\rho : \mathfrak{D}_s \to C(S^1)$ such that $\rho(u_n) = e^{-2\pi in\theta}z$ and $\rho(h_n) = 2\pi\theta$. Check that ρ is covariant:

$$\rho \circ \beta(u_n) = \rho(u_{n+1}) = e^{-2\pi i(n+1)\theta}z = \theta \circ \rho(u_n)$$
$$\rho \circ \beta(h_n) = \rho(h_{n+1}) = 2\pi\theta 1 = \theta \circ \rho(h_n),$$

so then ρ can be extended to $\rho^\sim : \mathfrak{D}_s \rtimes_\beta \mathbb{Z} \to C(S^1) \rtimes_\theta \mathbb{Z}$. In fact, $\rho^\sim = \varphi$. Applying the Pimsner-Voiculescu exact sequence, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_j(\mathfrak{D}_s) & \longrightarrow & K_j(\mathfrak{C}_s) & \longrightarrow & K_{j+1}(\mathfrak{D}_s) & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \rho_*} & & \| \\
0 & \longrightarrow & K_j(C(S^1)) & \longrightarrow & K_j(\mathbb{T}^2_\theta) & \longrightarrow & K_{j+1}(C(S^1)) & \longrightarrow & 0
\end{array}$$

for $j = 0, 1$ (mod 2). Since we have that ρ_* is an isomorphism, the Five Lemma completes the proof. □

It is known that $K_0(C(\mathbb{T}^2))$ is generated by the classes [1] and $[B]$, where B is the Bott projection. As for the map $\delta : K_0(C(\mathbb{T}^2)) \to K_1(C(S^1))$ in the P-V sequence, we have $\delta([B]) = [z]$. Also, $[B]$ generates the kernel of $\tau_* : K_0(C(\mathbb{T}^2)) \to \mathbb{Z}$, where τ_* is the map induced by a unital trace τ on $C(\mathbb{T}^2)$.

Let $s = 2\pi|\theta|$. Let $[B_0] = \psi_*^{-1}([B]) \in K_0(\mathfrak{C}_s)$, where $\psi = \varphi$ as $\theta = 0$, and $[B_\theta] = \varphi_*([B_0]) \in K_0(\mathbb{T}^2_\theta)$.

Proposition 4.2.7. *If $\delta : K_0(\mathbb{T}^2_\theta) \to K_1(C(S^1))$ is the map in the Pimsner-Voiculescu exact sequence, then $\delta([B_\theta]) = [z]$.*

Proof. Using that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0(C(S^1)) & \longrightarrow & K_0(C(\mathbb{T}^2)) & \xrightarrow{\delta} & K_1(C(S^1)) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \psi_*} & & \uparrow{\scriptstyle \varphi_*} & & \uparrow{\scriptstyle \psi_*} & & \\
0 & \longrightarrow & K_0(\mathfrak{D}_s) & \longrightarrow & K_0(\mathfrak{C}_s) & \xrightarrow{\delta} & K_1(\mathfrak{D}_s) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \rho_*} & & \\
0 & \longrightarrow & K_0(C(S^1)) & \longrightarrow & K_0(\mathbb{T}^2_\theta) & \xrightarrow{\delta} & K_1(C(S^1)) & \longrightarrow & 0
\end{array}$$

we compute that

$$\delta([B_0]) = \delta(\varphi_*^{-1}([B])) = \psi_*^{-1}(\delta([B])) = \psi_*^{-1}([z]) = [u_0], \quad \text{and so}$$
$$\delta([B_\theta]) = \delta(\varphi_*([B_0)) = \rho_*(\delta([B_0])) = \rho_*([u_0]) = [z].$$

□

Consider the trace τ on $\mathbb{T}^2_\theta$ given by $\tau(f) = \int_{S^1} f(z)dz$ for $f \in C(S^1)$. It is shown by Rieffel [20] that if $x \in [0,1] \cap (\mathbb{Z} + \theta\mathbb{Z})$ for θ irrational, then there exists a projection p_x such that $\tau(p_x) = x$. Furthermore, it shown by Pimsner and Voiculescu [19] that there exists $[q_\theta] \in K_0(\mathbb{T}^2_\theta)$ such that $\tau_*([q_\theta]) = \theta$ and $\delta([q_\theta]) = [z]$. Note that the classes [1] and $[q_\theta]$ are generators of $K_0(\mathbb{T}^2_\theta)$.

Proposition 4.2.8. *The map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \tau_*([B_x])$ is continuous, where each τ is the trace on $\mathbb{T}^2_x$.*

Corollary 4.2.9. *We have $f(x) = x$ for any $x \in \mathbb{R}$.*

Theorem 4.2.10. *If $s \geq 0$, then the positive cone $K_0(\mathfrak{C}_s)_+$ is contained in $\{(n,m) \in \mathbb{Z}^2 : |m|s \leq 2\pi n\}$.*

Proof. Take $y = n[1] + m[B_s] \in K_0(\mathfrak{C}_s)_+$. As for $\theta = s/2\pi \geq 0$, we have $\tau_* \circ \varphi_*(y) = n + m\theta \geq 0$, while for $-\theta$ we have $\tau_* \circ \varphi_*(y) = n - m\theta \geq 0$. Hence, $2\pi n + ms \geq 0$ and $2\pi n - ms \geq 0$. □

Proposition 4.2.11. *It $s \geq 0$, then we have*

(1) : *$K_0(\mathfrak{C}_{s'})_+ \subset K_0(\mathfrak{C}_s)_+$ for $s' \geq s$;*

(2) : *$\cap_{s \geq 0} K_0(\mathfrak{C}_s)_+ = \{(n,0) : n \geq 0\} \neq K_0(\mathfrak{C}_s)_+$;*

(3) : *If $(n,m) \in K_0(\mathfrak{C}_s)_+$, then $(n,-m) \in K_0(\mathfrak{C}_s)_+$;*

(4) : *$K_0(\mathfrak{C}_s)_+ \neq K_0(C(\mathbb{T}^2))_+$ for $s \neq 0$.*

Sketch of Proof. As for (1), the canonical $*$-homomorphism from $\mathfrak{C}_{s'}$ to $\mathfrak{C}_s$ induces the K-theory group homomorphism from $K_0(\mathfrak{C}_{s'})$ to $K_0(\mathfrak{C}_s)$, and moreover, the image of $K_0(\mathfrak{C}_{s'})_+$ is contained in $K_0(\mathfrak{C}_s)_+$, where this is true for C^*-algebras and their $*$-homomorphisms. In fact, the induced map is an isomorphism. Note also that if $s' \geq s$ and $|m|s' \leq 2\pi n$, then $|m|s \leq 2\pi n$.

As for (2), note that the equations $2\pi n + ms \geq 0$ and $2\pi n - ms \geq 0$ for any $s \geq 0$ require $m = 0$. Since $K_0(\mathfrak{C}_s)_+ - K_0(\mathfrak{C}_s)_+ = K_0(\mathfrak{C}_s)$, where this is true for a unital C^*-algebra, and $K_0(\mathfrak{C}_s) \cong \mathbb{Z}^2$ we can not have $\{(n,0) : n \geq 0\} = K_0(\mathfrak{C}_s)_+$.

As for (3), note that if $|m|s \leq 2\pi n$, then $|-m|s \leq 2\pi n$.

As for (4), note that $K_0(C(\mathbb{T}^2))_+ = \{(n,m) : n > 0\} \cup \{(0,0)\}$. Also, when $\theta = 0$, we have $\tau_* \circ \varphi_*(y) = n$. In fact, note that the class $[B_0]$ may be viewed as $[B_0] - [1] \neq 0$. □

Proposition 4.2.12. *We have* $K_0(C(\mathbb{T}^2))_+ = \cup_{s>0} K_0(\mathfrak{C}_s)_+$.

Sketch of Proof. Take $(n,m) \in K_0(C(\mathbb{T}^2))_+$. We may assume that $n > 0$ and $m > 0$. Since $(n,m) = (n-1,0) + (1,m)$ and $(n-1,0) \in K_0(\mathfrak{C}_s)_+$, it is enough to prove that $(1,m) \in K_0(\mathfrak{C}_s)_+$ for some $s > 0$.

There exists $\epsilon_0 \in (0,2]$ such that $(1,1) \in K_0(\mathbb{T}^2_{\varepsilon_0})_+$, and then $(1,m) \in K_0(\mathbb{T}^2_\varepsilon)_+$ for $\varepsilon = \varepsilon_0/m$. Hence, if $s = 2\arcsin(\varepsilon_0/2m)$, then $(1,m) \in K_0(\mathfrak{C}_s)_+$. □

Proposition 4.2.13. *For any $s_0 > 0$, there exists an increasing sequence of positive real numbers $(s_j)_{j\in\mathbb{N}}$ and an increasing sequence of positive integers $(n_j)_{j\in\mathbb{N}}$ such that $s_j > 2\pi n_j$, so that $(n_j,1) \notin K_0(\mathfrak{C}_{s_j})_+$, and $(n_j,1) \in K_0(\mathfrak{C}_{s_{j-1}})_+$ where $s_{j-1} \leq 2\pi n_j$, for all $j \in \mathbb{N}$.*

Sketch of Proof. For any $s > 0$, the class $[B_s]$ may be viewed as $[B_s] - [1_n] \neq 0$, where the Bott projection B_s is in $M_k(\mathfrak{C}_s)$ and 1_n is the $n \times n$ identity matrix. Thus, $(n,1) \in K_0(\mathfrak{C}_s)_+$. If $s' > 2\pi n \geq s$, then $(n,1) \notin K_0(\mathfrak{C}_{s'})_+$. □

Lemma 4.2.14. *Assume that $(n,m) \in K_0(\mathfrak{C}_s)_+$ and $s'm > 2\pi n$, so that $(n,m) \in K_0(\mathfrak{C}_{s'})_+$, where $s < s'$. Then $\mathfrak{C}_s$ is not homotopically equivalent to $\mathfrak{C}_{s'}$.*

Proof. Assume that there exists a homotopy equivalence $\phi : \mathfrak{C}_s \to \mathfrak{C}_{s'}$. Then ϕ_* on K_0 is an isomorphism such that $x \in K_0(\mathfrak{C}_s)_+$ if and only if $\phi_*(x) \in K_0(\mathfrak{C}_{s'})_+$. Since ϕ_* is viewed as a matrix $D \in GL_2(\mathbb{Z})$, we have

$$D = \begin{pmatrix} 1 & k \\ 0 & \pm 1 \end{pmatrix}$$

since $\phi(1) = 1$, where elements of $K_0(\mathfrak{C}_s)$ are viewed as $(n, m) = n[1]+m[B_s]$ with respect to the basis $\{[1], [B_s]\}$.

If $(n, m) \in K_0(\mathfrak{C}_s)_+$, then $ms \leq 2\pi n$. Therefore, we have $ms' \leq 2\pi(n \pm km)$. It follows that $ms' \leq 2\pi n$, which is contradiction. □

Consequently,

Theorem 4.2.15. *Let $s_0 \geq 0$. There exists an increasing sequence of positive real numbers $(s_j)_{j\in\mathbb{N}}$ such that the extended soft tori of the family $(\mathfrak{C}_{s_j})_{j\in\mathbb{N}}$ are not homotopically equivalent to each other.*

Proposition 4.2.16. *The unital traces of $\mathfrak{C}_s$ form a separating family of maps for $K_0(\mathfrak{C}_s)$, i.e., if $x, y \in K_0(\mathfrak{C}_s)$ with $x \neq y$, then there exists a trace τ of $\mathfrak{C}_s$ such that $\tau_*(x) \neq \tau_*(y)$.*

Sketch of Proof. Let $x = n[1] + m[B_s]$, $y = r[1] + s[B_s] \in K_0(\mathfrak{C}_s)$. Let $\theta \in (0, s/2\pi)$ irrational. Define the trace τ of $\mathfrak{C}_s$ to be the composite of the unique trace of $\mathbb{T}^2_\theta$ with the canonical $*$-homomorphism from $\mathfrak{C}_s$ to $\mathbb{T}^2_\theta$. Then $\tau_*(x) = n + m\theta$ and $\tau_*(y) = r + s\theta$. Therefore, $\tau_*(x) = \tau_*(y)$ if and only if $n = r$ and $m = s$, that is, $x = y$. □

Notes. The first subsection of this section is based on the paper [13] of Farsi. The second subsection of this section is based on the paper [4] of Cerri. In [24] of the author, it is shown that the stable rank of the isometric versions of the soft tori is equal to infinity.

5 Finite dimensional representations

5.1 Those of the soft torus

Remind that $\mathfrak{B}_\varepsilon$ is the universal C^*-algebra generated by unitaries u_n for $n \in \mathbb{Z}$ such that $\|u_n - u_{n+1}\| \leq \varepsilon \leq 2$ for all n.

Lemma 5.1.1. *For $\varepsilon < 2$, $\mathfrak{B}_\varepsilon$ is isomorphic to the universal C^*-algebra generated by a unitary v_0 and self-adjoint elements h_n for $n \in \mathbb{Z}$ such that*

$$\|h_n\| \leq \frac{2}{\pi} \arcsin(\frac{\varepsilon}{2}),$$

denoted by $\mathfrak{B}'_\varepsilon$.

Proof. Define a $*$-homomorphism $\varphi : \mathfrak{B}_\varepsilon \to \mathfrak{B}'_\varepsilon$ by

$$\varphi(u_n) = \begin{cases} e^{i\pi h_n} \cdots e^{i\pi h_1} v_0 & n > 0, \\ v_0 & n = 0, \\ e^{-i\pi h_n} \cdots e^{-i\pi h_{-1}} v_0 & n < 0. \end{cases}$$

Indeed, check that if $n > 0$,

$$\begin{aligned} \|\varphi(u_n) - \varphi(u_{n+1})\| &= \|e^{i\pi h_n} \cdots e^{i\pi h_1} v_0 - e^{i\pi h_{n+1}} \cdots e^{i\pi h_1} v_0\| \\ &\leq \|1 - e^{i\pi h_{n+1}}\| \\ &= \sup_{\lambda \in \mathrm{sp}(h_{n+1})} |1 - e^{i\pi\lambda}| \end{aligned}$$

by spectral theory, where $\mathrm{sp}(h_{n+1})$ is the spectrum of h_{n+1}, so that $|\lambda| \leq \|h_{n+1}\|$. Since $|1 - e^{ix}| = |2\sin(x/2)|$ as shown before, the supremum is estimated by

$$2\sin(\frac{\pi\|h_{n+1}\|}{2}) \leq 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Similarly, the norms in other cases can be estimated by ε. The universal property of $\mathfrak{B}_\varepsilon$ ensures of φ being defined.

On the other hand, define a $*$-homomorphism $\psi : \mathfrak{B}'_\varepsilon \to \mathfrak{B}_\varepsilon$ by

$$\psi(v_0) = u_0 \quad \text{and} \quad \psi(h_n) = (i\pi)^{-1} \log(u_n u_{n-1}^*)$$

for $n > 0$ and also

$$\psi(h_n) = (i\pi)^{-1} \log(u_{n+1} u_n^*)$$

for $n < 0$ (This should be correct). Indeed, check that

$$\begin{aligned} \psi(h_n)^* &= (-i\pi)^{-1} \log(u_{n-1} u_n^*) \\ &= (-i\pi)^{-1} \log((u_n u_{n-1}^*)^{-1}) = \psi(h_n) \end{aligned}$$

by spectral theory, and also

$$\begin{aligned} \|\psi(h_n)\| &= \pi^{-1} \|\log(u_n u_{n-1}^*)\| \\ &\leq \pi^{-1} \|\log(e^{i\theta})\|_\infty = \pi^{-1} \|i\theta\|_\infty \end{aligned}$$

where $e^{i\theta}$ is in the spectrum of $u_n u_{n-1}^*$ and $\|\cdot\|_\infty$ is the supremum norm over the spectrum. Since

$$\begin{aligned} \varepsilon &\geq \|1 - u_n u_{n-1}^*\| = \sup_{e^{i\theta}} |1 - e^{i\theta}| \\ &= \sup_{e^{i\theta}} |2\sin(\frac{\theta}{2})| \end{aligned}$$

where each θ for $e^{i\theta}$ in the spectrum is in $(-\pi, \pi)$. It follows that

$$|\theta| \leq 2\arcsin(\frac{\varepsilon}{2}).$$

Therefore, $\|\psi(h_n)\|$ is estimated by the above upper bound divided by π. The universal property of $\mathfrak{B}'_\varepsilon$ ensures of ψ being defined.

Clearly, φ and ψ are inverses each other. Indeed, check that for $n > 0$,

$$\begin{aligned}
\psi \circ \varphi(u_n) &= \psi(e^{i\pi h_n} \cdots e^{i\pi h_1} v_0) \\
&= (u_n u_{n-1}^*)(u_{n-1} u_{n-2}^*) \cdots (u_1 u_0^*) u_0 = u_n, \\
\psi \circ \varphi(u_{-n}) &= \psi(e^{-i\pi h_{-n}} \cdots e^{-i\pi h_{-1}} v_0) \\
&= (u_{-n} u_{-n+1}^*)(u_{-n+1} u_{-n+2}^*) \cdots (u_{-1} u_0^*) u_0 = u_{-n},
\end{aligned}$$

and also, for $n > 0$,

$$\begin{aligned}
\varphi \circ \psi(h_n) &= \varphi((i\pi)^{-1} \log(u_n u_{n-1}^*) = (i\pi)^{-1} \log(\varphi(u_n u_{n-1}^*)) \\
&= (i\pi)^{-1} \log(e^{i\pi h_n} \cdots e^{i\pi h_1} v_0 v_0^* e^{-i\pi h_1} \cdots e^{-i\pi h_{n-1}}) \\
&= (i\pi)^{-1} \log(e^{i\pi h_n}) = h_n, \\
\varphi \circ \psi(h_{-n}) &= \varphi((i\pi)^{-1} \log(u_{-n+1} u_{-n}^*) = (i\pi)^{-1} \log(\varphi(u_{-n+1} u_{-n}^*)) \\
&= (i\pi)^{-1} \log(e^{-i\pi h_{-n+1}} \cdots e^{-i\pi h_{-1}} v_0 v_0^* e^{i\pi h_{-1}} \cdots e^{i\pi h_{-n}}) \\
&= (i\pi)^{-1} \log(e^{i\pi h_{-n}}) = h_{-n}.
\end{aligned}$$

□

Remark. This characterization can be used to show that $\mathfrak{B}_\varepsilon$ is homotopic to $C(\mathbb{T})$ as shown before. To see this, define $*$-homomorphisms $\rho : \mathfrak{B}'_\varepsilon \to C(\mathbb{T})$ and $\lambda : C(\mathbb{T}) \to \mathfrak{B}'_\varepsilon$ by $\rho(v_0) = z$, $\rho(h_n) = 0$, and $\lambda(z) = v_0$. Clearly, $\rho \circ \lambda$ is the identity map on $C(\mathbb{T})$. A homotopy from $\lambda \circ \rho$ to the identity map on $\mathfrak{B}'_\varepsilon$ is given by $\chi_t : \mathfrak{B}'_\varepsilon \to \mathfrak{B}'_\varepsilon$ defined by $\chi_t(v_0) = v_0$ and $\chi_t(h_n) = th_n$.

Recall that a C^*-algebra is residually finite dimensional (or RFD for short) if it has a separating family of finite dimensional representations.

Proposition 5.1.2. *For any $\varepsilon < 2$, $\mathfrak{B}_\varepsilon$ is RFD. In fact, for any nonzero $b \in \mathfrak{B}_\varepsilon$ there exists $n \in \mathbb{N}$, an automorphism β of $M_n(\mathbb{C})$, and a representation $\rho : \mathfrak{B}_\varepsilon \to M_n(\mathbb{C})$ such that $\rho(b) \neq 0$ and $\beta \circ \rho = \rho \circ \alpha$, i.e., ρ is equivariant for the actions β and α, where $\alpha(u_n) = u_{n+1}$.*

Proof. Use the isomorphism $\mathfrak{B}_\varepsilon \cong \mathfrak{B}'_\varepsilon$. Let $\pi : \mathfrak{B}_\varepsilon \to \mathbb{B}(H)$ be a faithful non-degenerate representation, where $\mathbb{B}(H)$ is the C^*-algebra of all bounded operators on a separable Hilbert space.

Let p_m be projections with finite ranks $m \in \mathbb{N}$, converging strongly to the unit of $\mathbb{B}(H)$. Set

$$T_{0,m} = p_m \pi(v_0) p_m, \quad K_{n,m} = p_m \pi(h_n) p_m.$$

Define

$$V_{0,m} = \begin{pmatrix} T_{0,m} & \sqrt{p_m - T_{0,m}T_{0,m}^*} \\ \sqrt{p_m - T_{0,m}^*T_{0,m}} & -T_{0,m}^* \end{pmatrix},$$

$$H_{n,m} = \begin{pmatrix} K_{n,m} & 0 \\ 0 & K_{n,m} \end{pmatrix} \in M_2(p_m\mathbb{B}(H)p_m) \cong M_{2m}(\mathbb{C}).$$

Since

$$V_{0,m}^* = \begin{pmatrix} T_{0,m}^* & \sqrt{p_m - T_{0,m}^*T_{0,m}} \\ \sqrt{p_m - T_{0,m}T_{0,m}^*} & -T_{0,m} \end{pmatrix},$$

It follows that

$$V_{0,m}V_{0,m}^* = \begin{pmatrix} p_m & b_{0,m} \\ c_{0,m} & p_m \end{pmatrix} \quad \text{where}$$

$$b_{0,m} = T_{0,m}\sqrt{p_m - T_{0,m}^*T_{0,m}} - \sqrt{p_m - T_{0,m}T_{0,m}^*}T_{0,m},$$

$$c_{0,m} = \sqrt{p_m - T_{0,m}^*T_{0,m}}T_{0,m}^* - T_{0,m}^*\sqrt{p_m - T_{0,m}T_{0,m}^*}.$$

To show that $b_{0,m} = 0$, observe that

$$T_{0,m}(p_m - T_{0,m}^*T_{0,m}) = (p_m - T_{0,m}T_{0,m}^*)T_{0,m}$$

which implies that

$$T_{0,m}\sqrt{p_m - T_{0,m}^*T_{0,m}} = \sqrt{p_m - T_{0,m}T_{0,m}^*}T_{0,m}$$

because it is the fact that for T a positive operator, its square root $T^{1/2}$ is defined to be a uniform limit of polynomials in the variables 1 and T, so that the first commuting relation above implies the second, as desired. Similarly, $c_{0,m} = 0$. Hence $V_{0,m}V_{0,m}^* = 1$ in $M_{2m}(\mathbb{C})$.

Quite similarly, $V_{0,m}^*V_{0,m} = 1$. Also, $H_{n,m} = H_{n,m}^*$ with

$$\|H_{n,m}\| = \|K_{n,m}\| \le \|h_n\| \le \frac{2}{\pi}\arcsin(\frac{\varepsilon}{2}).$$

Therefore, we obtain by universality a representation $\pi_m : \mathfrak{B}_\varepsilon \to M_{2m}(\mathbb{C})$ by setting $\pi_m(v_0) = V_{0,m}$ and $\pi_m(h_n) = H_{n,m}$.

Consider the direct product representation:

$$\Pi_m \pi_m : \mathfrak{B}_\varepsilon \to \Pi_{m=1}^\infty M_{2m}(\mathbb{C})$$

defined by $\Pi_m\pi_m(b) = (\pi_m(b))_{m=1}^\infty$ for $b \in \mathfrak{B}_\varepsilon$. To show that $\Pi_m\pi_m$ is an isometry, we need to check that $\|\Pi_m\pi_m(x)\| \geq \|x\| - \eta$ for any $\eta > 0$ and any x in the (not necessarily closed) $*$-algebra X generated by v_0, $h_{-N}, \cdots,$ h_N. Now assume that $x = F(v_0, h_{-N}, \cdots, h_N)$ is a finite linear combination of finite words in the $2N+2$ variables and their adjoints. Since $V_{0,m}$ and $H_{0,m}$ converge strongly to

$$\begin{pmatrix} \pi(v_0) & 0 \\ 0 & -\pi(v_0)^* \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \pi(h_n) & 0 \\ 0 & \pi(h_n) \end{pmatrix}$$

respectively, in the unit ball of $M_2(\mathbb{B}(H))$ as $m \to \infty$, we have

$$\begin{aligned}
&\limsup_{m\to\infty} \|F(V_{0,m}, H_{-N,m}, \cdots, H_{N,m})\| \\
&\geq \| \lim_{m\to\infty} F(V_{0,m}, H_{-N,m}, \cdots, H_{N,m})\| \\
&= \| \begin{pmatrix} \pi(F(v_0, h_{-N}, \cdots, h_N)) & 0 \\ 0 & \pi(F(-v_0^*, h_{-N}, \cdots, h_N)) \end{pmatrix} \| \\
&\geq \|\pi(x)\| = \|x\|.
\end{aligned}$$

Hence we can find n such that

$$\|\Pi_m\pi_m(x)\| \geq \|\pi_n(x)\| = \|F(V_{0,n}, H_{-N,n}, \cdots, H_{N,n}\| \geq \|x\| - \eta.$$

It follows that $\mathfrak{B}_\varepsilon$ is RFD. Indeed, $\{\pi_m\}$ is a separating family of finite dimensional representations of $\mathfrak{B}_\varepsilon$, as wanted.

For the second claim, let $b \in \mathfrak{B}_\varepsilon$ with $\|b\| = 1$. Take a finite dimensional representation $\pi : \mathfrak{B}_\varepsilon \to M_m(\mathbb{C})$ such that $\|\pi(b)\| > 3/4$. There exists $c \in \mathfrak{B}_\varepsilon$ such that $\|b - c\| < 1/4$ and c is in the $*$-algebra generated by $u_{-N}, \cdots, u_N$. Choose $M > 0$ and unitaries $v_0^{\pm 1}, \cdots, v_M^{\pm 1} \in M_m(\mathbb{C})$ such that

$$\|v_{n+1}^\pm - v_n^\pm\| \leq \varepsilon, \quad v_0^\pm = \pi(u_{\pm N}), \quad v_M^\pm = 1.$$

Then there is a representation $\pi' : \mathfrak{B}_\varepsilon \to M_m(\mathbb{C})$ such that $\pi'(u_n) = \pi'(u_{n+2(N+M)}$ ($2(N+M)$-periodic) and

$$\pi'(u_n) = \begin{cases} v_{-N-n}^- & -M-N \leq n < -N, \\ \pi(u_n) & -N \leq n \leq N, \\ v_{n-N}^+ & N < n \leq N+M. \end{cases}$$

Note that $\pi'(c) = \pi(c)$. In particular, $\|\pi'(c)\| > 1/2$ because $\|\pi(b) - \pi(c)\| < 1/4$, so that $3/4 < \|\pi(b)\| < 1/4 + \|\pi(c)\|$.

Now let $n = 2(N + M)m$. Define β to be the backward cyclic shift in block form with period $2(N + M)$, and define a covariant representation ρ of $\mathfrak{B}_\varepsilon$ on $M_n(\mathbb{C})$ by

$$\rho(u_j) = \begin{pmatrix} \pi'(u_j) & & & 0 \\ & \pi'(u_{j+1}) & & \\ & & \ddots & \\ 0 & & & \pi'(u_{j+2(N+M)-1}) \end{pmatrix}$$

so that $\beta \circ \rho = \rho \circ \alpha$. We have

$$\|\rho(b)\| \geq \|\pi'(b)\| \geq \|\pi'(c)\| - 1/4 > 0.$$

□

The commutative C^*-algebra $C(\mathbb{T}^2)$ is obviously RFD and has a separating family of 1-dimensional representations (characters). The highly (or extremely !) noncommutative, group C^*-algebra $C^*(F_2)$ of the free group F_2 on two generators has been shown to be RFD by Choi [5] (a surprise at that time).

Theorem 5.1.3. *The soft torus $C(\mathbb{T}^2)_\varepsilon$ is RFD.*

Proof. Assume that $0 < \varepsilon < 2$. Let $0 \neq a \in C(\mathbb{T}^2)_\varepsilon$. Then $b = E_\alpha(a^*a) \neq 0$, for the conditional expectation $E_\alpha : \mathfrak{B}_\varepsilon \rtimes_\alpha \mathbb{Z} \to \mathfrak{B}_\varepsilon$ is faithful. Choose n, ρ, and β as in the proposition above, and define

$$\pi : C(\mathbb{T}^2)_\varepsilon = \mathfrak{B}_\varepsilon \rtimes_\alpha \mathbb{Z} \to M_n(\mathbb{C}) \rtimes_\beta \mathbb{Z}$$

as the extension to the crossed product of the covariant $*$-homomorphism ρ. Since ρ is equivariant with respect to the actions β and α, we have

$$E_\beta(\pi(a^*a)) = \pi(E_\alpha(a^*a)) = \rho(b) \neq 0,$$

where $E_\beta : M_n(\mathbb{C}) \rtimes_\beta \mathbb{Z} \to M_n(\mathbb{C})$ is the conditional expectation. Hence $\pi(a) \neq 0$. Since β is inner, we have

$$M_n(\mathbb{C}) \rtimes_\beta \mathbb{Z} \cong M_n(\mathbb{C}) \rtimes_{\mathrm{id}} \mathbb{Z} \cong M_n(\mathbb{C}) \otimes C(\mathbb{T}).$$

Composing π with an evaluation map of $C(\mathbb{T}) \otimes M_n(\mathbb{C}) \cong C(\mathbb{T}, M_n(\mathbb{C}))$ the C^*-algebra of all $M_n(\mathbb{C})$-valued continuous functions on $\mathbb{T}$, we obtain a finite dimensional representation of $C(\mathbb{T}^2)_\varepsilon$ that does not vanish at a. □

Recall that the matrix algebra $M_n(\mathbb{C})$ has a faithful tracial state and has the property that every matrix $x \in M_n(\mathbb{C})$ which is hyponormal, i.e., $x^*x \geq xx^*$ is in fact normal, that is, $x^*x = xx^*$. As shown in [5],

Corollary 5.1.4. *The soft torus $C(\mathbb{T}^2)_\varepsilon$ has a faithful tracial state, and any hyponormal operator in it is normal.*

Proof. Those properties for $M_n(\mathbb{C})$ pass to direct products $\Pi_{n\in\mathbb{N}} M_{m_n}(\mathbb{C})$ (and their sums), and also to their subalgebras.

Indeed, $C(\mathbb{T}^2)_\varepsilon$ can be embedded into a direct product of matrix algebras $M_{m_n}(\mathbb{C})$ via a separating family of finite dimensional representations, i.e., be identified with a subalgebra of the direct product. Let τ_n be the faithful tracial state of $M_{m_n}(\mathbb{C})$, that is, the usual trace on it divided by m_n. A faithful tracial state τ of $C(\mathbb{T}^2)_\varepsilon$ is defined by

$$\tau(x) = \sum_n \frac{1}{2^n}\tau_n(x_n), \quad x = (x_n) \in C(\mathbb{T}^2)_\varepsilon.$$

If $x = (x_n) \in C(\mathbb{T}^2)_\varepsilon$ is hyponormal, then each x_n is hyponormal in $M_{m_n}(\mathbb{C})$, so that each x_n is normal, hence x is normal, as shown. □

5.2 Those of free product C^*-algebras

Let $\mathfrak{A}$ be a C^*-algebra and H a Hilbert space. Denote by $\mathrm{Rep}(\mathfrak{A}, H)$ the set of all (possibly degenerate) representation of $\mathfrak{A}$ on H, equipped with the coarsest topology for which the maps: $\mathrm{Rep}(\mathfrak{A}, H) \ni \pi \mapsto \pi(a)\xi \in H$ are continuous for all $a \in \mathfrak{A}$ and $\xi \in H$.

A representation $\pi \in \mathrm{Rep}(\mathfrak{A}, H)$ is finite dimensional if it just acts on a finite dimensional subspace of H. We say that π is residually finite dimensional (or RFD for short) if it is in the closure of the set of finite dimensional representations of $\mathrm{Rep}(\mathfrak{A}, H)$.

A state is said to be finite dimensional if its GNS representation is finite dimensional.

Theorem 5.2.1. *Let $\mathfrak{A}$ be a C^*-algebra. The following are equivalent:*

(1) : *The set of finite dimensional states is dense in the state space of $\mathfrak{A}$.*
(2) : *Every cyclic representation of $\mathfrak{A}$ is residually finite dimensional.*
(3) : *Every representation of $\mathfrak{A}$ is residually finite dimensional.*
(4) : *$\mathfrak{A}$ admits a faithful residually finite dimensional representation.*
(5) : *$\mathfrak{A}$ is residually finite dimensional.*

To prove this we need two lemmas as follows:

Lemma 5.2.2. *Let H be a Hilbert space and $\{H_s\}_{s\in S}$ be a family of Hilbert spaces indexed by a directed set S. Suppose that given $\xi_1, \cdots, \xi_n \in H$, and for each $s \in S$ there exist vectors $\xi_1^s, \cdots, \xi_n^s \in H_s$ such that*

$$\lim_{s\to\infty} \langle \xi_i^s, \xi_j^s \rangle = \langle \xi_i, \xi_j \rangle, \quad i, j = 1, \cdots, n.$$

Then there is $s_0 \in S$ and, for each $s \geq s_0$ there is an isometry u_s from the subspace H_0 of H spanned by $\xi_1, \cdots, \xi_n$ into H_s such that

$$\lim_{s \to \infty} \|u_s(\xi_i) - \xi_i^s\| = 0, \quad i = 1, \cdots, n.$$

Proof. Let $v : \mathbb{C}^n \longrightarrow H_0$ be the linear map sending the canonical basis i-th vector e_i to ξ_i. Since v is surjective, choose a right inverse w to v. For each $s \in S$, let $v_s : \mathbb{C}^n \longrightarrow H_s$ be given by $v_s(e_i) = \xi_i^s$ for all i. Observe that $v_s^* v_s$ viewed as an element of $M_n(\mathbb{C})$ converges to $v^* v$ since

$$\lim_{s \to \infty} \langle v_s^* v_s(e_i), e_j \rangle = \lim_{s \to \infty} \langle \xi_i^s, \xi_j^s \rangle = \langle \xi_i, \xi_j \rangle = \langle v^* v(e_i), e_j \rangle$$

for all i, j. Let $u_s' : H_0 \longrightarrow H_s$ be defined by $u_s' = v_s w$. Then

$$\lim_{s \to \infty} (u_s')^* u_s' = \lim_{s \to \infty} w^* v_s^* v_s w = w^* v^* v w = \mathrm{id}_{H_0}.$$

Therefore, we can find s_0 such that, for $s \geq s_0$, $(u_s')^* u_s'$ is invertible. Set $u_s = u_s'((u_s')^* u_s')^{-1/2}$ for such s. We then have

$$\begin{aligned}
\lim_{s \to \infty} \|u_s' \xi_i - \xi_i^s\|^2 &= \lim_{s \to \infty} \{\langle (u_s')^* u_s' \xi_i, \xi_i \rangle - 2\mathrm{Re}(\langle v_s^* v_s w \xi_i, e_i \rangle) + \langle v_s e_i, v_s e_i \rangle\} \\
&= \langle \xi_i, \xi_i \rangle - 2\langle v^* v w \xi_i, e_i \rangle + \langle v e_i, v e_i \rangle \\
&= 2\langle \xi_i, \xi_i \rangle - 2\langle \xi_i, v e_i \rangle = 0
\end{aligned}$$

for $i = 1, \cdots, n$, from which the conclusion follows, and indeed,

$$\begin{aligned}
&\lim_{s \to \infty} \|u_s \xi_i - \xi_i^s\|^2 = \lim_{s \to \infty} \langle u_s \xi_i - \xi_i^s, u_s \xi_i - \xi_i^s \rangle \\
&= \lim_{s \to \infty} \{\langle u_s \xi_i, u_s \xi_i \rangle - 2\mathrm{Re}(\langle u_s \xi_i, \xi_i^s \rangle) + \langle \xi_i^s, \xi_i^s \rangle\} \\
&= \lim_{s \to \infty} \{\langle \xi_i, u_s^* u_s \xi_i \rangle - 2\mathrm{Re}(\langle ((u_s')^* u_s')^{-1/2} \xi_i, (u_s')^* \xi_i^s \rangle) + \langle \xi_i^s, \xi_i^s \rangle\} \\
&= \lim_{s \to \infty} \{\langle \xi_i, \xi_i \rangle - 2\mathrm{Re}(\langle ((u_s')^* u_s')^{-1/2} \xi_i, w^* v_s^* v_s e_i \rangle) + \langle v_s e_i, v_s e_i \rangle\} \\
&= \langle \xi_i, \xi_i \rangle - 2\langle \xi_i, w^* v^* v e_i \rangle + \langle v e_i, v e_i \rangle \\
&= 2\langle \xi_i, \xi_i \rangle - 2\langle \xi_i, \xi_i \rangle = 0.
\end{aligned}$$

□

Lemma 5.2.3. *Suppose that π is a cyclic representation of $\mathfrak{A}$ on H with cyclic vector ξ, and $\{\pi_s\}_{s \in S}$ is a net in* $\mathrm{Rep}(\mathfrak{A}, H)$. *If*

$$\lim_{s \to \infty} \pi_s(a)\xi = \pi(a)\xi, \quad \forall a \in \mathfrak{A},$$

then π_s converges to $\pi \in$ $\mathrm{Rep}(\mathfrak{A}, H)$.

Proof. For $\eta = \pi(b)\xi \in H$ for $b \in \mathfrak{A}$, we have

$$\begin{aligned}\|\pi_s(a)\eta - \pi(a)\eta\| &\leq \|\pi_s(a)\pi(b)\xi - \pi_s(a)\pi_s(b)\xi\| + \|\pi_s(ab)\xi - \pi(ab)\xi\| \\ &\leq \|a\|\|\pi(b)\xi - \pi_s(b)\xi\| + \|\pi_s(ab)\xi - \pi(ab)\xi\|\end{aligned}$$

which shows that $\pi_s(a)$ converges $\pi(a)$ pointwise over the dense subset of those vectors in H. The uniform boundedness of $\{\pi_s(a)|a \in \mathfrak{A}\}$ implies that $\pi_s(a)$ converges strongly to $\pi(a)$, for all a. □

Proof of the theorem above. (1) ⇒ (2). We first assume that $\mathfrak{A}$ is unital. Let π be a cyclic representation of $\mathfrak{A}$ on H with cyclic vector ξ and state f. By assumption, there is a net (f_s) of finite dimensional states converging to f and let (ρ_s, H_s, ξ_s) be the corresponding GNS representations. Given $\{a_0 = 1, a_1, \cdots, a_n\}$ a finite subset of $\mathfrak{A}$, observe that

$$\lim_{s\to\infty} \langle \rho_s(a_i)\xi_s, \rho_s(a_j)\xi_s\rangle = \langle \pi(a_i)\xi, \pi(a_j)\xi\rangle$$

for all $i, j = 1, \cdots, n$. There exists a net $(u_s)_{s\geq s_0}$ of isometries from H_0 spanned by $\pi(a_i)\xi$ for $i = 0, 1, \cdots, n$ into H_s such that

$$\lim_{s\to\infty} \|u_s\pi(a_i)\xi - \rho_s(a_i)\xi_s\| = 0.$$

Let π_s be the representation of $\mathfrak{A}$ on H given by $\pi_s(a) = u_s^*\rho_s(a)u_s$. Claim that we have $\lim_{s\to\infty} \pi_s(a_i)\xi = \pi(a_i)\xi$ for all i. Indeed, by taking $i = 0$ in the equation obtained above, we obtain $\lim_{s\to\infty} \|u_s\xi - \xi_s\| = 0$. Therefore,

$$\begin{aligned}\|\pi_s(a_i)\xi - \pi(a_i)\xi\| &= \|u_s^*\rho_s(a_i)u_s\xi - u_s^*u_s\pi(a_i)\xi\| \\ &\leq \|\rho_s(a_i)u_s\xi - u_s\pi(a_i)\xi\| \\ &\leq \|\rho_s(a_i)u_s\xi - \rho_s(a_i)\xi_s\| + \|\rho_s(a_i)\xi_s - u_s\pi(a_i)\xi\| \\ &\leq \|a_i\|\|u_s\xi - \xi_s\| + \|\rho_s(a_i)\xi_s - u_s\pi(a_i)\xi\|,\end{aligned}$$

which goes to zero as $s \to \infty$.

Set $\beta = \{a_0, a_1, \cdots, a_n\}$. For each such β and for $\varepsilon > 0$, choose $\pi_s = \pi_{\varepsilon,\beta}$ such that $\|\pi_{\varepsilon,\beta}(a_i)\xi - \pi(a_i)\xi\| < \varepsilon$. We thus obtain a net $\{\pi_{\varepsilon,\beta}\}$ of finite dimensional representations such that $\lim \pi_{\varepsilon,\beta}(a)\xi = \pi(a)\xi$ for all $a \in \mathfrak{A}$. Hence, this net converges to π in $\mathrm{Rep}(\mathfrak{A}, H)$.

In the case that $\mathfrak{A}$ is non-unital, let $\mathfrak{A}^\sim$ be the unitization of $\mathfrak{A}$. Since the state space of $\mathfrak{A}$ is included in that of $\mathfrak{A}^\sim$, if (1) holds for $\mathfrak{A}$, then it also does for $\mathfrak{A}^\sim$. Hence, (2) holds for $\mathfrak{A}^\sim$ so that it also for $\mathfrak{A}$.

(2) ⇒ (3). Given an representation of $\mathfrak{A}$ on a Hilbert space H, write $\pi = \oplus_{\lambda\in\Lambda}\pi_\lambda$ where each π_λ is a cyclic sub-representation of π. For each

finite subset F of Λ, let $\pi_F = \oplus_{\lambda\in F}\pi_\lambda$, viewed as a degenerate representation on H. Then the net so obtained converges to π. Since each π_λ is RFD by assumption, so is π_F. Therefore, π is RFD.

(3) $\Rightarrow$ (4). This is obvious.

(4) $\Rightarrow$ (5). Let π be a faithful RFD representation of $\mathfrak{A}$, so that $\pi = \lim \pi_s$ where each π_s is finite dimensional. If $a \in \mathfrak{A}$ is nonzero, then $\pi(a) \neq 0$. Since $\pi_s(a)$ converges strongly to $\pi(a)$, some $\pi_s(a)$ must be nonzero.

(5) $\Rightarrow$ (1). Assume that $\mathfrak{A}$ is unital. Denote by $F(\mathfrak{A})$ the set of all finite dimensional states of $\mathfrak{A}$. Note that $F(\mathfrak{A})$ is convex. In fact, if $f, g \in F(\mathfrak{A})$, then the GNS representation of a convex combination $h = (1-t)f + tg$ is equivalent to a sub-representation of the direct sum of the GNS representations for f and g. Indeed, note that $h = (1-t)f + tg \leq f + g = l$, that is, $l - h$ is positive, so that we have

$$h(a) = \langle \pi_l(a)v\xi_l, \xi_l\rangle = \langle v\pi_l(a)\xi_l, \xi_l\rangle, \quad a \in \mathfrak{A},$$

where π_l is the GNS representation for l with cyclic vector ξ_l, and v is in the commutant of $\pi_l(\mathfrak{A})$, with $0 \leq v \leq 1$ (see Murphy [16]). Also, note that for $a \in \mathfrak{A}$,

$$\begin{aligned}\langle \pi_l(a)\xi_l, \xi_l\rangle &= l(a) = f(a) + g(a)\\ &= \langle \pi_f(a)\xi_f, \xi_f\rangle + \langle \pi_g(a)\xi_g, \xi_g\rangle\\ &= \langle (\pi_f \oplus \pi_g)(a)(\xi_f \oplus \xi_g), \xi_f \oplus \xi_g\rangle,\end{aligned}$$

where π_f, and π_g are the GNS representations for f and g with cyclic vectors ξ_f and ξ_g, respectively, and $\pi_f \oplus \pi_g$ is their product representation of $\mathfrak{A}$.

Assume that there is a state g of $\mathfrak{A}$ not contained in the weak*-closure of $F(\mathfrak{A})$. Identify the set $\mathfrak{A}_{sa}$ of self-adjoint elements of $\mathfrak{A}$ with the corresponding elements of the dual of the set $\mathfrak{A}'_{sa}$ of self-adjoint continuous functionals on $\mathfrak{A}$ with the weak* topology. Use the Hahn-Banach (or Mazur) theorem to obtain an element $a \in \mathfrak{A}_h$ and a real number r such that $g(a) > r$ and $f(a) \leq r$ for all $f \in F(\mathfrak{A})$. This implies that for any finite dimensional representation π of $\mathfrak{A}$ and any unit vector ξ in the representation space of π, one has $\langle \pi(a)\xi, \xi\rangle \leq r$. Therefore, $\pi(a) \leq r$. By hypothesis, the direct sum of all finite dimensional representations of $\mathfrak{A}$ is faithful, so that we have $a \leq r$, which contradicts to that $g(a) > r$.

The non-unital case follows from the unital case. Note that the representation theory of the unitization of $\mathfrak{A}$ is the sum of that of $\mathfrak{A}$ and the identity representation. □

Lemma 5.2.4. *Let π be a non-degenerate representation of a C^*-algebra $\mathfrak{A}$ on a Hilbert space H. Suppose that π_s is a net in* $\mathrm{Rep}(\mathfrak{A}, H)$ *that converges to π. If ρ_s is another net on the same directed set in* $\mathrm{Rep}(\mathfrak{A}, H)$ *such that the restriction of each $\rho_s(a)$ to the representation space H_s of π_s coincides with $\pi_s(a)$, then ρ_s also converges to π.*

Proof. Let p_s denote the orthogonal projection onto H_s. Claim that p_s converges strongly to the identity operator. In fact, for $\xi \in H$, $a \in \mathfrak{A}$, and all s, we have

$$\begin{aligned}\|\pi(a)\xi - p_s(\pi(a)\xi)\| &= \mathrm{dist}(\pi(a)\xi, H_s) \\ &\leq \|\pi(a)\xi - p_s(\pi_s(a)\xi)\| = \|\pi(a)\xi - \pi_s(a)\xi\|\end{aligned}$$

which shows that p_s converges pointwise to the identity operator over the dense set $\{\pi(a)\xi | a \in \mathfrak{A}, \xi \in H\}$. Since $\{p_s\}_s$ is uniformly bounded, the claim follows.

Since $\rho_s(a)p_s = \pi_s(a)$ for all $a \in \mathfrak{A}$ and $\xi \in H$, we have

$$\begin{aligned}\|\rho_s(a)\xi - \pi(a)\xi\| &\leq \|\rho_s(a)\xi - \rho_s(a)p_s\xi\| + \|\pi_s(a)\xi - \pi(a)\xi\| \\ &\leq \|a\|\|\xi - p_s\xi\| + \|\pi_s(a)\xi - \pi(a)\xi\|\end{aligned}$$

from which we see that $\rho_s(a)\xi$ converges to $\pi(a)\xi$, that is, ρ_s converges to π. □

Theorem 5.2.5. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be C^*-algebras. Then their free product C^*-algebra $\mathfrak{A}_1 * \mathfrak{A}_2$ is RFD if and only if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are RFD. If both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unital, then their unital free product $\mathfrak{A}_1 *_{\mathbb{C}} \mathfrak{A}_2$ is RFD if and only if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are RFD.*

Proof. Since the RFD property passes to subalgebras, both forward implications are clear.

To prove the reverse implication in the unital case, let π be a faithful non-degenerate representation of $\mathfrak{A}_1 *_{\mathbb{C}} \mathfrak{A}_2$ on a Hilbert space H. Let π_i be the restriction of π to $\mathfrak{A}_i$ for $i = 1, 2$. Take a net $\{\pi_s^i\}_s$ in $\mathrm{Rep}(\mathfrak{A}_i, H)$ of finite dimensional representations converging to π_i, on a common directed set, if necessary, by replacing both directed sets by their product. Let H_s^i be the representation space for π_s^i.

For each s, choose a finite dimensional subspace K_s of H containing both H_s^1 and H_s^2 with dimension a common multiple of $\dim H_s^1$ and $\dim H_s^2$. Let ρ_s^i be any representation of $\mathfrak{A}_i$ on K_s as its representation space, whose restriction on H_s^i is π_s^i. For example, one may take an appropriate multiple of π_s^i.

Since each π_i is unital, and so nondegenerate, we obtain $\lim_s \rho_s^i = \pi_i$.

For each s, let $\rho_s = \rho_s^1 * \rho_s^2$, which is a well-defined, finite dimensional representation of $\mathfrak{A}_1 *_{\mathbb{C}} \mathfrak{A}_2$ since $\rho_s^1(1)$ and $\rho_s^2(1)$ are both equal to the orthogonal projection onto K_s. It follows that $\lim_s \rho_s = \pi$ which proves that π is RFD, so that $\mathfrak{A}_1 *_{\mathbb{C}} \mathfrak{A}_2$ is also RFD.

The proof of the non-unital case is similar. Take a faithful representation of $\mathfrak{A}_1 * \mathfrak{A}_2$ on H. Let π_i be the restriction of π to $\mathfrak{A}_i$. Write $\pi_i = \lim_s \pi_s^i$ as above. Set $\pi_s = \pi_s^1 * \pi_s^2$. Then (π_s) converges to π. □

Corollary 5.2.6. *Suppose that C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have $M_n(\mathbb{C})$ as their unital C^*-subalgebras. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are RFD, then the amalgam $\mathfrak{A}_1 *_{M_n(\mathbb{C})} \mathfrak{A}_2$ over $M_n(\mathbb{C})$ is RFD.*

Proof. Let $\mathfrak{B}_i = e_1 \mathfrak{A}_i e_1$, where e_1 is the canonical rank one projection for $M_n(\mathbb{C})$, with $e_1 + e_2 + \cdots + e_n = 1$. Since

$$\mathfrak{A}_1 *_{M_n(\mathbb{C})} \mathfrak{A}_2 \cong M_n(\mathfrak{B}_1 *_{\mathbb{C}} \mathfrak{B}_2)$$

and RFD property passes to subalgebras and matrix algebras, we are done by the theorem above.

As for the isomorphism above, check that since $e_1 + e_2 + \cdots + e_n = 1 \in \mathfrak{A}_j$, for $a \in \mathfrak{A}_j$ we have

$$a = (\sum_{k=1}^n e_k) a (\sum_{l=1}^n e_l) = \sum_{k,l=1}^n e_k a e_l$$

which is viewed as the following matrix:

$$\begin{pmatrix} e_1 a e_1 & e_1 a e_2 & \cdots & e_1 a e_n \\ e_2 a e_1 & e_2 a e_2 & \cdots & e_2 a e_n \\ \vdots & \vdots & \ddots & \vdots \\ e_n a e_1 & e_n a e_2 & \cdots & e_n a e_n \end{pmatrix}$$

in $M_n(\mathfrak{B}_j)$, where each $e_k \mathfrak{A}_j e_l$ is identified with $\mathfrak{B}_j$. Indeed, the product $ab = (\sum_{k=1}^n e_k) a (\sum_{l=1}^n e_l)(\sum_{k'=1}^n e_{k'}) b (\sum_{l'=1}^n e_{l'})$ as well as other operations correspond to the product and those of the corresponding matrices as above. Freeness and amalgamation over $M_n(\mathbb{C})$ will imply the isomorphism. □

Notes. The first subsection of this section is based on the paper [7] of Eilers and Exel. The second subsection of this section is based on the paper [11] of Exel and Loring. In [25] of the author, it is shown that the isometric versions of the soft tori are generalized RFD but not RFD.

6 Beginning noncommutative shape theory

6.1 Universal C^*-algebras

Let $\mathfrak{G} = \{x_\alpha\}$ be a set of generators and $\mathfrak{R}$ be their relations of the form:

$$\|p(x_{\alpha_1}, \cdots, x_{\alpha_n}, x^*_{\alpha_1}, \cdots, x^*_{\alpha_n})\| \leq \eta$$

where p is a polynomial of $2n$ variables with complex coefficients, generators $x_{\alpha_1}, \cdots, x_{\alpha_n} \in \mathfrak{G}$, and $\eta \geq 0$.

A representation of $(\mathfrak{G}, \mathfrak{R})$ is a set of bounded operators $\{y_\alpha\}$ on a Hilbert space H which satisfy the same relation as above by replacing $\{x_\alpha\}$ with $\{y_\alpha\}$. Such a representation extends uniquely to a representaion (i.e., $*$-homomorphism) from the free $*$-algebra $F(\mathfrak{G})$ generated by $\mathfrak{G}$ into $\mathbb{B}(H)$ of all bounded operators on H.

A pair $(\mathfrak{G}, \mathfrak{R})$ of generators and relations is admissible if there is a representation of $(\mathfrak{G}, \mathfrak{R})$, and if for representaions $\{y^\beta_\alpha\}$ of $(\mathfrak{G}, \mathfrak{R})$ on H^β, the direct sum $\oplus_\beta y^\beta_\alpha \in \mathbb{B}(\oplus_\beta H^\beta)$ for each α, and $\{\oplus_\beta y^\beta_\alpha\}$ is a representation of $(\mathfrak{G}, \mathfrak{R})$.

For any $z \in F(\mathfrak{G})$, define a C^*-seminorm on $F(\mathfrak{G})$ by

$$\|z\|_s = \sup\{\|\rho(z)\| : \quad \rho \text{ a representation of } F(\mathfrak{G})\}.$$

The universal C^*-algebra of $(\mathfrak{G}, \mathfrak{R})$, denoted by $C^*(\mathfrak{G}, \mathfrak{R})$, is defined to be the completion of the quotient of $F(\mathfrak{G})$ by $\{z : \|z\|_s = 0\}$.

Any representation of $(\mathfrak{G}, \mathfrak{R})$ extends uniquely to a representation of $C^*(\mathfrak{G}, \mathfrak{R})$, and any representaion of $C^*(\mathfrak{G}, \mathfrak{R})$ gives a representation of $(\mathfrak{G}, \mathfrak{R})$.

Example 6.1.1. Let $\mathfrak{A}$ be a C^*-algebra. Set $\mathfrak{G} = \mathfrak{A}$ and $\mathfrak{R}$ the set of all $*$-algebraic relations in $\mathfrak{A}$. Then $C^*(\mathfrak{G}, \mathfrak{R}) \cong \mathfrak{A}$.

Let $\mathfrak{A}$ be a C^*-algebra. Let $\mathfrak{G}$ be a dense $*$-subalgebra of $\mathfrak{A}$ over a dense subfield of $\mathbb{C}$, and $\mathfrak{R}$ the set of all $*$-algebraic relations and the scalar mutiple relations on $\mathfrak{G}$. Then $C^*(\mathfrak{G}, \mathfrak{R}) \cong \mathfrak{A}$. In particular, a separable C^*-algebra is the universal C^*-algebra on a countable set of generators and relations.

Let $\mathfrak{B}$ be a Banach $*$-algebra. Set $\mathfrak{G} = \mathfrak{B}$ and $\mathfrak{R}$ the $*$-algebraic relations. Then $C^*(\mathfrak{G}, \mathfrak{R})$ is the enveloping C^*-algebra of $\mathfrak{B}$. In particular, set $\mathfrak{B} = L^1(G)$ for a locally compact group G. Then $C^*(\mathfrak{G}, \mathfrak{R}) = C^*(G)$ the full group C^*-algebra of G. If G is discrete, $C^*(G) = C^*(G, \mathfrak{R})$.

Let $\mathfrak{G} = \{x\}$ and $\mathfrak{R} = \{x = x^*, \|x\| \leq 1, \|1 - x^2\| \leq 1\}$. Then $C^*(\mathfrak{G}, \mathfrak{R}) \cong C_0((0, 1])$ the C^*-algebra of all continuous functions on $(0, 1]$ vanishing at 0. Let $\mathfrak{G} = \{x, 1\}$ and

$$\mathfrak{R} = \{x = x^*, \|x\| \leq 1, \|1 - x^2\| \leq 1, 1 = 1^* = 1^2, x1 = 1x = x\}.$$

Then $C^*(\mathfrak{G}, \mathfrak{R}) \cong C([0,1])$ the C^*-algebra of all continuous functions on $[0,1]$. These are the universal positive contraction C^*-algebras.

Let $\mathfrak{G} = \{x\}$ and $\mathfrak{R} = \{x^*x = 1, xx^* = 1\}$. Then $C^*(\mathfrak{G}, \mathfrak{R}) \cong C(\mathbb{T}) \cong C^*(\mathbb{Z})$, the universal unitary algebra, where $\mathbb{T}$ is the 1-torus.

Let $\mathfrak{G} = \{x\}$ and $\mathfrak{R} = \{x^*x = xx^*, \|x\| \leq 1\}$. Then $C^*(\mathfrak{G}, \mathfrak{R}) \cong C_0(D_0)$, where D_0 is the punctured unit disk. Let $\mathfrak{G} = \{x, 1\}$ and

$$\mathfrak{R} = \{x^*x = xx^*, \|x\| \leq 1, 1 = 1^* = 1^2, x1 = 1x = x\}.$$

Then $C^*(\mathfrak{G}, \mathfrak{R}) \cong C(D)$, where D is the unit disk. These are the universal normal contraction C^*-algebras.

Let $\mathfrak{G} = \{x\}$ and $\mathfrak{R} = \{x^*x = 1\}$. Then $C^*(\mathfrak{G}, \mathfrak{R}) \cong C^*(S)$ the C^*-algebra generated by the unilateral shift S. This is the universal isometry algebra, or the Toeplitz algebra $\mathfrak{T}$.

Let $\mathfrak{G} = \{x\}$ and $\mathfrak{R} = \{\|x\| \leq 1\}$, or let $\mathfrak{G} = \{x, 1\}$ and $\mathfrak{R} = \{\|x\| \leq 1, 1 = 1^* = 1, x1 = 1x = x\}$. Then $C^*(\mathfrak{G}, \mathfrak{R})$ are the universal non-unital and unital contraction algebras respectively.

Let $A = (a_{ij})$ be an $n \times n$ matrix with components 0 or 1. Let $\mathfrak{G} = \{s_1, \cdots, s_n\}$ and

$$\mathfrak{R} = \{s_i^*s_i = (s_i^*s_i)^2, s_i^*s_i = \sum_{j=1}^n a_{ij}s_js_j^*, s_k^*s_i = 0 \quad \text{for all } i \text{ and } k \neq i\}.$$

Then $C^*(\mathfrak{G}, \mathfrak{R}) \cong O_A$ the Cuntz-Krieger algebra for A.

Let θ be an irrational number. Let $\mathfrak{G} = \{u, v\}$ and

$$\mathfrak{R} = \{u^*u = uu^* = v^*v = vv^* = 1, uv = e^{2\pi i\theta}vu\}.$$

Then $C^*(\mathfrak{G}, \mathfrak{R}) \cong R_\theta$ the irrational rotation algebra.

Let $\mathfrak{G} = \{x_{ij} : 1 \leq i, j \leq n\}$ and

$$\mathfrak{R} = \{x_{ij} = x_{ji}^*, x_{ij}x_{kl} = \delta_{jk}x_{il} : 1 \leq i, j, k, l \leq n\}.$$

Then $C^*(\mathfrak{G}, \mathfrak{R}) \cong M_n(\mathbb{C})$ the $n \times n$ matrix algebra over $\mathbb{C}$.

Let $\mathfrak{G} = \{1, x_{ij} : 1 \leq i, j \leq n\}$ and

$$\mathfrak{R} = \{1 = 1^* = 1^2, x_{ij} = x_{ji}^* = \sum_{k=1}^n x_{ik}x_{kj}, x_{ij}1 = 1x_{ij} = x_{ij} : 1 \leq i, j \leq n\}.$$

Then $C^*(\mathfrak{G}, \mathfrak{R})$ is said to be the noncommutative Grassmanian, denoted by G_n^{nc}.

Let $\mathfrak{G} = \{x_{ij} : 1 \leq i, j \leq n\}$ and

$$\mathfrak{R} = \{\sum_{k=1}^{n} x_{ki}^* x_{kj} = \delta_{ij} 1, \sum_{k=1}^{n} x_{ik} x_{jk}^* = \delta_{ij} 1 : 1 \leq i, j \leq n\}.$$

Then $C^*(\mathfrak{G}, \mathfrak{R})$ is said to be the noncommutative unitary group, denoted by U_n^{nc}. Note that $U_1^{nc} \cong C(\mathbb{T})$.

6.2 Projective C^*-algebras

We denote by SC the category of separable C^*-algebras as objects and their $*$-homomorphisms as morphisms. Denote by SC_1 the category of separable unital C^*-algebras and their unital $*$-homomorphisms. Let CC be the category of separable commutative C^*-algebras, that is equivalent to the category of pointed compact metrizable spaces, and CC_1 be the category of separable commutative unital C^*-algebras, which is equivalent to the category of compact metrizable spaces.

Now consider a subcategory SC_q of SC which is closed under quotients.

Definition 6.2.1. A morphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ in SC_q is said to be projective in SC_q if for any $\mathfrak{C} \in SC_q$, closed ideal $\mathfrak{J}$ of $\mathfrak{C}$, and morphism $\sigma : \mathfrak{B} \to \mathfrak{C}/\mathfrak{J}$, there is a morphism $\psi : \mathfrak{A} \to \mathfrak{C}$ such that $\pi \circ \psi = \sigma \circ \varphi$, where $\pi : \mathfrak{C} \to \mathfrak{C}/\mathfrak{J}$ is the quotient map, that is, the following diagram commutes:

$$\begin{array}{ccccccccc} & & & & \mathfrak{A} & \xrightarrow{\varphi} & \mathfrak{B} & & \\ & & & & {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\sigma} & & \\ 0 & \longrightarrow & \mathfrak{J} & \longrightarrow & \mathfrak{C} & \xrightarrow{\pi} & \mathfrak{C}/\mathfrak{J} & \longrightarrow & 0 \end{array}$$

A C^*-algebra $\mathfrak{A}$ is said to be projective in SC_q if the identity map on $\mathfrak{A}$ is projective.

If either $\mathfrak{A}$ or $\mathfrak{B}$ is projective, then any morphism from $\mathfrak{A}$ to $\mathfrak{B}$ is projective. A composition of a projective morphism with any other morphism is projective.

Example 6.2.2. (1) : $C_0((0, 1])$ is projective in SC. Note that $C_0((0, 1])$ is the universal C^*-algebra generated by a generator h such that $0 \leq h \leq 1$ with $h \neq 1$. Note also that the relation $0 \leq h \leq 1$ (i.e. h and $1 - h$ are positive in a quotient) is always liftable.

(2) : $\mathbb{C}$ is projective in SC_1 but not in SC. Indeed, consider the short exact sequence: $0 \to C_0((0, 1)) \to C_0((0, 1]) \to \mathbb{C} \to 0$. There is a morphism

from $\mathbb{C}$ to the quotient, but no lift from $\mathbb{C}$ to the non-unital $C_0((0,1])$. However, in SC_1 the unit in a quotient can be lifted to the unit of its extension.

(3) : $C([0,1]^2)$ is projective in CC_1 but not in SC_1. Indeed, the real and imaginary parts of the image of the unilateral shift S in the Calkin algebra give a homomorphism of $C([0,1]^2)$ into $C^*(S)/\mathbb{K}$ which cannot be lifted:

$$\begin{array}{ccc} C([0,1]^2) & \xrightarrow{\text{id}} & C([0,1]^2) \\ {\scriptstyle \text{no lift } \psi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ C^*(S) & \longrightarrow & C^*(S)/\mathbb{K} \cong C(\mathbb{T}) \end{array}$$

where φ sends the coordinate functions x and y on $[0,1]^2$ to the real and imaginary parts of the coordinate function $z \in C(\mathbb{T})$ respectively. Because if there exists such ψ, then there exists $f \in C([0,1]^2)$ such that $\psi(f) = S$, which implies that S commutes with S^*, but S is a proper isometry, that is, the contradiction.

Recall that the cone $C\mathfrak{A}$ over a C^*-algebra is defined to be $C_0((0,1]) \otimes \mathfrak{A} \cong C_0((0,1],\mathfrak{A})$ the C^*-algebra of continuous $\mathfrak{A}$-valued functions over $(0,1]$.

Proposition 6.2.3. *If a C^*-algebra is projective, then it is contractible, i.e., the identity map is homotopic to the zero map. In particular, a projective C^*-algebra in SC is nonunital.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & \mathfrak{A} & \xrightarrow{\text{id}} & \mathfrak{A} & & \\ & & & & {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \text{id}} & & \\ 0 & \longrightarrow & S\mathfrak{A} & \longrightarrow & C\mathfrak{A} & \xrightarrow{\delta_1} & \mathfrak{A} & \longrightarrow & 0 \end{array}$$

where the suspension $S\mathfrak{A}$ over $\mathfrak{A}$ is isomorphic to $C_0((0,1),\mathfrak{A})$, and δ_1 is the evaluation map at 1. Since $\text{id}_{C\mathfrak{A}}$ is homotopic to the zero map, the composition $\delta_1 \circ \text{id}_{C\mathfrak{A}} \circ \psi = \text{id}_{\mathfrak{A}}$ is also homotopic to the zero map.

Let $\mathfrak{A}$ be a unital C^*-algebra and $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a $*$-homomorphism. Note that $\|\varphi(1)\| = \|\varphi(1)\varphi(1)\| \leq \|\varphi(1)\|^2$, so that $\|\varphi(1)\| \geq 1$ if $\varphi(1) \neq 0$. Note also that the C^*-norm condition implies that $\|\varphi(1)\| = \|\varphi(1)^2\| = \|\varphi(1)\|^2$, so that $\|\varphi(1)\| = 1$ if $\varphi(1) \neq 0$. Hence, no unital homomorphisms connecting the identity map on $\mathfrak{A}$ and the zero map continuously. Thus, $\mathfrak{A}$ is not contractible. □

Remark. This fact implies that C^*-algebra invariants such as K-theory groups are always vanishing on projective C^*-algebras. Thus, it is rare that a C^*-algebra is projective.

Proposition 6.2.4. *Let $\mathfrak{A}$ be a projective C^*-algebra and $\varphi, \psi : \mathfrak{A} \to \mathfrak{B}$ $*$-homomorphisms. Then φ and ψ are homotopic, and write $\varphi \sim \psi$.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & \mathfrak{A} & \xrightarrow{\mathrm{id}} & \mathfrak{A} & & \\ & & & & {\scriptstyle (\varphi\oplus\psi)^{\sim}}\downarrow & & \downarrow{\scriptstyle \varphi\oplus\psi} & & \\ 0 & \longrightarrow & S\mathfrak{B} & \longrightarrow & C([0,1],\mathfrak{B}) & \xrightarrow{\delta_0\oplus\delta_1} & \mathfrak{B}\oplus\mathfrak{B} & \longrightarrow & 0 \end{array}$$

where $(\varphi \oplus \psi)^{\sim}$ is a lift of $\varphi \oplus \psi$. Then a homotopy between φ and ψ is given by $\{(\varphi \oplus \psi)_t^{\sim}\}_{t\in[0,1]}$, where $*$-homomorphisms $(\varphi \oplus \psi)_t^{\sim} : \mathfrak{A} \to \mathfrak{B}$ are defined by $(\varphi \oplus \psi)_t^{\sim}(a) = \delta_t((\varphi \oplus \psi)^{\sim}(a))$, where δ_t is the evaluation map at $t \in [0, 1]$. □

Proposition 6.2.5. *A C^* algebra is projective in SC if and only if its unitization is projective in SC_1.*

Proof. Given $\sigma : \mathfrak{A}^+ \to \mathfrak{C}/\mathfrak{J}$, consider its restriction $\sigma : \mathfrak{A} \to \mathfrak{C}/\mathfrak{J}$. Let $\psi : \mathfrak{A} \to \mathfrak{C}$ be its lift. Since $\mathfrak{C}$ is unital, we have a lift $\psi^+ : \mathfrak{A}^+ \to \mathfrak{C}$ for σ.

Conversely, given $\sigma : \mathfrak{A} \to \mathfrak{C}/\mathfrak{J}$. If $\mathfrak{C}/\mathfrak{J}$ is nonunital, there is a lift $\psi^+ : \mathfrak{A}^+ \to \mathfrak{C}^+$ for $\sigma^+ : \mathfrak{A}^+ \to (\mathfrak{C}/\mathfrak{J})^+ \cong \mathfrak{C}^+/\mathfrak{J}$. Its restriction to $\mathfrak{A}$ gives a lift of σ. If $\mathfrak{C}/\mathfrak{J}$ is unital, there is a lift $\psi^+ : \mathfrak{A}^+ \to \mathfrak{C}$ for $\sigma^+ : \mathfrak{A}^+ \to \mathfrak{C}/\mathfrak{J}$ the extension from σ. Its restriction to $\mathfrak{A}$ gives a lift of σ. □

Proposition 6.2.6. *If $\mathfrak{A}_n$ are projective C^*-algebras in SC (resp. SC_1), then their free product $*\mathfrak{A}_n$ (resp. unital free product $*_{\mathbb{C}}\mathfrak{A}_n$) is projective in there.*

Proof. Let $\sigma : *\mathfrak{A}_n \to \mathfrak{C}/\mathfrak{J}$ be a $*$-homomorphism. For each $\mathfrak{A}_n$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & \mathfrak{A}_n & \xrightarrow{\mathrm{id}} & \mathfrak{A}_n & & \\ & & & & {\scriptstyle \varphi_n}\downarrow & & \downarrow{\scriptstyle \sigma_n} & & \\ 0 & \longrightarrow & \mathfrak{J} & \longrightarrow & \mathfrak{C} & \xrightarrow{\pi} & \mathfrak{C}/\mathfrak{J} & \longrightarrow & 0 \end{array}$$

where σ_n is the restriction of σ to $\mathfrak{A}_n$, and φ_n is a lift of σ_n since each $\mathfrak{A}_n$ is projective. Then the map $*\varphi_n$ extended from φ_n for all n to $*\mathfrak{A}_n$ gives a lift of σ. Similarly, the case for the unital free product $*_{\mathbb{C}}\mathfrak{A}_n$ is proved. □

Recall that a subspace A of a topological space Y is said to be a retract of Y if for the identity map id_A on A there exists a continuous map f from Y to A such that the restriction map $f|_A$ to A is id_A, and f is called a retraction:

$$\begin{array}{ccc} Y & = & Y \\ {\scriptstyle \mathrm{id}_A}\uparrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{\mathrm{id}_A} & A. \end{array}$$

This is equivalent to say that any continuous map g from A to a topological space X can be extended to a continuous map f from Y to X:

$$\begin{array}{ccc} Y & = & Y \\ {\scriptstyle \mathrm{id}_A}\uparrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{g} & X. \end{array}$$

The extension problem is whether or not such an extension exists, for a not necessarily retract space.

A metric space X is said to be an absolute retract (or AR) if its image as a closed subset K of a metrix space Y is a retract of Y.

Proposition 6.2.7. *A C^*-algebra $\mathfrak{A}$ is projective in CC (resp. CC_1) if and only if $\mathfrak{A} = C_0(X)$ for a locally compact (resp. compact) absolute retract X.*

Proof. Since $\mathfrak{A}$ is a commutative C^*-algebra, $\mathfrak{A} \cong C_0(X)$ for a locally compact Hausdorff space X. Suppose that X is an absolute retract. We need to show that for any closed subspace K of a locally compact Hausdorff space Y and a $*$-homomorphism $\sigma : C_0(X) \to C_0(K)$, there exits its lift $\varphi : C_0(X) \to C_0(Y)$ such that

$$\begin{array}{ccccccccc} & & & & C_0(X) & \xrightarrow{\mathrm{id}} & C_0(X) & & \\ & & & & {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\sigma} & & \\ 0 & \longrightarrow & C_0(Y \setminus K) & \longrightarrow & C_0(Y) & \longrightarrow & C_0(K) & \longrightarrow & 0. \end{array}$$

Then there exists a continuous map $\sigma^\wedge : K \to X$ such that $f \circ \sigma^\wedge = \sigma(f)$ for $f \in C_0(X)$. In fact, by Gelfand representation the spaces K and X are identified with the spaces of maximal ideals of $C_0(K)$ and $C_0(X)$ respectively, that consist of the kernels $\ker(\chi)$ and $\ker(\psi)$ for characters χ and ψ of $C_0(K)$ and $C_0(X)$. Thus, the map $\sigma^\wedge$ is given by $\sigma^\wedge(\ker(\chi)) = \ker(\chi \circ \sigma)$. Since (or if) K is a retract of Y, there exists an extension

$\rho^\wedge : Y \to X$ from $\sigma^\wedge$. Define $\varphi : C_0(X) \to C_0(Y)$ by $\varphi(f) = f \circ \rho^\wedge$ for $f \in C_0(X)$. Since $\rho^\wedge|_K = \sigma^\wedge$, the diagram commutes.

Conversely, the commutative diagram, Gelfand representation, and the reverse argument imply that X is an absolute retract. □

Remark. It seems that our proof is natural but perhaps be wrong since we need to assume the if part, or it might be that the statement itself is wrong. Or the category should be replaced with the category with AR quotient spaces, or the definition of projectivity may be modified. Anyway, $\sigma : C_0(X) \to C_0(K)$ would not imply a continuous map from X to K. But note that we may replace $C_0(K)$ with the image of $C_0(X)$ under σ, so that we may assume that σ is surjective. Then K is viewed as a closed subset of X.

Let $\mathfrak{A}$ be a C^*-algebra. Denote by $[\mathfrak{A}, \mathfrak{A}]$ the commutator ideal of $\mathfrak{A}$ generated by elements $xy - yx$ for $x, y \in \mathfrak{A}$. Then the quotient $\mathfrak{A}/[\mathfrak{A}, \mathfrak{A}]$ is a commutative C^*-algebra and is called the abelianization of $\mathfrak{A}$. Any $*$-homomorphism φ from $\mathfrak{A}$ into a commutative C^*-algebra $\mathfrak{C}$ factors through $\mathfrak{A}/[\mathfrak{A}, \mathfrak{A}]$:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\varphi} & \mathfrak{C} \\ \downarrow & & \| \\ \mathfrak{A}/[\mathfrak{A}, \mathfrak{A}] & \xrightarrow{\psi} & \mathfrak{C} \end{array}$$

where the map ψ is defined by $\psi([a]) = \varphi(a)$ for $[a] \in \mathfrak{A}/[\mathfrak{A}, \mathfrak{A}]$. Any $*$-homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ induces its abelianization $\varphi_a : \mathfrak{A}/[\mathfrak{A}, \mathfrak{A}] \to \mathfrak{B}/[\mathfrak{B}, \mathfrak{B}]$.

Proposition 6.2.8. *If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is projective in SC (or SC_1), then φ_a is projective in CC (or CC_1 respectively). So it $\mathfrak{A}$ is projective in SC (or SC_1), then $\mathfrak{A}/[\mathfrak{A}, \mathfrak{A}]$ is projective in CC (or CC_1 respectively).*

Proof. Consider the following diagram:

$$\begin{array}{ccccc} \mathfrak{A} & \xrightarrow{\varphi} & \mathfrak{B} & & \mathfrak{C} \\ {\scriptstyle \pi_{\mathfrak{A}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathfrak{B}}} & & \downarrow \\ \mathfrak{A}/[\mathfrak{A}, \mathfrak{A}] & \xrightarrow{\varphi_a} & \mathfrak{B}/[\mathfrak{B}, \mathfrak{B}] & \xrightarrow{\sigma} & \mathfrak{C}/\mathfrak{I} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

where $\mathfrak{C}$ is a separable commutative C^*-algebra. By projectivety, there is a lift $\omega : \mathfrak{A} \to \mathfrak{C}$ of $\sigma \circ \pi_{\mathfrak{B}} \circ \varphi : \mathfrak{A} \to \mathfrak{C}/\mathfrak{J}$. Since $\mathfrak{C}$ is commutative, ω factors through $\mathfrak{A}/[\mathfrak{A}, \mathfrak{A}]$ to give $\psi : \mathfrak{A}/[\mathfrak{A}, \mathfrak{A}] \to \mathfrak{C}$. □

6.3 Semiprojective C^*-algebras

Definition 6.3.1. A morphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ in SC_q is said to be semiprojective in SC_q if for any $\mathfrak{C} \in SC_q$ and increasing sequences $\mathfrak{J}_n$ of closed ideals of $\mathfrak{C}$ with $\mathfrak{J}$ the closure of the union $\cup_n \mathfrak{J}_n$, and any morphism $\sigma : \mathfrak{B} \to \mathfrak{C}/\mathfrak{J}$, there is a morphism $\psi : \mathfrak{A} \to \mathfrak{C}/\mathfrak{J}_n$ for some n with $\pi \circ \psi = \sigma \circ \varphi$, where $\pi : \mathfrak{C}/\mathfrak{J}_n \to \mathfrak{C}/\mathfrak{J}$ is the quotient map:

$$\begin{array}{ccccccccc} & & & & \mathfrak{A} & \xrightarrow{\varphi} & \mathfrak{B} & & \\ & & & & {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\sigma} & & \\ 0 & \longrightarrow & \mathfrak{J}_n & \longrightarrow & \mathfrak{C}/\mathfrak{J}_n & \xrightarrow{\pi} & \mathfrak{C}/\mathfrak{J} & \longrightarrow & 0. \end{array}$$

A C^*-algebra $\mathfrak{A}$ is said to be semiprojective in SC_q if the identity map on $\mathfrak{A}$ is semiprojective.

If $\mathfrak{A}$ or $\mathfrak{B}$ is semiprojective, then any morphism from $\mathfrak{A}$ to $\mathfrak{B}$ is semiprojective. A composition of a semiprojective morphism with any other morphism is semiprojective. Any projective morphism and any projective C^*-algebra are semiprojective.

Recall that a subspace A of a topological space Y is said to be a neighbourhood retract of Y if A is a retract of some open subspace U of Y:

$$\begin{array}{ccccc} Y & \longleftarrow & U & = & U \\ \| & & \uparrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle f} \\ Y & \longleftarrow & A & \xrightarrow{\mathrm{id}_A} & A \end{array}$$

for some extension $f : U \to A$ from id_A, or equivalently,

$$\begin{array}{ccccc} Y & \longleftarrow & U & = & U \\ \| & & \uparrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle f} \\ Y & \longleftarrow & A & \xrightarrow{g} & X \end{array}$$

for any $g : A \to X$ and its lift $f : U \to X$.

A metric space X is said to be an absolute neighbourhood retract (or ANR) if its image as closed subset K of a metric space Y is necessarily a neighbourhood retract of Y.

Proposition 6.3.2. *A commutative C^*-algebra $C_0(X)$ is semiprojective in CC if and only if X is an ANR.*

A unital commutative C^*-algebra $C(X)$ is semiprojective in CC_1 if and only if X is a compact ANR.

Proof. Suppose that X is an ANR. We need to show that for any closed subspace K of a locally compact Hausdorff space Y and a $*$-homomorphism $\sigma : C_0(X) \to C_0(K)$, there exits its lift $\varphi : C_0(X) \to C_0(U)$ for some open subspace U of Y containing K such that

$$\begin{array}{ccccc} C_0(X) & \xrightarrow{\text{id}} & C_0(X) & & \\ \varphi \downarrow & & \downarrow \sigma & & \\ C_0(U) & \longrightarrow & C_0(K) & \longrightarrow & 0. \end{array}$$

Then there exists a continuous map $\sigma^\wedge : K \to X$ such that $f \circ \sigma^\wedge = \sigma(f)$ for $f \in C_0(X)$, given by $\sigma^\wedge(\ker(\chi)) = \ker(\chi \circ \sigma)$. Since (or if) K is a neighbourhood retract of Y, there exists an open subset U of Y containing K and an extension $\rho^\wedge : U \to X$ from $\sigma^\wedge$. Define $\varphi : C_0(X) \to C_0(U)$ by $\varphi(f) = f \circ \rho^\wedge$ for $f \in C_0(X)$. Since $\rho^\wedge|_K = \sigma^\wedge$, the diagram commutes.

Furthermore, since $U \cap (Y \setminus K)$ is open in Y and does not contain K and does be contained in $Y \setminus K$, in the diagram above we may replace $C_0(U)$ with:

$$\begin{aligned} C_0(Y)/C_0(U \cap (Y \setminus K)) &\cong C_0(Y \setminus (U \cap (Y \setminus K))) \\ &= C_0((Y \setminus U) \cup K) \end{aligned}$$

which has $C_0(K)$ as a quotient and just fits to the definition of semiprojectivity.

Conversely, the commutative diagram, Gelfand representation, and the reverse argument imply that X is an absolute retract. □

Remark. It seems that the our proof is natural but perhaps be wrong since we need to assume the if part, or it might be that the statement itself is wrong. Or the category should be replaced with the category with ANR quotient spaces, or the definition of semiprojectivity may be modified.

There exist contractible spaces which are not ANR, like the cone over the Cantor set.

Proposition 6.3.3. *If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is semiprojective in SC (or SC_1), then φ_a from $\mathfrak{A}/[\mathfrak{A}, \mathfrak{A}]$ to $\mathfrak{B}/[\mathfrak{B}, \mathfrak{B}]$ is semiprojective in CC (or CC_1 respectively). So it $\mathfrak{A}$ is semiprojective in SC (or SC_1), then $\mathfrak{A}/[\mathfrak{A}, \mathfrak{A}]$ is semiprojective in CC (or CC_1 respectively).*

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
\mathfrak{A} & \xrightarrow{\varphi} & \mathfrak{B} & & \mathfrak{C}/\mathfrak{J}_n \\
{\scriptstyle \pi_{\mathfrak{A}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathfrak{B}}} & & \downarrow \\
\mathfrak{A}/[\mathfrak{A},\mathfrak{A}] & \xrightarrow{\varphi_a} & \mathfrak{B}/[\mathfrak{B},\mathfrak{B}] & \xrightarrow{\sigma} & \mathfrak{C}/\mathfrak{J} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

where $\mathfrak{C}$ is a separable commutative C^*-algebra. By semiprojectivety, there is a lift $\omega : \mathfrak{A} \to \mathfrak{C}/\mathfrak{J}_n$ of $\sigma \circ \pi_{\mathfrak{B}} \circ \varphi : \mathfrak{A} \to \mathfrak{C}/\mathfrak{J}$. Since $\mathfrak{C}/\mathfrak{J}_n$ is commutative, ω factors through $\mathfrak{A}/[\mathfrak{A},\mathfrak{A}]$ to give $\psi : \mathfrak{A}/[\mathfrak{A},\mathfrak{A}] \to \mathfrak{C}/\mathfrak{J}_n$. □

Lemma 6.3.4. *Let $\mathfrak{C}$ be a C^*-algebra and $\{\mathfrak{J}_n\}$ an increasing sequences of closed ideals of $\mathfrak{C}$ with $\mathfrak{J}$ the closure of the union $\cup\mathfrak{J}_n$. Let $\pi_n : \mathfrak{C} \to \mathfrak{C}/\mathfrak{J}_n$ and $\pi : \mathfrak{C} \to \mathfrak{C}/\mathfrak{J}$ be the quotient maps. Then we have $\|\pi(x)\| = \inf_n \|\pi_n(x)\|$ for any $x \in \mathfrak{C}$.*

Proof. Indeed, from the definition of the quotient norm we have

$$\|\pi(x)\| = \inf_{b \in \mathfrak{J}} \|x + b\| \leq \inf_{b \in \mathfrak{J}_n} \|x + b\| = \|\pi_n(x)\|.$$

Hence $\|\pi(x)\| \leq \inf_n \|\pi_n(x)\|$. Conversely, for any $b \in \mathfrak{J}$ and $\varepsilon > 0$, there exists $b_n \in \mathfrak{J}_n$ for some n such that $\|b - b_n\| < \varepsilon$, so that

$$\|x + b_n\| = \|x + b_n - b + b\| \leq \|x + b\| + \varepsilon.$$

It follows that $\inf_n \|\pi_n(x)\| \leq \|x + b\| + \varepsilon$. Thus, $\inf_n \|\pi_n(x)\| \leq \|\pi(x)\| + \varepsilon$. Since ε is arbitrary, it is proved. □

Proposition 6.3.5. *With the same notation as in the lemma above, let p' be a projection in $\mathfrak{C}/\mathfrak{J}$. Then there is a projection $p \in \mathfrak{C}/\mathfrak{J}_n$ for some n such that $\pi(p) = p'$, where this π is the quotient map $\pi : \mathfrak{C}/\mathfrak{J}_n \to \mathfrak{C}/\mathfrak{J}$.*

Proof. Take a positive element $x \in \mathfrak{C}$ with $\pi(x) = p'$. Then $\pi(x - x^2) = 0$. Thus $\|\pi_n(x - x^2)\| < 1/4$ for some n. Hence the spectrum $\mathrm{sp}(\pi_n(x))$ of $\pi_n(x)$ is disconnected at $1/2$, and p is constructed from $\pi_n(x)$ by functional calculus. Indeed, the spectral mapping theorem implies that the spectrum of $\pi_n(x - x^2)$ is $\{\lambda - \lambda^2 \,|\, \lambda \in \mathrm{sp}(\pi_n(x))\}$. Since $|\lambda(1 - \lambda)| < 1/4$, if $\lambda < 1$, then we have $(\lambda - 1/2)^2 > 0$. By functional calculus, there is a projection $p(\lambda) \in C(\mathrm{sp}(\pi_n(x)))$ such that $\pi(p(\lambda)) = p(\pi(x)) = p'$. □

Corollary 6.3.6. *If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is semiprojective in SC_1 (resp. CC_1), then it is semiprojective in SC (resp. CC). So if $\mathfrak{A}$ is semiprojective in SC_1, it is semiprojective in SC. Conversely, if a morphism or a C^*-algebra is unital and semiprojective in SC, then it is semiprojective in SC_1.*

Proof. Given $\sigma : \mathfrak{B} \to \mathfrak{C}/\mathfrak{J}$, let $p' = \sigma(1_\mathfrak{B})$. Lift p' to $p \in \mathfrak{C}/\mathfrak{J}_n$. Replace $\mathfrak{C}$ by $p(\mathfrak{C}/\mathfrak{J}_n)p$, $\mathfrak{J}_k$ by $p(\mathfrak{J}_k/\mathfrak{J}_n)p$ for $k \geq n$, and $\mathfrak{J}$ by $p(\mathfrak{J}/\mathfrak{J}_n)p$. By the assumption,

$$\sigma : \mathfrak{B} \to p'(\mathfrak{C}/\mathfrak{J})p' \cong p(\mathfrak{C}/\mathfrak{J}_n)p/p(\mathfrak{J}/\mathfrak{J}_n)p$$

has a lift:

$$\psi : \mathfrak{A} \to p(\mathfrak{C}/\mathfrak{J}_n)p/p(\mathfrak{J}_k/\mathfrak{J}_n)p \cong p''(\mathfrak{C}/\mathfrak{J}_k)p'' \subset \mathfrak{C}/\mathfrak{J}_k$$

where $p'' = \pi_{n,k}(p)$ and $\pi_{n,k} : \mathfrak{C}/\mathfrak{J}_n \to \mathfrak{C}/\mathfrak{J}_k$.

Conversely, if φ is semiprojective in SC, $\mathfrak{C}$ is unital, and $\sigma : \mathfrak{B} \to \mathfrak{C}/\mathfrak{J}$ is unital, let $\psi : \mathfrak{A} \to \mathfrak{C}/\mathfrak{J}_k$ be a lift. Set $q = \psi(1_\mathfrak{A})$. Since $\pi(1_{\mathfrak{C}/\mathfrak{J}_k} - q) = 0$, we have $\|\pi_n(1_{\mathfrak{C}/\mathfrak{J}_k} - q)\| < 1$ for some n, where $\pi_n = \pi_{k,n}$ in the sense above. But $\pi_n(1_{\mathfrak{C}/\mathfrak{J}_k} - q)$ is a projection, so $\pi_n(q) = 1_{\mathfrak{C}/\mathfrak{J}_n}$, that is, $\pi_n \circ \psi$ is a unital lift. □

Remark. It follows that $\mathbb{C}$ is semiprojective in SC.

Corollary 6.3.7. *A C^*-algebra is semiprojective in SC if and only if its unitization is semiprojective in SC_1.*

Proposition 6.3.8. *The noncommutative Grassmanian G_n^{nc} defined in the subsection* 6.1 *is semiprojective.*

Proof. We may work in SC_1. If $\sigma : G_n^{nc} \to \mathfrak{C}/\mathfrak{J}$ is given, it has the canonical extension to the matrix algebra over them: $\sigma_n : M_n(G_n^{nc}) \to M_n(\mathfrak{C}/\mathfrak{J}) \cong M_n(\mathfrak{C})/M_n(\mathfrak{J})$. If $x = (x_{ij}) \in M_n(G_n^{nc})$ where x_{ij} are generators for G_n^{nc}, then x is a projection, that is, $x = x^* = x^2$ since $x_{ij} = x_{ji}^* = \sum_{k=1}^n x_{ik}x_{kj}$. Thus, $\sigma(x) = p'$ is a projection in $M_n(\mathfrak{C})/M_n(\mathfrak{J})$. Lift p' to a projection $p = (p_{ij}) \in M_n(\mathfrak{C})/M_n(\mathfrak{J}_k) \cong M_n(\mathfrak{C}/\mathfrak{J}_k)$ for some k. Then the map $x_{ij} \mapsto p_{ij}$ gives a lift of σ to $\mathfrak{C}/\mathfrak{J}_k$. □

Proposition 6.3.9. *Suppose that $p'_1, \cdots, p'_r$ are orthogonal projections in $\mathfrak{C}/\mathfrak{J}$. Then for some n, there are orthogonal projections $p_1, \cdots, p_r \in \mathfrak{C}/\mathfrak{J}_n$ with $\pi(p_j) = p'_j$ for all j.*

If $\mathfrak{C}$ is unital and $p'_1 + \cdots + p'_r = 1$, then we can choose p_j to have $p_1 + \cdots + p_r = 1$.

Proof. As before, lift p_1' to $p_1 \in \mathfrak{C}/\mathfrak{J}_{n_1}$. Replace $\mathfrak{C}$ by $(1-p_1)(\mathfrak{C}/\mathfrak{J}_{n_1})(1-p_1)$, $\mathfrak{J}_n$ by $(1-p_1)(\mathfrak{J}_n/\mathfrak{J}_{n_1})(1-p_1)$, and $\mathfrak{J}$ by $(1-p_1)(\mathfrak{J}/\mathfrak{J}_{n_1})(1-p_1)$. Note that

$$(1-p_1)(\mathfrak{C}/\mathfrak{J}_{n_1})(1-p_1)/(1-p_1)(\mathfrak{J}/\mathfrak{J}_{n_1})(1-p_1) \cong (1-p_1')(\mathfrak{C}/\mathfrak{J})(1-p_1')$$

which contains p_2'. Now lift p_2' to p_2 in

$$(1-p_1)(\mathfrak{C}/\mathfrak{J}_{n_1})(1-p_1)/(1-p_1)(\mathfrak{J}_{n_2}/\mathfrak{J}_{n_1})(1-p_1) \cong (1-p_1'')(\mathfrak{C}/\mathfrak{J}_{n_2})(1-p_1'')$$

for some $n_2 \geq n_1$, where $p_1'' = \pi_{n_1,n_2}(p_1)$ and $\pi_{n_1,n_2} : \mathfrak{C}/\mathfrak{J}_{n_1} \to \mathfrak{C}/\mathfrak{J}_{n_2}$. Then p_1'' and p_2 are orthogonal projections lifting p_1' and p_2' respectively. So replace p_1'' with p_1 by the same symbol. Continueing this process inductively. we obtain the projections lifted as desired.

If $\mathfrak{C}$ is unital and $p_1' + \cdots + p_r' = 1$, then $p_r' = 1 - p_1' - \cdots - p_{r-1}'$. In this case, stop the induction after $r-1$ steps and set $p_r = 1 - p_1 - \cdots - p_{r-1}$. □

Corollary 6.3.10. *If $\varphi_j : \mathfrak{A}_j \to \mathfrak{B}_j$ are morphisms in SC_q, then their direct sum $\oplus_{j=1}^r \varphi_j : \oplus_{j=1}^r \mathfrak{A}_j \to \oplus_{j=1}^r \mathfrak{B}_j$ is semiprojective if and only if each φ_j is semiprojective. So if $\mathfrak{A}_j$ are unital, then the direct sum $\oplus_{j=1}^r \mathfrak{A}_j$ is semiprojective if and only if each $\mathfrak{A}_j$ is semiprojective.*

Proof. First lift $p_j' = \sigma(1_{\mathfrak{B}_j})$ to $p_j \in \mathfrak{C}/\mathfrak{J}_k$, and then lift φ_i to

$$\pi_{k,n_j}(p_j)(\mathfrak{C}/\mathfrak{J}_{n_j})\pi_{k,n_j}(p_j) \cong \pi_{k,n_j}(p_j)[(\mathfrak{C}/\mathfrak{J}_k)/(\mathfrak{J}_{n_j}/\mathfrak{J}_k)]\pi_{k,n_j}(p_j)$$

for some $n_j \geq k$. Let $n = \max n_j$. It follows that $\oplus_{j=1}^r \varphi_j$ is semiprojective.

The converse is trivial. Indeed, given $\sigma : \mathfrak{B}_l \to \mathfrak{C}/\mathfrak{J}$, use semiprojectivity for $\sigma \circ pr_l : \oplus_{j=1}^r \mathfrak{B}_j \to \mathfrak{C}/\mathfrak{J}$, where $pr_l : \oplus_{j=1}^r \mathfrak{B}_j \to \mathfrak{B}_l$ is the canonical projection map. □

Remark. It might be still not known that a direct sum of nonunital semiprojective C^*-algebras is always semiprojective.

Proposition 6.3.11. *If s' is an isometry in $\mathfrak{C}/\mathfrak{J}$, then there is an isometry $s \in \mathfrak{C}/\mathfrak{J}_n$ for some n such that $\pi(s) = s'$, where $\pi : \mathfrak{C}/\mathfrak{J}_n \to \mathfrak{C}/\mathfrak{J}$ is the quotient map.*

If s' is unitary, then s can be chosen to be unitary.

Proof. Let $x \in \mathfrak{C}$ with $\pi(x) = s'$, where $\pi : \mathfrak{C} \to \mathfrak{C}/\mathfrak{J}$ is the quotient map. Then the norm $\|\pi_n(x)^*\pi_n(x) - 1\|$ converges to zero, so that $\pi_n(x^*x)$ is invertible for some n. Set $s_n = \pi_n(x)[\pi_n(x^*x)]^{-1/2}$. Then s_n is an isometry with $\pi(s_n) = s'$ for the π in the statement.

If s' is unitary, we also have that the norm $\|\pi_n(x)\pi_n(x)^* - 1\|$ converges to zero, so that $\pi_n(x)$ is invertible for some n large, and hence s_n is unitary for some n large. □

Corollary 6.3.12. *The Toeplitz algebra $\mathfrak{T}$, $C(S^1)$, and the noncommutative unitary group U_n^{nc} defined in the subsection* 6.1 *are semiprojective.*

Proof. It is because $\mathfrak{T}$ is generated by an isometry, which is mapped to an isometry by an morphism to a quotient, which can be lifted as above, and similarly note that $C(S^1)$ is generated by an unitary.

Let $\sigma : U_n^{nc} \to \mathfrak{C}/\mathfrak{J}$ be given. Let $\{x_{ij} : 1 \leq i, j \leq n\}$ be generators of U_n^{nc}. Then $x^*x = 1_n$ and $xx^* = 1_n$, where $x = (x_{ij}) \in M_n(U_n^{nc})$. Thus the matrix x is mapped to a unitary u' under the map $M_n(U_n^{nc}) \to M_n(\mathfrak{C}/\mathfrak{J})$ extended from σ. Lift u' to a unitary $u = (u_{ij}) \in M_n(\mathfrak{C}/\mathfrak{J}_k)$ for some k. Then the map $x_{ij} \mapsto u_{ij}$ gives a lift of σ to $\mathfrak{C}/\mathfrak{J}_k$. □

Proposition 6.3.13. *Let p' and q' be projections of $\mathfrak{C}/\mathfrak{J}$ and u' a partial isometry of $\mathfrak{C}/\mathfrak{J}$ such that $(u')^*u = p'$ and $u'(u')^* = q'$. Let p and q be projections of $\mathfrak{C}$ with $\pi(p) = p'$ and $\pi(q) = q'$. Then there is a partial isometry $u \in \mathfrak{C}/\mathfrak{J}_n$ for some n such that $\pi(u) = u'$ and $u^*u = \pi_n(p)$ and $uu^* = \pi_n(q)$.*

Proof. Let $x \in \mathfrak{C}$ with $\pi(x) = u'$. Then we have

$$\|\pi_n(x)^*\pi_n(x) - \pi_n(p)\| \quad \text{and} \quad \|\pi_n(x)\pi_n(x)^* - \pi_n(q)\|$$

are small for some n. Let f be a continuous function which is identically zero near 0 and $f(\lambda) = \lambda^{-1/2}$ for λ near 1. Then $w = \pi_n(x)f(\pi_n(x^*x))$ is a partial isometry in $\mathfrak{C}/\mathfrak{J}_n$ with $\|w^*w - \pi_n(p)\|$ and $\|ww^* - \pi_n(q)\|$ small. Indeed, we have

$$\begin{aligned} w^*w &= f(\pi_n(x^*x))\pi_n(x^*x)f(\pi_n(x^*x)), \\ ww^* &= \pi_n(x)f(\pi_n(x^*x))f(\pi_n(x^*x))\pi_n(x^*) \end{aligned}$$

and note that $\lambda^{-1/2}\lambda\lambda^{-1/2} = 1$ and $\lambda^{-1/2}\lambda^{-1/2} = \lambda^{-1}$ (and for w to be a partial isometry it is enough to check that w^*w is a projection). Set $v_j = z_j(z_j^*z_j)^{-1/2}$ for $j = 1, 2$, where

$$z_1 = (2w^*w - 1)(2\pi_n(p) - 1) + 1, \quad z_2 = (2\pi_n(q) - 1)(2ww^* - 1) + 1.$$

Then v_j are unitaries in $(\mathfrak{C}/\mathfrak{J}_n)^+$ which conjugate $\pi_n(p)$ and ww^* to w^*w and $\pi_n(q)$ respectively, and $\pi(v_j) = 1 \in (\mathfrak{C}/\mathfrak{J})^+$. Thus, $u = v_2wv_1$ is the desired partial isometry. □

Corollary 6.3.14. *The Cuntz-Krieger algebra O_A for any matrix A with components* 0 *or* 1 *is semiprojective.*

Proof. Recall that generators $s_1, \cdots s_n$ of O_A are partial isometries such that $s_i^* s_i = \sum_{j=1}^n a_{ij} s_j s_j^*$ and $s_k^* s_i =$ for all i, k with $i \neq k$, where $A = (a_{ij})_{i,j=1}^n$, which are mapped to partial isometries by a morphism to a quotient, which are lifted as above. □

Proposition 6.3.15. *The matrix algebra $M_2(\mathbb{C})$ is semiprojective.*

Proof. Let $e_{11}, e_{12}, e_{21}, e_{22}$ be matrix units in $\mathfrak{C}/\mathfrak{J}$, i.e., $e_{ij} = e_{ji}^*$ and $e_{ij}e_{kl} = \delta_{jk}e_{il}$ for $1 \leq i, j, k, l \leq 2$. Lift e_{11} to a projection p in $\mathfrak{C}/\mathfrak{J}_k$, and e_{12} to a partial isometry s in $\mathfrak{C}/\mathfrak{J}_n$ for some $n \geq k$ with $u^*u = \pi_n(p)$ and $uu^* = 1 - \pi_n(p)$. Then $\{\pi_n(p), u, u^*, 1 - \pi_n(p)\}$ is a system of matrix units in $\mathfrak{C}/\mathfrak{J}_n$ which lift the e_{ij}. □

Proposition 6.3.16. *If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is semiprojective in SC_1, then $\varphi_2 : M_2(\mathfrak{A}) \to M_2(\mathfrak{B})$ is semiprojective in SC_1. So if $\mathfrak{A}$ is unital and semiprojective, then $M_2(\mathfrak{A})$ is semiprojective.*

Proof. Let $\{f_{ij}\}$ be the matrix units of $M_2(\mathbb{C}) \subset M_2(\mathfrak{A})$ and $\{e'_{ij}\}$ their image in $\mathfrak{C}/\mathfrak{J}$ under $\sigma \circ \varphi_2$. Lift $\{e'_{ij}\}$ to the matrix units $\{e_{ij}\}$ in $\mathfrak{C}/\mathfrak{J}_k$ for some k. Replace $\mathfrak{C}$ by $e_{11}(\mathfrak{C}/\mathfrak{J}_k)e_{11}$, $\mathfrak{J}_n$ by $e_{11}(\mathfrak{J}_n/\mathfrak{J}_k)e_{11}$ for $n \geq k$, and $\mathfrak{J}$ by $e_{11}(\mathfrak{J}/\mathfrak{J}_k)e_{11}$. Since the restriction of φ_2 to $f_{11}M_2(\mathfrak{A})f_{11}$ is viewed as φ, the restriction of $\sigma \circ \varphi_2$ to $f_{11}M_2(\mathfrak{A})f_{11}$ lifts to a homomorphism:

$$\psi : f_{11}M_2(\mathfrak{A})f_{11} \to \pi_n(e_{11})[(\mathfrak{C}/\mathfrak{J}_k)/(\mathfrak{J}_n/\mathfrak{J}_k)]\pi_n(e_{11})$$

for some n.

If $x \in M_2(\mathfrak{A})$, write $x = \sum_{i,j=1}^2 f_{i1}x_{ij}f_{1j}$, where $x_{ij} \in f_{11}M_2(\mathfrak{A})f_{11}$. Set

$$\psi_2(x) = \sum_{i,j=1}^{2} \pi_n(e_{i1})\psi(x_{ij})\pi_n(e_{1j}).$$

Then ψ_2 is a lift of $\sigma \circ \varphi_2$. □

Proposition 6.3.17. *Let $\mathfrak{A}$ be a unital and semiprojective C^*-algebra and p a full projection in $\mathfrak{A}$. Then $p\mathfrak{A}p$ is semiprojective.*

Proof. Since p is full, we can find projections p' and q and a partial isometry v in $M_r(p\mathfrak{A}p)$ for some r such that $qM_r(p\mathfrak{A}p)q \cong \mathfrak{A}$, $p' \leq q$, $v^*v = p \oplus 0 \oplus \cdots \oplus 0$ (diagonal sum), and $vv^* = p'$. Indeed, since p is a full projection, there exist $x_j \in \mathfrak{A}$ such that $1 = \sum_{j=1}^r x_j p x_j^*$ for some r, so that $p = \sum_{j=1}^r (px_jp)(px_j^*p)$, which is viewed as:

$$p = \begin{pmatrix} px_1p & \cdots & px_rp \end{pmatrix} \begin{pmatrix} px_1^*p \\ \vdots \\ px_r^*p \end{pmatrix} = v^*v$$

where the row and column matrices v^* and v can be viewed as the corresponding matrices in $M_r(p\mathfrak{A}p)$, and $p = p\oplus 0\oplus\cdots\oplus 0$ in this sense. Moreover, we have

$$p' = vv^* = (p \oplus \cdots \oplus p)vv^*(p \oplus \cdots \oplus p) \leq p \oplus \cdots \oplus p = q$$

since $vv^* \leq 1 \oplus \cdots \oplus 1$.

Let $\sigma : p\mathfrak{A}p \to \mathfrak{C}/\mathfrak{J}$. Extend it to $\sigma_r : M_r(p\mathfrak{A}p) \to M_r(\mathfrak{C}/\mathfrak{J})$. Let $q' = \sigma_r(q)$. Lift q' to a projection $q'' \in M_r(\mathfrak{C}/\mathfrak{J}_k)$. Let w be a lift of the restriction $\sigma_r|qM_r(p\mathfrak{A}p)q$ to the image $\pi_n(q')M_r(\mathfrak{C}/\mathfrak{J}_n)\pi_n(q')$ for some $n \geq k$. Set $u' = \sigma_r(v)$. We find a partial isometry $u \in M_r(\mathfrak{C}/\mathfrak{J}_n)$ which lifts u', for which $u^*u = 1 \oplus 0 \oplus \cdots \oplus 0$ and $uu^* = w(p')$. Identify $p\mathfrak{A}p$ and $\mathfrak{C}/\mathfrak{J}_n$ with the upper left-hand corners in $M_r(p\mathfrak{A}p)$ and $M_r(\mathfrak{C}/\mathfrak{J}_n)$ respectively. Let $\psi(x) = u^*w(vxv^*)u$. Then ψ is a lift of σ to $\mathfrak{C}/\mathfrak{J}_n$. □

Corollary 6.3.18. *If $\mathfrak{A}$ is a unital and semiprojective C^*-algebra, then $M_n(\mathfrak{A})$ is semiprojective for all n. In particular, $M_n(\mathbb{C})$ is semiprojective for all n.*

Proof. Since $M_2(\mathfrak{A})$ is semiprojective, it follows by induction that $M_{2^k}(\mathfrak{A}) \cong M_2(M_{2^{k-1}}(\mathfrak{A}))$ is semiprojective for all k. Note that $M_n(\mathfrak{A})$ is a full corner in $M_{2^k}(\mathfrak{A})$ for some k. □

Corollary 6.3.19. *Suppose that unital C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are strongly Morita equivalent. Then $\mathfrak{A}$ is semiprojective if and only if $\mathfrak{B}$ is semiprojective.*

Proof. Note that $\mathfrak{A}$ and $\mathfrak{B}$ are each isomorphic to full corners in matrix algebras over the other. □

Remark. This can be false if $\mathfrak{A}$ or $\mathfrak{B}$ is nonunital.

Corollary 6.3.20. *If $\mathfrak{A}$ is a unital semiprojective C^*-algebra and F is a finite dimensional C^*-algebra, then their tensor product $\mathfrak{A} \otimes F$ is semiprojective.*

In particular, F and $C(S^1) \otimes F$ are semiprojective.

Proposition 6.3.21. *If $\mathfrak{A}_j$ $(1 \leq j \leq r)$ are semiprojective C^*-algebras in SC (resp. SC_1), then their free product $*\mathfrak{A}_j$ is semiprojective in SC (resp. their unital free product $*_{\mathbb{C}}\mathfrak{A}_j$ is semiprojective in SC_1.*

Proof. This can be proved similarly as in the projective case above. □

Proposition 6.3.22. *If $\mathfrak{A}_j$ $(1 \le j \le r)$ are semiprojective C^*-algebras and F is a finite dimensional C^*-subalgebra of $\mathfrak{A}_j$ for all j, then their amalgamated free product $*_F \mathfrak{A}_j$ over F is semiprojective.*

Note that an infinite free product of semiprojective C^*-algebras will not in general be semiprojective.

Example 6.3.23. The commutative C^*-algebra $C([0,1]^2)$ is semiprojective (in fact projective) in CC_1, but not semiprojective in SC_1. For let u be the unilateral shift, $\mathfrak{C}$ the C^*-algebra of all sequences in $C^*(u)$ converging to a scalar multiple of the identity,

$$\mathfrak{J}_n = \{(x_j) : x_j \in \mathbb{K} \subset C^*(u) \quad \text{for all } j, \text{ and } x_j = 0 \text{ for } j > n\}, \quad \text{and}$$
$$\mathfrak{J} = \{(x_j) : x_j \in \mathbb{K} \quad \text{for all } j, \text{ and } x_j \to 0\}.$$

Then $\mathfrak{J}$ is the closure of the union $\cup \mathfrak{J}_n$, and $\mathfrak{C}/\mathfrak{J}$ is isomorphic to the C^*-algebra of all sequences in $\pi(C^*(u))$ converging to a scalar multiple of the identity, where $\pi : C^*(u) \to C^*(u)/\mathbb{K} \cong C(S^1)$. Let $x = (x_j)$ and $y = (y_j)$ with $x_n = \mathrm{Re}(\pi(u))/n$ and $y_n = \mathrm{Im}(\pi(u))/n$. Then x and y are commuting self-adjoint contractions in $\mathfrak{C}/\mathfrak{J}$, so there is a $*$-homomorphism $\sigma : C([0,1]^2) \to C^*(x,y) \subset \mathfrak{C}/\mathfrak{J}$. But σ cannot be lifted to $\mathfrak{C}/\mathfrak{J}_n$ for any n.

Also, one can define a $*$-homomorphism from $C(S^1 \times S^1)$ into $\mathfrak{C}/\mathfrak{J}$ which cannot be lifted, by sending two unitary generators to e^{ix} and e^{iy}. Thus $C(S^1 \times S^1)$ is not semiprojective in SC_1. Similarly, it is shown that the rotation algebras are not semiprojective.

Remark. This shows that a universal C^*-algebra on a finite set of generators and relations need not be semiprojective. For such a C^*-algebra to be semiprojective, the relations must be partially liftable in the sense that if $x_1, \cdots, x_n \in \mathfrak{C}/\mathfrak{J}$ satisfy the relations, then suitable preimages in $\mathfrak{C}/\mathfrak{J}_k$ for some k also satisfy the same ones. It follows from the propositions above and elementary C^*-algebra theory that the relations such as $\|x_j\| \le \eta$, $x_j = x_j^*$, $x_j = x_j^* = x_j^2$, $x_j^* x_j = 1$, $x_j^* x_j = x_j x_j^* = 1$, and their matrix versions are partially liftable. But commutation relations among generators are not partially liftable.

Proposition 6.3.24. *Let $\mathfrak{A} = C^*(\mathfrak{G}, \mathfrak{R})$, where $\mathfrak{G} = \{x_1, \cdots, x_n\}$ and $\mathfrak{R} = \{\|p_1(\cdot)\| \le \eta_1, \cdots, \|p_k(\cdot)\| \le \eta_k\}$ with $\eta_j > 0$ for all $j = 1, \cdots, k$. If $\varphi : \mathfrak{A} \to \mathfrak{B}$ satisfies*

$$\|p_j(\varphi(x_1), \cdots, \varphi(x_n), \varphi(x_1^*), \cdots, \varphi(x_n^*))\| < \eta_j$$

for each j, then φ is semiprojective.

Proof. It follows from considering the quotient norm definition as before. □

It is known from topology that a compact retract of an open set in an ANR is an ANR.

Definition 6.3.25. A unital C^*-algebra $\mathfrak{A}$ is said to be a retract of a C^*-algebra $\mathfrak{B}$ if there is a unital homomorphism $\omega : \mathfrak{A} \to M(\mathfrak{B})$ the multiplier algebra of $\mathfrak{B}$ and a surjective homomorphism $\rho : \mathfrak{B} \to \mathfrak{A}$ such that $\rho^\sim \circ \omega = \mathrm{id}_\mathfrak{A}$, where $\rho^\sim$ is the canonical extension of ρ to a homomorphism from $M(\mathfrak{B})$ to $\mathfrak{A}$:

$$\begin{array}{ccccc}
M(\mathfrak{B}) & = & M(\mathfrak{B}) & = & M(\mathfrak{B}) \\
{\scriptstyle\omega}\uparrow & & \downarrow{\scriptstyle\rho^\sim} & & \uparrow \\
\mathfrak{A} & \xrightarrow{\mathrm{id}_\mathfrak{A}} & \mathfrak{A} & \xleftarrow{\rho} & \mathfrak{B} \\
 & & \downarrow & & \uparrow \\
 & & 0 & & 0.
\end{array}$$

Theorem 6.3.26. *Let $\mathfrak{D}$ be a semiprojective C^*-algebra in SC, $\mathfrak{K}$ a closed ideal of $\mathfrak{D}$, and $\mathfrak{A}$ a unital C^*-algebra which is a retract of $\mathfrak{K}$. Then $\mathfrak{A}$ is semiprojective.*

Proof. Let $\omega : \mathfrak{A} \to M(\mathfrak{K})$ and $\rho : \mathfrak{K} \to \mathfrak{A} \to 0$ be as above. Let $\mathfrak{C}$ be a unital C^*-algebra and $\sigma : \mathfrak{A} \to \mathfrak{C}/\mathfrak{J}$ unital. There is a homomorphism $\theta : \mathfrak{D} \to M(\mathfrak{K})$ which is the identity on $\mathfrak{K}$. Set $\alpha = \sigma \circ \rho^\sim \circ \theta$:

$$\mathfrak{D} \xrightarrow{\theta} M(\mathfrak{K}) \xrightarrow{\rho^\sim} \mathfrak{A} \xrightarrow{\sigma} \mathfrak{C}/\mathfrak{J}.$$

Then α lifts to a map $\beta : \mathfrak{D} \to \mathfrak{C}/\mathfrak{J}_k$ for some k. Since $\pi \circ \beta(\mathfrak{K}) = \sigma(\mathfrak{A})$ contains the identity of $\mathfrak{C}/\mathfrak{J}$, the image $\mathfrak{B}_n = \pi_n \circ \beta(\mathfrak{K})$ for some n contains the identity of $\mathfrak{C}/\mathfrak{J}_n$. For n such, $\pi_n \circ \beta$ extends to a unital homomorphism $\gamma : M(\mathfrak{K}) \to \mathfrak{B}_n \subset \mathfrak{C}/\mathfrak{J}_n$ which lifts $\sigma \circ \rho^\sim$, and so $\psi = \gamma \circ \omega$ gives a lift of $\sigma \circ \rho^\sim \circ \omega = \sigma$. □

Remark. It is known that every compact ANR is a retract of an open set in a compact AR. In fact, a metrizable compact ANR is a retract of an open set in the Hilbert cube.

From now on, we require that the category S be closed under quotients and countable inductive limits, and taking tensor product with $C([0,1])$.

Consider inductive limits:

$$\mathfrak{D} = \varinjlim(\mathfrak{D}_n, \gamma_{n,n+1})$$

where $\gamma_{n,n+1} : \mathfrak{D}_n \to \mathfrak{D}_{n+1}$ is a not necessarily injective $*$-homomorphism. For defining $\mathfrak{D}$, set $\mathfrak{J}_{n,m} = \ker(\gamma_{n,m}) \subset \mathfrak{D}_n$, where $\gamma_{n,m} : \mathfrak{D}_n \to \mathfrak{D}_m$ for $m > n$, constructed by composing from $\gamma_{n,n+1}, \gamma_{n+1,n+2}, \cdots$, to $\gamma_{m-1,m}$. We have $\mathfrak{J}_{n,m} \subset \mathfrak{J}_{n,m+1}$ for all $m > n$. Let $\mathfrak{J}_n$ be the closure of the union $\cup_m \mathfrak{J}_{n,m}$. Then $\gamma_{n,n+1}$ drops to an injective $*$-homomorphism $\gamma'_{n,n+1} : \mathfrak{D}_n/\mathfrak{J}_n \to \mathfrak{D}_{n+1}/\mathfrak{J}_{n+1}$, and $\mathfrak{D}$ is defined as the inductive limit $\varinjlim(\mathfrak{D}_n/\mathfrak{J}_n, \gamma'_{n,n+1})$ where the connected maps are injective. Note that the C^*-algebra $\mathfrak{C}/\mathfrak{J}$ considered above is an inductive limit $\varinjlim(\mathfrak{C}/\mathfrak{J}_n, \pi_{n,n+1})$, where the connecting maps are surjective. An inductive limit with injective connecting maps is said to be a faithful inductive limit. Denote by γ_n the canonical homomorphism from $\mathfrak{D}_n$ into $\mathfrak{D} = \varinjlim(\mathfrak{D}_n, \gamma_{n,n+1})$. If $\mathfrak{D}$ is unital, then $\mathfrak{D}_n$ is unital for n sufficiently large.

Theorem 6.3.27. *Let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be semiprojective in S, $\mathfrak{D} = \varinjlim(\mathfrak{D}_n, \gamma_{n,n+1})$, and $\beta : \mathfrak{B} \to \mathfrak{D}$ a morphism. Then for sufficiently large n, there are homomorphisms $\alpha_n : \mathfrak{A} \to \mathfrak{D}_n$ such that $\gamma_n \circ \alpha_n \sim \beta \circ \varphi$ and $\gamma_n \circ \alpha_n \to \beta \circ \varphi$ pointwise:*

$$\mathfrak{A} \xrightarrow{\alpha_n} \mathfrak{D}_n \xrightarrow{\gamma_n} \mathfrak{D} \quad \sim \quad \textit{and} \quad \to \quad \mathfrak{A} \xrightarrow{\varphi} \mathfrak{B} \xrightarrow{\beta} \mathfrak{D}.$$

Proof. Let $\mathfrak{C}$ be the C^*-subalgebra of $\Pi_n C([n, n+1], \mathfrak{D}_n)$ consisting of all sequences (f_n) for which

$$f_{n+1}(n+1) = \gamma_{n,n+1}(f_n(n+1)), \quad \forall n,$$

and

$$\lim_{t \geq s \to \infty} \|f_n(t) - \gamma_{m,n}(f_m(s))\| = 0, \quad m \leq s \leq m+1, n \leq t \leq n+1, m \leq n.$$

($\mathfrak{C}$ is said to be Brown's mapping telescope.) Let

$$\mathfrak{J}_k = \{(f_n) \in \mathfrak{C} \mid f_n \equiv 0 \text{ for } n > k\},$$
$$\mathfrak{J} = \{(f_n) \in \mathfrak{C} \mid \lim \|f_n\|_\infty = 0\}.$$

Then $\mathfrak{J}$ is the closure of the union $\cup \mathfrak{J}_k$, and $\mathfrak{C}/\mathfrak{J} \cong \mathfrak{D}$. Let σ be β, regarded as a morphism from $\mathfrak{B}$ to $\mathfrak{C}/\mathfrak{J}$. Lift $\sigma \circ \varphi : \mathfrak{A} \to \mathfrak{C}/\mathfrak{J}$ to $\psi : \mathfrak{A} \to \mathfrak{C}/\mathfrak{J}_k$. Let α_n be the composition of ψ with the evaluation map at $n > k$. The homotopy is given by composing ψ with the evaluation map at $t \geq n$ and then with γ_r $(r \leq t \leq r+1)$. □

Theorem 6.3.28. *Let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be semiprojective in S, $\mathfrak{D} = \varinjlim(\mathfrak{D}_n, \gamma_{n,n+1})$, and let $\beta_0, \beta_1 : \mathfrak{B} \to \mathfrak{D}_k$ for some k with $\gamma_k \circ \beta_0 \sim \gamma_k \circ \beta_1$:*

$$\mathfrak{B} \xrightarrow{\beta_0} \mathfrak{D}_k \xrightarrow{\gamma_k} \mathfrak{D} \quad \sim \quad \mathfrak{B} \xrightarrow{\beta_1} \mathfrak{D}_k \xrightarrow{\gamma_k} \mathfrak{D}.$$

Then for sufficiently large $n \geq k$, we have $\gamma_{k,n} \circ \beta_0 \circ \varphi \sim \gamma_{k,n} \circ \beta_1 \circ \varphi$:

$$\mathfrak{A} \xrightarrow{\varphi} \mathfrak{B} \xrightarrow{\beta_0} \mathfrak{D}_k \xrightarrow{\gamma_{k,n}} \mathfrak{D}_n \quad \sim \quad \gamma_{k,n} \circ \beta_1 \circ \varphi.$$

Proof. Let

$$E = \{(x, f, y) \in \mathfrak{D}_k \oplus C([0,1], \mathfrak{D}) \oplus \mathfrak{D}_k \mid f(0) = \gamma_k(x), f(1) = \gamma_k(y)\},$$

and for $n \geq k$ let

$$E_n = \{(x, f, y) \in \mathfrak{D}_k \oplus C([0,1], \mathfrak{D}_n) \oplus \mathfrak{D}_k \mid f(0) = \gamma_{k,n}(x), f(1) = \gamma_{k,n}(y)\}.$$

Then $E = \varinjlim(E_n, \theta_{n,n+1})$, where $\theta_{n,n+1}(x, f, y) = (x, \gamma_{n,n+1} \circ f, y)$ for $(x, f, y) \in E_n$. Indeed,

$$(\gamma_{n,n+1} \circ f)(0) = \gamma_{n,n+1}(f(0)) = \gamma_{n,n+1}(\gamma_{k,n}(x)) = \gamma_{k,n+1}(x).$$

Similarly, $(\gamma_{n,n+1} \circ f)(1) = \gamma_{k,n+1}(y)$. By the assumption, if ρ_t is a path of homomorphisms from $\mathfrak{B}$ to $\mathfrak{D}$ with $\rho_t = \gamma_k \circ \beta_t$ for $0, 1$, define $\sigma : \mathfrak{B} \to E$ by $\sigma(x) = (\beta_0(x), f, \beta_1(x))$, where $f(t) = \rho_t(x)$. Lift σ to a map $\alpha : \mathfrak{A} \to E_n$ with $\theta \circ \alpha \sim \sigma \circ \varphi$:

$$\mathfrak{A} \xrightarrow{\alpha} E_n \xrightarrow{\theta_n} E \quad \sim \quad \mathfrak{A} \xrightarrow{\varphi} \mathfrak{B} \xrightarrow{\sigma} E.$$

Thus, if π_n^0 and π^0 are the projections of E_n and E respectively onto their first coordinates $\mathfrak{D}_k$, we have

$$\delta_0 = \pi_n^0 \circ \theta_n \circ \alpha \sim \pi^0 \circ \sigma \circ \varphi = \beta_0 \circ \varphi.$$

Similarly, if π_n^1 and π^1 are the projections of E_n and E respectively onto their third coordinates, we have $\delta_1 \sim \beta_1 \circ \varphi$. The map α gives a homotopy from $\gamma_{k,n} \circ \delta_0 : \mathfrak{A} \to \mathfrak{D}_n$ to $\gamma_{k,n} \circ \delta_1$. Thus,

$$\gamma_{k,n} \circ \beta_0 \circ \varphi \sim \gamma_{k,n} \circ \delta_0 \sim \gamma_{k,n} \circ \delta_1 \sim \gamma_{k,n} \circ \beta_1 \circ \varphi$$

as maps from $\mathfrak{A}$ to $\mathfrak{D}_n$. □

Remark. In CC_1, semiprojective ANR's are locally projective. As a result, if X is a compact ANR, there is a finite open cover of X such that whenever continuous functions from X are close with respect to the cover, they are homotopic. Since there are simple semiprojective C^*-algebras which are not projective, the local projectivity result does not carry over to the commutative case.

Theorem 6.3.29. *If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is semiprojective in S, and $\beta_n, \beta : \mathfrak{B} \to \mathfrak{D}$ with $\beta_n \to \beta$ pointwise, then for sufficiently large n, we have $\beta_n \circ \varphi \sim \beta \circ \varphi$.*

Proof. Let $\mathfrak{C} = C([0,1], \mathfrak{D})$,

$$\begin{aligned} \mathfrak{J}_k &= \{f \in \mathfrak{C} \mid f(1/n) = 0 \text{ for all } n,\ f \equiv 0 \text{ on } [0, 1/k]\}, \\ \mathfrak{J} &= \{f \in \mathfrak{C} \mid f(1/n) = 0 \text{ for all } n\}. \end{aligned}$$

Then $\mathfrak{J}$ is the closure of the union $\cup \mathfrak{J}_k$, and $\mathfrak{C}/\mathfrak{J}$ is isomorphic to the C^*-algebra of all convergent sequences of elements of $\mathfrak{D}$, denoted by $C(\mathbb{N}, \mathfrak{D})$. Indeed, define a $*$-homomorphism φ from $\mathfrak{C}/\mathfrak{J}$ onto $C(\mathbb{N}, \mathfrak{D})$ by $\varphi([g]) = (g(1/n))_{n=1}^{\infty}$ for $[g] \in \mathfrak{C}/\mathfrak{J}$ since $(g(1/n) + f(1/n))_{n=1}^{\infty} = (g(1/n))_{n=1}^{\infty}$ for any $f \in \mathfrak{J}$, and $\lim_{n\to\infty} g(1/n) = g(0) \in \mathfrak{D}$. Moreover, for $f \in \mathfrak{J}$,

$$\|g + f\|_\infty = \sup_{t\in[0,1]} \|(g + f)(t)\| \geq \sup_n \|(g + f)(1/n)\| = \sup_n \|g(1/n)\|.$$

Hence, $\|[g]\| \geq \|(g(1/n))_{n=1}^{\infty}\|_\infty = \|\varphi([g])\|_\infty$, so that φ is continuous. Furthermore, for $g, h \in \mathfrak{C}$, suppose that $g(1/n) = h(1/n)$ for all n. Then $g - h \in \mathfrak{J}$, so that $[g] = [h]$. Hence φ is injective. It follows from C^*-algebra theory that φ is an isomorphism.

Let $\sigma : \mathfrak{B} \to \mathfrak{C}/\mathfrak{J}$ be defined by $\sigma(x) = (\beta_n(x))$. Lift $\sigma \circ \varphi$ to $\mathfrak{C}/\mathfrak{J}_k$. □

6.4 Noncommutative shape theory

Definition 6.4.1. Let $\mathfrak{A}$ be a C^*-algebra in S as before. A shape system for $\mathfrak{A}$ in S is an inductive system $(\mathfrak{A}_n, \gamma_{n,n+1})$ in S with $\mathfrak{A} \cong \varinjlim(\mathfrak{A}_n, \gamma_{n,n+1})$ and $\gamma_{n,n+1} : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ semiprojective in S. A strong shape system for $\mathfrak{A}$ is a shape system in which each $\mathfrak{A}_n$ is semiprojective. A faithful shape (resp. strong shape) system is a shape (resp. strong shape) system for which each $\gamma_{n,n+1}$ is injective.

Proposition 6.4.2. *Let $(\mathfrak{A}_n, \gamma_{n,n+1})$ be a shape (resp. strong shape) system for a C^*-algebra $\mathfrak{A}$ in SC (resp. SC_1). Then $(\mathfrak{A}_n/[\mathfrak{A}_n, \mathfrak{A}_n], [\gamma_{n,n+1}])$ is a shape (resp. strong shape) system for the abelianization $\mathfrak{A}/[\mathfrak{A}, \mathfrak{A}]$ in CC (resp. CC_1), where $[\gamma_{n,n+1}] : \mathfrak{A}_n/[\mathfrak{A}_n, \mathfrak{A}_n] \to \mathfrak{A}_{n+1}/[\mathfrak{A}_{n+1}, \mathfrak{A}_{n+1}]$.*

Proof. This follows as shown in the similar case of semiprojectivity. □

Theorem 6.4.3. *Every separable C^*-algebra has a shape system in SC. A unital C^*-algebra has a shape system in SC_1.*

Proof. Write a C^*-algebra $\mathfrak{A} = C^*(\mathfrak{G}, \mathfrak{R})$, where $\mathfrak{G} = \{x_1, x_2, \cdots\}$ is a countable set of generators and $\mathfrak{R} = \{\|p_1(\cdot)\| \leq \eta_1, \|p_2(\cdot)\| \leq \eta_2, \cdots\}$ as before. Set $\mathfrak{G}_n = \{x_1, \cdots, x_n\}$ and

$$\mathfrak{R}_n = \{\|p_i(\cdot)\| \leq \eta_i + 1/n, \|x_i\| \leq \|x_i\|_{\mathfrak{A}} + 1/n \quad (1 \leq i \leq n)\},$$

where each p_i here involves only $x_1, \cdots, x_n$. Set $\mathfrak{A}_n = C^*(\mathfrak{G}_n, \mathfrak{R}_n)$. There is a natural map $\gamma_{n,n+1} : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ since $\mathfrak{G}_n \subset \mathfrak{G}_{n+1}$ and the relations in $\mathfrak{R}_{n+1}$ include stronger forms of all of the relations in $\mathfrak{R}_n$. Indeed, this follows from universality of $\mathfrak{A}_n$ since the stronger relations of $\mathfrak{A}_{n+1}$ implies the weaker relations of $\mathfrak{A}_n$. Then $\gamma_{n,n+1}$ is semiprojective, and $\mathfrak{A} \cong \varinjlim(\mathfrak{A}_n, \gamma_{n,n+1})$. □

Remark. It is not clear whether every separable C^*-algebra has a strong shape system in SC. This is true in CC_1: every compact Hausdorff space is a projective limit of ANR's (in fact of polyhedra). It appears highly unlikely that a general separable C^*-algebra has a faithful shape system in SC. Probably, $C([0,1]^2)$ is a counterexample, and it is seen that $C([0,1]^2)$ has no faithful strong shape system in SC.

AF algebras, inductive limits of algebras of the form $C(S^1) \otimes F$ for F a finite dimensional C^*-algebra, and the Cuntz algebra O_∞ have natural faithful strong shape systems.

Question: does every separable nuclear C^*-algebra have a shape (or strongl shape) system in SC of nuclear C^*-algebras?

It is not even clear that commutative C^*-algebras have nuclear shape systems.

Definition 6.4.4. Two inductive systems $(\mathfrak{A}_n, \gamma_{n,n+1})$ and $(\mathfrak{B}_n, \theta_{n,n+1})$ of C^*-algebras in S are equivalent, and write $(\mathfrak{A}_n, \gamma_{n,n+1}) \sim_S (\mathfrak{B}_n, \theta_{n,n+1})$, if there are sequences of $*$-homomorphisms $\alpha_i : \mathfrak{A}_{k_i} \to \mathfrak{B}_{n_i}$ and $\beta_i : \mathfrak{B}_{n_i} \to \mathfrak{A}_{k_{i+1}}$ with $k_i < n_i < k_{i+1}$, such that $\beta_i \circ \alpha_i \sim \gamma_{k_i,k_{i+1}}$ and $\alpha_{i+1} \circ \beta_i \sim \theta_{k_i,k_{i+1}}$ for each i :

$$\begin{array}{ccccccccc} \mathfrak{A}_{k_i} & \xrightarrow{\alpha_i} & \mathfrak{B}_{n_i} & \xrightarrow{\beta_i} & \mathfrak{A}_{k_{i+1}} & \sim & \mathfrak{A}_{k_i} & \xrightarrow{\gamma_{k_i,k_{i+1}}} & \mathfrak{A}_{k_{i+1}} \\ \mathfrak{B}_{n_i} & \xrightarrow{\beta_i} & \mathfrak{A}_{k_{i+1}} & \xrightarrow{\alpha_{i+1}} & \mathfrak{B}_{n_{i+1}} & \sim & \mathfrak{B}_{n_i} & \xrightarrow{\theta_{k_i,k_{i+1}}} & \mathfrak{B}_{n_{i+1}}. \end{array}$$

If we have such α_i and β_i only with $\beta_i \circ \alpha_i \sim \gamma_{k_i,k_{i+1}}$, write $(\mathfrak{A}_n, \gamma_{n,n+1}) \precsim_S (\mathfrak{B}_n, \theta_{n,n+1})$ (subequivalence).

Remark. In fact, $\sim_S$ is an equivalence relation and $\precsim$ is transitive. The equivalence $\sim_S$ for two inductive systems implies that each system is subequivalent to the other. But its converse is not true.

Proposition 6.4.5. *If two inductive systems of C^*-algebras in SC are equivalent (resp. subequivalent), then two inductive systems of their abelianizations in CC are equivalent (resp. subequivalent).*

The same also holds for SC_1 and CC_1.

Proof. Abelianize the maps α_i and β_i and their homotopies. □

Theorem 6.4.6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras in S with shape systems $(\mathfrak{A}_n, \gamma_{n,n+1})$ and $(\mathfrak{B}_n, \theta_{n,n+1})$ respectively. If there exist inductive systems $(\mathfrak{C}_n, \omega_{n,n+1})$ and $(\mathfrak{D}_n, \delta_{n,n+1})$ in S which are equivalent in S and*

$$\mathfrak{A} \cong \varinjlim(\mathfrak{C}_n, \omega_{n,n+1}) \quad \textit{and} \quad \mathfrak{B} \cong \varinjlim(\mathfrak{D}_n, \delta_{n,n+1}),$$

then $(\mathfrak{A}_n, \gamma_{n,n+1}) \sim_S (\mathfrak{B}_n, \theta_{n,n+1})$.

Also, subequivalence for such two inductive systems implies subequivalence for such two shape systems.

Proof. Suppose that we have $\rho_j : \mathfrak{C}_{p_j} \to \mathfrak{D}_{q_j}$ and $\sigma_j : \mathfrak{D}_{q_j} \to \mathfrak{C}_{p_{j+1}}$ with $p_j < q_j < p_{j+1}$ and $\sigma_j \circ \rho_j \sim \omega_{p_j,p_{j+1}}$, $\rho_{j+1} \circ \sigma_j \sim \delta_{q_j,q_{j+1}}$. We use induction. Suppose that $\alpha_1, \beta_1, \cdots, \alpha_{r-1}$, and β_{r-1} have been chosen, so that (1) $\beta_{r-1} : \mathfrak{B}_{n_{r-1}} \to \mathfrak{A}_{k_r}$ satisfies $\beta_{r-1} = \beta^{\sim} \circ \theta_{n_{r-1},n_{r-1}+2}$ for some $\beta^{\sim} : \mathfrak{B}_{n_{r-1}+2} \to \mathfrak{A}_{k_r}$:

$$\begin{array}{ccc} \mathfrak{B}_{n_{r-1}} & \xrightarrow{\beta_{r-1}} & \mathfrak{A}_{k_r} \\ {\scriptstyle \theta_{n_{r-1},n_{r-1}+2}}\downarrow & & \| \\ \mathfrak{B}_{n_{r-1}+2} & \xrightarrow{\beta^{\sim}} & \mathfrak{A}_{k_r} \end{array}$$

(2) there are numbers q_{j-1} and p_j with $n_{r-1} + 2 < q_{j-1} < p_j < k_r$; (3) identifying $\mathfrak{A}$ with $\varinjlim \mathfrak{A}_n$ and with $\varinjlim \mathfrak{C}_n$, there is a map $\xi : \mathfrak{B}_{n_{r-1}+2} \to \mathfrak{D}_{q_{j-1}}$ such that $\gamma_{k_r} \circ \beta^{\sim} \sim \omega_{p_j} \circ \sigma_{j-1} \circ \xi$ as maps from $\mathfrak{B}_{n_{r-1}+2}$ to $\mathfrak{A}$:

$$\begin{array}{l} \mathfrak{B}_{n_{r-1}+2} \xrightarrow{\beta^{\sim}} \mathfrak{A}_{k_r} \xrightarrow{\gamma_{k_r}} \mathfrak{A} \quad \sim \\ \mathfrak{B}_{n_{r-1}+2} \xrightarrow{\xi} \mathfrak{D}_{q_{j-1}} \xrightarrow{\sigma_{j-1}} \mathfrak{C}_{p_j} \xrightarrow{\omega_{p_j}} \mathfrak{A} \end{array}$$

and $\delta_{q_{j-1}} \circ \xi \sim \theta_{n_{r-1}+2}$ as maps from $\mathfrak{B}_{n_{r-1}}$ to $\mathfrak{B} = \varinjlim \mathfrak{B}_n = \varinjlim \mathfrak{D}_n$:

$$\mathfrak{B}_{n_{r-1}+2} \xrightarrow{\xi} \mathfrak{D}_{q_{j-1}} \xrightarrow{\delta_{q_{j-1}}} \mathfrak{B} \quad \sim \quad \mathfrak{B}_{n_{r-1}+2} \xrightarrow{\theta_{n_{r-1}+2}} \mathfrak{B}.$$

We construct α_r with analogous properties (1) to (3) such that $\alpha_r \circ \beta_{r-1} \sim \theta_{n_{r-1},n_r}$:

$$\mathfrak{B}_{n_{r-1}} \xrightarrow{\beta_{r-1}} \mathfrak{A}_{k_r} \xrightarrow{\alpha_r} \mathfrak{B}_{n_r} \quad \sim \quad \mathfrak{B}_{n_{r-1}} \xrightarrow{\theta_{n_{r-1},n_r}} \mathfrak{B}_{n_r}.$$

The construction can then be repeated inductively to yield the equivalence. First, regarding

$$\gamma_{k_r+3} = \gamma_{k_r+4} \circ \gamma_{k_r+3,k_r+4} : \mathfrak{A}_{k_r+3} \to \mathfrak{A}_{k_r+4} \to \mathfrak{A}$$

as a map into $\mathfrak{A} = \varinjlim \mathfrak{C}_n$, by semiprojectivity of γ_{k_r+3,k_r+4} there is a map $\psi : \mathfrak{A}_{k_r+3} \to \mathfrak{C}_{p_s}$ for sufficiently large s with $\omega_{p_s} \circ \psi \sim \gamma_{k_r+3}$:

$$\mathfrak{A}_{k_r+3} \xrightarrow{\psi} \mathfrak{C}_{p_s} \xrightarrow{\omega_{p_s}} \mathfrak{A} \quad \sim \quad \mathfrak{A}_{k_r+3} \xrightarrow{\gamma_{k_r+3}} \mathfrak{A}.$$

Then

$$\begin{aligned} \omega_{p_s} \circ \psi \circ \gamma_{k_r,k_r+3} \circ \beta^{\sim} &\sim \gamma_{k_r+3} \circ \gamma_{k_r,k_r+3} \circ \beta^{\sim} = \gamma_{k_r} \circ \beta^{\sim} \\ &\sim \omega_{p_j} \circ \sigma_{j-1} \circ \xi = \omega_{p_s} \circ \omega_{p_j,p_s} \circ \sigma_{j-1} \circ \xi \end{aligned}$$

as maps from $\mathfrak{B}_{n_{r-1}+2}$ to $\mathfrak{A} = \varinjlim \mathfrak{C}_n$. So, the semiprojectivity of $\theta_{n_{r-1}+1,n_{r-1}+2}$ implies that by increasing s we obtain

$$f = \psi \circ \gamma_{k_r,k_r+3} \circ \beta^{\sim} \circ \theta_{n_{r-1}+1,n_{r-1}+2} \sim \omega_{p_j,p_s} \circ \sigma_{j-1} \circ \xi \circ \theta_{n_{r-1}+1,n_{r-1}+2} = g.$$

Now regard

$$h = \delta_{q_s} \circ \rho_s \circ \psi \circ \gamma_{k_r+2,k_r+3} : \mathfrak{A}_{k_r+2} \to \mathfrak{A}_{k_r+3} \to \mathfrak{B} = \varinjlim \mathfrak{D}_n.$$

By semiprojectivity of γ_{k_r+2,k_r+3} there is a map $\alpha^{\sim} : \mathfrak{A}_{k_r+2} \to \mathfrak{B}_l$ for sufficiently large $l > q_s$ with $\theta_l \circ \alpha^{\sim} \sim h$. Thus we have

$$\begin{aligned} &\theta_l \circ \alpha^{\sim} \circ \gamma_{k_r,k_r+2} \circ \beta^{\sim} \circ \theta_{n_{r-1}+1,n_{r-1}+2} \\ &\sim h \circ \gamma_{k_r,k_r+2} \circ \beta^{\sim} \circ \theta_{n_{r-1}+1,n_{r-1}+2} \\ &= \delta_{q_s} \circ \rho_s \circ \psi \circ \gamma_{k_r+2,k_r+3} \circ \gamma_{k_r,k_r+2} \circ \beta^{\sim} \circ \theta_{n_{r-1}+1,n_{r-1}+2} \\ &= \delta_{q_s} \circ \rho_s \circ \psi \circ \gamma_{k_r,k_r+3} \circ \beta^{\sim} \circ \theta_{n_{r-1}+1,n_{r-1}+2} = \delta_{q_s} \circ \rho_s \circ f \\ &\sim \delta_{q_s} \circ \rho_s \circ g = \delta_{q_s} \circ \rho_s \circ \omega_{p_j,p_s} \circ \sigma_{j-1} \circ \xi \circ \theta_{n_{r-1}+1,n_{r-1}+2} \\ &\sim \delta_{q_{j-1}} \circ \xi \circ \theta_{n_{r-1}+1,n_{r-1}+2} \sim \theta_{n_{r-1}+1} \end{aligned}$$

since $\delta_{q_s} \circ \rho_s \circ \omega_{p_j,p_s} \circ \sigma_{j-1} \sim \delta_{q_{j-1}}$ by assumption. Again we can increase l so that $\alpha_r \circ \beta_{r-1} \sim \theta_{n_{r-1},n_r}$, where $k_r = l$ and $\alpha_r = \alpha^{\sim} \circ \gamma_{k_r,k_r+2}$. For the next stage in the induction, the analog of ξ is $\psi \circ \gamma_{k_r+2,k_r+3}$. □

Corollary 6.4.7. *Any two shape systems for a C^*-algebra in S are equivalent.*

Proof. Note that for a C^*-algebra $\mathfrak{A}$, the trivial system $(\mathfrak{A}_n = \mathfrak{A}, \mathrm{id}_{\mathfrak{A}})$ with $\mathrm{id}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$ the identity map and itself are equivalent. □

Definition 6.4.8. We say that two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ have the same shape (in S), or are shape equivalent in S, written $\mathrm{Sh}(\mathfrak{A}) = \mathrm{Sh}(\mathfrak{B})$, if $(\mathfrak{A}_n, \gamma_{n,n+1}) \sim_S (\mathfrak{B}_n, \theta_{n,n+1})$ for some (hence any) shape systems for $\mathfrak{A}$ and $\mathfrak{B}$ in S. The shape of $\mathfrak{B}$ dominates the shape of $\mathfrak{A}$, written $\mathrm{Sh}(\mathfrak{A}) \leq \mathrm{Sh}(\mathfrak{B})$, if $(\mathfrak{A}_n, \gamma_{n,n+1}) \precsim_S (\mathfrak{B}_n, \theta_{n,n+1})$.

Remark. We have $\mathrm{Sh}(\mathfrak{A}) = \mathrm{Sh}(\mathfrak{B})$ if and only if $\mathfrak{A}$ and $\mathfrak{B}$ have equivalent inductive systems in S. If we have $\mathrm{Sh}(\mathfrak{A}) = \mathrm{Sh}(\mathfrak{B})$ in S, then so does in $S' \supset S$. This definition agrees with the topological definition: if X and Y are compact metrizable spaces, then $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ if and only if $\mathrm{Sh}(C(X)) = \mathrm{Sh}(C(Y))$ in CC_1, and also $\mathrm{Sh}(X) \leq \mathrm{Sh}(Y)$ if and only if $\mathrm{Sh}(C(X)) \leq \mathrm{Sh}(C(Y))$ in CC_1. There are spaces X and Y for which $\mathrm{Sh}(X) \leq \mathrm{Sh}(Y)$ and $\mathrm{Sh}(Y) \leq \mathrm{Sh}(X)$ but $\mathrm{Sh}(X) \neq \mathrm{Sh}(Y)$.

Corollary 6.4.9. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are homotopy equivalent in* S, *then* $\mathrm{Sh}(\mathfrak{A}) = \mathrm{Sh}(\mathfrak{B})$. *If* $\mathfrak{B}$ *homotopy dominates* $\mathfrak{A}$, *then* $\mathrm{Sh}(\mathfrak{A}) \leq \mathrm{Sh}(\mathfrak{B})$.

Proof. A homotopy equivalence between $\mathfrak{A}$ and $\mathfrak{B}$ induces an equivalence between the systems $(\mathfrak{A}, \mathrm{id}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathrm{id}_{\mathfrak{B}})$. □

Remark. The converse is not generally true, as given by the circle and Warsaw circle. So shape equivalence is a strictly weaker notion than homotopy equivalence.

Corollary 6.4.10. *Let X and Y be locally compact metrizable spaces. Then* $\mathrm{Sh}(C_0(X)) = \mathrm{Sh}(C_0(Y))$ *in SC if and only if* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

Corollary 6.4.11. *Let* $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, *and* $\mathfrak{D}$ *be separable C^*-algebras such that* $\mathrm{Sh}(\mathfrak{A}) = \mathrm{Sh}(\mathfrak{C})$ *and* $\mathrm{Sh}(\mathfrak{B}) = \mathrm{Sh}(\mathfrak{D})$ *both in S. Then*

$$\mathrm{Sh}(\mathfrak{A} \otimes_{\max} \mathfrak{B}) = \mathrm{Sh}(\mathfrak{C} \otimes_{\max} \mathfrak{D}),$$
$$\mathrm{Sh}(\mathfrak{A} \otimes_{\min} \mathfrak{B}) = \mathrm{Sh}(\mathfrak{C} \otimes_{\min} \mathfrak{D}),$$
$$\mathrm{Sh}(\mathfrak{A} * \mathfrak{B}) = \mathrm{Sh}(\mathfrak{C} * \mathfrak{D}),$$

all in S, and if they are unital, then in S,

$$\mathrm{Sh}(\mathfrak{A} *_{\mathbb{C}} \mathfrak{B}) = \mathrm{Sh}(\mathfrak{C} *_{\mathbb{C}} \mathfrak{D}).$$

Proof. Let $(\mathfrak{A}_n, \gamma_{n,n+1})$, $(\mathfrak{B}_n, \theta_{n,n+1})$, $(\mathfrak{C}_n, \omega_{n,n+1})$, and $(\mathfrak{D}_n, \delta_{n,n+1})$ be shape systems for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$ respectively. Then $(\mathfrak{A}_n \otimes \mathfrak{B}_n, \gamma_{n,n+1} \otimes \theta_{n,n+1})$ and $(\mathfrak{C}_n \otimes \mathfrak{D}_n, \omega_{n,n+1} \otimes \delta_{n,n+1})$ are equivalent systems for $\mathfrak{A} \otimes \mathfrak{B}$ and $\mathfrak{C} \otimes \mathfrak{D}$, where $\otimes$ means any C^*-tensor product. Moreover, the tensor product operation $\otimes$ can be replaced with the free product operation $*$ and with the unital free product operation $*_{\mathbb{C}}$ in the unital case. □

Remark. If $\mathfrak{A}$ and $\mathfrak{B}$ are AF algebras, then $\mathrm{Sh}(\mathfrak{A}) = \mathrm{Sh}(\mathfrak{B})$ in SC if and only if $\mathfrak{A} \cong \mathfrak{B}$.

Let $\mathfrak{A}$ be a C^*-algebra. Denote by $P(\mathfrak{A}\otimes\mathbb{K})$ the semigroup of equivalence classes of projections in $\mathfrak{A} \otimes \mathbb{K}$, with orthogonal addition. Let $K_0(\mathfrak{A})$ and $K_1(\mathfrak{A})$ be the K-groups of $\mathfrak{A}$. Recall that if $\mathfrak{A}$ is unital, $K_0(\mathfrak{A})$ is defined to be the Grothendieck group of stably equivalence classes of projections in $\mathfrak{A} \otimes \mathbb{K}$. There is a canonical homomorphism from $P(\mathfrak{A} \otimes \mathbb{K})$ into $K_0(\mathfrak{A})$. Denote by $K_0(\mathfrak{A})_+$ the image under this map. If we have

$$(1): \quad K_0(\mathfrak{A})_+ - K_0(\mathfrak{A})_+ = K_0(\mathfrak{A}), \quad (2): \quad K_0(\mathfrak{A})_+ \cap (-K_0(\mathfrak{A})_+) = \{0\},$$

then $(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+)$ is called an ordered group, and in other words, $K_0(\mathfrak{A})$ is identified with the Grotendieck group of $P(\mathfrak{A} \otimes \mathbb{K})$. If $\mathfrak{A} \otimes \mathbb{K}$ has an approximate identity of projections, i.e., $\mathfrak{A}$ is stably unital, then the condition (1) holds. In addition, if $\mathfrak{A} \otimes \mathbb{K}$ contains no infinite projections, i.e., $\mathfrak{A}$ is stably finite, then the condition (2) holds.

Denote by $P(\mathfrak{A})$ the subset of $P(\mathfrak{A} \otimes \mathbb{K})$ (or its image in $K_0(\mathfrak{A})$) corresponding to projections of $\mathfrak{A}$, and called the scale of $\mathfrak{A}$. Even if $\mathfrak{A}$ is simple, stably unital, and stably finite, we can have $P(\mathfrak{A}) = \{0\}$. Even if $\mathfrak{A}$ is simple, unital, and stably finite, $P(\mathfrak{A})$ does not in general generate $K_0(\mathfrak{A})$. It is said to be that $(P(\mathfrak{A} \otimes \mathbb{K}), P(\mathfrak{A}))$ is the scaled semigroup for a C^*-algebra $\mathfrak{A}$, and $(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, P(\mathfrak{A}))$ is the scaled pre-ordered K_0-group for $\mathfrak{A}$.

Proposition 6.4.12. *Let $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \gamma_{n,n+1})$ as before and $\mathfrak{J}_n$ the kernel of $\gamma_{n,n+1}$. Then $(P(\mathfrak{A} \otimes \mathbb{K}), P(\mathfrak{A}))$ is the algebraic inductive limit of $((P((\mathfrak{A}_n/\mathfrak{J}_n)\otimes\mathbb{K}), P(\mathfrak{A}_n/\mathfrak{J}_n)), [\gamma_{n,n+1}]_*)$ with $[\gamma_{n,n+1}]_*$ the induced map from $[\gamma_{n,n+1}] : \mathfrak{A}_n/\mathfrak{J}_n \to \mathfrak{A}_{n+1}/\mathfrak{J}_{n+1}$. Similarly, for $K_0(\mathfrak{A})$ and $K_1(\mathfrak{A})$. That is:*

$$(P(\mathfrak{A} \otimes \mathbb{K}), P(\mathfrak{A})) \cong \varinjlim((P((\mathfrak{A}_n/\mathfrak{J}_n) \otimes \mathbb{K}), P(\mathfrak{A}_n/\mathfrak{J}_n)), [\gamma_{n,n+1}]_*),$$

$$K_j(\mathfrak{A}) \cong \varinjlim(K_j(\mathfrak{A}_n/\mathfrak{J}_n), [\gamma_{n,n+1}]_*) \quad \text{for } j = 0, 1.$$

Proof. This follows in a way similar to the case of faithful inductive limits, i.e., with injective connecting maps, but one needs to handle non-injective connecting maps as considered in lifting projections and their equivalence. Indeed, note that $\mathfrak{A} \cong \varinjlim(\mathfrak{A}_n/\mathfrak{J}_n, [\gamma_{n,n+1}])$ with the connecting maps $[\gamma_{n,n+1}]$ injective. □

Remark. This is not the same as in [1] and should be the right statement.

Proposition 6.4.13. *If* $\mathrm{Sh}(\mathfrak{A}) = \mathrm{Sh}(\mathfrak{B})$ *in SC, then*

$$(P(\mathfrak{A} \otimes \mathbb{K}), P(\mathfrak{A})) \cong (P(\mathfrak{B} \otimes \mathbb{K}), P(\mathfrak{B})),$$
$$(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, P(\mathfrak{A})) \cong (K_0(\mathfrak{B}), K_0(\mathfrak{B})_+, P(\mathfrak{B}))$$

as scaled semigroups and scaled pre-ordered K_0*-groups respectively, and also* $K_1(\mathfrak{A}) \cong K_1(\mathfrak{B})$.

If $\mathrm{Sh}(\mathfrak{A}) \leq \mathrm{Sh}(\mathfrak{B})$ *in SC, then* $P(\mathfrak{A} \otimes \mathbb{K})$, $K_0(\mathfrak{A})$, *and* $K_1(\mathfrak{A})$ *are direct summands of* $P(\mathfrak{B} \otimes \mathbb{K})$, $K_0(\mathfrak{B})$, *and* $K_1(\mathfrak{B})$ *respectively, with the induced order and scale.*

Proof. Let $(\mathfrak{A}_n, \gamma_{n,n+1})$ and $(\mathfrak{B}_n, \theta_{n,n+1})$ be shape systems for C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$. An equivalence between the systems gives the following diagram:

$$\begin{array}{ccccccccc}
\mathfrak{A}_{k_1} & \to \cdots \to & \mathfrak{A}_{k_2} & \longrightarrow \cdots \longrightarrow & \cdots \longrightarrow & \mathfrak{A} \\
\alpha_1 \searrow & \nearrow \beta_1 & \searrow \alpha_2 & \nearrow \beta_2 & & \\
\cdots \to \mathfrak{B}_{n_1} & \to \cdots \longrightarrow & \mathfrak{B}_{n_2} & \longrightarrow \cdots \to & & \mathfrak{B}
\end{array}$$

where the triangles commute up to homotopy. This diagram induces the following diagram:

$$\begin{array}{cccccc}
P(\mathfrak{A}_{k_1} \otimes \mathbb{K}) & \to \cdots \to & P(\mathfrak{A}_{k_2} \otimes \mathbb{K}) & \longrightarrow \cdots \longrightarrow & \cdots \longrightarrow & P(\mathfrak{A} \otimes \mathbb{K}) \\
(\alpha_1)_* \searrow & \nearrow (\beta_1)_* & \searrow (\alpha_2)_* & \nearrow (\beta_2)_* & & \\
\cdots \to P(\mathfrak{B}_{n_1} \otimes \mathbb{K}) & \to \cdots \longrightarrow & P(\mathfrak{B}_{n_2} \otimes \mathbb{K}) & \longrightarrow \cdots \to & & P(\mathfrak{B} \otimes \mathbb{K})
\end{array}$$

and similarly, $P(\cdot \otimes \mathbb{K})$ can be replaced with $K_0(\cdot)$ and $K_1(\cdot)$. The induced diagram commutes, so we obtain an isomorphism between the inductive limits $P(\mathfrak{A} \otimes \mathbb{K})$ and $P(\mathfrak{B} \otimes \mathbb{K})$. Since all the induced homomorphisms preserve order and scale, the order and scale of the inductive limits are the inductive limit order and scale respectively.

In the case that $\mathrm{Sh}(\mathfrak{A}) \leq \mathrm{Sh}(\mathfrak{B})$, only the odd triangles commute, we obtain scaled homomorphisms $\alpha_* : P(\mathfrak{A} \otimes \mathbb{K}) \to P(\mathfrak{B} \otimes \mathbb{K})$ and $\beta : P(\mathfrak{B} \otimes \mathbb{K}) \to P(\mathfrak{A} \otimes \mathbb{K})$ with $\beta_* \circ \alpha_* = \mathrm{id}_{P(\mathfrak{A} \otimes \mathbb{K})}$. Similarly, $P(\cdot \otimes \mathbb{K})$ can be replaced with $K_0(\cdot)$ and $K_1(\cdot)$. □

Corollary 6.4.14. *If* $\mathrm{Sh}(\mathfrak{A}) = \mathrm{Sh}(\mathfrak{C})$ *and* $\mathrm{Sh}(\mathfrak{B}) = \mathrm{Sh}(\mathfrak{D})$ *both in SC, then*

$$K_0(\mathfrak{A} \otimes_{\max} \mathfrak{B}) \cong K_*(\mathfrak{C} \otimes_{\max} \mathfrak{D}),$$
$$K_0(\mathfrak{A} \otimes_{\min} \mathfrak{B}) \cong K_*(\mathfrak{C} \otimes_{\min} \mathfrak{D}),$$
$$K_0(\mathfrak{A} * \mathfrak{B}) \cong K_*(\mathfrak{C} * \mathfrak{D})$$

as scaled preordered groups.

If C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are stably shape equivalent, i.e., $\mathrm{Sh}(\mathfrak{A}\otimes\mathbb{K}) = \mathrm{Sh}(\mathfrak{B}\otimes\mathbb{K})$, *then* $K_0(\mathfrak{A}) \cong K_0(\mathfrak{B})$ *as preordered groups.*

Proof. As shown above, the shape equivalences induce the shape equivalences for the maximal and minimal C^*-tensor products and free products of those C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$, $\mathfrak{D}$.

Note that $K_0(\mathfrak{A}) \cong K_0(\mathfrak{A}\otimes\mathbb{K})$ for a C^*-algebra. □

Remark. The assumption is not the same as in [1] and should be the right one.

Proposition 6.4.15. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be stably unital C^*-algebras. If we have* $\mathrm{Sh}(\mathfrak{A}\otimes\mathbb{K}) \leq \mathrm{Sh}(\mathfrak{B}\otimes\mathbb{K})$ *in SC, and $\mathfrak{B}$ is stably finite, then $\mathfrak{A}$ is stably finite.*

Proof. Let $(\mathfrak{A}_n, \gamma_{n,n+1})$ and $(\mathfrak{B}_n, \theta_{n,n+1})$ be shape systems for $\mathfrak{A}\otimes\mathbb{K}$ and $\mathfrak{B}\otimes\mathbb{K}$ respectively. We may assume that $\mathfrak{A}_n$ and $\mathfrak{B}_n$ are unital for each n although the connecting maps are not unital in general. Then

$$(\mathfrak{A}\otimes\mathbb{K})^+ = \varinjlim(\mathfrak{A}_n^+, \gamma_{n,n+1}^+), \quad (\mathfrak{B}\otimes\mathbb{K})^+ = \varinjlim(\mathfrak{B}_n^+, \theta_{n,n+1}^+)$$

with unital connecting maps between the unitizations of $\mathfrak{A}_n$ (and those of $\mathfrak{B}_n$), and $(\mathfrak{A}_n^+, \gamma_{n,n+1}^+) \precsim (\mathfrak{B}_n^+, \theta_{n,n+1}^+)$ in SC_1. By assumption, $\mathfrak{B}\otimes\mathbb{K}$ contains no infinite projections. Hence $\theta_n(\mathfrak{B}_n)$ contains no infinite projections. The same is true for $\theta_n^+(\mathfrak{B}_n^+) \cong \theta_n(\mathfrak{B}_n) \oplus \mathbb{C}$, and also for $(\mathfrak{B}\otimes\mathbb{K})^1 \cong \varinjlim \theta_n^+(\mathfrak{B}_n^+)/\ker(\theta_n^+)$. Thus $(\mathfrak{B}\otimes\mathbb{K})^+$ contains no non-unitary isometries.

Note that a unital inductive limit $\mathfrak{D} = \varinjlim(\mathfrak{D}_n, \delta_{n,n+1})$ contains no non-unitary isometries if and only if, for any k and any isometry $s \in \mathfrak{D}_k$, there is an $n > k$ such that $\delta_{k,n}(s)$ is unitary in $\mathfrak{D}_n$. If s is an isometry in $\mathfrak{A}_m^+$, choose i with $k_i > m$, then $\alpha_i \circ \gamma_{m,k_i}^+(s)$ is an isometry v in $\mathfrak{B}_{n_i}^+$. Thus, for sufficiently large j, we have $\theta_{n_i,n_j}^+(v)$ unitary in $\mathfrak{B}_{n_j}^+$. Then $\beta_j \circ \theta_{n_i,n_j}^+(v)$ is a unitary in $\mathfrak{A}_{k_{j+1}}^+$, which is connected by a path of unitaries to $\gamma_{m,k_{j+1}}^+(s)$. It follows that $\gamma_{m,k_{j+1}}^+(s)$ is unitary. Hence $(\mathfrak{A}\otimes\mathbb{K})^+$ can not contain non-unitary isometries. □

Remark. It follows that if C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are stably shape equivalent and if they have shape systems in a suitably nice class of C^*-algebras, then we have the Kasparov's KK-group isomorphisms:

$$KK(\mathfrak{A}\otimes\mathfrak{C}, \mathfrak{D}) \cong KK(\mathfrak{B}\otimes\mathfrak{C}, \mathfrak{D}),$$
$$KK(\mathfrak{C}, \mathfrak{A}\otimes\mathfrak{D}) \cong KK(\mathfrak{C}, \mathfrak{B}\otimes\mathfrak{D})$$

for all suitably nice C^*-algebras $\mathfrak{C}$ and $\mathfrak{D}$. However, it is difficult to write down an explicit invertible element of $KK(\mathfrak{A}, \mathfrak{B})$, even when $\mathfrak{A}$ and $\mathfrak{B}$ are AF algebras, or when $\mathfrak{A} = C(WS^1)$ and $\mathfrak{B} = C(S^1)$, where WS^1 means the Warsaw circle.

Note that Kasparov equivalence, i.e., existence of an invertible element in $KK(\mathfrak{A}, \mathfrak{B})$ is much weaker than stable shape equivalence. For example, if $\mathfrak{A}$ and $\mathfrak{B}$ are AF algebras, then

$$KK(\mathfrak{A}, \mathfrak{B}) \cong \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{B}))$$

and the Kasparov (or intersection) product corresponds to composition of homomorphisms, so $\mathfrak{A}$ and $\mathfrak{B}$ are Kasparov equivalent if and only if $K_0(\mathfrak{A}) \cong K_0(\mathfrak{B})$ as groups, ignoring the order structure completely. But $\mathfrak{A}$ and $\mathfrak{B}$ are stably shape equivalent if and only if they are stably isomorphic.

Notes. This section of four subsections is based on the paper [1] of Blackadar. This is just the beginning of the story of the noncommutative shape theory. More investigation about the theory would be continued in somewhere else in the future. It is hoped that our effort here will not be in vain.

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