The theory of C^⊠-algebra unit ball extreme points, extremally rich C^⊠-algebras, and the λ-function : a review

Ryukyu Math. J., 23(2010), 19-116

THE THEORY OF C^*-ALGEBRA UNIT BALL EXTREME POINTS, EXTREMALLY RICH C^*-ALGEBRAS, AND THE λ-FUNCTION — A REVIEW

TAKAHIRO SUDO

Dedicated to Professor Masamichi Takesaki on his seventyseventh birthday

Abstract

In this paper we review, rebuild, and study the theory of the set of all extremal points of the unit ball of a C^*-algebra, consisting of partial isometries with a certain condition, and also the theory of extremally rich C^*-algebras, and moreover, the theory of the λ-function in operator algebras, as well.

Primary 46-02, 46L05, 46L80.
Keywords: C*-algebra, Unit ball, Extreme point, Extremally rich C*-algebra, λ-function, Partial isometry, Quasi-invertible, Spectral theory.

Introduction

This paper is devoted to reviewing, rebuilding, and studying the theory of the set of all extremal points of the unit ball of a C^*-algebra, consisting of partial isometries with a certain condition, and also the theory of extremally rich C^*-algebras, mainly, in the first section. The definition for C^*-algebras to be extremally rich is introduced by Brown and Pedersen [3]. With careful reading the item and some efforts, revealing lines and secret, more detailed full proofs are provided and could be useful for reference.

Received November 30, 2010.

In the middle of the first and third sections, we give a review on higher extremal richness for C^*-algebras, introduced by the author [21]

In the third section, with the same mind as above, we review, rebuild, and study the λ-function in operator algebras, studied by Pedersen [17].

Contents

1 Geomerty for the unit ball of a C^*-algebra

This section is taken from Brown and Pedersen [3].

1.1 Quasi-invertible elements

Let $\mathfrak{A}$ be a unital C^*-algebra and $\mathfrak{A}^{-1}$ be the group of all invertible elements of $\mathfrak{A}$. Regard $\mathfrak{A}$ as an operator algebra on a Hilbert space H. Let $\mathfrak{A}''$ be the weak closure of $\mathfrak{A}$ in $\mathbb{B}(H)$ of all bounded operators on H, called the enveloping von Neumann algebra of $\mathfrak{A}$, isomorphic to the second dual of $\mathfrak{A}$ as a Banach space.

Definition 1.1.1. For each $T \in \mathfrak{A}$, define

$$m(T) = \inf\{\|T\xi\| \mid \xi \in H, \|\xi\| = 1\}, \quad m(|T|) = \sup\{\varepsilon \geq 0 \mid \varepsilon I \leq |T|\},$$

where I is the identity operator. We may call $m(T)$ the spherical distance of T (from zero).

It follows that $m(T) = m(|T|)$. Indeed, note that

$$\|T\xi\|^2 = \langle T\xi, T\xi\rangle = \langle T^*T\xi, \xi\rangle = \langle (T^*T)^{1/2}\xi, (T^*T)^{1/2}\xi\rangle = \||T|\xi\|^2$$

and if for $\xi \in H$ with $1 = \|\xi\|^2 = \langle \xi, \xi\rangle$ (the inner product), we have

$$\varepsilon = \langle \varepsilon\xi, \xi\rangle \leq \langle |T|\xi, \xi \rangle,$$

since $\langle |T|\xi, \xi\rangle \leq \||T|\xi\|$, then $\varepsilon \leq m(T)$, so that $m(|T|) \leq m(T)$. Conversely,

$$m(T)^2 \leq \|T\xi\|^2 = \||T|\xi\|^2 = \langle |T|^2\xi, \xi\rangle = \langle T^*T\xi, \xi\rangle.$$

Hence $0 \leq m(T)^2 I \leq T^*T$. It follows by a fact of C^*-algebra theory (see [10]) (or Löwner-Heinz inequality (see [7])) that $m(T)I \leq |T|$, Therefore, $m(T) \leq m(|T|)$.

Thus, $m(T) > 0$ if and only if $|T|$ is invertible. Since $|T| = U^*T$ where $T = U|T|$ is the polar decomposition of T, that is equivalent to say that T is left invertible in $\mathbb{B}(H)$ and also in $\mathfrak{A}$.

Let $\mathfrak{A}_1$ denote the closed unit ball of $\mathfrak{A}$ and $\mathfrak{A}_e$ the set of all extreme points in the convex set $\mathfrak{A}_1$. Recall that elements of $\mathfrak{A}_e$ consist of partial isometries V of $\mathfrak{A}$ such that

$$(I - VV^*)\mathfrak{A}(I - V^*V) = 0$$

(see [13]). Thus, the defect projections $I - VV^*$ and $I - V^*V$ for V are (said to be) centrally orthogonal. In particular, we have

$$(I - VV^*)(I - V^*V) = (I - VV^*) - (I - VV^*)V^*V = 0$$

and $I = VV^* + V^*V - V(V^*)^2V$. Moreover, for any $A, B \in \mathfrak{A}$,

$$(I - VV^*)A^*B(I - V^*V) = (A(I - VV^*))^*(B(I - V^*V)) = 0.$$

Theorem 1.1.2. *For an element T of a unital C^*-algebra $\mathfrak{A}$, the following conditions are equivalent:*

(i) $T \in \mathfrak{A}^{-1}\mathfrak{A}_e\mathfrak{A}^{-1}$.

(ii) *There are orthogonal closed ideals $\mathfrak{J}$, $\mathfrak{K}$ of $\mathfrak{A}$ such that $T + \mathfrak{J}$ is left invertible in $\mathfrak{A}/\mathfrak{J}$ and $T + \mathfrak{K}$ is right invertible in $\mathfrak{A}/\mathfrak{K}$.*

(iii) *There is an $\varepsilon > 0$ such that $m(T + \mathfrak{J}) \geq \varepsilon$ and $m(T^* + \mathfrak{K}) \geq \varepsilon$ in $\mathfrak{A}/\mathfrak{J}$ and $\mathfrak{A}/\mathfrak{K}$ respectively.*

(iv) *There is an $\varepsilon > 0$ such that $\max\{m(\pi(T)), m(\pi(T^*))\} \geq \varepsilon$ for any irreducible representation π of $\mathfrak{A}$.*

(v) *T and also T^* have closed ranges, and the projections on the orthogonal complements. i.e., the kernel projections of T^* and T are centrally orthogonal in $\mathfrak{A}$.*

(vi) *There is an element $V \in \mathfrak{A}_e$ with $\ker V = \ker T$ such that $T = V|T|$, and 0 is an isolated point in the spectrum $\mathrm{sp}(|T|)$.*

(vii) $T \in \mathfrak{A}_e\mathfrak{A}_+^{-1}$.

Proof. (i) $\Rightarrow$ (ii). Let $T = AVB$ for $A, B \in \mathfrak{A}^{-1}$ and $V \in \mathfrak{A}_e$. Let $\mathfrak{J}$ and $\mathfrak{K}$ be the closed ideals of $\mathfrak{A}$ generated by the defect projections $I - V^*V$ and $I - VV^*$ of V, respectively. Since $V \in \mathfrak{A}_e$, we have $\mathfrak{J} \cap \mathfrak{K} = \{0\}$. Indeed, If $a \in \mathfrak{J} \cap \mathfrak{K}$, then $a = \lim_{n\to\infty} \sum_{j=1}^n a_{j,n}(I - V^*V)b_{j,n}$ for some $a_{j,n}, b_{j,n} \in \mathfrak{A}$ and $a = \lim_{m\to\infty} \sum_{k=1}^m c_{k,m}(I - VV^*)d_{k,m}$ for some $c_{k,m}, d_{k,m} \in \mathfrak{A}$, so that

$$\begin{aligned} a^2 &= (\lim_{n\to\infty} \sum_{j=1}^n a_{j,n}(I - V^*V)b_{j,n})(\lim_{m\to\infty} \sum_{k=1}^m c_{k,m}(I - VV^*)d_{k,m}) \\ &= \lim_{n,m\to\infty} (\sum_{j=1}^n a_{j,n}(I - V^*V)b_{j,n} \sum_{k=1}^m c_{k,m}(I - VV^*)d_{k,m}) \\ &= \lim_{n,m\to\infty} (\sum_{j=1}^n \sum_{k=1}^m a_{j,n}(I - V^*V)b_{j,n}c_{k,m}(I - VV^*)d_{k,m}) = 0. \end{aligned}$$

Since $\mathfrak{J}\mathfrak{K} = \mathfrak{J} \cap \mathfrak{K}$, we have $\mathfrak{J}$ and $\mathfrak{K}$ orthogonal.

Let $\rho : \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$ be the quotient map. Note that $\rho(V)$ is an isometry because

$$\rho(V)^*\rho(V) = (V^* + \mathfrak{J})(V + \mathfrak{J}) = V^*V + \mathfrak{J} = I + \mathfrak{J} = \rho(I).$$

Since $A^*A \geq \varepsilon I$ and $B^*B \geq \varepsilon I$ for some $\varepsilon > 0$, we have

$$\begin{aligned} \rho(T^*T) &= \rho(B^*V^*A^*AVB) \geq \varepsilon\rho(B^*V^*VB) \\ &= \varepsilon\rho(B^*B) \geq \varepsilon^2\rho(I). \end{aligned}$$

It follows that $\rho(|T|)$ is invertible in $\mathfrak{A}/\mathfrak{J}$, whence $\rho(T)$ is left invertible in $\mathfrak{A}/I$.

Let $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{K}$ be the quotient map. Then $\pi(V^*)$ is an isometry. Since $AA^* \geq \delta I$ and $BB^* \geq \delta I$ for some $\delta > 0$,

$$\begin{aligned}\pi(TT^*) &= \pi(AVBB^*V^*A^*) \geq \delta\pi(AVV^*A^*)\\ &= \delta\pi(AA^*) \geq \delta^2\pi(I).\end{aligned}$$

Hence $\pi(|T^*|)$ is invertible. Since $|T^*| = TU^*$ for $T = U|T|$ the polar decomposition of T, $\pi(T)$ is right invertible in $\mathfrak{A}/\mathfrak{K}$.

(ii) $\Leftrightarrow$ (iii). This is evident from the definition of $m(T)$ given above.

(iii) $\Rightarrow$ (iv). If π is an irreducible representation of $\mathfrak{A}$, then $\ker\pi$ is a prime ideal, i.e., if $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are closed ideals of $\mathfrak{A}$ such that $\mathfrak{J}_1\mathfrak{J}_2 \subset \ker\pi$, then $\mathfrak{J}_1 \subset \ker\pi$ or $\mathfrak{J}_2 \subset \ker\pi$. Since $\mathfrak{J} \cap \mathfrak{K} = 0$, it follows that either $\mathfrak{J} \subset \ker\pi$ or $\mathfrak{K} \subset \ker\pi$. In the first case,

$$m(\pi(T)) \geq m(T + \mathfrak{J}) \geq \varepsilon$$

because $\mathrm{sp}(\pi(|T|)) \subset \mathrm{sp}(|T| + \mathfrak{J})$ since there is a $*$-homomorphism from $A/\mathfrak{J}$ to $A/\ker\pi$. Indeed, first note that

$$|\pi(T)|^2 = \pi(T)^*\pi(T) = \pi(T^*T) = \pi(|T|^2) = \pi(|T|)^2.$$

Hence $|\pi(T)| = \pi(|T|)$. Similarly, we have $|T + \mathfrak{J}| = |T| + \mathfrak{J}$. Now suppose that $m(\pi(T)) < m(T + \mathfrak{J})$. Therefore,

$$m(\pi(|T|)) = m(|\pi(T)|) = m(\pi(T)) < m(T+\mathfrak{J}) = m(|T+\mathfrak{J}|) = m(|T|+\mathfrak{J}).$$

Thus, $m(\pi(|T|))I < m(|T| + \mathfrak{J})I \leq |T| + \mathfrak{J}$. It follows that $m(\pi(|T|)) \notin \mathrm{sp}(|T| + \mathfrak{J})$. On the other hand, since $|T| + \mathfrak{J}$ is invertible, $\pi(|T|)$ is also invertible. Hence, $m(\pi(|T|)) > 0$ and $m(\pi(|T|))I \leq \pi(|T|)$, and since the set of all invertible elements of $\pi(\mathfrak{A})$ is open in it, $m(\pi(|T|)) \in \mathrm{sp}(\pi(|T|))$. This is the contradiction.

In the second case that $\mathfrak{K} \subset \ker\pi$, we obtain $m(\pi(T^*)) \geq \varepsilon$ similarly.

(iv) $\Rightarrow$ (iii). Let $\mathrm{Prim}(\mathfrak{A})$ denote the primitive ideal space of $\mathfrak{A}$ equipped with the Jacobson topology. Elements of $\mathrm{Prim}(\mathfrak{A})$ are kernels of irreducible representations of $\mathfrak{A}$. We have

$$m(\pi(T)) = m(\pi(|T|)) = \alpha - \|\pi(\alpha I - |T|)\|$$

for any $\alpha \geq \|T\|$, because $\|T\| = \||T|\|$ since

$$\|T\|^2 = \|T^*T\| = \||T|^2\| = \||T|\|^2,$$

and $0 \leq A \leq \|A\|I$ for $A \in \mathfrak{A}$ since

$$\langle A\xi, \xi\rangle \leq \|A\xi\|\|\xi\| \leq \|A\|\|\xi\|^2 = \langle \|A\|\xi, \xi\rangle,$$

and $|T| \leq \||T|\|I \leq \alpha I$ implies

$$0 \leq \alpha I - \pi(|T|) \leq \|\pi(\alpha I - |T|)\|I$$

so that $(\alpha - \|\pi(\alpha I - |T|)\|)I \leq \pi(|T|)$. Hence $\alpha - \|\pi(\alpha I - |T|)\| \leq m(\pi(|T|))$. Conversely, for $0 \leq \varepsilon I \leq \pi(|T|)$, it follows that $\alpha I + \varepsilon I \leq \pi(|T|) + \alpha I$ for $\alpha \geq \|T\|$. Therefore,

$$0 \leq \alpha I - \pi(|T|) \leq (\alpha - \varepsilon)I, \quad \text{which implies} \quad \|\alpha I - \pi(|T|)\| \leq \alpha - \varepsilon.$$

Thus, $\varepsilon \leq \alpha - \|\pi(\alpha I - |T|)\|$. Hence, $m(\pi(|T|)) \leq \alpha - \|\pi(\alpha I - |T|)\|$.

Also, the function $A^\wedge : \ker\pi \mapsto \|\pi(A)\|$ for each positive element $A \in \mathfrak{A}$ is lower semi-continuous on $\mathrm{Prim}(\mathfrak{A})$. Indeed, we have for each $\alpha \geq 0$,

$$(A^\wedge)^{-1}([0,\alpha]) = \{\ker\pi \in \mathrm{Prim}(\mathfrak{A}) \mid \mathrm{sp}(\pi(A)) \subset [0,\alpha]\}$$

is shown to be closed in $\mathrm{Prim}(\mathfrak{A})$ (see [13]). Therefore, the inverse image $(A^\wedge)^{-1}((\alpha,\infty))$ is open there. It follows that both functions f_T and f_{T^*} on $\mathrm{Prim}(\mathfrak{A})$ defined by

$$f_T(\ker\pi) = m(\pi(T)) \quad \text{and} \quad f_{T^*}(\ker\pi) = m(\pi(T^*))$$

are upper semi-continuous. Indeed, for $\beta \geq m(\pi(T)) = m(\pi(|T|))$, take $\alpha \geq \max\{\beta, \|T\|\}$ so that $m(\pi(T)) = \alpha - \|\pi(\alpha I - |T|)\|$. Hence f_T is upper semi-continuous since $f_T^{-1}([\beta,\infty))$ is closed so that $f_T^{-1}([0,\beta))$ open, and so is f_{T^*}. Since we have

$$\mathrm{sp}(|T|) \cup \{0\} = \sqrt{\mathrm{sp}(T^*T) \cup \{0\}} = \sqrt{\mathrm{sp}(TT^*) \cup \{0\}} = \mathrm{sp}(|T^*|) \cup \{0\},$$

we see that $f_T(\mathfrak{L}) = f_{T^*}(\mathfrak{L})$ for every $\mathfrak{L} \in \mathrm{Prim}(\mathfrak{A})$ if non-zero. Indeed, let $\mathfrak{L} = \ker\pi$ and $m(\pi(|T|)) \neq 0$. Then

$$\begin{aligned} m(\pi(|T|) &= \min\{\lambda \in \mathrm{sp}(\pi(|T|)) \setminus \{0\}\} \\ &= \min\{\lambda \in \mathrm{sp}(\pi(|T^*|)) \setminus \{0\}\} = m(\pi(|T^*|)). \end{aligned}$$

Let $P = \{\mathfrak{L} \in \mathrm{Prim}(\mathfrak{A}) \mid f_T(\mathfrak{L}) < \varepsilon\}$ and $Q = \{\mathfrak{L} \in \mathrm{Prim}(\mathfrak{A}) \mid f_{T^*}(\mathfrak{L}) < \varepsilon\}$, which are open in $\mathrm{Prim}(\mathfrak{A})$, and disjoint by the assumption (because if $P \cap Q \neq \emptyset$, then the contradiction). Then the corresponding closed ideals

$$\begin{aligned} \mathfrak{I} &= \ker(\mathrm{Prim}(\mathfrak{A}) \setminus P) = \cap\{\mathfrak{L} \in \mathrm{Prim}(\mathfrak{A}) \setminus P\}, \\ \mathfrak{K} &= \ker(\mathrm{Prim}(\mathfrak{A}) \setminus Q) = \cap\{\mathfrak{L} \in \mathrm{Prim}(\mathfrak{A}) \setminus Q\} \end{aligned}$$

are orthogonal. Indeed, if $\mathfrak{J}\mathfrak{K} \neq \{0\}$, there are $a \in \mathfrak{J}$ and $b \in \mathfrak{K}$ with $ab \neq 0$. Then there is an irreducible representation π of $\mathfrak{A}$ such that $\pi(ab) \neq 0$, so that $\pi(\mathfrak{J}) \neq \{0\}$ and $\pi(\mathfrak{K}) \neq \{0\}$. Hence $\pi \in P$ and $\pi \in Q$. This is the contradiction. Let $\rho : \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$ be the quotient map. Then

$$m(\rho(T)) = \inf\{f_T(\mathfrak{L}) \mid \mathfrak{L} \in \mathrm{Prim}(\mathfrak{A}) \setminus P\} \geq \varepsilon.$$

Indeed, the first equality holds as follows. Since $\mathfrak{J} \subset \mathfrak{L}$ for $\mathfrak{L} \in \mathrm{Prim}(\mathfrak{A}) \setminus P$ with $\mathfrak{L} = \ker \pi$, there is the quotient map from $\mathfrak{A}/\mathfrak{J}$ to $\mathfrak{A}/\mathfrak{L}$, so that $\|\rho(a)\| \geq \|\pi(a)\|$ for $a \in \mathfrak{A}$. Therefore, for $\alpha \geq \|T\|$,

$$m(\pi(T)) = \alpha - \|\pi(\alpha I - |T|)\| \geq \alpha - \|\rho(\alpha I - |T|)\| = m(\rho(T))$$

where the last equlity for the $*$-homomorphism ρ is shown as the same way exactly as above. Hence $m(\rho(T)) \leq$ the infimum above. Furthermore, now assume that $m(\rho(T)) <$ the infimum above. There is an irreducible representation μ of $\mathfrak{A}/\mathfrak{J}$ such that $\|\rho(\alpha I - |T|)\| = \|\mu(\rho(\alpha I - |T|))\|$, which corresponds to the irreducible representation $\bar{\mu}$ of $\mathfrak{A}$ such that $\bar{\mu}(\mathfrak{J}) = 0$ and $\bar{\mu} = \mu \circ \rho$. Hence $\bar{\mu} \in \mathrm{Prim}(\mathfrak{A}) \setminus P$, which implies the contradiction.

Similarly, $m(T^* + \mathfrak{K}) \geq \varepsilon$ in $\mathfrak{A}/\mathfrak{K}$.

(iii) $\Rightarrow$ (v). Since

$$\mathrm{sp}(|T| + \mathfrak{J}) \subset [\varepsilon, \|T\|], \quad \text{and} \quad \mathrm{sp}(|T^*| + \mathfrak{K}) \subset [\varepsilon, \|T\|],$$

it follows that

$$\mathrm{sp}(|T^*| + \mathfrak{J}) \subset [\varepsilon, \|T\|] \cup \{0\}, \quad \text{and} \quad \mathrm{sp}(|T| + \mathfrak{K}) \subset [\varepsilon, \|T\|] \cup \{0\}$$

using the spectrum identity for $|T|$ and $|T^*|$ checked above. Since $\mathfrak{J} \cap \mathfrak{K} = \{0\}$, the canonical map from $\mathfrak{A}$ to $\mathfrak{A}/\mathfrak{J} \oplus \mathfrak{A}/\mathfrak{K}$ is an injection, i.e., there is a $*$-homomorphism from the image of $\mathfrak{A}$ to $\mathfrak{A}$. Thus, we see that

$$\mathrm{sp}(|T|) \subset [\varepsilon, \|T\|] \cup \{0\}, \quad \text{and} \quad \mathrm{sp}(|T^*|) \subset [\varepsilon, \|T\|] \cup \{0\}.$$

If $|T|$ is invertible, then its range is closed clearly, and the range of T is also closed via polar decomposition. By the inclusions above, in particular, if $|T|$ is not invertible, 0 is an isolated point in $\mathrm{sp}(|T|)$. This is equivalent with $|T|$ and also T having closed ranges. In fact, 0 is an eigenvalue for $|T|$, and there is a projection p (given below) such that $|T| + p$ is invertible. Hence the range of $|T|$ is closed. Conversely, if the range of $|T|$ is closed, we assume that 0 is not an isolated point of $\mathrm{sp}(|T|)$. Since $|T|$ is normal and has its range closed, the residue and continuous spectrums are empty, so that there is a sequence (λ_n) conversing to 0, each of which is in the point spectrum of $|T|$. Anyway, spectral theory and functional calculus imply that there is $f_n(|T|)$ invertible such that $\||T| - f_n(|T|)\| < \varepsilon$, where $f_n(\cdot)$ is an invertible continuous function on $\mathrm{sp}(|T|)$ that approximtates unifomly the coordinate function $t \mapsto t$, for example, $f_n(t) = 1/n$ for $t \leq 1/n$ and $f_n(t) = t$ for $t \geq 1/n$. Then $\||T|\xi - f_n(|T|)\xi\| < \varepsilon$ for any $\xi \in H$. Let $\xi = f_n(|T|)^{-1}\eta$ for any $\eta \in H$. Then $\||T|\xi - \eta\| < \varepsilon$. Therefore, the closure of the range of $|T|$ is H, but the range of $|T|$ is not equal to H. This is the contradiction.

The projections on the orthogonal complements of $|T|(H)$ and $|T^*|(H)$ are given by $h(|T|)$ and $h(|T^*|)$, where h is any continuous function on $\mathbb{R}_+$ such that $h(0) = 1$ and $h(t) = 0$ for $t \geq \varepsilon$ via spectral theorem. It follows by spectral theorem that

$$\{0\} = h(\mathrm{sp}(|T| + \mathfrak{J})) = \mathrm{sp}(h(|T| + \mathfrak{J})) = \mathrm{sp}(h(|T|) + \mathfrak{J}).$$

Therefore, $h(|T|) \in \mathfrak{J}$. Similarly, $h(|T^*|) \in \mathfrak{K}$. Thus,

$$h(|T|)\mathfrak{A}\, h(|T^*|) \subset \mathfrak{J} \cap \mathfrak{K} = \{0\}.$$

(v) $\Rightarrow$ (vi). Since $|T|$ and $|T^*|$ have closed ranges, 0 is an isolated point in both $\mathrm{sp}(|T|)$ and $\mathrm{sp}(|T^*|)$. Thus we have a continuous function on $\mathrm{sp}(|T|)$ defined by $e(0) = 0$ and $e(t) = 1/t$ for $t \geq \varepsilon > 0$ (small enough). Let $V = Te(|T|) \in \mathfrak{A}$. Then V is a partial isometry in $\mathfrak{A}$ with $\ker V = \ker T$, and $V|T| = T$. Indeed,

$$V^*V = e(|T|)T^*Te(|T|) = e(|T|)|T|^2e(|T|)$$

is a projection since $e(t)t^2e(t) = 0$ if $t = 0$ and $= 1$ if $t \geq \varepsilon$. Also, $V|T| = Te(|T|)|T| = T$ since $e(t)t = 0$ if $t = 0$ and $= 1$ if $t \geq \varepsilon$.

From the polar decompositon above, the range projections of T and T^* are VV^* and V^*V respectively. By the assumption, $(I - VV^*)\mathfrak{A}(I - V^*V) = \{0\}$. Therefore, $V \in \mathfrak{A}_e$.

(vi) $\Rightarrow$ (vii). If $T = V|T|$ for some $V \in \mathfrak{A}_e$ with $\ker V = \ker T$, then $I - V^*V$ is the projection on the kernel of $|T|$. Indeed, V^*V is the projection to the closure of the range of $|T|$. Hence its complement is that one. Since 0 is an isolated point of $\mathrm{sp}(|T|)$, it follows that $A = |T| + I - V^*V$ is an invertible element of $\mathfrak{A}_+$, and

$$VA = V|T| + V - VV^*V = V|T|.$$

Hence $T \in \mathfrak{A}_e\mathfrak{A}_+^{-1}$.

(vii) $\Rightarrow$ (i). If $T = VA \in \mathfrak{A}_e\mathfrak{A}_+^{-1}$, then $T = IVA \in \mathfrak{A}^{-1}\mathfrak{A}_e\mathfrak{A}^{-1}$. $\square$

Remark. Let $\mathfrak{L}$ be a modular left ideal of an algebra $\mathfrak{A}$, i.e., $\mathfrak{A}\mathfrak{L} \subset \mathfrak{L}$, and for some $u \in \mathfrak{A}$, $a - au$, $a - ua \in \mathfrak{L}$ for all $a \in \mathfrak{A}$, that is, $\mathfrak{A}/\mathfrak{L}$ unital (see [10]). The ideal of $\mathfrak{A}$ defined by $\mathfrak{J} = \{a \in \mathfrak{A} \,|\, a\mathfrak{A} \subset \mathfrak{L}\}$ for $\mathfrak{L}$ maximal is called the primitive ideal of $\mathfrak{A}$ associated to $\mathfrak{L}$. This ideal is just the kernel of an irreducible representation of $\mathfrak{A}$. An irreducible representation π of $\mathfrak{A}$ is unitarily equivalent to the GNS representaion π_φ accociated to a pure (cyclic vector) state φ. A primitive ideal of a C^*-algebra is always prime. For this, note that $\ker \pi_\varphi = N_\varphi + N_\varphi^*$ as $\mathfrak{J}$, where N_φ as $\mathfrak{L}$ is a closed left ideal of $\mathfrak{A}$ defined by $\{a \in \mathfrak{A} \,|\, \varphi(a^*a) = 0\}$, and $\pi_\varphi(a)(b + N_\varphi) = ab + N_\varphi$ for $a \in \mathfrak{A}$ and $b + N_\varphi \in H_\varphi$ the Hilbert space completion of A/N_φ, with the inner product:

$$\langle a + N_\varphi, b + N_\varphi \rangle = \varphi(b^*a).$$

As we observed above,

$$m(T) = m(|T|) = \min\{\lambda \in \mathrm{sp}(|T|)\},$$

so that we may call it the spectrum distance of $|T|$ from zero.

Proposition 1.1.3. *If the primitive ideal space* $\mathrm{Prim}(\mathfrak{A})$ *of a unital* C^**-algebra* $\mathfrak{A}$ *is a Hausdorff space, then the conditions in the theorem above are equivalent to the following condition:*

(viii) *For any irreducible representation* π *of* $\mathfrak{A}$, $\pi(T)$ *is either left or right invertible.*

Proof. (iv) $\Rightarrow$ (viii). If $m(\pi(T)) > 0$, then $\pi(T)$ is left invertible since $|T| = U^*T$ via polar decompositon $T = U|T|$, and if $m(\pi(T^*)) > 0$, then $\pi(T)$ is right invertible since $|T^*| = TU^*$.

(viii) $\Rightarrow$ (iv). If $\mathrm{Prim}(\mathfrak{A})$ is a Hausdorff space, the function

$$\ker \pi \mapsto \max\{m(\pi(T)), m(\pi(T^*))\}$$

is continuous, because all norm-valued functions on $\mathrm{Prim}(\mathfrak{A})$ such as $\ker \pi \mapsto \|\pi(|T|)\|$ are continuous, and we have $m(\pi(T)) = \alpha - \|\pi(\alpha I - |T|)\|$ for any $\alpha \geq \|T\|$ as shown above. By assumption, the function is never zero. Since $\mathrm{Prim}(\mathfrak{A})$ is compact, the function has the minimum positive. Indeed, its compactness can be proved by considering a family of closed subsets that have finite intersections non-empty to show that the intersection of the family is non-empty by using the property of hull-kernel topology. $\square$

Example 1.1.4. There is a C^*-algeba $\mathfrak{A}$ with an element T that does not satisfy the conditions in the theorem above; the image of T under any irreducible representation of $\mathfrak{A}$ is left or right invertible. Let $H = l^2(\mathbb{Z})$

and $\mathbb{B}(H) \otimes c(\mathbb{N})$ the C^*-algebra of all convergent sequences $S = (S_n)$ with each $S_n \in \mathbb{B}(H)$. Let P denote the projection on the subspace $l^2(\mathbb{N})$ of H. Define

$$\mathfrak{A} = \{S = (S_n) \in \mathbb{B}(H) \otimes c(\mathbb{N}) \mid (I - P)S_\infty P = PS_\infty(I - P) = 0\},$$

where $S_\infty = \lim_{n\to\infty} S_n \in \mathbb{B}(H)$. Then any irreducible representation of $\mathfrak{A}$ either passes through one of copies of $\mathbb{B}(H)$ obtained by evaluating $S \mapsto S_n$ for some $n \in \mathbb{N}$, or it passes through one of two copies at infinity obtained by the evaluations $S \mapsto S_\infty P$ and $S \mapsto S_\infty(I - P)$. Indeed, if it does pass through more than one such copy, it does have non-trivial invariant subspaces. This is the contradiction to its irreducibility.

Let (e_k) be the standard basis for H. Let T_n be the weighted bilateral shift defined by $T_n e_k = e_{k+1}$ for $k \neq 0$, and $T_n e_0 = n^{-1}e_1$. Let $T_\infty e_k = e_{k+1}$ for $k \neq 0$, but $T_\infty e_0 = 0$. Then $T_n \to T_\infty$ in $\mathbb{B}(H)$. Indeed, it follows by direct computation that

$$\|T_n\xi - T_\infty\xi\| = \|n^{-1}\alpha_0 e_1\| = n^{-1}|\alpha_0| \leq n^{-1}$$

for $\xi = \sum_{k=-\infty}^{\infty} \alpha_k e_k \in H$ with $\|\xi\| = 1$. Hence $\|T_n - T_\infty\| \leq 1/n$. Moreover, $T_\infty^* e_k = e_{k-1}$ for $k \neq 1$, but $T_\infty^* e_1 = 0$. Indeed, for $\xi = \sum \alpha_k e_k$, $\eta = \sum \beta_l e_l \in H$,

$$\begin{aligned}
\langle T_\infty\xi, \eta\rangle &= \langle \sum_{k\neq 0} \alpha_k e_{k+1}, \sum \beta_l e_l\rangle = \langle \sum_{k\neq 1} \alpha_{k-1} e_k, \sum \beta_l e_l\rangle \\
&= \sum_{k\neq 1} \alpha_{k-1}\bar{\beta}_k = \sum_{k\neq 0} \alpha_k \bar{\beta}_{k+1} \\
&= \langle \xi, \sum_{l\neq 0} \beta_{l+1} e_l\rangle = \langle \xi, \sum_{l\neq 1} \beta_l e_{l-1}\rangle = \langle \xi, T_\infty^*\eta\rangle.
\end{aligned}$$

Thus, T_∞ and T_∞^* commute with P. Indeed,

$$\langle T_\infty P\xi, \eta\rangle = \langle P\xi, T_\infty^*\eta\rangle = \langle \sum_{k\geq 1} \alpha_k e_k, \sum_{l\geq 2} \beta_l e_{l-1}\rangle = \sum_{k=1}^{\infty} \alpha_k \bar{\beta}_{k+1},$$

$$\langle PT_\infty\xi, \eta\rangle = \langle T_\infty\xi, P\eta\rangle = \langle \sum_{k\geq 1} \alpha_k e_{k+1}, \sum_{l\geq 1} \beta_l e_l\rangle = \sum_{k=2}^{\infty} \alpha_{k-1}\bar{\beta}_k = \sum_{k=1}^{\infty} \alpha_k \bar{\beta}_{k+1}.$$

Consequently, $T = (T_n)$ belongs to $\mathfrak{A}$ to be non-empty.

For every n, T_n is invertible. Moreover, $T_\infty P$ is left invertible in $\mathfrak{A}P$, being the unilateral shift on $l^2(\mathbb{N})$, whereas $T_\infty(I - P)$ is right invertible in $\mathfrak{A}(I-P)$, being the adjoint shift on $l^2(\mathbb{Z}\setminus\mathbb{N})$. Indeed, $P = T_\infty^* T_\infty P = T_\infty^* P T_\infty P$ and

$$(I-P) = T_\infty T_\infty^*(I-P) = T_\infty(I-P)T_\infty^*(I-P).$$

It follows that the image of T under any irreducible representation of $\mathfrak{A}$ is either invertible, left invertible, or right invertible. Nevertheless, T does not satisfy the condition (vi) above, because $\mathrm{sp}(|T_n|) = \{n^{-1}, 1\}$. Indeed, direct computation shows that $T_n^* e_k = e_{k-1}$ for $k \neq 1$ and $T_n^* e_1 = n^{-1}e_0$, so that $T_n^* T_n e_k = e_k$ for $k \neq 0$ and $T_n^* T_n e_0 = n^{-2} e_0$. Hence

$$\mathrm{sp}(|T_n|) = \sqrt{\mathrm{sp}(T_n^* T_n)} = \sqrt{\{n^{-2}, 1\}} = \{n^{-1}, 1\}.$$

Remark that a separable example can be constructed, as given by the C^*-subalgebra of $\mathfrak{A}$ generated by T, together with all sequences of compact operators tending to zero at infinity.

We denote by $\mathfrak{A}_q^{-1}$ the set of all elements of a unital C^*-algebra $\mathfrak{A}$ satisfying the equivalent conditions above, and the elements are called quasi-invertible. It follows from the condition (iv) that $\mathfrak{A}_q^{-1}$ is open. In the case where $\mathfrak{A}$ is a prime C^*-algebra, i.e., if the zero ideal of $\mathfrak{A}$ is prime, equivalently, if every pair of non-zero closed ideals of $\mathfrak{A}$ has the intersection non-zero, in particular if $\mathfrak{A}$ is simple, then the elements of $\mathfrak{A}_q^{-1}$ are either left or right invertible (by the condition (viii)).

Definition 1.1.5. For $T \in \mathfrak{A}_q^{-1}$, define

$$\begin{aligned} m_q(T) &= \max\{\varepsilon \mid (0,\varepsilon) \cap \mathrm{sp}(|T|) = \emptyset\} \\ &= \inf\{\max\{m(\pi(T)), m(\pi(T^*))\} \mid \pi \text{ an irreducible representation of } \mathfrak{A}, \}. \end{aligned}$$

where the second equality follows from the proof of the theorem above. If $T \notin \mathfrak{A}_q^{-1}$, set $m_q(T) = 0$, so that $\mathfrak{A}_q^{-1} = \{T \in \mathfrak{A} \mid m_q(T) > 0\}$. For any $T \in \mathfrak{A}$, define

$$\alpha_q(T) = \mathrm{dist}(T, \mathfrak{A}_q^{-1})$$

so that α_q vanishes on the closure of $\mathfrak{A}_q^{-1}$.

We may call $m_q(T)$ the spectral distance (from zero) for T, and $\alpha_q(T)$ the distance of T to $\mathfrak{A}_q^{-1}$.

Proposition 1.1.6. *For any $T \in \mathfrak{A}$, we have*

$$m_q(T) = \mathrm{dist}(T, \mathfrak{A} \setminus \mathfrak{A}_q^{-1}).$$

Proof. Since $\mathfrak{A}_q^{-1}$ is open, if $T \notin \mathfrak{A}_q^{-1}$, both sides give zero to be equal.

Let $T \in \mathfrak{A}_q^{-1}$ with $T = V|T|$ and $V \in \mathfrak{A}_e$. Let

$$T_0 = V(|T| - m_q(T)V^*V)) = V|T_0|.$$

Indeed, we have

$$\begin{aligned}T_0^*T_0 &= (|T| - m_q(T)V^*V)V^*V(|T| - m_q(T)V^*V) \\ &= |T|V^*V|T| - m_q(T)V^*V|T| - m_q|T|V^*VV^*V + m_q(T)^2V^*VV^*V \\ &= |T|^2 - 2m_q(T)|T| + m_q(T)^2V^*V = (|T| - m_q(T)V^*V)^2.\end{aligned}$$

Clearly, $\|T - T_0\| = \|m_q(T)V\| = m_q(T)$. It follows from the condition (vi) above that $T_0 \notin \mathfrak{A}_q^{-1}$ since spectral theory implies that 0 is not isolated in the spectrum of $|T| - m_q(T)V^*V$ via functional calculus, where V^*V is viewed as a function p defined as $p(0) = 0$ and $p(t) = 1$ for $t \in \mathrm{sp}(|T|)$ nonzero. Therefore,

$$\mathrm{dist}(T, \mathfrak{A} \setminus \mathfrak{A}_q^{-1}) \leq m_q(T).$$

Conversely, if $B \in \mathfrak{A}$, then for any irreducible representation π of $\mathfrak{A}$ we have

$$m(\pi(T + B)) \geq m(\pi(T)) - \|\pi(B)\| \geq m(\pi(T)) - \|B\|.$$

Indeed, note that for any $\xi \in H_\pi$ the representation space for π with norm one,

$$\|\pi(T)\xi\| \leq \|\pi(T + B)\xi\| + \|\pi(B)(-\xi)\| \leq \|\pi(T + B)\xi\| + \|\pi(B)\|.$$

Similarly, we get $m(\pi((T + B)^*)) \geq m(\pi(T^*)) - \|B\|$. Thus,

$$m(\pi(T + B)) \vee m(\pi((T + B)^*)) \geq m(\pi(T)) \vee m(\pi(T^*)) - \|B\|,$$

where $\vee$ means maximum. From the condition (iv) in the theorem above, if $\|B\| < m_q(T)$, then $T + B \in \mathfrak{A}_q^{-1}$. It follows that

$$m_q(T) \leq \mathrm{dist}(T, \mathfrak{A} \setminus \mathfrak{A}_q^{-1}).$$

Indeed, if $C = T + B \notin \mathfrak{A}_q^{-1}$, then $m_q(T) \leq \|B\| = \|T - C\|$. $\square$

Corollary 1.1.7. *The spectral distance function $m_q(\cdot)$ on $\mathfrak{A}_q^{-1}$ and the distance function $\alpha_q(\cdot)$ on $\mathfrak{A}$ to $\mathfrak{A}_q^{-1}$ satisfy the following: for $S, T \in \mathfrak{A}$ and $z \in \mathbb{C}$,*

$$\begin{aligned}&m_q(zT) = |z|m_q(T) \quad \textit{and} \quad |m_q(S) - m_q(T)| \leq \| S - T\|; \\ &\alpha_q(zT) = |z|\alpha_q(T) \quad \textit{and} \quad |\alpha_q(S) - \alpha_q(T)| \leq \| S - T\|.\end{aligned}$$

Proof. Homogeneity follows from the fact that $\mathfrak{A}_q^{-1}$ is stable under multiplication with non-zero complex numbers. Note that $|zT| = |z||T|$, and

$$\operatorname{dist}(zT, \mathfrak{A}_q^{-1}) = \inf_B \|zT - B\| = |z| \inf_B \|T - z^{-1}B\| = |z|\alpha_q(T).$$

For the inequalities as continuity, note also that

$$\|S\xi\| \leq \|S - T\| + \|T\xi\|; \quad \|S - B\| \leq \|S - T\| + \|T - B\|$$

for $\xi \in H$ and $B \in \mathfrak{A}_q^{-1}$. $\square$

Remark. The set $\mathfrak{A}_e$ of extreme points is described as the inner core in $\mathfrak{A}_q^{-1}$:

$$\mathbb{R}_+\mathfrak{A}_e = (\mathbb{C} \setminus \{0\})\mathfrak{A}_e = \{T \in \mathfrak{A}_q^{-1} \,|\, m_q(T) = \|T\|\}.$$

Indeed, if $m_q(T) = \|T\|$, then $\operatorname{sp}(|T|) = \{0, \|T\|\}$, so that $|T| = \|T\|P$ with P a projection via functional calculus. Thus, $T = \|T\|VP$ with $V \in \mathfrak{A}_e$ since $T \in \mathfrak{A}_q^{-1}$. Note also that

$$(VP)(VP)^* = VPV^* = VV^*, \quad (VP)^*(VP) = PV^*VP = P = V^*V.$$

Hence $VP \in \mathfrak{A}_e$ since $(I - VV^*)\mathfrak{A}(I - V^*V) = \{0\}$. Thus, $T \in \mathbb{R}_+\mathfrak{A}_e \subset (\mathbb{C} \setminus \{0\})\mathfrak{A}_e$. Conversely, let $zV \in (\mathbb{C} \setminus \{0\})\mathfrak{A}_e$. Then $m_q(zV) = |z|m_q(V)$. Since V^*V and also $|V|$ are projections, we have $\operatorname{sp}(|V|) = \{0, 1\}$, so that $m_q(V) = 1 = \|V\|$. Hence $m_q(zV) = \|zV\|$.

The two functions α_q and m_q measure the distance to the boundary $\partial\mathfrak{A}_q^{-1}$ of $\mathfrak{A}_q^{-1}$ from the outside of $\mathfrak{A}_q^{-1}$ and from the inside, respectively.

Proposition 1.1.8. *Let $T \in \mathfrak{A}$. If $\|T - A\| < \beta$ for some $A \in \mathfrak{A}_q^{-1}$ with $A = W|A|$ and $W \in \mathfrak{A}_e$, then $T + \beta W \in \mathfrak{A}_q^{-1}$ with $m_q(T + \beta W) \geq \beta - \|T - A\|$. Moreover, $W^*T + \beta I \in \mathfrak{A}^{-1}$ and $TW^* + \beta I \in \mathfrak{A}^{-1}$.*

Proof. We have

$$\begin{aligned} T + \beta W &= \beta W + A + T - A \\ &= W(\beta W^*W + |A|) + T - A. \end{aligned}$$

The spectrum of $\beta W^*W + |A|$ is disjoint from the interval $(0, \beta)$. Indeed, let g be a continuous function defined by $g(0) = 0$ and $g(t) = \beta 1 + t$ for $t \in \operatorname{sp}(|A|) \setminus \{0\}$. Functional calculus and spectral theory imply that $\beta W^*W + |A| = g(|A|)$ and

$$\operatorname{sp}(g(|A|)) = g(\operatorname{sp}(|A|)) = \{0\} \cup \{\beta + t \,|\, t \in \operatorname{sp}(|A|) \setminus \{0\}\}.$$

Therefore,

$$\begin{aligned} m_q(T+\beta W) &\geq m_q(W(\beta W^*W+|A|)) - \|T-A\| \\ &= m_q(|W(\beta W^*W+|A|)|) - \|T-A\| \\ &\geq \beta - \|T-A\| > 0, \end{aligned}$$

where note that

$$(\beta W^*W+|A|)W^*W(\beta W^*W+|A|) = (\beta W^*W+|A|)^2$$

and hence $|W(\beta W^*W+|A|)| = \beta W^*W+|A|$.

For the other statement, put $B = W^*(T-A)$. Then

$$W^*T+\beta I = B+|A|+\beta I = (B(|A|+\beta I)^{-1}+I)(|A|+\beta I).$$

We have

$$\|B\| \leq \|T-A\| < \beta \leq \|(|A|+\beta I)^{-1}\|^{-1}$$

since $0 < \beta I \leq |A|+\beta I$ so that $(|A|+\beta I)^{-1} \leq \beta^{-1}I$, which implies $\|(|A|+\beta I)^{-1}\| \leq \beta^{-1}$. It then follows that $B(|A|+\beta I)^{-1}+I \in \mathfrak{A}^{-1}$ since $\|B(|A|+\beta I)^{-1}\| < 1$, whence $W^*T+\beta I \in \mathfrak{A}^{-1}$. Thus, $-\beta \notin \mathrm{sp}(W^*T)$, so $-\beta \notin \mathrm{sp}(TW^*)$ either, since $\beta \neq 0$. □

Denote by $(\mathfrak{A}_q^{-1})^-$ the norm closure of $\mathfrak{A}_q^{-1}$.

Theorem 1.1.9. *If $T \in (\mathfrak{A}_q^{-1})^- \setminus \mathfrak{A}_q^{-1}$, there is an irreducible representation π of $\mathfrak{A}$ such that $\pi(T)$ is neither left nor right invertible in $\mathbb{B}(H_\pi)$.*

Proof. Since $T \notin \mathfrak{A}_q^{-1}$, there is a universal net $(\mathfrak{B}_i)$ in $\mathrm{Prim}(\mathfrak{A})$ such that

$$\max\{m(T+\mathfrak{B}_i), m(T^*+\mathfrak{B}_i)\} \to 0,$$

where universality means that for any subset Y of $\mathrm{Prim}(\mathfrak{A})$, the net is either eventually in Y (i.e., $\mathfrak{B}_i \in Y$ for any $i \geq i_0$ some) or eventually in $\mathrm{Prim}(\mathfrak{A}) \setminus Y$. For each i, choose pure states φ_i and ψ_i whose GNS representations have kernel $\mathfrak{B}_i$, such that

$$\varphi_i(|T|) \leq 2\,m(T+\mathfrak{B}_i) \quad \text{and} \quad \psi_i(|T^*|) \leq 2\,m(T^*+\mathfrak{B}_i),$$

because via GNS $(\pi_{\varphi_i}, \xi_{\varphi_i})$ for φ_i,

$$\varphi_i(|T|) = \langle \pi_{\varphi_i}(|T|)\xi_{\varphi_i}, \xi_\varphi \rangle \leq \|\pi_{\varphi_i}(|T|)\xi_{\varphi_i}\| \leq 2\,m(\pi_{\varphi_i}(|T|)) = 2\,m(T+\mathfrak{B}_i),$$

and similarly for ψ_i. Since the net is universal, there are states φ and ψ of $\mathfrak{A}$, such that $\varphi_i \to \varphi$ and $\psi_i \to \psi$ in weak* topology. Note that for a unital

C^*-algebra $\mathfrak{A}$, its state space is the weak* closed convex hull of pure states of $\mathfrak{A}$. In particular, it follows from those estimates that $\varphi(|T|) = \psi(|T^*|) = 0$.

Let $\mathfrak{F}$ denote the closed set of limit points (= accumulation points) of $(\mathfrak{B}_i)$ in $\mathrm{Prim}(\mathfrak{A})$. Put $\mathfrak{J} = \ker \mathfrak{F}$ the intersection of elements of $\mathfrak{F}$. If $A \in \mathfrak{J}$, then $A \in \mathfrak{B}$ for every $\mathfrak{B} \in \mathfrak{F}$, so that $\|A + \mathfrak{B}\| = \|0 + \mathfrak{B}\| = 0$, whence

$$\mathfrak{C} = \{\mathfrak{B} \in \mathrm{Prim}(\mathfrak{A}) \mid \|A + \mathfrak{B}\| \geq \varepsilon\}$$

is a compact subset of $\mathrm{Prim}(\mathfrak{A})$, disjoint from $\mathfrak{F}$ for every $\varepsilon > 0$. If $(\mathfrak{B}_i)$ were frequently in $\mathfrak{C}$ it would have a limit point in $\mathfrak{C}$ by compactness, which is impossible. Thus $(\mathfrak{B}_i)$ belongs to $\mathrm{Prim}(\mathfrak{A}) \setminus \mathfrak{C}$ eventually, and since this happens for every ε we conclude that $\|A + \mathfrak{B}_i\| \to 0$. It follows that

$$\begin{aligned}|\varphi(A)| = \lim |\varphi_i(A)| &= \lim |\langle \pi_{\varphi_i}(A)\xi_{\varphi_i} | \xi_{\varphi_i}\rangle| \\ &\leq \lim \|\pi_{\varphi_i}(A)\| = \lim \|A + \mathfrak{B}_i\| = 0.\end{aligned}$$

Similarly, $\psi(A) = 0$. Since this holds for every $A \in \mathfrak{J}$, the states φ and ψ are viewed as those of $\mathfrak{A}/\mathfrak{J}$. Since $\varphi(|T|) = \psi(|T^*|) = 0$ it follows that neither $|T| + \mathfrak{J}$ nor $|T^*| + \mathfrak{J}$ are invertible in $\mathfrak{A}/\mathfrak{J}$. Indeed, if $|T| + \mathfrak{J}$ were invertible, $|T| + \mathfrak{J} \geq \delta I$ for some $\delta > 0$. Then $\varphi(|T| + \mathfrak{J}) = \varphi(|T|) = 0 \geq \delta\varphi(I) = \delta > 0$, the contradiction. Therefore, there are irreducible representations of $\mathfrak{A}/\mathfrak{J}$ with kernels $\mathfrak{D}/\mathfrak{J}$ and $\mathfrak{R}/\mathfrak{J}$ such that

$$m(|T| + \mathfrak{D}) = m(|T^*| + \mathfrak{R}) = 0,$$

where $(\mathfrak{A}/\mathfrak{J})/(\mathfrak{D}/\mathfrak{J}) \cong \mathfrak{A}/\mathfrak{D}$ and $(\mathfrak{A}/\mathfrak{J})/(\mathfrak{R}/\mathfrak{J}) \cong \mathfrak{A}/\mathfrak{R}$. Indeed, if $m(\pi(|T| + \mathfrak{J})) > 0$ for any irreducible representation π of $\mathfrak{A}/\mathfrak{J}$, then $\pi(|T| + \mathfrak{J})$ is invertible, so that $|T| + \mathfrak{J}$ is also invertible via universal representation.

Assume, to obtain a contradiction, that $|T| + \mathfrak{R}$ and $|T^*| + \mathfrak{D}$ are both invertible in $\mathfrak{A}/\mathfrak{R}$ and $\mathfrak{A}/\mathfrak{D}$, respectively, so that

$$m(T + \mathfrak{R}) \geq \varepsilon \quad \text{and} \quad m(T^* + \mathfrak{D}) \geq \varepsilon$$

for some $\varepsilon > 0$. Now use the fact that $T \in (\mathfrak{A}_q^{-1})^-$ to find $S \in \mathfrak{A}_q^{-1}$ with $\|S - T\| < 1/3\varepsilon$. It follows by norm continuity that

$$\begin{aligned}m(S + \mathfrak{D}) \leq 1/3\varepsilon \quad &\text{and} \quad m(S^* + \mathfrak{R}) \leq 1/3\varepsilon; \\ m(S + \mathfrak{R}) \geq 2/3\varepsilon \quad &\text{and} \quad m(S^* + \mathfrak{D}) \geq 2/3\varepsilon,\end{aligned}$$

where note that

$$\varepsilon - m(S + \mathfrak{R}) \leq m(T + \mathfrak{R}) - m(S + \mathfrak{R}) \leq |m(T + \mathfrak{R}) - m(S + \mathfrak{R})| < 1/3\varepsilon.$$

However, since $\mathrm{sp}(|S|)$ and $\mathrm{sp}(|S^*|)$ are the same, zero apart, we have $m(S+\mathfrak{D}) = m(S^* + \mathfrak{D})$, unless one of the values is zero. It is therefore concluded that

$$m(S+\mathfrak{D}) = 0 = m(S^* + \mathfrak{R}).$$

For every $\delta > 0$, the set $\{\mathfrak{B} \in \mathrm{Prim}(\mathfrak{A}) \,|\, m(S+\mathfrak{B}) < \delta\}$ is open and a neighbourhood of $\mathfrak{D}$. Since $\mathfrak{B}_i \to \mathfrak{D}$ because $\mathfrak{D} \in \mathrm{hull}(\ker \mathfrak{F}) = \bar{\mathfrak{F}}$ the closure of $\mathfrak{F}$ in hull-kernel topology, equal to $\mathfrak{F}$, it follows that $m(S+\mathfrak{B}_i) \to 0$. The same argument, applied to S^* and $\mathfrak{R}$, shows that also $m(S^*+\mathfrak{B}_i) \to 0$. But this is the contradiction, as $S \in \mathfrak{A}_q^{-1}$. Thus we obtain that either $m(T+\mathfrak{R}) = 0$ or $m(T^* + \mathfrak{D}) = 0$. Therefore, there is an irreducible representation π with kernel $\mathfrak{D}$ or $\mathfrak{R}$, such that

$$0 = m(\pi(T)) = m(|\pi(T)|) = m(|\pi(T)^*|) = m(\pi(T^*)),$$

which says that neighther $\pi(T)$ nor $\pi(T^*)$ is invertible in $\mathbb{B}(H_\pi)$. $\square$

Corollary 1.1.10. *If $\mathfrak{A}_q^{-1}$ is dense in $\mathfrak{A}$, then the condition* (viii) *above is equivalent to those conditions* (i) *to* (vii) *above.*

Proof. The implication (iv) $\Rightarrow$ (viii) is checked before. We need to check that (viii) $\Rightarrow$ those equivalent conditions, i.e., $T \in \mathfrak{A}_q^{-1}$. Suppose that $T \notin \mathfrak{A}_q^{-1}$, so that $T \in (\mathfrak{A}_q^{-1})^- \setminus \mathfrak{A}_q^{-1}$. By the theorem above, there is an irreducible representation π of $\mathfrak{A}$ such that $\pi(T)$ is neither left nor right invertible, which contradicts to (viii) that for each irreducible representation π of $\mathfrak{A}$, $\pi(T)$ is either left or right invertible. $\square$

Corollary 1.1.11. *The element $T = (T_n) \in \mathbb{B}(l^2(\mathbb{Z})) \otimes c(\mathbb{N})$ such that $(I-P)T_\infty P = PT_\infty(I-P) = 0$, where P is the projecttion to $l^2(\mathbb{N})$, in the example above does not belong to the closure $(\mathfrak{A}_q^{-1})^-$.*

1.2 Extremal extensions of partial isometries

Proposition 1.2.1. *Let E_1, E_2, F_1, F_2 be projections on a Hilbert space H such that $E_1 + E_2 = F_1 + F_2 = I$. Suppose that $T \in \mathbb{B}(H)$ such that $F_1TE_2 = 0$ and F_1TE_1 is a bijection of E_1H onto F_1H. If T is left or right invertible in $\mathbb{B}(H)$, the same is true for any operator*

$$S = F_1TE_1 + F_2RE_1 + F_2TE_2, \quad \textit{where } R \in \mathbb{B}(H).$$

Proof. On the orthogonal decomposition $H = E_1H \oplus E_2H = F_1H \oplus F_2H$, we have

$$\begin{aligned} T &= (F_1+F_2)T(E_1+E_2) = F_1TE_1 + F_2TE_1 + F_2TE_2 \\ &= \begin{pmatrix} F_1TE_1 & 0 \\ F_2TE_1 & F_2TE_2 \end{pmatrix} \equiv \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}; \quad S = \begin{pmatrix} A & 0 \\ F_2RE_1 & B \end{pmatrix}. \end{aligned}$$

If T has a right inverse U in $\mathbb{B}(H)$, we have the equation:

$$TU = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \begin{pmatrix} K & L \\ M & N \end{pmatrix} = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix},$$

where $TU = (F_1 + F_2)T(E_1 + E_2)(E_1 + E_2)U(F_1 + F_2)$ viewed. Since A is assumed to be invertible, $K = A^{-1}$. Thus $AL = 0$ forces $L = 0$, whence N becomes a right inverse for B. it follows that

$$\begin{pmatrix} A & 0 \\ F_2RE_1 & B \end{pmatrix} \begin{pmatrix} K & 0 \\ -NF_2RE_1K & N \end{pmatrix} = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix},$$

where $NF_2RE_1K = NRK$, so that S is right invertible in $\mathbb{B}(H)$.

If T has a left inverse V in $\mathbb{B}(H)$, we have the equation:

$$VT = \begin{pmatrix} K & L \\ M & N \end{pmatrix} \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix},$$

where $VT = (E_1 + E_2)V(F_1 + F_2)(F_1 + F_2)T(E_1 + E_2)$ viewed. Then N is a left inverse for B. We then have

$$\begin{pmatrix} A^{-1} & 0 \\ -NF_2RE_1A^{-1} & N \end{pmatrix} \begin{pmatrix} A & 0 \\ F_2RE_1 & B \end{pmatrix} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix},$$

where $NF_2RE_1A^{-1} = NRA^{-1}$, which shows that S is left invertible in $\mathbb{B}(H)$. □

Let $\mathfrak{A}$ be a unital C^*-algebra and $\mathfrak{A}''$ the enveloping von Neumann algebra for $\mathfrak{A}$. If $T \in \mathfrak{A}$, we have $T = V|T|$ in $\mathfrak{A}''$, but $V \notin \mathfrak{A}$ in general. However, it follows from Stone-Weierstrass theorem that $Vf(|T|) \in \mathfrak{A}$ for every continuous function f on $\mathrm{sp}(|T|)$ with $f(0) = 0$. Indeed, f can be approximated closely (or uniformly) by polynomials without constant terms and with variable $|T|$. For every $\delta > 0$, let E_δ and F_δ denote the spectral projections of $|T|$ and $|T^*| = V|T|V^*$ in $\mathfrak{A}''$, respectively, corresponding to the open inverval (δ, ∞).

Theorem 1.2.2. *Let $T \in \mathfrak{A}$ a unital C^*-algebra, with polar decomposition $T = V|T|$ in $\mathfrak{A}''$. For each $\delta > \alpha_q(T) = \mathrm{dist}(T, \mathfrak{A}_q^{-1})$, there is an extreme, partial isometry U in $\mathfrak{A}_e$ such that*

$$UE_\delta = F_\delta U = VE_\delta = F_\delta V.$$

Namely,

$$U\chi_{(\delta,\infty)}(|T|) = V\chi_{(\delta,\infty)}(|T|) \quad \textit{and equal to} \quad \chi_{(\delta,\infty)}(|T^*|)U = \chi_{(\delta,\infty)}(|T^*|)V.$$

For $\delta < \alpha_q(T)$, there is no extremal extension of VE_δ in $\mathfrak{A}$.

Proof. If $\delta > \alpha_q(T)$ we can find $A = W|A|$ in $\mathfrak{A}_q^{-1}$, so that $W \in \mathfrak{A}_e$, with $\|T - A\| < \delta$. Choose β and γ such that $\|T - A\| < \beta < \gamma < \delta$. Define continuous functions f and g on $\mathbb{R}_+$ by

$$f(t) = \begin{cases} 1/\gamma, & 0 \le t \le \gamma, \\ 1/t, & \gamma \le t \end{cases}; \quad g(t) = \begin{cases} t/\gamma^2, & 0 \le t \le \gamma, \\ 1/t, & \gamma \le t. \end{cases}$$

Now let $B = (T + \beta W)(I + \beta g(|T|)V^*W)^{-1} f(|T|)$, where since $\|\beta g\|_\infty = \beta/\gamma < 1$ and $Vg(|T|) \in \mathfrak{A}$ because $g(0) = 0$, we see that $I + \beta g(|T|)V^*W$ is indeed invertible in $\mathfrak{A}$, because $\|\beta g(|T|)V^*W\| \le \|\beta g(|T|)\| = \|\beta g\|_\infty < 1$ via Gelfand transform, where the supremum norm is taken on the spectrum of $|T|$. Evidently, $f(|T|) \in \mathfrak{A}^{-1}$. Since $T + \beta W \in \mathfrak{A}_q^{-1}$, it follows from $\mathfrak{A}_q^{-1}$ being stable under multiplication by elements of $\mathfrak{A}^{-1}$ that B is also in $\mathfrak{A}_q^{-1}$.

Calculate that

$$\begin{aligned} F_\gamma(T + \beta W) &= VV^*\chi_{(\gamma,\infty)}(|T^*|)VV^*(V|T| + \beta W) \\ &= V\chi_{(\gamma,\infty)}(V^*|T^*|V)V^*(V|T| + \beta W) \\ &= VE_\gamma|T| + \beta VE_\gamma V^*W \\ &= VE_\gamma|T|(I + \beta g(|T|)V^*W), \end{aligned}$$

where $\chi_{(\gamma,\infty)}(\cdot)$ means the characteristic function corresponding to the open interval (γ, ∞), and note that $|T| = V^*|T^*|V$ and $p(|T|) = p(V^*|T^*|V) = V^*p(|T^*|)V$ for any polynomial $p(\cdot)$ with one variable, and $\chi_{(\gamma,\infty)}(t) = \chi_{(\gamma,\infty)}(t) \cdot t \cdot g(t)$ for $t \ge \gamma$. It follows that

$$F_\gamma B = VE_\gamma|T|f(|T|) = VE_\gamma,$$

where note that $\chi_{(\gamma,\infty)}(t) = \chi_{(\gamma,\infty)}(t) \cdot t \cdot f(t)$ for $t \ge \gamma$. Also, $VE_\gamma = F_\gamma V$, because

$$V\chi_{(\gamma,\infty)}(|T|) = V\chi_{(\gamma,\infty)}(|T|)V^*V = \chi_{(\gamma,\infty)}(V|T|V^*)V = \chi_{(\gamma,\infty)}(|T^*|)V.$$

Write B in the following matrix form:

$$\begin{aligned} B &= (F_\delta + (F_\gamma - F_\delta) + (I - F_\gamma))B(E_\delta + (E_\gamma - E_\delta) + (I - E_\gamma)) \\ &= \begin{pmatrix} F_\delta BE_\delta & F_\delta B(E_\gamma - E_\gamma) & F_\delta B(I - E_\gamma) \\ (F_\gamma - F_\delta)BE_\delta & (F_\gamma - F_\delta)B(E_\gamma - E_\gamma) & (F_\gamma - F_\delta)B(I - E_\gamma) \\ (I - F_\gamma)BE_\delta & (I - F_\gamma)B(E_\gamma - E_\gamma) & (I - F_\gamma)B(I - E_\gamma) \end{pmatrix} \\ &\equiv (B_{ij}). \end{aligned}$$

Since we have

$$\begin{aligned} (F_\delta + (F_\gamma - F_\delta))B = F_\gamma B = F_\gamma V &= (F_\delta + (F_\gamma - F_\delta))V \\ &= VE_\gamma = V(E_\delta + (E_\gamma - E_\delta)), \end{aligned}$$

it follows that

$$B = \begin{pmatrix} F_\delta V E_\delta & F_\delta V(E_\gamma - E_\gamma) & 0 \\ (F_\gamma - F_\delta)VE_\delta & (F_\gamma - F_\delta)V(E_\gamma - E_\gamma) & 0 \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$$

and moreover,

$$\begin{aligned} F_\delta V E_\delta &= \chi_{(\delta,\infty)}(V|T|V^*)VE_\delta = V\chi_{(\delta,\infty)}(|T|)V^*VE_\delta = VE_\delta, \\ F_\delta V(E_\gamma - E_\delta) &= \chi_{(\delta,\infty)}(V|T|V^*)V(E_\gamma - E_\delta) = VE_\delta E_\gamma - VE_\delta = 0; \\ (F_\gamma - F_\delta)VE_\delta &= (\chi_{(\gamma,\infty)}(V|T|V^*) - \chi_{(\delta,\infty)}(V|T|V^*))VE_\delta \\ &= VE_\gamma E_\delta - VE_\delta^2 = VE_\delta - VE_\delta = 0, \\ (F_\gamma - F_\delta)V(E_\gamma - E_\delta) &= V(E_\gamma - E_\delta)^2 = V(E_\gamma - E_\delta) \end{aligned}$$

so that B has the lower triangular form.

Choose decreasing continuous functions h_1 and h_2 from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that

$$h_i(t) = 1 \quad (0 \le t \le \gamma), \quad h_i(t) = 0 \quad (\delta \le t); \quad h_1(t)h_2(t) = h_1(t) \quad (\forall t),$$

where $i = 1, 2$. For example, since $h_1(t)(h_2(t) - 1) = 0$ for $\gamma < t < \delta$, let $\gamma < \omega < \delta$, define

$$\begin{aligned} h_1(t) &= \frac{-1}{\omega - \gamma}(t - \omega), \quad h_2(t) = 1 \quad (\gamma < t < \delta); \\ h_1(t) &= 0, \quad h_2(t) = \frac{-1}{\delta - \omega}(t - \delta), \quad (\omega \le t < \delta). \end{aligned}$$

Define $C = B - h_1(|T^*|)B(I - h_2(|T|))$. Since we have

$$\begin{aligned} &F_\delta h_i(|T^*|) = (\chi_{(\delta,\infty)} \cdot h_i)(|T^*|) = 0, \\ &h_i(|T|)E_\delta = (h_i \cdot \chi_{(\delta,\infty)})(|T|) = 0, \\ &(I - F_\gamma)(I - h_i(|T^*|)) = ((1 - \chi_{(\gamma,\infty)})(1 - h_i))(|T^*|) = 0, \\ &(I - h_i(|T|))(I - E_\gamma) = ((1 - h_i)(1 - \chi_{(\gamma,\infty)}))(|T|) = 0, \end{aligned}$$

it follows that

$$\begin{aligned} &h_1(|T^*|)B(I - h_2(|T|)) \\ &= (0 + h_1(|T^*|)(F_\gamma - F_\delta) + (I - F_\gamma))B(E_\delta + (E_\gamma - E_\delta)(I - h_2(|T|)) + 0) \end{aligned}$$

so that we have

$$h_1(|T^*|)B(I - h_2(|T|)) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & h_1(|T^*|)V(E_\gamma - E_\delta)(I - h_2(|T|)) & 0 \\ B_{31} & B_{32}(I - h_2(|T|)) & 0 \end{pmatrix}$$

where note that

$$
\begin{aligned}
&h_1(|T^*|)V(E_\gamma - E_\delta)(I - h_2(|T|)) \\
&= (F_\gamma - F_\delta)h_1(|T^*|)V(E_\gamma - E_\delta)(I - h_2(|T|))(E_\gamma - E_\delta), \\
&B_{32}(I - h_2(|T|)) = B_{32}(I - h_2(|T|))(E_\gamma - E_\delta).
\end{aligned}
$$

Moreover, observe that

$$
\begin{aligned}
&h_1(|T^*|)V(E_\gamma - E_\delta)(I - h_2(|T|)) \\
&= h_1(V|T|V^*)V(\chi_{(\gamma,\infty)}(|T|) - \chi_{(\delta,\infty)}(|T|))(I - h_2(|T|)) \\
&= Vh_1(|T|)(\chi_{(\gamma,\infty)}(|T|) - \chi_{(\delta,\infty)}(|T|))(I - h_2(|T|)) \\
&= V(\chi_{(\gamma,\infty)}(|T|) - \chi_{(\delta,\infty)}(|T|))(h_1(1 - h_2))(|T|) = 0.
\end{aligned}
$$

Therefore, the matrix form of C becomes:

$$
C = B - h_1(|T^*|)B(I - h_2(|T|)) = \begin{pmatrix} VE_\delta & 0 & 0 \\ 0 & V(E_\gamma - E_\delta) & 0 \\ 0 & B_{32}h_2(|T|) & B_{33} \end{pmatrix}.
$$

If B is left or right invertible in $\mathbb{B}(H)$, then it follows from the proposition above that C also is left or right invertible, respectively, since $V(E_\delta + (E_\gamma - E_\delta))$ is a bijection of $(E_\delta + (E_\gamma - E_\delta))H$ onto $(F_\delta + (F_\gamma - F_\delta))H$. Indeed, for $\xi \in H$, we have $VE_\gamma^2\xi = F_\gamma VE_\gamma\xi \in F_\gamma H$, with $\|VE_\gamma\xi\| = \|E_\gamma\xi\|$, and for $\eta \in H$,

$$
F_\gamma\eta = \chi_{(\gamma,\infty)}(|T^*|)\eta = V\chi_{(\gamma,\infty)}(|T|)V^*\eta = VE_\gamma V^*\eta \in VE_\gamma H.
$$

Note that with $B_{21} = (B_{31}, B_{32})$, $B_{21}' = (0, B_{32}h_2(|T|))$, and $0_2 = (0,0)^t$,

$$
B = \begin{pmatrix} VE_\delta & 0 & 0 \\ 0 & V(E_\gamma - E_\delta) & 0 \\ B_{31} & B_{32} & B_{33} \end{pmatrix} = \begin{pmatrix} VE_\gamma & 0_2 \\ B_{21} & B_{33} \end{pmatrix}, \quad C = \begin{pmatrix} VE_\gamma & 0_2 \\ B_{21}' & B_{33} \end{pmatrix}.
$$

Such simultaneous left or right invertibility also holds for quotient images of the operators B and C. Therefore, it follows from $B \in \mathfrak{A}_q^{-1}$ that $C \in \mathfrak{A}_q^{-1}$ also, by the condition (ii).

Evidently, $CE_\delta = F_\delta C = VE_\delta = F_\delta V$ using the (first) matrix form for C. Let $C = U|C|$ be the polar decomposition of C in $\mathfrak{A}$, with $U \in \mathfrak{A}_e$ by the condition (vi). We have

$$
\begin{aligned}
C^*CE_\delta &= C^*(CE_\delta) = C^*F_\delta V = (F_\delta C)^*V = (F_\delta V)^*V \\
&= V^*F_\delta V = V^*\chi_{(\delta,\infty)}(V|T|V^*)V = E_\delta,
\end{aligned}
$$

whence $|C|E_\delta = E_\delta$, because

$$E_\delta = E_\delta|C|^2E_\delta = (|C|E_\delta)^*|C|E_\delta = E_\delta^2,$$

so that $|C|E_\delta = E_\delta$ because $|C|E_\delta$ is positive from $|C|E_\delta = E_\delta|C|$, i.e., $\langle|C|E_\delta\xi, \xi\rangle = \langle|C|E_\delta\xi, E_\delta\xi\rangle \geq 0$. Therefore, $UE_\delta = U|C|E_\delta = CE_\delta = VE_\delta$. Similarly,

$$\begin{aligned} F_\delta CC^* &= F_\delta(F_\delta C)C^* = F_\delta(CE_\delta)C^* = (F_\delta C)(CE_\delta)^* \\ &= F_\delta V(F_\delta V)^* = \chi_{(\delta,\infty)}(|T^*|)VV^*\chi_{(\delta,\infty)}(|T^*|) = F_\delta, \end{aligned}$$

whence $F_\delta|C^*| = F_\delta$. Thus, $F_\delta U = F_\delta|C^*|U = F_\delta U|C|U^*U = F_\delta C = F_\delta V$. These relations also follow from the matrix form of C.

Now assume that $VE_\delta = UE_\delta$ for some $U \in \mathfrak{A}_e$ and some $\delta > 0$. Define $e \in C(\mathbb{R}_+)$ by $e(t) = \max\{t - \delta, 0\}$ for $t \in \mathbb{R}_+$. For $\varepsilon > 0$,

$$T_\varepsilon \equiv U(e(|T|) + \varepsilon I) = V(e(|T|) + \varepsilon U,$$

since $e = \chi_{(\delta,\infty)} \cdot e$, so that $e(|T|) = E_\delta e(|T|)$. Evidently, $T_\varepsilon \in \mathfrak{A}_e\mathfrak{A}_+^{-1} = \mathfrak{A}_q^{-1}$. Moreover,

$$\|T - T_\varepsilon\| \leq \||T| - e(|T|)\| + \varepsilon \leq \delta + \varepsilon.$$

Since ε is arbitrary, we conclude that $\alpha_q(T) = \operatorname{dist}(T, \mathfrak{A}_q^{-1}) \leq \delta$. □

Corollary 1.2.3. *If $T = V|T|$ is the polar decomposition of an element of a unital C^*-algebra $\mathfrak{A}$, then each $Vf(|T|)$ in $\mathfrak{A}$ has an extremal decomposition $Uf(|T|) = Vf(|T|)$, with $U \in \mathfrak{A}_e$, provided that f is a continuous function on* $\operatorname{sp}(|T|)$ *vanishing on $[0, \delta]$ for some $\delta > \alpha_q(T)$.*

Proof. Since $f = \chi_{(\delta,\infty)} \cdot f$, we have

$$Uf(|T|) = UE_\delta f(|T|) = VE_\delta f(|T|) = Vf(|T|).$$

□

Corollary 1.2.4. *Every element T in $\mathfrak{A}\backslash\mathfrak{A}_q^{-1}$ has a canonical approximant T_0 in $(\mathfrak{A}_q^{-1})^-$, i.e., $\|T - T_0\| = \alpha_q(T) = \operatorname{dist}(T, \mathfrak{A}_q^{-1})$.*

Proof. Define $e_0(t) = \max\{t - \alpha_q(T), 0\}$. Let $T = V|T|$ be the polar decomposition for T. Put $T_0 = Ve_0(|T|)$. Evidently, using the C^*-norm condition,

$$\|T - T_0\|^2 = \|V(|T| - e_0(|T|))\|^2 = \|(|T| - e_0(|T|))^2\| = \||T| - e_0(|T|)\|^2,$$

so that $\|T - T_0\| = \||T| - e(|T|)\| = \|t - e(t)\|_\infty = \alpha_q(T)$. It also follows that $T_0(\mathfrak{A}_q^{-1})^-$, because $Ve_\varepsilon(|T|) = Ue_\varepsilon(|T|)$ for some $U \in \mathfrak{A}_e$, where $e_\varepsilon(t) = \max\{t - \varepsilon - \alpha_q(T), 0\}$. Set $T_\varepsilon = U(e_\varepsilon(|T|) + \varepsilon I)$, so that $T_\varepsilon \in \mathfrak{A}_q^{-1}$, converging uniformly to T_0. □

Proposition 1.2.5. *For each T in $\mathfrak{A}$ a unital C^*-algebra, and $\varepsilon > 0$, $\delta > 0$, there is $S \in \mathfrak{A}_q^{-1}$ such that $m_q(S) \geq \varepsilon$ and $\|S - T\| \leq \alpha_q(T) + \varepsilon + \delta$.*

Proof. Define e in $C(\mathbb{R}_+)$ by $e(t) = \max\{t - \alpha_q(T) - \delta, 0\}$. Let $T = V|T|$ be the polar decomposition of T. Choose an extreme partial isometry $U \in \mathfrak{A}_e$ such that $Ve(|T|) = Ue(|T|)$. Set

$$S = U(e(|T|) + \varepsilon I) = Ve(V^*|T^*|V) + \varepsilon U = (e(|T^*|) + \varepsilon I)U.$$

We see that

$$\begin{aligned}\|S - T\| &= \|V(e(|T|) - |T|) + \varepsilon U\| \\ &\leq \|e - \mathrm{id}\|_\infty + \varepsilon \leq \alpha_q(T) + \delta + \varepsilon.\end{aligned}$$

Moreover, $m_q(S) \geq \varepsilon$ by the condition (iv). Indeed, check that

$$\begin{aligned}S^*S &= (e(|T|) + \varepsilon I)U^*UU^*U(e(|T|) + \varepsilon I) \\ &= (U(e(|T|) + \varepsilon I))^*UU^*(e(|T^*|) + \varepsilon I)U \\ &= ((e(|T^*|) + \varepsilon I)U)^*UU^*(e(|T^*|) + \varepsilon I)U = (U^*(e(|T^*|) + \varepsilon I)U)^2.\end{aligned}$$

Hence $|S| = U^*(e(|T^*|) + \varepsilon I)U$, and $\varepsilon U^*U \leq |S|$. Note that every irreducible representation of $U^*U\mathfrak{A}U^*U$ a hereditary C^*-subalgebra of $\mathfrak{A}$ can be extended to that of $\mathfrak{A}$, and its converse by restriction is also true. $\square$

Proposition 1.2.6. *If V is a partial isometry in a unital C^*-algebra $\mathfrak{A}$, then either $\alpha_q(V) = 1$, or else $\alpha_q(V) = 0$, in which case $V = UV^*V = VV^*U$ for some $U \in \mathfrak{A}_e$.*

Proof. Put $P = V^*V = |V|$ since $(V^*V)^2 = V^*V$. Let $\alpha_q(V) < \delta < 1$, and let $e(t) = (1 - \delta)^{-1} \max\{t - \delta, 0\}$. Then there is $U \in \mathfrak{A}_e$ such that

$$V = VP = Ve(P) = Ue(P) = UP = UV^*V,$$

because $\mathrm{sp}(P) = \{0, 1\}$ and $\mathrm{id}(1) = 1 = e(1)$. Moreover, note that

$$VV^*UP = VV^*VP = VV^*V = V.$$

Hence $VV^*U(P\xi) = V\xi$ for $\xi \in H$. Since V is a partial isometry, VV^*U is a partial isometry from $P(H)$ to $V(H)$. Hence, $VV^*U = V$. Note also that $T = U(P + \varepsilon I) = VP + \varepsilon U \in \mathfrak{A}_q^{-1}$ for every $\varepsilon > 0$, and that $\|T - V\| \leq \varepsilon$, whence $\alpha_q(V) = 0$.

On the other hand, note that $\alpha_q(0) = \mathrm{dist}(0, \mathfrak{A}_q^{-1}) = 0$, because $U(\varepsilon I) \in \mathfrak{A}_q^{-1}$ for any $U \in \mathfrak{A}_e$ and $\varepsilon > 0$. Therefore, $\alpha_q(V) \leq \|V\| + \varepsilon = 1 + \varepsilon$, so that $\alpha_q(V) \leq 1$, or using the estimate $\alpha_q(V) \leq \|V\|$ shown above. $\square$

1.3 Extremally rich C^*-algebras

We say that a unital C^*-algebra $\mathfrak{A}$ is extremally rich if the set $\mathfrak{A}_q^{-1}$ of all quasi-invertible elements is dense in $\mathfrak{A}$.

If $\mathfrak{A}$ is finite in the sense that every extreme point is unitary, then $\mathfrak{A}$ is extremally rich if it has stable rank one (of Rieffel [18]), i.e., if $\mathfrak{A}^{-1}$ of all invertible elements is dense in $\mathfrak{A}$. In particular, if $\mathfrak{A}$ is commutative, i.e., $\mathfrak{A} = C(X)$ of all continuous functions on some compact Hausdorff space X, then extremal richness of $\mathfrak{A}$ is equivalent to that $\dim X \leq 1$. However, every von Neumann algebra is extremally rich, whereas the stable rank of a von Neumann algebra is infinite unless the algebra is finite, in which case that of is one. Thus the definition of being extremally rich, like the concept of real rank zero (of Brown and Pedersen [2]), is viewed as generalization of being stable rank one in a sense (or an infinite analogue of it). As we shall see, the class of extremally rich C^*-algebras includes stable rank one algebras, von Neumann algebras (as well as AW^*-algebras), and purely infinite (simple) C^*-algebras, and even Toeplitz algebras.

A non-unital C^*-algebra $\mathfrak{A}$ is said to be extremally rich if the unitization $\mathfrak{A}^+ = \mathfrak{A} + \mathbb{C}I$ is extremally rich.

Lemma 1.3.1. *Let A and B be elements of a unital C^*-algebra $\mathfrak{A}$ such that $A^*A + B^*B = I$. If $A, B \in \mathfrak{A}_q^{-1}$ with polar decompositions $A = U|A|$ and $B = V|B|$, then UV^* is a partial isometry in $\mathfrak{A}$. Further, if W is an extremal extension of UV^*, i.e., $W|UV^*| = |VU^*|W = UV^*$, then $W^*A + B \in \mathfrak{A}_q^{-1}$ with $m_q(W^*A + B) \geq 1$.*

Proof. Let $P = U^*U$ and $Q = V^*V$. Since $|A|^2 + |B|^2 = I$ we have $|B| = (I - |A|^2)^{1/2}$, so that $|A| + |B| \geq I$, because $t + \sqrt{1-t^2} \geq 1$ for $0 \leq t \leq 1$. Moreover, P and Q are the spectral projections of $|A|$, corresponding to the intervals $(0,1]$ and $[0,1)$. Note that since $|A|^2 \leq I$, we have $1 \geq \||A|^2\| = \|A^*A\| = \|A\|^2$, and $(\chi_{(0,1]} \cdot \mathrm{id})(t) = t = \mathrm{id}(t)$ for $0 \leq t \leq 1$, and $\chi_{[0,1)}(t)\sqrt{1-t^2} = \sqrt{1-t^2}$ for $0 \leq t \leq 1$. In particular, $PQ = QP$ and $I = P \vee Q$ which corresponds to the union of $P(H)$ and $Q(H)$. Thus UV^* is a partial isometry with

$$(UV^*)^*(UV^*) = VU^*UV^* = VV^*VPV^* = VPQV^* = (VPQV^*)^2,$$
$$(UV^*)(UV^*)^* = UV^*VU^* = UU^*UQU^* = UPQU^* = (UPQU^*)^2.$$

Therefore, note also that $(UV^*)((UV^*)^*\xi) = UPQU^*\xi$ for $\xi \in H$. Hence $UPQU^*$ is the range projection of UV^*, so that $WW^* = UPQU^*$.

Assume now that W is an extremal extension of UV^*, and consider the element $C = W^*A + B$. Note that we have

$$\begin{aligned} CPQ &= (W^*U|A| + V|B|)PQ = (W^*WW^*U|A| + V|B|)PQ \\ &= (W^*(UPQU^*)U|A| + V|B|)PQ \\ &= (W^*|VU^*|^2(UPQU^*)U|A| + V|B|)PQ \\ &= (VU^*|VU^*|(UPQU^*)U|A| + V|B|)PQ \\ &= (VU^*(UPQU^*)U|A| + V|B|)PQ \\ &= (VQP|A| + V|B|)PQ \\ &= V(|A| + |B|)PQ \\ &= VQP(|A| + |B|) = VP(|A| + |B|), \end{aligned}$$

where note that $(t+\sqrt{1-t^2})\chi_{(0,1]}(t)\chi_{[0,1)}(t) = \chi_{[0,1)}(t)\chi_{(0,1]}(t)(t+\sqrt{1-t^2})$ for $0 \leq t \leq 1$.

To show that $m_q(C) \geq 1$, it suffices to verify this inequality in every irreducible representation of $\mathfrak{A}$. Passing to an irreducible representation of $\mathfrak{A}$ without changing the notation, there are the following 4 cases:

(i): $P = Q = I$. In this case $C = V(|A| + |B|)$, whence

$$C^*C = (|A| + |B|)^2 = |A|^2 + |A||B| + |B||A| + |B|^2 \geq |A|^2 + |B|^2 = I,$$

where note that the sum of positive elements is also positive. Thus $m_q(C) \geq 1$.

(ii): $P = I$, $Q \neq I$. Now $VV^* = I$ because $V \in \mathfrak{A}_e$. Indeed, we have $(I - VV^*)a(I - Q)\xi = 0$ for $a \in \mathfrak{A}$ and ξ in the representation space under an irreducible representation, so that $I - VV^* = 0$, because there is $(I - Q)\xi \neq 0$, which is always a cyclic vector by irreducibility. Thus, UV^* is an isometry, whence $W = UV^*$ since $|UV^*| = 1$. Consequently,

$$C = W^*U|A| + V|B| = VU^*U|A| + V|B| = V(|A| + |B|),$$

so that $CC^* \geq I$ and $m_q(C) \geq 1$.

(iii): $P \neq I$, $Q = I$. Now $UU^* = I$ because $U \in \mathfrak{A}_e$, so $UV^*(UV^*)^* = UV^*VU^* = UQU^* = I$, i.e., UV^* a co-isometry, or VU^* an isometry, whence $W = UV^*$ since $|VU^*| = 1$. The rest is the same as above.

(iv): $P \neq I$, $Q \neq I$. Consider

$$\begin{aligned} CQ &= CPQ + C(I-P)Q \\ &= VP(|A| + |B|) + V(I-P)|B| \\ &= VP(|A| + |B|) + V(I-P)(|A| + |B|) = V(|A| + |B|), \end{aligned}$$

where note that $1-\chi_{(0,1]}(t) = \chi_{[0]}(t)(= \delta_0(t))$ for $0 \le t \le 1$. Since $VV^* = 1$,

$$CC^* \ge CQC^* = V(|A| + |B|)^2 V^* \ge I,$$

where $I \ge Q = QQ^*$ and $(|A| + |B|)^2 \ge I$. $\square$

Theorem 1.3.2. *For a unital C^*-algebra $\mathfrak{A}$, the following conditions are equivalent:*

(i) *$\mathfrak{A}$ is extremally rich.*

(ii) *For any $T \in \mathfrak{A}$ and $\varepsilon > 0$ there is a $W \in \mathfrak{A}_e$ such that $T+\varepsilon W \in \mathfrak{A}_q^{-1}$.*

(iii) *For any $T \in \mathfrak{A}$ and $\varepsilon > 0$ there is a $W \in \mathfrak{A}_e$ such that $W^*T + \varepsilon I \in \mathfrak{A}^{-1}$.*

(iv) *For every pair $S, T \in \mathfrak{A}$ with $S^*S + T^*T \in \mathfrak{A}^{-1}$ there is a $W \in \mathfrak{A}_e$ such that $W^*S + T \in \mathfrak{A}_q^{-1}$. If $S^*S + T^*T > \delta^2 I$, we can assume also that $m_q(W^*S + T) > \delta$.*

Proof. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) both follow as: As shown before, if $\|T - A\| < \varepsilon$ for some $A \in \mathfrak{A}_q^{-1}$, with $A = W|A|$ and $W \in \mathfrak{A}_e$, then $T + \varepsilon W \in \mathfrak{A}_q^{-1}$ and $W^*T + \varepsilon I \in \mathfrak{A}^{-1}$.

Clearly, (ii) $\Rightarrow$ (i). If $T + \varepsilon W \in \mathfrak{A}_q^{-1}$, $\|T - (T + \varepsilon W)\| \le \varepsilon$.

(iv) $\Rightarrow$ (i). Let $S = \varepsilon I$. Since $T' = W^*S + T \in \mathfrak{A}_q^{-1}$, then $\|T - T'\| \le \varepsilon$.

(iii) $\Rightarrow$ (ii). If $G = W^*T + \varepsilon I \in \mathfrak{A}^{-1}$ for some $W \in \mathfrak{A}_e$, i.e., $-\varepsilon \notin \mathrm{sp}(W^*T)$, then $-\varepsilon \notin \mathrm{sp}(TW^*)$, so that $H = TW^* + \varepsilon I \in \mathfrak{A}^{-1}$. It follows that $m(H) \ge \delta$ and $m(G^*) \ge \delta$ for some $\delta > 0$. Let π be an irreducible representation of $\mathfrak{A}$. If $\pi(W)$ is an isometry we have

$$\begin{aligned} m(\pi(T + \varepsilon W)) &= m(\pi((TW^* + \varepsilon I)W) \\ &= m(\pi(HW)) \ge \delta m(\pi(W)) = \delta. \end{aligned}$$

Indeed, since $\delta I \le |H|$, we have $\delta^2 I \le H^*H$, so that $\delta^2\pi(I) = \delta^2\pi(W^*W) \le \pi(W^*H^*HW) = |\pi(HW)|^2$, hence $\delta\pi(I) \le |\pi(HW)|$. Note also that $|\pi(W)| = \pi(I)$. Similarly, if $\pi(W)$ is a co-isometry we have

$$\begin{aligned} m(\pi(T^* + \varepsilon W^*)) &= m(\pi((T^*W + \varepsilon I)W^*)) \\ &= m(\pi(G^*W^*)) \ge \delta m(\pi(W^*)) = \delta. \end{aligned}$$

It follows from the condition (iv) (in the theorem above) that $T+\varepsilon W \in \mathfrak{A}_q^{-1}$.

(i) $\Rightarrow$ (iv). Put $H = S^*S + T^*T > \delta^2 I$ and choose $\varepsilon > 0$ (small enough) so that we have $m(H)^{1/2}(1 - 5\varepsilon) > \delta$. Now take $A, B \in \mathfrak{A}_q^{-1}$ such that

$$\|SH^{-1/2} - A\| < \varepsilon, \quad \|TH^{-1/2} - B\| < \varepsilon.$$

Using that

$$H^{-1/2}(S^*S+T^*T)H^{-1/2}=\frac{1}{\sqrt{H}}H\frac{1}{\sqrt{H}}=I,$$

we compute, with $K=A^*A+B^*B$, that

$$\begin{aligned}\|I-K\|&=\|H^{-1/2}HH^{-1/2}-(A^*A+B^*B)\|\\&\leq\|H^{-1/2}S^*SH^{-1/2}-A^*A\|+\|H^{-1/2}T^*TH^{-1/2}-B^*B\|\end{aligned}$$

and note that

$$\begin{aligned}\varepsilon^2&>\|SH^{-1/2}-A\|^2\\&=\|H^{-1/2}S^*SH^{-1/2}-S^*H^{-1/2}A-A^*SH^{-1/2}+A^*A\|;\\\varepsilon^2&>\|TH^{-1/2}-A\|^2\\&=\|H^{-1/2}T^*TH^{-1/2}-T^*H^{-1/2}A-A^*TH^{-1/2}+A^*A\|,\end{aligned}$$

so that we estimate

$$\begin{aligned}&\|H^{-1/2}S^*SH^{-1/2}-A^*A\|\\&\leq\varepsilon^2+\|S^*H^{-1/2}A+A^*SH^{-1/2}-2A^*A\|\\&\leq\varepsilon^2+\|(S^*H^{-1/2}-A^*)A\|+\|A^*(SH^{-1/2}-A)\|\\&\leq\varepsilon^2+2\varepsilon\|A\|<\varepsilon^2+2\varepsilon(1+\varepsilon)=3\varepsilon^2+2\varepsilon,\end{aligned}$$

where $\|A\|<\varepsilon+\|SH^{-1/2}\|$, and $S^*S\leq H$, so that $H^{-1/2}S^*SH^{-1/2}\leq I$, so that $\|SH^{-1/2}\|^2\leq 1$. Similarly, $\|H^{-1/2}T^*TH^{-1/2}-B^*B\|\leq 3\varepsilon^2+2\varepsilon$. Therefore, it is concluded that

$$\|I-K\|\leq 6\varepsilon^2+4\varepsilon,$$

(this should be correct).

Since $AK^{-1/2},BK^{-1/2}\in\mathfrak{A}_q^{-1}$, we find an $W\in\mathfrak{A}_e$ such that

$$m_q((W^*A+B)K^{-1/2})\geq 1.$$

Indeed, note that

$$(AK^{-1/2})^*AK^{-1/2}+(BK^{-1/2})^*BK^{-1/2}=K^{-1/2}(A^*A+B^*B)K^{-1/2}=I$$

and it is shown above that every partial isometry in $\mathfrak{A}$ has an extremal extension. Assuming that $\varepsilon<2/15$ (corrected) we have

$$K^{1/2}\geq\sqrt{1-4\varepsilon-6\varepsilon^2}I>(1-3\varepsilon)I,$$

because $I-K \le \|I-K\|I \le (4\varepsilon+6\varepsilon^2)I$, so that $(1-4\varepsilon-6\varepsilon^2)I \le K$, and

$$1-4\varepsilon-6\varepsilon^2 > 1-6\varepsilon+9\varepsilon^2 \quad \Leftrightarrow \quad \varepsilon(2-15\varepsilon) > 0.$$

It follows that $m_q(W^*A+B) \ge 1-3\varepsilon$. Note that for any $\omega > 0$,

$$|(W^*A+B)K^{-1/2}|^2 = K^{-1/2}(W^*A+B)^*(W^*A+B)K^{-1/2} \ge (1-\omega)I,$$

so that we have

$$(W^*A+B)^*(W^*A+B) \ge (1-\omega)K,$$

and hence $|W^*A+B| \ge (1-\omega)^{1/2}K^{1/2} > (1-\omega)^{1/2}(1-3\varepsilon)I$. Since ω is arbitrary, we obtain $|W^*A+B| \ge (1-3\varepsilon)I$. Moreover,

$$\begin{aligned}&\|W^*A+B-(W^*S+T)H^{-1/2}\| \\ &\le \|W^*(A-SH^{-1/2})\| + \|B-TH^{-1/2}\| \le 2\varepsilon,\end{aligned}$$

which implies that by continuity of m_q,

$$m_q(W^*A+B) - m_q((W^*S+T)H^{-1/2} \le 2\varepsilon,$$

so that $m_q((W^*S+T)H^{-1/2}) \ge 1-5\varepsilon$. Since $H \ge m(H)I$, we conclude that

$$m_q(W^*S+T) \ge (1-5\varepsilon)m(H)^{1/2} > \delta.$$

Indeed, we have

$$H^{-1/2}|W^*S+T|^2H^{-1/2} \ge (1-5\varepsilon)^2 I,$$

so that $|W^*S+T|^2 \ge (1-5\varepsilon)^2 H \ge (1-5\varepsilon)^2 m(H)I > \delta^2 I$. □

Remark. In general, we can not conclude that $W^*S+T \in \mathfrak{A}^{-1}$, even though $S^*S+T^*T \in \mathfrak{A}^{-1}$. Take $S=0$ and T a non-unitary isometry. The weaker assumption that $AS+BT \in \mathfrak{A}_q^{-1}$ for some $A, B \in \mathfrak{A}$ dose not imply W^*S+T. To see this, take $S=0$ and $T=B^*B$ for some non-unitary co-isometry B. Then $A0+BB^*B = B \in \mathfrak{A}_q^{-1}$ since $I-BB^*=0$, so $B \in \mathfrak{A}_e$, but $W^*0+B^*B \notin \mathfrak{A}_q^{-1}$ for all W, because $(B^*B)B^*B = B^*B$ is a nonzero projection $\notin \mathfrak{A}_e$, since $0 \neq I-B^*B \in (I-B^*B)\mathfrak{A}(I-B^*B) \neq \{0\}$.

Theorem 1.3.3. *For extremally rich C^*-algebras, their quotients, direct sums, direct products, and hereditary C^*-subalgebras are all extremally rich again.*

Proof. As for quotients, note that surjective $*$-homomorphisms map extreme points to extreme points. Indeed, for $V \in \mathfrak{A}_e$, $(I - VV^*)\mathfrak{A}(I - V^*V) = \{0\}$ implies that $(\pi(I) - \pi(V)\pi(V)^*)\pi(\mathfrak{A})(\pi(I) - \pi(V)^*\pi(V)) = \{0\}$ for any quotient map π from a unital C^*-algebra $\mathfrak{A}$, so that $\pi(V) \in \pi(\mathfrak{A})_e$. For any $\pi(A) \in \pi(\mathfrak{A})$ and $\varepsilon > 0$, there is $B \in \mathfrak{A}_e$ with $\|A - B\| < \varepsilon$, which implies that $\|\pi(A) - \pi(B)\| < \varepsilon$, with $\pi(B) \in \pi(\mathfrak{A})_e$.

Let $\{\mathfrak{A}_\lambda \,|\, \lambda \in \Lambda\}$ be a family of extremally rich C^*-algebras. Let $\mathfrak{A} = \Pi_\lambda \mathfrak{A}_\lambda$ be their direct product C^*-algebra. Assume first that all $\mathfrak{A}_\lambda$ are unital. Let $T = (T_\lambda) \in \mathfrak{A}$ and $\varepsilon > 0$. As shown before, for every λ, there is $S_\lambda \in (\mathfrak{A}_\lambda)_q^{-1}$ such that $m_q(S_\lambda) \geq \varepsilon/2$ and $\|T_\lambda - S_\lambda\| \leq (\varepsilon/2) + (\varepsilon/2) = \varepsilon$. With $S = (S_\lambda)$, then $\|S - T\| \leq \varepsilon$ and $m_q(S) \geq \varepsilon/2$. This shows that $\mathfrak{A}$ is extremally rich. If some of $\mathfrak{A}_\lambda$ are non-unital we consider $\mathfrak{A}_\lambda^+$ and $\Pi_\lambda \mathfrak{A}_\lambda^+$ being extremally rich, which contains $\mathfrak{A}$ as a closed ideal, in particular a hereditary C^*-subalgebra. It follows from the hereditary case shown below that $\mathfrak{A}$ is extremally rich. The same reasoning applies to the direct sum $\sum_\lambda \mathfrak{A}_\lambda$ of $\mathfrak{A}_\lambda$, which is also a closed ideal of $\mathfrak{A}$.

Now let $\mathfrak{B}$ be a hereditary C^*-subalgebra of an extremally rich C^*-algebra $\mathfrak{A}$. We may assume $\mathfrak{A}$ to be unital. Let $\mathfrak{B}^+ = \mathfrak{B} + \mathbb{C}I$ and $I + B \in \mathfrak{B}^+$ with $B \in \mathfrak{B}$. Given $\varepsilon > 0$ choose $\delta > 0$ such that $2\delta < 1$, $4\delta\|B\| < 1$, and $4\delta\|B\|^2 < \varepsilon$. Since $\mathfrak{A}$ is extremally rich, there is $I + A \in \mathfrak{A}_q^{-1}$ such that $\|A - B\| < \delta$. Put $D = I - (I + A - B)^{-1}$, which is permissible since $\|A - B\| < 1$. Let

$$
\begin{aligned}
C &= (I + A - B)^{-1}(I + A)(I - DB)^{-1} \\
&= (I - D)((I + A - B) + B)(I - DB)^{-1} \\
&= (I + (I - D)B)(I - DB)^{-1} \\
&= ((I - DB) + B)(I - DB)^{-1} = I + B(I - DB)^{-1},
\end{aligned}
$$

where note that $(I + A - B)^{-1} = \sum_{k=0}^{\infty}(I - (I + A - B))^k = \sum_{k=0}^{\infty}(B - A)^k$ (Neumann series), so that

$$
\begin{aligned}
\|D\| &= \|I - (I + A - B)^{-1}\| \\
&= \|\sum_{k=1}^{\infty}(B - A)^k\| \leq \sum_{k=1}^{\infty}\delta^k = \delta/(1 - \delta),
\end{aligned}
$$

which implies that

$$
\|I - (I - DB)\| = \|DB\| \leq \frac{\delta}{1 - \delta} \cdot \frac{1}{4\delta} < 1,
$$

where the last inequality is equivalent to that $\delta < 3/4$, and hence $I - DB$

is invertible. Note also that

$$\|D\| = \|(B-A)\sum_{k=0}^{\infty}(B-A)^k\|$$
$$\leq \|A-B\|(1-\|A-B\|)^{-1} < \delta(1-\delta)^{-1} < 2\delta,$$

where the last inequality is equivalent to that $\delta(2\delta-1) < 0$, i.e., $0 < \delta < 1/2$, so that, in particular, $\|DB\| \leq \|D\|\|B\| < 2\delta(1/4\delta) = 1/2$. By construction, $C \in \mathfrak{A}_q^{-1}$ since $I + A \in \mathfrak{A}_q^{-1}$ stable under multiplication by elements of $\mathfrak{A}^{-1}$. Moreover,

$$\|C-(I+B)\| = \|I + B(I-DB)^{-1} - (I-B)\|$$
$$= \|B((I-DB)^{-1} - I)\| = \|B(\sum_{k=0}^{\infty}(DB)^k - I)\|$$
$$= \|BDB(\sum_{k=0}^{\infty}(DB)^k\| \leq \|B\|^2\|D\|\|(1-\|DB\|)^{-1}$$
$$< 2\|B\|^2\|D\| < 4\delta\|B\|^2 < \varepsilon.$$

Finally, $C - I = B(I-DB)^{-1} = \sum_{k=0}^{\infty} B(DB)^k \in \mathfrak{B}$, because $\mathfrak{B}$ is closed and $\mathfrak{B}\mathfrak{A}\mathfrak{B} \subset \mathfrak{B}$. Thus $C = I + \mathfrak{B}_0$ for some $\mathfrak{B}_0 \in \mathfrak{B}$.

Since scalar multiples of elements of the form $I+B$, $B \in \mathfrak{B}$ are dense in $\mathfrak{B}^+$, we have now proved $\mathfrak{B}^+ \cap \mathfrak{A}_q^{-1}$ is dense in $\mathfrak{B}^+$. But if $T \in \mathfrak{B}^+ \cap \mathfrak{A}_q^{-1}$, and $T = V|T|$ with $V \in \mathfrak{A}_e$, then $|T| \in \mathfrak{B}^+$ with a gap in its spectrum, so that

$$V = \lim_{n\to\infty} T(|T| + \frac{1}{n}I)^{-1} \in \mathfrak{B}^+,$$

where note that $\lambda/(\lambda + (1/n)) \to 1$ as $n \to \infty$, for λ in the spectrum of $|T|$. Evidently, $\mathfrak{A}_e \cap \mathfrak{B}^+ \subset \mathfrak{B}_e^+$. Indeed, for $V \in \mathfrak{A}_e \cap \mathfrak{B}^+$, we have $\{0\} = (I - VV^*)\mathfrak{A}(I - V^*V) \supset (I - VV^*)\mathfrak{B}^+(I - V^*V)$. It follows that $\mathfrak{B}^+$ is extremally rich. If $\mathfrak{B}$ is non-unital, this means that $\mathfrak{B}$ is extremally rich. If $\mathfrak{B}$ is unital, with a unit $P \neq I$, then $\mathfrak{B}^+ = \mathfrak{B} + \mathbb{C}(I-P)$, so $\mathfrak{B}$ as a quotient of $\mathfrak{B}^+$ is again extremally rich. □

Corollary 1.3.4. *Every hereditary C^*-subalgebra of a C^*-algebra with topological stable rank one has topological stable rank.*

Proof. Replace the term quasi-invertible with invertible in the proof above. □

1.4 Imprimitivity bimodules and matrices

Let $\mathfrak{A}$ be a C^*-algebra with two projections P, Q. As pointed out by Sakai, an element U in $P\mathfrak{A}_1Q$ belongs to $(P\mathfrak{A}_1Q)_e \equiv (P\mathfrak{A}Q)_e$ if and only if U is a partial isometry such that

$$(P - UU^*)\mathfrak{A}(Q - U^*U) = 0.$$

Of course, $(P\mathfrak{A}_1Q)_e$ may be empty, even though $\mathfrak{A}$ is unital, so we may assume that $(P\mathfrak{A}_1Q)_e \neq \emptyset$. With appropriate modifications, then, all the results so far remain true.

For C^*-algebras $\mathfrak{B}$ and $\mathfrak{C}$, a $\mathfrak{B}$-$\mathfrak{C}$ Hilbert bimodule is a $\mathfrak{B}$-$\mathfrak{C}$ bimodule $\mathfrak{X}$ together with $\mathfrak{B}$-valued and $\mathfrak{C}$-valued inner products $\langle\cdot,\cdot\rangle_{\mathfrak{B}}$, $\langle\cdot,\cdot\rangle_{\mathfrak{C}}$, such that several natural axioms are satisfied as:

1. $\langle x_1 + x_2, y_1\rangle_{\mathfrak{B}} = \langle x_1, y_1\rangle + \langle x_2, y_1\rangle_{\mathfrak{B}}$,
   $\langle x_1, y_1 + y_2\rangle_{\mathfrak{C}} = \langle x_1, y_1\rangle + \langle x, y_2\rangle_{\mathfrak{C}}, \quad x_1, x_2, y_1, y_2 \in \mathfrak{X}$,
2. $\langle bx, y\rangle_{\mathfrak{B}} = b\langle x, y\rangle_{\mathfrak{B}}, \quad (\langle x, by\rangle_{\mathfrak{B}} = \langle x, y\rangle_{\mathfrak{B}} b^*$ implied$), \quad x, y \in \mathfrak{X}, b \in \mathfrak{B}$,
   $\langle x, yc\rangle_{\mathfrak{C}} = \langle x, y\rangle_{\mathfrak{C}} c, \quad (\langle xc, y\rangle_{\mathfrak{C}} = c^*\langle x, y\rangle_{\mathfrak{C}}$ implied$), \quad x, y \in \mathfrak{X}, c \in \mathfrak{C}$,
3. $\langle zx, y\rangle_{\mathfrak{B}} = z\langle x, y\rangle_{\mathfrak{B}}, \quad \langle x, zy\rangle_{\mathfrak{C}} = z\langle x, y\rangle_{\mathfrak{C}}, \quad x, y \in \mathfrak{X}, z \in \mathbb{C}$,
4. $\langle x, y\rangle = \langle y, x\rangle^*, \quad x, y \in \mathfrak{X}$,
5. $\langle x, x\rangle \geq 0, \quad x \in \mathfrak{X}$, and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$,

so that $\langle\cdot,\cdot\rangle_{\mathfrak{B}}$ conjugate linear in the second variable and $\langle\cdot,\cdot\rangle_{\mathfrak{C}}$ conjugate linear in the first variable. In particular, for $X \in \mathfrak{X}$ we have

$$\|\langle X, X\rangle_{\mathfrak{B}}\| = \|\langle X, X\rangle_{\mathfrak{C}}\| = \|X\|^2$$

such that $\mathfrak{X}$ is a Banach bimodule. If $\mathfrak{B}_0$ and $\mathfrak{C}_0$ are the closed linear spans of the sets $\langle\mathfrak{X}, \mathfrak{X}\rangle_{\mathfrak{B}}$ and $\langle\mathfrak{X}, \mathfrak{X}\rangle_{\mathfrak{C}}$, then $\mathfrak{B}_0$ and $\mathfrak{C}_0$ are closed ideals of $\mathfrak{B}$ and $\mathfrak{C}$; and $\mathfrak{X}$ is a $\mathfrak{B}$-$\mathfrak{C}$ imprimitivity bimodule precisely when $\mathfrak{B}_0 = \mathfrak{B}$ and $\mathfrak{C}_0 = \mathfrak{C}$. In the general case we have that $\mathfrak{X}$ is a $\mathfrak{B}_0$-$\mathfrak{C}_0$ imprimitivity bimodule. If $\mathfrak{B}$ or $\mathfrak{C}$ are non-unital, the bimodule action can be extended to the multiplier algebras $M(\mathfrak{B})$ and $M(\mathfrak{C})$, so that $\mathfrak{X}$ becomes a unital $M(\mathfrak{B})$-$M(\mathfrak{C})$ Hilbert bimodule. Thus any Hilbert bimodule can be extended to a unital Hilbert bimodule in such a way that the underlying imprimitivity structure remains unchanged. In other ways, for example, we can use the unitized algebras $\mathfrak{B}^+$ and $\mathfrak{C}^+$ instead of $M(\mathfrak{B})$ and $M(\mathfrak{C})$.

Recall that for a C^*-algebra $\mathfrak{A}$, $M(\mathfrak{A})$ consists of double centralisers (L, R) that are bounded linear maps on $\mathfrak{A}$ such that

$$L(ab) = L(a)b, \quad R(ab) = aR(b), \quad \text{and} \quad R(a)b = aL(b)$$

for $a, b \in \mathfrak{A}$. For example, if $c \in \mathfrak{A}$ and (L_c, R_c) defined by $L_c(a) = ca$ and $R_c(a) = ac$ for $a \in \mathfrak{A}$ is a double centraliser on $\mathfrak{A}$. Multiplication and involution of $M(\mathfrak{A})$ are defined by

$$(L_1, R_1)(L_2, R_2) = (L_1 L_2, R_2 R_1), \quad (L, R)^* = (R^*, L^*),$$

where $L^*(a) = L(a^*)^*$ and $R^*(a) = R(a^*)^*$.

If $\mathfrak{X}$ is a unital $\mathfrak{B}$-$\mathfrak{C}$ Hilbert bimodule, then $\mathfrak{X}$, $\mathfrak{B}$, and $\mathfrak{C}$ can be embedded in a unital C^*-algebra $\mathfrak{A}$ containing projections P and Q such that

$$\mathfrak{B} = P\mathfrak{A}P, \quad \mathfrak{C} = Q\mathfrak{A}Q, \quad \mathfrak{X} = P\mathfrak{A}Q,$$

the bimodule action is given by multiplication in $\mathfrak{A}$, and

$$\langle X, Y\rangle_{\mathfrak{B}} = XY^*, \quad \langle X, Y\rangle_{\mathfrak{C}} = X^*Y.$$

Thus, $XY^*Z = \langle X, Y\rangle_{\mathfrak{B}} Z = X\langle Y, Z\rangle_{\mathfrak{C}}$. One such embedding, characterized by the relation $P + Q = I$, is given by the linking algebra construction for imprimitivity bimodules and generalized to Hilbert bimodules. Conversely, given any two projections P and Q in a C^*-algebra $\mathfrak{A}$, $P\mathfrak{A}Q$ becomes a $P\mathfrak{A}P$-$Q\mathfrak{A}Q$ Hilbert bimodule in such a way.

Note that for $T = V|T|$, with $\delta > \alpha_q(T)$ and $U \in \mathfrak{A}_e$,

$$UE_\delta = F_\delta U = VE_\delta,$$

where $E_\delta = \chi_{(\delta,\infty)}(|T|)$ and $F_\delta = \chi_{(\delta,\infty)}(|T^*|)$, is equivalent to that

$$U|T|f(|T|) = |T^*|f(|T^*|)U = Tf(|T|)(= V|T|f(|T|))$$

for $f \in C(\mathrm{sp}(|T|))$ vanishing on $[0, \delta]$. Indeed, note that

$$\begin{aligned} U|T|f(|T|) &= Uf(|T|)|T| = Vf(|T|)|T| = V|T|f(|T|) = Tf(|T|); \\ |T^*|f(|T^*|)U &= |T^*|f(|T^*|)F_\delta U = |T^*|f(|T^*|)VE_\delta \\ &= V|T|V^*Vf(|T|)V^*VE_\delta = V|T|f(|T|)E_\delta = Tf(|T|). \end{aligned}$$

Conversely, note also that E_δ and F_δ are respectively, weak limits of $f(|T|)$ and $f(|T^*|)$ as $f(t) \to \chi_{(\delta,\infty)}(t)$ pointwise.

As a convenient summary of the results shown in previous sections, and as the definition for $\mathfrak{X}$ to be extremally rich,

Theorem 1.4.1. *Let $\mathfrak{X}$ be either a C^*-algebra, or a space $P\mathfrak{A}Q$ with P, Q projections in the multiplier algebra of a C^*-algebra $\mathfrak{A}$, or a $\mathfrak{B}_0$-$\mathfrak{C}_0$ imprimitivity bimodule, where $\mathfrak{B}_0 = \mathrm{span}\langle \mathfrak{X}, \mathfrak{X}\rangle_{\mathfrak{B}}$ and $\mathfrak{C}_0 = \mathrm{span}\langle \mathfrak{X}, \mathfrak{X}\rangle_{\mathfrak{C}}$; and* *assume that the unit ball $\mathfrak{X}_1$ of $\mathfrak{X}$ has an extreme point. Then the following conditions are equivalent:*

(i) *The set $\mathfrak{X}_q^{-1}$ of all quasi-invertible elements is dense in $\mathfrak{X}$.*

(ii) *For each $T \in \mathfrak{X}$ and every continuous function f on $\mathbb{R}_+$ vanishing on a neighbourhood of zero, there is a $W \in \mathfrak{X}_e$ the set of all extreme points in the unit ball $\mathfrak{X}_1$, such that*

$$W|T|f(|T|) = Tf(|T|) = f(|T^*|)T$$

with $f(|T|) = f(\langle T, T\rangle_{\mathfrak{C}_0}^{1/2})$, etc.

(iii) *For every $T \in \mathfrak{X}$ and $\varepsilon > 0$, there is a $W \in \mathfrak{X}_e$ such that $T + \varepsilon W \in \mathfrak{X}_q^{-1}$.*

(iv) *For every $T \in \mathfrak{X}$ and $\varepsilon > 0$, there is a $W \in \mathfrak{X}_e$ such that $W^*T + \varepsilon I$ is invertible in $\mathfrak{C}_0^+$, with $W^*T = \langle W, T\rangle_{\mathfrak{C}_0}$.*

As a note, the zero space is extremally rich.

Proposition 1.4.2. *Let $\mathfrak{A}$ be an extremally rich C^*-algebra and P, Q projections in $\mathfrak{A}$ such that $I - P \sim I - Q$ Murray-von Neumann equivalence in $\mathfrak{A}^+$. Then $P\mathfrak{A}_1Q$ has an extreme point if $P\mathfrak{A}Q \neq 0$ and $P\mathfrak{A}Q$ is extremally rich.*

Proof. Assuming that $P\mathfrak{A}Q \neq 0$, we see that $P\mathfrak{A}^+Q = P\mathfrak{A}Q$ (since $PIQ = PQ \in \mathfrak{A}$); so we may also assume that $\mathfrak{A}$ is unital. Choose $V \in \mathfrak{A}$ such that $V^*V = I - Q$ and $VV^* = I - P$. Then the set $\mathfrak{F} = V + P\mathfrak{A}_1Q$ is a closed face of $\mathfrak{A}_1$. Actually, every closed face $\mathfrak{F}$ of $\mathfrak{A}_1$ arises in this manner for a unique partial isometry $V \in \mathfrak{A}''$ belonging locally to $\mathfrak{A}$, which means that $V = TV^*V$ for some $T \in \mathfrak{A}_1$.

Recall that a face $\mathfrak{F}$ of a convex set $\mathfrak{C}$ is a convex closed subset of $\mathfrak{C}$ such that for $x, y \in \mathfrak{C}$, if $tx + (1-t)y \in \mathfrak{F}$ for $0 < t < 1$, then $x, y \in \mathfrak{F}$. Note that $(V + P\mathfrak{A}_1Q)(I - Q) = V$ and $(V + P\mathfrak{A}_1Q)Q = P\mathfrak{A}_1Q$, so that $V + P\mathfrak{A}_1Q \subset \mathfrak{A}_1$. For $x, y \in \mathfrak{A}_1$, if $tx + (1-t)y \in \mathfrak{F}$ for $0 < t < 1$, so that $tx + (1-t)y = V + Pa_tQ$ for some $a_t \in \mathfrak{A}_1$, then $Pa_tQ - Pa_sQ = (t-s)x + (s-t)y$ goes to zero as both t and $s \to 0$ or 1, i.e., Cauchy sequence, and thus,

$$\lim(tx + (1-t)y) = \lim(V + Pa_tQ) = V + P(\lim Pa_tQ)Q \in V + P\mathfrak{A}_1Q$$

where the limit means that $t \to 0$ or $t \to 1$.

If $T \in P\mathfrak{A}_1Q$ then $S = V + T \in \mathfrak{F}$ with $|S| = V^*V + |T|$, because letting $T = PaQ$ for some $a \in \mathfrak{A}_1$ we have

$$\begin{aligned} S^*S &= (V^* + Qa^*P)(V + PaQ) \\ &= V^*V + T^*T = (V^*V + |T|)(V^*V + |T|) \end{aligned}$$

since $V = VV^*V = (I-P)V$ so that $PV = 0 = V^*P$, and we have

$$V^*V|T|^2 = V^*VQa^*PaQ = 0$$

since $V^* = V^*VV^* = (I-Q)V^*$ so that $QV^* = 0 = VQ$, and hence

$$0 = V^*V|T|^2 = (I-Q)|T|^2 = V^*V|T|^2V^*V = (|T|V^*V)^*|T|V^*V$$

so that $V^*V|T| = 0$.

If $f \in C(\mathbb{R}_+)$ vanishing in a neighbourhood of zero and with $f(1) = 1$, since $\mathfrak{A}$ is extremally rich, there is a $W \in \mathfrak{A}_e$ such that $W|S|f(|S|) = Sf(|S|)$ (by the equivalent condition above in the theorem). Since we have

$$\begin{aligned} W|S|f(|S|) &= W(V^*V + |T|)f(V^*V + |T|) \\ &= W(V^*Vf(V^*V + |T|) + |T|f(V^*V + |T|)) \\ &= W(V^*V + |T|f(|T|)) \end{aligned}$$

because

$$\begin{aligned} V^*Vf(V^*V + |T|) &= V^*VV^*Vf(V^*V + |T|)V^*V \\ &= V^*Vf(V^*V(V^*V + |T|)V^*V) = V^*Vf(V^*V) = V^*V \end{aligned}$$

from that V^*V commutes with $V^*V + |T|$ and $f(1) = 1$, and

$$\begin{aligned} |T|f(V^*V + |T|) &= |T|^{1/2}f(V^*V + |T|)|T|^{1/2} \\ &= f(|T|^{1/2}(V^*V + |T|)|T|^{1/2}) = f(|T|(V^*V + |T|)) \\ &= f(|T|^2) = f(|T|^{1/2}|T||T|^{1/2}) \\ &= |T|^{1/2}f(|T|)|T|^{1/2} = |T|f(|T|) \end{aligned}$$

from that $|T|$ commutes with $V^*V + |T|$, and on the other hand, since

$$\begin{aligned} Sf(|S|) &= (V + T)f(V^*V + |T|) \\ &= Vf(V^*V + |T|) + Tf(V^*V + |T|) \\ &= VV^*Vf(V^*V + |T|) + R|T|f(V^*V) \\ &= Vf(V^*V) + Rf(|T|^2) = V + Tf(|T|), \end{aligned}$$

where $R|T|$ means the polar decomposition for T, it follows that

$$W(V^*V + |T|f(|T|)) = V + Tf(|T|),$$

whence, by orthogonality, $WV^*V = V$ (by multiplying the equation by V^*V from the right) and $W|T|f(|T|) = Tf(|T|)$ (by multiplying the equation by Q from the right). But then $W \in \mathfrak{F}$, since $W = V + W(I - V^*V) =$

$V + WQ$, and $V^*V = W^*V$ so that $V^*V = V^*W$ and hence $V = VV^*W$, and it follows that

$$PWQ = (I - VV^*)W(I - V^*V) = (W - V)(I - V^*V) = WQ.$$

Thus, by the facial property, $W \in \mathfrak{F}_e = V + (P\mathfrak{A}_1 Q)_e = V + (P\mathfrak{A}Q)_e$. Note that W is an extreme point in $\mathfrak{A}_1$ i.e., a one-point face, so that so is in $\mathfrak{F}$. It follows that $W = V + U$ for some $U \in (P\mathfrak{A}Q)_e$ and $U|T|f(|T|) = Tf(|T|)$ since $PV = 0$ and $PU = U$ and $PT = T$, so that $P\mathfrak{A}Q$ is extremally rich by the theorem above. □

As shown in the C^*-algebra case in the previous section,

Corollary 1.4.3. *If $\mathfrak{X}$ is an extremally rich, C^*-algebra $\mathfrak{A}$, or space of the form $P\mathfrak{A}Q$, or Hilbert bimodule, so that $\mathfrak{X}_e \neq \emptyset$, then every isometry in $\mathfrak{X}$ has an extension in $\mathfrak{X}_e$.*

Proposition 1.4.4. *If $\mathfrak{A}$ is an extremally rich C^*-algeba and $U, V \in \mathfrak{A}_e$, let $P = I - UU^*$ and $Q = I - V^*V$ be the defect projections. Then either $P\mathfrak{A}Q = 0$, or else $(P\mathfrak{A}Q)_e \neq \emptyset$ and $P\mathfrak{A}Q$ is extremally rich.*

Proof. Assuming that $P\mathfrak{A}Q \neq 0$ we let $\mathfrak{J}$ be the closed ideal of $\mathfrak{A}$ generated by $P_0 = I - VV^*$ and $Q_0 = I - U^*U$, whereas $\mathfrak{K}$ is the closed ideal generated by $P\mathfrak{A}Q$. Since $P\mathfrak{A}Q_0 = 0$ and $P_0\mathfrak{A}Q = 0$ from $U, V \in \mathfrak{A}_e$, it follows that $\mathfrak{J} \cap \mathfrak{K} = \mathfrak{J}\mathfrak{K} = 0$. Indeed, for $f = f(a_jP_0b_j, c_jQ_0d_j) \in \mathfrak{J}$ and $g = g(s_kPx_kQt_k) \in \mathfrak{K}$ with some $a_j, b_j, c_j, d_j, s_k, x_k, t_k \in \mathfrak{A}$, where $f(\cdot,\cdot)$ and $g(\cdot)$ are polynomials with two variables and one variable respectively, we have $fg = 0$, because

$$\begin{aligned}
(a_jP_0b_j)(s_kPx_kQt_k) &= a_jP_0(b_js_kPx_k)Qt_k = 0, \\
(c_jQ_0d_j)(s_kPx_kQt_k) &= c_j(Q_0(d_js_k)P)x_kQt_k = 0, \quad \text{and also} \\
(s_kPx_kQt_k)(a_jP_0b_j) &= s_kPx_k(Q(t_ka_j)P_0)b_j = 0, \\
(s_kPx_kQt_k)(c_jQ_0d_j) &= s_kP(x_kQt_kc_j)Q_0d_j = 0.
\end{aligned}$$

Let $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$ be the quotient map. We see that $\pi(UV)$ is a partial isometry with support projection $\pi(V^*V) = I - \pi(Q)$ and range projection $\pi(UU^*) = I - \pi(P)$. Indeed, check that

$$\begin{aligned}
\pi(UV)^*\pi(UV) &= V^*U^*UV + \mathfrak{J} = V^*V - V^*Q_0V + \mathfrak{J} = \pi(V^*V), \\
\pi(UV)\pi(UV)^* &= UVV^*U^* + \mathfrak{J} = UU^* - UP_0U^* + \mathfrak{J} = \pi(UU^*).
\end{aligned}$$

Since the quotient $\pi(\mathfrak{A})$ is extremally rich, it follows that $\pi(P\mathfrak{A}Q) = \pi(P)\pi(\mathfrak{A})\pi(Q)$ is extremally rich. Since $\mathfrak{J} \cap \mathfrak{K} = 0$, the restriction of π

to $\mathfrak{K}$ is an isometric isomorphism, so that $P\mathfrak{A}Q$, being isometrically isomorphic to $\pi(P\mathfrak{A}Q)$, is also extremally rich. Note that $\mathfrak{K} = \mathfrak{K}/(\mathfrak{J} \cap \mathfrak{K}) \cong (\mathfrak{K} + \mathfrak{J})/\mathfrak{J} = \pi(\mathfrak{K})$. □

Theorem 1.4.5. *If $\mathfrak{A}$ is an extremally rich C^*-algebra, then so is the matrix algebra $M_n(\mathfrak{A})$ over $\mathfrak{A}$ for any n.*

Proof. We may assume that $\mathfrak{A}$ is unital. Note that if $\mathfrak{A}$ is non-unital, then $M_n(\mathfrak{A})$ is a closed ideal (so a hereditary C^*-subalgebra) of $M_n(\mathfrak{A}^+)$. If $M_n(\mathfrak{A}^+)$ is extremally rich, then so is $M_n(\mathfrak{A})^+$. Moreover, it suffices to prove the theorem for $n = 2$ since by iteration this gives the result for all 2^k, where note that $M_{2^k}(\mathfrak{A}) \cong M_2(M_{2^{k-1}}(\mathfrak{A}))$, and for a given n, $M_n(\mathfrak{A})$ is a hereditary C^*-subalgebra of $M_{2^k}(\mathfrak{A})$ for k large, with $n \leq 2^k$.

Let $T \in M_2(\mathfrak{A})$, with

$$T = \begin{pmatrix} A & D \\ C & B \end{pmatrix} \quad A, B, C, D \in \mathfrak{A}.$$

Since $\mathfrak{A}$ is extremally rich, we can approximate A with $UH \in \mathfrak{A}_q^{-1}$, where $U \in \mathfrak{A}_e$ and $H \in \mathfrak{A}^{-1}$. Thus,

$$\begin{pmatrix} UH & D \\ C & B \end{pmatrix} \begin{pmatrix} H^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} U & D \\ C_1 & B \end{pmatrix} \equiv T_1,$$

where $C_1 = CH^{-1} \in \mathfrak{A}$. Furthermore, since U is a partial isometry,

$$\begin{pmatrix} I & 0 \\ -C_1U^* & I \end{pmatrix} T_1 \begin{pmatrix} I & -U^*D \\ 0 & I \end{pmatrix} = \begin{pmatrix} U & D \\ C_1(I - U^*U) & B_1 \end{pmatrix} \begin{pmatrix} I & -U^*D \\ 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} U & (I - UU^*)D \\ C_1(I - U^*U) & B_1 \end{pmatrix} \equiv T_2,$$

where $B_1 = B - C_1U^*D \in \mathfrak{A}$. Now we approximate B_1 with an element $VK \in \mathfrak{A}_q^{-1}$, where $V \in \mathfrak{A}_e$ and $K \in \mathfrak{A}^{-1}$, to obtain an element $T_3 \in M_2(\mathfrak{A})$. Thus,

$$T_3 \begin{pmatrix} I & 0 \\ 0 & K^{-1} \end{pmatrix} = \begin{pmatrix} U & (I - UU^*)D_1 \\ C_1(I - U^*U) & V \end{pmatrix} \equiv T_4,$$

where $\mathfrak{D}_1 = DK^{-1} \in \mathfrak{A}$. Since $U \in \mathfrak{A}_e$ and V is a partial isometry, we have

$$\begin{pmatrix} I & -(I - UU^*)D_1V^* \\ 0 & I \end{pmatrix} T_4 \begin{pmatrix} I & 0 \\ -V^*C_1(I - U^*U) & I \end{pmatrix}$$
$$= \begin{pmatrix} U & (I - UU^*)D_1(I - V^*V) \\ C_1(I - U^*U) & V \end{pmatrix} \begin{pmatrix} I & 0 \\ -V^*C_1(I - U^*U) & I \end{pmatrix}$$
$$= \begin{pmatrix} U & (I - UU^*)D_1(I - V^*V) \\ (I - VV^*)C_1(I - U^*U) & V \end{pmatrix} \equiv T_5.$$

where $-(I-UU^*)D_1V^*C_1(I-U^*U) = 0$ and $V^* = V^*VV^*$. If $(I-VV^*)\mathfrak{A}(I-U^*U) \neq 0$ and $(I-UU^*)\mathfrak{A}(I-V^*V) \neq 0$, then these Hilbert bimodules, denoted by $\mathfrak{X}_1$ and $\mathfrak{X}_2$, are extremally rich by the proposition above. We can therefore approximate T_5 with an element of the form:

$$\begin{pmatrix} U & YN \\ XM & V \end{pmatrix}, \quad X \in (X_1)_e, Y \in (\mathfrak{X}_2)_e,$$

and M and N are invertible in $(I-U^*U)\mathfrak{A}(I-U^*U)$ and $(I-V^*V)\mathfrak{A}(I-V^*V)$ respectively. Thus,

$$\begin{pmatrix} U & YN \\ XM & V \end{pmatrix}\begin{pmatrix} (M+U^*U)^{-1} & 0 \\ 0 & (N+V^*V)^{-1} \end{pmatrix} = \begin{pmatrix} U & Y \\ X & V \end{pmatrix} \equiv Z,$$

because $U = UU^*U = U(UU^*+M)$, $YN = Y(N+V^*V)$, $XM = X(M+U^*U)$, and $V = VV^*V = V(V^*V+N)$. It follows that Z is an extreme partial isometry in $M_2(\mathfrak{A})_1$ with defect projections:

$$I_2 - ZZ^* = \begin{pmatrix} I-UU^*-YY^* & 0 \\ 0 & I-VV^*-XX^* \end{pmatrix} \equiv P,$$

$$I_2 - Z^*Z = \begin{pmatrix} I-U^*U-X^*X & 0 \\ 0 & I-V^*V-Y^*Y \end{pmatrix} \equiv Q,$$

where $I_2 = I \oplus I$ the diagonal sum of I is the unit of $M_2(\mathfrak{A})$. Check that

$$ZZ^* = \begin{pmatrix} UU^*+YY^* & UX^*+YV^* \\ XU^*+VY^* & XX^*+VV^* \end{pmatrix} = \begin{pmatrix} UU^*+YY^* & 0 \\ 0 & XX^*+VV^* \end{pmatrix}$$

$$Z^*Z = \begin{pmatrix} U^*U+X^*X & U^*Y+X^*V \\ Y^*U+V^*X & Y^*Y+V^*V \end{pmatrix} = \begin{pmatrix} U^*U+X^*X & 0 \\ 0 & Y^*Y+V^*V \end{pmatrix}.$$

Also, $PM_2(\mathfrak{A})Q = \{0\}$ follows from that $PKQ = 0$ for any $K \in M_2(\mathfrak{A})$ by computing directly all the components in PKQ.

If one or both of the above mentioned Hilbert bimodules are zero, we replace X or Y or both by zero in Z, which then again is an extreme partial isometry.

We have shown that T can be approximated by an element RZS, with R and S invertible in $M_2(\mathfrak{A})$. This shows that $M_2(\mathfrak{A})$ is extremally rich. □

1.5 Inductive limits and Morita equivalence

Definition 1.5.1. Let $\mathfrak{B}$ be a C^*-subalgebra of a unital C^*-algebra $\mathfrak{A}$, such that $I \in \mathfrak{B}$. We say that the embedding $\mathfrak{B} \subset \mathfrak{A}$ is preserving extreme points if $\mathfrak{B}_e \subset \mathfrak{A}_e$. If $\mathfrak{A}$ is non-unital, we consider the unitized $\mathfrak{A}^+$ and put

$\mathfrak{B}^+ = \mathfrak{B} + \mathbb{C}I$. We then say that $\mathfrak{B}$ is embedded preserving extreme points if $\mathfrak{B}^+$ is embedded in $\mathfrak{A}^+$ preserving extreme points. Note that if $\mathfrak{B}$ has a unit P, then $(\mathfrak{B}^+)_e = \mathfrak{B}_e + (I-P)\mathbb{T}$. Note further that if $\mathfrak{B}$ is finite or prime or these combinations like continuous functions over a space with values in a prime algebra, then elements of $(\mathfrak{B}^+)_e$ are unitaries or isometries or co-isometries; and then $\mathfrak{B}$ is always embedded in $\mathfrak{A}$ preserving extreme points.

Let $\mathfrak{A} = \varinjlim \mathfrak{A}_j$ be an inductive limit of a net of C^*-algebras $(\mathfrak{A}_j)_{j\in J}$. We say that the limit is preserving extreme points if each $\mathfrak{A}_j$ is embedded in $\mathfrak{A}$ preserving extreme points.

Remark. Let $V + (I-P)z \in \mathfrak{B}_e + (I-P)\mathbb{T}$. Then

$$(V^* + (I-P)\bar{z})(V + (I-P)z) = V^*V + (I-P) = I$$

since $V = PV$ and $V^* = V^*P$. Check that

$$\begin{aligned}&(I - (V + (I-P)z)(V + (I-P)z)^*\mathfrak{B}^+(I - (V + (I-P)z)^*(V + (I-P)z)\\ &= (I - \{VV^* + (I-P)\})\mathfrak{B}^+(I - \{V^*V + (I-P)\})\\ &= (P - VV^*)(\mathfrak{B} + \mathbb{C}I)(P - V^*V)\\ &= (P - VV^*)\mathfrak{B}(P - V^*V) + \mathbb{C}(P - VV^*)P(P - V^*V) = 0.\end{aligned}$$

Hence $\mathfrak{B}_e + (I-P)\mathbb{T} \subset (\mathfrak{B}^+)_e$. Note also that $\mathfrak{B}^+ = \mathfrak{B} + \mathbb{C}(I-P)$. If $W = V + \mu(I-P) \in (\mathfrak{B}^+)_e$, then $(I - WW^*)\mathfrak{B}^+(I - W^*W) = \{0\}$, so that it follows from the same computations above that $V \in \mathfrak{B}_e$ and $|\mu| = 1$.

Remark. Note that if $\mathfrak{B}$ is prime, so is $\mathfrak{B}^+$. Let $W \in \mathfrak{B}_e^+$ and let $\mathfrak{B}^+(I - WW^*)\mathfrak{B}^+$ and $\mathfrak{B}^+(I - W^*W)\mathfrak{B}^+$ be the closed ideals of $\mathfrak{B}^+$ generated by $I - WW^*$ and $I - W^*W$ respectively. If both of the closed ideals are non-zero, we must have

$$\mathfrak{B}^+(I - WW^*)\mathfrak{B}^+\mathfrak{B}^+(I - W^*W)\mathfrak{B}^+ \neq \{0\}$$

since $\mathfrak{B}^+$ is prime, but $(I - WW^*)\mathfrak{B}^+(I - W^*W) = \{0\}$. Thus, we have $\mathfrak{B}^+(I - WW^*)\mathfrak{B}^+ = \{0\}$ or $\mathfrak{B}^+(I - W^*W)\mathfrak{B}^+ = \{0\}$. It follows that $WW^* = I$ or $W^*W = I$.

Remark. As checked above, $\mathfrak{B}_e^+ = \mathfrak{B}_e + (I-P)\mathbb{T}$ and if $W = V + (I-P)z \in \mathfrak{B}_e^+$, then $W^*W = V^*V + (I-P)$ and $WW^* = VV^* + (I-P)$. If $\mathfrak{B}$ is prime in the sense that every extreme point is unitary (or in the sense that $\mathfrak{B}$ contains no proper isometries), then $V^*V = P$ and $VV^* = P$, so that $W^*W = I$ and $WW^* = I$.

Proposition 1.5.2. *If $\mathfrak{A} = \varinjlim \mathfrak{A}_j$ is an inductive limit of extremally rich C^*-algebras $\mathfrak{A}_j$ preserving extreme points, then $\mathfrak{A}$ is extremally rich.*

Proof. For the unital case, note that $\cup(\mathfrak{A}_j)_q^{-1} \subset \mathfrak{A}_q^{-1}$ by preserving extreme points, and any element of $\mathfrak{A}$ can be approximated arbitrary by elements of $\cup \mathfrak{A}_j$.

For the non-unital case, note that if $\mathfrak{A}_j$ is unital and extremally rich, so is $\mathfrak{A}_j^+ = \mathfrak{A}_j + \mathbb{C}I$, which can be shown as checked before in showing the invariance of being extremally rich for taking hereditary C^*-subalgebas. $\square$

Example 1.5.3. Let $H = l^2(\mathbb{Z})$ and P the projection of H onto $l^2(\mathbb{N})$. Let

$$\mathfrak{A}_n = (\oplus^n \mathbb{B}(H)) \oplus \mathfrak{D}, \quad \mathfrak{D} = \mathbb{B}(PH) \oplus \mathbb{B}((I-P)H),$$

where elements of $\mathfrak{D}$ are regarded as block diagonal operators on H. We may consider each $\mathfrak{A}_n$ as embedded in $\mathbb{B}(H) \otimes c_1(\mathbb{N})$ the algebra of all convergent sequences of $\mathbb{B}(H)$, writing every element $T \in \mathfrak{A}_n$ as an eventually constant sequence: $T = (T_1, T_2, \cdots, T_n, D, D, \cdots)$ with $T_j \in \mathbb{B}(H)$ and $D \in \mathfrak{D}$. Note that the embedding $\mathfrak{A}_n \to \mathfrak{A}_{n+1}$ obtained in this way is not p. e. p., because the element:

$$V = (I, \cdots, I, D, D, \cdots), \quad D = S \oplus T^* \in \mathfrak{D}$$

for S and T non-unitary isometries on PH and $(I-P)H$ respectively, is extremal in $\mathfrak{A}_n$, but not in $\mathfrak{A}_{n+1}$, since $\mathfrak{D}$ is not an extreme partial isometry in $\mathbb{B}(H)$. Indeed, check that $(I - SS^*)\mathbb{B}(PH)(I - S^*S) = \{0\}$ and $(I - TT^*)\mathbb{B}((I-P)H)(I - T^*T) = \{0\}$, from which $D = S \oplus T^*$ is extremal in $\mathbb{B}(PH) \oplus \mathbb{B}((I-P)H)$, and also

$$\begin{aligned}
&I - \begin{pmatrix} S & 0 \\ 0 & T^* \end{pmatrix} \begin{pmatrix} S^* & 0 \\ 0 & T \end{pmatrix} = \begin{pmatrix} P - SS^* & 0 \\ 0 & 0 \end{pmatrix}, \\
&I - \begin{pmatrix} S^* & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & T^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I - P - TT^* \end{pmatrix}, \\
&\begin{pmatrix} P - SS^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} PAP & PA(I-P) \\ (I-P)AP & (I-P)A(I-P) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I - P - TT^* \end{pmatrix} \\
&= \begin{pmatrix} 0 & (P - SS^*)A(I - P - TT^*) \\ 0 & 0 \end{pmatrix}, \quad A \in \mathbb{B}(H).
\end{aligned}$$

Since there is a non-zero operator from $(I - P - TT^*)H$ to $(P - SS^*)H$ in $\mathbb{B}((I-P)H, PH)$, the diagonal operator $D = S \oplus T^*$ is not extremal in $\mathbb{B}(H)$.

The inductive limit $\mathfrak{A} = \varinjlim \mathfrak{A}_n$ of $\mathfrak{A}_n$ is just the C^*-algebra considered in the previous example, and it contains an element which does not belong to the closure of $\mathfrak{A}_q^{-1}$, so that $\mathfrak{A}$ is not extremally rich, even though each $\mathfrak{A}_n$ certainly is, being a von Neumann algebra.

Proposition 1.5.4. *If $\mathfrak{A} = \varinjlim \mathfrak{A}_j$ is an inudctive limit of extremally rich C^*-algebras such that each $\mathfrak{A}_j$ is a hereditary C^*-subalgebra of $\mathfrak{A}$, then $\mathfrak{A}$ is extremally rich.*

Proof. Except in the trivial case, $\mathfrak{A}_j$ does not contain the unit for $\mathfrak{A}$. Thus we consider $\mathfrak{A}^+$ and $\mathfrak{A}_j^+ = \mathfrak{A}_j + \mathbb{C}I$. If $U \in (\mathfrak{A}_j^+)_e$ of the form $U = \theta I + A$, with $|\theta| = 1$ and $A \in \mathfrak{A}_j$, then $I - UU^* \in \mathfrak{A}_j$ and $I - U^*U \in \mathfrak{A}_j$. Indeed, note that

$$UU^* = I + \theta A^* + \bar{\theta} A + AA^*, \quad U^*U = I + \bar{\theta} A + \theta A^* + A^*A.$$

Since $I - UU^*$ and $I - U^*U$ are projections of $\mathfrak{A}_j$, we have

$$\begin{aligned}(I - UU^*)\mathfrak{A}(I - U^*U) &= (I - UU^*)^2\mathfrak{A}(I - U^*U)^2 \\ &\subset (I - UU^*)\mathfrak{A}_j(I - U^*U) = \{0\},\end{aligned}$$

because $\mathfrak{A}_j\mathfrak{A}\mathfrak{A}_j \subset \mathfrak{A}_j$ being hereditary. Thus $U \in \mathfrak{A}_e$, which shows that the inductive limit is preserving extreme points. Hence $\mathfrak{A}$ is extremally rich. $\square$

Corollary 1.5.5. *Let $\mathbb{K}$ be the C^*-algebra of all compact operators on l^2. If $\mathfrak{A}$ is an extremally rich C^*-algebra, then so is $\mathfrak{A} \otimes \mathbb{K}$.*

Proof. Assume that $\mathfrak{A}$ is unital. Since $\mathfrak{A}$ is extremally rich, so is every $M_n(\mathfrak{A})$. Note that $M_n(\mathfrak{A}) \cong 1 \otimes p(\mathfrak{A} \otimes \mathbb{K})1 \otimes p$ a hereditary C^*-subalgebra of $\mathfrak{A} \otimes \mathbb{K}$, where p is the canonical projection of rank k. It follows from $\mathfrak{A} \otimes \mathbb{K} = \varinjlim M_n(\mathfrak{A})$ that $\mathfrak{A} \otimes \mathbb{K}$ is extremally rich. Even if $\mathfrak{A}$ is non-unital, we have $(\mathfrak{A} \otimes M_n(\mathbb{C}))(\mathfrak{A} \otimes \mathbb{K})(\mathfrak{A} \otimes M_n(\mathbb{C})) \subset \mathfrak{A} \otimes M_n(\mathbb{C})$, which is equivalent to say that $M_n(\mathfrak{A}) \cong \mathfrak{A} \otimes M_n(\mathbb{C})$ is hereditary in $\mathfrak{A} \otimes \mathbb{K}$. $\square$

Lemma 1.5.6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with separable C^*-subalgebras $\mathfrak{A}_0$ and $\mathfrak{B}_0$. Let $\mathfrak{X}$ be an $\mathfrak{A}$-$\mathfrak{B}$ imprimitivity bimodule. If $\mathfrak{A}$ is extremally rich, there are separable C^*-subalgebras $\mathfrak{A}_\infty$ and $\mathfrak{B}_\infty$, with $\mathfrak{A}_0 \subset \mathfrak{A}_\infty \subset \mathfrak{A}$ and $\mathfrak{B}_0 \subset \mathfrak{B}_\infty \subset \mathfrak{B}$ and a separable subspace $\mathfrak{X}_\infty$ of $\mathfrak{X}$ which is an $\mathfrak{A}_\infty$-$\mathfrak{B}_\infty$ imprimitivity bimodule, such that $\mathfrak{A}_\infty$ is extremally rich and $\mathfrak{J}\mathfrak{B}\mathfrak{K} = 0$ for any pari $\mathfrak{J}, \mathfrak{K}$ of orthogonal ideals in $\mathfrak{B}_\infty$.*

Proof. By convention we let $\mathfrak{A}^+ = \mathfrak{A}$ if $I \in \mathfrak{A}$, otherwise $\mathfrak{A}^+ = \mathfrak{A} + \mathbb{C}I$. By a recursive construction, we are going to define increasing sequences

$(\mathfrak{A}_n)$, $(\mathfrak{B}_n)$, and $(\mathfrak{X}_n)$ of separable C^*-subalgebras and submodules such that, among other things,

$$\mathfrak{A}_n \subset \mathrm{span}\langle \mathfrak{X}_n, \mathfrak{X}_n\rangle_{\mathfrak{A}} \subset \mathfrak{A}_{n+1}, \quad \mathfrak{B}_n \subset \mathrm{span}\langle \mathfrak{X}_n, \mathfrak{X}_n\rangle_{\mathfrak{B}} \subset \mathfrak{B}_{n+1},$$

and $\mathfrak{A}_n\mathfrak{X}_n + \mathfrak{X}_n\mathfrak{B}_n \subset \mathfrak{X}_{n+1}$ for all n. As the first step, choose a separable subspace $\mathfrak{X}_0$ of $\mathfrak{X}$ such that $\mathfrak{A}_0 \subset \mathrm{span}\langle \mathfrak{X}_0, \mathfrak{X}_0\rangle_{\mathfrak{A}}$ and $\mathfrak{B}_0 \subset \mathrm{span}\langle \mathfrak{X}_0, \mathfrak{X}_0\rangle_{\mathfrak{B}}$. We may assume that $I \in \mathfrak{A}_n$ for all n. Recall that the imprimitivity for $\mathfrak{X}$ is that

$$\mathrm{span}\langle \mathfrak{X}, \mathfrak{X}\rangle_{\mathfrak{A}} = \mathfrak{A} \quad \text{and} \quad \mathrm{span}\langle \mathfrak{X}, \mathfrak{X}\rangle_{\mathfrak{B}} = \mathfrak{B}.$$

Since $\mathfrak{A}$ is extremally rich and $\mathfrak{A}_n$ is separable, we can choose a separable C^*-subalgebra $\mathfrak{A}_{n+1}$ of $\mathfrak{A}$ such that the norm closure of $\mathfrak{A}_{n+1}^+ \cap (\mathfrak{A}^+)_q^{-1}$ contains $\mathfrak{A}_n$. Indeed, assume that $\{x_k\}$ is a countable dense subset of $\mathfrak{A}_n$. Define $\mathfrak{A}_{n+1}$ to be the C^*-algebra generated by the set $\{x_k\}$ and the set $\{b_{k,n} \mid k \in \mathbb{N}, n \in \mathbb{N}\}$, where each $b_{k,n} \in (\mathfrak{A}^+)_q^{-1}$ satisfies $\|x_k - b_{k,n}\| < 1/n$ for every k, n.

Enlarging if necessary $\mathfrak{A}_{n+1}$ by a separable subset we may also assume that $\langle \mathfrak{X}_n, \mathfrak{X}_n\rangle_{\mathfrak{A}} \subset \mathfrak{A}_{n+1}$.

Since $\mathfrak{B}_n$ is separable, its primitive ideal space $\mathrm{Prim}(\mathfrak{B}_n)$ is second countable, and we can choose a countable open basis (Ω_n) for $\mathrm{Prim}(\mathfrak{B}_n)$. For each disjoint pair (Γ, Δ) in $(\Omega_n) \times (\Omega_n)$ we let $(\mathfrak{J}(\Gamma), \mathfrak{J}(\Delta))$ be the corresponding pair of closed, orthogonal ideals of $\mathfrak{B}_n$. Indeed, as for their complements, $\Gamma^c \cup \Delta^c = \mathrm{Prim}(\mathfrak{B}_n)$. By the hull-kernel topology definition, $\mathrm{hull}(\mathfrak{J}(\Gamma)) = \Gamma^c$ and $\mathrm{hull}(\mathfrak{J}(\Delta)) = \Delta^c$, and therefore, $\mathfrak{J}(\Gamma)\mathfrak{J}(\Delta) = \ker(\Gamma^c)\ker(\Delta^c) = \ker(\mathrm{Prim}(\mathfrak{B}_n)) = \{0\}$, i.e., being orthogonal.

Let $\mathfrak{K}(\Gamma)$ and $\mathfrak{K}(\Delta)$ denote the closed ideals of $\mathfrak{B}$ generated by $\mathfrak{J}(\Gamma)$ and $\mathfrak{J}(\Delta)$, respectively. If $\mathfrak{K}(\Gamma) \cap \mathfrak{K}(\Delta) \neq 0$, choose $B_{\Gamma,\Delta} \in \mathfrak{B}$ such that $\mathfrak{J}(\Gamma)B_{\Gamma,\Delta}\mathfrak{J}(\Delta) \neq 0$. Indeed, note that

$$\mathfrak{K}(\Gamma)\mathfrak{K}(\Delta) = (\mathfrak{B}\mathfrak{J}(\Gamma)\mathfrak{B})(\mathfrak{B}\mathfrak{J}(\Delta)\mathfrak{B}) = \mathfrak{B}(\mathfrak{J}(\Gamma)\mathfrak{B}\mathfrak{J}(\Delta))\mathfrak{B},$$

and thus, if $\mathfrak{J}(\Gamma)\mathfrak{B}\mathfrak{J}(\Delta) = 0$, then so is $\mathfrak{K}(\Gamma)\mathfrak{K}(\Delta) = 0$. Let $\mathfrak{B}_{n+1}$ be the C^*-subalgebra of $\mathfrak{B}$ generated by $\mathfrak{B}_n$ together with the (at most countable) set of such new elements, and also a countable dense set from $\langle \mathfrak{X}_n, \mathfrak{X}_n\rangle_{\mathfrak{B}}$.

Finally choose $\mathfrak{X}_{n+1}$ as a separable subspace of $\mathfrak{X}$ large enough so that

$$\mathfrak{A}_{n+1} \subset \mathrm{span}\langle \mathfrak{X}_{n+1}, \mathfrak{X}_{n+1}\rangle_{\mathfrak{A}}, \quad \mathfrak{B}_{n+1} \subset \mathrm{span}\langle \mathfrak{X}_{n+1}, \mathfrak{X}_{n+1}\rangle_{\mathfrak{B}},$$

and $\mathfrak{A}_n\mathfrak{X}_n + \mathfrak{X}_n\mathfrak{B}_n \subset \mathfrak{X}_{n+1}$.

By induction, for every n: the norm closure of $\mathfrak{A}_{n+1}^+ \cap (\mathfrak{A}^+)_q^{-1}$ contains $\mathfrak{A}_n^+$, and if $\mathfrak{J}$ and $\mathfrak{K}$ are orthogonal closed ideals of $\mathfrak{B}_n$, such that $\mathfrak{J}\mathfrak{B}\mathfrak{K} \neq 0$, then $\mathfrak{J}\mathfrak{B}_{n+1}\mathfrak{K} \neq 0$.

Let $\mathfrak{A}_\infty$, $\mathfrak{B}_\infty$, and $\mathfrak{X}_\infty$ be the closures of the unions $\cup_n \mathfrak{A}_n$, $\cup_n \mathfrak{B}_n$, and $\cup_n \mathfrak{X}_n$, respectively. It follows that X_∞ is an $\mathfrak{A}_\infty$-$\mathfrak{B}_\infty$ imprimitivity bimodule, with $\mathrm{span}\langle \mathfrak{X}_\infty, \mathfrak{X}_\infty \rangle_{\mathfrak{A}} = \mathfrak{A}_\infty$ and $\mathrm{span}\langle \mathfrak{X}_\infty, \mathfrak{X}_\infty \rangle_{\mathfrak{B}} = \mathfrak{B}_\infty$, and that $\mathfrak{A}_\infty^+ \cap (\mathfrak{A}^+)_q^{-1}$ is dense in $\mathfrak{A}_\infty^+$, which impies that $\mathfrak{A}_\infty$ is extremally rich. Note that any irreducible representation of a C^*-subalgebra $\mathfrak{A}_\infty$ is extended to that of $\mathfrak{A}$, so that using one of criterions obtained before for an element to belong to $(\mathfrak{A}^+)_q^{-1}$ we see that $\mathfrak{A}_\infty^+ \cap (\mathfrak{A}^+)_q^{-1} \subset (\mathfrak{A}_\infty^+)_q^{-1}$.

Finally, if $\mathfrak{I}$ and $\mathfrak{K}$ are orthogonal ideals in $\mathfrak{B}_\infty$, then with $\mathfrak{I}_n = \mathfrak{I} \cap \mathfrak{B}_n$ and $\mathfrak{K}_n = \mathfrak{K} \cap \mathfrak{B}_n$, we have a pair of orthogonal ideals in $\mathfrak{B}_n$. If $\mathfrak{I}_n \mathfrak{B} \mathfrak{K}_n \neq 0$, then we have $\mathfrak{I}_{n+1}\mathfrak{K}_{n+1} = \mathfrak{I}\mathfrak{B}_{n+1}\mathfrak{K} \neq 0$, a contradiction. Thus $\mathfrak{I}_n \mathfrak{B} \mathfrak{K}_n = 0$ for all n, whence $\mathfrak{I}\mathfrak{B}\mathfrak{K}$, as claimed. □

Theorem 1.5.7. *If $\mathfrak{A}$ and $\mathfrak{B}$ are strongly Morita equivalent C^*-algebras and $\mathfrak{A}$ is extremally rich, then so is $\mathfrak{B}$.*

Proof. By assumption there is an $\mathfrak{A}$-$\mathfrak{B}$ imprimitivity bimodule $\mathfrak{X}$. Given $B \in \mathfrak{B}$, let $\mathfrak{B}_0 = C^*(B)$ the C^*-algebra generated by B and put $\mathfrak{A}_0 = 0$. Using the lemma just above we find $\mathfrak{A}_\infty$, $\mathfrak{B}_\infty$, and $\mathfrak{X}_\infty$ as described there. Since $\mathfrak{X}_\infty$ is an $\mathfrak{A}_\infty$-$\mathfrak{B}_\infty$ imprimitivity bimodule, $\mathfrak{A}_\infty$ and $\mathfrak{B}_\infty$ are strongly Morita equivalent. It follows that $\mathfrak{A}_\infty \otimes \mathbb{K}$ and $\mathfrak{B}_\infty \otimes \mathbb{K}$, both being separable, are isomorphic, i.e., $\mathfrak{A}_\infty$ and $\mathfrak{B}_\infty$ are stably isomorphic. Since $\mathfrak{A}_\infty$ is extremally rich, so is $\mathfrak{A}_\infty \otimes \mathbb{K}$, thus also $\mathfrak{B}_\infty \otimes \mathbb{K}$. Since $\mathfrak{B}_\infty$ is a hereditary C^*-subalgebra of $\mathfrak{B}_\infty \otimes \mathbb{K}$, it follows that $\mathfrak{B}_\infty$ is extremally rich. Thus $I + B$ can be approximated by elements from $(\mathfrak{B}_\infty^+)_q^{-1}$. Note now that if $U \in (\mathfrak{B}_\infty^+)_e$, then U is a partial isometry satisfying $(I - UU^*)\mathfrak{B}_\infty^+(I - U^*U) = 0$.

By construction of $\mathfrak{B}_\infty$ it follows that also

$$(I - UU^*)\mathfrak{B}^+(I - U^*U) = 0,$$

whence $U \in \mathfrak{B}_e^+$. Indeed, let $\mathfrak{I}_1 = \mathfrak{B}_\infty^+(I - UU^*)\mathfrak{B}_\infty^+$ and $\mathfrak{I}_2 = \mathfrak{B}_\infty^+(I - U^*U)\mathfrak{B}_\infty^+$, and $\mathfrak{K}_1 = \mathfrak{B}^+(I - UU^*)\mathfrak{B}^+$ and $\mathfrak{K}_2 = \mathfrak{B}^+(I - U^*U)\mathfrak{B}^+$. Then we have $\mathfrak{I}_1\mathfrak{I}_2 = 0$. Now assume that $(I - UU^*)\mathfrak{B}^+(I - U^*U) \neq 0$. Then it follows that $\mathfrak{K}_1\mathfrak{B}^+\mathfrak{K}_1 \neq 0$, from which there must exist an element $B \in \mathfrak{B}_\infty^+$ with $\mathfrak{I}_1 B \mathfrak{I}_2 \neq 0$, as shown before, so that $\mathfrak{I}_1\mathfrak{I}_2 \neq 0$, a contradiction. Consequently, $(\mathfrak{B}_\infty^+)_q^{-1} \subset (\mathfrak{B}^+)_q^{-1}$ and thus $I + B$ is in the norm closure of $(\mathfrak{B}^+)_q^{-1}$. Hence $\mathfrak{B}$ is extremally rich. □

Corollary 1.5.8. *If $\mathfrak{A}$ and $\mathfrak{B}$ are strongly Morita equivalent C^*-algebras and $\mathfrak{A}$ has stable rank one, then so does $\mathfrak{B}$.*

Proof. Replace the term extremally rich by stable rank one in those lemma and theorem just above. Note that stable rank one is preserved under stable isomorphism. □

1.6 Extensions and extremal richness

Theorem 1.6.1. *Let $\mathfrak{I}$ be a closed ideal in a unital C^*-algebra $\mathfrak{A}$. Assume that both $\mathfrak{I}$ and the quotient $\mathfrak{A}/\mathfrak{I}$ are extremally rich. Then the following conditions are equivalent:*

(i) *$\mathfrak{A}$ is extremally rich.*

(ii) *Extreme partial isometries lift, i.e., $(\mathfrak{A}/\mathfrak{I})_e = \mathfrak{A}_e/\mathfrak{I}$, and $P\mathfrak{A}Q$ is extremally rich for each defect projection P from $\mathfrak{A}_e$ and each defect projection Q from $\mathfrak{I}_e^+$. Note that $P\mathfrak{A}Q = P\mathfrak{I}Q$, since $Q \in \mathfrak{I}$.*

(iii) *For every $E \in (\mathfrak{A}/\mathfrak{I})_e$ there is a lift $U \in \mathfrak{A}_e$, i.e., $E = U + \mathfrak{I}$, such that $(I - UU^*)\mathfrak{A}Q_1$ and $P_1\mathfrak{A}(I - U^*U)$ are extremally rich for all defect projections P_1 and Q_1 from $\mathfrak{I}_e^+$.*

(iv) *$(\mathfrak{A}/\mathfrak{I})_e = \mathfrak{A}_e/\mathfrak{I}$ and $\mathfrak{A}_e + \mathfrak{I} \subset \mathrm{cl}(\mathfrak{A}_q^{-1})$, where $\mathrm{cl}(\cdot)$ means the norm closure.*

Proof. As for (ii), check that if $Q = I - UU^*$ for $U = \theta I + V \in \mathfrak{I}_e^+$ with $|\theta| = 1$ and $V \in \mathfrak{I}$, then $Q = -\theta V^* - \bar{\theta} V - VV^* \in \mathfrak{I}$.

(i) $\Rightarrow$ (ii). If $\mathfrak{A}$ is extremally rich, then so is $P\mathfrak{A}Q$ for any choice of defect projections P and Q. Let $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{I}$ denote the quotient map and consider $T \in \mathfrak{A}$ such that $\pi(T) \in (\mathfrak{A}/\mathfrak{I})_e$. If $T = V|T|$ is the polar decomposition of T in $\mathfrak{A}''$, and f is a continuous function on $\mathbb{R}_+$ vanishing in a neighbourhood of zero, such that $f(1) = 1$, then

$$Vf(|T|) = Uf(|T|) = f(|T^*|)U$$

for some $U \in \mathfrak{A}_e$. Since the defect projections of $\pi(T)$ are centrally orthogonal,

$$\begin{aligned} \pi(U) &= \pi(U|T|) + \pi(U(I - |T|)) \\ &= \pi(Uf(|T|)) + \pi(|T^*|U(I - |T|)) \\ &= \pi(Vf(|T|)) + \pi(f(|T^*|)U(I - |T|)) \\ &= \pi(V|T|) + \pi(V|T|(I - |T|)) = \pi(T), \end{aligned}$$

so that U is a lift of $\pi(T)$. Check that $\pi(I - TT^*)\pi(U)\pi(I - T^*T) = 0$ implies that

$$\pi(U(I - T^*T)) = \pi(U(I - |T|^2)) = \pi(|T^*|^2U(I - |T|^2)),$$

and since $I - |T|^2 = (I - |T|)(I + |T|)$ and $I + |T|$ is invertible, we have $\pi(U(I - |T|)) = \pi(|T^*|^2U(I - |T|))$, and moreover, since $\pi(TT^*)$ is a projection, we have $\pi(|T^*|^2) = \pi(TT^*)^2$, which implies $\pi(|T^*|) = \pi(TT^*)$.

(ii) $\Rightarrow$ (iii). This is immediate, because $I - UU^*$ and $I - U^*U$ are defect projections from $\mathfrak{A}_e$.

(iii) $\Rightarrow$ (iv). Given $U \in \mathfrak{A}_e$ and $K \in \mathfrak{J}$, we let $P = I - UU^*$ and $Q = I - U^*U$ be the defect projections. Replacing if necessary K with another element from $\mathfrak{J}$ we may assume that P and Q have the properties specified in the condition (iii). Now we have

$$\begin{aligned}
U + K &= (I - P + P)(U + K)(I - Q + Q) \\
&= (I - P)(U + K)U^*U + (I - P)(U + K)Q + P(U + K) \\
&= (I - P)(U + KU^*U) + (I - P)UQ + (I - P)KQ + PU + PK \\
&= (I - P)(I + KU^*)U + UQ + KQ - PKQ + PU + PK \\
&= (I - P)(I + KU^*)(I - P)U + PK + KQ,
\end{aligned}$$

where $U = (I - P)U$, $UQ = 0$, and $PU = 0$, and $PKQ = 0$ since $U \in \mathfrak{A}_e$. Since $(I-P)\mathfrak{J}^+(I-P)$ is extremally rich because $\mathfrak{J}$ is so, we can approximate the first term by an element of the form HWU, where $H \in ((I - P)\mathfrak{J}^+(I - P))_+^{-1}$ and $W \in ((I - P)\mathfrak{J}^+(I - P))_e$. Thus

$$\begin{aligned}
(H + P)^{-1}(U + K) &\approx (H^{-1} + P)HWU + (H^{-1} + P)PK + (H + P)^{-1}KQ \\
&= (I - P)WU + PK + (H + P)^{-1}KQ \\
&= WU + PK + K_1Q,
\end{aligned}$$

with $K_1 = (H + P)^{-1}K$ in $\mathfrak{J}$, where $(H + P)^{-1} = H^{-1} + P$.

Since $W + P \in I + \mathfrak{J}$, the defect projections of W are the same as those for the extreme point $W + P$ in $\mathfrak{J}^+$, namely

$$P_1 = I - P - WW^* \quad \text{and} \quad Q_1' = I - P - W^*W.$$

Note that for some $B \in \mathfrak{J}$ and $\lambda \in \mathbb{T}$,

$$W = (I - P)(B + \lambda I)(I - P) = (I - P)B(I - P) + \lambda(I - P),$$

so that $\bar{\lambda}W \in I - P + \mathfrak{J}$, and we may replace W with $\bar{\lambda}W$, and also

$$(W + P)(W + P)^* = WW^* + WP^* + PW + P = WW^* + P$$

since $PW = 0 = WP^*$, and similarly, $(W + P)^*(W + P) = W^*W + P$, and

$$P_1\mathfrak{J}^+Q_1' = P_1(I - P)\mathfrak{J}^+(I - P)Q_1' + P_1P\mathfrak{J}^+Q_1' + P_1\mathfrak{J}^+PQ_1' = 0.$$

We define $Q_1 = U^*Q_1'U = I - Q - (WU)^*(WU)$, and note that it is a defect projection of the extreme point $Q + U^*WU \in \mathfrak{J}_e^+$. Check that

$$\begin{aligned}(Q + U^*WU)(Q + U^*WU)^* &= QQ^* + QU^*W^*U \\ &\quad + U^*WUQ^* + U^*WUU^*W^*U \\ &= Q^2 + U^*W(I-P)W^*U \\ &= Q + (W^*U)^*W^*U \equiv I - Q_1'', \\ (Q + U^*WU)^*(Q + U^*WU) &= Q^*Q + U^*W^*UU^*WU \\ &= Q + (WU)^*WU = I - Q_1,\end{aligned}$$

and

$$\begin{aligned}Q_1''\mathfrak{J}^+Q_1 &= Q_1''(I-Q)\mathfrak{J}^+(I-Q)Q_1 + Q_1''Q\mathfrak{J}^+Q_1 + Q_1''\mathfrak{J}^+QQ_1 \\ &= Q_1''U^*U\mathfrak{J}^+U^*UQ_1 \\ &= (U^*U + (W^*U)^*W^*UU^*U)\mathfrak{J}^+(U^*U + (WU)^*WU) \\ &= U^*(I + WW^*)UU^*U\mathfrak{J}^+U^*UU^*(I + W^*W)U \\ &= U^*(I + WW^*)(I-P)U\mathfrak{J}^+U^*(I-P)(I + W^*W)U = \{0\},\end{aligned}$$

where note that $U\mathfrak{J}^+U^* \subset \mathfrak{J} + \mathbb{C}(I-P) \subset \mathfrak{J}^+ + \mathbb{C}P$ for the last equality.

Note also that both P_1 and Q_1' belong to $\mathfrak{J}$, because $(W+P)(W+P)^*, (W+P)^*(W+P) \in I + \mathfrak{J}$, and also $Q_1 = U^*Q_1'U \in \mathfrak{J}$, so that

$$\begin{aligned}P_1\mathfrak{A}Q_1 &= P_1(P_1\mathfrak{A}Q_1)Q_1 \subset P_1\mathfrak{J}Q_1 \\ &= (I - P - WW^*)\mathfrak{J}U^*(I - P - W^*W)U \\ &\subset (I - P - WW^*)\mathfrak{J}(I - P - W^*W)U = \{0\}\end{aligned}$$

since $W \in ((I-P)\mathfrak{J}^+(I-P))_e$.

Since $PKU^*W^* \in P\mathfrak{A}(I-P)$ and $U^*W^*K_1Q \in (I-Q)\mathfrak{A}Q$, with $U^* = (I-Q)U^*$, and thus nilpotent (under the matrix decompositions with respect to $P + (I-P)$ and $Q + (I-Q)$), both elements $I - PKU^*W^*$ and $I - U^*W^*K_1Q$ are invertible. We let

$$\begin{aligned}T &= (I - PKU^*W^*)(WU + PK + K_1Q)(I - U^*W^*K_1Q) \\ &= (WU + PK + K_1Q - PK(I - Q - Q_1))(I - U^*W^*K_1Q) \\ &\quad (\text{with} \quad U^*W^*WU = I - Q - Q_1, \qquad U^*W^*PK = 0, \quad \text{and} \\ &\qquad PKU^*W^*K_1Q = (I - UU^*)KU^*W^*K_1(I - U^*U) = 0), \\ &= WU + PK(Q + Q_1) + K_1Q - (I - P - P_1)K_1Q \\ &\quad (\text{with} \quad WUU^*W^*K_1Q = WW^*K_1Q = (I - P - P_1)K_1Q, \\ &\qquad K_1QU^*W^*K_1Q = 0), \\ &= WU + PK(Q + Q_1) + (P + P_1)K_1Q = WU + PKQ_1 + P_1K_1Q.\end{aligned}$$

To complete the argument that $U + K \in \mathrm{cl}(\mathfrak{A}_q^{-1})$, it suffices to show that T can be approximated by quasi-invertible elements. However, both $P\mathfrak{A}Q_1$ and $P_1\mathfrak{A}Q$ are extremally rich by assumption, since P_1 and Q_1 are defect projections from $\mathfrak{I}_e^+$, so T can be approximated by an element of the form $S = WU + R_1 + R_2$, where $R_1 \in (P\mathfrak{A}Q_1)_q^{-1}$ and $R_2 \in (P_1\mathfrak{A}Q)_q^{-1}$. To show that $S \in \mathfrak{A}_q^{-1}$, consider the following matrix decomposition:

$$((I - P - P_1) + P + P_1)S \begin{pmatrix} I - Q - Q_1 \\ Q \\ Q_1 \end{pmatrix} = \begin{pmatrix} WU & 0 & 0 \\ 0 & 0 & R_1 \\ 0 & R_2 & 0 \end{pmatrix}.$$

It is clear that S has closed range, because this holds for all three components, since W and U are partial isometries and R_1 and R_2 are quasi-invertible. Moreover,

$$\ker S^* = \ker R_1^* + \ker R_2^*, \quad \ker S = \ker R_2 + \ker R_1$$

as a kernel projection of an operator, since $(WU)^*WU = I - Q - Q_1$ and $I - P - P_1 = WW^* = W(I - P)W^* = WU(WU)^*$. Therefore,

$$\begin{aligned}(\ker S^*)\mathfrak{A}(\ker S) &= (\ker R_1^* + \ker R_2^*)\mathfrak{A}(\ker R_2 + \ker R_1) \\ &\subset (\ker R_1^*)P\mathfrak{A}Q_1(\ker R_1) + (\ker R_2^*)P_1\mathfrak{A}Q(\ker R_2) + P\mathfrak{A}Q + P_1\mathfrak{A}Q_1 = 0,\end{aligned}$$

as desired, where $\ker R_1 \leq Q_1$, $\ker R_2 \leq Q$, $\ker R_1^* \leq P$, and $\ker R_2^* \leq P_1$. It follows from the central orthogonality in $\mathfrak{A}$ by the kernel projections of S and S^* that $S \in \mathfrak{A}_q^{-1}$.

(iv) $\Rightarrow$ (i). Given $A \in \mathfrak{A}$ and $\varepsilon > 0$, find $B \in \mathfrak{A}$ such that $\|A - B\| < \varepsilon/2$ and $\pi(B) \in \pi(\mathfrak{A})_q^{-1}$, since $\mathfrak{A}/\mathfrak{I} = \pi(\mathfrak{A})$ is extremally rich. By assumption we can find $U \in \mathfrak{A}_e$ and $H \in \mathfrak{A}_+^{-1}$ such that $\pi(UH) = \pi(B)$. Thus, $B - UH \in \mathfrak{I}$. Consequently $BH^{-1} = U + K$ for some $K \in \mathfrak{I}$. Such element belongs to $\mathrm{cl}(\mathfrak{A}_q^{-1})$ by assumption. Thus there is $C \in \mathfrak{A}_q^{-1}$ with $\|BH^{-1} - C\| < \varepsilon/2\|H\|$, so that $\|B - CH\| = \|BH^{-1}H - CH\| \leq \|BH^{-1} - C\|\|H\| < \varepsilon/2$. Therefore, $\|A - CH\| < \varepsilon$ with $CH \in \mathfrak{A}_q^{-1}$, i.e. $\mathfrak{A}_q^{-1}$ is dense in $\mathfrak{A}$. □

Remark. Note that if an extreme partial isometry in $(\mathfrak{A}/\mathfrak{I})_e$ can be lifted to a quasi-invertible element in $\mathfrak{A}$, then it can also be lifted to an element in $\mathfrak{A}_e$. Thus in the conditions (ii) and (iv) it suffices to demand that quasi-invertibles lift, i.e. $(\mathfrak{A}/\mathfrak{I})_q^{-1} = \mathfrak{A}_q^{-1}/\mathfrak{I}$. Namely, we have $(\mathfrak{A}/\mathfrak{I})_e = \mathfrak{A}_e/\mathfrak{I} \Leftrightarrow (\mathfrak{A}/\mathfrak{I})_q^{-1} = \mathfrak{A}_q^{-1}/\mathfrak{I}$. Clear is $\Rightarrow$. As for $\Leftarrow$ as mentioned, let $\pi(T) \in (\mathfrak{A}/\mathfrak{I})_e$ with $\pi(T) = \pi(A)$ for some $A = U|A| \in \mathfrak{A}_q^{-1}$ with $U \in \mathfrak{A}_e$. It follows exactly by the same way as in (i) $\Rightarrow$ (ii) that $\pi(U) = \pi(T)$.

Corollary 1.6.2. *Let $\mathfrak{J}$ be a closed ideal of a C^*-algebra $\mathfrak{A}$, with stable rank one* $\mathrm{sr}(\mathfrak{J}) = 1$. *Then $\mathfrak{A}$ is extremally rich if and only if $\mathfrak{A}/\mathfrak{J}$ is extremally rich and extreme partial isometries lift.*

Proof. We may assume that $\mathfrak{A}$ is unital. Since $\mathfrak{J}^+$ has stable rank one, $\mathfrak{J}^+_e = \mathfrak{U}(\mathfrak{J}^+)$ of all unitaries of $\mathfrak{J}^+$. So there are no non-zero defect projections from $\mathfrak{J}^+_e$. Thus the result is immediate from condition (ii) in the theorem just above. $\square$

Proposition 1.6.3. *If $\mathfrak{J}$ is a closed ideal of a C^*-algebra $\mathfrak{A}$, then $\mathfrak{A}$ has stable rank one if and only if both $\mathfrak{J}$ and $\mathfrak{A}/\mathfrak{J}$ have stable rank one and invertibles lift, i.e.* $(\mathfrak{A}^+/\mathfrak{J})^{-1} = (\mathfrak{A}^+)^{-1}/\mathfrak{J}$.

Proof. If $\mathrm{sr}(\mathfrak{A}) = 1$, then we have $\mathrm{sr}(\mathfrak{A}/\mathfrak{J}) = 1 = \mathrm{sr}(\mathfrak{J})$. Having stable rank one implies being extremally rich. Note also that invertibles are quasi-invertibles. Thus invertibles lift by the remark just above.

To prove the converse we may assume that $\mathfrak{A}$ is unital. Assume that $\mathrm{sr}(\mathfrak{A}/\mathfrak{J}) = 1 = \mathrm{sr}(\mathfrak{J})$ and that invertibles lift, equivalently $\Leftrightarrow$, unitaries lift. Clear is $\Rightarrow$. As for $\Leftarrow$, note that an invertible operator $\pi(A) \in \mathfrak{A}/\mathfrak{J}$ has the polar decompositon $\pi(A) = U|\pi(A)|$ with U unitary. Since unitaries as well as positive invertibles lift, so does $\pi(A)$, because if $|\pi(A)| \geq \varepsilon\pi(I)$ for some $\varepsilon > 0$, then $\pi(|A| - \varepsilon I) \geq 0$, from which there is a positive $B \in \mathfrak{A}$ such that $\pi(B) = \pi(|A| - \varepsilon I)$, so that $\pi(B + \varepsilon I) = |\pi(A)|$ with $B + \varepsilon I \geq \varepsilon I$.

It follows from the corollary just above that $\mathfrak{A}$ is extremally rich, i.e. $\mathrm{cl}(\mathfrak{A}_q^{-1}) = \mathfrak{A}$, and we also know that for every $V \in \mathfrak{A}_e$ there is a unitary $U \in \mathfrak{A}$ such that $V - U \in \mathfrak{J}$. Then $U^*V - I \in \mathfrak{J}$, so that $U^*V \in \mathfrak{J}^+_e$. Check that

$$(I - (U^*V)(V^*U))\mathfrak{J}^+(I - V^*UU^*V) = U^*(I - VV^*)U\mathfrak{J}^+(I - V^*V) = \{0\}.$$

Since $\mathrm{sr}(\mathfrak{J}^+) = 1$, we have U^*V unitary, hence V unitary. Thus $\mathfrak{A}_e = \mathfrak{A}_u$ of all unitaries of $\mathfrak{A}$, so that $\mathfrak{A}_q^{-1} = \mathfrak{A}^{-1}$, and hence $\mathrm{sr}(\mathfrak{A}) = 1$. $\square$

Remark. It is shown by Rørdam [19] that

$$\alpha(T) = \mathrm{dist}(T + \mathfrak{J}, \mathfrak{A}^{-1} + \mathfrak{J})$$

for every $T \in \mathfrak{A}$ if $\mathrm{sr}(\mathfrak{J}) = 1$, where $\alpha(T)$ is the distance between T and $\mathfrak{A}^{-1}$. The proposition above is an immediate consequence of this formula. Indeed, if $\mathrm{sr}(\mathfrak{A}) = 1$, then $\alpha(T) = 0$ for every $T \in \mathfrak{A}$, so that $\mathrm{sr}(\mathfrak{A}/\mathfrak{J}) = 1$ by the formula, but it seems that lifting invertibles does not follow directly from this formula. However, as for the converse, for every $T \in \mathfrak{A}$, we have $T + \mathfrak{J}$ approximated by invertibles in $\mathfrak{A}/\mathfrak{J}$ which are lifted to $\mathfrak{A}$, so that $\alpha(T) = 0$ by the formula.

Corollary 1.6.4. *Every split extension of C^*-algebras with stable rank one has stable rank one.*

Proof. If $\mathrm{sr}(\mathfrak{I}) = 1 = \mathrm{sr}(\mathfrak{A}/\mathfrak{I})$, and if there is a C^*-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{B} \cap \mathfrak{I} = 0$ and $\mathfrak{B} + \mathfrak{I} = \mathfrak{A}$, then every invertible element of $\mathfrak{A}^+/\mathfrak{I}$ lifts to an invertible element of $\mathfrak{B}^+$. Since $(\mathfrak{B}^+)^{-1} \subset (\mathfrak{A}^+)^{-1}$, the result follows from the previous result. □

Remark. Check that the (injective) lifting map γ from $\mathfrak{A}/\mathfrak{I}$ to $\mathfrak{A}$ is defined by $\gamma(b + \mathfrak{I}) = b$ for $b \in \mathfrak{B}$. If $b_1 - b_2 \in \mathfrak{I}$ for $b_1, b_2 \in \mathfrak{B}$, then $b_1 = b_2$. For $b_1, b_2 \in \mathfrak{B}$ and $c_1, c_2 \in \mathfrak{I}$, we have $(b_1+c_1)(b_2+c_2) = b_1b_2+c_1c_2$. For $a \in \mathfrak{A}$, consider $a = (a - \gamma \circ \pi(a)) + \gamma \circ \pi(a)$, through which we have $\mathfrak{A} \cong \mathfrak{I} \oplus \mathfrak{B}$, where $\pi(a) = b + \mathfrak{I}$ for some $b \in \mathfrak{B}$.

Proposition 1.6.5. *Let $\mathfrak{I}$ be an extremally rich, closed ideal of a C^*-algebra $\mathfrak{A}$, such that $\mathfrak{A}/\mathfrak{I}$ has stable rank one. Then $\mathfrak{A}$ is extremally rich if invertibles lift.*

Proof. Since $\mathfrak{A}^+/\mathfrak{I}$ has stable rank one, we have $(\mathfrak{A}^+/\mathfrak{I})_e = (\mathfrak{A}^+/\mathfrak{I})_u$. By assumption, every unitary of $(\mathfrak{A}^+/\mathfrak{I})_u$ lifts to $\mathfrak{A}^+_u$. Hence $\mathfrak{A}$ is extremally rich by condition (iii) in the theorem above. □

Remark. It has been mentioned that the condition that invertibles lift in many cases is equivalent to surjectivity of the natural map from $K_1(\mathfrak{A})$ to $K_1(\mathfrak{A}/\mathfrak{I})$, equivalently, injectivity of the map from $K_0(\mathfrak{I})$ into $K_0(\mathfrak{A})$.

Corollary 1.6.6. *If $\mathfrak{I}$ is an extremally rich ideal of a C^*-algebra $\mathfrak{A}$, and $\mathfrak{B}$ a C^*-subalgebra of $\mathfrak{A}$ with stable rank one, such that $\mathfrak{A} = \mathfrak{I} + \mathfrak{B}$, then $\mathfrak{A}$ is extremally rich.*

Proof. We may assume that $\mathfrak{A}$ and $\mathfrak{B}$ are both unital, with a common unit, since adjoining $\mathbb{C}I$ to $\mathfrak{A}$ and $\mathfrak{B}$ does not change the assumption. Since $\mathrm{sr}(\mathfrak{B}) = 1$ and $\mathfrak{I} \cap \mathfrak{B}$ is a closed ideal of $\mathfrak{B}$, we have $\mathrm{sr}(\mathfrak{I} \cap \mathfrak{B}) = 1$ and that invertibles lift from $\mathfrak{B}/(\mathfrak{I} \cap \mathfrak{B})$ to $\mathfrak{B}$. As $\mathfrak{B}^{-1} \subset \mathfrak{A}^{-1}$ and $\mathfrak{A}/\mathfrak{I} \cong \mathfrak{B}/(\mathfrak{B} \cap \mathfrak{I})$, we have that invertibles lift from $\mathfrak{A}/\mathfrak{I}$ to $\mathfrak{A}$, whence $\mathfrak{A}$ is extremally rich by the proposition just above. □

Corollary 1.6.7. *Let $\mathfrak{A}$ be a C^*-algebra of the form $\mathfrak{A} = \mathfrak{I} + \mathfrak{B}$ for some closed ideal $\mathfrak{I}$ with stable rank one and some extremally rich C^*-subalgebra $\mathfrak{B} \subset \mathfrak{A}$ for which the inclusion is preserving extreme points. Then $\mathfrak{A}$ is extremally rich.*

Proof. We pass to the unitized algebras $\mathfrak{A}^+$ and $\mathfrak{B}^+$. Since $\mathfrak{J} \cap \mathfrak{B}^+$ has stable rank one and $\mathfrak{B}^+$ is extremally rich, it follows that every extreme partial isometry of $\mathfrak{A}^+/\mathfrak{J} \cong \mathfrak{B}^+/(\mathfrak{J} \cap \mathfrak{B}^+)$, being extremally rich, has the form $U + \mathfrak{J}$ for some extreme partial isometry $U \in \mathfrak{B}_e^+$. Then $U \in \mathfrak{A}_e^+$ by assumption. We have shown that all extreme partial isometies lift from $\mathfrak{A}^+/\mathfrak{J}$ to $\mathfrak{A}^+$, whence $\mathfrak{A}^+$ (and $\mathfrak{A}$) is extremally rich. □

Let $\mathfrak{T}$ denote the Toeplitz algebra on l^2, i.e. the C^*-algeba generated by the unilateral shift S. It is known that $\mathfrak{T}$ is an extension of the C^*-algebra $\mathbb{K}$ of all compact operators by $C(\mathbb{T})$ the C^*-algebra of all continuous functions on the 1-torus $\mathbb{T}$:

$$0 \longrightarrow \mathbb{K} \xrightarrow{\iota} \mathfrak{T} \xrightarrow{\rho} C(\mathbb{T}) \longrightarrow 0.$$

Let θ denote the $*$-automorphism of $C(\mathbb{T})$ of order two given by $\theta f(t) = f(t^{-1})$ for $t \in \mathbb{T}$ and $f \in C(\mathbb{T})$. Define the extended Toeplitz algebra by

$$\mathfrak{T}_\theta = \{B \oplus C \in \mathfrak{T} \oplus \mathfrak{T} \,|\, \rho(B) = \theta(\rho(C))\}.$$

It is shown by Pedersen that $\mathfrak{T}_\theta$ is isomorphic to the C^*-algebra on $l^2 \oplus l^2$ generated by $S \oplus S^*$ (which may be identified with the weighted bilateral shift T on $l^2(\mathbb{Z})$, for which $Te_n = e_{n+1}$ for all $n \neq 0$, but $Te_0 = 0$, where $\{e_n\}$ is the standard basis for $l^2(\mathbb{Z})$), and that $\mathfrak{T}_\theta$ is an extension of $\mathbb{K}\oplus\mathbb{K}$ by $C(\mathbb{T})$. Moreover, $\mathfrak{T}_\theta$ is extremally rich, and every extreme partial isometry W in $(\mathfrak{T}_\theta)_e$ has the form

$$W = U(S \oplus S^*)^n V, \quad U, V \in \mathfrak{U}(\mathfrak{T}_\theta), \quad n \in \mathbb{Z},$$

where $(S \oplus S^*)^n = (S^* \oplus S)^{-n}$ if $n < 0$. In particular, every isometry of $\mathfrak{T}_\theta$ is unitary, yet $(\mathfrak{T}_\theta)_e \neq \mathfrak{U}(\mathfrak{T}_\theta)$.

In the case of extremally rich C^*-algebras $\mathfrak{A}$,

$$\mathrm{sr}(\mathfrak{A}) = 1 \quad \Leftrightarrow \quad \mathfrak{A}_e = \mathfrak{A}_u \quad \Leftrightarrow$$

$\mathfrak{A}$ is residually finite in the sense that no infinite projections in any primitive quotients of $\mathfrak{A}$. Note that $\mathrm{sr}(\mathfrak{T}) = 2 = \mathrm{sr}(\mathfrak{T}_\theta)$.

Proposition 1.6.8. *The universal C^*-algebra generated by a partial isometry U, such that $(I - UU^*)f(U, U^*)(I - U^*U) = 0$ for any polynomial f in two variables, is isomorphic to the extended Toeplitz algebra $\mathfrak{T}_\theta$.*

Proof. Let U be any partial isometry on a Hilbert space H satisfying the condition $(I - UU^*)f(U, U^*)(I - U^*U) = 0$. Let $\mathfrak{A}$ be the C^*-algebra generated by U, and also by $f(U, U^*)$. Assume that $I - UU^* \neq 0$ and

$I - U^*U \neq 0$. Then the closed ideals $\mathfrak{J}$ and $\mathfrak{K}$ of $\mathfrak{A}$ generated by these two projections are nonzero and orthogonal. Check that

$$\mathfrak{J}\mathfrak{K} = \mathfrak{A}(I - UU^*)\mathfrak{A}(I - U^*U)\mathfrak{A} = \mathfrak{A}(I - UU^*)C^*(U, U^*)(I - U^*U)\mathfrak{A} = 0$$

since $\mathfrak{A} = C^*(U, U^*)$ is the closure of all linear spans of $f(U, U^*)$.

Let P denote the projection on the closure $\mathrm{cl}(\mathfrak{J}H)$. Then $P \in \mathfrak{A}'$ the commutant of $\mathfrak{A}$ and $0 \neq P \neq I$. Indeed, APH and A^*PH are contained in PH for any $A \in \mathfrak{A}$, so that $PAP = AP$ and $PA^*P = A^*P$, and hence $AP = PA$.

Moreover, $(I-UU^*)(I-P) = 0$ and $(I-U^*U)P = 0$, because note that $(I - UU^*)P = P(I - UU^*) = I - UU^*$ and $(I - U^*U)\mathfrak{J} = 0$. Thus PU is a non-unitary isometry on PH and $(I - P)U$ is a non-unitary co-isometry on $(I - P)H$. Check that

$$\begin{aligned}
(PU)^*(PU) &= U^*PU = U^*UP = P, \\
(PU)(PU)^* &= PUU^*P = P(I - (I - UU^*)) \\
&= P - (I - UU^*) \neq P; \\
((I - P)U)^*((I - P)U) &= U^*(I - P)U = U^*U(I - P) = 0 \neq I - P, \\
((I - P)U)((I - P)U)^* &= (I - P)UU^*(I - P) = I - P.
\end{aligned}$$

Then $P\mathfrak{A}$, being generated by PU (and PU^*), is isomorphic to the Toeplitz algebra $\mathfrak{T}$, and the isomorphism $\pi_1 : \mathfrak{T} \to P\mathfrak{A}$ can be chosen such that $\pi_1(S) = PU$. Note that $\pi_1(S^2) = PU^2 = PUPU = \pi_1(S)\pi_1(S)$ and $\pi_1(S^*) = PU^* = (PU)^* = \pi_1(S)^*$. Similarly, we can find an isomorphism $\pi_2 : \mathfrak{T} \to (I - P)\mathfrak{A}$ such that $\pi_2(S) = (I - P)U^*$. Since $\mathfrak{T}_\theta$ is a C^*-subalgebra of $\mathfrak{T} \oplus \mathfrak{T}$, this gives an injective homomorphism: $\pi : \mathfrak{T}_\theta \to P\mathfrak{A} \oplus (I - P)\mathfrak{A}$ given by $\pi(B \oplus C) = \pi_1(B) \oplus \pi_2(C)$. Note now that $\pi(S \oplus S^*) = PU \oplus (I - P)U = U$. Since $\mathfrak{T}_\theta$ is generated by $S \oplus S^*$ and $\mathfrak{A}$ is by U, it follows that π is an isomorphism, so that $\mathfrak{T}_\theta \cong \mathfrak{A}$.

If $I - UU^* = 0$ but $I - U^*U \neq 0$, then $\mathfrak{A} \cong \mathfrak{T}$, which is the quotient of $\mathfrak{T}_\theta$ by the map $\pi(S \oplus S^*) = S^*$, and if $I - U^*U = 0$ but $I - UU^* \neq 0$, we similarly get $\mathfrak{A} = \pi(\mathfrak{T}_\theta)$ by the map $\pi(S \oplus S^*) = S$. Finally, if U is unitary, then $\mathfrak{A} \cong \varphi(C(\mathbb{T}))$, where φ is the transposed map of the inclusion map $\mathrm{sp}(U) \subset \mathbb{T}$, so $\mathfrak{A} \cong \pi(\mathfrak{T}_\theta)$, where π is the composition of the quotient map $\mathfrak{T}_\theta \to \mathfrak{T}_\theta / \oplus^2 \mathbb{K}$ with φ.

We have shown that for any choice of U satisfying the condition, there is a homomorphism $\pi : \mathfrak{T}_\theta \to \mathfrak{A}$ such that $\pi(S \oplus S^*) = U$, which proves that $\mathfrak{T}_\theta$ is the universal C^*-algebra for partial isometries satisfying the condition. $\square$

Corollary 1.6.9. *Let $\mathfrak{A}$ be a unital C^*-algebra generated by a closed ideal with stable rank one and an extreme partial isometry W. Then $\mathfrak{A}$ is extremally rich.*

Proof. The C^*-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by W is a quotient $\mathfrak{T}_\theta$ being extremally rich, since $(I - WW^*)\mathfrak{A}(I - W^*W) = 0$. Thus $\mathfrak{B}$ is extremally rich. Moreover, every $W_0 \in \mathfrak{B}_e$ has the form $W_0 = UW^nV$ for some $U, V \in \mathfrak{B}_u$ and $n \in \mathbb{Z}$, where $W^n = (W^*)^{-n}$ if $n < 0$, because extreme partial isometries lift from every quotient of $\mathfrak{T}_\theta$. Thus the embedding of $\mathfrak{B}$ in $\mathfrak{A}$ is preserving extreme points, whence $\mathfrak{A}$ is extremally rich by the corollary above. □

Example 1.6.10. Let $\mathfrak{A}_0$ be an infinite matroid C^*-algebra, e.g. $\mathfrak{A}_0 = \mathfrak{F} \otimes \mathbb{K}$, where $\mathfrak{F} = \otimes^\infty M_2(\mathbb{C})$ is the Fermion algebra. Choose two isometries $U, V \in M(\mathfrak{A}_0)$, such that the defect projections $P = I - UU^*$ and $Q = I - VV^*$ satisfy $P \in \mathfrak{A}_0$, $Q \notin \mathfrak{A}_0$, and $\mathrm{tr}(P) = 1 = \mathrm{tr}(Q)$, where $\mathrm{tr}(\cdot)$ refers to the unique trace on $\mathfrak{A}_0$, extended to $M(\mathfrak{A}_0)$. Define $\mathfrak{A}_1 = C^*(\mathfrak{A}_0, U)$ and $\mathfrak{A}_2 = C^*(\mathfrak{A}_0, V)$. Let $\mathfrak{A}$ be the C^*-subalgebra of $M(\mathfrak{A}_0) \otimes M_2(\mathbb{C})$ consisting of matrices of the form:

$$\begin{pmatrix} A_1 & C \\ B & A_2 \end{pmatrix}, \quad A_i \in \mathfrak{A}_i \quad (i = 1, 2), \quad B, C \in \mathfrak{A}_0.$$

Consider the partial isometry $W \in \mathfrak{A}$, together with its defect projections:

$$W = \begin{pmatrix} U & 0 \\ 0 & V^* \end{pmatrix}, \quad I - WW^* = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}, \quad I - W^*W = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}.$$

If $\mathfrak{A}$ were extremally rich, W would have an extremal extension, and this, as we see, would imply the existence of an extreme point W_0 in the unit ball of $P\mathfrak{A}_0Q$, because note that

$$\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 & C \\ B & A_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} 0 & PCQ \\ 0 & 0 \end{pmatrix}$$

so that $(I - WW^*)\mathfrak{A}(I - W^*W) = P\mathfrak{A}_0Q$, being extremally rich. Since $\mathfrak{A}_0$ is prime (even simple) the equation $(P - W_0W_0^*)\mathfrak{A}_0(Q - W_0^*W_0) = 0$ implies $P = W_0W_0^*$ or $Q = W_0^*W_0$. Note that it follows from that equation that the $(\mathfrak{A}_0(P - W_0W_0^*)\mathfrak{A}_0)(\mathfrak{A}_0(Q - W_0^*W_0)\mathfrak{A}_0) = 0$, which implies that $\mathfrak{A}_0(P - W_0W_0^*)\mathfrak{A}_0 = 0$ or $\mathfrak{A}_0(Q - W_0^*W_0)\mathfrak{A}_0 = 0$ by primeness of $\mathfrak{A}_0$, where $P - W_0W_0^* \in \mathfrak{A}_0$ since $W_0 \in P\mathfrak{A}_0Q \subset \mathfrak{A}$, and note that the C^*-algebra generated by $\mathfrak{A}_0$ and $Q - W_0^*W_0$ is also prime. The last possibility $Q = W_0^*W_0$ is ruled out, since $Q \notin \mathfrak{A}_0$. This means, in particular, that

$$\mathrm{tr}(W_0W_0^*) = \mathrm{tr}(W_0^*W_0) < \mathrm{tr}(Q) = 1 = \mathrm{tr}(P),$$

from which we see that the first possibility $P = W_0 W_0^*$ also is impossible. We conclude that no W_0 exists, so $\mathfrak{A}$ is not extremally rich.

If $E = I \otimes e_{11}$ in $\mathfrak{A}$, then we have $E\mathfrak{A}E = \mathfrak{A}_1$ and $(I-E)\mathfrak{A}(I-E) = \mathfrak{A}_2$. Both of these C^*-algebras are extremally rich by the corollary above, yet $\mathfrak{A}$ is not. This behaviour contrasts with the positive results for stable rank one and real rank zero.

If $\mathfrak{J}$ and $\mathfrak{K}$ denote the closed ideals in $\mathfrak{A}$ generated by $\mathfrak{A}_1$ and $\mathfrak{A}_2$, then these algebras are extremally rich, being stably isomorphic to $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Now $\mathfrak{A}/\mathfrak{K} \cong \mathfrak{J}/(\mathfrak{K} \cap \mathfrak{J})$, which is extremally rich; and quasi-invertibles lift from $\mathfrak{J}^+/(\mathfrak{K} \cap \mathfrak{J})$ to $\mathfrak{J}^+$, hence to $\mathfrak{A}$, because $\mathfrak{J}$ is hereditary in $\mathfrak{A}$ (even an ideal). Note that $\mathfrak{A} = \mathfrak{J} + \mathfrak{K}$ by direct matrix computation. It follows that we can not relax the condition that $\mathfrak{J}$ having stable rank one, to that $\mathfrak{J}$ being extremally rich, in the statement that if $\mathrm{sr}(\mathfrak{J}) = 1$, then $\mathfrak{A}$ is extremally rich if and only if $\mathfrak{A}/\mathfrak{J}$ is extremally rich and extreme partial isometries lift, in the corollary above.

Note that $\mathfrak{A}_0$ is simple and $\mathfrak{A}_0 \cong \mathfrak{A}_0 \otimes M_2(\mathbb{C}) \cong \mathfrak{J} \cap \mathfrak{K}$. It follows that

$$\mathfrak{A}/\mathfrak{K} \cong \mathfrak{J}/(\mathfrak{J} \cap \mathfrak{K}) \cong \mathfrak{A}_1/\mathfrak{A}_0 \cong C(\mathbb{T}),$$

which has stable rank one; and every unitary in $\mathfrak{A}/\mathfrak{K}$ lifts to an isometry or a co-isometry in $\mathfrak{A}$. Therefore, we can not relax the condition lifting invertibles to lifting quasi-invertibles, in the statement that for an extremally rich ideal $\mathfrak{J}$ in $\mathfrak{A}$, with $\mathrm{sr}(\mathfrak{A}/\mathfrak{J}) = 1$, if invertibles lift, then $\mathfrak{A}$ is extremally rich, in the proposition above. Moreover,

$$\mathfrak{A}/\mathfrak{J} \cong \mathfrak{K}/(\mathfrak{K} \cap \mathfrak{J}) \cong \mathfrak{A}_2/\mathfrak{A}_0 = C^*(V + \mathfrak{A}_0).$$

Now both V and $V + \mathfrak{A}_0$ are proper isometries, so that $C^*(V)$ and $C^*(V + \mathfrak{A}_0)$ are both isomorphic to the Toeplitz algebra. Consequently, $C^*(V) \cap \mathfrak{J} = 0$ and $C^*(V) + \mathfrak{J} = \mathfrak{A}$, so that we have a split extension. This shows that we can not relax the conditions that $\mathfrak{A}/\mathfrak{J}$ or $\mathfrak{B}$ have stable rank one to just being extremally rich, in the statement that if $\mathfrak{A} = \mathfrak{J} + \mathfrak{B}$ for an extremally rich ideal $\mathfrak{J}$ and a C^*-subalgebra $\mathfrak{B}$ with $\mathrm{sr}(\mathfrak{B}) = 1$, then $\mathfrak{A}$ is extremally rich, in the corollary above.

2 Higher extremal richness for C^*-algebras

Abstract. We introduce a notion generalizing the extremal richness of Brown and Pedersen for C^*-algebras, and consider its basic properties.

This section is taken from the author [21], and is slightly modified.

2.1 Introduction

The extremal richness for C^*-algebras is introduced by Brown and Pedersen [3] as an infinite analogue of the (topological) stable rank of Rieffel [18] for C^*-algebras (especially for finite ones). More precisely, the extremal richness (especially for infinite C^*-algebras) is an infinite analogue of stable rank 1. Our starting point is to consider an infinite analogue of higher stable rank. To do this, we need to generalize the extremal richness again in the general setting for the higher stable rank as considered by Rieffel [18]. Namely, we introduce higher extremal richness for C^*-algebras, and consider its basic properties by borrowing some methods for extremally rich C^*-algebras from [3].

2.2 Higher extremal richness for C^*-algebras

Recall from Brown and Pedersen [3] the following:

Definition 2.2.1. Let $\mathfrak{A}$ be a unital C^*-algebra. Denote by $\mathfrak{A}_1$ the closed unit ball of $\mathfrak{A}$, i.e., $\mathfrak{A}_1 = \{a \in \mathfrak{A} \mid \|a\| \leq 1\}$. Denote by $\partial\mathfrak{A}_1$ the set of extreme points of the convex set $\mathfrak{A}_1$. Let $\mathfrak{A}^{-1}$ be the set of all invertible elements of $\mathfrak{A}$. We say that $\mathfrak{A}$ is extremally rich (or has extremal richness) if the set $\mathfrak{A}^{-1}\partial\mathfrak{A}_1\mathfrak{A}^{-1}$ of all quasi-invertible elements of $\mathfrak{A}$ is dense in $\mathfrak{A}$. For a nonunital C^*-algebra $\mathfrak{A}$, we say that $\mathfrak{A}$ is extremally rich if so is its unitization $\mathfrak{A}^+$ by $\mathbb{C}$.

Remark. It is known that elements of $\partial\mathfrak{A}_1$ are partial isometries v of $\mathfrak{A}$ such that $(1 - vv^*)\mathfrak{A}(1 - v^*v) = 0$.

We give the following:

Definition 2.2.2. Let $\mathfrak{A}$ be a unital C^*-algebra. Denote by $L_n(\mathfrak{A})$ the set of all elements $(a_j)_{j=1}^n \in \mathfrak{A}^n$ (n-direct sum) such that for each $(a_j)_{j=1}^n$, there exists an element $(b_j)_{j=1}^n \in \mathfrak{A}^n$ such that $\sum_{j=1}^n b_j a_j$ is invertible in $\mathfrak{A}$, and by $R_n(\mathfrak{A})$ the set of all elements $(x_j)_{j=1}^n \in \mathfrak{A}^n$ such that for each $(x_j)_{j=1}^n$, there exists an element $(y_j)_{j=1}^n \in \mathfrak{A}^n$ such that $\sum_{j=1}^n x_j y_j$ is invertible in $\mathfrak{A}$. We say that $\mathfrak{A}$ has extremal richness n (denoted by $\mathrm{exr}(\mathfrak{A}) = n$) if there exists the smallest positive integer n such that the set $L_n(\mathfrak{A})(\partial\mathfrak{A}_1)^n R_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$. If no such integer, let $\mathrm{exr}(\mathfrak{A}) = \infty$. Set $X_n(\mathfrak{A}) = L_n(\mathfrak{A})(\partial\mathfrak{A}_1)^n R_n(\mathfrak{A})$. For a nonunital C^*-algebra $\mathfrak{A}$, we say that $\mathfrak{A}$ has extremal richness n if so does its unitization $\mathfrak{A}^+$ by $\mathbb{C}$.

To confirm the above definition for higher extremal richness,

Lemma 2.2.3. *Let $\mathfrak{A}$ be a unital C^*-algebra. For an integer $n \geq 1$, if $X_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$, then $X_{n+1}(\mathfrak{A})$ is dense in $\mathfrak{A}^{n+1}$.*

Proof. Let $(a_j)_{j=1}^{n+1} \in \mathfrak{A}^{n+1}$. Since $(a_j)_{j=1}^{n} \in \mathfrak{A}^n$, by assumption, for $\varepsilon > 0$ there exists $(b_j)_{j=1}^{n} \in X_n(\mathfrak{A})$ such that $\|a_j - b_j\| < \varepsilon/2$ $(1 \leq j \leq n)$. Since $(b_j)_{j=1}^{n} \in X_n(\mathfrak{A})$, let $(b_j)_{j=1}^{n} = (c_j d_j f_j)_{j=1}^{n}$, where $(c_j)_{j=1}^{n} \in L_n(\mathfrak{A}), (f_j)_{j=1}^{n} \in R_n(\mathfrak{A})$ and $(d_j)_{j=1}^{n} \in (\partial\mathfrak{A}_1)^n$. Since $(b_2, \cdots, b_n, a_{n+1}) \in \mathfrak{A}^n$, there exists $(x_j)_{j=1}^{n} \in X_n(\mathfrak{A})$ such that $\|b_j - x_j\| < \varepsilon/2$ $(2 \leq j \leq n)$ and $\|a_{n+1} - x_n\| < \varepsilon/2$. Since $(x_j)_{j=1}^{n} \in X_n(\mathfrak{A})$, let $(x_j)_{j=1}^{n} = (c'_j d'_j f'_j)_{j=1}^{n}$, where $(c'_j)_{j=1}^{n} \in L_n(\mathfrak{A}), (f'_j)_{j=1}^{n} \in R_n(\mathfrak{A})$ and $(d'_j)_{j=1}^{n} \in (\partial\mathfrak{A}_1)^n$. Note that $(c_1, c'_1, \cdots, c'_n) \in L_{n+1}(\mathfrak{A})$, $(f_1, f'_1, \cdots, f'_n) \in R_{n+1}(\mathfrak{A})$. Indeed, there exist (r'_j), $(s'_j) \in \mathfrak{A}^n$ such that $\sum_{j=1}^{n} r'_j c'_j$, $\sum_{j=1}^{n} f'_j s'_j$ are invertible in $\mathfrak{A}$. We take $(0, r'_1, \cdots, r'_n)$, $(0, s'_1, \cdots, s'_n)$ for $(c_1, c'_1, \cdots, c'_n), (f_1, f'_1, \cdots, f'_n)$ to be in $L_{n+1}(\mathfrak{A})$, $R_{n+1}(\mathfrak{A})$ respectively. Thus, we obtain that $(b_1, x_1, \cdots, x_n) \in X_{n+1}(\mathfrak{A})$ such that $\|a_1 - b_1\| < \varepsilon$ and $\|a_{j+1} - x_j\| < \varepsilon$ $(1 \leq j \leq n)$. □

Corollary 2.2.4. *Let $\mathfrak{A}$ be a C^*-algebra. If $\mathfrak{A}$ is extremally rich, then $\mathfrak{A}$ has extremal richness* 1.

If $\mathfrak{A}$ has stable rank one, then $\mathfrak{A}$ is extremally rich if and only if $\mathfrak{A}$ has extremal richness 1.

Proof. Note that $\mathfrak{A}^{-1}\partial\mathfrak{A}_1\mathfrak{A}^{-1} \subset L_1(\mathfrak{A})\partial\mathfrak{A}_1 R_1(\mathfrak{A})$. Also, if $\mathfrak{A}$ has stable rank one, we in fact have $\mathfrak{A}^{-1} = L_1(\mathfrak{A}) = R_1(\mathfrak{A})$. □

Remark. This is a subtle point. It is desirable that we have such an equivalence without stable rank one, or we may expect the similar theory as for extremal richness 1, by replacing extremally richness with it.

Now recall from Rieffel [18] the following:

Definition 2.2.5. Let $\mathfrak{A}$ be a unital C^*-algebra. We say that $\mathfrak{A}$ has stable rank n (denoted by $\mathrm{sr}(\mathfrak{A}) = n$) if there exists the smallest positive integer n such that $L_n(\mathfrak{A})$ (or $R_n(\mathfrak{A})$) is dense in $\mathfrak{A}^n$. If no such integer, let $\mathrm{sr}(\mathfrak{A}) = \infty$. For a nonunital C^*-algebra $\mathfrak{A}$, we say that $\mathfrak{A}$ has stable rank n if so does $\mathfrak{A}^+$.

It is easy to see that

Proposition 2.2.6. *Let $\mathfrak{A}$ be a C^*-algebra. Then*

$$\mathrm{exr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}).$$

Proof. Note that $L_n(\mathfrak{A}) \subset X_n(\mathfrak{A})$ (and $R_n(\mathfrak{A}) \subset X_n(\mathfrak{A})$), because of $1 \in \partial\mathfrak{A}_1$. Thus, if $L_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$, then so is $X_n(\mathfrak{A})$. □

Remark. This says that higher extremal richness can be estimated by higher stable rank. Also, the stable rank for many C^*-algebras has been calculated in the literature (after Rieffel [18] but we will not mention about this here).

Example 2.2.7. If $\mathfrak{A}$ has stable rank 1, then it has extremal richness 1. For example, the $n \times n$ matrix algebras $M_n(\mathbb{C})$ over $\mathbb{C}$ and the C^*-algebra $\mathbb{K}$ of all compact operators on a Hilbert space have stable rank 1. Let $C(X)$ be the C^*-algebra of all continuous functions on a compact Hausdorff space X. If $\dim X \leq 1$, then $\mathrm{sr}(C(X)) = 1$. Let $\mathfrak{B}$ be either an infinite von Neumann algebra or a purely infinite C^*-algebra. Then $\mathrm{sr}(\mathfrak{B}) = \infty$ but $\mathrm{exr}(\mathfrak{B}) = 1$. See [3] and [18].

Proposition 2.2.8. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{D}$ its quotient C^*-algebra. Then* $\mathrm{exr}(\mathfrak{D}) \leq \mathrm{exr}(\mathfrak{A})$.

Proof. Let π be a surjective $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{D}$. It is clear that $\pi(L_n(\mathfrak{A})) \subset L_n(\mathfrak{D})$ and $\pi(R_n(\mathfrak{A})) \subset R_n(\mathfrak{D})$. Note that $\pi(\partial\mathfrak{A}_1) \subset \partial\mathfrak{D}_1$. In fact, if $(1 - vv^*)\mathfrak{A}(1 - v^*v) = 0$, then $(1 - \pi(v)\pi(v)^*)\mathfrak{D}(1 - \pi(v)^*\pi(v)) = 0$. Therefore, $\pi(X_n(\mathfrak{A}))$ is contained in $X_n(\mathfrak{D})$. Hence if $\mathrm{exr}(\mathfrak{A}) \leq n$, then $\mathrm{exr}(\mathfrak{D}) \leq n$. □

Proposition 2.2.9. *Let $\mathfrak{A}$ be a finite or infinite direct product of C^*-algebras $\mathfrak{A}_j$. Then* $\mathrm{exr}(\mathfrak{A}) = \max_j \mathrm{exr}(\mathfrak{A}_j)$ *or* $\sup_j \mathrm{exr}(\mathfrak{A}_j)$ *(respectively).*

Proposition 2.2.10. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{B}$ its hereditary C^*-subalgebra (in particular, its closed ideal). Then* $\mathrm{exr}(\mathfrak{B}) \leq \mathrm{exr}(\mathfrak{A})$.

Proof. We can use the similar argument as given in the proof of [3, Theorem 3.5], where we consider n-tuples $(a_1, \cdots, a_n) \in \mathfrak{A}$ or $\mathfrak{B}$ instead of single elements $a \in \mathfrak{A}$ or $\mathfrak{B}$. Note that $L_n(\mathfrak{A})$ and $R_n(\mathfrak{A})$ are invariant under the left or right multiplication by elements of $GL_n(\mathfrak{A})$ respectively, in particular, by diagonal matrices with invertible elements on the diagonal. □

Proposition 2.2.11. *Let $\mathfrak{A}$ be a C^*-algebra and $M_n(\mathfrak{A})$ the $n \times n$ matrix algebra over $\mathfrak{A}$. Then* $\mathrm{exr}(\mathfrak{A}) = \mathrm{exr}(M_n(\mathfrak{A}))$.

Proof. To show $\mathrm{exr}(\mathfrak{A}) \geq \mathrm{exr}(M_n(\mathfrak{A}))$, we can modify the proof of [3, Theorem 4.5] in the same way, where we consider n-tuples instead of single elements. Since $\mathfrak{A}$ can be viewed as a hereditary C^*-sublagebra of $M_n(\mathfrak{A})$ by a standard rank 1 projection, we have $\mathrm{exr}(\mathfrak{A}) \leq \mathrm{exr}(M_n(\mathfrak{A}))$ by the above proposition. □

Proposition 2.2.12. *Let $\mathfrak{A}$ be an inductive limit of C^*-algebras $\mathfrak{A}_j$, where we assume that each $\partial(\mathfrak{A}_j)_1$ is embedded in $\partial(\mathfrak{A}_{j+1})_1$. Then* $\mathrm{exr}(\mathfrak{A}) \leq \sup_j \mathrm{exr}(\mathfrak{A}_j)$.

If in addition $\mathfrak{A}_j$ are hereditary C^*-subalgebras of $\mathfrak{A}$, then $\mathrm{exr}(\mathfrak{A}) = \sup_j \mathrm{exr}(\mathfrak{A}_j)$.

In particular, we obtain $\mathrm{exr}(\mathfrak{A}) = \mathrm{exr}(\mathfrak{A} \otimes \mathbb{K})$.

Proof. Note that the union of $X_n(\mathfrak{A}_j)$ is contained in $X_n(\mathfrak{A})$. For the second, use one of the propositions above. For the third, note that $\mathfrak{A} \otimes \mathbb{K}$ is an inductive limit of $M_n(\mathfrak{A})$ $(n \geq 1)$ satisfying the above assumptions. □

Remark. More properties concerning higher extremal richness would be deduced somewhere else in the future.

3 The λ-function in operator algebras

This section is taken from Pedersen [17].

3.1 Introduction

Let $\mathfrak{B}$ be a normed space and $\mathfrak{B}_1$ the closed unit ball in $\mathfrak{B}$. Denote by $\mathfrak{B}_e$ the set of all extreme points of the convex set $\mathfrak{B}_1$. Aron and Lohman [1] investigate the λ-function, defined on elements $T \in \mathfrak{B}_1$ to be the supremum $\lambda(T)$ of numbers $\lambda \in [0,1]$ for which there exists a pair $(V, B) \in \mathfrak{B}_e \times \mathfrak{B}_1$ such that $T = \lambda V + (1-\lambda)B$. They show that when $\mathfrak{B}$ has the λ-property that $\lambda(T) > 0$ for any $T \in \mathfrak{B}$, then every closed face of $\mathfrak{B}_1$ contains extreme points, so that any convex function on $\mathfrak{B}_1$ that attains its maximum must do so on $\mathfrak{B}_e$. Moreover, if $\mathfrak{B}$ has the uniform λ-property that $\lambda(T) \geq \varepsilon > 0$ for any $T \in \mathfrak{B}_1$, then $\mathfrak{B}$ has the Krein-Milman-like property that the closure of the convex hull of $\mathfrak{B}_e \cap \mathfrak{F}$: $\mathrm{cl}(\mathrm{conv}(\mathfrak{B}_e \cap \mathfrak{F})) = \mathfrak{F}$ for every closed face $\mathfrak{F}$ of $\mathfrak{B}_1$. Refer also to Lohman [9]. A larger class of C^*-algebras (AW^*-algebras of Kaplansky) have the same property, where recall that a C^*-algebra $\mathfrak{A}$ is said to be monotone complete if each bounded incresing net of self-adjoint elements in $\mathfrak{A}$ has a self-adjoint, least upper bound, and $\mathfrak{A}$ is an AW^*-algebra if each maximal commutative C^*-subalgebra of $\mathfrak{A}$ is monotone complete.

3.2 Notation and Preliminaries

Let $\mathfrak{A}$ denote a $*$-algebra of bounded operators on a Hilbert space H, closed in the norm topology on $\mathbb{B}(H)$. Abstractly this means that $\mathfrak{A}$ is a C^*-algebra. i.e. a Banach algebra with involution and with the norm condition $\|T^*T\| = \|T\|^2$ for $T \in \mathfrak{A}$. If we further assume that $\mathfrak{A}$ is closed in the weak operator topology on $\mathbb{B}(H)$, in which case $\mathfrak{A}$ is a von Neumann algebra. Abstractly this means that $\mathfrak{A}$ is a dual space. In a von Neumann algebra $\mathfrak{A}$ the unit ball is weakly compact, so that the Krein-Milman theorem applies, and also $\mathfrak{A}$ is generated by its projections, in the strong sense that the spectral resolution of every normal operator in $\mathfrak{A}$ belongs to $\mathfrak{A}$, and the set of all projections in $\mathfrak{A}$ forms a complete lattice: a sublattice of the set of closed subspaces of H.

Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{A}_1$ the closed unit ball of $\mathfrak{A}$ and $\mathfrak{A}_e$ the set of all extreme points of $\mathfrak{A}_1$. Assume throughout that $\mathfrak{A}$ is unital, i.e. $I \in \mathfrak{A}$, otherwise $\mathfrak{A}_e = \emptyset$. It is shown by Kadison in 1951 [8] (cf. [13]) that

$$\mathfrak{A}_e = \{V \in \mathfrak{A}_1 \mid (I - VV^*)\mathfrak{A}(I - V^*V) = 0\}.$$

If $\mathfrak{A}$ is prime, or in particular simple, then $V \in \mathfrak{A}_e$ are either isometries or co-isometries, i.e. $V^*V = I$ or $VV^* = I$.

Denote by $\mathfrak{A}_u$ the subgroup of all unitary elements of $\mathfrak{A}$ in the group $\mathfrak{A}^{-1}$ of all invertible elements of $\mathfrak{A}$ under multiplication. Then $\mathfrak{A}_u \subset \mathfrak{A}_e$.

A C^*-algebra $\mathfrak{A}$ is finite, if $T^*T = I$ implies $TT^* = I$ for $T \in \mathfrak{A}$, i.e. if every isometry is unitary. If $\mathfrak{A}$ is a von Neumann algebra, then this implies that $\mathfrak{A}_e = \mathfrak{A}_u$. For if $V \in \mathfrak{A}_e$, there is a central projection $Z \in \mathfrak{A}$ such that ZV is an isometry in $Z\mathfrak{A}$, and $(I - Z)V$ is a co-isometry in $(I - Z)\mathfrak{A}$ (see the subsection 3.4 below); and finiteness of $\mathfrak{A}$ implies that $ZV + (I - Z)V^*$ is unitary, whence $V \in \mathfrak{A}_u$. The same argument will work if $\mathfrak{A}$ is a finite AW^*-algebra.

The stable rank one $\mathrm{sr}(\mathfrak{A}) = 1$, i.e. density of $\mathfrak{A}^{-1}$ in $\mathfrak{A}$, implies that $\mathfrak{A}_e = \mathfrak{A}_u$, and thus $\mathfrak{A}$ is finite.

The fact that the convex hull of $\mathfrak{A}_u$: $\mathrm{conv}\,\mathfrak{A}_u$ is dense in $\mathfrak{A}_1$ is the Russo-Dye theorem [20]. It is proved by Rørdam [19] that

Theorem 3.2.1. *If T is a non-invertible element of $\mathfrak{A}_1$ with $\alpha(T) \equiv \mathrm{dist}(T, \mathfrak{A}^{-1}) < 1$, there is for every $\beta > 2/(1-\alpha(T))$ unitaries $U_1, \cdots, U_n \in \mathfrak{A}_u$, where $n - 1 < \beta \leq n$, such that*

$$T = \frac{1}{\beta}(\sum_{j=1}^{n-1} U_j) + \frac{\beta + 1 - n}{\beta} U_n.$$

When $\mathfrak{A}_e = \mathfrak{A}_u$, this allows us to determine the λ-function (see the subsection 3.5 below).

3.3 Polar decompositions

As von Neumann showed, every operator $T \in \mathbb{B}(H)$ has a polar decomposition $T = V|T|$, where $|T| = (T^*T)^{1/2}$ and V is a (unique) partial isometry such that $\ker V = \ker T$. The construction: $V = \lim T(n^{-1}I + |T|)^{-1}$, where the limit is taken in the strong operator topology, shows that if $T \in \mathfrak{A}$ a von Neumann algebra, then $|T| \in \mathfrak{A}$ and $V \in \mathfrak{A}$. If $\mathfrak{A}$ is a C^*-algebra, we can not be certain that $V \in \mathfrak{A}$, rather, V belongs to the von Neumann algebra generated by $\mathfrak{A}$, equal to the double commutant $\mathfrak{A}''$ of $\mathfrak{A}$. But if $T \in \mathfrak{A}^{-1}$, then $T = U|T|$ for $U = T|T|^{-1}$ a unitary of $\mathfrak{A}$.

For $\delta > 0$, denote by E_δ the spectral projection of $|T|$ corresponding to the open interval (δ, ∞), i.e. $E_\delta = \chi_{(\delta,\infty)}(|T|)$, where $\chi_{(\delta,\infty)}(\cdot)$ is the characteristic function on the open interval.

Theorem 3.3.1. ([15] or [19].) *If T is an element of a C^*-algebra $\mathfrak{A}$, with polar decomposition $V|T|$, then for each $\delta > \operatorname{dist}(T, \mathfrak{A}^{-1})$ there is a unitary $U \in \mathfrak{A}_e$ such that $UE_\delta = VE_\delta$.*

For $\delta < \operatorname{dist}(T, \mathfrak{A}^{-1})$ there is no unitary extension of VE_δ in $\mathfrak{A}$.

Corollary 3.3.2. *Each element of the form $Vf(|T|)$, where f is a continuous function on $\operatorname{sp}(|T|)$ such that $f(t) = 0$ for $t \le \delta$, and $\delta > \operatorname{dist}(T, \mathfrak{A}^{-1})$, has a unitary polar decomposition $Uf(|T|) = Vf(|T|)$ in $\mathfrak{A}$.*

Proposition 3.3.3. *If $V \in \mathfrak{A}_e$ with $\operatorname{dist}(V, \mathfrak{A}^{-1}) < 1$, then $V \in \mathfrak{A}_u$.*

Proof. Let $P = V^*V$ and $Q = VV^*$ be the support and range projections of V, respectively. Then $V = VP$ is the polar decomposition of V, so that $P = E_\delta$ for any $\delta \in (0,1)$. By the theorem above there is a unitary $U \in \mathfrak{A}_u$, such that $UP = VP = V$. Consequently, $Q = VV^* = UPU^*$.

By assumption, $(I - Q)\mathfrak{A}(I - P) = 0$, hence

$$0 = (I - Q)U(I - P) = U(I - P)U^*U(I - P) = U(I - P).$$

It follows that $U = UP = V$, so $V \in \mathfrak{A}_u$. □

Corollary 3.3.4. *If $\mathfrak{A}^{-1}$ is dense in $\mathfrak{A}$, then $\mathfrak{A}_e = \mathfrak{A}_u$.*

Proposition 3.3.5. ([15] or [19].) *If $T \notin \mathfrak{A}^{-1}$, then*

$$\operatorname{dist}(T, \mathfrak{A}_u) = \max\{\|T\| - 1, \operatorname{dist}(T, \mathfrak{A}^{-1}) + 1\}.$$

Sketch of Proof from [19]. Clearly $\|T - U\| \ge \|T\| - 1$ for every $U \in \mathfrak{A}_u$, with $\|U\| = 1$. Since T is non-invertible, $\|T - U\| \ge 1$. Set $S_0 = \|T - U\|^{-1}(T - U) + U$. Then S_0 is in $\operatorname{cl}(\mathfrak{A}^{-1})$ since $\|S_0 - U\| = 1$. Thus

$$\begin{aligned}\alpha(T) = \operatorname{dist}(T, \mathfrak{A}^{-1}) \le \|T - S_0\| &= \|T - U + U - S_0\| \\ &= \|T - U - \|T - U\|^{-1}(T - U)\| \\ &= (1 - \|T - U\|^{-1})\|T - U\| = \|T - U\| - 1,\end{aligned}$$

which implies that $\operatorname{dist}(T, \mathfrak{A}_u) \ge \max\{\alpha(T) + 1, \|T\| - 1\}$.

Let $S \in \mathfrak{A}^{-1}$, with polar decomposition $S = U|S|$ with $U \in \mathfrak{A}_u$. Since $-I \leq |S| - I \leq (\|S\| - 1)I$, we have

$$\operatorname{dist}(S, \mathfrak{A}_u) \leq \|S - U\| = \||S| - I\| \leq \max\{1, \|S\| - 1\}.$$

By continuity, this also holds for elements in $\operatorname{cl}(\mathfrak{A}^{-1})$. Since there is $S_0 \in \operatorname{cl}(\mathfrak{A}^{-1})$, with $\|T - S_0\| = \alpha(T)$ and $\|S_0\| = \|T\| - \alpha(T)$, we have

$$\begin{aligned}\operatorname{dist}(T, \mathfrak{A}_u) &\leq \|T - S_0\| + \operatorname{dist}(S_0, \mathfrak{A}_u)\\ &\leq \alpha(T) + \max\{1, \|S_0\| - 1\} = \max\{\alpha(T) + 1, \|T\| - 1\}.\end{aligned}$$

□

Proposition 3.3.6. *If $\mathfrak{A}$ is a von Neumann algebra and $T \in \mathfrak{A}$, there is an extreme point $W \in \mathfrak{A}_e$ such that $T = W|T|$.*

Proof. The set $\mathfrak{C} = \{W \in \mathfrak{A}_1 \mid T = W|T|\}$ is a non-empty, convex, weakly closed subset of the weakly compact unit ball in $\mathbb{B}(H)$. Note that if $W_1, W_2 \in \mathfrak{C}$, then for $0 \leq t \leq 1$, $(tW_1 + (1-t)W_2)|T| = tT + (1-t)T = T$; and if $W_n \in \mathfrak{C}$ with $W_n \to W$ weakly, then $\langle W|T|\xi, \eta\rangle = \lim_n \langle W_n|T|\xi, \eta\rangle = \langle T\xi, \eta\rangle$ for $\xi, \eta \in H$.

By Krein-Milman's theorem we can find an extreme point $W \in \mathfrak{C}$. Since $\|W\| = 1$ we have $W^*W|T| = |T|$, because

$$|T|(I - W^*W)|T| = |T|^2 - T^*T = T^*T - T^*T = 0,$$

and $W^*W \leq \|W^*W\|I = I$, so $I - W^*W \geq 0$, and hence $\sqrt{I - W^*W}|T| = 0$, which implies $(I - W^*W)|T| = 0$.

Both $W|W|$ and $W(2 - |W|)$ belongs to $\mathfrak{A}_1$, where note that

$$\|W|W|\|^2 = \||W|W^*W|W|\| = \|(W^*W)^2\| = \|W\|^4 = 1,$$

and we have

$$\|W(2 - |W|)\|^2 = \|(2 - |W|)W^*W(2 - |W|)\| = \||W|^4 - 4|W|^3 + 4|W|^2\|,$$

where we consider the real-valued function $f(x) = x^4 - 4x^3 + 4x^2$ for $x \in \mathbb{R}$, with $f'(x) = 4x(x-1)(x+1)$, it follows from which, by functional calculus that $\|W(2 - |W|)\|^2 = 1$ since $f([0,1]) = [0,1]$.

We also have

$$\begin{aligned}T = W|W||T|, \quad T = W(2 - |W|)|T|,\\ W = \frac{1}{2}(W|W| + W(2 - |W|)),\end{aligned}$$

it follows from the third equality easily checked that

$$T = W|T| = \frac{1}{2}(W|W||T| + W(2 - |W|)|T|),$$

so that if the first equality holds, the second equality is deduced. Check that $T = W|T| = W|W|^2|T| = W|W||T|$, because

$$(|W||T|)^*(|W||T|) = |T||W|^2|T| = |T|W^*W|T| = |T|^2$$

and $|T|$ commutes with W^*W, so that $|W|$ and $|W|^{1/2}$ also commute with $|T|$, and therefore,

$$\langle |W||T|\xi, \xi\rangle = \langle |T||W|^{1/2}\xi, |W|^{1/2}\xi\rangle \geq 0, \quad \xi \in H,$$

i.e. $|W||T| \geq 0$, and hence $|W||T| = ||W||T|| = |T|$.

We conclude from the third equality (a convex combination) above and extremality of W that $W = W|W|$, i.e. $|W|$ is a projection and W is a partial isometry. Indeed, it follows that $W^*W = W^*W|W|$, which implies $|W|^2 = |W|^3 = |W|^4$, hence $|W| = |W|^2 = W^*W$ a projection.

If $A \in (I - WW^*)\mathfrak{A}_1(I - W^*W)$, then $A|T| = 0$. Thus $T = (W \pm A)|T|$. Then $\|W \pm A\| \leq 1$. Indeed, let $A = PBQ$ for some $B \in \mathfrak{A}_1$, with $P = I - WW^*$ and $Q = I - W^*W$, and we have

$$\begin{aligned}
\|W \pm A\|^2 &= \|(I - P)W(I - Q) \pm PBQ\|^2 \\
&= \|\{(I - Q)W^*(I - P) \pm QB^*P)\}\{(I - P)W(I - Q) \pm PBQ\}\| \\
&= \|(I - Q)W^*(I - P)W(I - Q) + QB^*PBQ\| \\
&= \max\{\|W^*(I - P)W\|, \|B^*PB\|\} \leq 1.
\end{aligned}$$

Therefore, $W \pm A \in \mathfrak{C}$, with $W = 2^{-1}((W + A) + (W - A))$. It follows from the extremality of W that $W = W + A$, hence $A = 0$, whence $W \in \mathfrak{A}_e$. □

Remark. Another proof is as follows. Let $T = V|T|$ be the polar decomposition of T. Note that the set $\mathfrak{D} = V + (I - VV^*)\mathfrak{A}_1(I - V^*V)$ is a weakly closed face of $\mathfrak{A}_1$. In fact, as shown by Edwards and Rüttimann, every weakly closed face in $\mathfrak{A}_1$ has this form. An extreme point W of $\mathfrak{D}$ therefore belongs to $\mathfrak{C}$, and let $W = V + (I - VV^*)B(I - V^*V)$. It follows that $W|T| = V|T| = T$ since $V^*V|T| = |T|$.

3.4 Von Neumann algebras

For an operator $T \in \mathbb{B}(H)$, define

$$m(T) = \inf\{\|Tx\| \,|\, x \in H, \|x\| = 1\}.$$

Since we have $\|Tx\|^2 = \langle T^*Tx, x\rangle = \langle |T|x, |T|x\rangle = \||T|x\|^2$ for $x \in H$, it follows that $m(T) = m(|T|)$, and moreover, if $|T| \in \mathfrak{A}^{-1}$, then

$$m(T) = \min\{\varepsilon > 0 \,|\, \varepsilon \in \mathrm{sp}(|T|)\} = m_q(T),$$
$$m(T) = \max\{\varepsilon \geq 0 \,|\, \varepsilon I \leq |T|\},$$

and $m(T) = \||T|^{-1}\|^{-1}$, where $m_q(T)$ is defined in the subsection 1.1. As for the second equality, see the subsection 1.1.

As for the equivalence between the first left value N and the second value M, note that for any $0 < \varepsilon < M$, we have $|T| - \varepsilon I \geq (M - \varepsilon)I$, so that $|T| - (M - \varepsilon)I \geq \varepsilon I$, which implies $M - \varepsilon \notin \mathrm{sp}(|T|)$. Hence $M - \varepsilon \leq N$, so that $M \leq N$. Conversely, for any $N > \varepsilon > 0$, we have $N - \varepsilon \notin \mathrm{sp}(|T|)$. Then $|T| - (N - \varepsilon)I$ is invertible, and is positive since $\mathrm{sp}(|T| - (N - \varepsilon)I) = \mathrm{sp}(|T|) - (N - \varepsilon)$. Thus, there is $\delta > 0$ such that $|T| - (N - \varepsilon)I \geq \delta I$. Hence $|T| \geq (N - \varepsilon + \delta)I$. Therefore, $M \geq (N - \varepsilon + \delta)$, so that $M \geq N$.

As for the third equality, we check that $m(|T|) \cdot \||T|^{-1}\| = 1$. Since

$$1 = \|x\| = \||T|^{-1}|T|x\| \leq \||T|^{-1}\|\||T|x\|,$$

we have $1 \leq \||T|^{-1}\| \cdot m(|T|)$. Conversely,

$$\begin{aligned} 1 = \|I\| = \||T||T|^{-1}\| &= \sup_{\|x\|=1} \||T||T|^{-1}x\| \\ &\geq \||T||T|^{-1}x\| = \||T|^{-1}x\| \cdot \||T|\frac{|T|^{-1}x}{\||T|^{-1}x\|}\| \geq \||T|^{-1}x\| \cdot m(|T|), \end{aligned}$$

so that $m(|T|)^{-1} \geq \||T|^{-1}x\|$. Therefore, $1 \geq m(|T|) \cdot \||T|^{-1}\|$.

By the open mapping theorem the condition $m(T) > 0$ is equivalent to T being injective with closed range. In this case $T = U|T|$ with $U = T|T|^{-1}$ a unique isometry.

Now consider T as an element of some von Neumann algebra $\mathfrak{A}$ in $\mathbb{B}(H)$ with center $\mathfrak{Z} = \mathfrak{A} \cap \mathfrak{A}'$. Denote by $\mathfrak{Z}_p$ the set of projections in $\mathfrak{Z}$. Define

$$m_p(T) = \sup\{m(ZT + (I - Z)T^*) \,|\, Z \in \mathfrak{Z}_p\}.$$

If we decompose $T = H + iK$ in real and imaginary parts, then

$$m_p(T) = \sup\{m(H + iSK) \,|\, S \in \mathfrak{Z}_s\},$$

where $\mathfrak{Z}_s$ denotes the set of symmetries $S \in \mathfrak{Z}$ of the form $S = 2Z - I$ for some $Z \in \mathfrak{Z}_p$. Check that

$$\begin{aligned} T = H + iK &= \frac{T+T^*}{2} + iS \cdot \frac{T-T^*}{2i} \\ &= \frac{I+S}{2} \cdot T + \left(I - \frac{I+S}{2}\right) T^* = ZT + (I-Z)T^*. \end{aligned}$$

If $\mathfrak{A} = \mathbb{B}(H)$ (in particular), then we have

$$m_p(T) = \max\{m(T), m(T^*)\}.$$

Indeed, If $Z = I$, then $m(T) \leq m_p(T)$. If $Z = 0$, then $m(T^*) \leq m_p(T)$. Conversely, for $x \in H$ with $\|x\| = 1$,

$$\begin{aligned} m(ZT + (I-Z)T^*)^2 &\leq \|(ZT + (I-Z)T^*)x\|^2 = \|ZTx\|^2 + \|(I-Z)T^*x\|^2 \\ &= \begin{cases} \|ZT\xi\|^2, & x = \xi \oplus 0 \in ZH \oplus (I-Z)H, \\ \|(I-Z)T^*\eta\|^2, & x = 0 \oplus \eta \in ZH \oplus (I-Z)H. \end{cases} \\ &\leq \begin{cases} \|T\xi\|^2, \\ \|T^*\eta\|^2. \end{cases} \end{aligned}$$

It follows that $m(ZT+(I-Z)T^*) \leq m(T)$ and $m(ZT+(I-Z)T^*) \leq m(T^*)$. Hence $m_q(T) \leq \max\{m(T), m(T^*)\}$.

Lemma 3.4.1. *If $\mathfrak{A}$ is a von Neumann algebra and $T \in \mathfrak{A}$, there is a central projection $Z \in \mathfrak{A}$ such that $m_p(T) = m(ZT + (I-Z)T^*)$.*

Proof. If Y and Z both belong to $\mathfrak{Z}_p$, and

$$\varepsilon I \leq Y|T| + (I-Y)|T^*|, \quad \varepsilon I \leq Z|T| + (I-Z)|T^*|,$$

then by spectral theory, $\varepsilon X \leq X|T|$ and $\varepsilon(I-X) \leq (I-X)|T^*|$ for $X = Y \vee Z$ and also for $X = Y \wedge Z$, because these statements only involve Y, Z and $|T|$ commutaing, respectively Y, Z and $|T^*|$. Note that it follows from multiplying the inequalities by $Y, I-Y$, and $Z, I-Z$ respectively that $\varepsilon Y \leq Y|T|$, $\varepsilon(I-Y) \leq (I-Y)|T^*|$, and $\varepsilon Z \leq Z|T|$, $\varepsilon(I-Z) \leq (I-Z)|T^*|$. Moreover, $\varepsilon YZ \leq YZ|T|$ and $\varepsilon(I-Y)(I-Z) \leq (I-Y)(I-Z)|T^*|$, with $YZ = Y \wedge Z$ and $(I-Y)(I-Z) = I - (Y+Z-YZ) = I - (Y \vee Z)$. Furthermore, we have $\varepsilon(Z - ZY) \leq (Z - ZY)|T^*|$, to which by adding $\varepsilon(I-Z) \leq (I-Z)|T^*|$ we get $\varepsilon(I - YZ) \leq (I - YZ)|T^*|$. Also, we have $\varepsilon(Z - ZY) \leq (Z - ZY)|T|$, to which by adding $\varepsilon Y \leq Y|T|$ we get $\varepsilon(Y + Z - ZY) \leq (Y + Z - ZY)|T|$, as completed. Therefore, we obtain

$$\varepsilon X \leq X|T| + (I-X)|T^*|.$$

We have

$$m(ZT+(I-Z)T^*) = \max\{\varepsilon \mid \varepsilon I \le |ZT+(I-Z)T^*|\}$$
$$= \max\{\varepsilon \mid \varepsilon I \le Z|T| + (I-Z)|T^*|\},$$

because

$$(ZT+(I-Z)T^*)^*(ZT+(I-Z)T^*) = ZT^*T + (I-Z)TT^*$$
$$= Z|T|^2 + (I-Z)|T^*|^2 = (Z|T| + (I-Z)|T^*|)^2$$

and therefore, $|ZT+(I-Z)T^*| = Z|T| + (I-Z)|T^*|$. This means that if (Z_n) is a sequence of $\mathfrak{Z}_p$ such that the sequence $\{\varepsilon_n = m(Z_nT+(I-Z_n)T^*)\}$ increases to $m_p(T)$, then with $Y_k = \vee_{n\ge k} Z_n \in \mathfrak{Z}_p$ we have $\varepsilon_m \le m(Y_kT + (I-Y_k)T^*)$ for every $k \ge m$. Indeed, for $m \in \mathbb{Z}$ fixed and for any $n \ge m$ we have $\varepsilon_m \le m(Z_nT + (I-Z_n)|T^*|)$, so that $\varepsilon_m I \le Z_n|T| + (I-Z_n)|T^*|$ for $n \ge m$. Thus, for every $k \ge m$, we deduce that $\varepsilon_m \le Y_k|T| + (I-Y_k)|T^*|$ by using the estimate for X obtained above inductively and taking weak limit.

Arguing in the same way on the decreasing sequence (Y_k) in $\mathfrak{Z}_p$, we see that if $Z = \wedge Y_k \in \mathfrak{Z}_p$, then $\varepsilon_k \le m(ZT+(I-Z)T^*)$ for every k. Indeed, we have $\varepsilon_k I = m(Z_kT + (I-Z_k)|T^*|)I \le Z_k|T| + (I-Z_k)|T^*|$, which implies $\varepsilon_k \le Z|T| + (I-Z)|T^*|$. It follows that $m_p(T) \le m(ZT + (I-Z)T^*)$. By definition, $m_p(T) \ge m(ZT+(I-Z)T^*)$. □

Recall that the λ-function for $T \in \mathfrak{A}_1$ of an operator algebra $\mathfrak{A}$ in $\mathbb{B}(H)$ is defined by

$$\lambda(T) = \sup\{\lambda \in [0,1] \mid T = \lambda V + (1-\lambda)B, \quad V \in \mathfrak{A}_e, B \in \mathfrak{A}_1\}.$$

Theorem 3.4.2. *If $\mathfrak{A}$ is a von Neumann algebra, and $T \in \mathfrak{A}_1$, then*

$$\lambda(T) = \frac{1}{2}(1 + m_p(T)).$$

Moreover, if $\frac{1}{2} \le \lambda \le \lambda(T)$, there are extreme points $V, W \in \mathfrak{A}_1$, such that $T = \lambda V + (1-\lambda)W$.

Proof. If $T = \lambda V + (1-\lambda)B$ for some $V \in \mathfrak{A}_e$ and $B \in \mathfrak{A}_1$, put $P = V^*V$ and $Q = VV^*$. We can find a central projection $Z \in \mathfrak{Z}_p$ such that $I - Q \le Z \le P$. To see this, note that $I-P$ and $I-Q$ are centrally orthogonal since $V \in \mathfrak{A}_e$, so for every unitary $U \in \mathfrak{A}_u$, $I-Q$ is orthogonal to $U(I-P)U^* = I - UPU^*$, that is, $I-Q = (I-Q)UPU^*$, i.e. $I-Q \le UPU^*$. Take $Z = \wedge UPU^*$, the infimum being taken over $U \in \mathfrak{A}_u$, so that $I-Q \le Z$

and $Z \le P$. Evidently $uZu^* = Z$ for every $u \in \mathfrak{A}_u$, i.e. $uZ = Zu$. Indeed, $uZu^* = u(\wedge UPU^*)u^* \le uUPU^*u^*$, and replacing U with u^*U yields $uZu^* \le UPU^*$, thus $uZu^* \le Z$, and conversely, $Z \le UPU^*$ for $U \in \mathfrak{A}_u$, and in particular, $Z \le uUPU^*u^*$, which implies $u^*Zu \le UPU^*$, and hence $u^*Zu \le Z$, and thus $Z \le uZu^*$. Also $\mathfrak{A} = \mathrm{span}(\mathfrak{A}_u)$ (in fact, every element is a linear combination of 4 (even 3) unitaries). It follows that $Z \in \mathfrak{Z}_p$.

Since $Z \le P$ and $I - Z \le Q$, we have

$$(ZV)^*(ZV) = V^*ZV = ZP = Z,$$
$$((I-Z)V^*)^*(I-Z)V^* = V(I-Z)V^* = (I-Z)Q = I - Z.$$

It follows that $W = ZV + (I-Z)V^*$ is an isometry in $\mathfrak{A}$. Let $T_0 = ZT + (I-Z)T^*$ and $B_0 = ZB + (I-Z)B^*$. Rewrite the equation $T = \lambda V + (1-\lambda)B$ as $T_0 = \lambda W + (1-\lambda)B_0$. Check that

$$\begin{aligned} T_0 &= ZT + (I-Z)T^* \\ &= \lambda(ZV + (I-Z)V^*) + (1-\lambda)(ZB + (I-Z)B^*) = \lambda W + (1-\lambda)B_0. \end{aligned}$$

Since $B_0 \in \mathfrak{A}_1$, indeed, $\|B_0\| = \max\{\|ZB\|, \|(I-Z)B^*\|\} \le 1$, we compute

$$\begin{aligned} m_p(T) &\ge m(T_0) = m(\lambda W + (1-\lambda)B_0) \\ &= \inf\{\|(\lambda W + (1-\lambda)B_0)x\| \mid \|x\| = 1\} \\ &\ge \inf\{\lambda\|Wx\| - (1-\lambda)\|B_0\| \mid \|x\| = 1\} = 2\lambda - 1. \end{aligned}$$

This inequality holds for any decomposition $T = \lambda V + (1-\lambda)B$. Therefore, we conclude that $m_p(T) \ge 2 \cdot \lambda(T) - 1$.

To prove the reverse inequality, as shown in the lemma above we take a projection $Z \in \mathfrak{Z}_p$ such that $m_p(T) = m(ZT + (I-Z)T^*)$, and set $m_p(T) = \varepsilon$. Set $A = |ZT + (I-Z)T^*| = Z|T| + (I-Z)|T^*|$. Then $\varepsilon I \le A$. It is shown that this implies that for any $\lambda \in [\frac{1}{2}, \frac{1}{2}(1+\varepsilon)]$ we can find unitaries $U_1, U_2 \in \mathfrak{A}_u$ such that $A = \lambda U_1 + (1-\lambda)U_2$. This fact is verified by writing $U_1 = B + i(1-\lambda)D$ and $U_2 = C - i\lambda D$, where B, C, and D are self-adjoint elements of $\mathfrak{A}$ given by

$$\begin{aligned} B &= \frac{1}{2\lambda}(A + (2\lambda - 1)A^{-1}), \\ C &= \frac{1}{2(1-\lambda)}(A - (2\lambda - 1)A^{-1}), \\ D &= \frac{1}{1-\lambda}(I - B^2)^{\frac{1}{2}} = \frac{1}{\lambda}(I - C^2)^{\frac{1}{2}}. \end{aligned}$$

Here $(2\lambda-1)A^{-1}$ should be interpreted as 0 when $\lambda=\frac{1}{2}$ (if $m_p(T)=0$ this may be the only choice), and if $\lambda=1$ (so that $A=I$) the formulae for C and D should be interpreted as 0. Note that

$$U_1=B+i(I-B^2)^{1/2},\quad U_2=C-i(I-C^2)^{1/2}.$$

so that direct computation implies $U_1^*U_1=U_1U_1^*=I$ and $U_2^*U_2=U_2U_2^*=I$, and

$$\lambda U_1+(1-\lambda)U_2=\lambda B+(1-\lambda)C+i(\lambda(I-B^2)^{1/2}-(1-\lambda)(I-C^2)^{1/2})$$

with $\lambda B+(1-\lambda)C=2^{-1}(A+(2\lambda-1)A^{-1})+2^{-1}(A-(2\lambda-1)A^{-1})=A$, and

$$\begin{aligned}
I-B^2&=I-\frac{1}{4\lambda^2}(A+(2\lambda-1)A^{-1})^2\\
&=I-\frac{1}{4\lambda^2}(A^2+2(2\lambda-1)I+(2\lambda-1)^2(A^{-1})^2)\\
&=\frac{1}{4\lambda^2}((4\lambda^2-4\lambda+2)I-A^2-(2\lambda-1)^2(A^{-1})^2);\\
I-C^2&=I-\frac{1}{4(1-\lambda)^2}(A-(2\lambda-1)A^{-1})^2\\
&=I-\frac{1}{4(1-\lambda)^2}(A^2-2(2\lambda-1)I+(2\lambda-1)^2(A^{-1})^2)\\
&=\frac{1}{4(1-\lambda)^2}((4\lambda^2-4\lambda+2)I-A^2-(2\lambda-1)^2(A^{-1})^2),
\end{aligned}$$

so that $\lambda(I-B^2)^{1/2}=(1-\lambda)(I-C^2)^{1/2}$, where since $0\le 2\lambda-1\le\varepsilon$, we have for $1\ge t\ge 2\lambda-1$,

$$\begin{aligned}
-t^2+(2\lambda-1)^2+1-(2\lambda-1)^2\frac{1}{t^2}&=\frac{-t^4+((2\lambda-1)^2+1)t^2-(2\lambda-1)^2}{t^2}\\
&=\frac{-(t^2-(2\lambda-1)^2)(t^2-1)}{t^2}\ge 0,
\end{aligned}$$

so that $I-B^2\ge 0$ and $I-C^2\ge 0$ by functional calculus.

Therefore, for $\frac{1}{2}\le\lambda\le\frac{1}{2}(1+m_p(T))$ we obtain

$$Z|T|+(I-Z)|T^*|=A=\lambda U_1+(1-\lambda)U_2.$$

As shown in the proposition above, we can choose extreme points W_1 and W_2 in $\mathfrak{A}_e$ such that $T=W_1|T|$ and $T^*=W_2|T|$. Then

$$\begin{aligned}
T&=ZT+((I-Z)T^*)^*=W_1Z|T|+(I-Z)|T^*|W_2^*\\
&=W_1Z(Z|T|+(I-Z)|T^*|)+(I-Z)(Z|T|+(I-Z)|T^*|)W_2^*\\
&=W_1Z(\lambda U_1+(1-\lambda)U_2)+(I-Z)(\lambda U_1+(1-\lambda)U_2)W_2^*\\
&=\lambda(ZW_1U_1+(I-Z)U_1W_2^*)+(1-\lambda)(ZW_1U_2+(I-Z)U_2W_2^*).
\end{aligned}$$

Evidently, $V = ZW_1U_1 + (I - Z)U_1W_2^*$ and $W = ZW_1U_2 + (I - Z)U_2W_2^*$ are extreme points. Indeed, check that

$$\begin{aligned}
I - VV^* &= I - ZW_1W_1^* - (I - Z)U_1W_2^*W_2U_1^* \\
&= (I - Z)U_1(I - W_2^*W_2)U_1^* + Z(I - W_1W_1^*); \\
I - V^*V &= I - ZU_1W_1^*W_1U_1 - (I - Z)W_2W_2^* \\
&= (I - Z)(I - W_2W_2^*) + ZU_1^*(I - W_1^*W_1)U_1,
\end{aligned}$$

it follows from which that $(I - VV^*)\mathfrak{A}(I - V^*V)$. Similarly, done for W. Thus we have $T = \lambda V + (1-\lambda)W$, as desired. Choosing $\lambda = \frac{1}{2}(1+m_p(T))$, we get $\lambda(T) \geq \frac{1}{2}(1 + m_p(T))$. □

3.5 C^*-algebras and the unitary λ-function

For each $T \in \mathfrak{A}_1$ the unit ball in a C^*-algebra $\mathfrak{A}$, define

$$\lambda_u(T) = \sup\{\lambda \in [0,1] \,|\, T = \lambda U + (1 - \lambda)B, \quad U \in \mathfrak{A}_u, B \in \mathfrak{A}_1\}.$$

We call it the unitary λ-function. Clearly $\lambda_u(T) \leq \lambda(T)$, since $\mathfrak{A}_u \subset \mathfrak{A}_e$. The λ-function and the unitary λ-function agree when $\mathfrak{A}_e = \mathfrak{A}_u$. Set $\alpha(T) = \operatorname{dist}(T, \mathfrak{A}^{-1})$.

Theorem 3.5.1. *If $\mathfrak{A}$ is a C^*-algebra and T is a non-invertible element of $\mathfrak{A}_1$, then*

$$\lambda_u(T) = \frac{1}{2}(1 - \alpha(T)).$$

If T is invertible, then $\lambda_u(T) = \frac{1}{2}(1 + \|T^{-1}\|^{-1})$.

Proof. Assume that $T \notin \mathfrak{A}^{-1}$. If $T = \lambda U + (1 - \lambda)B$ for some $U \in \mathfrak{A}_u$ and $B \in \mathfrak{A}_1$, then $\lambda \leq \frac{1}{2}$, since otherwise $T = \lambda U(I + \lambda^{-1}(1 - \lambda)U^*B) \in \mathfrak{A}^{-1}$, because $\|\lambda^{-1}(1 - \lambda)U^*B\| < 1$, if $1 - \lambda < \lambda$, i.e. $\frac{1}{2} < \lambda$. Now,

$$\|T - \lambda(U + B)\| = \|(1 - 2\lambda)B\| \leq 1 - 2\lambda.$$

Since $U + sB = U(I + sU^*B) \in \mathfrak{A}^{-1}$ for every $s < 1$, we see that $U + B \in \operatorname{cl}(\mathfrak{A}^{-1})$, whence $\alpha(T) \leq 1 - 2\lambda$. Since this holds for all decompositions as above, we conclude that $\alpha(T) \leq 1 - 2\lambda_u(T)$.

Another argument estimating

$$\|T - U\| = \|(1 - \lambda)B - (1 - \lambda)U\| \leq 2(1 - \lambda)$$

is also available, since $\operatorname{dist}(T, \mathfrak{A}_u) \geq \operatorname{dist}(T, \mathfrak{A}^{-1}) + 1$ as given before.

When $\alpha(T) = 1$ the result above shows that $\lambda_u(T) = 0$, so in order to prove the reverse inequality we may assume that $\alpha(T) < 1$. Then there is for every $\beta > 2(1-\alpha(T))^{-1}$ a convex combination

$$T = \frac{1}{\beta}(U_1 + \cdots + U_{n-1}) + \frac{\beta + 1 - n}{\beta} U_n,$$

with $U_j \in \mathfrak{A}_u$ and $n - 1 < \beta \leq n$, by the theorem in 3.1. Taking

$$B = \frac{1}{\beta - 1}(U_2 + \cdots + U_{n-1} + (\beta + 1 - n)U_n),$$

we have $T = \frac{1}{\beta}U_1 + (1 - \frac{1}{\beta})B$, with $B \in \mathfrak{A}_1$, so that $\lambda_u(T) \geq \frac{1}{\beta}$. It follows that $\lambda_u(T) \geq \frac{1}{2}(1 - \alpha(T))$, giving the desired equation.

If $T \in \mathfrak{A}^{-1}$ we have $T = U|T|$ with $U \in \mathfrak{A}_u$. Thus $T^{-1} = |T|^{-1}U^*$ and $\|T^{-1}\| = \||T|^{-1}U^*\| = \||T|^{-1}\|$. Thus we see that $m(T) = \|T^{-1}\|^{-1}$. Since $|T| \geq m(T)I$ there are unitaries U_1 and $U_2 \in \mathfrak{A}_u$ such that with $\lambda_0 = \frac{1}{2}(1 + m(T))$ we have

$$|T| = \lambda_0 U_1 + (1 - \lambda_0)U_2.$$

Multiplying this equation with U we see $\lambda_u(T) \geq \frac{1}{2}(1 + m(T))$.

Conversely, if $T = \lambda U + (1-\lambda)B$ with $U \in \mathfrak{A}_u$ and $B \in \mathfrak{A}_1$ we get

$$\begin{aligned} m(T) &= \inf\{\|\lambda U x + (1-\lambda)Bx\| \mid \|x\| = 1\} \\ &\geq \inf\{\lambda\|Ux\| - (1-\lambda)\|B\| \mid \|x\| = 1\} = 2\lambda - 1. \end{aligned}$$

This holds for any decomposition for T, so that $m(T) \geq 2\lambda_u(T) - 1$.

Combined with, we obtain $\lambda_u(T) = \frac{1}{2}(1 + m(T)) = \frac{1}{2}(1 + \|T^{-1}\|^{-1})$. $\square$

Remark. When T is invertible the number $m(T)$ in the formulas serves as a measure of the negative distance from T to $\mathfrak{A}^{-1}$. If $T \in \mathfrak{A}^{-1}$ with $T = U|T|$, then U is an approximant to T in $\mathfrak{A}_u$, and

$$\operatorname{dist}(T, \mathfrak{A}_u) = \|T - U\| = \||T| - I\| = \max\{\|T\| - 1, 1 - m(T)\}.$$

Indeed, the last equality follows from functional calculus, and for the first equality, we have $\|T - V\| \geq \max\{\|T\| - 1, 1 - m(T)\}$ for any $V \in \mathfrak{A}_u$, because $\|T - V\| \geq \|T\| - \|V\| = \|T\| - 1$, and we have $\|T - V\| \geq m(T - V)$, so that for any $\varepsilon > 0$, we have $\|T - V\| + \varepsilon > \|(T - V)x\|$ for some $x \in H$ with $\|x\| = 1$, and $\|(V - T)x\| \geq \|Vx\| - \|Tx\| = 1 - \|Tx\|$, therefore,

$$\|Tx\| + \varepsilon > 1 - \|T - V\|$$

which implies $m(T) + \varepsilon \geq 1 - \|T - V\|$, hence $\|T - V\| + \varepsilon \geq 1 - m(T)$, thus, we obtain $\|T - V\| \geq 1 - m(T)$.

Proposition 3.5.2. *If T is an invertible element of the unit ball $\mathfrak{A}_1$ of a C^*-algebra $\mathfrak{A}$, then $\lambda_u(T) = \lambda(T)$.*

Proof. We have $\lambda_u(T) = \frac{1}{2}(1 + \|T^{-1}\|^{-1}) > \frac{1}{2}$. If we had $\lambda_u(T) < \lambda(T)$, there would be $V \in \mathfrak{A}_e$ and $B \in \mathfrak{A}_1$ such that $T = \lambda V + (1-\lambda)B$ with $\lambda > \lambda_u(T) > \frac{1}{2}$. With $A = \lambda^{-1}T$ we have $A \in \mathfrak{A}^{-1}$ and

$$\|V - A\| = \|(\lambda^{-1} - 1)B\| \leq \lambda^{-1} - 1 < 1.$$

Then $V \in \mathfrak{A}_u$, so $\lambda \leq \lambda_u(T)$, a contradiction. Consequently $\lambda(T) \leq \lambda_u(T)$, whence $\lambda(T) = \lambda_u(T)$. □

Theorem 3.5.3. *Let $\mathfrak{A}$ be a C^*-algebra. The following conditions are equivalent:*

(i) *For every $T \in \mathfrak{A}_1$ and $0 < \varepsilon < \frac{1}{2}$ there are unitaries $U_1, U_2, U_3 \in \mathfrak{A}_u$ such that*

$$T = \frac{1}{2}(1-\varepsilon)U_1 + \frac{1}{2}(1-\varepsilon)U_2 + \varepsilon U_3.$$

(ii) $\lambda_u(T) \geq \frac{1}{2}$ *for every $T \in \mathfrak{A}_1$.*

(iii) $\mathfrak{A}$ *has the uniform λ_u-property, i.e. $\lambda_u(T) \geq \delta > 0$ for any $T \in \mathfrak{A}_1$.*

(iv) $\mathfrak{A}$ *has the λ_u-property, i.e. $\lambda_u(T) > 0$ for any $T \in \mathfrak{A}_1$.*

(v) $\operatorname{dist}(T, \mathfrak{A}^{-1}) < 1$ *for every $T \in \mathfrak{A}_1$.*

(vi) $\mathfrak{A}$ *has stable rank one, $\operatorname{sr}(\mathfrak{A}) = 1$, i.e. $\mathfrak{A}^{-1}$ is dense in $\mathfrak{A}$.*

Proof. (i) $\Rightarrow$ (ii). Note that

$$T = \varepsilon U_3 + (1-\varepsilon)\left(\frac{1}{2}U_1 + \frac{1}{2}U_2\right), \quad \text{with } \|2^{-1}U_1 + 2^{-1}U_2\| \leq 1.$$

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial.

(iv) $\Rightarrow$ (v). Because if $T = \lambda U + (1 - \lambda B)$ with $\lambda > 0$, $U \in \mathfrak{A}_u$, and $B \in \mathfrak{A}_1$, then $\lambda U \in \mathfrak{A}^{-1}$, so

$$\operatorname{dist}(T, \mathfrak{A}^{-1}) \leq \|T - \lambda U\| \leq 1 - \lambda < 1.$$

The implications (v) $\Rightarrow$ (vi) and (vi) $\Rightarrow$ (i) are due to Rørdam ([19]).

(v) $\Rightarrow$ (vi). Suppose that $\operatorname{sr}(\mathfrak{A}) \neq 1$. If $T \in \mathfrak{A}_1$ such that $\alpha = \operatorname{dist}(T, \mathfrak{A}^{-1}) > 0$, define $S = Vf(|T|)$, where $T = V|T|$ is the polar decomposition of T in $\mathbb{B}(H)$ and $f(t) = \min\{1, \alpha^{-1}t\}$. Since $f(0) = 0$ and f is continuous it follows that $S \in \mathfrak{A}$ and $S \in \mathfrak{A}_1$. But if $\operatorname{dist}(S, \mathfrak{A}^{-1}) < 1$, then with E_δ the spectral projection of $|S|$ corresponding to the interval (δ, ∞) there is a unitary $U \in \mathfrak{A}_u$ such that $VE_\delta = UE_\delta$ for some $\delta < 1$. Now, since $S = Vf(|T|) = Uf(|T|)$ and $T = V|T|$, it follows that E_δ is also a spectral projection of $|T|$, but corresponding to the interval $(\alpha\delta, \infty)$. Check that

$$|S|^2 = S^*S = f(|T|)U^*Uf(|T|) = f^2(|T|),$$

so that $|S| = f(|T|)$, hence, since $\frac{t}{\alpha} = \delta$ implies $t = \alpha\delta$,

$$E_\delta = \chi_{(\delta,\infty)}(|S|) = (\chi_{(\delta,\infty)} \circ f)(|T|) = \chi_{(\alpha\delta,\infty)}(|T|).$$

Since $\alpha\delta < \alpha = \operatorname{dist}(T, \mathfrak{A}^{-1})$, this is a contradiction to not being of such unitary extension U for T as proved before. Thus, $\operatorname{dist}(S, \mathfrak{A}^{-1}) \geq 1$, as desired.

(vi) $\Rightarrow$ (i). By assumption, $\operatorname{dist}(T, \mathfrak{A}^{-1}) = 0$ for every $T \in \mathfrak{A}$. If $T \in \mathfrak{A}_1$ is non-invertible, then for every $\beta > 2$ there are unitaries $U_1, U_2, U_3 \in \mathfrak{A}_u$, where $2 < \beta \leq 3$, such that

$$\begin{aligned} T &= \frac{1}{\beta}(U_1 + U_2) + \frac{\beta - 2}{\beta}U_3 \\ &= \frac{1}{2}(1 - (1 - \frac{2}{\beta}))(U_1 + U_2) + (1 - \frac{2}{\beta})U_3 \end{aligned}$$

with $0 < 1 - \frac{2}{\beta} \leq \frac{1}{3}$ since $2 < \beta \leq 3$.

Rørdam [19] also concluded as a corollary of his main result that if $\operatorname{sr}(\mathfrak{A}) = 1$, then for each $T \in \mathfrak{A}_1$ and each $\varepsilon \in (0, 1]$, there are unitaries $U_1, U_2, U_3 \in \mathfrak{A}_u$ such that

$$\begin{aligned} T &= \frac{1}{2 + \varepsilon}(U_1 + U_2 + \varepsilon U_3) \\ &= \frac{1}{2}(1 - \delta)(U_1 + U_2) + \delta U_3, \quad \text{with } 0 < \delta = \frac{\varepsilon}{2 + \varepsilon} \leq \frac{1}{3}. \end{aligned}$$

□

Remark. Note that if $\mathfrak{A}$ is a von Neumann algebra, then $T \in \mathfrak{A}_1$ is invertible if and only if $T = \frac{1}{2}(U_1 + U_2)$ with $U_1, U_2 \in \mathfrak{A}_u$ (see Olsen and Pedersen [12]). Indeed, let $T = U|T|$ be the polar decomposition of T with $U \in \mathfrak{A}_u$. Let $W = |T| + i(I - |T|^2)^{1/2}$. Then $|T| = \frac{1}{2}(W + W^*)$ with $W \in \mathfrak{A}_u$. Hence $T = \frac{1}{2}(UW + UW^*)$ with $UW, UW^* \in \mathfrak{A}_u$. Conversely,

$$T = \frac{1}{2}(U_1(I + U_1^*U_2)) = \frac{1}{2}U_1U_0|I + U_1^*U_2| = U|T|$$

with $U = U_1U_0$, where $I + U_1^*U_2 = U_0|I + U_1^*U_2|$ with a certain $U_0 \in \mathfrak{A}_u$. Note that $T^*T = \frac{1}{4}|I + U_1^*U_2|^2$, so that $|T| = \frac{1}{2}|I + U_1^*U_2|$.

Corollary 3.5.4. *Let X be a compact Hausdorff space and $C(X)$ be the C^*-algebra of all continuous functions on X. Then $C(X)$ has the λ-property if and only if the covering dimension of X is at most one, in which case $C(X)$ has the uniform λ-property for $\lambda = \frac{1}{2}$.*

Proof. Note that $\mathrm{sr}(C(X)) = 1$ if and only if $\dim X \leq 1$. □

3.6 Attaining the λ-function values

Proposition 3.6.1. *It T is an invertible element of the unit ball $\mathfrak{A}_1$ of a C^*-algebra $\mathfrak{A}$, then there exist V and W in $\mathfrak{A}_u$ such that*

$$T = \lambda(T)V + (1 - \lambda(T))W.$$

Proof. We have constructed a decomposition:

$$T = U|T| = \lambda_0 UU_1 + (1 - \lambda_0)UU_2$$

with $U, U_1, U_2 \in \mathfrak{A}_u$ and $\lambda_0 = \frac{1}{2}(1 + m(T))$. We further showed that $\lambda_0 = \lambda_u(T)$ for $T \in \mathfrak{A}^{-1}$. Since $\lambda_u(T) = \lambda(T)$ for $T \in \mathfrak{A}^{-1}$, we are done. □

Remark. We have shown in the subsection 3.4 that if $\mathfrak{A}$ is a von Neumann algebra, then the λ-function values are attained.

Proposition 3.6.2. *If X is a compact metric space, such that for every $f \in C(X)$, the number $\lambda(f)$ is attained in a decomposition for f, then X is finite.*

Proof. If X is infinite, there is a convergent sequence (x_n) in X with $x_n \neq x_m$ for $n \neq m$. Passing if necessary to a subsequence we may assume that $\mathrm{dist}(x_n, x_0) < \frac{2}{n\pi}$ for all n, where $x_0 = \lim x_n$, where $\mathrm{dist}(\cdot,\cdot)$ denotes the metric on X. Put $Y = \{x_n \,|\, n \geq 0\}$ and define $h \in C(Y)$ by $h(x_0) = 0$ and $h(x_n) = \frac{2}{n\pi}$ for $n \geq 1$. By Tietze's extension theorem, the function h extends to an element of $C(X)$, again denoted by h, with $0 \leq h \leq 1$. We may assume that $h(x) > 0$ for all $x \neq x_0$, replacing otherwise h with

$$\min\{1, \max\{h(x), \mathrm{dist}(x, x_0)\}\}$$

which does not change h on Y. Now define $f \in C(X)$ by

$$f(x) = h(x)e^{\frac{i}{h(x)}}, \quad x \in X \setminus \{x_0\}, \quad \text{and } f(x_0) = 0.$$

By construction, $f(x_n) = \frac{2}{n\pi} i^n$ for all n, with $i = \sqrt{-1}$. We claim that f can be approximated by invertible functions. Indeed, with

$$f_n(x) = h(x) e^{i(\min\{\frac{1}{h(x)}, n\})}$$

we have elements $f_n \in C(X)$ with $\operatorname{dist}(f_n, C(X)^{-1}) = 0$, because f_n is a product of a positive function with an invertible function. Also $f_n(x) = f(x)$ if $h(x) \geq \frac{1}{n}$, and

$$|f_n(x) - f(x)| \leq 2h(x) \leq \frac{2}{n}, \quad \text{otherwise.}$$

It follows that $\lambda_u(f) = \frac{1}{2}(1 - \alpha(T)) = \frac{1}{2}$. Thus, if we have a decomposition $f = \frac{1}{2}(u + b)$ with $u \in C(X)_u$ a circle-valued function and $b \in C(X)$ with norm ≤ 1, then

$$2\operatorname{Re}(u^* f) = \operatorname{Re}(1 + u^* b) \geq 0.$$

Consequently, we have $\operatorname{Re}(\bar{u}(x_n) i^n) \geq 0$ for all n, which is impossible since $(u(x_n))$ converges to $u(x_0)$. □

A sub-Stonean space is a compact Hausdorff space X such that any two disjoint open σ-compact subsets of X have disjoint closures.

If Y is an open subset of a compact Hausdorff space X, there is a continuous map $\Phi : \beta(Y) \to \overline{Y}$ from the Stone-Čech compactification $\beta(Y)$ of Y onto the closure $\overline{Y}$ of Y in X, extending the embedding map of Y into X.

Lemma 3.6.3. *The map $\Phi : \beta(Y) \to \overline{Y}$ is a homeomorphism for every open σ-compact subset Y of X if and only if X is a sub-Stonean space.*

Proof. It is shown by Grove and Pedersen [5] that if X is a sub-Stonean, then the map Φ for each Y is a homeomorphism. Since Φ is continuous and surjective, it is shown that Φ is injective, using $C(\beta(Y)) \cong C^b(Y)$, by reductio ad absurdum.

Conversely, take open σ-compact disjoint subsets Y and Z in X. Then $\beta(Y \cup Z) = \beta(Y) \oplus \beta(Z)$ (topological direct sum) and since $\Phi : \beta(Y) \oplus \beta(Z) \to \overline{Y} \cup \overline{Z}$ is a homeomorphism. it follows that $\overline{Y}$ and $\overline{Z}$ are disjoint. □

Remark. For any Tychonoff (or completely regular) space X, which is a Hausforff space such that for any $x \in X$ and any open neighbourhood U of x there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f = 0$ on $X \setminus U$, there is a Hausdorff compactification $\beta(X)$ with the property that every continuous function $f : X \to Y$, where Y is a compact Hausdorff space, extends to a continuous function from $\beta(X)$ to Y. Note that $C(\beta(X)) \cong C^b(X)$ the C^*-algebra of all bounded continuous functions on X, and thus,

$$C(\beta(Y \cup Z)) \cong C^b(Y \cup Z) \cong C^b(Y) \oplus C^b(Z) \cong C(\beta(Y)) \oplus C(\beta(Z)),$$

and hence $\overline{Y} \oplus \overline{Z} \approx \beta(Y \cup Z) \approx \overline{Y \cup Z} = \overline{Y} \cup \overline{Z}$.

Proposition 3.6.4. *If X is a compact Hausdorff space, the following are equivalent:*

(i) *X is sub-Stonean and* $\dim X \leq 1$.

(ii) *Every element f of $C(X)$ has a unitary polar decomposition $f = u|f|$ with $u \in C(X)_u$.*

(iii) $C(X)_1 = \frac{1}{2}(C(X)_u + C(X)_u)$.

Proof. (i) $\Rightarrow$ (ii). Given $f \in C(X)$, let $Y = \{x \in X \,|\, f(x) \neq 0\}$. Define $w(x) = \frac{f(x)}{|f(x)|}$ for $x \in Y$. Since Y is open and σ-compact, $\bar{Y}$ is homeomorphic to $\beta(Y)$, and as $w \in C^b(Y) = C(\beta(Y))$, the function w extends by continuity to a unitary function on $\bar{Y}$. Since $\dim X \leq 1$, every unitary function w (extended) on a closed subset extends to an element $u \in C(X)_u$, and $f = u|f|$.

(ii) $\Rightarrow$ (iii). If $f \in C(X)$ with $\|f\| \leq 1$, we have $f = u|f|$ for some $u \in C(X)_u$. Define $v = |f| + i\sqrt{1 - |f|^2}$, which is unitary, and

$$\frac{1}{2}(uv + uv^*) = u|f| = f \quad \text{with } uv, uv^* \in C(X)_u.$$

(iii) $\Rightarrow$ (i). If $f = \frac{1}{2}(u + v)$ with $u, v \in C(X)$ unitaries, then for $\varepsilon > 0$,

$$g = \frac{1}{2}(1 + \varepsilon)u + \frac{1}{2}(1 - \varepsilon)v \in C(X)^{-1}$$

because for any $x \in X$,

$$|g(x)| \geq \begin{cases} \frac{1}{2}(1+\varepsilon) - \frac{1}{2}(1-\varepsilon) = \varepsilon > 0, & \text{if } 0 < \varepsilon \leq 1, \\ \frac{1}{2}(1+\varepsilon) - \frac{1}{2}(\varepsilon - 1) = 1 > 0, & \text{if } \varepsilon \geq 1, \end{cases}$$

and also $\|f - g\| \leq \varepsilon$. It follows that invertible elements of $C(X)$ is dense in $C(X)$, whence $\dim(X) \leq 1$.

Now let Y and Z be disjoint open σ-compact subsets of X. Choose positive $f, g \in C(X)$ with norm < 1 and Y and Z as the complements of their zero sets, respectively. By assumption we have unitary functions u and v, such that $f - ig = \frac{1}{2}(u + v)$. If x is a point on the boundary of Z, it follows that $u(x)$ equal ± 1. Indeed, let $x_n \in Z$ such that $\lim x_n = x$, for which we have $-ig(x_n) = \frac{1}{2}(u(x_n)+v(x_n))$ and $ig(x_n) = \frac{1}{2}(\overline{u(x_n)}+\overline{v(x_n)})$. Thus, $\mathrm{Re}(u(x_n))+\mathrm{Re}(v(x_n)) = 0$ and $\mathrm{Im}(u(x_n))+\mathrm{Im}(v(x_n)) = -2g(x_n) < 0$. It follows from plane geometry that $u(x) = \pm 1$ and $v(x) = \mp 1$ as $n \to \infty$. Similarly, if we take $y_n \in Y$ such that $\lim y_n = y$ a point on the boundary of Y, then we have $\mathrm{Re}(u(y_n)) + \mathrm{Re}(v(y_n)) = 2f(y_n) > 0$ and $\mathrm{Im}(u(y_n)) + \mathrm{Im}(v(y_n)) = 0$. It follows that $u(y) = \pm i$ and $v(y) = \mp i$. Consequently we must have $\overline{Y} \cap \overline{Z} = \emptyset$, so that X is sub-Stonean. □

We say that a compact Hausdorff space X is sub-Stonean at most of order 2 if for each open σ-compact subset Y of X, the map $\Phi : \beta(Y) \to \overline{Y}$ is at most of order 2 at any point, which means that a point on the boundary of Y can be reached as a limit of at most two distinct universal nets in Y.

Lemma 3.6.5. *A compact Hausdorff space X is sub-Stonean at most of order 2 if and only if given any pairwise disjoint, open σ-compact subsets Y_1, Y_2, Y_3 of X we have $\overline{Y}_1 \cap \overline{Y}_2 \cap \overline{Y}_3 = \emptyset$.*

Proof. If $x \in \overline{Y}_1 \cap \overline{Y}_2 \cap \overline{Y}_3$, put $Y = Y_1 \cup Y_2 \cup Y_3$. Then the map $\Phi : \beta(Y) \to \overline{Y}$ have order 3 at the point x, i.e., $\Phi^{-1}(x)$ consists of at least 3 points, because

$$\beta(Y) = \beta(Y_1) \oplus \beta(Y_2) \oplus \beta(Y_3)$$

since $C(\beta(Y)) = C^b(Y_1 \cup Y_2 \cup Y_3) \cong C(\beta(Y_1)) \oplus C(\beta(Y_2)) \oplus C(\beta(Y_3))$.

Conversely, assume that X is not sub-Stonean at most of order 2. Thus for some open σ-compact subset Y of X we have distinct points $\gamma_1, \gamma_2, \gamma_3 \in \beta(Y)$, such that $\Phi(\gamma_i) = x$ $(i = 1, 2, 3)$ for some $x \in \overline{Y}$. Choose $f \in C^b(Y) = C(\beta(Y))$ such that $f(\gamma_i)$ $(i = 1, 2, 3)$ are three distinct points in $\mathbb{C}$. Let U_i be pairwise disjoint, open neighbourhoods of $f(\gamma_i)$ $(i = 1, 2, 3)$. Put $Y_i = f^{-1}(U_i)$, as subsets of Y, which are pairwise disjoint, open σ-compact subsets of X. Since $\beta(Y_i)$ is the closure of Y_i in $\beta(Y)$ and $Y_i \subset \beta(Y_i)$, it follows that $\gamma_i \in \beta(Y_i)$, whence $x \in \Phi(\beta(Y_i)) = \overline{Y}_i$ for $i = 1, 2, 3$. Note the following inclusions: $C_0(Y_i) \subset C^b(Y_i) = C(\beta(Y_i)) \leftarrow C(\beta(Y)) = C^b(Y)$, where $C_0(Y_i)$ the C^*-algebra of all continuous functions on Y_i vanishing at infinity and the quotient map is the restriction to Y_i, so that $\beta(Y_i) \subset \beta(Y)$ as a closed subset. □

Theorem 3.6.6. *If $\mathfrak{A} = C(X)$ is a unital commutative C^*-algebra, then*

$$(*): \quad \mathfrak{A}_1 (= C(X)_1) = \frac{1}{2}(\mathfrak{A}_u + \mathfrak{A}_1)(= \frac{1}{2}(C(X)_u + C(X)_1))$$

if X is a sub-Stonean space with $\dim X \leq 1$. Conversely, if the condition $(*)$ is satisfied, then X is sub-Stonean at most of order 2, with $\dim X \leq 1$.

Proof. It is shown by the propositon above that X is sub-Stonean and $\dim X \leq 1$ if and only if $C(X)_1 = \frac{1}{2}(\mathfrak{A}_u + \mathfrak{A}_u)$. This implies that $\mathfrak{A}_1 = \frac{1}{2}(\mathfrak{A}_u + \mathfrak{A}_1)$.

To prove the second, note first that if $u \in \mathfrak{A}_u$ and $b \in \mathfrak{B}$, then $(1+\varepsilon)u + (1-\varepsilon)b$ is invertible for every $\varepsilon > 0$ and close to $u + b$. Check that for any $x \in X$,

$$|(1+\varepsilon)u(x) + (1-\varepsilon)b(x)| \geq \begin{cases} (1+\varepsilon) - (1-\varepsilon) = 2\varepsilon > 0 & \text{if } 0 < \varepsilon \leq 1, \\ (1+\varepsilon) - (\varepsilon - 1) = 2 > 0 & \text{if } \varepsilon \geq 1. \end{cases}$$

It follows that the condition $(*)$ implies that $\mathfrak{A}^{-1}$ is dense in $\mathfrak{A}$ (i.e. $\mathrm{sr}(\mathfrak{A}) = 1$), so that $\dim X \leq 1$. Note that $\mathrm{sr}(C(X)) = \lfloor \frac{\dim X}{2} \rfloor + 1$, where $\lfloor x \rfloor$ means the maximum integer $\leq x \in \mathbb{R}$.

To prove that X is sub-Stonean at most of order 2, using the lemma above, let Y_1, Y_2, Y_3 be pairwise disjoint, open σ-compact subsets of X, choose positive functions $f_1, f_2, f_3 \in C(X)$ with norm less than 2 and with complements of their zero sets Y_1, Y_2, Y_3 respectively. Let $\theta = \exp(\frac{2}{3}\pi i)$, and define $f = \theta f_1 + \theta^2 f_2 + f_3$. If the condition $(*)$ is satisfied, we have $f = u + b$ for some $u \in \mathfrak{A}_u$ and $b \in \mathfrak{A}_1$. For any $x \in Y_3$ we have $u(x) + b(x) = f_3(x) > 0$, and it follows from plane geometry that $\mathrm{Re}(u(x)) \geq \frac{1}{2} f_3(x)$. Indeed, we have $\mathrm{Re}(u(x)) + \mathrm{Re}(b(x)) = f_3(x)$ and $\mathrm{Im}(u(x)) + \mathrm{Im}(b(x)) = 0$. Since $f_3(x) > 0$, we have $-\mathrm{Re}(u(x)) < \mathrm{Re}(b(x)) \leq \mathrm{Re}(u(x))$, so that $2\mathrm{Re}(u(x)) \geq f_3(x)$. Therefore, if x belongs to the boundary of Y_3 we see from the continuity of u that $\mathrm{Re}(u(x)) \geq 0$.

Similar arguments show that if x belongs to the boundary of Y_1, then we have $\mathrm{Re}(\theta^2 u(x)) \geq 0$, and if x belongs to the boundary of Y_2, then $\mathrm{Re}(\theta u(x)) \geq 0$. Indeed, note that for any $x \in Y_1$, we have $\theta^2 u(x) + \theta^2 b(x) = f_1(x) > 0$, and for any $x \in Y_2$, we have $\theta u(x) + \theta b(x) = f_2(x) > 0$.

If now $x \in \overline{Y}_1 \cap \overline{Y}_2 \cap \overline{Y}_3$, then $u(x)$ should belong to the 3 half spaces $\mathbb{C}_+$, $\theta\mathbb{C}_+$, and $\theta^2\mathbb{C}_+$, where $\mathbb{C}_+ = \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \geq 0\}$, whose intersection is $\{0\}$. But $|u(x)| = 1$, and we have reached a contradiction. Therefore, the intersection $\overline{Y}_1 \cap \overline{Y}_2 \cap \overline{Y}_3$ is empty. This shows that X is sub-Stonean at most of order 2. □

Proposition 3.6.7. *There exists a compact Hausdorff space X, which is not sub-Stonean but only sub-Stonean at most of order 2, such that the condition $(*) : C(X)_1 = \frac{1}{2}(C(X)_u + C(X)_1)$ holds.*

Proof. Let $X_1 = \beta(\mathbb{R}_+) \setminus \mathbb{R}_+$. It follows from [5] that X_1 is a connected sub-Stonean space of dimension 1. Choose any non-trivial open σ-compact subset Y of X_1, and let x_0 be a point in $\overline{Y} \setminus Y$. (If none existed, Y would be closed as well as open, contradicting the fact that X_1 is connected.) Let

$X_2 = X_1$. Define X to be the topological union of X_1 and X_2, glued together at x_0. Then X is a compact connected Hausdorff space of dimension 1, but it is not sub-Stonean, because the two copies of Y in X_1 and X_2 are disjoint open σ-compact subsets of X with a common boundary point x_0.

We wish to prove that λ-function values are attained for any element of $C(X)_1$. Note that

$$C(X) = \{(f_1, f_2) \in C(X_1) \oplus C(X_2) \mid f_1(x_0) = f_2(x_0)\},$$

and take $f = (f_1, f_2) \in C(X)_1$. If f is invertible, its λ-function value is attained with $\lambda(f) = \lambda_u(f)$, as shown before. We may therefore assume that f is not invertible, whence $\lambda_u(f) \leq \frac{1}{2}$. Indeed, suppose that $\lambda_u(f) > \frac{1}{2}$. Then there is $1 > \lambda > \frac{1}{2}$ such that $T = \lambda U + (1-\lambda)B$ for some $U \in C(X)_u$ and $B \in C(X)_1$, so that

$$\|T - \lambda U\| = (1-\lambda)\|B\| \leq 1 - \lambda < \frac{1}{2},$$

and hence $\|\lambda^{-1}T - U\| < \frac{1}{2\lambda} < 1$, which implies that $\|\lambda^{-1}U^*T - I\| < 1$, which implies that T is invertible.

However, since $\dim X = 1$, we have $\mathrm{sr}(C(X)) = 1$, so that $\lambda(f) \geq \lambda_u(f) \geq \frac{1}{2}$ for every $f \in C(X)_1$. Thus, $\lambda_u(f) = \frac{1}{2}$.

If $f(x_0) \neq 0$, we choose unitaries u_1, u_2 in $C(X_1)_u$ and $C(X_2)_u$ respectively such that $f_i = u_i|f_i$ $(i = 1, 2)$, where each f_i is the restriction of f to X_i. Then

$$u_1(x_0) = \frac{f_1(x_0)}{|f_1(x_0)|} = \frac{f_2(x_0)}{|f_2(x_0)|} = u_2(x_0),$$

so that $u = (u_1, u_2)$ is a unitary in $C(X)$ with $f = u|f|$. With $v = |f| + i\sqrt{1 - |f|^2}$, we obtain $f = \frac{1}{2}(uv + uv^*)$, with $uv, uv^* \in C(X)_u$.

We are left with the case where $f(x_0) = 0$. Find $w_1 \in C(X_1)_u$ such that $f_1 = w_1|f_1|$, and extend it from the closed subset X_1 to a continuous, circle-valued function w on the 1-dimensional space X. Replacing f with w^*f we see that it suffices to consider the case where $f = (f_1, f_2)$, $f_1(x_0) = f_2(x_0) = 0$ and $f_1 \geq 0$. Find elements $v_1, v_2 \in C(X_2)_u$ such that $f_2 = \frac{1}{2}(v_1 + v_2)$. The remaining task is to find suitable continuous extensions of these functions on X_1. Without loss of generality, we may assume that $\mathrm{Re}(v_1(x_0)) \geq 0$. Indeed, $v_1(x_0) + v_2(x_0) = 0$ and $\overline{v_1(x_0)} + \overline{v_2(x_0)} = 0$, so that $\mathrm{Re}(v_1(x_0)) + \mathrm{Re}(v_2(x_0)) = 0$, and also $\mathrm{Im}(v_1(x_0)) + \mathrm{Im}(v_2(x_0)) = 0$. Furthermore we may assume that $\mathrm{Im}(v_1(x_0)) \geq 0$, since the argument for $\mathrm{Im}(v_1(x_0)) \leq 0$ is quite symmetric.

Let $Z = \{x \in X_1 \mid f_1(x) \leq \mathrm{Re}(v_1(x_0))\}$. This is a closed subset of X_1 containing x_0. For each $x \in Z$, define $v(x) = v_1(x_0)$ and $b(x) =$

$2f_1(x) - v_1(x_0)$. Then $f_1 = \frac{1}{2}(v+b)$ on Z, and $|b(x)| \leq 1$. Check that $0 \leq 2f_1(x) \leq f_1(x) + \mathrm{Re}(v_1(x_0))$, so that $2f_1(x) - \mathrm{Re}(v_1(x_0)) \leq f_1(x) \leq \mathrm{Re}(v_1(x_0))$, and note that

$$\mathrm{Re}(2f_1(x) - v_1(x_0)) = 2f_1(x) - \mathrm{Re}(v_1(x_0)), \quad \mathrm{Im}(2f_1(x) - v_1(x_0)) = \mathrm{Im}(v_1(x_0)),$$

and hence $|b(x)| \leq |\, v_1(x_0)| = 1$.

Moreover, v and b are continuous extensions of v_1 and v_2 from the union $X_2 \cup_{x_0} Z$ glued at x_0, because $b(x_0) = -v_1(x_0) = v_2(x_0)$.

For each $x \in X_1 \setminus Z$ we define

$$v(x) = f_1(x) + i\sqrt{1 - |f_1(x)|^2}, \quad b(x) = f_1(x) - i\sqrt{1 - |f_1(x)|^2}.$$

These functions are unitary and continuous on $X_1 \setminus Z$, with $f_1 = \frac{1}{2}(v+b)$. To see that v and b are continuous on X, consider any point x on the boundary of Z. By definition of Z we must have $f_1(x) = \mathrm{Re}(v_1(x_0))$, which implies that

$$v(x) = v_1(x_0) = f_1(x) + i\sqrt{1 - |f_1(x)|^2}.$$

Moreover, $b(x) = 2f_1(x) - v_1(x_0) = f_1(x) - i\sqrt{1 - |f_1(x)|^2}$. We see that v and b on the boundary of Z agree with the definitions of v and b given on $X_1 \setminus Z$, and thus v and b are continuous on X_1. Put $w = (v, v_1)$ and $c = (b, v_2)$. Then w and c belong to $C(X)$, and w is unitary and $\|c\| \leq 1$, and $f = \frac{1}{2}(w+c)$ as:

$$\frac{1}{2}(w+c) = (\frac{1}{2}(v+b), \frac{1}{2}(v_1+v_2)) = (f_1, f_2).$$

$\square$

Remark. For C^*-algebras $\mathfrak{A}$ that are σ-finite in the sense that any family of non-zero pairwise orthogonal elements is countable, Haagerup and Rørdam [6] showed that the condition $\mathfrak{A}_1 = \frac{1}{2}(\mathfrak{A}_u + \mathfrak{A}_u)$ implies that $\mathfrak{A}$ is a finite AW^*-algebra. It had been known from Pedersen [16] that the condition implies that $\mathfrak{A}_1 = \mathfrak{A}_u\mathfrak{A}_{sa}$, so their aim was to provide a unitary polar decomposition for every self-adjoint element in $\mathfrak{A}_1$.

A C^*-algebra $\mathfrak{A}$ is an SAW^*-algebra (sub-AW^*) if for any two elements $S, T \in \mathfrak{A}$ with $ST = 0$, there is an $E \in \mathfrak{A}$ with $0 \leq E \leq I$, such that $SE = 0 = (I-E)T$. As shown by Pedersen [14], a commutative C^*-algebra $\mathfrak{A} = C(X)$ is an SAW^*-algebra if and only if X is a sub-Stonean space. If every element in a C^*-algebra $\mathfrak{A}$ has a unitary polar decomposition, i.e., $\mathfrak{A} = \mathfrak{A}_u\mathfrak{A}_+$, then $\mathfrak{A}$ is an SAW^*-algebra with $\mathrm{sr}(\mathfrak{A}) = 1$. The converse holds under the additional hypothesis that also $M_2(\mathfrak{A})$ is an SAW^*-algebra.

Conjecture. (This might be still open.) *For a unital C^*-algebra $\mathfrak{A}$ the following conditions are equivalent:*

(i) $\mathfrak{A}_1 = \frac{1}{2}(\mathfrak{A}_u + \mathfrak{A}_u)$.

(ii) $\mathfrak{A} = \mathfrak{A}_u\mathfrak{A}_+$.

(iii) $\mathfrak{A}$ *is an SAW^*-algebra with stable rank one.*

Moreover, if $\mathfrak{A}$ acts on a separable Hilbert space, the condition (iv) $\mathfrak{A}_1 = \frac{1}{2}(\mathfrak{A}_u + \mathfrak{A}_1)$ *implies that $\mathfrak{A}$ is a von Neumann algebra.*

3.7 Left invertible elements

For a C^*-algebra $\mathfrak{A}$, we have the following inclusions:

$$\mathfrak{A}_1^{-1} \subset \mathfrak{A}_u\mathfrak{A}_{1,+} \subset \frac{1}{2}(\mathfrak{A}_u + \mathfrak{A}_u) \subset \frac{1}{2}(\mathfrak{A}_u + \mathfrak{A}_1) \subset \mathrm{cl}(\mathfrak{A}_1^{-1}),$$

where $\mathfrak{A}_1^{-1} = \mathfrak{A}_1 \cap \mathfrak{A}^{-1}$ and $\mathfrak{A}_{1,+} = \mathfrak{A}_1 \cap \mathfrak{A}_+$ by definition. Check (again) that if $T = U|T|$ is the polar decomposition of $T \in \mathfrak{A}_1^{-1}$, then $U \in \mathfrak{A}_u$ and $1 \geq \|T\| = \||T|\|$. The second inclusion is equivalent to another $\mathfrak{A}_{1,+} \subset \frac{1}{2}(\mathfrak{A}_u + \mathfrak{A}_u)$. This is true by the decomposition

$$|T| = \frac{1}{2}((|T| + i\sqrt{1-|T|^2}) + (|T| - i\sqrt{1-|T|^2})).$$

For the last one, note that $\frac{1}{2}((1+\varepsilon)\mathfrak{A}_u + (1-\varepsilon)\mathfrak{A}_1) \subset \mathfrak{A}_1^{-1}$ for every $0 < \varepsilon \leq 1$.

Define $\mathfrak{A}_l^{-1} = \{A \in \mathfrak{A} \mid \mathfrak{A}A = \mathfrak{A}\}$ to be the multiplicative semigroup of left invertible elements. Thus $A \in \mathfrak{A}_l^{-1}$ if $BA = I$ for some $B \in \mathfrak{A}$. It follows from the open mapping theorem that $A \in \mathfrak{A}_l^{-1}$if and only if A is injective with closed range, so that

$$\mathfrak{A}_l^{-1} = \{A \in \mathfrak{A} \mid m(A) > 0\}$$

(which is checked before). Indeed, if $BA = I$, then we have that if $A\xi = A\eta$, then $\xi = BA\xi = BA\eta = \eta \in H$, and also if $A\xi_n \to \eta \in H$, then $\xi_n = BA\xi_n \to B\eta$, so that $A\xi_n \to AB\eta$, and thus $\eta = AB\eta \in AH$. Conversely, if A is injective with closed range, the open mapping theorem implies the existence of a bounded left inverse to A, and also $A \in \mathfrak{A}_l^{-1}$.

Using $BA = I$ we have

$$I = A^*B^*BA \leq \|B\|^2 A^*A = \|B\|^2|A|^2$$

and that $|A|$ is invertible. Moreover, $V = A|A|^{-1}$ is an isometry since $V^*V = I$. Consequently, $A \in \mathfrak{A}_l^{-1}$ if and only if it has a polar decomposition

$A = V|A|$ with V an isometry and $|A|$ invertible. In symbols, $\mathfrak{A}_l^{-1} = \mathfrak{A}_{is}\mathfrak{A}_+^{-1}$, where $\mathfrak{A}_{is}$ denotes the set of all isometries in $\mathfrak{A}$, so that $\mathfrak{A}_{is} \subset \mathfrak{A}_e$.

Setting $\mathfrak{A}_{1,l}^{-1} = \mathfrak{A}_1 \cap \mathfrak{A}_l^{-1}$ we have the following inclusions:

$$\mathfrak{A}_{1,l}^{-1} \subset \mathfrak{A}_{is}\mathfrak{A}_{1,+} \subset \frac{1}{2}(\mathfrak{A}_{is} + \mathfrak{A}_{is}) \subset \frac{1}{2}(\mathfrak{A}_{is} + \mathfrak{A}_1) \subset \mathrm{cl}(\mathfrak{A}_{1,l}^{-1}).$$

For the second, since $\mathfrak{A}_{1,+} \subset \frac{1}{2}(\mathfrak{A}_u + \mathfrak{A}_u) \subset \frac{1}{2}(\mathfrak{A}_{is} + \mathfrak{A}_{is})$, we have $\mathfrak{A}_{is}\mathfrak{A}_{1,+} \subset \frac{1}{2}(\mathfrak{A}_{is} + \mathfrak{A}_{is})$. For the last, observe that if $V \in \mathfrak{A}_{is}$ and $B \in \mathfrak{A}_1$, then for $t < 1$,

$$V + tB = (I + tBV^*)V \in \mathfrak{A}^{-1}V \subset \mathfrak{A}_l^{-1},$$

with $\|\frac{1}{2}(V + tB)\| \leq 1$ for $t \leq 1$, or

$$m(V + tB) \geq 1 - \|tB\| \geq 1 - t > 0,$$

where note that for $\xi \in H$ with $\|\xi\| = 1$,

$$\|(I + tBV^*)V\xi\| \geq \|V\xi\| - \|tBV^*V\xi\| \geq \|\xi\| - \|tB\|\|\xi\| = 1 - \|tB\|.$$

Denote by $\mathfrak{A}_r^{-1}$ the set of all right invertible elements in $\mathfrak{A}$. Note that $\mathfrak{A}_r^{-1} = (\mathfrak{A}_l^{-1})^*$.

If $\mathfrak{A}$ is a finite C^*-algebra, then $\mathfrak{A}_l^{-1} = \mathfrak{A}^{-1}$. Indeed, if $A \in \mathfrak{A}_l^{-1}$, then $A = U|A|$ with $|A| \in \mathfrak{A}^{-1}$ and $U^*U = I$, because $U = A|A|^{-1}$ and $U^*U = |A|^{-1}A^*A|A|^{-1} = I$. By finiteness, $UU^* = I$, so U is unitary and $A \in \mathfrak{A}^{-1}$. On the other hand, if $\mathfrak{A}_l^{-1}$ is dense in $\mathfrak{A}$, then $\mathfrak{A}$ is finite. For if $A \in \mathfrak{A}_l^{-1}$ and $BA = I$, we can find $C \in \mathfrak{A}_l^{-1}$ close to B such that $\|I - CA\| < 1$. Then $CA \in \mathfrak{A}^{-1}$, and if $DC = I$ we have

$$A = DCA \in D\mathfrak{A}^{-1} \in \mathfrak{A}_r^{-1},$$

whence $A \in \mathfrak{A}_l^{-1} \cap \mathfrak{A}_r^{-1} = \mathfrak{A}^{-1}$. Thus, if $A^*A = I$ in $\mathfrak{A}$, then $A^* = A^{-1}$, i.e., $A \in \mathfrak{A}_u$. As shown before, if $\mathfrak{A}^{-1}$ is dense in $\mathfrak{A}$, then $\mathfrak{A}_e = \mathfrak{A}_u$, so that if $A^*A = I$ in $\mathfrak{A}$, then $A \in \mathfrak{A}_e = \mathfrak{A}_u$.

By contrast, consider $\mathfrak{A} = \mathbb{B}(H)$ and let S be the unilateral shift on $l^2 = H$. Then $S \in \mathfrak{A}_l^{-1}$ since $S^*S = I$, but $\|S - A\| \geq 1$ for every $A \in \mathfrak{A}_r^{-1} \cap \mathfrak{A}_1$. Indeed, if $\|S - A\| < 1$ for $A \in \mathfrak{A}_r^{-1} \cap \mathfrak{A}_1$ with $AB = I$, then

$$\|SB - I\| \leq \|S - A\|\|B\| < 1,$$

so that $SB \in \mathfrak{A}^{-1}$, and thus $S \in \mathfrak{A}^{-1}$, a contradiction. Yet $\mathfrak{A}_l^{-1} \cup \mathfrak{A}_r^{-1}$ is dense in $\mathfrak{A}$ (for any factorial von Neumann algebra $\mathfrak{A}$), because for any $A \in \mathfrak{A}$, there is an extreme point $W \in \mathfrak{A}_e$ such that $A = W|A|$. Since $\mathfrak{A}$ is prime, i.e., the zero ideal of $\mathfrak{A}$ is prime, i.e., if $\mathfrak{J}\mathfrak{K} = \{0\}$ for closed ideals $\mathfrak{J}$ and $\mathfrak{K}$

of $\mathfrak{A}$, then $\mathfrak{J} = \{0\}$ or $\mathfrak{K} = \{0\}$, we have that every element V of $\mathfrak{A}_e$ is either an isometry or a co-isometry. Indeed, since $(I - VV^*)\mathfrak{A}(I - V^*V) = \{0\}$, the closed ideals $\mathfrak{J}$ and $\mathfrak{K}$ generated by $I - VV^*$ and $I - V^*V$ respectively satisfy $\mathfrak{J}\mathfrak{K} = \{0\}$, so that $\mathfrak{J} = \{0\}$ or $\mathfrak{K} = \{0\}$, equivalently, $I - VV^* = 0$ or $I - V^*V = 0$. Thus, if $W^*W = I$, then $|A|$ is closely approximated by an invertible element S in $\mathfrak{A}$, such that $\|A - WS\| < \varepsilon$, with $WS \in \mathfrak{A}_l^{-1}$, and if $WW^* = I$, then the same estimate holds, with $WS \in \mathfrak{A}_r^{-1}$.

For any element T in a C^*-algebra $\mathfrak{A}$ we define

$$\alpha_l(T) = \mathrm{dist}(T, \mathfrak{A}_l^{-1}).$$

Let $T = V|T|$ be the polar decomposition of T in $\mathbb{B}(H)$, and denote by E_δ the spectral projection of $|T|$ corresponding to the interval (δ, ∞) for $\delta > 0$, i.e., $E_\delta = \chi_{(\delta,\infty)}(|T|)$.

Theorem 3.7.1. *For each $\delta > \alpha_l(T)$ there is an isometry $U \in \mathfrak{A}_{is}$, such that $UE_\delta = VE_\delta$. For $\delta < \alpha_l(T)$ there is no isometric extension of VE_δ in $\mathfrak{A}$.*

Proof. If $\delta > \alpha_l(T)$ there is an $A \in \mathfrak{A}_l^{-1}$ such that $\|T - A\| < \delta$. Write $A = W|A|$, where $W \in \mathfrak{A}_{is}$ and $|A| \in \mathfrak{A}^{-1}$. Then

$$\|TW^* - AW^*\| \leq \|T - A\| < \delta,$$

and since $AW^* \in \mathrm{cl}(\mathfrak{A}^{-1})$ because $AW^* + \varepsilon I = W|A|W^* + \varepsilon I \in \mathfrak{A}^{-1}$ for every $\varepsilon > 0$, it follows that $\alpha(TW^*) = \mathrm{dist}(TW^*, \mathfrak{A}^{-1}) < \delta$.

Note now that TW^* has the polar decomposition $VW^*(W|T|W^*)$. Check that

$$(TW^*)^*(TW^*) = W|T|^2W^* = (W|T|W^*)^2,$$

and thus $|TW^*| = W|T|W^*$, and

$$(VW^*)^*VW^*(VW^*)^*VW^* = (VW^*)^*VV^*VW^* = (VW^*)^*VW^*,$$

and $\ker(TW^*) = \ker(VW^*)$, because if $VW^*\xi = 0$, then we have $TW^*\xi = V|T|V^*VW^*\xi = 0$, and conversely, if $TW^*\xi = 0$, then $VW^*\xi = VV^*VW^*\xi = 0$ since $W^*\xi \in \ker(T)$ and V^*V is the orthogonal projection to $\ker(T)^\perp$ that is the orthogonal complement of $\ker(T)$.

Moreover, if f is a polynomial without constant term, then $f(W|T|W^*) = Wf(|T|)W^*$ since $|T| = |T|W^*W$. The relation therefore holds when f is a Borel function wiht $f(0) = 0$, so if $E_\delta = \chi_{(\delta,\infty)}(|T|)$ with $\delta > 0$, then $WE_\delta W^*$ is the corresponding spectral projection for $W|T|W^*$, i.e., $\chi_{(\delta,\infty)}(W|T|W^*)$.

For $TW^* = VW^*(W|T|W^*)$, we find a unitary $U' \in \mathfrak{A}_u$ (as obtained before) such that

$$U'\chi_{(\delta,\infty)}(W|T|W^*) = VW^*\chi_{(\delta,\infty)}(W|T|W^*),$$

so that

$$U'WE_\delta W^* = VW^*(WE_\delta W^*) = VE_\delta W^*.$$

Therefore, $U = U'W \in \mathfrak{A}_{is}$ and $UE_\delta = VE_\delta$, as desired.

Conversely, if $U \in \mathfrak{A}_{is}$ such that $UE_\delta = VE_\delta$ for some $\delta > 0$, define $f(t) = \max\{t - \delta, 0\}$. Then $S = Vf(|T|) \in \mathfrak{A}$. In fact, since $f(t) = 0$ for $t \leq \delta$, we have

$$S = Vf(|T|)E_\delta = UE_\delta f(|T|) = Uf(|T|) \in \mathrm{cl}(\mathfrak{A}_l^{-1}),$$

because $K_\varepsilon \equiv f(|T|) + \varepsilon I \in \mathfrak{A}^{-1}$ for every $\varepsilon > 0$, with $UK_\varepsilon \in \mathfrak{A}_l^{-1}$. Since

$$\|T - S\| = \||T| - f(|T|)\| \leq \delta,$$

it follows that $\alpha_l(T) \leq \delta$. Note that

$$\|T - S\|^2 = \|(|T| - f(|T|))V^*V(|T| - f(|T|)\| = \||T| - f(|T|)\|^2$$

and $t - f(t) \leq \delta$ for $t \geq 0$. □

Corollary 3.7.2. *Each element of the form $Vf(|T|)$ with $T = V|T| \in \mathfrak{A}$, where f is a continuous function on $\mathrm{sp}(|T|)$ such that $f(t) = 0$ for $t \leq \delta$, for some $\delta > \alpha_l(T)$, has a polar decomposition $Uf(|T|) = Vf(|T|)$, where U is an isometry in $\mathfrak{A}$.*

Proof. Since $f = \chi_{(\delta,\infty)}f$, the functional calculus implies that

$$Uf(|T|) = U\chi_{(\delta,\infty)}(|T|)f(|T|) = UE_\delta f(|T|) = VE_\delta f(|T|) = Vf(|T|).$$

□

Proposition 3.7.3. *If a C^*-algebra $\mathfrak{A}$ contains an element T with $\alpha_l(T) > 0$, then there is an $S \in \mathfrak{A}_1$ with $\alpha_l(S) = 1$. If, moreover, $\alpha_l(T^*) \geq \alpha_l(T)$ we may assume that also $\alpha_l(S^*) = 1$.*

Proof. We regard $\mathfrak{A}$ as a C^*-subalgebra of $\mathbb{B}(H)$ and let $T = V|T|$ be the polar decomposition of T. Assuming, as we may, that $\|T\| = 1$ we let $S = Vf(|T|)$, where $f(t) = \min\{1, \frac{t}{\alpha_l(T)}\}$ for $0 \leq t \leq 1$.

If $\alpha_l(S) < 1$, then with $E_\delta = \chi_{(\delta,\infty)}(|S|)$, there is an isometry $U \in \mathfrak{A}_{is}$ such that $UE_\delta = VE_\delta$ for some $\alpha_l(S) < \delta < 1$. Since $S = Vf(|T|)$ and $T =$

$V|T|$, it follows that E_δ is also a spectral projection for $|T|$, corresponding to the interval $(\delta\alpha_l(T), \infty)$. Indeed, check that since $f(0) = 0$, we have $S^*S = f(|T|)V^*Vf(|T|) = f(|T|)^2$, so that $|S| = f(|T|)$, and

$$E_\delta = \chi_{(\delta,\infty)}(|S|) = \chi_{(\delta,\infty)}(f(|T|)) = \chi_{(\delta\alpha_l(T),\infty)}(|T|) \equiv E_{\delta\alpha_l(T)},$$

where $\chi_{(\delta,\infty)} \circ f(t) = \chi_{(\delta\alpha_l(T),\infty)}(t)$ for $0 \le t \le 1$, by solving $\frac{t}{\alpha_l(T)} = \delta$. Since $\delta\alpha_l(T) < \alpha_l(T)$, this contradicts with $UE_{\delta\alpha_l(T)} = VE_{\delta\alpha_l(T)}$. Thus $\alpha_l(S) = 1$. In fact, note also that $\|S\| = \|f(|T|)\| = \|f\|_\infty = 1$, and for every $\varepsilon > 0$, we have $\|S - \varepsilon I\| \le 1 + \varepsilon$, so that $\alpha_l(S) \le 1 + \varepsilon$, and hence $\alpha_l(S) \le 1$.

If $\alpha_l(T^*) \ge \alpha_l(T)$ and $\alpha_l(S^*) < 1$, then since $S^* = V^*(V|S|V^*)$, with

$$SS^* = Vf(|T|)^2V^* = (Vf(|T|)V^*)^2 = (V|S|V^*)^2,$$

so that $|S^*| = V|S|V^*$, and $\ker(S^*) = \ker(V^*)$, because if $S^*\xi = 0$, then $V^*\xi = V^*VV^*\xi = 0$ since VV^* is the orthogonal projection to $\ker(S^*)^\perp$, and if $V^*\xi = 0$, then $S^*\xi = f(|T|)V^*\xi = 0$, therefore, we can find an isometry $U \in \mathfrak{A}_{is}$ such that

$$U(VE_\delta V^*) = V^*(VE_\delta V^*)$$

with some $\alpha_l(S^*) < \delta < 1$, where $\chi_{(\delta,\infty)}(V|S|V^*) = VE_\delta V^*$ since this also holds for polynomials without constant term. Since $VE_\delta V^*$ is the spectral projection of $|T^*| = V|T|V^*$ corresponding to the interval $(\delta\alpha_l(T), \infty)$, indeed,

$$VE_\delta V^* = V\chi_{(\delta,\infty)}(f|T|)V^* = V\chi_{(\delta\alpha_l(T),\infty)}(|T|)V^* = \chi_{(\delta\alpha_l(T),\infty)}(|T^*|),$$

we conclude that $\alpha_l(T^*) \le \delta\alpha_l(T) < \alpha_l(T)$. We have reached a contradiction with $\alpha_l(T^*) \ge \alpha_l(T)$. Thus $\alpha_l(S^*) = 1$. $\square$

Lemma 3.7.4. (cf. [11]). *For every element T in a C^*-algebra $\mathfrak{A}$ and any isometry U in $\mathfrak{A}$, the spectrum of TU^* contains the disc with center 0 and radius $\alpha_l(T)$.*

Proof. If $\lambda \in \mathbb{C}$ with $|\lambda| < \alpha_l(T)$, but $\lambda \notin \mathrm{sp}(TU^*)$, then $TU^* - \lambda I = A \in \mathfrak{A}^{-1}$. Then

$$\|T - AU\| = \|\lambda U\| = |\lambda| < \alpha_l(T),$$

a contradiction, since $AU \in \mathfrak{A}_l^{-1}$. $\square$

Theorem 3.7.5. *Let $\mathfrak{A}$ be a C^*-algebra and $T \in \mathfrak{A}$. If $T \notin \mathfrak{A}_l^{-1}$, then*

$$\mathrm{dist}(T, \mathfrak{A}_{is}) = \max\{\|T\| - 1, \alpha_l(T) + 1\}.$$

Otherwise we have an approximant V in $\mathfrak{A}_{is}$ with

$$\mathrm{dist}(T, \mathfrak{A}_{is}) = \|T - V\| = \max\{\|T\| - 1, 1 - m(T)\}.$$

Proof. If $T \notin \mathfrak{A}_l^{-1}$, then for any $U \in \mathfrak{A}_{is}$,

$$\|T - U\| \geq \|U^*(T - U)\| = \|U^*T - I\| \geq r(U^*T - I) \geq 1 + \alpha_l(T),$$

because the spectral radius $r(U^*T-I)$ of U^*T-I must be at least $1+\alpha_l(T)$ by the lemma above, and note that $\mathrm{sp}(U^*T - I) = \mathrm{sp}(U^*T) - 1$. Clearly we also have $\|T - U\| \geq \|T\| - 1$, so we have established the following inequality:

$$\mathrm{dist}(T, \mathfrak{A}_{is}) \geq \max\{\|T\| - 1, \alpha_l(T) + 1\}.$$

To prove the reverse inequality, consider $\delta > \alpha_l(T)$. Then there is an isometry $W \in \mathfrak{A}_{is}$ such that $\alpha(TW^*) < \delta$, as in the proof of the theorem above. There is for any $\varepsilon > 0$, a unitary $U \in \mathfrak{A}_u$ such that

$$\|TW^* - U\| < \max\{\|TW^*\| - 1 + \varepsilon, \delta + 1\}.$$

because if $T \notin \mathfrak{A}^{-1}$, then $\mathrm{dist}(T, \mathfrak{A}_u) = \max\{\|T\| - 1, \mathrm{dist}(T, \mathfrak{A}^{-1}) + 1\}$, as shown before, and note that if $TW^* \in \mathfrak{A}^{-1}$, then $WT^* \in \mathfrak{A}^{-1}$, so that W must be in $\mathfrak{A}_u$, and thus T^* and also T are in $\mathfrak{A}^{-1}$, a contradiction.

It follows that $UW \in \mathfrak{A}_{is}$ with

$$\|T - UW\| = \|(TW^* - U)W\| \leq \|TW^* - U\| < \max\{\|T\| - 1 + \varepsilon, \delta + 1\}.$$

Since ε and δ are arbitrary we get

$$\mathrm{dist}(T, \mathfrak{A}_{is}) \leq \max\{\|T\| - 1, \alpha_l(T) + 1\}.$$

If $T \in \mathfrak{A}_l^{-1}$, we have a polar decomposition $T = V|T|$ with $V \in \mathfrak{A}_{is}$ and $|T| \in \mathfrak{A}^{-1}$. Thus, we have $m(T) = \frac{1}{\||T|^{-1}\|}$ as obtained before, and evidently

$$\|T - V\| = \||T| - I\| = \max\{\|T\| - 1, 1 - m(T)\}$$

by functional calculus, and note that using the C^*-norm condition,

$$\|T - V\|^2 = \|V(|T| - I)\|^2 = \|(|T| - I)^2\| = \||T| - I\|^2.$$

Therefore, we get $\mathrm{dist}(T, \mathfrak{A}_{is}) \leq \|T - V\| = \max\{\|T\| - 1, 1 - m(T)\}$.

Conversely, if $U \in \mathfrak{A}_{is}$, then $\|T - U\| \geq \|T\| - 1$, and moreover,

$$\begin{aligned}\|U - T\| &= \sup_x \|(U - T)x\| \geq \sup_x(\|Ux\| - \|Tx\|) \\ &= \sup_x(1 - \|Tx\|) = 1 - \inf_x \|Tx\| = 1 - m(T),\end{aligned}$$

where x ranges over the set of unit vectors in H. In fact, check that since $1 - \|T\xi\| \leq 1 - \inf_x \|Tx\|$ for $\xi \in H$ with $\|\xi\| = 1$, we have $\sup_\xi(1 - \|T\xi\|) \leq 1 - \inf_x \|Tx\|$, and conversely, for any $\varepsilon > 0$, there is $\xi \in H$ with norm 1 such that $\inf_x \|Tx\| + \varepsilon > \|T\xi\|$, so that

$$\sup_\xi(1 - \|T\xi\|) + \varepsilon \geq 1 - \|T\xi\| + \varepsilon > 1 - \inf_x \|Tx\|,$$

from which, the reverse inequality holds. It follows from the first and second inequalities obtained above that

$$\mathrm{dist}(T, \mathfrak{A}_{is}) \geq \max\{\|T\| - 1, 1 - m(T)\}.$$

□

3.8 Prime C^*-algebras

For an element T in a C^*-algebra $\mathfrak{A}$, we define

$$\alpha_p(T) = \mathrm{dist}(T, \mathfrak{A}_l^{-1} \cup \mathfrak{A}_r^{-1}) = \min\{\alpha_l(T), \alpha_l(T^*)\}.$$

Note that a C^*-algebra $\mathfrak{A}$ is prime, if and only if for any $S, T \in \mathfrak{A}$, $S\mathfrak{A}T = 0$ implies $S = 0$ or $T = 0$. Indeed, if $S\mathfrak{A}T = 0$, then $(\mathfrak{A}S\mathfrak{A})(\mathfrak{A}T\mathfrak{A}) = 0$, so that primeness implies $\mathfrak{A}S\mathfrak{A} = 0$ or $\mathfrak{A}T\mathfrak{A} = 0$, so that $S = 0$ or $T = 0$. Conversely, if two closed ideals $\mathfrak{J}$ and $\mathfrak{K}$ of $\mathfrak{A}$ satisfy $\mathfrak{J}\mathfrak{K} = 0$, then for any $S \in \mathfrak{J}$ and $T \in \mathfrak{K}$, we have $S\mathfrak{A}T = 0$, which implies $S = 0$ or $T = 0$. Thus if $T \neq 0$, then $\mathfrak{J} = 0$, and if $S \neq 0$, then $\mathfrak{K} = 0$. In a prime C^*-algebra $\mathfrak{A}$, extreme points of the convex set $\mathfrak{A}_1$ are either isometries or co-isometries: $\mathfrak{A}_e = \mathfrak{A}_{is} \cup \mathfrak{A}_{is}^*$.

Theorem 3.8.1. *If $\mathfrak{A}$ is a prime C^*-algebra and $T \in \mathfrak{A}_1$, then*

$$\lambda(T) = \frac{1}{2}(1 - \alpha_p(T))$$

if $T \notin \mathfrak{A}_l^{-1} \cup \mathfrak{A}_r^{-1}$. Otherwise, with $m_p(T) = \max\{m(T), m(T^*)\}$,

$$\lambda(T) = \frac{1}{2}(1 + m_p(T)).$$

Proof. If $\alpha > \alpha_p(T)$, there is an element $A \in \mathfrak{A}_l^{-1} \cup \mathfrak{A}_r^{-1}$ with $\|T - A\| < \alpha$. We may assume that $A = V|A|$ with $V \in \mathfrak{A}_{is}$ and $|A| \in \mathfrak{A}^{-1}$, passing otherwise to T^*. Consequently, we have

$$\alpha(TV^*) = \mathrm{dist}(TV^*, \mathfrak{A}^{-1}) \leq \|TV^* - V|A|V^*\| = \|(T - A)V^*\| < \alpha$$

because $AV^* = V|A|V^*$ is in the norm closure $\mathrm{cl}(\mathfrak{A}^{-1})$, as checked before, so that for any $\varepsilon > 0$, there is $A_\varepsilon \in \mathfrak{A}^{-1}$, with $\|TV^* - A_\varepsilon\| < \varepsilon + \|TV^* - AV^*\|$.

There are $U \in \mathfrak{A}_u$ and $B \in \mathfrak{A}_1$ such that

$$TV^* = \frac{1}{2}(1 - \alpha)U + \frac{1}{2}(1 + \alpha)B$$

since if $S \notin \mathfrak{A}^{-1}$ with norm ≤ 1, then $\lambda_u(S) = \frac{1}{2}(1 - \alpha(S))$, as obtained before, and note that if $TV^* \in \mathfrak{A}^{-1}$, then $VT^* \in \mathfrak{A}^{-1}$, so that $V \in \mathfrak{A}_u$, and thus T^* and also T are in $\mathfrak{A}^{-1} = \mathfrak{A}_l^{-1} \cap \mathfrak{A}_r^{-1}$, a contradiction, with $\|TV^*\| \leq 1$, and note also that $\lambda_u(TV^*) = \frac{1}{2}(1 - \alpha(T)) > \frac{1}{2}(1 - \alpha)$ (and see the remark below).

Since $V^*V = I$, we have

$$T = \frac{1}{2}(1 - \alpha)UV + \frac{1}{2}(1 + \alpha)BV$$

with $UV \in \mathfrak{A}_e$ and $BV \in \mathfrak{A}_1$. Hence $\lambda(T) \geq \frac{1}{2}(1 - \alpha)$ for any $\alpha > \alpha_p(T)$. If $\frac{1}{2}(1 - \alpha_p(T)) > \lambda(T)$, then we have a contradiction, so that

$$\lambda(T) \geq \frac{1}{2}(1 - \alpha_p(T)).$$

Conversely, if $T = \lambda V + (1 - \lambda)B$ for some $V \in \mathfrak{A}_e$ and $B \in \mathfrak{A}_1$, we may assume that $V^*V = I$ (passing otherwise to T^*). Assume for the moment that $\lambda \leq \frac{1}{2}$ and $\|B\| < 1$. Then

$$V + B = (I + BV^*)V \in \mathfrak{A}_l^{-1},$$

because $I + BV^* \in \mathfrak{A}^{-1}$ since $\|BV^*\| < 1$, so that

$$V^*(1 + BV^*)^{-1}(V + B) = I.$$

Since $T - \lambda(V + B) = (1 - 2\lambda)B$, we conclude that $\alpha_l(T) \leq 1 - 2\lambda$, with $\lambda \leq \lambda(T)$ and $1 - 2\lambda \geq 1 - 2\lambda(T)$. If $1 - 2\lambda(T) < \alpha_l(T)$, then we have a contradiction (where by continuity, the condition $\|B\| < 1$ can be removed), so that

$$1 - 2\lambda(T) \geq \alpha_l(T) \geq \alpha_p(T)$$

provided that $\lambda(T) \leq \frac{1}{2}$. But if $\lambda > \frac{1}{2}$, the same argument shows that

$$T = \lambda(V + \frac{1-\lambda}{\lambda}B) = \lambda(I + \frac{1-\lambda}{\lambda}BV^*)V \in \mathfrak{A}_l^{-1}$$

with $\frac{1-\lambda}{\lambda} < 1$. Thus if $T \notin \mathfrak{A}_l^{-1} \cup \mathfrak{A}_r^{-1}$, it follows that

$$\lambda(T) = \frac{1}{2}(1 - \alpha_p(T)).$$

Now if $T \in \mathfrak{A}_l^{-1}$ we have $T = V|T|$ with $V \in \mathfrak{A}_{is}$ and $|T| \in \mathfrak{A}^{-1}$. Thus $m(T) = \frac{1}{\||T|^{-1}\|}$. This implies that

$$|T| = \lambda_0 U + (1 - \lambda_0)B$$

for some $U \in \mathfrak{A}_u$ and $B \in \mathfrak{A}_1$ (in fact, we can choose $B \in \mathfrak{A}_u$), where $\lambda_0 = \frac{1}{2}(1 + m(T)) = \lambda_u(|T|)$, as shown before. It follows that $T = \lambda_0 VU + (1 - \lambda_0)VB$, so that $\lambda(T) \geq \frac{1}{2}(1 + m(T))$.

Conversely, if $T = \lambda V + (1 - \lambda)B$ with $V \in \mathfrak{A}_{is}$ and $B \in \mathfrak{A}_1$, then

$$\|Tx\| \geq \lambda\|Vx\| - (1 - \lambda)\|Bx\| \geq 2\lambda - 1$$

for any $x \in H$ with $\|x\| = 1$. Therefore, $m(T) \geq 2\lambda - 1$, so that $m(T) \geq 2\lambda(T) - 1$. If $T = \lambda V^* + (1 - \lambda)B$ since $\mathfrak{A}$ is prime, then $m(T^*) \geq 2\lambda(T) - 1$ similarly, Therefore, we always have

$$m_p(T) \geq 2\lambda(T) - 1.$$

Similarly, if $T^* \in \mathfrak{A}_l^{-1}$, i.e., $T \in \mathfrak{A}_r^{-1}$, then we have

$$T^* = \lambda_1 VU_1 + (1 - \lambda_1)VU_2,$$

where $\lambda_1 = \frac{1}{2}(1 + m(T^*))$. Therefore, $T = \lambda_1 U_1^* V^* + (1 - \lambda_1)U_2^* V^*$ with $U_1^* V^* \in \mathfrak{A}_e$ and $U_2^* V^* \in \mathfrak{A}_1$. Hence $\lambda(T) \geq \frac{1}{2}(1 + m(T^*))$.

It follows that if $T \in \mathfrak{A}_l^{-1} \cup \mathfrak{A}_l^{-1}$, then

$$\lambda(T) \geq \frac{1}{2}(1 + m_p(T)),$$

so that consequently, we get

$$\lambda(T^*) = \frac{1}{2}(1 + m(T^*)).$$

□

Remark. If $0 \leq \mu \leq \lambda_u(T) = \lambda \leq 1$, then there are $U \in \mathfrak{A}_u$ and $B \in \mathfrak{A}_1$ such that $T = \lambda U + (1-\lambda)B$, so that

$$\begin{aligned} T &= (\mu + \lambda - \mu)U + (1-\lambda)B \\ &= \mu U + (1-\mu)\{\frac{\lambda-\mu}{1-\mu}U + \frac{1-\lambda}{1-\mu}B\} \end{aligned}$$

with $\|\frac{\lambda-\mu}{1-\mu}U + \frac{1-\lambda}{1-\mu}B\| \leq \frac{\lambda-\mu+1-\lambda}{1-\mu} = 1$.

Corollary 3.8.2. *If T is a left or right invertible element of a prime C^*-algebra $\mathfrak{A}$, then $T = \lambda(T)V + (1-\lambda(T))W$ for some $V, W \in \mathfrak{A}_e$.*

Proof. The reason for this is in the proof above. □

Theorem 3.8.3. *A prime C^*-algebra $\mathfrak{A}$ has the λ-property if and only if*

$$\mathfrak{A} = \mathrm{cl}(\mathfrak{A}_l^{-1} \cup \mathfrak{A}_r^{-1}) \quad (\textit{norm closure}),$$

in which case it has the uniform λ-property for $\lambda = \frac{1}{2}$.

Proof. By the theorem above, we have $\lambda(T) \geq \frac{1}{2}$ if and only if $\alpha_p(T) = 0$ for all $T \in \mathfrak{A}_1$, i.e., if and only if $\mathfrak{A}_l^{-1} \cup \mathfrak{A}_r^{-1}$ is dense in $\mathfrak{A}$. Moreover, $\mathfrak{A}$ has the λ-property, i.e., $\lambda(T) > 0$ for every $T \in \mathfrak{A}_1$, if and only if $\alpha_p(S) < 1$ for every $S \in \mathfrak{A}_1$. But if $\alpha_p(T) = \min\{\alpha_l(T), \alpha_l(T^*)\} > 0$ for some $T \in \mathfrak{B}_1$, we may assume that $\alpha_l(T^*) \geq \alpha_l(T) > 0$, whence $\alpha_p(S) = 1$ for some $S \in \mathfrak{A}_1$ as shown before. □

3.9 Examples of infinite C^*-algebras

On a separable Hilbert space H $(= l^2)$, we let $\mathbb{K}$ denote the C^*-algebra of all compact operators on H, and $\mathfrak{F}$ denote the set of Fredholm operators in $\mathbb{B}(H)$ whose images in the Calkin algebra $\mathbb{B}(H)/\mathbb{K}$ are invertible.

Theorem 3.9.1. *Let $\mathfrak{A}$ be a C^*-subalgebra of $\mathbb{B}(H)$ containing $\mathbb{K}$, such that $\mathfrak{F} \cap \mathfrak{A}$ is dense in $\mathfrak{A}$. Then $\lambda(T) \geq \frac{1}{2}$ for every $T \in \mathfrak{A}_1$.*

Proof. Since $\mathbb{K}$ is a minimal ideal of $\mathfrak{A}$, we have $\mathfrak{A}$ is prime. In fact, if $\mathfrak{I}$ and $\mathfrak{K}$ are nonzero closed ideals of $\mathfrak{A}$, then $\mathfrak{I}\mathfrak{K}$ is nonzero, because it contains $\mathbb{K}$. Thus, by the theorem above, it suffices to show that left or right invertible elements of $\mathfrak{A}$ are dense in $\mathfrak{A}$.

Given $T \in \mathfrak{A}_1$ and $\varepsilon > 0$, by assumption we can find $F \in \mathfrak{F} \cap \mathfrak{A}$ such that $\|T - F\| < \varepsilon$. Since $\lambda(T) = \lambda(T^*)$ we may assume, without loss of generality, that the index n of F is ≤ 0 :

$$\mathrm{index}(F) = \dim\ker F - \dim\ker F^* = n \leq 0,$$

considering otherwise T^* and F^*. Note that if $V \in \mathfrak{A}_e$, i.e., $(I - VV^*)\mathfrak{A}(I - V^*V) = 0$, then $(I - V^*V)\mathfrak{A}^*(I - VV^*) = 0$ with $\mathfrak{A} = \mathfrak{A}^*$, namely $V^* \in \mathfrak{A}_e$.

Since $\mathbb{K} \subset \mathfrak{A}$, we can choose a partial isometry A of finite rank from $\ker F$ to $\ker F^*$. As $\ker F^* = F(H)^\perp$, the operator $F + A$ is an injection of H onto a closed subspace $F(H) \oplus A(H)$ of co-dimension $-n$. Indeed, if $(F + A)\xi = 0$, then $F\xi = 0$ and $A\xi = 0$. Thus, $\xi = 0$. By the open mapping theorem $(F + A)^*(F + A)$ is invertible, so that $F + A = V|F + A|$ for some isometry $V(= (F + A)|F + A|^{-1})$ in $\mathfrak{A}$. Since $F^*A = FA^* = 0$, it follows that $F = V|F|$. In fact, note that

$$|F + A|^2 = (F^* + A^*)(F + A) = F^*F + A^*A = |F|^2 + |A|^2,$$

and on the domain of F, we have

$$F = F + A = V|F + A| = V|F|.$$

Likewise, $F + \varepsilon A = V|F + \varepsilon A|$, where $|F + \varepsilon A| \in \mathfrak{A}^{-1}$. Thus $F + \varepsilon A \in \mathfrak{A}_l^{-1}$, and $\|T - (F + \varepsilon A)\| < 2\varepsilon$. Since ε is arbitray, it follows that $\alpha_p(T) = 0$, whence $\lambda(T) \geq \frac{1}{2}$ by the theorem above. □

Let S denote the unilateral shift on l^2, i.e. $S(\alpha_1, \alpha_2, \cdots) = (0, \alpha_1, \alpha_2, \cdots)$. Thus S is an isometry in $\mathfrak{F}$ with index -1. Since $S^n(1 - SS^*)$ is the rank one projection that takes the first basis vector to the n-th, it follows that the C^*-algebra $\mathfrak{T}$ generated by S, called the Toeplitz algebra, contains the C^*-algebra $\mathbb{K}$ of compact operators. Since the image of S in the Calkin algebra is a unitary with full spectrum, we have a short exact sequence:

$$0 \longrightarrow \mathbb{K} \xrightarrow{\ i\ } \mathfrak{T} \xrightarrow{\ q\ } C(\mathbb{T}) \longrightarrow 0.$$

The identification of $\mathfrak{T}/\mathbb{K}$ with $C(\mathbb{T})$ is given by $q(T_f) = f$ for $f \in C(\mathbb{T})$, where T_f is the Toeplitz operator on the Hardy space H^2, identified with l^2, so that $T_f = PM_fP$, where P is the projection of $L^2(\mathbb{T})$ onto H^2, and M_f is the multiplication operator on $L^2(\mathbb{T})$.

Corollary 3.9.2. *The Toeplitz algebra $\mathfrak{T}$ has the uniform λ-property for $\lambda = \frac{1}{2}$.*

Proof. Since $\mathbb{K} \subset \mathfrak{T}$ and invertible elements are dense in $\mathfrak{F}/\mathbb{K} = C(\mathbb{T})$, we have that $\mathfrak{F} \cap \mathfrak{T}$ is dense in $\mathfrak{T}$. Thus, the theorem above implies the conclusion. In fact, if $T \in \mathfrak{T}$, then $q(T)$ is approximated closely by an invertible element $U = q(u) \in \mathfrak{T}/\mathbb{K}$, with $u \in \mathfrak{F}$. Then the norm of $T - u + v$ for some $v \in \mathbb{K}$ is small and, still $u - v \in \mathfrak{F}$ by the invariance of Fredholm index for compact perturbation. □

Define the $*$-automorphism θ of $C(\mathbb{T})$ of order two, by $\theta f(t) = f(t^{-1})$ for $t \in \mathbb{T}$.

Proposition 3.9.3. *If $\mathfrak{A}$ is the C^*-algebra generated by $T = S \oplus S^*$ on the Hilbert space $H = l^2 \oplus l^2$, where S is the unilateral shift on l^2, then $\mathfrak{A}$ consists of all elements in $\mathbb{B}(H)$ of the form $B \oplus C$ in $\mathfrak{T} \oplus \mathfrak{T}$, such that $q(B) = \theta(q(C))$.*

Proof. Define

$$\mathfrak{A}_0 = \{B \oplus C \in \mathfrak{T} \oplus \mathfrak{T} \,|\, q(B) = \theta(q(C))\},$$

which is a normclosed $*$-subalgebra of $\mathbb{B}(H)$, i.e. a C^*-algebra. Indeed, if $B_j \oplus C_j \in \mathfrak{A}_0$ with $B_j \oplus C_j \to B \oplus C$, then

$$\begin{aligned}
&q(B_j + B_k) = \theta(q(C_j)) + \theta(q(C_k)) = \theta(q(C_j + C_k)),\\
&q(B_j B_k) = \theta(q(C_j))\theta(q(C_k)) = \theta(q(C_j C_k)),\\
&q(B_j^*) = \theta(q(C_j))^* = \theta(q(C_j^*)), \quad \text{and}\\
&q(B) = \lim_j q(B_j) = \lim_j \theta(q(C_j)) = \theta(q(C)).
\end{aligned}$$

Since $q(S) = \mathrm{id}$ and $q(S^*) = \overline{\mathrm{id}} = \mathrm{id}^{-1}$, where $\mathrm{id}(z) = z \in \mathbb{T}$, we see that $T = S \oplus S^* \in \mathfrak{A}_0$, whence $\mathfrak{A} \subset \mathfrak{A}_0$.

To prove the converse inclusion, note that

$$\begin{aligned}
T^*T - TT^* &= I \oplus SS^* - SS^* \oplus I\\
&= (I \oplus I) - (0 \oplus P_1) - ((I \oplus I) - (P_1 \oplus 0)) = P_1 \oplus -P_1,
\end{aligned}$$

where $P_1 = I - SS^*$ is the rank one projection on the first basis vector. Since $(T^*T - TT^*)^2 = P_1 \oplus P_1$, it follows that $P_1 \oplus 0 \in \mathfrak{A}$ and $0 \oplus P_1 \in \mathfrak{A}$. Moreover, as $T^n(P_1 \oplus 0) = S^n P_1 \oplus 0$ is the rank one projection on the n-th basis vector, we see that $\mathfrak{A}$ contains $\mathbb{K} \oplus 0$. Similarly, $0 \oplus \mathbb{K} \subset \mathfrak{A}$. The projection $Z = I \oplus 0$ in $\mathbb{B}(H)$ commutes with $\mathfrak{A}$ because it commutes with T, so the map $\mathfrak{A} \ni A \mapsto AZ$ is a $*$-homomorphism of $\mathfrak{A}$. Since $TZ = S \oplus 0$, we see that $\mathfrak{A}Z = \mathfrak{T} \oplus 0$. Now take any element $B \oplus C \in \mathfrak{A}_0$. There is an element $A \in \mathfrak{A}$ such that $AZ = B$. Since $\mathfrak{A} \subset \mathfrak{A}_0$, we know that $A = B \oplus D$, where $\theta(q(D)) = q(B)$. Also, $\theta(q(C)) = q(B)$, so that $q(D) = q(C)$, i.e., $C - D = K \in \mathbb{K}$. As $0 \oplus \mathbb{K} \subset \mathfrak{A}$,

$$B \oplus C = B \oplus (D + K) = A + (0 \oplus K) \in \mathfrak{A},$$

whence $\mathfrak{A}_0 \subset \mathfrak{A}$. □

Proposition 3.9.4. *Let $\mathfrak{A} = C^*(T)$ be the C^*-algebra generated by $T = S \oplus S^*$. Then*

$$\mathfrak{A}_e = \cup_{n \in \mathbb{Z}} \mathfrak{A}_u T^n \mathfrak{A}_u,$$

where T^{-n} means $(T^*)^n$ and $T^0 = I$. In particular, $\mathfrak{A}_e$ contains no non-unitary isometries, so $\mathfrak{A}$ is Murray-von Neumann finite.

Proof. If $V \in \mathfrak{A}_e$, then $V = U \oplus W$ for some partial isometries $U, W \in \mathfrak{T}$. In fact, since we have $\mathfrak{A}Z = \mathfrak{T} \oplus 0$ and $\mathfrak{A}(I - Z) = 0 \oplus \mathfrak{T}$, where $Z = I \oplus 0$ and $I = I \oplus I$ is the unit of $\mathfrak{A}$, both U and W must be extreme in $\mathfrak{T}$. Indeed, since $(I - VV^*)\mathfrak{A}(I - V^*V) = 0$, we have

$$\begin{aligned} 0 &= (I - VV^*)\mathfrak{A}(I - V^*V)Z = (I - UU^*)\mathfrak{T}(I - U^*U), \\ 0 &= (I - VV^*)\mathfrak{A}(I - V^*V)(I - Z) = (I - WW^*)\mathfrak{T}(I - W^*W). \end{aligned}$$

Therefore, U and W are either isometries or co-isometries, and in particular they belong to $\mathfrak{F}$, because $q(U)$ and $q(W)$ are invertible in $C(\mathbb{T}) = \mathfrak{T}/\mathbb{K}$. Since the winding number of the function $q(U) \in C(\mathbb{T})$ is $-\operatorname{index} U$, and since θ reverses the direction of its path, by $q(U) = \theta(q(W))$ we see that

$$-\operatorname{index} U = -(-\operatorname{index} W) = \operatorname{index} W.$$

Assume now that $\operatorname{index} U = n \geq 0$. Thus W is an isometry, U a co-isometry, and $S^n U$ and $(S^*)^n W$ are partial isometries of index zero, because

$$\operatorname{index}(S^n U) = \operatorname{index}(S^n) + \operatorname{index}(U) = -n + n = 0.$$

We choose partial isometries A and B of finite rank from $\ker(S^n U)$ to $\ker(U^*(S^*)^n)$ and from $\ker((S^*)^n W)$ to $\ker(W^* S^n)$, respectively. Then

$$U_1 = (S^n U + A) \oplus ((S^*)^n W + B) \in \mathfrak{A}_u$$

because both summands are unitaries in $\mathfrak{T}$, and

$$\theta(q(S^n U + A)) = \theta(q(S^n))\theta(q(U)) = q((S^*)^n)q(W) = q((S^*)^n W + B).$$

We have

$$\begin{aligned} (T^*)^n U_1 &= (S^*)^n (S^n U + A) \oplus S^n ((S^*)^n W + B) \\ &= (U + (S^*)^n A) \oplus (S^n (S^*)^n W + S^n B) \\ &= U \oplus S^n ((S^*)^n W + B), \end{aligned}$$

because $A(l^2) = \ker(U^*(S^*)^n) = \ker((S^*)^n)$ since U^* is an isometry, and hence $(S^*)^n A = 0$. Both W and $S^n((S^*)^n W + B)$ are isometries in $\mathfrak{T}$ with index $-n$, so

$$V_2 = W(W^* S^n + B^*)(S^*)^n + C \in \mathfrak{T}_u$$

for a partial isometry C of finite rank. Since

$$\ker C = S^n((S^*)^n W + B)l^2 = S^n l^2,$$

it follows that $CS^n = 0$, so

$$\begin{aligned} &V_2 S^n((S^*)^n W + B) \\ &= (W(W^* S^n + B^*)(S^*)^n + C)S^n((S^*)^n W + B) \\ &= W(W^* S^n + B^*)((S^*)^n W + B) = W. \end{aligned}$$

Finally, $WW^* = I - Q$ and $S^n(S^*)^n = I - P$ for some projections P and Q of rank n, so

$$V_2 = (I - Q)(I - P) + WB^*(S^*)^n + C = I + K$$

where $K = QP - P - Q + WB^*(S^*)^n + C \in \mathbb{K}$. Consequently, $U_2 = I \oplus V_2 = (I \oplus I) + (0 \oplus K)$ is a unitary in $\mathfrak{A}$, and as desired,

$$U_2(T^*)^n U_1 = (I \oplus V_2)(U \oplus S^n((S^*)^n W + B)) = U \oplus W.$$

The case where $\operatorname{index} U < 0$ follows by considering $V^* = U^* \oplus W^*$. $\square$

Proposition 3.9.5. *The C^*-algebra $\mathfrak{A} = C^*(T)$ with $T = S \oplus S^*$ has the uniform λ-property with $\lambda = \frac{1}{2}$.*

Proof. We see that there is the following short exact sequence:

$$0 \longrightarrow \mathbb{K} \oplus \mathbb{K} \xrightarrow{\ i\ } \mathfrak{A} \xrightarrow{\ q\ } C(\mathbb{T}) \longrightarrow 0,$$

where $q(B \oplus C) = q(B)$ for every $B \oplus C \in \mathfrak{A}$, and $q : \mathfrak{T} \to C(\mathbb{T}) \to 0$ as above. Therefore, we can use almost the same arguments as in the theorem above.

If $A = B \oplus C \in \mathfrak{A}$ (and may assume $A \in \mathfrak{A}_1$) and $\varepsilon > 0$, we can find $E = F \oplus G \in \mathfrak{A}$ such that $p(E) = q(F) = \theta(q(G))$ is invertible in $C(\mathbb{T})$, and

$$\max\{\|B - F\|, \|C - G\|\} = \|(B - F) \oplus (C - G)\| = \|A - E\| < \varepsilon.$$

Regarding F and G as elements in $\mathfrak{T} \cap \mathfrak{F}$ we may assume that $\operatorname{index} F = -\operatorname{index} G = n \leq 0$ (considering otherwise A^* and E^*). As shown above, this implies that $F = V|F|$ and $G = W^*|G|$, where V and W are isometries in $\mathfrak{T}$. In fact, $G^* = W|G^*|$, so $G = |G^*|W^* = W^*(W|G^*|W^*) = W^*|G|$. Since

$$\theta(q(V)) = \theta(q(F)q(F^*F)^{-\frac{1}{2}}) = q(G)q(G^*G)^{-\frac{1}{2}} = q(W^*),$$

where $\theta^2 = \mathrm{id}_{\mathfrak{A}}$, it follows that $V^* \oplus W$ and $V \oplus W^*$ are elements of $\mathfrak{A}$. For each element $X \oplus Y$ in $\mathfrak{A}$, define

$$\rho(X \oplus Y) = XV^* \oplus WY, \quad \text{and} \quad \sigma(X \oplus Y) = XV \oplus W^*Y.$$

Since we have $\theta(q(XV^*)) = q(Y)q(W) = q(W)q(Y) = q(WY)$ and similarly, $\theta(q(XV)) = q(W^*Y)$, it follows that ρ and σ are norm-decreasing linear maps of $\mathfrak{A}$ into itself. Check that

$$\begin{aligned} \|\rho(X \oplus Y)\| &= \max\{\|XV^*\|, \|WY\|\} \leq \|X \oplus Y\|, \\ \rho((X_1 + X_2) \oplus (Y_1 + Y_2)) &= (X_1 + X_2)V^* \oplus W(Y_1 + Y_2) \\ &= \rho(X_1 \oplus Y_1) + \rho(X_2 \oplus Y_2). \end{aligned}$$

Moreover, $\sigma \circ \rho = \mathrm{id}_{\mathfrak{A}}$. Now

$$\begin{aligned} \|\rho(A) - (|F^*| \oplus |G|)\| &= \|(BV^* \oplus WC) - (FV^* \oplus WG)\| \\ &= \|(B - F)V^* \oplus W(C - G)\| \\ &= \|\rho(A - E)\| < \varepsilon, \end{aligned}$$

and since $q(|F^*|) = q(V|F|V^*) = q(|F|)$ we see that $|F^*| \oplus |G| = \rho(E) \in \mathfrak{A}$. Since $|F^*| \oplus |G|$ (positive) is in $\mathrm{cl}(\mathfrak{A}^{-1})$, we have $\mathrm{dist}(\rho(A), \mathfrak{A}^{-1}) < \varepsilon$. Therefore,

$$\lambda_u(\rho(A)) > \frac{1}{2}(1 - \varepsilon)$$

since $\lambda_u(\rho(A)) = \frac{1}{2}(1 - \alpha(\rho(A)))$, as shown before, if $\rho(A) \in \mathfrak{A}_1 \setminus \mathfrak{A}^{-1}$. So there are some $U \in \mathfrak{A}_u$ and $D \in \mathfrak{A}_1$ such that

$$\rho(A) = \frac{1}{2}(1 - \varepsilon)U + \frac{1}{2}(1 + \varepsilon)D.$$

Consequently,

$$A = \sigma(\rho(A)) = \frac{1}{2}(1 - \varepsilon)\sigma(U) + \frac{1}{2}(1 + \varepsilon)\sigma(D),$$

with $\sigma(U) \in \mathfrak{A}_1$, whereas, if $U = U_1 \oplus U_2$, we have $\sigma(U) = U_1 V \oplus W^* U_2 \in \mathfrak{A}_e$. It follows that $\lambda(A) \geq \frac{1}{2}(1 - \varepsilon)$, and since ε is arbitrary we obtain $\lambda(A) \geq \frac{1}{2}$.

When $\rho(A) \in \mathfrak{A}_1 \cap \mathfrak{A}^{-1}$, we have $\lambda_u(\rho(A)) = \frac{1}{2}(1 + \frac{1}{\|\rho(A)^{-1}\|}) > \frac{1}{2}$, as shown before. It follows from the same argument above that $\lambda(A) > \frac{1}{2}$. □

Remark. The six-term exact sequence in K-theory for the short exact sequence for $\mathfrak{A}$ is given by

$$\begin{array}{ccccc} \mathbb{K}\oplus\mathbb{K} & \xrightarrow{i_*} & K_0(\mathfrak{A}) & \xrightarrow{q_*} & \mathbb{Z} \\ {\scriptstyle\delta}\uparrow & & & & \downarrow \\ \mathbb{Z} & \xleftarrow{q^*} & K_1(\mathfrak{A}) & \xleftarrow{i_*} & 0 \end{array}$$

which implies that $K_1(\mathfrak{A}) = 0$ and $K_0(\mathfrak{A}) = \mathbb{Z}\oplus\mathbb{Z}$, one copy of $\mathbb{Z}$ for finite projections and one copy for the co-finite projections, where $\delta(n) = (n, n)$, on the K_0-level, $i_*(n, m) = (n - m, 0)$ and $q_*(n, m) = m$.

3.10 Purely infinite C^*-algebras

Recall that a C^*-algebra $\mathfrak{A}$ has real rank zero if for every $S, T \in \mathfrak{A}$ such that $ST = 0$ and every $\varepsilon > 0$, there is a projection $P \in \mathfrak{A}$ such that $SP = 0$ and $\|(I - P)T\| < \varepsilon$. This condition has a number of equivalent formulations. One is that $\mathfrak{A}^{-1}\cap\mathfrak{A}_{sa}$ is dense in $\mathfrak{A}_{sa}$. Another is that the set of self-adjoint elements of $\mathfrak{A}$ with finite spectra is dense in $\mathfrak{A}_{sa}$.

A simple C^*-algebra $\mathfrak{A}$ is said to be purely infinite if it has real rank zero and every non-zero projection of $\mathfrak{A}$ is infinite, i.e., Murray-von Neumann equivalent to a proper subprojection. This implies that for any P, Q of non-zero projections of $\mathfrak{A}$, there is a partial isometry $V \in \mathfrak{A}$ such that $V^*V = P$ and $VV^* \leq Q$ (a famous non-trivial fact by Cuntz [4]).

Theorem 3.10.1. *If $\mathfrak{A}$ is a purely infinite, simple C^*-algebra, the set of elements T of the form $T = V|T|$, where V is an isometry or a co-isometry in $\mathfrak{A}$, is dense in $\mathfrak{A}$. Thus,* $\mathrm{cl}(\mathfrak{A}_e\mathfrak{A}_+) = \mathfrak{A}$.

Proof. If $T \in \mathfrak{A}$, then it has a polar decomposition $T = V|T|$, with $V \in \mathfrak{A}''$. It follows from the Stone-Weierstrass theorem that $Vf(|T|) \in \mathfrak{A}$ whenever f is a continuous function on $\mathrm{sp}(|T|)$ vanishing at zero. Note also that $T^* = V^*|T^*|$ is the polar decomposition of T^*, with $|T^*| = V|T|V^*$.

If $|T| \in \mathfrak{A}^{-1}$, then $V = T|T|^{-1}$ is an isometry in $\mathfrak{A}$. Similarly, if $|T^*| \in \mathfrak{A}^{-1}$, then V^* is an isometry, so V is a co-isometry in $\mathfrak{A}$.

If 0 is an isolated point both in $\mathrm{sp}(|T|)$ and $\mathrm{sp}(|T^*|)$, let $e(t) = 1$ if $t \in \mathrm{sp}(|T|)\setminus\{0\}$ and $e(0) = 0$. Then $P = e(|T|)$ and $Q = e(|T^*|)$ are projections in $\mathfrak{A}$, and $V = VP = QV$. In fact, $|T|(I - P) = \mathrm{id}(|T|)(1 - e)(|T|) = 0$, so that $(I-P)V^*V(I-P) = 0$ and hence $V(I-P) = 0$. Also, $(I-Q)|T^*| = 0$. Thus, $(I - Q)VV^*(I - Q) = 0$ and hence $(I - Q)V = 0$. As $I - P$ and $I - Q$ are non-zero projections in $\mathfrak{A}$, and $\mathfrak{A}$ is purely infinite, there is a partial isometry $W \in \mathfrak{A}$ such that $W^*W = I - P$ and $WW^* \leq I - Q$. Then $U = W + V$ is an isometry in $\mathfrak{A}$, and $T = U|T|$. Check that

$$U^*U = (W^* + V^*)(W + V) = (I - P) + W^*V + V^*W + V^*V = I - P + V^*V$$

since $V^*WW^*V \leq V^*(I - Q)V = 0$ so that $W^*V = 0$, and

$$U|T| = (W + V)|T| = WP|T| + VP|T| = V|T| = T.$$

We are left with the case where 0 is an accumulation point both in $\mathrm{sp}(|T|)$ and in $\mathrm{sp}(|T^*|)$. Given $\varepsilon > 0$ we define

$$f_1(t) = \max\{t - \varepsilon, 0\}, \quad f_2(t) = \max\{t - 2\varepsilon, 0\},$$
$$g_1(t) = \max\{1 - \frac{t}{\varepsilon}, 0\}, \quad g_2(t) = \max\{1 - \frac{t}{2\varepsilon}, 0\},$$

for $t \geq 0$. Assuming, as we may, that $\|T\| = 1$, we see that

$$f_1(|T^*|)g_1(|T^*|) = f_2(|T|)g_2(|T|) = 0,$$

because $f_i g_i = 0$ for $1 \leq i \leq 2$. Since $\mathfrak{A}$ has real rank zero we can find nonzero projections P and Q in $\mathfrak{A}$ such that

$$(I - P)g_2(|T|) = 0, \quad \|Pf_2(|T|)\| \leq \varepsilon,$$
$$\|(I - Q)g_1(|T^*|)\| \leq \varepsilon, \quad Qf_1(|T^*|) = 0,$$

since $g_1(0) = 1$ and $g_2(0) = 1$ and $0 \in \mathrm{sp}(|T|) \cap \mathrm{sp}(|T^*|)$. Since $\mathfrak{A}$ is purely infinite, we can find a partial isometry $W \in \mathfrak{A}$ such that $W^*W = P$ and $WW^* \leq Q$. Now define $S = \varepsilon W + Vf_1(|T|)$. Then $S \in \mathfrak{A}$ with

$$\|T - S\| \leq \varepsilon + \|V(|T| - f_1(|T|))\| \leq \varepsilon + \|\mathrm{id} - f_1\|_\infty = 2\varepsilon.$$

On the other hand,

$$\begin{aligned} S^*S &= \varepsilon^2 P + f_1^2(|T|) + \varepsilon(W^*Vf_1(|T|) + f_1(|T|)V^*W) \\ &= \varepsilon^2 P + f_1^2(|T|) + 2\varepsilon \mathrm{Re}(W^*(I - Q + Q)f_1(|T^*|)V) \\ &= \varepsilon^2 P + f_1^2(|T|) \geq \varepsilon^2 g_2(|T|) + f_1^2(|T|), \end{aligned}$$

where note that $(I - Q)WW^*(I - Q) \leq (I - Q)Q(I - Q) = 0$. Since $\varepsilon^2 g_2(t) + f_1^2(t) > 0$ for $0 \leq t \leq 1$, we see that $|S|$ is invertible, whence $S = U|S|$ for $U = S|S|^{-1}$ an isometry in $\mathfrak{A}$. Thus, $T \in \mathrm{cl}(\mathfrak{A}_e\mathfrak{A}_+)$. $\square$

Corollary 3.10.2. *Every purely infinite, simple C^*-algebra has the uniform λ-property for $\lambda = \frac{1}{2}$.*

Proof. Since $\mathfrak{A}_+ \subset \mathrm{cl}(\mathfrak{A}^{-1})$, it follows from the theorem above that

$$\mathfrak{A} = \mathrm{cl}(\mathfrak{A}_e \mathfrak{A}^{-1}) = \mathrm{cl}(\mathfrak{A}_l^{-1} \cup \mathfrak{A}_r^{-1}).$$

Note that $\mathfrak{A}$ is prime. Indeed, if two closed ideals $\mathfrak{J}$ and $\mathfrak{K}$ of $\mathfrak{A}$, i.e. both 0 or $\mathfrak{A}$, satisfy $\mathfrak{J}\mathfrak{K} = 0$, then $\mathfrak{J} = 0$ or $\mathfrak{K} = 0$. Thus, $\mathfrak{A}_e = \mathfrak{A}_{is} \cup \mathfrak{A}_{is}^*$. Hence $\mathfrak{A}_e \mathfrak{A}^{-1} = \mathfrak{A}_l^{-1} \cup \mathfrak{A}_r^{-1}$.

Therefore, $\alpha_p(T) = 0$ for every $T \in \mathfrak{A}_1$, whence $\lambda(T) \geq \frac{1}{2}$ as obtained before. □

Conjecture. (This might be still open.) *For a unital C^*-algebra $\mathfrak{A}$, the following conditions are equivalent:*

(i) $\mathfrak{A}_1 = \frac{1}{2}(\mathfrak{A}_e + \mathfrak{A}_e)$.

(ii) $\mathfrak{A} = \mathfrak{A}_e \mathfrak{A}_+$.

(iii) $\mathfrak{A}$ *is an SAW^*-algebra with something?.*

Moreover, if $\mathfrak{A} \subset \mathbb{B}(H)$, with H separable, then already the condition

(iv) $\mathfrak{A}_1 = \frac{1}{2}(\mathfrak{A}_e + \mathfrak{A}_1)$

implies that $\mathfrak{A}$ is a von Neumann algebra.

Brief. (ii) $\Rightarrow$ (i). Obvious. Because if $A \in \mathfrak{A}_1$ with $A = V|A|$ a polar decomposition with $V \in \mathfrak{A}_e$ by condition (ii), then

$$A \in \mathfrak{A}_e \mathfrak{A}_{1,+} \subset \mathfrak{A}_e \frac{1}{2}(\mathfrak{A}_u + \mathfrak{A}_u) \subset \frac{1}{2}(\mathfrak{A}_e + \mathfrak{A}_e),$$

where the first inclusion is checked before.

The job will be to show (i) $\Rightarrow$ (ii). For condition (iii), note that any sort of weak polar decomposition implies that $\mathfrak{A}$ is an SAW^*-algebra. Indeed, if $ST = 0$ in $\mathfrak{A}$, we may assume that $S, T \in \mathfrak{A}_+$, replacing them otherwise with S^*S and TT^*, still $S^*STT^* = 0$. Now let $R = S - T$, and assume that we have a decomposition $R = V|R|$ for some $V \in \mathfrak{A}$ with $\|V\| \leq 1$. Then

$$|R|(I - V^*V)|R| = |R|^2 - R^*R = 0,$$

whence $(I - V^*V)|R| = 0$ because $V^*V \leq \|V^*V\| I \leq I$.

We have $|R| = S + T$ because $|R|^2 = (S+T)^*(S+T) = (S+T)^2$ with $S + T \in \mathfrak{A}_+$. Thus, $S - T = V(S+T)$, i.e., $(I - V)S = (I + V)T$. It follows that

$$S(I - V)^*(I - V)S = S(I + V^*V - V - V^*)S =$$
$$T(I + V)^*(I + V)T = T(I + V^*V + V + V^*)T,$$

which equals zero, since both of the right sides are orthogonal. Hence $|I - V|S = 0 = |I + V|T$, and also $|I - V|^2 S = 0 = |I + V|^2 T$.

Note also that $(I - V^*V)(S + T) = 0$ implies $(I - V^*V)T^2 = 0$. Thus, $(I - V^*V)T^2(I - V^*V) = 0$. Hence $T(I - V^*V) = 0$, so that $V^*VT = T$. Therefore, also $V^*VS = S$.

It follows that

$$(2I - V - V^*)S = 0 = (2I + V + V^*)T.$$

Let $E = \frac{1}{2}I - \frac{1}{4}(V + V^*)$. Then we have $0 \leq E \leq I$ since $\frac{1}{4}(V + V^*) \leq \frac{1}{4}\|V + V^*\|I \leq \frac{1}{2}I$ and $I - E = \frac{1}{2}I + \frac{1}{4}(V + V^*) \geq \frac{1}{2}I - \frac{1}{2}I = 0$, and we have

$$ES = 0 = (I - E)T.$$

Therefore, $\mathfrak{A}$ is an SAW^*-algebra.

We see that condition (ii) is much stronger than that $\mathfrak{A}$ is an SAW^*-algebra. □

Recall from [2] that a unital C^*-algebra $\mathfrak{A}$ has real rank n (in symbols $\mathrm{RR}(\mathfrak{A}) = n$), if there is a non-negative least integer $n \geq 0$, such that for any $n + 1$ self-adjoint elements $A_1, \cdots, A_{n+1}$ of $\mathfrak{A}$ and $\varepsilon > 0$, there are $B_1, \cdots, B_{n+1} \in \mathfrak{A}_{sa}$ such that $\|A_j - B_j\| < \varepsilon$ $(1 \leq j \leq n + 1)$ and $\sum_{j=1}^{n+1} B_j^2 \in \mathfrak{A}^{-1}$. If $\mathfrak{A}$ is commutative, i.e., $\mathfrak{A} = C(X)$, then $\mathrm{RR}(C(X)) = \dim X$. Moreover, we always have $\mathrm{RR}(\mathfrak{A}) \leq 2\,\mathrm{sr}(\mathfrak{A}) - 1$. Thus, if the stable rank of $\mathfrak{A}$ is 1, then the real rank of $\mathfrak{A}$ is 0 or 1. The converse is definitely false, since every von Neumann algebra has real rank zero, but stable rank infinite, unless it is finite.

Theorem 3.10.3. *If $\mathfrak{A}$ is a C^*-algebra with the uniform λ-property for $\lambda = \frac{1}{2}$, then the real rank of $\mathfrak{A}$ is at most one.*

Proof. Given $A_1, A_2 \in \mathfrak{A}_{sa}$ and $\varepsilon > 0$, we let $T = A_1 + iA_2$. By scaling the elements we may assume that $\|T\| \leq 1$. By assumption we can find $V \in \mathfrak{A}_e$ and $B \in \mathfrak{A}_1$ such that

$$T = \frac{1}{2}(1 - \frac{\varepsilon}{2})V + \frac{1}{2}(1 + \frac{\varepsilon}{2})B, \quad \text{and let}$$
$$T_0 = \frac{1}{2}(1 + \frac{\varepsilon}{2})V + \frac{1}{2}(1 - \frac{\varepsilon}{2})B.$$

Then $\|T_0 - T\| = \|\frac{\varepsilon}{2}(V - B)\| \leq \varepsilon$, so if we write $T_0 = B_1 + iB_2$, with $B_1, B_2 \in \mathfrak{A}_{sa}$, then $\|B_j - A_j\| \leq \varepsilon$ for $j = 1, 2$. Moreover,

$$\begin{aligned}\frac{1}{2}(T_0^*T_0 + T_0T_0^*) &= \frac{1}{2}((B_1 - iB_2)(B_1 + iB_2) + (B_1 + iB_2)(B_1 - iB_2)) \\ &= B_1^2 + B_2^2.\end{aligned}$$

To show that the element above is invertible, consider the following multiple S of T_0:

$$S = \left(\frac{1}{2}(1+\frac{\varepsilon}{2})\right)^{-1} T_0 = V + tB,$$

where $t = (1+\frac{\varepsilon}{2})^{-1}(1-\frac{\varepsilon}{2}) < 1$. Realizing $\mathfrak{A}$ as an operator algebra on some Hilbert space H, we let Z denote the projection onto the closure of the subspace $\mathfrak{A}(I - V^*V)H$. Since $(I - VV^*)\mathfrak{A}(I - V^*V) = 0$, it follows that Z belongs to the center of $\mathfrak{A}''$ the von Neumann algebra generated by $\mathfrak{A}$, and that

$$(I - VV^*)Z = 0 = (I - V^*V)(I - Z).$$

Note that $I - V^*V \leq Z$ implies that $(I - Z)(I - V^*V)(I - Z) = 0$, so that $(I - V^*V)(I - Z) = 0$. Note also that for any $A \in \mathfrak{A}$, we have $ZA(I - V^*V) = A(I - V^*V)$ and $A(I - V^*V) = AZ(I - V^*V)$, so that $ZA = AZ$ on $(I - V^*V)H$, and on the other hand, $Z = 0$ on V^*VH since $(I - V^*V)V^*V = 0$, so that $AZ = 0$ and $A^*Z = 0$, and thus also $AZ = ZA$ on V^*VH. This centerness also extends to $\mathfrak{A}''$ by weak continuity.

Thus $V(I - Z)$ is an isometry on $(I - Z)H$, and VZ is a co-isometry on ZH. Note that $I - Z = (I - Z)V^*V(I - Z)$ and $Z = ZVV^*Z$ by the equation above. Therefore,

$$\begin{aligned}
S^*S(I - Z) &= (V^* + tB^*)(V + tB)(I - Z) \\
&= V^*V(V^* + tB^*)(V + tB)V^*V(I - Z) \\
&= (V^*V + tV^*BV^*V + tV^*VB^*V + t^2V^*VB^*BV^*V)(I - Z) \\
&= V^*(I + tBV^* + tVB^* + t^2VB^*BV^*)V(I - Z) \\
&= V^*(I + tVB^*)(I + tBV^*)V(I - Z) \\
&\geq V^*(1 - t)^2V(I - Z) = (1 - t)^2(I - Z).
\end{aligned}$$

Similarly,

$$\begin{aligned}
SS^*Z &= (V + tB)(V^* + tB^*)Z \\
&= VV^*(V + tB)(V^* + tB^*)VV^*Z \\
&= V(I + tV^*B)(I + tB^*V)V^*Z \\
&\geq V(1 - t)^2V^*Z = (1 - t)^2Z.
\end{aligned}$$

Consequently, $S^*S + SS^* \geq S^*S(I - Z) + SS^*Z \geq (1 - t)^2I$, which proves that $B_1^2 + B_2^2$ is invertible. □

Remark. Note that what we have is

$$\begin{aligned}
&V^*(I + t(BV^* + VB^*) + t^2 VB^*BV^*)V(I - Z) \\
&\geq (V^*V - t\|V^*BV^* + VB^*V\|V^*V + t^2 V^*VB^*BV^*V)(I - Z) \\
&\geq ((1 - 2t)I + t^2 B^*B)(I - Z) \\
&= ((1 - t)^2 I - t^2(I - B^*B))(I - Z) \\
&= (1 - t)^2(I - Z) - t^2(I - B^*B - Z + Z) \\
&= (1 - t)^2(I - Z) - t^2(I - B^*B),
\end{aligned}$$

with $B^*B \leq I$, where this estimate is not sufficient to prove the estimates in the proof above, that is only the part we can not recover at this moment. Probably, other reasons to deduce those estimates in the proof would be hidden. A reason might be that the estimate $|I + tBV^*| \geq I - t\|BV^*\|I$ holds, with $I - t\|BV^*\|I \geq (1 - t)I$, where the plausible estimate certainly holds if $VB^*BV^* - \|BV^*\|^2 I \geq 0$, but we always have $VB^*BV^* - \|BV^*\|^2 I \leq 0$. Finally, we certainly have $\|I + tBV^*\|^2 \geq (1 - t)^2$.

References

[1] R. M. Aron and R. H. Lohman, *A geometric function determined by extreme points in the unit ball of a normed space*, Pacific J. Math. **127** (1987), 209-231.

[2] Lawrence G. Brown and Gert K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131-149.

[3] Lawrence G. Brown and Gert K. Pedersen, *On the geometry of the unit ball of a C^*-algebra*, J. reine angew. Math. **469** (1995), 113-147.

[4] J. Cuntz, *K-theory for certain C^*-algebras*, Ann. of Math. **113** (1981), 181-197.

[5] K. Grove and G. K. Pedersen, *Sub-Stonean spaces and corona sets*, J. Funct. Anal. **56** (1984), 124-143.

[6] U. Haagerup and M. Rørdam, *C^*-algebras of unitary rank two*, J. Operator Theory, **30** (1993), 161-171.

[7] F. Hiai and K. Yanagi, *Hilbert spaces and linear operators* (in Japanese), Makino Bookstore (1995).

[8] R. V. KADISON, *Isometries of operator algebras*, Ann. of Math. **54** (1951), 325-338.

[9] R. H. LOHMAN, *The λ-function in Banach spaces*, Contemp. Math. **85** (1989), 345-353.

[10] G. J. MURPHY, *C^*-algebras and Operator theory*, Academic Press, (1990).

[11] C. L. OLSEN, *Unitary approximation*, J. Funct. Anal. **85** (1989), 392-419.

[12] C. L. OLSEN AND G. K. PEDERSEN, *Convex combinations of unitary operators in von Neumann algebras*, J. Funct. Anal. **66** (1986), 365-380.

[13] G. K. PEDERSEN, *C^*-Algebras and their Automorphism Groups*, Academic Press (1979).

[14] G. K. PEDERSEN, *SAW^*-algebras and corona C^*-algebras, contributions to non-commutative topology*, J. Operator Theory **15** (1986), 15-32.

[15] G. K. PEDERSEN, *Unitary extensions and polar decompositions in a C^*-algebra*, J. Operator Theory **17** (1987), 357-364.

[16] G. K. PEDERSEN, *Three quavers on unitary elements in C^*-algebras*, Pacific J. Math. **137** (1989), 169-180.

[17] G. K. PEDERSEN, *The λ-function in operator algebras*, J. Operator Theory **26** (1991), 345-381.

[18] M. A. RIEFFEL, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. **46** (1983), 301-333.

[19] M. RØRDAM, *Advances in the theory of unitary rank and regular approximation*, Ann. of Math. **128** (1988), 153-172.

[20] B. RUSSO AND H. A. DYE, *A note on unitary operators in C^*-algebras*, Duke Math. J. **33** (1966), 413-416.

[21] T. SUDO, *Higher extremal richness for C^*-algebras*, Far East J. Math., Vol. **1**, no. 3-4 (June-Dec. 2007) 4 pp.

Department of Mathematical Sciences, Faculty of Science, University of the Ryukyus, Senbaru 1, Nishihara, Okinawa 903-0213, Japan
Email: sudo@math.u-ryukyu.ac.jp
(Visit also www.math.u-ryukyu.ac.jp/~sudo)