The cyclic cohomology theory for smooth algebra crossed products by the group of integers in C^⊠-algebra crossed products : a review

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THE CYCLIC COHOMOLOGY THEORY FOR SMOOTH ALGEBRA CROSSED PRODUCTS BY THE GROUP OF INTEGERS IN C^*-ALGEBRA CROSSED PRODUCTS — A REVIEW

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Dedicated to Professor Yoshiomi Nakagami on his seventieth birthday

Abstract

We review and study the cyclic cohomology theory for smooth algebra crossed products by the group of integers in C^*-algebra crossed products, which is studied by Ryszard Nest.

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Introduction

We begin to study the cyclic cohomology theory for smooth algebras like differential algebras in C^*-algebras, which is one of important and useful theories, such as K-theory and index theory, in noncommutative (differential or topological) geometry initiated by Alain Connes ([1]). For this, as the first step toward this program, we review and study the cyclic cohomology theory for smooth algebra crossed products by $\mathbb{Z}$ the group of integers in C^*-algebra crossed products, which is studied by Ryszard Nest ([3]) following Connes.

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As a plan, we intended to study the theory for smooth algebra crossed products by several other groups, obtained by several other authors, but we have limited time and effort to do this, so that this project to be postponed would be continued in elsewhere, probably.

This paper exactely based on Nest [3] is organized as in the contents below, and becomes more detailed in not few parts in each subsection by our certain effort, adding elementary computaions or helpful proofs for the readers, and be corrected (or interpreted) in some parts, possibly, from misprints. Several notaions are changed from origianl ones by our taste.

Contents

1 Cyclic cohomology for smooth algebra crossed products by $\mathbb{Z}$

This one section of 13 subsections is taken from Ryszard Nest [3].

1.1 Introduction

Let A be a C^*-algebra. We say that a dense subalgebra $\mathfrak{A}$ of A is smooth if $\mathfrak{A}$ is a Fréchet algebra in some nuclear topology stronger than the norm topology from A.

Throughout this section, we assume that A is unital, $\mathfrak{A}$ is a smooth subalgebra of A containing the unit, and α is an automorphism of A mapping $\mathfrak{A}$ onto $\mathfrak{A}$ such that the restrictions of α and α^{-1} to $\mathfrak{A}$ are continuous with respect to each of the seminorms defining the topology of $\mathfrak{A}$.

In this situation, in the subsection 1.2 we give a construction of a smooth crossed product of $\mathfrak{A}$ by α, denoted by $\mathfrak{A}\rtimes_\alpha\mathbb{Z}$, which is a smooth subalgebra of the C^*-crossed product $A\rtimes_\alpha\mathbb{Z}$ and contains the algebraic crossed product of $\mathfrak{A}$ by α.

The rest of this section is devoted to the study of the cyclic cohomology of the smooth crossed product and the main results in the subsections 1.12 and 1.13, as follows.

Theorem A. *There is a linear map*

$$\# : H_\lambda^n(\mathfrak{A}) \to H_\lambda^{n+1}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}),$$

compatible with the boundary map in the Pimsner-Voiculescu six-term exact sequence of the K-theory for the C^*-crossed product $A \rtimes_\alpha \mathbb{Z}$ by $\mathbb{Z}$ and such that the following diagram is exact:

$$\begin{array}{ccccc} HC^{\mathrm{ev}}(\mathfrak{A}) & \xrightarrow{\#} & HC^{\mathrm{odd}}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) & \longrightarrow & HC^{\mathrm{odd}}(\mathfrak{A}) \\ {\scriptstyle 1-\alpha}\uparrow & & & & \downarrow{\scriptstyle 1-\alpha} \\ HC^{\mathrm{ev}}(\mathfrak{A}) & \longleftarrow & HC^{\mathrm{ev}}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) & \xleftarrow{\#} & HC^{\mathrm{odd}}(\mathfrak{A}). \end{array}$$

For the proof, given in the subsections 1.3 to 1.6 are the construction of a representation for the Hochschild cohomology of $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ and an analysis of the $\mathbb{E}_1$-term of the spectral sequence associated to the exact couple:

$$\begin{array}{ccc} H_\lambda(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) & \longrightarrow & H_\lambda(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & \downarrow \\ H(\mathfrak{A} \rtimes_\alpha \mathbb{Z}, (\mathfrak{A} \rtimes_\alpha \mathbb{Z})^*) & = & H(\mathfrak{A} \rtimes_\alpha \mathbb{Z}, (\mathfrak{A} \rtimes_\alpha \mathbb{Z})^*). \end{array}$$

On the other hand, it is obtained in the subsection 1.6 that

Proposition B. *The periodic cyclic cohomology of $\mathfrak{A}\rtimes_\alpha\mathbb{Z}$ can be computed using only the homogeneous cochains φ on $\mathfrak{A}\rtimes_\alpha\mathbb{Z}$ such that for $a_0, \cdots, a_n \in$*

$\mathfrak{A}$,

$$\varphi(u^{m_0}a_0, \cdots, u^{m_n}a_n) = \varphi(u^{m_0}a_0, \cdots, u^{m_n}a_n)\delta_{m_0+\cdots+m_n,0}.$$

In the subsections 1.9 and 1.11, given are a construction of the map # and a direct proof for the following:

Theorem C. *The following long cohomology sequence is exact:*

$$\cdots \to H_\lambda^{n-1}(\mathfrak{A}) \xrightarrow{1-\alpha} H_\lambda^{n-1}(\mathfrak{A}) \xrightarrow{\#} H_\lambda^n(\mathfrak{A} \rtimes_\alpha \mathbb{Z})_{\mathrm{hom}} \to H_\lambda^n(\mathfrak{A}) \xrightarrow{1-\alpha} \cdots,$$

where the cohomology groups with the subscript hom *are computed with the help of the homogeneous cochains above.*

This result is applied to the construction of the six-term exact sequence for the periodic cyclic cohomology of the smooth crossed product in the subsection 1.12.

In the subsection 1.13, given are a construction of the pairing between cyclic cohomology of the smooth crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ and the K-theory groups of the C^*-crossed product $A \rtimes_\alpha \mathbb{Z}$, under a condition that $\mathfrak{A}$ is sufficiently large to detect all the K-theory classes of A, and the proof of the compatibility of the six-term exact sequence in cyclic cohomology of $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ with the six-term exact sequence in the K-theory of $A \rtimes_\alpha \mathbb{Z}$.

1.2 Smooth crossed product

Let $\mathfrak{A}$ be a unital topological algebra with a topology given by an increasing sequence of seminorms $\|\cdot\|_k$ for $k = 1, 2, \cdots$. Let α be an automorphism of $\mathfrak{A}$.

We assume that $\mathfrak{A}$ is a complete, nuclear vector space and both α and α^{-1} are continuous with respect to each of the seminorms $\|\cdot\|_k$.

Define the sequence of functions $\rho_k : \mathbb{Z} \to \mathbb{R}^+$, $k \geq 1$, by

$$\rho_k(n) = \sup_{1\leq i\leq k} \Big(\sum_{t=-n}^{n} \|\alpha^t\|_i\Big)^k, \quad n \in \mathbb{Z}.$$

Remark. A seminorm p on a vector space V over $\mathbb{C}$ is defined by satisfying $0 \leq p(x) < \infty$ for $x \in V$, $p(x+y) \leq p(x) + p(y)$ for $x, y \in V$, and $p(cx) = |c|p(x)$ for $x \in V$ and $c \in \mathbb{C}$. A $*$-seminorm p on a $*$-algebra $\mathfrak{A}$ is a seminorm p on $\mathfrak{A}$ as a vector space such that $p(ab) \leq p(a)p(b)$, $p(a^*) = p(a)$, and $p(a^*a) = p(a)^2$ for $a, b \in \mathfrak{A}$. The topology given by seminorms $(\|\cdot\|_k)$ on $\mathfrak{A}$ has a basic system of neighbourhoods of $0 \in \mathfrak{A}$ defined by

$$\{a \in \mathfrak{A} \mid \|a\|_{k_j} < \varepsilon, (1 \leq j \leq n)\}.$$

We now define a version of operator norm with respect to the seminorms:

$$\|\alpha^t\|_i = \sup_{\|a\|_i \le 1} \|\alpha^t(a)\|_i, \quad a \in \mathfrak{A}.$$

Note that $\|1\|_i \le \|1\|_i^2$, so that $\|1\|_i \ge 1$. Hence $\|\alpha^t\|_i \ge 1$. Since $\|a\|_i \le \|a\|_{i+1}$ for $a \in \mathfrak{A}$, we have $\|\alpha^t\|_i \le \|\alpha^t\|_{i+1}$.

Lemma 1.2.1. (1) : $|n|\rho_k(n) \le \rho_{k+1}(n)$; (2) : $(\sum_{t=-n}^{n} \|\alpha^t\|_k)\rho_k(n) \le \rho_{k+1}(n)$; (3) : $\rho_k(m) \le \rho_k(n)\rho_k(m-n)$; (4) : $\rho_k(n)^m \le \rho_{km}(n)$.

Proof. As for (1), check that for $n \ge 0$,

$$(\sum_{t=-n}^{n} \|\alpha^t\|_i)^{k+1} = (\sum_{t=-n}^{n} \|\alpha^t\|_i)(\sum_{t=-n}^{n} \|\alpha^t\|_i)^k \ge (2n+1)(\sum_{t=-n}^{n} \|\alpha^t\|_i)^k.$$

Thus, $\rho_{k+1}(n) \ge (2n+1)\rho_k(n) \ge n \cdot \rho_k(n)$.

The first equality above implies the claim (2).

As for (3), note that $\|\alpha^{t+s}\|_i \le \|\alpha^t\|_i\|\alpha^s\|_i$, because

$$\|\alpha^{t+s}(a)\|_i = \|\alpha^t(\alpha^s(a))\|_i \le \|\alpha^t\|_i\|\alpha^s(a)\|_i \le \|\alpha^t\|_i\|\alpha^s\|_i\|a\|_i.$$

It follows that

$$\begin{aligned}
&(\sum_{t=-n}^{n} \|\alpha^t\|_i)(\sum_{t=-(m-n)}^{m-n} \|\alpha^t\|_i) \\
&\ge (\sum_{t=-n}^{n} \|\alpha^t\|_i)\|\alpha^0\|_i + (\sum_{t=-n}^{n} \|\alpha^t\|_i)(\sum_{t=-(m-n),\ne 0}^{m-n} \|\alpha^t\|_i) \\
&\ge \sum_{t=-(m-n)}^{m-n} \|\alpha^t\|_i + \sum_{t=n+1}^{m} \|\alpha^t\|_i + \sum_{t=-m}^{-n-1} \|\alpha^t\|_i = \sum_{t=-m}^{m} \|\alpha^t\|_i.
\end{aligned}$$

Therefore, we obtain the claim (3).

As for (4), since $(\sum_{-n}^{n} \|\alpha^t\|_i)^{km} \le \rho_{km}(n)$, we have $(\sum_{-n}^{n} \|\alpha^t\|_i)^k \le \rho_{km}(n)^{\frac{1}{m}}$. Thus, $\rho_k(n) \le \rho_{km}(n)^{\frac{1}{m}}$. □

Let $\mathfrak{A} \rtimes_\alpha [u, u^{-1}]$ denote the algebraic crossed product of $\mathfrak{A}$ by an action α of $\mathbb{Z}$, where elements of $\mathfrak{A} \rtimes_\alpha [u, u^{-1}]$ are given by finite sums: $\sum_n a_n u^n$ for $n \in \mathbb{Z}$ and $a_n \in \mathfrak{A}$, and the algebra has the covariance relation: $uau^{-1} = \alpha(a)$ for $a \in \mathfrak{A}$.

We now define an increasing sequence of seminorms on $\mathfrak{A} \rtimes_\alpha [u, u^{-1}]$ by

$$\|\sum_n a_n u^n\|_k = \sup_n \rho_k(n)\|a_n\|_k.$$

Indeed, check that for $c \in \mathbb{C}$,

$$\|c\sum_n a_n u^n\|_k = \sup_n \rho_k(n)\|ca_n\|_k = |c|\|\sum_n a_n u^n\|_k,$$

and

$$\|\sum_n a_n u^n + \sum_n b_n u^n\|_k = \sup_n \rho_k(n)\|a_n + b_n\|_k \le \|\sum_n a_n u^n\|_k + \|\sum_n b_n u^n\|_k,$$

and

$$\|(\sum_n a_n u^n)^*\|_k = \sup_n \rho_k(-n)\|a_n^*\|_k = \sup_n \rho_k(n)\|a_n\|_k = \|\sum_n a_n u^n\|_k,$$

and

$$\begin{aligned}\|\sum_{n=-t}^{t} a_n u^n \sum_{m=-t}^{t} b_m u^m\|_k &= \|\sum_{n,m=-t}^{t} a_n u^n b_m u^m\|_k = \|\sum_{n,m=-t}^{t} a_n \alpha^n(b_m) u^{n+m}\|_k \\ &= \sup_{n,m} \rho_k(n+m)\|a_n \alpha^n(b_m)\|_k \\ &\le \sup_{n,m} \rho_k(n)\rho_k(m)\|a_n\|_k \|u\|_k^n \|b_m\|_k \|u^*\|_k^n \\ &= \sup_n \rho_k(n)\|a_n\|_k \cdot \sup_m \rho_k(m)\|b_m\|_k \\ &= \|\sum_{n=-t}^{t} a_n u^n\|_k \cdot \|\sum_{m=-t}^{t} b_m u^m\|_k\end{aligned}$$

where $\|1\|_k = \|1\|_k^2$ so that $\|1\|_k = 1$ and $\|1\|_k = \|u^* u\|_k = \|u\|_k^2$, and finally,

$$\|\sum_n a_n u^n\|_k = \sup_n \rho_k(n)\|a_n\|_k \le \sup_n \rho_{k+1}(n)\|a_n\|_{k+1} = \|\sum_n a_n u^n\|_{k+1}.$$

It is shown by the computations above that the algebraic operations in $\mathfrak{A} \rtimes_\alpha [u, u^*]$ are continuous in the topology defined by the sequence of seminorms $\|\cdot\|_k$, $k = 1, 2, \cdots$. Hence we can make the following:

Definition 1.2.2. Define the smooth crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ to be the topological algebra obtained by the completion of $\mathfrak{A} \rtimes_\alpha [u, u^*]$ in the topology defined by the sequence of seminorms $\|\cdot\|_k$.

We note that the coefficient maps $c_m : \mathfrak{A} \rtimes_\alpha [u, u^*] \to \mathfrak{A}$ defined by $c_m(\sum_n a_n u^n) = a_m$ are continuous and hence extend to the completion $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$. Indeed, note that

$$\|\sum_n a_n u^n\|_k \ge \rho_k(m)\|a_m\|_k \ge (\sum_{t=-m}^{m} \|\alpha^t\|_k)^k \|a_m\|_k \ge (2m+1)^k \|a_m\|_k.$$

Note also that each element x of the smooth crossed product has a unique representation: $x = \sum_n a_n u^n$ (an infinite sum), $a_n \in \mathfrak{A}$.

Define the pairing between elements $x = \sum_n a_n u^n \in \mathfrak{A} \rtimes_\alpha \mathbb{Z}$ and two-sided sequences $\{\varphi_n\}_{n\in\mathbb{Z}}$ of $\varphi_n \in \mathfrak{A}^*$ the dual space of $\mathfrak{A}$ by

$$\langle \sum_n a_n u^n, \{\varphi_n\}\rangle = \sum_n \varphi_n(a_n).$$

Proposition 1.2.3. *The topological dual* $(\mathfrak{A} \rtimes_\alpha \mathbb{Z})^*$ *can be identified with the space of two-sided sequences* $\{\varphi_n\}$ *of* $\varphi_n \in \mathfrak{A}^*$, *satisfying the following condition* $(*)$: *that there exist constants* $c, k \geq 0$ *such that*

$$\sup_n \frac{\|\varphi_n\|_k}{\rho_k(n)} \leq c.$$

Proof. Suppose that $\{\varphi_n\}_{n\in\mathbb{Z}}$ defines a continuous linear functional on $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$. Then for some c, k, we have

$$|\sum_n \varphi_n(a_n)| \leq c\|\sum_n a_n u^n\|_k = c \cdot \sup_n \rho_k(n)\|a_n\|_k$$

for all $\sum_n a_n u^n \in \mathfrak{A} \rtimes_\alpha \mathbb{Z}$. Choosing monomials au^n we get

$$|\varphi_n(a)| \leq c \cdot \sup_n \rho_k(n)\|a\|_k.$$

Hence $\|\varphi_n\|_k \leq c \cdot \sup_n \rho_k(n)$. Thus,

$$\frac{\|\varphi_n\|_k}{\rho_k(n)} \leq \frac{\|\varphi_n\|_k}{\sup_n \rho_k(n)} = c,$$

which implies the condition $(*)$.

Conversely, suppose that the condition $(*)$ holds. Then, by the lemma above,

$$\begin{aligned}
|\sum_n \varphi_n(a_n)| &\leq \sum_n \|\varphi_n\|_k \|a_n\|_k \leq c \sum_n \rho_k(n)\|a_n\|_k \\
&\leq c \sum_n \frac{1}{n^2} \rho_{k+2}(n)\|a_n\|_{k+2} \\
&\leq c \sum_n \frac{1}{n^2} \sup_m \rho_{k+2}(m)\|a_m\|_{k+2} \\
&= (c \sum_n \frac{1}{n^2})\|\sum_n a_n u^n\|_{k+2},
\end{aligned}$$

where $\|\varphi_n\|_k$ is defined to be

$$\sup_{\|a\|_k \leq 1} |\varphi_n(a)|, \quad a \in \mathfrak{A}.$$

□

Proposition 1.2.4. *The smooth crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ is nuclear as a topological vector space.*

Proof. Given any vector space E and a seminorm $\|\cdot\|_k$ on E, we denote by $(E)_k^-$ the Banach space given by completing the quotient space $E/\ker(\|\cdot\|_k)$ in the induced norm. Then the claim that $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ is nuclear can be formulated as in the following.

For each i, we can find $\lambda_k \in \mathbb{C}$, $\varphi_k \in \mathfrak{A}^*$, and $b_k \in (\mathfrak{A})_i^-$, $k \in \mathbb{N}$, with $\sum_k |\lambda_k| < \infty$, $\sup_k \|b_k\|_i < \infty$, and $\sup_k \|\varphi_k\|_j < \infty$ for some j, such that the induced map π from $\mathfrak{A}$ to $(\mathfrak{A})_i^-$ has the representaion given by

$$\pi(a) = \sum_k \lambda_k \varphi_k(a) b_k, \quad a \in \mathfrak{A}.$$

What we have to show is that the same holds for $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$. Fix an index i. For each $x \in \mathfrak{A} \rtimes_\alpha \mathbb{Z}$, we have

$$\begin{aligned} x = \sum_n a_n u^n \mapsto & \sum_{n,k} \lambda_k \frac{1}{n^2} \psi_{n,k}(x) y_{n,k} \\ &= \sum_{n,k} \lambda_k \varphi_k(a_n) b_k u^n = \sum_n (\sum_k \lambda_k \varphi_k(a_n) b_k) u^n \\ &= \sum_n \pi(a_n) u^n \in (\mathfrak{A} \rtimes_\alpha \mathbb{Z})_i^-, \end{aligned}$$

where $y_{n,k} = (1/\rho_i(n)) b_k u^n$ and $\psi_{n,k}(x) = n^2 \rho_i(n) \varphi_k(a_n)$. Now we have

$$\|y_{n,k}\|_i = (1/\rho_i(n)) \|b_k u^n\|_i \leq (1/\rho_i(n)) \|b_k\|_i \leq \sup_k \|b_k\|_i < \infty.$$

Hence

$$\sup_{n,k} \|y_{n,k}\|_i \leq \sup_k \|b_k\|_i < \infty, \quad \text{and also,}$$

$$\sum_{n,k} |\lambda_k \frac{1}{n^2}| = \sum_k |\lambda_k| \sum_n \frac{1}{n^2} < \infty..$$

Moreover, we have

$$\|b_k\|_i = \|\rho_i(n) y_{n,k} u^{-n}\|_i \leq \rho_i(n) \|y_{n,k}\|_i \leq \rho_i(n) \sup_{n,k} \|y_{n,k}\|_i.$$

In particular, as $\rho_i(0) = 1$, it follows that $\sup_k \|b_k\|_i \leq \sup_{n,k} \|y_{n,k}\|_i$. Let $l = \max\{i, j\}$. Then we obtain

$$\sup_{n,k} \|\psi_{n,k}\|_{l+2} = \sup_{n,k} \frac{n^2 \rho_i(n) \|\varphi_k\|_{l+2}}{\rho_{l+2}(n)} < \infty.$$

Indeed, check that since

$$\|x\|_{l+2} = \|\sum_n a_n u^n\|_{l+2} = \sup_n \rho_{l+2}(n)\|a_n\|_{l+2} \geq \rho_{l+2}(n)\|a\|_{l+2},$$

we have

$$\begin{aligned}|\psi_{n,k}(x)| &= |n^2 \rho_i(n) \varphi_k(a_n)| \leq n^2 \rho_i(n) \|\varphi_k\|_{l+2} \|a_n\|_{l+2} \\ &\leq \frac{n^2 \rho_i(n) \|\varphi_k\|_{l+2}}{\rho_{l+2}(n)} \|x\|_{l+2},\end{aligned}$$

which implies that $\|\psi_{n,k}\|_{l+2} \leq n^2 \rho_i(n) \|\varphi_k\|_{l+2} (\rho_{l+2}(n))^{-1}$. Hence

$$\sup_{n,k} \|\psi_{n,k}\|_{l+2} \leq \sup_{n,k} \frac{n^2 \rho_i(n) \|\varphi_k\|_{l+2}}{\rho_{l+2}(n)} \leq \sup_k \ \|\varphi_k\|_{l+2} < \infty,$$

because $\rho_{l+2}(n) \geq n^2 \rho_l(n)$ by the lemma above. Conversely,

$$|n^2 \rho_i(n) \varphi_k(a_n)| = |\psi_{n,k}(x)| \leq \| \psi_{n,k}\|_{l+2} \|x\|_{l+2} \leq \sup_{n,k} \|\varphi_{n,k}\|_{l+2} \cdot \|x\|_{l+2}.$$

In particular, if we take $x = a_n$, then

$$|\varphi_k(a_n)| \leq \frac{1}{n^2 \rho_i(n)} \sup_{n,k} \|\varphi_{n,k}\|_{l+2} \cdot \rho_{l+2}(n) \|a_n\|_{l+2}.$$

Therefore, $\|\varphi_k\|_{l+2} \leq (n^2 \rho_i(n))^{-1} \sup_{n,k} \|\varphi_{n,k}\|_{l+2} \cdot \rho_{l+2}(n)$, which implies

$$\sup_{n,k} \frac{n^2 \rho_i(n) \|\varphi_k\|_{l+2}}{\rho_{l+2}(n)} \leq \sup_{n,k} \|\psi_{n,k}\|_{l+2}.$$

□

Example 1.2.5. Let $\mathfrak{A} = C^\infty(\mathbb{T})$ and suppose that α is the automorphism of $\mathfrak{A}$ induced by a rotation on $\mathbb{T}$. The algebra $\mathfrak{A}$ consists of functions $\sum_n a_n z^n$, $\{a_n\} \in S(\mathbb{Z})$, where $S(\mathbb{Z})$ denotes the space of rapidly decreasing sequences, topologized by the norms:

$$\|\{a_n\}\|_k = \sup_n (1 + n^2)^{k/2} |a_n|, \quad k = 1, 2, \cdots,$$

and also $\mathfrak{A}$ by $\|\sum a_n z^n\|_k = \|\{a_n\}\|_k$. Since $\alpha(z) = e^{i\theta}z$, where θ is the rotation angle, we have $\|\alpha^k\|_i = \|\alpha^{-k}\|_i = 1$ for all k, i, because

$$\|\alpha(\sum a_n z^n)\|_i = \|\sum a_n e^{in\theta} z^n\|_i = \|\sum a_n z^n\|_i,$$

and hence $\rho_k(n) = (2|n|+1)^k$. The smooth crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ is the algebra of double two-sided sums:

$$\sum_{n,m} a_{n,m} z^n u^m, \quad \{a_{n,m}\} \in S(\mathbb{Z}^2),$$

where $uzu^{-1} = \alpha(z)$, topologized by the norms:

$$\begin{aligned}\|\sum_{n,m} a_{n,m} z^n u^m\|_k &= \sup_m (2|m|+1)^k \|\{a_{n,m}\}\|_k \\ &= \sup_m (2|m|+1)^k \sup_n (1+n^2)^{k/2} |a_{n,m}|.\end{aligned}$$

This is the dense subalgebra of the rotation C^*-algebra, considered by Connes in Noncommutative differential geometry. In general, one cannot choose the functions ρ_k to be of polynomial growth.

Remark. The topology of $\mathfrak{A} = C^\infty(\mathbb{T}^l)$ may be given by the norms:

$$p_n(f) = \sum_{|\alpha| \le n} \sup_{z \in \mathbb{T}^l} |\partial^\alpha f(z)|, \quad f \in \mathfrak{A}.$$

1.3 Projective resolution of smooth crossed products

Let $\mathfrak{C} = \mathfrak{A} \rtimes_\alpha \mathbb{Z}$. Denote by $\mathfrak{A}^{\mathrm{op}}$ the opposite algebra of $\mathfrak{A}$, obtained by reversing multiplication in $\mathfrak{A}$. Let $\mathfrak{B} = \mathfrak{A} \otimes \mathfrak{A}^{\mathrm{op}}$ and $\mathfrak{D} = \mathfrak{C} \otimes \mathfrak{C}^{\mathrm{op}}$, where all the tensor products considered are the projective tensor products, check that, which are the completions of their algebraic tensor products under the greatest (or projective) cross (semi)norm(s):

$$\|x\|_{\gamma,k} = \inf\{\sum_j \|x_j\|_\gamma \|y_j\|_k \,|\, x = \sum_j x_j \otimes y_j\},$$

with $\|x_j \otimes y_j\|_{\gamma,k} = \|x_j\|_\gamma \|y_j\|_k$.

Recall the standard projective resolutions:

$$(M_n, b) \to \mathfrak{A} \quad \text{and} \quad (L_n, b) \to \mathfrak{C},$$

where $M_n = \mathfrak{B}\otimes(\otimes^n\mathfrak{A})$ and $L_n = \mathfrak{D}\otimes(\otimes^n\mathfrak{C})$, with the boundaray operators $b : M_n \to M_{n-1}$ and $b : L_n \to L_{n-1}$ defined by $b(x_0 \otimes x_1 \otimes \cdots \otimes x_n) =$

$$\sum_{i=0}^{n-1}(-1)^i x_0 \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes x_i x_{i+1} \otimes x_{i+2} \otimes \cdots \otimes x_n$$
$$+ (-1)^n x_0 x_n^\circ \otimes x_1 \otimes \cdots \otimes x_{n-1},$$

where $x^\circ \in \mathfrak{A}^{\mathrm{op}}$ for $x \in \mathfrak{A}$ and $xy^\circ = x \otimes y^\circ \in \mathfrak{B}$ (resp. for $\mathfrak{C}$ and $\mathfrak{D}$).

Remark. Check that the map $b : M_1 = \mathfrak{B} \otimes \mathfrak{A} \to \mathfrak{A}$ is defined by

$$b(x_0 \otimes x_1) = x_0 x_1 - x_0 x_1^\circ \in \mathfrak{A}$$

and the map $b : M_2 \to M_1$ is defined by

$$b(x_0 \otimes x_1 \otimes x_2) = x_0 x_1 \otimes x_2 - x_0 \otimes x_1 x_2 + x_2 x_2^\circ \otimes x_1,$$

so that $b^2(x_0 \otimes x_1 \otimes x_2) =$

$$(x_0x_1x_2 - x_0x_1x_2^\circ) - (x_0x_1x_2 - x_0(x_1x_2)^\circ) + (x_0x_2^\circ x_1 - x_0x_2^\circ x_1^\circ) = 0,$$

where $(x_1x_2)^\circ = x_2^\circ x_1^\circ$ and $x_0x_1x_2^\circ = x_0x_2^\circ x_1$ since $x_0 \in \mathfrak{B}$.

Now denote $\mathfrak{D} \otimes (\otimes^n\mathfrak{A})$ by $\mathfrak{D}M_n$. Note the following inclusions:

$$\mathfrak{D}M_n \subset L_n \quad \text{and} \quad b(\mathfrak{D}M_n) \subset \mathfrak{D}M_{n-1} \subset L_{n-1}.$$

This says that $(\mathfrak{D}M_n, b)$ is a subcomplex of (L_n, b) and thus we have an exact sequence of complexes:

$$0 \longrightarrow (\mathfrak{D}M_n, b) \xrightarrow{\ i\ } (L_n, b) \xrightarrow{\ \pi\ } (L_n/\mathfrak{D}M_n, b) \longrightarrow 0.$$

Let us now define a subspace $Q_n \subset L_n$ by

$$Q_n = \oplus_{i\neq 0}\ker(\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes c_0 \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}),$$

where the map c_0 at each i-th position is the coefficient map from $\mathfrak{C}$ to $\mathfrak{A}$ at the constant terms.

Lemma 1.3.1. *The image* $\mathrm{Im}(i)$ *of* $\mathfrak{D}M_n$ *is closed in* L_n *and has* Q_n *as a closed complement. The short exact sequence above splits topologically.*

Proof. Since the coefficient maps $c_m : \mathfrak{C} \to \mathfrak{A}$ are continuous, the maps:

$$\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes c_m \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} : L_n \to L_n = \mathfrak{D} \otimes (\otimes^n\mathfrak{C})$$

are continuous as well, where c_m is at the j-th position $(1 \leq j \leq n)$. We can write

$$\mathrm{Im}(i) = \cap_{m\neq 0, j\neq 0} \ker(\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes c_m \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id})$$

because we have

$$\begin{aligned}
&(\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes c_m \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id})(x_0 \otimes (\otimes_{k=1}^n x_k)) \\
&= x_0 \otimes (x_1 \otimes \cdots \otimes x_{j-1} \otimes c_m(x_j) \otimes x_{j+1} \otimes \cdots \otimes x_n),
\end{aligned}$$

so that $\mathfrak{D}M_n = \mathfrak{D} \otimes (\otimes^n \mathfrak{A}) \subset \ker(\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes c_m \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id})$ for any $m \neq 0$ and $j \neq 0$. Thus $\mathrm{Im}(i)$ is contained in the intersection above. The converse is also clear. It follows from the continuity of the maps c_m that $\mathrm{Im}(i)$ and Q_n also are closed in L_n. We have the following decompositon:

$$\begin{aligned}
&L_n \ni x_0 \otimes (\otimes_{k=1}^n x_k) \\
&= x_0 \otimes (\otimes_{k=1}^n c_0(x_k)) + (\cdots\cdots + x_0 \otimes (\otimes_{k=1}^n \sum_{m\neq 0} c_m(x_k)u^m)) \in \mathrm{Im}(i) \oplus Q_n
\end{aligned}$$

since each $x_k = \sum c_m(x_k)u^m = c_0(x_k) + \sum_{m\neq 0} c_m(x_k)u^m$, from which the splitting of the exact sequence above follows as well. □

Applying the functor $(\cdot)^{\mathrm{hom}} = \mathrm{Hom}_{\mathfrak{D}}(\cdot, \mathfrak{C}^*)$ to the exact sequence above we get

Proposition 1.3.2. *There is a long exact cohomology sequence:*

$$\cdots \xrightarrow{\delta} H^q(Q^{\mathrm{hom}}) \xrightarrow{\pi} H^q(\mathfrak{C}, \mathfrak{C}^*) \xrightarrow{i} H^q(\mathfrak{D}M^{\mathrm{hom}}) \xrightarrow{\delta} H^{q+1}(Q^{\mathrm{hom}}) \xrightarrow{\pi} \cdots,$$

with $H^q(\mathfrak{C}, \mathfrak{C}^*)$ the q-th Hochschild cohomology group of $\mathfrak{C}$ with coefficients in $\mathfrak{C}^*$.

Proof. Using the lemma above we obtain the following splitting exact sequence:

$$0 \longrightarrow (\mathfrak{D}M_n, b) \xrightarrow{i} (L_n, b) \xrightarrow{\pi} (Q_n, b) \longrightarrow 0.$$

Applying the functor $(\cdot)^{\mathrm{hom}}$ we get

$$0 \longrightarrow (Q_n^{\mathrm{hom}}, b) \xrightarrow{\pi} (L_n^{\mathrm{hom}}, b) \xrightarrow{i} (\mathfrak{D}M_n^{\mathrm{hom}}, b) \longrightarrow 0$$

equivalently, for convenience,

$$0 \to \mathrm{Hom}_{\mathfrak{D}}((Q_n, b), \mathfrak{C}^*) \to \mathrm{Hom}_{\mathfrak{D}}((L_n, b), \mathfrak{C}^*) \to \mathrm{Hom}_{\mathfrak{D}}((\mathfrak{D}M_n, b), \mathfrak{C}^*) \to 0,$$

where for simplicity we use the same symbols as b, π, i to denote their transposes. The long exact cohomology sequence corresponding to the exact sequence obtained above:

$$\begin{aligned}
&\cdots \xrightarrow{i} H^{q-1}((\mathfrak{D}M_n, b), \mathfrak{C}^*) \\
&\xrightarrow{\delta} H^q((Q_n, b), \mathfrak{C}^*) \xrightarrow{\pi} H^q((L_n, b), \mathfrak{C}^*) \xrightarrow{i} H^q((\mathfrak{D}M_n, b), \mathfrak{C}^*) \\
&\xrightarrow{\delta} H^{q+1}((Q_n, b), \mathfrak{C}^*) \xrightarrow{\pi} \cdots,
\end{aligned}$$

where $H^q(\cdot, \mathfrak{C}^*) = \ker(b_q(\cdot)^{\mathrm{hom}})/\mathrm{Im}(b_{q-1}(\cdot)^{\mathrm{hom}})$, and for short,

$$\begin{aligned}
&\cdots \xrightarrow{i} H^{q-1}(\mathfrak{D}M, \mathfrak{C}^*) \\
&\xrightarrow{\delta} H^q(Q, \mathfrak{C}^*) \xrightarrow{\pi} H^q(L, \mathfrak{C}^*) \xrightarrow{i} H^q(\mathfrak{D}M, \mathfrak{C}^*) \xrightarrow{\delta} H^{q+1}(Q, \mathfrak{C}^*) \xrightarrow{\pi} \cdots,
\end{aligned}$$

gives the desired result, where $H^q(L, \mathfrak{C}^*)$ is just the q-th Hochschild cohomology group $H^q(\mathfrak{C}, \mathfrak{C}^*)$ of $\mathfrak{C}$ with coefficients in $\mathfrak{C}^*$. □

Remark. Following [1], we may define the Hochschild cohomology of $\mathfrak{A}$ with coefficients in a bimodule M to be the cohomology $H^n(\mathfrak{A}, M)$ of the complex $(C^n(\mathfrak{A}, M), b)$, where $C^n(\mathfrak{A}, M)$ is the space of all n-linear maps from $\mathfrak{A}$ to M, and for $T \in C^n(\mathfrak{A}, M)$, $bT \in C^{n+1}(\mathfrak{A}, M)$ is defined by

$$\begin{aligned}
(bT)(a_1, \cdots, a_{n+1}) &= a_1 T(a_2, \cdots, a_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1)^i T(a_1, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) \\
&\quad + (-1)^{n+1} T(a_1, \cdots, a_n) a_{n+1}.
\end{aligned}$$

The space $\mathfrak{A}^*$ of all linear functionals on $\mathfrak{A}$ is a bimodule over $\mathfrak{A}$ by the equality $(a\varphi b)(c) = \varphi(bca)$ for $a, b, c \in \mathfrak{A}$ and $\varphi \in \mathfrak{A}^*$.

1.4 Preliminary computations

Now set (to be corrected as)

$$M_{n,k,l} = (\mathfrak{A}u^k \otimes \mathfrak{A}^{\mathrm{op}} u^l) \otimes (\otimes^n \mathfrak{A}).$$

Then $\oplus_{k,l} M_{n,k,l}$ is a dense subspace of $\mathfrak{D}M_n = \mathfrak{D} \otimes (\otimes^n \mathfrak{A})$, with $\mathfrak{D} = \mathfrak{C} \otimes \mathfrak{C}^{\mathrm{op}}$ and $\mathfrak{C} = \mathfrak{A} \rtimes_\alpha \mathbb{Z}$. We represent the elements of $(\oplus_{k,l} M_{n,k,l})^*$ as sequences $\{\varphi_{k,l}\}_{k,l \in \mathbb{Z}}$, $\varphi_{k,l} \in M_{n,k,l}^*$. We have the following slightly corrected:

Lemma 1.4.1. *A sequence* $\{\varphi_{k,l}\}_{k,l\in\mathbb{Z}}$, $\varphi_{k,l} \in M^*_{n,k,l}$ *extends to an element of* $(\mathfrak{D}M_n)^*$ *if and only if, given* $y, x_1, \cdots, x_n \in \mathfrak{A}$, *we can find* $i, j_0, j_1, \cdots, j_n \in \mathbb{N}$ *and constants* C *and* C' *such that*

$$\|\varphi_{k,l}(\cdot, y, x_1, \cdots, x_n)\|_i \leq C\rho_i(k)\rho_i(l)\|y\|_{j_0}\|x_1\|_{j_1}\cdots\|x_n\|_{j_n}, \quad k = 1, 2, \cdots,$$

and

$$\|\varphi_{k,l}(y, \cdot, x_1, \cdots, x_n)\|_i \leq C'\rho_i(k)\|y\|_{j_0}\rho_i(l)\|x_1\|_{j_1}\cdots\|x_n\|_{j_n}, \quad l = 1, 2, \cdots,$$

where each variable in the functionals corresponds to each variable in tensor factors.

Proof. This follows from the characterization of the dual space $\mathfrak{C}^*$ as in the proposition in the previous section. □

Definition 1.4.2. A sequence $\{\varphi_{k,l}\}_{k,l\in\mathbb{Z}}$, $\varphi_{k,l} \in M^*_{n,k,l}$ is called tempered if it satisfies the inequalities above. Denote by $\oplus^\wedge_{k,l}M^*_{n,k,l}$ the vector space of all such sequences.

Note that the decomposition

$$(\mathfrak{D}M_n)^* \to \oplus^\wedge_{k,l}M^*_{n,k,l}$$

is preserved by b. Indeed, this means that for $\varphi \in (\mathfrak{D}M_n)^*$ decomposed to $\{\varphi_{k,l}\}$ by restriction, the image $b\varphi \in (\mathfrak{D}M_{n+1})^*$ under b is decomposed to $\{b\varphi_{k,l}\}$ equal to $b\{\varphi_{k,l}\} \in \oplus^\wedge_{k,l}M^*_{n+1,k,l}$. We let $b_{k,l} = b|_{M^*_{n,k,l}}$ the restriction,

$$H^q_{k,l}(\mathfrak{A}, \mathfrak{A}^*) = H^q(M^*_{n,k,l}, b_{k,l}),$$

$\oplus^\wedge_{k,l}H^q_{k,l}(\mathfrak{A}, \mathfrak{A}^*)$ the vector space of all sequences $\{\xi_{k,l}\}_{k,l\in\mathbb{Z}}$, $\xi_{k,l} \in H^q_{k,l}(\mathfrak{A}, \mathfrak{A}^*)$, represented by a tempered sequence $\{\varphi_{k,l}\} \in \oplus^\wedge_{k,l}M^*_{q,k,l}$, and H^q_{res} the quotient of the vector space of all sequences $\{\varphi_{k,l}\} \in\oplus^\wedge_{k,l}M^*_{q,k,l}$, $\varphi_{k,l} \in \text{Im}(b_{k,l})$, by the image $\text{Im}(b)$. Note that $b(\oplus^\wedge_{k,l}M^*_{q,k,l}) \ni b\{\varphi_{k,l}\} = \{b_{k,l}\varphi_{k,l}\}$.

Lemma 1.4.3. *We have*

$$H^q((\mathfrak{D}M_n)^*, b) = (\oplus^\wedge_{k,l}H^q_{k,l}(\mathfrak{A}, \mathfrak{A}^*)) \oplus H^q_{\text{res}}.$$

Proof. Since the map by restriction

$$(\mathfrak{D}M_n)^* \to \oplus^\wedge_{k,l}M^*_{n,k,l}, \quad \varphi \mapsto \varphi|_{\oplus_{k,l}M_{n,k,l}}$$

is an isomorphism commuting with the coboundary operator b, the result follows from the definitions of the vector spaces in the statement. Note that each $\varphi_{k,l} = b_{k,l}\psi_{k,l}$ of $\{\varphi_{k,l}\}$ for certain $\{\psi_{k,l}\}$, in the preimage of H^q_{res} corresponds to the zero class of $H^q_{k,l}(\mathfrak{A}, \mathfrak{A}^*)$, but the class of φ in $H^q((\mathfrak{D}M_n)^*, b)$ may live. □

Next tackle the complex (Q_n, b). Define a $\mathfrak{D}$-module map $h : \mathfrak{D}M_n \to Q_{n+1} \subset L_{n+1} = \mathfrak{D} \otimes (\otimes^{n+1}\mathfrak{C})$ by

$$h((-1)^n \otimes x_1 \otimes \cdots \otimes x_x)$$
$$= \sum_{i=1}^{n} (-1)^{i-1} u^{-1} \otimes \alpha(x_1) \otimes \cdots \otimes \alpha(x_{i-1}) \otimes u \otimes x_i \otimes \cdots \otimes x_n,$$

where the i-th term is mapped to zero under $\mathrm{id} \otimes \cdots \otimes c_0 \otimes \cdots \otimes \mathrm{id}$ with c_0 at the i-th position, so that the sum belongs to Q_{n+1}.

A straightforward computation in L_n gives

$$bh(1 \otimes x_1 \otimes \cdots \otimes x_n) = hb(1 \otimes x_1 \otimes \cdots \otimes x_n)$$
$$+ (-1)^n (1 \otimes x_1 \otimes \cdots \otimes x_n - u^\circ u^{-1} \otimes \alpha(x_1) \otimes \cdots \otimes \alpha(x_n)).$$

Remark. This is not clear for us. For example, we compute

$$bh(1 \otimes x_1 \otimes x_2) = b(u^{-1} \otimes (u \otimes x_1 \otimes x_2) - u^{-1} \otimes (\alpha(x_1) \otimes u \otimes x_2))$$
$$= u^{-1}u \otimes (x_1 \otimes x_2) - u^{-1} \otimes (ux_1 \otimes x_2) + u^{-1}x_2^\circ \otimes (u \otimes x_1)$$
$$- u^{-1}\alpha(x_1) \otimes (u \otimes x_2) + u^{-1} \otimes (\alpha(x_1)u \otimes x_2) - u^{-1}x_2^\circ \otimes (\alpha(x_1) \otimes u)$$
$$= u^{-1}u \otimes (x_1 \otimes x_2) - u^{-1} \otimes (ux_1 \otimes x_2) + u^{-1}x_2^\circ \otimes (u \otimes x_1)$$
$$- x_1u^{-1} \otimes (u \otimes x_2) + u^{-1} \otimes (ux_1 \otimes x_2) - u^{-1}x_2^\circ \otimes (\alpha(x_1) \otimes u)$$

while

$$hb(1 \otimes x_1 \otimes x_2) = h(x_1 \otimes (x_2) - 1 \otimes (x_1x_2) + x_2^\circ \otimes (x_1))$$
$$= u^{-1}x_1 \otimes (u \otimes x_2) - u^{-1} \otimes (u \otimes x_1x_2) + u^{-1}x_2^\circ \otimes (u \otimes x_1),$$

where we view h as a right $\mathfrak{D}$-module map. Hence

$$bh(1 \otimes x_1 \otimes x_2) - hb(1 \otimes x_1 \otimes x_2)$$
$$= 1 \otimes (x_1 \otimes x_2) - x_1u^{-1} \otimes (u \otimes x_2) - u^{-1}x_2^\circ \otimes (\alpha(x_1) \otimes u)$$
$$- u^{-1}x_1 \otimes (u \otimes x_2) + u^{-1} \otimes (u \otimes x_1x_2).$$

Lemma 1.4.4. *The map $h : (\mathfrak{D}M_n, b) \to (Q_{n+1}, b)$ is a morphism of complexes.*

Proof. Note that b acts on Q_{n+1} modulo $\mathfrak{D}M_{n+1}$ via the isomorphism from Q_{n+1} to $L_{n+1}/\mathfrak{D}M_{n+1}$, given as above, and $bh = hb \bmod \mathfrak{D}M_n = \mathfrak{D} \otimes (\otimes^n \mathfrak{A})$, where

$$\begin{array}{ccc} \mathfrak{D}M_n & \xrightarrow{h} & Q_{n+1} \cong L_{n+1}/\mathfrak{D}M_{n+1} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b} \\ \mathfrak{D}M_{n-1} & \xrightarrow{h} & Q_n \cong L_n/\mathfrak{D}M_n \end{array}$$

(whose commutativity is not yet checked.) □

Note that both $(\mathfrak{D}M_n, b)$ and (Q_n, b) are acyclic. In fact, define an $\mathfrak{A}^{\mathrm{op}}$-module map $\rho : \mathfrak{D}M_n \to \mathfrak{D}M_{n+1}$ by

$$(u^k x_0 \otimes 1) \otimes x_1 \otimes \cdots \otimes x_n \mapsto (u^k \otimes 1) \otimes x_0 \otimes x_1 \otimes \cdots \otimes x_n.$$

Check that $\rho b + b\rho = \mathrm{id}$, so that $(\mathfrak{D}M_n, b)$ is acyclic. In particular, $b\rho b = b$:

$$\begin{array}{ccc} \mathfrak{D}M_n & = & \mathfrak{D}M_n \\ {\scriptstyle b}\downarrow & & \uparrow{\scriptstyle \rho} \\ \mathfrak{D}M_{n-1} & = & \mathfrak{D}M_{n-1}. \end{array}$$

For example, we compute

$$\begin{aligned} & b\rho((u^k x_0 \otimes 1) \otimes x_1) = b((u_k \otimes 1) \otimes x_0 \otimes x_1) \\ & = (u_k \otimes 1)x_0 \otimes x_1 - (u_k \otimes 1) \otimes x_0 x_1 + (u_k \otimes 1)x_1^\circ \otimes x_0, \end{aligned}$$

while

$$\begin{aligned} & \rho b((u^k x_0 \otimes 1) \otimes x_1) \\ & = \rho((u^k x_0 \otimes 1)x_1 - (u^k x_0 \otimes 1)x_1^\circ) \\ & = (u^k \otimes 1) \otimes x_0 x_1 - (u^k \otimes 1)x_1^\circ \otimes x_0. \end{aligned}$$

Hence $(b\rho + \rho b)((u_k x_0 \otimes 1) \otimes x_1) = (u_k x_0 \otimes 1) \otimes x_1$ sure.

As for (Q_n, b), take $x \in L_n$ with $bx \in \mathfrak{D}M_{n-1}$. Thus, $[bx] = 0$ in $Q_{n-1} \cong L_{n-1}/\mathfrak{D}M_{n-1}$. Then $bx = (\rho b + b\rho)bx$, i.e., $b(x - \rho bx) = 0$. Since (L_n, b) is acyclic, we can find $x' \in L_{n+1}$ with $x - \rho bx = bx'$. But then $x = bx' + \rho bx \equiv bx' \bmod \mathfrak{D}M_n$ since $\rho bx \in \mathfrak{D}M_n$. Thus $[x] = [bx']$ in $Q_n \cong L_n/\mathfrak{D}M_n$, and hence (Q_n, b) is acyclic as well.

Now define a (right) $\mathfrak{D}$-module map $k : Q_m \to \mathfrak{D}M_{m-1}$ for $m \geq 1$ by

$$(-1)^m \otimes u^{n_1}x_1 \otimes \cdots \otimes u^{n_m}x_m \mapsto$$
$$\begin{cases} \sum_{i=1}^{n_1} u^{n_1+\cdots+n_m}u^i\alpha^{n_1-i}(x_1) \otimes \cdots \otimes \alpha^{n_1+\cdots+n_m-1}(x_m) & \text{for } n_1 > 0, \\ 0 & \text{for } n_1 = 0, \\ -\sum_{i=n_1+1}^{0} u^{n_1+\cdots+n_m}u^i\alpha^{n_1-i}(x_1) \otimes \cdots \otimes \alpha^{n_1+\cdots+n_m-i}(x_m) & \text{for } n_1 < 0. \end{cases}$$

It holds that $kb = bk$. This can be established by a direct computation using the identity

$$k(1 \otimes u^{n_1}x_1 \otimes \cdots \otimes u^{n_m}x_m) = \rho kb(1 \otimes u^{n_1}x_1 \otimes \cdots \otimes u^{n_m}x_m)$$
$$- b(\sum_{i=1}^{n_1} u^i(u^\circ)^{n_1+\cdots n_m-i} \otimes \alpha^{n_1-i}(x_1) \otimes \cdots \otimes \alpha^{n_1+\cdots n_m-i}(x_m))$$

and induction together with the contracting homotopy property of ρ.

Check the above by: when $m = 2$, $n_1 = 2$, $n_2 = 1$ we compute

$$k((-1) \otimes u^2x_1 \otimes ux_2) = u^4\alpha(x_1) \otimes \alpha^2(x_2) + u^5x_1 \otimes \alpha(x_2),$$

while

$$\begin{aligned}
&\rho kb((-1) \otimes u^2x_1 \otimes ux_2) \\
&= \rho k((-1)u^2x_1 \otimes ux_2 + 1 \otimes u^2x_1ux_2 + (-1)(ux_2)^\circ) \otimes u^2x_1) \\
&= \rho k((-1)u^2x_1 \otimes ux_2 + 1 \otimes u^3\alpha^{-1}(x_1)x_2 + (-1)x_2^\circ u^\circ \otimes u^2x_1) \\
&= \rho(u^2(ux_2)(-1)u^2x_1 \\
&+ u^4\alpha^2(\alpha^{-1}(x_1)x_2) + u^5\alpha(\alpha^{-1}(x_1)x_2) + u^6(\alpha^{-1}(x_1)x_2) \\
&+ u^3\alpha(x_1)(-1)x_2^\circ u^\circ + u^4x_1(-1)x_2^\circ u^\circ) \\
&= \rho(u^5\alpha^{-2}(-x_2)x_1 \\
&+ u^4\alpha(x_1)\alpha^2(x_2) + u^5x_1\alpha(x_2) + u^6(\alpha^{-1}(x_1)x_2) \\
&+ u^3\alpha(x_1)(-1)x_2^\circ u^\circ + u^4x_1(-1)x_2^\circ u^\circ) \\
&= u^5 \otimes \alpha^{-2}(-x_2)x_1 \\
&+ u^4 \otimes \alpha(x_1)\alpha^2(x_2) + u^5 \otimes x_1\alpha(x_2) + u^6 \otimes \alpha^{-1}(x_1)x_2 \\
&+ u^3(-1)x_2^\circ u^\circ \otimes \alpha(x_1) + u^4(-1)x_2^\circ u^\circ) \otimes x_1.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\rho kb((-1) \otimes u^2x_1 \otimes ux_2) - k((-1) \otimes u^2x_1 \otimes ux_2) \\
&= u^5 \otimes \alpha^{-2}(-x_2)x_1 + u^4 \otimes \alpha(x_1)\alpha^2(x_2) + u^5 \otimes x_1\alpha(x_2) + u^6 \otimes \alpha^{-1}(x_1)x_2 \\
&+ u^3(-1)x_2^\circ u^\circ \otimes \alpha(x_1) + u^4(-1)x_2^\circ u^\circ \otimes x_1 - u^4\alpha(x_1) \otimes \alpha^2(x_2) - u^5x_1 \otimes \alpha(x_2),
\end{aligned}$$

while

$$
\begin{aligned}
&b(u(u^\circ)^2 \otimes \alpha(x_1) \otimes \alpha^2(x_2) + u^2u^\circ \otimes x_1 \otimes \alpha(x_2)) \\
&= u(u^\circ)^2\alpha(x_1) \otimes \alpha^2(x_2) - u(u^\circ)^2 \otimes \alpha(x_1)\alpha^2(x_2) + u(u^\circ)^2(\alpha^2(x_2))^\circ \otimes \alpha(x_1) \\
&+ u^2u^\circ x_1 \otimes \alpha(x_2) - u^2u^\circ \otimes x_1\alpha(x_2) + u^2u^\circ(\alpha(x_2))^\circ \otimes x_1 \\
&= u(u^\circ)^2\alpha(x_1) \otimes \alpha^2(x_2) - u(u^\circ)^2 \otimes \alpha(x_1)\alpha^2(x_2) + u(u^\circ)^2(\alpha^2(x_2))^\circ \otimes \alpha(x_1) \\
&+ u^2u^\circ x_1 \otimes \alpha(x_2) - u^2u^\circ \otimes x_1\alpha(x_2) + u^2u^\circ(\alpha(x_2))^\circ \otimes x_1
\end{aligned}
$$

(so that it probably fails to have the identity).

We have $\mathrm{Im}(b)|_{\mathfrak{D}M_1} = \mathrm{Im}(kb)|_{Q_2}$, but since $bkb = kb^2 = 0$ and $(\mathfrak{D}M_n, b)$ is acyclic, it is enough to show the inclusion $b(\mathfrak{D}M_1) \subset kb(Q_2)$ $(\subset \mathfrak{D}M_0)$. In fact, $b(\mathrm{Im}(kb)|_{Q_2}) = 0$ implies that $\mathrm{Im}(kb)|_{Q_2} \subset \mathrm{Im}(b)|_{\mathfrak{D}M_1}$ $(\subset \mathfrak{D}M_0)$. But the left-hand side is generated by elements of the form $x - x^\circ$, $x \in \mathfrak{A}$, and note that $x - x^\circ = kb(u^{-1} \otimes u \otimes x)$. Check this by:

$$b(1 \otimes x) = x - x^\circ,$$

but

$$
\begin{aligned}
&kb(u^{-1} \otimes u \otimes x) \\
&= k(u^{-1}u \otimes x - u^{-1} \otimes ux + u^{-1}x^\circ \otimes u) \\
&= 0 - u^{-1}u^2x + u^{-1}x^\circ u^2 \\
&= -ux + x^\circ u = -u(x - x^\circ),
\end{aligned}
$$

where $k(1 \otimes x) = 0$ since $1 \otimes x \notin Q_1$.

Lemma 1.4.5. *We have $H^q(Q^{\mathrm{hom}}) = H^{q-1}(\mathfrak{D}M^{\mathrm{hom}})$.*

Proof. We have the following diagram:

$$
\begin{array}{ccccccc}
\cdots & \xrightarrow{b} & Q_3 & \xrightarrow{b} & Q_2 & \xrightarrow{kb} & \mathfrak{D} \\
 & & \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle h} & & \| \\
\cdots & \xrightarrow{b} & \mathfrak{D}M_2 & \xrightarrow{b} & \mathfrak{D}M_1 & \xrightarrow{b} & \mathfrak{D}
\end{array}
$$

with both rows free acyclic. □

1.5 Hochschild cohomology of the smooth crossed product

Theorem 1.5.1. *The Hochschild cohomology $H^q(\mathfrak{C}, \mathfrak{C}^*)$ of the smooth crossed product $\mathfrak{C} = \mathfrak{A} \rtimes_\alpha \mathbb{Z}$ fits into the long exact sequence as:*

$$\begin{aligned}
\cdots \xrightarrow{\delta} \; & (\oplus^\wedge_{k,l} H^q_{k,l}(\mathfrak{A}, \mathfrak{A}^*)) \oplus H^q_{\text{res}} \xrightarrow{\pi} H^{q+1}(\mathfrak{C}, \mathfrak{C}^*) \\
\xrightarrow{i} \; & (\oplus^\wedge_{k,l} H^{q+1}_{k,l}(\mathfrak{A}, \mathfrak{A}^*)) \oplus H^{q+1}_{\text{res}} \\
\xrightarrow{\delta} \; & (\oplus^\wedge_{k,l} H^{q+1}_{k,l}(\mathfrak{A}, \mathfrak{A}^*)) \oplus H^{q+1}_{\text{res}} \xrightarrow{\pi} H^{q+2}(\mathfrak{C}, \mathfrak{C}^*) \xrightarrow{i} \cdots,
\end{aligned}$$

where δ is defined by $\delta\varphi = \varphi - \varphi \circ \alpha$.

Proof. Recall the long exact cohomology sequence obtained above:

$$\begin{aligned}
\cdots \xrightarrow{\delta} \; & H^q(Q^{\text{hom}}) \xrightarrow{\pi} H^q(\mathfrak{C}, \mathfrak{C}^*) \\
\xrightarrow{i} \; & H^q(\mathfrak{D}M^{\text{hom}}) \xrightarrow{\delta} H^{q+1}(Q^{\text{hom}}) \xrightarrow{\pi} \cdots.
\end{aligned}$$

Using the lemmas above we have the space identifications:

$$H^{q+1}(Q^{\text{hom}}) \cong H^q(\mathfrak{D}M^{\text{hom}}) \cong (\oplus^\wedge_k H^q_k(\mathfrak{A}, \mathfrak{A}^*)) \oplus H^q_{\text{res}},$$

and hence obtain the exact sequence in the statement. To compute δ recall the definition of the connecting homomorphism as follows.

First, start with $\{\varphi_k\}_{k\in\mathbb{Z}}$ representing a class $\xi \in H^q(\mathfrak{D}M^{\text{hom}})$.
Second, pull it back to an element of L^{hom}_q, say
$\varphi^\sim_{m_0}(x_0 u^{m_0}, \cdots, x_q u^{m_q}) = \varphi_{m_0}(x_0 u^{m_0}, x_1, \cdots, x_q)\delta_{m_1,0} \cdots \delta_{m_q,0}$.
Third, now $b\varphi^\sim = \pi\psi$ for some $\psi \in Q^{\text{hom}}_{q+1}$, and
$\delta\xi = [\psi] \in H^{q+1}(Q^{\text{hom}}) \cong H^q(\mathfrak{D}M^{\text{hom}})$.

Indeed, use the diagram:

$$\begin{array}{ccccccc}
Q^{\text{hom}}_{q+1} & \xrightarrow{\pi} & L^{\text{hom}}_{q+1} & & & & \\
{\scriptstyle b}\uparrow & & {\scriptstyle b}\uparrow & & & & \\
Q^{\text{hom}}_q & \xrightarrow{\pi} & L^{\text{hom}}_q & \xrightarrow{i} & \mathfrak{D}M^{\text{hom}}_q & \longrightarrow & 0.
\end{array}$$

But since the isomorphism $H^{q+1}(Q^{\text{hom}}) \cong H^q(\mathfrak{D}M^{\text{hom}})$ is obtained by composing cocycles on Q_{q+1} with h and $(\pi\psi) \circ h = \varphi$ (possibly, we may identify this φ with ψ as in the text), we then have

$$\begin{aligned}
\varphi &= (\pi\psi) \circ h = (b\varphi^\sim) \circ h = \varphi^\sim \circ bh \\
&= \varphi^\sim \circ (hb + (-1)^n(\text{id} - \alpha) \\
&= (-1)^n(\varphi^\sim - \varphi^\sim \circ \alpha)
\end{aligned}$$

since $\varphi^{\sim} \circ h = 0$. Hence it follows that

$$(-1)^n \delta[\varphi] = \delta[\varphi^{\sim} - \varphi^{\sim} \circ \alpha].$$

Possibly, $(-1)^n \delta^{-1}[\psi] = [\varphi^{\sim} - \varphi^{\sim} \circ \alpha]$ as in the text. □

1.6 The $\mathbb{E}_1$-term of the spectral sequence

Let $\mathfrak{C} = \mathfrak{A} \rtimes_\alpha \mathbb{Z}$ as before. Consider the following diagram in Connes [1]:

$$\begin{array}{ccc} H^*_\lambda(\mathfrak{C}) = HC^*(\mathfrak{C}) & \xrightarrow{S} & H^*_\lambda(\mathfrak{C}) = HC^*(\mathfrak{C}) \\ B\uparrow & & \downarrow I \\ H^*(\mathfrak{C}, \mathfrak{C}^*) & = & H^*(\mathfrak{C}, \mathfrak{C}^*), \end{array}$$

with the long exact sequence:

$$\cdots HC^n(\mathfrak{C}) \xrightarrow{I} H^n(\mathfrak{C}, \mathfrak{C}^*) \xrightarrow{B} HC^{n-1}(\mathfrak{C}) \xrightarrow{S} HC^{n+1}(\mathfrak{C}) \cdots$$

where $HC^n(\mathfrak{C})$ is the n-th cyclic (or Connes) cohomology group of the subcomplex $(C^n_\lambda(\mathfrak{C}), b)$ of all cyclic cochains in the Hochschild complex $(C^n(\mathfrak{C}, \mathfrak{C}^*), b)$, and

$$0 \longrightarrow C^n_\lambda(\mathfrak{C}) \xrightarrow{I} C^n(\mathfrak{C}, \mathfrak{C}^*) \longrightarrow C^n/C^n_\lambda \longrightarrow 0$$

implies $HC^n(\mathfrak{C}) = H^n_\lambda(\mathfrak{C}) \xrightarrow{I} H^n(\mathfrak{C}, \mathfrak{C}^*)$, with $H^n(C/C_\lambda) = H^{n-1}(C_\lambda)$, and $B : C^{n+1} \to C^n_\lambda$ defined by $B\varphi(x_0, x_1, \cdots, x_n)$

$=$ cyclic antisymmetrization of $\varphi(1, x_0, \cdots, x_n) + (-1)^n \varphi(x_0, x_1, \cdots, x_n, 1)$

implies $B : H^{n+1}(\mathfrak{C}, \mathfrak{C}^*) \to HC^n(\mathfrak{C})$, and $S : C^n_\lambda(\mathfrak{C}) \to C^{n+2}_\lambda(\mathfrak{C})$ defined by $S\varphi$

$=$ cyclic antisymmetrization of the cup product of φ with

the 2-cocycle as a generator of $HC^2(\mathbb{C})$, implies $S : HC^n(\mathfrak{C}) \to HC^{n+2}(\mathfrak{C})$.

The $\mathbb{E}_1$-term is given by the homology of the complex $(H^n(\mathfrak{C}, \mathfrak{C}^*), d_0)$ with $d_0 = IB : H^n(\mathfrak{C}, \mathfrak{C}^*) \to H^{n-1}(\mathfrak{C}, \mathfrak{C}^*)$.

Lemma 1.6.1. *Given any n-cochain φ, let*

$$\varphi_{(k)}(x_0, x_1, \cdots, x_{n-1}) = \varphi(x_0, x_1, \cdots, x_{k-1}, 1, x_k, \cdots, x_{n-1}),$$
$$\varphi_{(k,k+1)}(x_0, x_1, \cdots, x_{n-2}) = \varphi(x_0, x_1, \cdots, x_{k-1}, 1, 1, x_k, \cdots, x_{n-2}).$$

Denote by N the cyclic antisymmetrization operator. Then

$$\sum_{k>0}(b\varphi)_{(k,k+1)} = b(\sum_{k>0} \varphi_{(k,k+1)}) + \sum_{k>0}(-1)^{k-1}\varphi_{(k)},$$
$$N((b\varphi)_{(n,n+1)}) = bN(\varphi_{(n-1,n)}) + (-1)^{n-1}N(\varphi_{(n)}).$$

Proof. Check that the case where $n = 2$ as follows.

$$\begin{aligned}&(b\varphi)(x_0, x_1, x_2, x_3) = \varphi(b(x_0, x_1, x_2, x_3))\\ &= \varphi(x_0x_1, x_2, x_3) - \varphi(x_0, x_1x_2, x_3) + \varphi(x_0, x_1, x_2x_3) - \varphi(x_0x_3^\circ, x_1, x_2),\end{aligned}$$

so that

$$\begin{aligned}&(b\varphi)_{(1,2)}(x_0, x_1) = (b\varphi)(x_0, 1, 1, x_1)\\ &= \varphi(x_0, 1, x_1) - \varphi(x_0, 1, x_1) + \varphi(x_0, 1, x_1) - \varphi(x_0x_1^\circ, 1, 1),\\ &(b\varphi)_{(2,3)}(x_0, x_1) = (b\varphi)(x_0, x_1, 1, 1)\\ &= \varphi(x_0x_1, 1, 1) - \varphi(x_0, x_1, 1) + \varphi(x_0, x_1, 1) - \varphi(x_0, x_1, 1)\end{aligned}$$

and hence,

$$\begin{aligned}&\sum_{k>0}(b\varphi)_{(k,k+1)}(x_0, x_1) = (b\varphi)_{(1,2)}(x_0, x_1) + (b\varphi)_{(2,3)}(x_0, x_1)\\ &= \varphi(x_0, 1, x_1) - \varphi(x_0x_1^\circ, 1, 1) + \varphi(x_0x_1, 1, 1) - \varphi(x_0, x_1, 1).\end{aligned}$$

On the other hand,

$$\varphi_{(1)}(x_0, x_1) = \varphi(x_0, 1, x_1), \quad \varphi_{(2)}(x_0, x_1) = \varphi(x_0, x_1, 1), \quad \varphi_{(1,2)}(x_0) = \varphi(x_0, 1, 1),$$

so that

$$\begin{aligned}&(b(\sum_{k>0}\varphi_{(k,k+1)}) + \sum_{k>0}(-1)^{k-1}\varphi_{(k)})(x_0, x_1)\\ &= b(\varphi_{(1,2)})(x_0, x_1) + \varphi_{(1)}(x_0, x_1) - \varphi_{(2)}(x_0, x_1)\\ &= \varphi_{(1,2)}(b(x_0, x_1)) + \varphi(x_0, 1, x_1) - \varphi(x_0, x_1, 1)\end{aligned}$$

and

$$\begin{aligned}\varphi_{(1,2)}(b(x_0, x_1)) &= \varphi_{(1,2)}(x_0, x_1 - x_0x_1^\circ)\\ &= \varphi(x_0x_1, 1, 1) - \varphi(x_0x_1^\circ, 1, 1).\end{aligned}$$

As for the second equality, we compute

$$\begin{aligned}&N((b\varphi)_{(2,3)})(x_0, x_1) = (b\varphi)_{(2,3)}(x_0, x_1) - (b\varphi)_{(2,3)}(x_1, x_0)\\ &= (b\varphi)(x_0, x_1, 1, 1) - (b\varphi)(x_1, x_0, 1, 1)\\ &= \varphi(x_0x_1, 1, 1) - \varphi(x_0, x_1, 1) - \varphi(x_1x_0, 1, 1) + \varphi(x_1, x_0, 1),\end{aligned}$$

and on the other hand,

$$\begin{aligned}
&bN(\varphi_{(1,2)})(x_0,x_1) - N\varphi_{(2)}(x_0,x_1) \\
&= N(\varphi_{(1,2)})(b(x_0,x_1)) - \varphi_{(2)}(x_0,x_1) + \varphi_{(2)}(x_1,x_0) \\
&= N\varphi_{(1,2)}(x_0x_1 - x_0x_1^\circ) - \varphi(x_0,x_1,1) + \varphi(x_1,x_0,1)
\end{aligned}$$

and $N\varphi_{(1,2)}(x_0x_1 - x_0x_1^\circ) = \varphi(x_0x_1,1,1) - \varphi(x_0x_1^\circ,1,1)$ (from which the second identity seems to slightly fail). □

Recall that

$$\begin{array}{ccccccc}
\xrightarrow{\delta} (\oplus^\wedge_{k,l} H^{n-1}_{k,l}(\mathfrak{A},\mathfrak{A}^*)) \oplus H^{n-1}_{\mathrm{res}} & \xrightarrow{\pi} & H^n(\mathfrak{C},\mathfrak{C}^*) & \xrightarrow{i} & (\oplus^\wedge_{k,l} H^n_{k,l}(\mathfrak{A},\mathfrak{A}^*)) \oplus H^n_{\mathrm{res}} \\
 & & \downarrow{\scriptstyle d_0} & & \\
\xrightarrow{\delta} (\oplus^\wedge_{k,l} H^{n-2}_{k,l}(\mathfrak{A},\mathfrak{A}^*)) \oplus H^{n-2}_{\mathrm{res}} & \xrightarrow{\pi} & H^{n-1}(\mathfrak{C},\mathfrak{C}^*) & \xrightarrow{i} & (\oplus^\wedge_{k,l} H^{n-1}_{k,l}(\mathfrak{A},\mathfrak{A}^*)) \oplus H^{n-1}_{\mathrm{res}}
\end{array}$$

and note that the map i is just the restriction map from $H^n(\mathfrak{C},\mathfrak{C}^*) = H^n(L^{\mathrm{hom}})$ to $H^n(\mathfrak{D}M^{\mathrm{hom}})$, while the map π is given by the composition of cochains with the map $k : Q_n \to \mathfrak{D}M_{n-1}$. Consider the following decomposition:

$$\begin{aligned}
&(\oplus^\wedge_{k,l} H^n_k(\mathfrak{A},\mathfrak{A}^*)) \oplus H^n_{\mathrm{res}} \\
&= H^n_0(\mathfrak{A},\mathfrak{A}^*) \oplus [(\oplus^\wedge_{(k,l)\neq(0,0)} H^n_{k,l}(\mathfrak{A},\mathfrak{A}^*)) \oplus H^n_{\mathrm{res}}]
\end{aligned}$$

with $H^n_0(\mathfrak{A},\mathfrak{A}^*) = H^n(\mathfrak{A},\mathfrak{A}^*)$ called the homogeneous part and the second direct summand $[\cdots]$ the non-homogeneous part.

(A) First consider the map d_0 at the non-homogeneous part.

Consider the map $id_0\pi = i \circ d_0 \circ \pi : \mathrm{coker}(\delta) \to \ker(\delta)$, where

$$\begin{aligned}
&\mathrm{coker}(\delta_{n-2}) = (\oplus^\wedge_{k,l} H^{n-1}_{k,l}(\mathfrak{A},\mathfrak{A}^*) \oplus H^{n-1}_{\mathrm{res}} / \delta_{n-2}((\oplus^\wedge_{k,l} H^{n-1}_{k,l}(\mathfrak{A},\mathfrak{A}^*) \oplus H^{n-1}_{\mathrm{res}}) \\
&\to \ker(\delta_{n-1}) \subset (\oplus^\wedge_{k,l} H^{n-1}_{k,l}(\mathfrak{A},\mathfrak{A}^*) \oplus H^{n-1}_{\mathrm{res}}
\end{aligned}$$

is well defined by the long exact sequence in the previous section. Let $\varphi = \{\varphi_{k,l}\} \in \oplus^\wedge_{k,l} M^*_{n,k,l}$. Set $\varphi_k = \varphi_{k,0}$.

Lemma 1.6.2. *$i \circ d_0\pi[\{\varphi_k\}_{k\in\mathbb{Z}}] = [\{\sum_{i=1}^k \varphi_k \circ \alpha^{k-i}\}_{k\in\mathbb{Z}}]$, where the sum is the summation with zero at zero and positive and negative signs as before.*

Proof. We compute

$$\begin{aligned}
&(i \circ d_0\pi\varphi)_l(1, x_1u^l, x_2, \cdots, x_n) = (iIB(\varphi \circ k))_l(1, x_1u^l, x_2, \cdots, x_n) \\
&= N((\varphi(k(1, x_1u^l, x_2, \cdots, x_n))))
\end{aligned}$$

where

$$\begin{aligned}
k(1, x_1u^l, x_2, \cdots, x_n) &= k(1, u^l\alpha^{-l}(x_1), x_2, \cdots, x_n)\\
&= \sum_{i=1}^{l} u^{l+i}\alpha^{l-i}(\alpha^{-l}(x_1)) \otimes \alpha^{l-i}(x_2) \otimes \cdots \ \alpha^{l-i}(x_n)\\
&= \sum_{i=1}^{l} u^l x_1 u^i \otimes \alpha^{l-i}(x_2) \otimes \cdots \ \alpha^{l-i}(x_n)\\
&= \sum_{i=1}^{l} \alpha^{l-i}(u^i x_1 u^l, x_2, \cdots, x_n)
\end{aligned}$$

(probably, something in the definition is necessary to be changed). □

Corollary 1.6.3. (I) $\delta[\{\varphi_l\}_{l\in\mathbb{Z}}] = 0 \Rightarrow i \circ d_0\pi[\{\varphi_l\}_{l\in\mathbb{Z}}] = [\{l\varphi_l\}_{l\in\mathbb{Z}}]$.
(II) $d_0\pi[\{\varphi_l\}_{l\in\mathbb{Z}}] = 0 = \varphi_0 \Rightarrow [\{\varphi_l\}_{l\in\mathbb{Z}}] \in \mathrm{Im}(\delta)$.
(III) $i \circ d_0\pi$ *is onto* $\ker(\delta) \ominus H_0^n(\mathfrak{A}, \mathfrak{A}^*)$.
(IV) $(i \circ d_0\pi\varphi)_0 = 0$ *for any cocycle* φ.

Proof. (I) $\delta[\{\varphi_k\}] = 0$ means that $\varphi_k - \varphi_k \circ \alpha = b\omega_k$ for $k \in \mathbb{Z}$, for some tempered sequence $\{\omega_k\}_{k\in\mathbb{Z}}$. Note that the diagram:

$$H^{n-1}(\mathfrak{D}M^{\mathrm{hom}}) \xrightarrow{b} H^n(\mathfrak{D}M^{\mathrm{hom}}) \xrightarrow{\delta} H^{n+1}(Q^{\mathrm{hom}})$$

with $H^{n-1}(\mathfrak{D}M^{\mathrm{hom}}) = (\oplus^{\wedge}_{k,l} H^{n-1}_{k,l}(\mathfrak{A}, \mathfrak{A}^*)) \oplus H^{n-1}_{\mathrm{res}}$. Then for $l > 0$,

$$\begin{aligned}
\sum_{i=1}^{l} \varphi_l \circ \alpha^{l-i} &= l\varphi_l - \sum_{i=1}^{l}\sum_{t=1}^{l-i}(\varphi_l - \varphi_l \circ \alpha) \circ \alpha^{l-i-t}\\
&= l\varphi_l - b(\sum_{i=1}^{l}\sum_{t=1}^{l-i} \omega_l \circ \alpha^{l-i-t}).
\end{aligned}$$

Indeed, check that

$$\sum_{i=1}^{l} \varphi_l \circ \alpha^{l-i} = -\sum_{i=1}^{l}(\varphi_l - \varphi_l \circ \alpha^{l-i}) + \sum_{i=1}^{l} \varphi_l = l\varphi_l - \sum_{i=1}^{l}(\varphi_l - \varphi_l \circ \alpha^{l-i})$$

and

$$\begin{aligned}
&\sum_{t=1}^{l-i}(\varphi_l - \varphi_l \circ \alpha) \circ \alpha^{l-i-t} =\\
&(\varphi_l - \varphi_l \circ \alpha) \circ \alpha^{l-i-1} + (\varphi_l - \varphi_l \circ \alpha) \circ \alpha^{l-i-2} + \cdots + (\varphi_l - \varphi_l \circ \alpha)\\
&= (\varphi_l - \varphi_l \circ \alpha) + (\varphi_l - \varphi_l \circ \alpha) \circ \alpha + \cdots + (\varphi_l - \varphi_l \circ \alpha) \circ \alpha^{l-i-1}\\
&= \varphi_l - \varphi_l \circ \alpha^{l-i}.
\end{aligned}$$

Since the term $\{\sum_{i=1}^{l}\sum_{t=1}^{l-i}\omega_l\circ\alpha^{l-i-t}\}_{l\in\mathbb{Z}}$ defines a tempered sequence, we get

$$[\{l\varphi_l\}] = [\{\sum_{i=1}^{l}\varphi_l\circ\alpha^{l-i}\}] = i\circ d_0\pi[\{\varphi_l\}].$$

(II) It follows from the first equation above that

$$i\circ d_0\pi[\{\varphi_l\}] = [\{l\varphi_l\}] + \delta[\{\psi_l\}],$$

where $\psi_l = -\sum_{i=1}^{l}\sum_{t=1}^{l-i}\varphi_l\circ\alpha^{l-i-t}$, because $\delta[\{\psi_l\}] = [\{\psi_l - \psi_l\circ\alpha\}]$. If $\varphi_0 = 0$, then also $\psi_0 = 0$, and $\{\psi_k\}$ is a tempered sequence. Thus we can write $l\varphi_l + \delta\psi_l = b\omega_l$ for some tempered sequence $\{\omega_l\}_{l\in\mathbb{Z}}$ with $\omega_0 = 0$, since the equation above is zero from the assumption. But then

$$\{\varphi_l\} = b\{\frac{1}{l}\omega_l\} - \delta\{\frac{1}{l}\psi_l\},$$

and both $\{(1/l)\omega_l\}$ and $\{(1/l)\psi_l\}$ are tempered. Hence $[\{\varphi_l\}] = \delta[\{(-1/l)\psi_l\}]$ as a cohomology class.

(III) This follows immediately from (I) and the fact that given a tempered cochain $\{\varphi_l\}$ with $\varphi_0 = 0$, $\{(1/l)\varphi_l\}$ is also tempered, so that if $\delta[\{(1/l)\varphi_l\}] = 0$, then $i\circ d_0\pi[\{(1/l)\varphi_l\}] = [\{\varphi_l\}]$, indeed.

(IV) This follows from the lemma just above, because the zero-th term of the sum is zero. □

(B) Second consider the map d_0 at the homogeneous part $H^n(\mathfrak{A},\mathfrak{A}^*)$.

Note that the definition of d_0 on $H^n(\mathfrak{A},\mathfrak{A}^*)$ is viewed as the derivative of the spectral sequence, relating Hochschild and cyclic cohomology of $\mathfrak{A}$.

Lemma 1.6.4. (I) *Given a cocycle φ on $\mathfrak{C}$, we have $d_0(i\varphi)_0 = (i\circ d_0\varphi)_0$.*

(II) *Given a cocycle φ on $\mathfrak{A}$, we have $d_0\pi\{\varphi\cdot\delta_{k,0}\} = \pi\{d_0\varphi\cdot\delta_{k,0}\}$.*

Proof. (I) This follows immediately from the fact that i is the restriction map. Note that the left hand side is the 0-th term of the cyclic antisymmetrization of $i\varphi(1,\cdot,\dots,\cdot) + (-1)^n i\varphi(\cdot,\dots,\cdot,1)$, and the right hand side is the 0-th term of the restriction of the cyclic antisymmetrization of $\varphi(1,\cdot,\dots,\cdot) + (-1)^n\varphi(\cdot,\dots,\cdot,1)$.

(II) Since $i\circ d_0\pi\{\varphi\cdot\delta_{k,0}\} = 0$ by (IV) of the corollary above, we can write $d_0\pi\{\varphi\cdot\delta_{k,0}\} = \pi\{\psi\cdot\delta_{k,0}\}$ for some cocycle ψ on $\mathfrak{A}$, using the exactness of the long cohomology sequence. Since the composition with $h : \mathfrak{D}M_{n-1}\to Q_n$ inverts π, i.e. $(\pi\psi)\circ h = \psi$ (under the identification as before), we have $\psi = (d_0\pi\varphi)\circ h$. This is given by

$$\psi(x_0, x_1, \cdots, x_n) = \varphi(k(1\otimes Nh(x_0\otimes\cdots x_n)))$$

since $(d_0\pi\varphi)\circ h = N(\varphi\circ k\circ h) = \varphi\circ k\circ (Nh)$. Look at a typical term

$$k(1\otimes x_i\otimes\cdots x_n\otimes x_0u^{-1}\otimes\alpha(x_1)\otimes\cdots\otimes\alpha(x_k)\otimes u\otimes x_{k+1}\otimes\cdots\otimes x_{i-1}).$$

It can be non-zero only if we have either

$$\begin{aligned}
&k(1\otimes x_0u^{-1}\otimes\alpha(x_1)\otimes\cdots\otimes\alpha(x_k)\otimes u\otimes x_{k+1}\otimes\cdots\otimes x_n)\\
&= -x_0\otimes x_1\otimes\cdots\otimes x_k\otimes 1\otimes x_{k+1}\otimes\cdots\otimes x_n, \quad \text{or}\\
&k(1\otimes u\otimes x_{k+1}\otimes\cdots\otimes x_n\otimes x_0u^{-1}\otimes\alpha(x_1)\otimes\cdots\otimes\alpha(x_k))\\
&= 1\otimes x_{k+1}\otimes\cdots\otimes x_n\otimes x_0\otimes x_1\otimes\cdots\otimes x_k.
\end{aligned}$$

Check that the first equality can be computed by definition

$$\begin{aligned}
&-u^{-1+1}\alpha^{-1}(ux_0u^{-1})\otimes\alpha^{-1}\alpha(x_1)\otimes\cdots\otimes\alpha^{-1}\alpha(x_k)\\
&\otimes\alpha^{-1+1}(1)\otimes\alpha^{-1+1}(x_{k+1})\ \otimes\cdots\otimes\alpha^{-1+1}(x_n)\\
&= -x_0\otimes x_1\otimes\cdots\otimes x_k\otimes 1\otimes x_{k+1}\otimes\cdots\otimes x_n
\end{aligned}$$

and the second equatlity should be

$$\begin{aligned}
&u^{1-1}u\alpha^0(1)\otimes x_{k+1}\otimes\cdots\otimes x_n\otimes\alpha^{-1}(ux_0u^{-1})\otimes\alpha^{-1}\alpha(x_1)\otimes\cdots\otimes\alpha^{-1}\alpha(x_k))\\
&= u\otimes x_{k+1}\otimes\cdots\otimes x_n\otimes x_0\otimes x_1\otimes\cdots\otimes x_k.
\end{aligned}$$

Combining the signs from cyclic permutation and from the position of u in $h(x_0\otimes x_1\otimes\cdots\otimes x_n)$ we get $\psi(x_0,x_1,\cdots,x_n) =$

$$[\text{cyclic anti-symm of } \varphi(1,x_0,\cdots,x_n)] + \sum_{j>0}(-1)^j\varphi_{(j)}(x_0,x_1,\cdots,x_n).$$

Then the first lemma in this subsection gives that $\psi = d_0\varphi + \text{coboundary}$, and hence the proof is completed. □

(C) Computation of the $\mathbb{E}_1$-term.

Consider a similar decomposition of $H^n(\mathfrak{C},\mathfrak{C}^*)$ as the above decomposition into homogeneous and non-homogeneous parts:

$$H^n(\mathfrak{C},\mathfrak{C}^*) = H^n_{\mathrm{hom}}(\mathfrak{C},\mathfrak{C}^*)\oplus H^n_e(\mathfrak{C},\mathfrak{C}^*), \quad \varphi = \varphi_{\mathrm{hom}}+\varphi_e,$$

where $\varphi_{\mathrm{hom}}(x_0u^{m_0},\cdots,x_nu^{m_n}) = \varphi(x_0u^{m_0},\cdots,x_nu^{m_n})\cdot\delta_{m_0+\cdots+m_n,0}$. Since the operator d_0 preserves this splitting, we can write

$$\mathbb{E}_1(\mathfrak{C}) = \mathbb{E}_1(\mathfrak{C})_{\mathrm{hom}}\oplus\mathbb{E}_1(\mathfrak{C})_e.$$

Check that $d_0\varphi_{\text{hom}}(x_0u^{m_0}, x_1u^{m_1}, \cdots, x_nu^{m_n}) =$ cyclic anti-symm of

$$\begin{aligned}
&\varphi_{\text{hom}}(1, x_0u^{m_0}, \cdots, x_nu^{m_n}) + (-1)^n\varphi_{\text{hom}}(x_0u^{m_0}, x_1u^{m_1}, \cdots, x_nu^{m_n}, 1)\\
&= \varphi(1, x_0u^{m_0}, \cdots, x_nu^{m_n}) \cdot \delta_{0+m_0+\cdots+m_n,0}\\
&+ (-1)^n\varphi(x_0u^{m_0}, \cdots, x_nu^{m_n}, 1) \cdot \delta_{m_0+\cdots+m_n+0,0}\\
&= [\varphi(1, x_0u^{m_0}, \cdots, x_nu^{m_n}) + (-1)^n\varphi(x_0u^{m_0}, \cdots, x_nu^{m_n}, 1)] \cdot \delta_{m_0+\cdots+m_n,0}.
\end{aligned}$$

Theorem 1.6.5. *We have* $\mathbb{E}_1(\mathfrak{C})_e = \mathbb{E}_1^*(\mathfrak{C})_e = \oplus_n\mathbb{E}_1^n(\mathfrak{C})_e = 0$.

Proof. Let $\ker(\delta)_e^n$ and $\operatorname{coker}(\delta)_e^n$ be the kernel and cokernel of the restriction of δ to $(\oplus_{j\neq 0}^{\wedge}H_j^n(\mathfrak{A}, \mathfrak{A}^*)) \oplus H_{\text{res}}^n$ respectively. Look at the diagram:

$$\begin{array}{ccccc}
0 \to \operatorname{coker}(\delta)_e^n & \xrightarrow{\pi} & H_e^{n+1}(\mathfrak{C}, \mathfrak{C}^*) & \xrightarrow{i} & \ker(\delta)_e^{n+1} \to 0\\
& & \big\downarrow d_0 & & \\
0 \to \operatorname{coker}(\delta)_e^{n-1} & \xrightarrow{\pi} & H_e^{n}(\mathfrak{C}, \mathfrak{C}^*) & \xrightarrow{i} & \ker(\delta)_e^{n} \to 0\\
& & \big\downarrow d_0 & & \\
0 \to \operatorname{coker}(\delta)_e^{n-2} & \xrightarrow{\pi} & H_e^{n-1}(\mathfrak{C}, \mathfrak{C}^*) & \xrightarrow{i} & \ker(\delta)_e^{n-1} \to 0,
\end{array}$$

where the exactness of the rows follows from the long exact cohomology sequence obtained in the previous section. Suppose that we are given $\varphi \in H_e^n(\mathfrak{C}, \mathfrak{C}^*)$ such that $d_0\varphi = 0$. It follows from (III) in the above corollary that there is $\psi_1 \in \operatorname{coker}(\delta)_e^n$ such that $i\varphi = i\circ d_0\pi\psi_1$. Since the middle row is exact, we can find an element $\psi_2 \in \operatorname{coker}(\delta)_e^{n-1}$ such that $\varphi = d_0\pi\psi_1 + \pi\psi_2$, because $i(\varphi - d_0\pi\psi_1) = 0$. But then

$$d_0\pi\psi_2 = d_0(\varphi - d_0\pi\psi_1) = 0 - 0 = 0,$$

and hence, from (II) in the corollary above, $\operatorname{Im}(\delta) \ni \psi_2 = 0$ in $\operatorname{coker}(\delta)_e^{n-1}$. Thus, $\varphi = d_0\pi\psi_1$. Hence $\mathbb{E}_1^n(\mathfrak{C})_e = \ker(d_0|_{H_e^n(\mathfrak{C},\mathfrak{C}^*)})/d_0H_e^{n+1}(\mathfrak{C}, \mathfrak{C}^*) = 0$, and $\mathbb{E}_1(\mathfrak{C})_e = \mathbb{E}_1^*(\mathfrak{C})_e = \oplus_n\mathbb{E}_1^n(\mathfrak{C})_e = 0$. □

Note that the decomposition of cocycles given above works equally well in the cyclic case, so that we can write

$$HC^n(\mathfrak{C}) = HC_\lambda^n(\mathfrak{C}) = H_\lambda^n(\mathfrak{C})_{\text{hom}} \oplus H_\lambda^n(\mathfrak{C})_e.$$

Therefore, we obtain

Corollary 1.6.6. $H_\lambda^n(\mathfrak{C})_e \subset \ker(S)$.

Proof. Since S preserves the above decomposition of cyclic cocycles and since for a cyclic cocycle we have

$$\varphi \in \mathrm{Im}(d_0) \Leftrightarrow \varphi \in \mathrm{Im}(S) + \ker(S),$$

so that $S\varphi \in \mathrm{Im}(S^2)$, and we can conclude from the theorem above that

$$SH^n_\lambda(\mathfrak{C})_e \subset S^2 H^{n-2}_\lambda(\mathfrak{C})_e.$$

Indeed, recall that $S\varphi = N(\sigma \# \varphi)$, where the cup product $\sigma\#\varphi$ with $[\sigma] \in HC^2(\mathbb{C})$ is defined by $\sigma\#\varphi = (\sigma \otimes \varphi) \circ \pi$, where $\pi : \Omega^{n+2}(\mathbb{C} \otimes \mathfrak{C}) \to \Omega^2(\mathbb{C}) \otimes \Omega^n(\mathfrak{C})$ is a natural homomorphism of differential graded algebras. Recall that $d_0 = IB$ and suppose that $\varphi = d_0\psi = IB\psi$ for some ψ, and if $IB\psi \neq 0$, then φ is viewed in $\mathrm{Im}(B) = \ker(S)$, and if $IB\psi = 0$, then $B\psi \in \ker(I) = \mathrm{Im}(S)$, identified with φ. Note also that the non-exact sequence is:

$$\cdots \to HC^{n-2}(\mathfrak{C})_e \xrightarrow{\;S\;} HC^n(\mathfrak{C})_e \xrightarrow{\;S\;} HC^{n+2}(\mathfrak{C})_e \to \cdots .$$

Conversely, if $S\varphi = 0$, then $\varphi = B\psi$ for some ψ, and if $\varphi = S\rho$ for some ρ, then $d_0\varphi = IBS\rho = 0$, hence $\varphi \in \mathrm{Im}(d_0)$, as checked.

Iterating the inclusion above we get $SH^n_\lambda(\mathfrak{C})_e \subset S^k H^{n-2k}_\lambda(\mathfrak{C})_e$ for $k = 1, 2, \cdots$, and choosing $k > n/2$ we get the result desired. □

To describe $\mathbb{E}_1(\mathfrak{C})_{\mathrm{hom}}$ we set

$$H^n_{\mathrm{eq}}(\mathfrak{A}) = \text{homology of } (\ker(\delta|_{H(\mathfrak{A},\mathfrak{A}^*)}), d_0),$$
$$H^n_{\mathrm{coeq}}(\mathfrak{A}) = \text{homology of } (\mathrm{coker}(\delta|_{H(\mathfrak{A},\mathfrak{A}^*)}), d_0).$$

Then the following holds:

Theorem 1.6.7. *The $\mathbb{E}_1$-term of the spectral sequence of the smooth crossed product $\mathfrak{C} = \mathfrak{A} \rtimes_\alpha \mathbb{Z}$ fits into a long exact sequence:*

$$\cdots \xrightarrow{\Delta} H^{n-1}_{\mathrm{coep}}(\mathfrak{A}) \xrightarrow{\pi} \mathbb{E}^n_1(\mathfrak{C}) \xrightarrow{i} H^n_{\mathrm{eq}}(\mathfrak{A}) \xrightarrow{\Delta} H^{n-2}_{\mathrm{coeq}}(\mathfrak{A}) \xrightarrow{\pi} \cdots .$$

Proof. It follows from the homogeneous part of the long exact sequence in the previous subsection that the rows of the following diagram are exact:

$$\begin{array}{ccccccccc}
 & & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \\
0 \to & & \mathrm{coker}(\delta|_{H^{n-1}(\mathfrak{A},\mathfrak{A}^*)}) & \xrightarrow{\pi} & H^n(\mathfrak{C},\mathfrak{C})_{\mathrm{hom}} & \xrightarrow{i} & \ker(\delta|_{H^n(\mathfrak{A},\mathfrak{A}^*)}) & \to 0 & \\
 & & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \\
0 \to & & \mathrm{coker}(\delta|_{H^{n-2}(\mathfrak{A},\mathfrak{A}^*)}) & \xrightarrow{\pi} & H^{n-1}(\mathfrak{C},\mathfrak{C})_{\mathrm{hom}} & \xrightarrow{i} & \ker(\delta|_{H^{n-1}(\mathfrak{A},\mathfrak{A}^*)}) & \to 0 & \\
 & & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & &
\end{array}$$

The diagram is commutative by the lemma above, and according to the theorem above, the homology of the middle column is equal to $\mathbb{E}_1(\mathfrak{C})$. Applying the long exact homology sequence to the short exact sequence of complexes given by this diagram, we obtain

$$0 \to (\mathrm{coker}(\delta), d_0) \to (H^n(\mathfrak{C}, \mathfrak{C}^*)_{\mathrm{hom}}, d_0) \to (\ker(\delta), d_0) \to 0.$$

Note that the connecting homomorphism Δ is defined as a map:

$$\begin{aligned} H^n_{\mathrm{eq}}(\mathfrak{A}) &= \ker(d_0|_{\ker(\delta|_{H^n})})/d_0(\ker(\delta|_{H^{n+1}})) \xrightarrow{\Delta} \\ H^{n-2}_{\mathrm{coeq}}(\mathfrak{A}) &= \ker(d_0|_{\mathrm{coker}(\delta|_{H^{n-2}})})/d_0(\mathrm{coker}(\delta|_{H^{n-1}})), \end{aligned}$$

where $H^k = H^k(\mathfrak{A}, \mathfrak{A}^*)$, and see the lemma (corrected) below for its definition. □

Lemma 1.6.8. *The connecting homomorphism of the theorem above is given as follows. For a class* $[\varphi] \in H^n_{\mathrm{eq}}(\mathfrak{A})$, *if* $\varphi^\sim$ *is the lifting of the cochain* φ *to the cochain on* $\mathfrak{C}$ *as described in the previous subsection, then we have* $\varphi^\sim = \pi\rho$ *and* $d_0\varphi^\sim = \pi\gamma$ *for some cochains* ρ *and* γ *on* $\mathfrak{A}$, *and then*

$$\Delta[\varphi] = [\gamma] = [d_0(\varphi^\sim - \pi\rho) \circ h].$$

Proof. Since $[\varphi] \in H^n_{\mathrm{eq}}(\mathfrak{A})$ with $\varphi \in H^n$, we have $d_0\varphi = 0$. The commutative diagram in the theorem above implies that

$$\begin{array}{ccccc} \rho & \xrightarrow{\pi} & \varphi^\sim & \xrightarrow{i} & \varphi \\ \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 \\ \gamma & \xrightarrow{\pi} & d_0\varphi^\sim & \xrightarrow{i} & d_0\varphi \\ \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 \\ 0 & \xrightarrow{\pi} & 0 & \xrightarrow{i} & 0 \end{array}$$

for some ρ and γ. Therefore, we get

$$\begin{aligned} [\gamma] &= d_0[\rho] \\ &= d_0[\varphi^\sim \circ h] = [d_0\varphi^\sim \circ h] \\ &= [d_0(\varphi^\sim - \pi\rho) \circ h] \end{aligned}$$

and hence the conclusion should be $\Delta[\varphi] = [\gamma]$. □

1.7 Example by a diffeomorphism of a compact C^∞-manifold

We refer to [3] and also [1]. Suppose that $\mathfrak{A} = C^\infty(X)$, where X is a compact C^∞-manifold. Then any automorphism α of $\mathfrak{A}$ is induced by a diffeomorphism of X. Give $\mathfrak{A}$ the C^∞-topology of uniform convergence of derivatives, where the (semi)norms p_n on $\mathfrak{A}$ are given by $p_n(f) = \sum_{|\alpha|\le n} \sup_{x\in X} |\partial^\alpha f(x)|$ for $f \in \mathfrak{A}$. Then the assumption in the subsection 1.2 (for $\mathfrak{A}$ to be nuclear) is satisfied. Apply the preceding results to the smooth crossed product $\mathfrak{C} = C^\infty(X) \rtimes_\alpha \mathbb{Z}$.

Denote by $\mathfrak{D}'_n(X)$ the space of the de Rham n-currents on X. Recall that $M_n = \mathfrak{B} \otimes (\otimes^n \mathfrak{A})$ with $\mathfrak{B} = \mathfrak{A} \otimes \mathfrak{A}^{\mathrm{op}}$. In this case $\mathfrak{B} \cong C^\infty(X \times X)$ and $\otimes^n \mathfrak{A} \cong C^\infty(X^n)$. Note that the map from M_n^* to $\mathfrak{D}'_n(X)$ by antisymmetrization A_S induces an isomorphism $A_S : H^n(\mathfrak{A}, \mathfrak{A}^*) \to \mathfrak{D}'_n(X)$. Indeed as in [1, p. 207],

$$\frac{1}{n!} \sum_{\sigma\in S_n} \varepsilon(\sigma)\varphi(f_0, f_{\sigma(1)}, \cdots, f_{\sigma(n)}) = \langle C_\varphi, f_0 df_1 \wedge \cdots \wedge df_n \rangle$$

for $f_j \in \mathfrak{A}$ $(0 \le j \le n)$ with $A_S(\varphi) = C_\varphi$.

The operator $d_0 = IB : H^n(\mathfrak{A}, \mathfrak{A}^*) \to H^{n-1}(\mathfrak{A}, \mathfrak{A}^*)$ is just the standard de Rham boundary operator $d_0 : \mathfrak{D}'_n(X) \to \mathfrak{D}'_{n-1}(X)$ for currents, that is induced by exterior differentiation for differential forms.

The complexes defining $H^*_{\mathrm{eq}}(\mathfrak{A})$ and $H^*_{\mathrm{coep}}(\mathfrak{A})$ become, respectively,

$$0 \to \mathfrak{D}'_N(X)^\alpha \xrightarrow{d_0} \cdots \xrightarrow{d_0} \mathfrak{D}'_1(X)^\alpha \xrightarrow{d_0} \mathfrak{D}'_0(X)^\alpha \to 0,$$
$$0 \to \mathrm{coker}(\delta|_{\mathfrak{D}'_N(X)^\alpha}) \xrightarrow{d_0} \cdots \xrightarrow{d_0} \mathrm{coker}(\delta|_{\mathfrak{D}'_0(X)^\alpha}) \to 0,$$

where $N = \dim X$ and $\delta = \mathrm{id} - \alpha$ acts on n-currents in the sense that $\delta(C_\varphi) = C_{\delta\varphi} = C_{\varphi-\varphi\circ\alpha}$ (or in other one), and $\mathfrak{D}'_n(X)^\alpha = \ker(\delta|_{\mathfrak{D}'_n(X)})$, and $\mathfrak{D}'_n(X) = 0$ for $n \ge N+1$ since $\mathfrak{D}_n(X) = 0$.

Lemma 1.7.1. *The connecting homomorphism* $\Delta : H^n_{\mathrm{eq}}(\mathfrak{A}) \to H^{n-2}_{\mathrm{coeq}}(\mathfrak{A})$ *is zero, with* $\mathfrak{A} = C^\infty(X)$.

Proof. Given a cochain φ on $\mathfrak{A}$ representing a class in $H^n_{\mathrm{eq}}(\mathfrak{A})$, we may suppose that φ is an α-invariant n-current on X. Set

$$\varphi^\sim(u^{m_0}x_0, \cdots, u^{m_n}x_n) = \begin{cases} \varphi(\alpha^{m_0}(x_0), \alpha^{m_0+m_1}(x_1), \cdots, \alpha^{m_0+\cdots+m_n}(x_n)) & m_0 + \cdots + m_n = 0, \\ 0 & m_0 + \cdots + m_n \ne 0. \end{cases}$$

Then $\varphi^{\sim}$ is a cyclic cocycle on $\mathfrak{C}$, and hence

$$\Delta[\varphi] = [(d_0\varphi^{\sim}) \circ h] = 0$$

because $[d_0\varphi^{\sim}] = IB[\varphi^{\sim}] = [0]$ by exactness of cyclic cohomology long exact sequence. Check also that

$$\begin{aligned}
&(\varphi^{\sim})^{\sigma}(u^{m_0}x_0, \cdots, u^{m_n}x_n) \\
&= \varphi^{\sim}(u^{m_{\sigma(0)}}x_{\sigma(0)}, \cdots, u^{m_{\sigma(n)}}x_{\sigma(n)}) \\
&= \varphi(\alpha^{m_{\sigma(0)}}(x_{\sigma(0)}), \cdots, \alpha^{m_{\sigma(0)}+\cdots+m_{\sigma(n)}}(x_{\sigma(n)})) \\
&= \varphi(x_{\sigma(0)}, \cdots, x_{\sigma(n)}) \\
&= \varepsilon(\sigma)\varphi^{\sim}(u^{m_0}x_0, \cdots, u^{m_n}x_n),
\end{aligned}$$

so that $[\varphi] \in HC^n(\mathfrak{C})$. □

Lemma 1.7.2. *It holds that for $\mathfrak{C} = \mathfrak{A} \rtimes_{\alpha} \mathbb{Z}$,*

$$\mathbb{E}_1^n(\mathfrak{C}) \cong H^{n-1}_{\text{coep}}(\mathfrak{A}) \oplus H^n_{\text{eq}}(\mathfrak{A}),$$

where the splitting is given by $\varphi = \varphi_1 + \varphi_2$ with $\varphi_2 = A_S(\varphi|_{\mathfrak{A}})$ and $\varphi_1 = A_S((\varphi - \varphi_2^{\sim}) \circ h)$.

Proof. It follows from the lemma that

$$0 \longrightarrow H^{n-1}_{\text{coeq}}(\mathfrak{A}) \xrightarrow{\ \pi\ } \mathbb{E}_1^n(\mathfrak{C}) \xrightarrow{\ i\ } H^n_{\text{eq}}(\mathfrak{A}) \xrightarrow{\ \Delta\ } 0.$$

Check that $\pi((\varphi - \varphi_2^{\sim}) \circ h) = \varphi - \varphi_2^{\sim}$ and $i(\varphi - \varphi_2^{\sim}) = \varphi_2 - \varphi_2 = 0$, and also $\varphi = (\varphi - \varphi_2^{\sim}) + \varphi_2^{\sim}$. For any $[\psi] \in H^n_{\text{eq}}(\mathfrak{A})$, there is a class $[\varphi] \in \mathbb{E}_1^n(\mathfrak{C})$ such that $i[\varphi] = [\psi]$. Define the splitting morphism c by $c[\psi] = [\varphi - \pi(\varphi_1)]$. Then $i \circ c[\psi] = [\varphi_2] = [\psi]$. □

Before going on, for φ an n-cochain on $\mathfrak{C}$, we set

$$\begin{aligned}
&T\varphi(x_0, x_1, \cdots, x_n) = (-1)^n\varphi(x_n, x_0, \cdots, x_{n-1}), \\
&R\varphi = \frac{1}{n+1}(n + 1 + nT + (n-1)T^2 + \cdots + 2T^{n-1} + T^n)\varphi, \\
&N\varphi = \frac{1}{n+1}(1 + T + \cdots + T^n)\varphi, \\
&b'\varphi(x_0, x_1, \cdots, x_{n+1}) = \sum_{i=0}^{n}(-1)^i\varphi(x_0, \cdots, x_ix_{i+1}, \cdots, x_{n+1}).
\end{aligned}$$

The operators T, R, and N map L_n^* to L_n^* and M_n^* to M_n^*, while b' maps L_n^* to L_{n+1}^* and M_n^* to M_{n+1}^*, respectively, where $L_n = \mathfrak{D} \otimes (\otimes^n \mathfrak{C})$. Define a map $\bar{\pi}$ from M_n^* to $(L_n^*)_{\text{hom}}$ by

$$\bar{\pi}\varphi(x_0, x_1, \cdots, x_n) = \varphi(k(1 \otimes x_0 \otimes \cdots \otimes x_n)), \quad x_j \in \mathfrak{C}$$

with $k((-1)^m \otimes u^{n_1}x_1 \otimes \cdots \otimes u^{n_m}x_m) = \sum_{i=1}^{n_1} u^{n_1+\cdots+n_m}u^i\alpha^{n_1-i}(x_1) \otimes \cdots \otimes \alpha^{n_1+\cdots n_m-i}(x_m)$ if $n_1 > 0$, which is zero if $n_1 = 0$, and is the negative sum $-\sum_{i=n_1+1}^{0}$ if $n_1 < 0$. (Possibly, as before, multiplying by $\delta_{n_1+\cdots+n_m,0}$ with that functional $\bar{\pi}\varphi$ is necessary to have it in $(L_n^*)_{\text{hom}}$.)

Considering π as a map from M_{n-1}^* to $(L_n^*)_{\text{hom}}$, we set for $\varphi \in M_{n-1}^*$,

$$\#^\sim(\varphi) = \pi\varphi - bR\bar{\pi}\varphi - R\bar{\pi}b\varphi \in (L_n^*)_{\text{hom}}.$$

Note that

$$\begin{array}{ccccc} M_{n-1}^* & \xrightarrow{\bar{\pi}} & (L_{n-1}^*)_{\text{hom}} & \xrightarrow{R} & (L_{n-1}^*)_{\text{hom}} \\ {\scriptstyle b}\downarrow & & & & \downarrow{\scriptstyle b} \\ M_n^* & \xrightarrow{\bar{\pi}} & (L_n^*)_{\text{hom}} & \xrightarrow{R} & (L_n^*)_{\text{hom}}. \end{array}$$

Proposition 1.7.3. (I) *$bN\varphi = Nb'\varphi$ and $(1-T)b\varphi = b'(1-T)\varphi$.*

(II) *$(1-T)\varphi = b'(\varphi|_0) + (b\varphi)|_0$ and $(1-T)R = 1-N$.*

(III) *$N\bar{\pi}N\varphi = 0$.*

(IV) *The map $\#^\sim$ maps cyclic cocycles to cyclic cocycles, cyclic coboundaries to cyclic coboundaries, and commutes with b.*

Proof. (I) Check that for instance, for φ an 1-cochain on $\mathfrak{C}$,

$$\begin{aligned} &bN\varphi(x_0, x_1, x_2) = b(\frac{1}{2}(1+T)\varphi)(x_0, x_1, x_2) \\ &= 2^{-1}(1+T)\varphi(b(x_0, x_1, x_2)) \\ &= 2^{-1}(1+T)\varphi(x_0x_1, x_2) - 2^{-1}(1+T)\varphi(x_0, x_1x_2) + 2^{-1}(1+T)\varphi(x_0x_2^\circ, x_1) \\ &= 2^{-1}[(\varphi(x_0x_1, x_2) - \varphi(x_2, x_0x_1)) - (\varphi(x_0, x_1x_2) - \varphi(x_1x_2, x_0)) \\ &+ (\varphi(x_0x_2^\circ, x_1) - \varphi(x_1, x_0x_2^\circ))] \end{aligned}$$

while we have

$$\begin{aligned} &N(b'\varphi)(x_0, x_1, x_2) = \frac{1}{3}(1+T+T^2)(b'\varphi)(x_0, x_1, x_2) \\ &= 3^{-1}[(b'\varphi)(x_0, x_1, x_2) + (b'\varphi)(x_2, x_0, x_1) + (b'\varphi)(x_1, x_2, x_0)] \\ &= 3^{-1}[(\varphi(x_0x_1, x_2) - \varphi(x_0, x_1x_2)) + (\varphi(x_2x_0, x_1) - \varphi(x_2, x_0x_1)) \\ &+ (\varphi(x_1x_2, x_0) - \varphi(x_1, x_2x_0))]. \end{aligned}$$

(Possbily, the scalar multiple in the definition of N need to be changed to have the equality, or the equality should be changed as $(n+1)bN\varphi = (n+2)Nb'\varphi$ for φ an n-cochain, where we need to have $x_0 x_2^\circ = x_2 x_0$.)

Check also that for φ an 1-cochain on $\mathfrak{C}$,

$$\begin{aligned}
&(1-T)b\varphi(x_0,x_1,x_2) = (b\varphi)(x_0,x_1,x_2) - (b\varphi)(x_2,x_0,x_1) \\
&= (\varphi(x_0x_1,x_2) - \varphi(x_0,x_1x_2) + \varphi(x_0x_2^\circ,x_1)) \\
&- (\varphi(x_2x_0,x_1) - \varphi(x_2,x_0x_1) + \varphi(x_2x_1^\circ,x_0)),
\end{aligned}$$

while we have

$$\begin{aligned}
&b'(1-T)\varphi(x_0,x_1,x_2) = b'\varphi(x_0,x_1,x_2) - b'(T\varphi)(x_0,x_1,x_2) \\
&= (\varphi(x_0x_1,x_2) - \varphi(x_0,x_1x_2)) \\
&- [(T\varphi)(x_0x_1,x_2) - (T\varphi)(x_0,x_1x_2)] \\
&= \varphi(x_0x_1,x_2) - \varphi(x_0,x_1x_2) + \varphi(x_2,x_0x_1) - \varphi(x_1x_2,x_0).
\end{aligned}$$

Hence $(1-T)b\varphi(x_0,x_1,x_2) - b'(1-T)\varphi(x_0,x_1,x_2)$ should be

$$\varphi(x_0x_2^\circ,x_1) - \varphi(x_2x_0,x_1) - \varphi(x_2x_1^\circ,x_0) + \varphi(x_1x_2,x_0),$$

which can be zero if $x_0x_2^\circ = x_2x_0$ and $x_2x_1^\circ = x_1x_2$.)

(II) Compute that $(1-T)\varphi(x_0,x_1,\cdots,x_n) =$

$$\varphi(x_0,x_1,\cdots,x_n) - (-1)^n\varphi(x_n,x_0,\cdots,x_{n-1}), \quad \text{and}$$

$$\begin{aligned}
&b'(\varphi|_0)(x_0,x_1,\cdots,x_n) + (b\varphi)|_0(x_0,\cdots,x_n) \\
&= \sum_{i=0}^{n-1}(-1)^i(\varphi|_0)(x_0,\cdots,x_ix_{i+1},\cdots,x_n) + (b\varphi)(1,x_0,\cdots,x_n) \\
&= \sum_{i=0}^{n-1}(-1)^i\varphi(1,x_0,\cdots,x_ix_{i+1},\cdots,x_n) + \varphi(1x_0,\cdots,x_n) \\
&+ \sum_{i=1}^{n}(-1)^i\varphi(1,x_0,\cdots,x_{i-1}x_i,\cdots x_n) + (-1)^{n+1}\varphi(x_n^\circ,x_0,\cdots,x_{n-1}) \\
&= \varphi(x_0,\cdots,x_n) + (-1)^{n+1}\varphi(x_n^\circ,x_0,\cdots,x_{n-1}),
\end{aligned}$$

so that we need to have $x_n = x_n^\circ$ to have the identity.

Complute also that $(1-T)R\varphi(x_0, x_1, \cdots x_n) = (R\varphi - TR\varphi)(x_0, x_1, \cdots, x_n) =$

$$\frac{1}{n+1}(n+1+\sum_{j=1}^{n}(n+1-j)T^j)\varphi(x_0, x_1, \cdots, x_n)$$
$$- (-1)^n(R\varphi)(x_n, x_0, \cdots, x_{n-1})$$
$$= \varphi(x_0, \cdots, x_n) + \sum_{j=1}^{n}(1 - \frac{j}{n+1})(-1)^{nj}\varphi(x_{n-j+1}, \cdots, x_n, x_0, \cdots, x_{n-j})$$
$$- (-1)^n\varphi(x_n, x_0 \cdots, x_{n-1})$$
$$- \sum_{j=1}^{n}(1 - \frac{j}{n+1})(-1)^{n(j+1)}\varphi(x_{n-j}, \cdots, x_n, x_0, \cdots, x_{n-j-1})$$
$$= \varphi(x_0, \cdots, x_n) - \frac{(-1)^{n+1}}{n+1}\varphi(x_n, x_0 \cdots, x_{n-1})$$
$$- \sum_{j=2}^{n}(\frac{(-1)^{nj}}{n+1})\varphi(x_{n-j+1}, \cdots, x_n, x_0, \cdots, x_{n-j}) - \frac{(-1)^{n(n+1)}}{n+1}\varphi(x_0, \cdots, x_n),$$

which can be $= (1 - N)\varphi(x_0, x_1, \cdots, x_n)$ if we define N to have the last identity.

(III) First note the following identity:

$$\sum_{i=1}^{n_1}\varphi \circ \alpha^{-i} + \sum_{i=1}^{m}\varphi \circ \alpha^{-i-n} = \sum_{i=1}^{n+m}\varphi \circ \alpha^{-i}.$$

Suppose that φ is a cyclic n-cochain on $\mathfrak{A}$, so that $\varphi = N\psi$ for some ψ, and set $x_i = u^{m_i}a_i$ for $a_i \in \mathfrak{A}$ $(i = 1, \cdots, n)$ and $a_i^{\sim} = \alpha^{m_0+m_1+\cdots+m_i}(x_i)$ (corrected), where $m_0 + m_1 + \cdots + m_n = 0$. We have

$$T^{-i}\bar{\pi}\varphi(x_0, x_1, \cdots, x_n)$$
$$= (-1)^{ni}\varphi(k(1, x_i, x_{i+1}, \cdots, x_n, x_0, x_1, \cdots, x_{i-1}))$$
$$= (-1)^{ni}\sum_{j=1}^{m_i}\varphi(u^j\alpha^{-j}(\alpha^{m_i}(x_i), \alpha^{m_i+m_{i+1}}(x_{i+1}), \cdots, \alpha^{m_i+m_{i+1}+\cdots+m_{i-1}}(x_{i-1})))$$
$$= (-1)^{ni}\sum_{j=1}^{m_i}\varphi(u^j\alpha^{-j-m_0-m_1-\cdots-m_{i-1}}(a_i^{\sim}, a_{i+1}^{\sim}, \cdots, a_n^{\sim}, a_1^{\sim}, \cdots, a_{i-1}^{\sim}))$$
$$= (-1)^{ni}\sum_{j=1}^{m_i}(-1)^{i(n-i+1)}\varphi(u^j\alpha^{-j-m_0-m_1-\cdots-m_{i-1}}(a_0^{\sim}, a_1^{\sim}, \cdots, a_n^{\sim}))$$
$$= \sum_{j=1}^{m_i}\varphi(u^j\alpha^{-j-m_0-m_1-\cdots-m_{i-1}}(a_0^{\sim}, a_1^{\sim}, \cdots, a_n^{\sim})).$$

But then

$$
\begin{aligned}
&(n+1)N\bar{\pi}\varphi(x_0, x_1, \cdots, x_n) \\
&= \sum_{i=0}^{n}(T^{-i}\bar{\pi}\varphi)(x_0, x_1, \cdots, x_n) \\
&= \left(\sum_{i=0}^{n}\sum_{l=1}^{m_i}\varphi\circ\alpha^{-l-m_0-\cdots-m_{i-1}}\right)(a_0^{\sim}, a_1^{\sim}, \cdots, a_n^{\sim}),
\end{aligned}
$$

where u^j should probably be dropped in the definition of k, and the double sum gives zero by the first equation and by $m_0 + \cdots + m_n = 0$. Indeed, note that

$$
\begin{aligned}
&\sum_{i=0}^{n}\sum_{l=1}^{m_i}\varphi\circ\alpha^{-l-m_0-\cdots-m_{i-1}} = \sum_{l=1}^{m_0}\varphi\circ\alpha^{-l} + \sum_{l=1}^{m_1}\varphi\circ\alpha^{-l-m_0} \\
&+\sum_{l=1}^{m_2}\varphi\circ\alpha^{-l-m_0-m_1} + \cdots + \sum_{l=1}^{m_n}\varphi\circ\alpha^{-l-m_0-\cdots-m_{n-1}} \\
&= \sum_{l=1}^{m_0+m_1}\varphi\circ\alpha^{-l} + \sum_{l=1}^{m_2}\varphi\circ\alpha^{-l-m_0-m_1} + \cdots + \sum_{l=1}^{m_n}\varphi\circ\alpha^{-l-m_0-\cdots-m_{n-1}} \\
&= \cdots = \sum_{l=1}^{m_0+m_1+\cdots+m_n}\varphi\circ\alpha^{-l}
\end{aligned}
$$

which is nothing but zero (sum).

(IV) Note that $\pi b\varphi = b\pi\varphi$ and, moreover, $\bar{\pi}\varphi = (\pi\varphi)|_0$ since π is given by the compositon with k, so that $(\pi\varphi)|_0(x_0, x_1, \cdots, x_n) = \pi\varphi(1, x_0, \cdots, x_n) = \varphi\circ k(1, x_0, \cdots, x_n) = \bar{\pi}\varphi(x_0, x_1, \cdots, x_n)$. Hence (II) gives an identity as a consequence:

$$
\begin{aligned}
(1-T)\pi\varphi &= b'((\pi\varphi)|_0) + (b\pi\varphi)|_0 \\
&= b'((\pi\varphi)|_0) + (\pi b\varphi)|_0 = b'\bar{\pi}\varphi + \bar{\pi}b\varphi.
\end{aligned}
$$

Check also the identities as a consequence, using the above identities and (I), (II), and (III):

$$
\begin{aligned}
(1-T)\#^{\sim}\varphi &= (1-T)(\pi\varphi - bR\bar{\pi}\varphi - R\bar{\pi}b\varphi) \\
&= b'\bar{\pi}\varphi + \bar{\pi}b\varphi - b'(1-T)R\bar{\pi}\varphi - (1-N)\bar{\pi}b\varphi \\
&= b'\bar{\pi}\varphi - b'(1-N)\bar{\pi}\varphi + N\bar{\pi}b\varphi \\
&= b'N\bar{\pi}\varphi + N\bar{\pi}b\varphi, \quad \text{and}
\end{aligned}
$$

$$\begin{aligned}
\#^{\sim} bN\varphi &= \pi bN\varphi - bR\bar{\pi}bN\varphi - R\bar{\pi}b^2N\varphi = \pi bN\varphi \quad \text{while} \\
bN\#^{\sim} N\varphi &= bN(\pi N\varphi - bR\bar{\pi}N\varphi - R\bar{\pi}bN\varphi) \\
&= b\pi N\varphi - bNbR\bar{\pi}N\varphi - bRN\bar{\pi}Nb'\varphi = \pi bN\varphi
\end{aligned}$$

where note that $NR = RN$ by definition and $b(\cdot)b = 0$, and we have

$$\begin{aligned}
\#^{\sim} b\varphi &= \pi b\varphi - bR\bar{\pi}b\varphi - R\bar{\pi}b^2\varphi = \pi b\varphi, \quad \text{while} \\
b\#^{\sim}\varphi &= b(\pi\varphi - bR\bar{\pi}\varphi - R\bar{\pi}b\varphi) = \pi b\varphi.
\end{aligned}$$

Those equations show that the map $\#^{\sim}$ has the desired properties. Note that the first equation implies that

$$(1-T)\#^{\sim}N\varphi = b'N\bar{\pi}N\varphi + N\bar{\pi}Nb'\varphi = 0,$$

so that we get $\#^{\sim}N\varphi = T\#^{\sim}N\varphi$, which implies that $\#^{\sim}N\varphi$ is cyclic, i.e. $N\#^{\sim}N\varphi = \#^{\sim}N\varphi$. □

Theorem 1.7.4. *When $\mathfrak{A} = C^\infty(X)$ and $\mathfrak{C} = C^\infty(X) \rtimes_\alpha \mathbb{Z}$, we have*

$$\mathrm{E}_\infty^n(\mathfrak{C}) \cong H^{n-1}_{\mathrm{coeq}}(\mathfrak{A}) \oplus H^n_{\mathrm{eq}}(\mathfrak{A}).$$

Proof. Using the splitting of $\mathrm{E}_1^n(\mathfrak{C})$ as shown in the lemma above:

$$\mathrm{E}_1^n(\mathfrak{C}) \cong H^{n-1}_{\mathrm{coeq}}(\mathfrak{A}) \oplus H^n_{\mathrm{ep}}(\mathfrak{A}), \quad \varphi = \varphi_1 + \varphi_2,$$

the fact that $\#^{\sim}$ is homotopic to π, and the equality $(\pi\varphi)\circ h = \varphi$, we can write $\varphi = \#^{\sim}\varphi_1 + \varphi_2^{\sim}$ in $\mathrm{E}_1^n(\mathfrak{C})$ for any cocycle φ representing an element of $\mathrm{E}_1^n(\mathfrak{C})$. Since both φ_1 and φ_2 are cyclic, so are $\#^{\sim}\varphi_1$ ((IV) in the proposition above) and $\varphi_2^{\sim}$. This means that every element of $\mathrm{E}_1^n(\mathfrak{C})$ can be represented by a cyclic cocycle. Since all the boundary operators $d_1, d_2, \cdots$ kill cyclic cocycles, we get the isomorphisms:

$$\mathrm{E}_1^n(\mathfrak{C}) \cong \mathrm{E}_2^n(\mathfrak{C}) \cong \cdots \cong \mathrm{E}_\infty^n(\mathfrak{C}).$$

□

Remark. Review quickly from a text book [2] of Hattori that for an exact couple of modules:

$$\begin{array}{ccc}
A & \xrightarrow{\ f\ } & A \\
{\scriptstyle h}\uparrow & & \downarrow{\scriptstyle g} \\
E & = & E,
\end{array}$$

set $d = g \circ h : E \to E$. Then $d^2 = 0$ since $h \circ g = 0$. Then the homology group $H(E)$ with respect to (E, d) is defined by $H(E) = Z(E)/B(E)$, where $Z(E) = \ker(d)$ and $B(E) = \mathrm{Im}(d)$. Note that $x \in Z(E) \Leftrightarrow d(x) = g(h(x)) = 0 \Leftrightarrow h(x) \in \ker(g) = \mathrm{Im}(f) \Leftrightarrow x \in h^{-1}(\mathrm{Im}(f))$, and $y = d(x) = g(h(x)) \in B(E) \Leftrightarrow y \in g(\mathrm{Im}(h)) = g(\ker(f))$. The derived (exact) couple is then defined by

$$\begin{array}{ccc} A^1 = \mathrm{Im}(f) & \xrightarrow{f^1 = f} & A^1 = \mathrm{Im}(f) \\ {\scriptstyle h^1}\uparrow & & \downarrow{\scriptstyle g^1} \\ E^1 = H(E) & = & E^1 = H(E), \end{array}$$

where check by definition that $f(x) \in \ker(f^1) \Leftrightarrow f(f(x)) = 0 \Leftrightarrow f(x) \in \mathrm{Im}(h) \Leftrightarrow f(x) = h(z) = h^1[z]$ with $[z] \in E^1$, and $g^1(f(x)) = [g(f^{-1}(f(x))] = [g(x - f(y))] = 0$ for some $y \in A \Leftrightarrow g(x - f(y)) \in \mathrm{Im}(d) = g(h(E)) \Leftrightarrow x - f(y) \in h(E) = \ker(f) \Leftrightarrow f(x) = f(f(y))$, and $h^1[z] = h(z) = 0 \Leftrightarrow z \in \ker(h) = \mathrm{Im}(g) \Leftrightarrow z = g(x)$ for some $x \in A \Leftrightarrow [z] = [g(x)] = [g(f^{-1}(f(x))]$. The homology spectral sequence E^n with $E^0 = E$ is then defined by deriving inductively as $E^n = H(E^{n-1})$ with respect to $(E^{n-1}, d^{(n-1)} = g^{(n-1)} \circ h^{(n-1)})$, with $g^{(1)} = g^1$ and $h^{(1)} = h^1$. Furthermore,

$$E^\infty = Z^\infty / B^\infty = (\cap_n Z^{(n)})/(\cup_n B^{(n)}),$$

where $E^r = Z^{(r)}/B^{(r)}$, with $Z^{(r)} = h^{-1}(\mathrm{Im}(f^r))$ and $B^{(r)} = g(\ker(f^r))$.

If $d^r = 0$ for every $r \geq n$ (≥ 2), then the spectral sequence (E^r, d^r) is said to be collapsed, and then $E^n \cong E^{n+1} \cong \cdots \cong E^\infty$.

In the case of the theorem above,

$$\begin{array}{ccc} \mathrm{Im}(S) & \xrightarrow{S^1} & \mathrm{Im}(S) \\ {\scriptstyle B^1}\uparrow & & \downarrow{\scriptstyle I^1} \\ E_1 = H(H(\mathfrak{C}, \mathfrak{C}^*)) & = & E_1 = H(H(\mathfrak{C}, \mathfrak{C}^*)), \end{array}$$

with $d_1 = d^1 = I^1 \circ B^1$, and

$$\begin{array}{ccc} \mathrm{Im}(S^2) & \xrightarrow{S^{(2)}} & \mathrm{Im}(S^2) \\ {\scriptstyle B^{(2)}}\uparrow & & \downarrow{\scriptstyle I^{(2)}} \\ E_2 = H(H(H)) & = & E_2 = H(H(H)), \end{array}$$

with $d_2 = d^{(2)} = I^{(2)} \circ B^{(2)}$, and so on. Note also that

$$\begin{array}{ccccccc} \xrightarrow{d_j} & E_j^n & \xrightarrow{d_j} & E_j^{n-1} & \xrightarrow{d_j} \\ & \| & & \| & \\ \xrightarrow{d_{j+1}} & E_{j+1}^n & \xrightarrow{d_{j+1}} & E_{j+1}^{n-1} & \xrightarrow{d_{j+1}} \end{array}$$

provided that $d_j = 0$ since $E_{j+1}^n = \ker(d_j|_{E_j^n})/d_j(E_j^{n+1})$.

Corollary 1.7.5. *Suppose that $X = \mathbb{T}$ and α preserves orientation. Then*

$$\dim HC^{\mathrm{ev}}(\mathfrak{C}) = \dim HC^{\mathrm{odd}}(\mathfrak{C}) = 2,$$

where $\mathfrak{C} = \mathfrak{A} \rtimes_\alpha \mathbb{Z}$ with $\mathfrak{A} = C^\infty(X)$.

Proof. Denote by $H_n(X)$ the homology groups of a compact C^∞-manifold X computed as

$$H_n(X) = \frac{\ker(d_0|_{\mathfrak{D}'_n(X)})}{\mathrm{Im}(d_0|_{\mathfrak{D}'_{n+1}(X)})}.$$

Since the groups $H_n(X)$, $H^n_{\mathrm{eq}}(\mathfrak{A})$, $H^n_{\mathrm{coeq}}(\mathfrak{A})$ are all computed by complexes of n-currents on X, it is easy to see that the identity map $\varphi \mapsto \varphi$ on the cochain level in cohomology descends to the maps

$$\alpha_n : H^n_{\mathrm{eq}}(\mathfrak{A}) \to H_n(X)^\alpha \quad \text{and} \quad \beta_n : H_n(X)/\delta H_n(X) \to H^n_{\mathrm{coep}}(\mathfrak{A}).$$

Note that $\mathfrak{D}'_n(X)^\alpha = \ker(\delta|_{\mathfrak{D}'_n(X)})$, which corresponds to $\ker(\delta|_{H^n(\mathfrak{A},\mathfrak{A}^*)})$, as checked before, so that if $[\varphi] \in H^n_{\mathrm{eq}}(\mathfrak{A}) \cong \ker(d_0|_{\mathfrak{D}'_n(X)^\alpha})/d_0(\mathfrak{D}'_{n+1}(X)^\alpha)$, then $\delta(\varphi) = 0$, which implies that $\varphi - \varphi \circ \alpha = 0$, so that $[\varphi] = [\varphi \circ \alpha]$ in $H_n(X)$, which means that $[\varphi] \in H_n(X)^\alpha$. Note also that $H^n_{\mathrm{coeq}}(\mathfrak{A})$ is identifed with the homology of the complex $(\mathrm{coker}(\delta|_{\mathfrak{D}'_n(X)}), d_0)$ and that

$$\mathrm{coker}(\delta|_{\mathfrak{D}'_n(X)}) = \frac{\mathfrak{D}'_n(X)}{\ker(\delta|_{\mathfrak{D}'_n(X)})} = \frac{\mathfrak{D}'_n(X)}{\mathfrak{D}'_n(X)^\alpha},$$

and there is a quotient map from $\mathfrak{D}'_n(X)$ to $\mathfrak{D}'_n(X)/\mathfrak{D}'_n(X)^\alpha$, which induces a map from $H_n(X)$ to $H^n_{\mathrm{coeq}}(\mathfrak{A})$, and if $[\varphi] \in \delta H_n(X)$, then the class $\delta[\varphi] = [\delta\varphi]$ is mapped to zero, and hence the map from $H_n(X)/\delta H_n(X)$ to $H^n_{\mathrm{coeq}}(\mathfrak{A})$ is deduced.

Define the following maps:

$$\mathrm{coker}(\alpha_{n-1}) \xrightarrow{s_1} \mathrm{coker}(\beta_n) \xrightarrow{s_2} \ker(\alpha_{n-2}) \xrightarrow{s_3} \ker(\beta_{n-1}),$$

where the maps s_1, s_2, and s_3 are given as follows.

(1) Starting with $[\varphi] \in H_{n-1}(X)^\alpha$, we have $\delta\varphi = d_0\omega$ for some ω with $[\omega] \in H_n(X)^\alpha$ since $d_0\varphi = 0$, $d_0(\varphi \circ \alpha) = 0$ and thus $d_0(\delta\varphi) = 0$, because $[\varphi] = [\varphi \circ \alpha]$, and we set $s_1(\varphi) =$ the class of ω in $\mathrm{coker}(\beta_n)$. Note that

$$\mathrm{coker}(\alpha_{n-1}) = H^{n-1}_{\mathrm{eq}}(\mathfrak{A})/\ker(\alpha_{n-1}) \cong H_{n-1}(X)^\alpha,$$

and also $\mathrm{coker}(\beta_n) = (H_n(X)/\delta H_n(X))/\ker(\beta_n) \cong H^n_{\mathrm{coeq}}(\mathfrak{A})$ and $\delta[\omega] = [\delta(\omega)] = [\omega - \omega \circ \alpha] = 0$.

(2) Given $[\varphi] \in H^n_{\mathrm{coep}}(\mathfrak{A})$, then $d_0\varphi = \delta\omega$ for some $\omega \in H^{n-1}$, because $d_0\varphi \in H^{n-1}_{\mathrm{coeq}}(\mathfrak{A}) = \ker(d_0|_{\mathrm{coker}(\delta_{H^{n-1}})})/d_0(\mathrm{coker}(\delta|_{H^n}))$ with $\mathrm{coker}(\delta_{H^{n-1}}) = H^{n-1}/\ker(\delta) \cong \delta(H^{n-1})$. Then $\delta(d_0\omega) = d_0\omega - (d_0\omega \circ \alpha) = d_0(\delta\omega) = d_0(d_0\varphi) = 0$. Hence $d_0\omega \in \ker(\delta|_{H^{n-2}})$. This says that $[d_0\omega] \in H^{n-2}_{\mathrm{eq}}(\mathfrak{A})$ and we set $s_2(\varphi) =$ the class of $d_0\omega$ in $\ker(\alpha_{n-2})$ since $0 = \delta[d_0\omega] = [d_0\omega] - [d_0\omega \circ \alpha]$ with $[d_0\omega] = 0$ in $H_{n-2}(X)$.

(3) Given $[\varphi] \in \ker(\alpha_{n-2}) \subset H^{n-2}_{\mathrm{eq}}(\mathfrak{A}) = \ker(d_0|_{\ker(\delta|_{H^{n-2}})})/d_0(\ker(\delta|_{H^{n-1}}))$, then $d_0\varphi = \delta\varphi = 0$ and $\varphi = d_0\omega$ for some $n-1$ current ω. Then $\delta\omega$ gives an $n-1$ current on X and moreover $d_0\delta\omega = \delta d_0\omega = \delta\varphi = 0$, and hence $[\delta\omega] \in H_{n-1}(X)$ and $\in \delta H_{n-1}(X)$, which is mapped to zero under β_{n-2}. We set $s_3(\varphi) =$ the class of $\delta\omega$ in $\ker(\beta_{n-1})$.

The above short sequence obtained in that way is exact and furthermore, s_1 is injective and s_3 is surjective. In fact, the injectiveness of s_1: if $s_1([\varphi]) = [\omega] = 0$, then $w = d_0(\rho)$ for some ρ, and hence $\delta\varphi = d_0(d_0\rho) = 0$, and thus $\varphi \in \ker(\delta)$ and then $[\varphi] \in H^{n-1}_{\mathrm{eq}}(\mathfrak{A})$. Also, if $\ker(\beta_n) = \delta H_n(X)$, then $[\omega] = [\delta\rho]$ for some ρ, and thus $[\delta\varphi] = [d_0\omega] = [\delta d_0\rho]$, which may imply $[\varphi] = [d_0\rho]$ if δ is injective at the class, and thus $[\varphi] = 0 \in \mathrm{coker}(\alpha_{n-1})$. The surjectiveness of s_3 follows if $\ker(\beta_{n-1}) = \delta H_{n-1}(X)$. The exactness at $\mathrm{coker}(\beta_n)$: if $s_2([\varphi]) = [d_0\omega] = 0$ for some $n-1$ current ω, then $d_0\varphi = \delta\omega$ and hence $[\varphi] = s_1([\omega])$. The exactness at $\ker(\alpha_{n-2})$: if $s_3([\varphi]) = [\delta\omega] = 0$ with $\varphi = d_0\omega$, then $d_0(\delta\omega) = 0$ so that $\delta\omega = d_0(\psi)$ for some ψ, and thus $s_2([\psi]) = [d_0\omega] = [\varphi]$.

When $X = \mathbb{T}$ we get the following:

(4) As $n = 0$, $\beta_0 : H_0(\mathbb{T})/\delta H_0(\mathbb{T}) \to H^0_{\mathrm{coeq}}(\mathfrak{A})$ is surjective, i.e., $H^0_{\mathrm{coeq}}(\mathfrak{A}) \cong H_0(\mathbb{T}) = \mathbb{C}$. Also, $\alpha_0 : H^0_{\mathrm{eq}}(\mathfrak{A}) \to H_0(\mathbb{T})$ is surjective since $H_0(\mathbb{T}) = \mathbb{C}$ and the generator can be represented by any α-invariant measure on the circle.

(5) As $n = 1$, $\beta_1 : H_1(\mathbb{T})/\delta H_1(\mathbb{T}) \to H^1_{\mathrm{coeq}}(\mathfrak{A})$ is surjective. Also $\alpha_1 : H^1_{\mathrm{eq}}(\mathfrak{A}) \to H_1(\mathbb{T})$ is surjective, in fact α preserves orientation of $\mathbb{T}$.

It follows that

$$0 \to \mathrm{coker}(\alpha_1) \xrightarrow{s_1} \mathrm{coker}(\beta_2) = 0 \xrightarrow{s_2} \ker(\alpha_0) \xrightarrow{s_3} \ker(\beta_1) \to 0$$

and

$$\begin{array}{ccccc} 0 & \longrightarrow & \operatorname{coker}(\alpha_0) & \xrightarrow{s_1} & \operatorname{coker}(\beta_1) \\ & & \| & & \| \\ & & H_0(\mathbb{T}) & & H^1_{\text{coeq}}(\mathfrak{A}). \end{array}$$

Let μ denote an α-invariant probability measure on the unit circle and let τ be the fundamental class of $\mathbb{T}$: $\tau(f,g) = \int f dg$. We can write $H_0(\mathbb{T}) = H^0_{\text{coeq}}(\mathfrak{A}) = \mathbb{C}\mu$ and $H_1(\mathbb{T}) = H^1_{\text{eq}}(\mathfrak{A}) = \mathbb{C}\tau$. There are two possibilities as follows.

If τ is non-zero in $H^1_{\text{coeq}}(\mathfrak{A})$, then $H^0_{\text{eq}}(\mathfrak{A}) = H_0(\mathbb{T}) = \mathbb{C}\mu$, because $\ker(\beta_1) = 0$, so that $\ker(\alpha_0) = 0$. This gives that $\dim \mathbb{E}^0_\infty(\mathfrak{C}) = \dim H^0_{\text{eq}}(\mathfrak{A}) = 1$ and $\dim \mathbb{E}^2_\infty(\mathfrak{C}) = \dim H^1_{\text{coep}}(\mathfrak{A}) = 1$.

If $\tau = \delta\omega$ for a 1-current ω on $\mathbb{T}$, then $H^1_{\text{coeq}}(\mathfrak{A}) = 0$, $H^0_{\text{eq}}(\mathfrak{A}) = \mathbb{C}\mu \oplus \mathbb{C}d_0\omega$, and $\dim \mathbb{E}^0_\infty(\mathfrak{C}) = \dim H^0_{\text{eq}}(\mathfrak{A}) = 2$ and $\dim \mathbb{E}^2_\infty(\mathfrak{C}) = \dim H^1_{\text{coeq}}(\mathfrak{A}) = 0$.

In both cases,

$$\dim \mathbb{E}^1_\infty(\mathfrak{C}) = \dim H^0_{\text{coeq}}(\mathfrak{A}) + \dim H^1_{\text{eq}}(\mathfrak{A}) = 2.$$

Since $\mathbb{E}^n_\infty(\mathfrak{A})$ are the graded groups of the filtration of $HC(\mathfrak{C})$ by dimension, the result now follows from the last theorem in the subsection 1.6 and the fact that $H^n_{\text{eq}}(\mathbb{T}) = 0$ and $H^n_{\text{coeq}}(\mathbb{T}) = 0$ for $n \geq 2$. □

Remark. A filtration of a chain complex C_* is a sequence $\{F_nC_*\}_{n\in\mathbb{Z}}$ of subcomlexes of C_* such that $F_nC_* \subset F_{n+1}C_*$ for $n \in \mathbb{Z}$, $\cup_n F_nC_* = C_*$, and $\cap_n F_nC_* = \{0\}$ (see [2]). According to [1], for the double complex $(C^{n,m} = C^{n-m}(\mathfrak{A}, \mathfrak{A}^*), (b, B))$, the $\mathbb{E}_2$ term of the spectral sequence associated to the first filtration $F_pC = \sum_{n\geq p} C^{n,m}$ is zero, and the second filtration with $F^q = \sum_{m\geq q} C^{n,m}$ yields the same filtration of $H^*(\mathfrak{A})$ as the filtration by dimensions of cycles, and that $H^p(F^qC) = HC^n(\mathfrak{A})$ for $n = p - 2q$, and $H^n(C^{*,*}) = H^{\text{ev}}(\mathfrak{A})$ if n is even and $= H^{\text{odd}}(\mathfrak{A})$ if n is odd, and the associated spectral sequence converges to the associated graded $\sum F^qH^*(\mathfrak{A})/F^{q+1}H^*(\mathfrak{A})$.

1.8 Cyclic cohomology of the smooth crossed product: Computation outline

As the first step, we consider the following diagram: for $\mathfrak{C} = \mathfrak{A} \rtimes_\alpha \mathbb{Z}$,

$$\begin{array}{ccccccc}
\xrightarrow{\delta} H^{n-1}(\mathfrak{A},\mathfrak{A}^*) & \xrightarrow{\pi} & H^n(\mathfrak{C},\mathfrak{C}^*)_{\text{hom}} & \xrightarrow{i} & H^n(\mathfrak{A},\mathfrak{A}^*) \xrightarrow{\delta} \\
\uparrow I & & \uparrow I & & \uparrow I \\
\xrightarrow{\delta} H^{n-1}_\lambda(\mathfrak{A}) & \xrightarrow{\#_\alpha} & H^n_\lambda(\mathfrak{C})_{\text{hom}} & \xrightarrow{i} & H^n_\lambda(\mathfrak{A}) \xrightarrow{\delta} \\
\uparrow S & & \uparrow S & & \uparrow S \\
\xrightarrow{\delta} H^{n-3}_\lambda(\mathfrak{A}) & \xrightarrow{\#_\alpha} & H^{n-2}_\lambda(\mathfrak{C})_{\text{hom}} & \xrightarrow{i} & H^{n-2}_\lambda(\mathfrak{A}) \xrightarrow{\delta}
\end{array}$$

where the map $\#_\alpha$ will be constructed so as to make the above diagram commutative and then be proved that the middle row is exact.

The main ingredient of the proof is a cochain map $\eta : C^{n+1}(\mathfrak{C},\mathfrak{C}^*)_{\text{hom}} \to C^n(\mathfrak{C},\mathfrak{C}^*)$, which satisfies $b\eta = \eta b + \delta$ and $N\eta N = \eta N$. Such a map η induces two maps as follows by passing to quotients and restriction to $\mathfrak{A}$: from the quotient of cocycles vanishing on $\mathfrak{A}$ by coboundaries to $H(\mathfrak{A},\mathfrak{A}^*)$, and from the quotient of cyclic cocycles vanishing on $\mathfrak{A}$ by cyclic coboundaries to $H_\lambda(\mathfrak{A})$. Both of these maps are defined on $\ker(i)$ in cohomology:

$$H^n(\mathfrak{C},\mathfrak{C}^*)_{\text{hom}} \supset \ker(i) \xrightarrow{\eta} H^{n-1}(\mathfrak{A},\mathfrak{A}^*),$$
$$H^n_\lambda(\mathfrak{C})_{\text{hom}} \supset \ker(i) \xrightarrow{\eta} H^{n-1}_\lambda(\mathfrak{A}).$$

Next it will be shown that $\eta\pi = \text{id}$ in Hochshild cohomology. This, together with the commutativity of the diagram above allow to conclude that $\ker(i) \subset \text{Im}(\#_\alpha)$ in $H_\lambda(\mathfrak{C})_{\text{hom}}$. It will be shown that the sequence is exact:

$$\cdots \xrightarrow{\delta} H^{n-1}_\lambda(\mathfrak{A}) \xrightarrow{\#_\alpha} H^n_\lambda(\mathfrak{C})_{\text{hom}} \xrightarrow{i} H^n_\lambda(\mathfrak{A}) \xrightarrow{\delta} \cdots .$$

As the final step, using this we construct a six-term exact sequence of the periodic cyclic cohomology of $\mathfrak{C}$.

1.9 Construction of a map in cyclic cohomology

Start with a differential graded algebra (E,d) defined as $E = E_0 \oplus E_1$, where

$$E_0 = \{\sum_n a_n u^n : n \in \mathbb{Z}, a_n \in \mathbb{C}\} \quad \text{and}$$
$$E_1 = \{\sum_{n,m} a_{n,m} u^n (du) u^m : n,m \in \mathbb{Z}, a_{n,m} \in \mathbb{C}\},$$

where the sums are finite sums, and the product structure is given by $u^0 = 1$, $u^i u^j = u^{i+j}$, $u^i(u^j(du)u^k)u^l = u^{i+j}(du)u^{k+l}$, and $(du)u^i(du) = 0$, and where the graded differential d is defined by $du^n = \sum_{i=1}^n u^i(u^{-1}du)u^{n-i}$ if n positive, and if n is negative, the sum is changed to $-\sum_{i=n+1}^0$, and $d1 = 0$, and $d|_{E_1} = 0$.

The fact that (E, d) becomes a differential graded algebra with the above definitions follows from its representation as a quotient of the universal differential graded algebra $\Omega(\mathbb{C}[u, u^{-1}])$ by the graded ideal generated by $\oplus_{i\geq 2}\Omega_i$ together with $d1 \in \Omega_1$ and $1 \in \Omega_0$ the unit of $\mathbb{C}[u, u^{-1}]$. Recall that the algebra $\Omega(\mathbb{C}[u, u^{-1}])$ is generated by finite linear combinations of symbols: $g_0dg_1dg_2\cdots dg_n \in \Omega_n$ with $g_i = u^{k_i}$ for some $k_i \in \mathbb{Z}$, and has product and differential given by

$$
\begin{aligned}
&(g_0dg_1\cdots dg_n)(g_{n+1}dg_{n+2}\cdots dg_m)\\
&= \sum_{j=1}^{n}(-1)^{n-j}g_0dg_1\cdots d(g_jg_{j+1})\cdots dg_ndg_{n+1}\cdots dg_m\\
&\text{and}\quad d(g_0dg_1\cdots dg_n) = dg_0dg_1\cdots dg_n.
\end{aligned}
$$

Now suppose that we are given a cycle $(\Omega, d^\sim, \varphi)$ (such as $\varphi(g_0dg_1\cdots dg_n) = 0$ unless $n = k$ and $g_0g_1\cdots g_n = 1$ and $\varphi(g_0dg_1\cdots dg_k) = c(g_1, \cdots, g_k)$ for some k-dimensional cycle c) and an action α of $\mathbb{Z}$ on Ω, i.e., an automorphism of Ω commuting with $d^\sim$. Define a crossed product cycle $(E \otimes_\alpha \Omega, d, \#_\alpha\varphi)$ as follows.

(1) $E \otimes_\alpha \Omega = E \otimes \Omega$ as an algebraic tensor product of graded vector spaces;

(2) the product structure is induced by the relations: $(1 \otimes \omega)(u \otimes 1) = u \otimes \alpha^{-1}(\omega)$ and $(1 \otimes w)(du \otimes 1) = (-1)^{\deg\omega}du \otimes \alpha^{-1}(\omega)$;

(3) the differential d is given by $d(\omega_1 \otimes \omega_2) d\omega_1 \otimes \omega_2 + (-1)^{\deg\omega}\omega_1 \otimes d^\sim\omega_2$;

(4) the closed graded trace $\#_\alpha\varphi$ is given by $\#_\alpha\varphi(u^i(u^{-1}du)u^j \otimes \omega) = \varphi(\alpha^j(\omega))\delta_{i+j,0}$ and $\#_\alpha\varphi|_{E_0\otimes\Omega} = 0$.

Let us check that $\#_\alpha\varphi$ is indeed a closed graded trace as: for n positive,

$$
\begin{aligned}
\#_\alpha\varphi(d(u^n \otimes \omega)) &= \#_\alpha\varphi(\sum_{i=1}^{n} u^i(u^{-1}du)u^{n-i} \otimes \omega + u^n \otimes d^\sim\omega)\\
&= \sum_{i=1}^{n} \varphi(\alpha^{n-i}(\omega))\delta_{n,0} + \varphi(u^n \otimes d^\sim\omega) = 0 + 0 = 0,\\
\#_\alpha\varphi(d(u^i(u^{-1}du)u^j \otimes \omega)) &= \#_\alpha\varphi(0 \otimes \omega - u^i(u^{-1}du)u^j \otimes d^\sim\omega)\\
&= -\varphi(\alpha^j(d^\sim\omega))\delta_{i+j,0} = -\varphi(d^\sim\alpha^j(\omega))\delta_{i+j,0} = 0
\end{aligned}
$$

where the sum $\sum_{i=1}^{n}$ is zero if $n = 0$ and it is replaced with $-\sum_{i=n+1}^{0}$ if n is negative, and φ is closed, which implies the last equality. Hence $(\#_\alpha\varphi) \circ d = 0$, i.e., $\#_\alpha\varphi$ is closed.

To check the graded trace property it is sufficient to show that

$$\begin{aligned}
&\#_\alpha\varphi(u^{-n-m-1}(du)u^n \otimes \omega)(u^m \otimes \omega')) \\
&= \#_\alpha\varphi(u^{-n-m-1}(du)u^{n+m} \otimes \alpha^{-m}(\omega)\omega') = \varphi(\alpha^n(\omega)\alpha^{n+m}(\omega'))\delta_{0,0}; \\
&\#_\alpha\varphi((u^m \otimes \omega')(u^{-n-m-1}(du)u^n \otimes \omega)) \\
&= (-1)^{\deg\omega'}\#_\alpha\varphi(u^{-n-1}(du)u^n \otimes \alpha^m(\omega')\omega) = (-1)^{\deg\omega'}\varphi(\alpha^{n+m}(\omega')\alpha^n(\omega))\delta_{0,0},
\end{aligned}$$

and since φ is a graded trace on Ω, we get

$$\begin{aligned}
\varphi(\alpha^n(\omega)\alpha^{n+m}(\omega')) &= (-1)^{(\deg\omega)(\deg\omega')}\varphi(\alpha^{n+m}(\omega')\alpha^n(\omega)) \\
&= (-1)^{(\deg\omega+1)(\deg\omega')}[(-1)^{\deg\omega'}\varphi(\alpha^{n+m}(\omega')\alpha^n(\omega))].
\end{aligned}$$

Since $\deg(u^{-n-m-1}(du)u^n \otimes \omega) = 1 + \deg\omega$, the equality combined and obtained shows that $\#_\alpha\varphi$ is a graded trace.

Now let φ be a cyclic cocycle on $\mathfrak{A}$ and consider the cycle $(\Omega(\mathfrak{A}), d, \varphi^\wedge)$, with the action α of $\mathbb{Z}$ defined by $\alpha(x_0dx_1\cdots dx_n) = \alpha(x_0)d\alpha(x_1)\cdots d\alpha(x_n)$, and where $\varphi^\wedge$ denotes the associated graded trace defined by $\varphi^\wedge(x_0dx_1\cdots dx_n) = \varphi(x_0, x_1, \cdots, x_n)$. Then $\#_\alpha\varphi$ is a closed graded trace on $E \otimes_\alpha \Omega(\mathfrak{A})$.

Define a homomorphism $\rho : \mathfrak{A} \rtimes_\alpha [u, u^{-1}] \to E \otimes_\alpha \Omega(\mathfrak{A})_0$ by $\rho(u^m x) = u^m \otimes x$ for $x \in \mathfrak{A}$. Check that

$$\begin{aligned}
\rho(u^m x u^n y) &= \rho(u^{m+n}\alpha^{-n}(x)y) = u^{m+n} \otimes \alpha^{-n}(x)y, \\
\rho(u^m x)\rho(u^n y) &= (u^m \otimes x)(u^n \otimes y) = u^{m+n} \otimes \alpha^{-n}(x)y.
\end{aligned}$$

Then $\#_\alpha\varphi(x_0, x_1, \cdots, x_n) = \#_\alpha\varphi(\rho(x_0)d\rho(x_1)\cdots d\rho(x_n))$, where note that for $x_i \in \mathfrak{A}$, we have $\rho(x_i) = 1 \otimes x_i$ and $d\rho(x_i) = d1 \otimes x_i + 1 \otimes dx_i = 1 \otimes dx_i$ and $d\rho(u^{m_i}x_i) = d(u^{m_i} \otimes x_i) = du^{m_i} \otimes x_i + u^{m_i} \otimes dx_i$ with $du^{m_i} = \sum_{j=1}^{m_i} u^j(u^{-1}du)u^{m_i-j}$ if m_i positive. Note that we use the same symbol $\#_\alpha\varphi$ to denote both the closed graded trace on $E \otimes_\alpha \Omega(\mathfrak{A})$ and the corresponding cyclic cocycle on $\mathfrak{C}$.

Now let us fix the notation as: $x_i = u^{m_i}a_i$ $(i = 1, \cdots, n, m_i \in \mathbb{Z}, a_i \in \mathfrak{A})$, $D(e\otimes\omega) = de\otimes\omega$ for $e \in E$ and $\omega \in \Omega(\mathfrak{A})$, and $\gamma = u^{-1}du\otimes 1 \in E\otimes_\alpha\Omega(\mathfrak{A})$. Note that γ is closed and D is a derivation of $E \otimes_\alpha \Omega(\mathfrak{A})$ anticommuting with d. Check that

$$\begin{aligned}
d\gamma &= d(u^{-1}du \otimes 1) = d(u^{-1}du) \otimes 1 + u^{-1}du \otimes d1 \\
&= (du^{-1}du - u^{-1}d^2u) \otimes 1 = -u^{-1}(du)u^{-1}du \otimes 1 = 0
\end{aligned}$$

since $0 = d1 = d(u^{-1}u) = (du^{-1})u + u^{-1}du$, and also that

$$\begin{aligned}
&D((e\otimes\omega)(u\otimes\mu)) = D(eu\otimes\alpha^{-1}(\omega)\mu)\\
&= d(eu)\otimes\alpha^{-1}(\omega)\mu = (de)u\otimes\alpha^{-1}(\omega)\mu + edu\otimes\alpha^{-1}(\omega)\mu\\
&= D(e\otimes\omega)(u\otimes\mu) + (-1)^{\deg\omega}(e\otimes\omega)D(u\otimes\mu),\\
&D((e\otimes\omega)(du\otimes\mu)) = (-1)^{\deg\omega}D(edu\otimes\alpha^{-1}(\omega)\mu)\\
&= (-1)^{\deg\omega}d(edu)\otimes\alpha^{-1}(\omega)\mu = (-1)^{\deg\omega}(de)du\otimes\alpha^{-1}(\omega)\mu\\
&= D(e\otimes\omega)(du\otimes\mu) + (-1)^{\deg\omega}(e\otimes\omega)D(du\otimes\mu),
\end{aligned}$$

and further that

$$\begin{aligned}
(D\circ d)(e\otimes\omega) &= D(de\otimes\omega + (-1)^{\deg e}e\otimes d\omega) = (-1)^{\deg e}de\otimes d\omega,\\
(d\circ D)(e\otimes\omega) &= d(de\otimes\omega) = d^2e\otimes\omega + (-1)^{\deg de}de\otimes d\omega\\
&= (-1)(D\circ d)(e\otimes\omega).
\end{aligned}$$

Let us set that

$$\begin{aligned}
\pi_i\varphi(x_0,x_1,\cdots,x_n) &= \#_\alpha\varphi(x_0dx_1\cdots dx_{i-1}Dx_idx_{i+1}dx_n),\\
\bar{\pi}_i\varphi(x_0,x_1,\cdots,x_{n-1}) &= \pi_{i+1}\varphi(1,x_0,x_1,\cdots,x_{n-1});\\
\rho_i\varphi(x_0,x_1,\cdots,x_{n-1}) &= \#_\alpha\varphi(x_0dx_1\cdots dx_{i-1}\gamma dx_i\cdots dx_{n-1}).
\end{aligned}$$

Lemma 1.9.1. *Suppose that φ is a cyclic cocycle on $\mathfrak{A}$. Then the following identities hold:* (1) $\#_\alpha\varphi = \sum_{i=1}^n \pi_i\varphi$, (2) $b\bar{\pi}_i\varphi = \pi_{i+1}\varphi - \pi_i\varphi$ *for* $i > 0$, *and* $b\bar{\pi}_0\varphi = \pi_1\varphi - \pi_n\varphi$,

$$(3)\quad b\rho_i\varphi(x_0,x_1,\cdots,x_n) = (-1)^i\#_\alpha\varphi(x_0dx_1\cdots dx_{i-1}[x_i,\gamma]dx_{i+1}\cdots dx_n),$$

and (4) $T\rho_i\varphi = -\rho_{i-1}\varphi$.

Proof. (1) From the definition of the product in $E\otimes_\alpha\Omega$ we get $x_0dx_1dx_2\cdots dx_n$

$$\begin{aligned}
&= x_0d(u^{m_1}\otimes a_1)dx_2\cdots dx_n\\
&= x_0(du^{m_1}\cdot a_1)dx_2\cdots dx_n + x_0(u^{m_1}da_1)dx_2\cdots dx_n\\
&= x_0(du^{m_1}\cdot a_1)(du^{m_2}\cdot a_2 + u^{m_2}da_2)dx_3\cdots dx_n\\
&+ x_0(u^{m_1}da_1)(du^{m_2}\cdot a_2 + u^{m_2}da_2)dx_3\cdots dx_n\\
&= x_0(du^{m_1}\cdot a_1)(u^{m_2}da_2)dx_3\cdots dx_n + x_0(u^{m_1}da_1)(du^{m_2}\cdot a_2)dx_3\cdots dx_n\\
&+ x_0(u^{m_1}da_1)(u^{m_2}da_2)dx_3\cdots dx_n = \cdots\cdots\\
&= \sum_{i=1}^n x_0(u^{m_1}da_1)\cdots(u^{m_{i-1}}da_{i-1})(du^{m_i}\cdot a_i)(u^{m_{i+1}}da_{i+1})\cdots(u^{m_n}da_n)\\
&+ x_0(u^{m_1}da_1)(u^{m_2}da_2)\cdots(u^{m_n}da_n),
\end{aligned}$$

which is also equal to

$$\sum_{i=1}^{n} x_0 dx_1 \cdots dx_{i-1} Dx_i dx_{i+1} \cdots dx_n + x_0(u^{m_1} da_1)(u^{m_2} d_2) \cdots (u^{m_n} da_n)$$

by the same computation, and $\#_\alpha \varphi$ is zero on the last term because $\#_\alpha \varphi|_{E_0 \otimes_\alpha \Omega} = 0$. Hence the first identity (1) holds.

(2) and (3). By a straightforward computation we check that

$$\begin{aligned}
&b(\bar{\pi}_0 \varphi)(x_0, x_1, \cdots, x_n) \\
&= \sum_{i=0}^{n-1} (-1)^i \bar{\pi}_0 \varphi(x_0, x_1, \cdots, x_i x_{i+1}, \cdots, x_n) + (-1)^n \bar{\pi}_0 \varphi(x_0 x_n^\circ, x_1, \cdots, x_{n-1}) \\
&= \sum_{i=0}^{n-1} (-1)^i \pi_1 \varphi(1, x_0, x_1, \cdots, x_i x_{i+1}, \cdots, x_n) + (-1)^n \pi_1 \varphi(1, x_0 x_n^\circ, x_1, \cdots, x_{n-1}) \\
&= \sum_{i=0}^{n-1} (-1)^i \#_\alpha \varphi(1 D x_0 dx_1 \cdots d(x_i x_{i+1}) \cdots dx_n) \\
&+ (-1)^n \#_\alpha \varphi(1 D(x_0 x_n^\circ) dx_1 \cdots dx_{n-1}),
\end{aligned}$$

while we have

$$\begin{aligned}
&(\pi_1 \varphi - \pi_n \varphi)(x_0, x_1, \cdots, x_n) \\
&= \#_\alpha \varphi(x_0 D x_1 dx_2 \cdots dx_n) - \#_\alpha \varphi(x_0 dx_1 \cdots dx_{n-1} D x_n).
\end{aligned}$$

To get the equality between those we use the facts that both D and d are derivations of $E \otimes_\alpha \Omega$ and that $\#_\alpha \varphi$ is a graded trace.

Check also that

$$\begin{aligned}
&b(\rho_k \varphi)(x_0, x_1, \cdots, x_n) \\
&= \sum_{i=0}^{n-1} (-1)^i \rho_k \varphi(x_0, x_1, \cdots, x_i x_{i+1}, \cdots, x_n) + (-1)^n \rho_k \varphi(x_0 x_n^\circ, x_1, \cdots, x_{n-1}) \\
&= \sum_{i=0}^{n-1} (-1)^i \#_\alpha \varphi(x_0 dx_1 \cdots dx_{k-1} \gamma dx_k \cdots d(x_i x_{i+1}) \cdots dx_n) \\
&+ (-1)^n \#_\alpha \varphi(x_0 x_n^\circ dx_1 \cdots dx_{k-1} \gamma dx_k \cdots dx_{n-1}),
\end{aligned}$$

and in particular, note that $d(x_{k-1} x_k)\gamma = ((dx_{k-1}) x_k + (-1)^{\deg x_{k-1}} x_{k-1} dx_k)\gamma$ and $\gamma d(x_k x_{k+1}) = \gamma((dx_k) x_{k+1} + (-1)^{\deg x_k} x_k dx_{k+1})$.

(4) We have

$$
\begin{aligned}
&T\rho_i\varphi(x_0, x_1, \cdots, x_{n-1}) = (-1)^{n-1}\rho_i\varphi(x_{n-1}, x_0, \cdots, x_{n-2})\\
&= (-1)^{n-1}\#_\alpha\varphi(x_{n-1}dx_0 \cdots dx_{i-2}\gamma dx_{i-1}\cdots dx_{n-2})\\
&= (-1)^{n-1}\#_\alpha\varphi(dx_0 \cdots dx_{i-2}\gamma dx_{i-1}\cdots dx_{n-2}x_{n-1})\\
&= (-1)^{n-1}(-1)^n\#_\alpha\varphi(x_0dx_1 \cdots dx_{i-2}\gamma dx_{i-1}\cdots dx_{n-2}dx_{n-1})\\
&= -\rho_{i-1}\varphi(x_0, x_1, \cdots, x_{n-1}),
\end{aligned}
$$

where we use the identity $d(x_0(dx_1 \cdots dx_{i-2}\gamma dx_{i-1}\cdots dx_{n-2})x_{n-1}) =$

$$(dx_0dx_1 \cdots dx_{i-2}\gamma dx_{i-1}\cdots dx_{n-2})x_{n-1} + (-1)^{n-1}x_0dx_1 \cdots dx_{i-2}\gamma dx_{i-1}\cdots dx_{n-1},$$

together with the closedness of $\#_\alpha\varphi$. □

Recall now the map $\#^\sim : M^*_{n-1} \to (L^*_n)_{\text{hom}}$ defined by $\#^\sim(\varphi) = \pi\varphi - bR\bar{\pi}\varphi - R\bar{\pi}b\varphi$ in the subsection 1.7.

Proposition 1.9.2. (1) $\#_\alpha = (-1)^n n\#^\sim$ *on* $H^{n-1}_\lambda(\mathfrak{A})$.
(2) $\#_\alpha$ *commutes with* S.
(3) $\#_\alpha\delta = 0$ *in cyclic cohomology.*

Proof. (1) Let φ be a cyclic cocycle on $\mathfrak{A}$. Using the trace property of $\#_\alpha\varphi$ we note that the identity $T\bar{\pi}_i\varphi = \bar{\pi}_{i-1}\varphi$ (i mod n (from $i = 1$ to n)). Indeed,

$$
\begin{aligned}
&T\bar{\pi}_i\varphi(x_0, x_1, \cdots, x_{n-1}) = (-1)^{n-1}\bar{\pi}_i\varphi(x_{n-1}, x_0, x_1, \cdots, x_{n-2})\\
&= (-1)^{n-1}\pi_{i+1}\varphi(1, x_{n-1}, x_0, \cdots x_{n-2})\\
&= (-1)^{n-1}\#_\alpha\varphi(1dx_{n-1}dx_0 \cdots Dx_{i-1}\cdots dx_{n-2}), \quad \text{while}\\
&\bar{\pi}_{i-1}\varphi(x_0, x_1, \cdots, x_{n-1}) = \pi_i\varphi(1, x_0, x_1, \cdots, x_{n-1})\\
&= \#_\alpha\varphi(1dx_0dx_1 \cdots Dx_{i-1}\cdots dx_{n-1}).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
nbR\bar{\pi}_0\varphi &= nb\frac{1}{n}(n\bar{\pi}_0\varphi + (n-1)T\bar{\pi}_0\varphi + \cdots T^{n-1}\bar{\pi}_0\varphi)\\
&= b(n\bar{\pi}_0\varphi + (n-1)\bar{\pi}_{n-1}\varphi + \cdots \bar{\pi}_1\varphi)\\
&= n(\pi_1\varphi - \pi_n\varphi) + (n-1)(\pi_n\varphi - \pi_{n-1}\varphi) + \cdots + (\pi_2\varphi - \pi_1\varphi)\\
&= n\pi_1\varphi - \sum_{i=1}^{n}\pi_i\varphi = n\pi_1\varphi - \#_\alpha\varphi.
\end{aligned}
$$

Therefore

$$\begin{aligned}\#_\alpha\varphi &= n(\pi_1\varphi - bR\bar{\pi}_0\varphi)\\ &= (-1)^n n((-1)^n\pi_1\varphi - bR(-1)^n\bar{\pi}_0\varphi)\\ &= (-1)^n n(\pi\varphi - bR\bar{\pi}\varphi - R\bar{\pi}b\varphi) = (-1)n\#^{\sim}(\varphi),\end{aligned}$$

where $\pi = (-1)^n\pi_1$ and $\bar{\pi} = (-1)^n\bar{\pi}_0$ on $(n-1)$-cochains (and probably, using $b\varphi$ is zero as a cohomology class).

(2) Let us consider the following diagram:

$$\begin{array}{ccc}\Omega(\mathfrak{A}\rtimes_\alpha[u,u^{-1}]) & = & \Omega(\mathfrak{A}\rtimes_\alpha[u,u^{-1}])\\ \downarrow\cong & & \downarrow\cong\\ \Omega(\mathfrak{A}\otimes\mathbb{C}\rtimes_\alpha[u,u^{-1}]) & & \Omega(\mathfrak{A}\rtimes_\alpha[u,u^{-1}]\otimes\mathbb{C})\\ \downarrow & & \downarrow\\ E\otimes_\alpha\Omega(\mathfrak{A}\otimes\mathbb{C}) & & \Omega(\mathfrak{A}\rtimes_\alpha[u,u^{-1}])\otimes\Omega(\mathbb{C})\\ \downarrow & & \downarrow\\ E\otimes_\alpha(\Omega(\mathfrak{A})\otimes\Omega(\mathbb{C})) & \xrightarrow{\cong} & (E\otimes_\alpha\Omega(\mathfrak{A}))\otimes\Omega(\mathbb{C}),\end{array}$$

where the vertical arrows are given by the universality of the $\Omega(\cdot)$-construction, and the obvious isomorphism at the bottom gives $\#_\alpha(\varphi\#\omega) = (\#_\alpha\varphi)\#\omega$ for any closed graded trace ω on $\Omega(\mathbb{C})$ and $\#$ the cup product. If we take ω as the generator of $H^2_\lambda(\mathbb{C})$ given by $\omega(1d1d1) = 2\pi i$, then we compute $\#_\alpha S\varphi = \#_\alpha(\omega\#\varphi) = \#_\alpha(\varphi\#\omega)$ in the left-hand column, while we compute $S\#_\alpha\varphi = \omega\#(\#_\alpha\varphi) = (\#_\alpha\varphi)\#\omega$ in the right-hand column, i.e., we get $S\#_\alpha = \#_\alpha S$.

(3) Let us compute $\#_\alpha\delta\varphi$ for a cyclic cocycle φ. We have

$$\begin{aligned}\pi_i(\delta\varphi)(x_0,x_1,\cdots,x_n) &= \#_\alpha(\varphi - \varphi\circ\alpha)(x_0dx_1,\cdots,dx_{i-1}Dx_idx_{i+1}\cdots dx_n)\\ &= \#_\alpha\varphi(x_0dx_1,\cdots,dx_{i-1}(du^{m_i})a_idx_{i+1}\cdots dx_n)\\ &\quad - \#_\alpha\varphi(x_0dx_1\cdots dx_{i-1}u^{-1}(du^{m_i})ua_idx_{i+1}\cdots dx_n).\end{aligned}$$

Note that $\#_\alpha(\varphi\circ\alpha)(u^{i-1}(du)u^j\otimes\omega) = \#_\alpha\varphi(u^{i-1}(du)u^j\otimes\alpha(\omega))$. We also have $(du^{m_i})a_i - u^{-1}(du^{m_i})ua_i$

$$\begin{aligned}&= \sum_{i=1}^{m_i}(u^{i-1}(du)u^{m_i-i} - u^{i-2}(du)u^{m_i-i+1})a_i = (u^{m_i}u^{-1}du - u^{-1}(du)u^{m_i})a_i\\ &= [u^{m_i},u^{-1}du]a_i = [x_i,\gamma],\end{aligned}$$

where note that $u^{-1}dua_i = u^{-1}du\alpha^{-1}(\alpha(a_i)) = u^{-1}\alpha(a_i)du = a_iu^{-1}du$, and hence we get

$$\#_\alpha\delta\varphi = \sum_{i=1}^n \pi_i\delta\varphi = \sum_{i=1}^n(-1)^i b\rho_i\varphi = b(\sum_{i=1}^n(-1)^i\rho_i\varphi),$$

with $\sum_{i=1}^n(-1)^i\rho_i\varphi$ is a cyclic cochain on $\mathfrak{C}$, since $T\rho_i\varphi = -\rho_{i-1}\varphi$ obtained above. This completes the proof. □

Proposition 1.9.3. *All the cochains (defined on the algebraic part) such as $\pi_i\varphi$ constructed in this subsection extend by continuity to all of $\mathfrak{C}$.*

Proof. Consider the case of $\pi_1\varphi$. All the other cases follow by the same way.

Suppose that $x_0, x_1, \cdots, x_n \in \mathfrak{C}$ are the monomials $x_i = u^{m_i}a_i$ with $a_i \in \mathfrak{A}$ and $m_0 + \cdots + m_n = 0$, and set $a_j^\sim = \alpha^{m_0+m_1+\cdots+m_j}(a_j)$. Then

$$\pi_1\varphi(x_0, x_1, \cdots, x_n) = \sum_{j=1}^{m_1} \varphi\circ\alpha^{-j}(a_0^\sim a_1^\sim, a_2^\sim, \cdots, a_n^\sim).$$

But our computation shows that for m_1 positive, $\pi_1\varphi(x_0, x_1, \cdots, x_n)$

$$= \#_\alpha\varphi(x_0Dx_1dx_2\cdots dx_n) = \#_\alpha\varphi(x_0(du^{m_1})a_1dx_2\cdots dx_n)$$

$$= \sum_{i=1}^{m_1}\#_\alpha\varphi(x_0u^i(u^{-1}du)u^{m_1-i}a_1dx_2\cdots dx_n)$$

$$= \sum_{i=1}^{m_1}\#_\alpha\varphi(u^{m_0}a_0u^i(u^{-1}du)u^{m_1-i}a_1u^{m_2}da_2\cdots u^{m_n}da_n)$$

$$= \sum_{i=1}^{m_1}\#_\alpha\varphi(u^{m_0}a_0u^i(u^{-1}du)u^{m_1-i}a_1u^{m_2+m_3+\cdots+m_n}\alpha^{-m_3-\cdots-m_n}(da_2)$$
$$\cdots\alpha^{-m_{n-1}-m_n}(da_{n-2})\alpha^{-m_n}(dx_{n-1})da_n)$$

$$= \sum_{i=1}^{m_1}\#_\alpha\varphi(u^{m_0}a_0u^i(u^{-1}du)u^{m_1+m_2+\cdots+m_n-i}\alpha^{-m_2-m_3+\cdots-m_n}(a_1)\alpha^{-m_3-\cdots-m_n}(da_2)$$
$$\cdots\alpha^{-m_{n-1}-m_n}(da_{n-2})\alpha^{-m_n}(da_{n-1})da_n)$$

$$= \sum_{i=1}^{m_1}\#_\alpha\varphi(u^{m_0+i}(u^{-1}du)u^{m_1+m_2+\cdots+m_n-i}\alpha^{-m_1-\cdots-m_n}(a_0)\alpha^{-m_2-m_3+\cdots-m_n}(a_1)$$
$$\alpha^{-m_3-\cdots-m_n}(da_2)\cdots\alpha^{-m_{n-1}-m_n}(da_{n-2})\alpha^{-m_n}(da_{n-1})da_n)$$

$$= \sum_{i=1}^{m_1}\varphi\circ\alpha^{m_1+m_2+\cdots+m_n-i}(\alpha^{-m_1-\cdots-m_n}(a_0)\alpha^{-m_2-m_3+\cdots-m_n}(a_1)$$

$\alpha^{-m_3-\cdots-m_n}(da_2)\cdots\alpha^{-m_{n-1}-m_n}(da_{n-2})\alpha^{-m_n}(da_{n-1})da_n)$

$$= \sum_{i=1}^{m_1} \varphi\circ\alpha^{-i}(a_0\alpha^{m_1}(a_1)$$

$\alpha^{m_1+m_2}(da_2)\cdots\alpha^{m_1+\cdots+m_{n-2}}(da_{n-2})\alpha^{m_1+\cdots+m_{n-1}}(da_{n-1})\alpha^{m_1+\cdots+m_n}(da_n))$.

Given that φ satisfies an estimate of the form:

$$\varphi(a_0,a_1,\cdots,a_{n-1})\leq c\|a_0\|_k\|a_1\|_k\cdots\|a_{n-1}\|_k,$$

we get $|\pi_1\varphi(x_0,x_1,\cdots,x_n)|$

$$\leq c\sum_{i=1}^{m_1}\|\alpha^{-i}\|_k\|\alpha^{m_1}\|_k^n\|\alpha^{m_2}\|_k^{n-1}\cdots\|\alpha^{m_n}\|_k\|a_0\|_{k'}\|a_1\|_{k'}\|da_2\|_k\cdots\|da_n\|_k$$

(corrected partly, but possibly, we can write $\|da_j\|_k\leq\|a_j\|_{k+1}$), where we use a fact that $\|ab\|_k\leq\|a\|_{k'}\|b\|_{k'}$ for some $k'\in\mathbb{N}$. Choosing $k''=\max\{k+1,k'\}+n+1$, then we have $|\pi_1\varphi(x_0,x_1,\cdots,x_n)|$

$$\leq c\rho_{k''}(m_0)\rho_{k''}(m_1)\cdots\rho_{k''}(m_n)\|a_0\|_{k''}\|a_1\|_{k''}\cdots\|a_n\|_{k''},$$

where $\rho_k(m)=\max_{1\leq i\leq k}(\sum_{t=-m}^{m}\|\alpha^t\|_i)^k$. If now x_i are given as finite sums of monomials: $x_i=\sum_{m_i}u^{m_i}a_{i,m_i}$, then we get $|\pi_1\varphi(x_0,x_1,\cdots,x_n)|$

$$\leq C\sum_{m_0}\cdots\sum_{m_n}\frac{1}{m_0^2}\cdots\frac{1}{m_n^2}\rho_{k''+2}(m_0)\cdots\rho_{k''+2}(m_n)\|a_{0,m_0}\|_{k''}\cdots\|a_{n,m_n}\|_{k''}$$

$$\leq C\cdot 2^n\left(1+\sum_{m>0}\frac{1}{m^2}\right)^n\|x_0\|_{k''+2}\cdots\|x_n\|_{k''+2},$$

where $|n|\rho_k(n)\leq\rho_{k+1}(n)$ and $|n|\rho_{k+1}(n)\leq\rho_{k+2}(n)$, and the multiple 2 in 2^n corresponds to the subsums of $\sum_{m_j}$ for m_j positive or negative, and $1+$ corresponds to $m_j=0$, since $\rho_k(0)=1$. Hence the continuity of $\pi_1\varphi$ is established. □

Definition 1.9.4. Define the cochain map $\#: C^{n-1}(\mathfrak{A},\mathfrak{A}^*)\to C^n(\mathfrak{C},\mathfrak{C}^*)_{\mathrm{hom}}$ by $\#\varphi=(-1)^n n\#^{\sim}\varphi$.

1.10 For the cochain map

(A) Given $\varphi\in L_n^*$ with $L_n=\mathfrak{D}\otimes(\otimes^n\mathfrak{C})$ and $x_0,x_1,\cdots,x_{n-1},a\in\mathfrak{C}$, we set

$$h_a\varphi(x_0,x_1,\cdots,x_{n-1})=\sum_{i=0}^{n-1}(-1)^{n-i-1}\varphi(x_0,\cdots,x_i,a,x_{i+1},\cdots,x_{n-1}).$$

We have $bh_a\varphi(x_0, x_1, \cdots, x_n) = h_a\varphi \circ b(x_0, x_1, \cdots, x_n)$

$$= \sum_{j=0}^{n-1}(-1)^j h_a\varphi(x_0, x_1, \cdots, x_jx_{j+1}, \cdots, x_n) + (-1)^n h_a\varphi(x_0x_n^\circ, x_1, \cdots, x_{n-1})$$
$$= \sum_{j=0}^{n-1}\sum_{i=0}^{n-1}(-1)^{n+j-i-1}\varphi(x_0, x_1, \cdots, x_i, a, x_{i+1}, \cdots x_jx_{j+1}, \cdots, x_n)$$
$$+ \sum_{i=0}^{n-1}(-1)^{2n-i-1}\varphi(x_0x_n^\circ, x_1, \cdots, x_i, a, x_{i+1}, \cdots, x_{n-1}),$$

while we have $h_ab\varphi(x_0, x_1, \cdots, x_n)$

$$= \sum_{i=0}^{n-1}(-1)^{n-i-1}b\varphi(x_0, \cdots, x_i, a, x_{i+1}, \cdots, x_n)$$
$$= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}(-1)^{n-i-1+j}\varphi(x_0, \cdots, x_jx_{j+1}, \cdots, x_i, a, x_{i+1}, \cdots, x_n)$$
$$+ \sum_{i=0}^{n-1}(-1)^{2n-i-1}\varphi(x_0x_n^\circ, \cdots, x_i, a, x_{i+1}, \cdots, x_{n-1})$$

where in the first term there are the cases where $x_jx_{j+1} = x_ja$ or ax_{j+1} when $j = i$ or $j = i + 1$ respectively, and other cases in the first and second terms are the same as the first computation result, so that we get $bh_a\varphi(x_0, x_1, \cdots, x_n)$

$$= h_ab\varphi(x_0, x_1, \cdots, x_n) + (-1)^{n-1}\sum_{i=0}^{n}\varphi(x_0, \cdots, [a, x_i], \cdots, x_n).$$

Check indeed that $bh_a\varphi(x_0, x_1, x_2) = h_a\varphi \circ b(x_0, x_1, x_2)$

$$= h_a\varphi(x_0x_1, x_2) - h_a\varphi(x_0, x_1x_2) + h_a\varphi(x_0x_2^\circ, x_1)$$
$$= (-\varphi(x_0x_1, a, x_2) + \varphi(x_0x_1, x_2, a)) - (-\varphi(x_0, a, x_1x_2) + \varphi(x_0, x_1x_2, a))$$
$$+ (-\varphi(x_0x_2^\circ, a, x_1) + \varphi(x_0x_2^\circ, x_1, a))$$

while we have $h_ab\varphi(x_0, x_1, x_2)$

$$= b\varphi(x_0, a, x_1, x_2) - b\varphi(x_0, x_1, a, x_2) + b\varphi(x_0, x_1, x_2, a)$$
$$= (\varphi(x_0a, x_1, x_2) - \varphi(x_0, ax_1, x_2) + \varphi(x_0, a, x_1x_2) - \varphi(x_0x_2^\circ, a, x_1))$$
$$- (\varphi(x_0x_1, a, x_2) - \varphi(x_0, x_1a, x_2) + \varphi(x_0, x_1, ax_2) - \varphi(x_0x_2^\circ, x_1, a))$$
$$+ (\varphi(x_0x_1, x_2, a) - \varphi(x_0, x_1x_2, a) + \varphi(x_0, x_1, x_2a) - \varphi(x_0a^\circ, x_1, x_2))$$

Hence we obtain $bh_a\varphi(x_0,x_1,x_2) - h_a b\varphi(x_0,x_1,x_2) =$

$$
\begin{aligned}
&- \{\varphi(x_0a,x_1,x_2) - \varphi(x_0a^\circ,x_1,x_2)) + \varphi(x_0,x_1a-ax_1,x_2) + \varphi(x_0,x_2,x_2a-ax_2)\} \\
&= -\{\varphi(x_0a,x_1,x_2) - \varphi(x_0a^\circ,x_1,x_2)) + \varphi(x_0,[x_1,a],x_2) + \varphi(x_0,x_2,[x_2,a])\}, \\
&= -\{\varphi([a,x_0],x_1,x_2) + \varphi(x_0,[a,x_1],x_2) + \varphi(x_0,x_1,[a,x_2])\}.
\end{aligned}
$$

We also have $(1-T)h_a\varphi(x_0,x_1,\cdots,x_{n-1})$

$$
\begin{aligned}
&= h_a\varphi(x_0,x_1,\cdots,x_{n-1}) - (-1)^{n-1}h_a\varphi(x_{n-1},x_0,\cdots,x_{n-2}) \\
&= \sum_{i=0}^{n-1}(-1)^{n-i-1}\varphi(x_0,\cdots,x_i,a,a_{i+1},\cdots,x_{n-1}) \\
&+ \sum_{i=0}^{n-1}(-1)^{2n-i-1}\varphi(x_{n-1},x_0,\cdots,x_{i-1},a,x_i,\cdots,x_{n-2}),
\end{aligned}
$$

while we have $(h_a(1-T)\varphi)(x_0,x_1,\cdots,x_{n-1})$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1}(-1)^{n-i-1}(1-T)\varphi(x_0,\cdots,x_i,a,x_{i+1},\cdots,x_{n-1}) \\
&= \sum_{i=0}^{n-1}(-1)^{n-i-1}\varphi(x_0,\cdots,x_i,a,x_{i+1},\cdots,x_{n-1}) \\
&+ \sum_{i=0}^{n-1}(-1)^{2n-i}\varphi(x_{n-1},x_0,\cdots,x_i,a,x_{i+1},\cdots,x_{n-2}),
\end{aligned}
$$

where in the last term, x_{n-1} is replaced by a when $i=n-1$, and that x_0 can not be replaced by a. Hence we get $\{(1-T)h_a\varphi - h_a(1-T)\varphi\}(x_0,x_1,\cdots,x_{n-1})$

$$
\begin{aligned}
&= (-1)^{2n-1}\varphi(x_{n-1},a,x_0,\cdots,x_{n-2}) - (-1)^{n-1}\varphi(a,x_0,\cdots,x_{n-1}) \\
&= (-1)^{n-1}T\varphi(a,x_0,\cdots,x_{n-1}) + (-1)^n\varphi(a,x_0,\cdots,x_{n-1}) \\
&= (-1)^n(1-T)\varphi(a,x_0,x_1,\cdots,x_{n-1})
\end{aligned}
$$

(where the index of (-1) is corrected).

(B) Consider $\mathfrak{C}\otimes M_n(\mathbb{C})$ and replace φ by $\varphi\#\mathrm{Tr}$. Set

$$
U = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}, \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
$$

and $\alpha_\theta = \mathrm{Ad}(UR_\theta) \in \mathrm{Aut}(\mathfrak{C}\otimes M_2(\mathbb{C}))$. Then we have

$$
\frac{d}{d\theta}\alpha_\theta(x) = \alpha_\theta([J,x]).
$$

Check this equation as follows: for

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathfrak{C}),$$

it follows by direct matrix computation that

$$\alpha_\theta(x) = \mathrm{Ad}(UR_\theta)x = UR_\theta x R_\theta^* U^* =$$

$$\begin{pmatrix} a\sin^2\theta + b\frac{\sin 2\theta}{2} + c\frac{\sin 2\theta}{2} + d\cos^2\theta & (a\frac{\sin 2\theta}{2} - b\sin^2\theta + c\cos^2\theta - d\frac{\sin 2\theta}{2})u^* \\ u(a\frac{\sin 2\theta}{2} + b\cos^2\theta - c\sin^2\theta - d\frac{\sin 2\theta}{2}) & u(a\cos^2\theta - b\frac{\sin 2\theta}{2} - c\frac{\sin 2\theta}{2} + d\sin^2\theta)u^* \end{pmatrix}$$

where

$$R_\theta^* = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad J^* = \begin{pmatrix} 0 & u^* \\ 1 & 0 \end{pmatrix},$$

and it is shown that the element-wise derivative of $\alpha_\theta(x)$ with respect to θ, i.e., $\frac{d}{d\theta}\alpha_\theta(x)$ is equal to

$$\alpha_\theta([J,x]) = \alpha_\theta(Jx - xJ) = \alpha_\theta \begin{pmatrix} -b-c & a-d \\ a-d & b+c \end{pmatrix} =$$

$$\begin{pmatrix} (a-d)\sin 2\theta + (b+c)\cos 2\theta & ((a-d)\cos 2\theta - (b+c)\sin 2\theta)u^* \\ u((a-d)\cos 2\theta - (b+c)\sin 2\theta) & u((d-a)\sin 2\theta - (b+c)\cos 2\theta)u^* \end{pmatrix}.$$

We thus have, for any n-cochain ψ on $\mathfrak{C} \otimes M_2(\mathbb{C})$,

$$\begin{aligned} &\psi \circ \alpha_{\pi/2}(x_0, x_1, \cdots, x_n) - \psi \circ \alpha_0(x_0, x_1, \cdots, x_n) \\ &= \int_0^{\pi_2} \frac{d}{d\theta}\psi \circ \alpha_\theta(x_0, x_1, \cdots, x_n) d\theta \\ &= \sum_{i=0}^n \int_0^{\pi/2} \psi \circ \alpha_\theta(x_0, x_1, \cdots, [J, x_i], \cdots, x_n) d\theta \end{aligned}$$

by the fundamental theorem of calculus, where note that

$$\begin{aligned} &\frac{d}{d\theta}\alpha_\theta(x_0, x_1, \cdots, x_n) = \frac{d}{d\theta}(\alpha_\theta(x_0), \alpha_\theta(x_1), \cdots, \alpha_\theta(x_n)) \\ &= \sum_{i=0}^n (\alpha_\theta(x_0), \alpha_\theta(x_1), \cdots, \frac{d}{d\theta}\alpha_\theta(x_i), \cdots, \alpha_\theta(x_n)) \\ &= \sum_{i=0}^n \alpha_\theta(x_0, x_1, \cdots, [J, x_i], \cdots, x_n). \end{aligned}$$

(C) Define, given $\varphi \in (L_n^*)_{\mathrm{hom}}$,

$$\bar{\eta}\varphi = \int_0^{\pi/2} h_J((\varphi \# \mathrm{Tr}) \circ \alpha_\theta) d\theta|_{\mathfrak{C}\otimes e_{11}},$$

where e_{ij} is the matrix unit of $M_2(\mathbb{C})$. Applying (A) and (B) above we get

$$b\bar{\eta}\varphi(x_0,\cdots,x_n) = \int_0^{\pi/2} bh_J((\varphi\#\mathrm{Tr})\circ\alpha_\theta)d\theta|_{\mathfrak{C}\otimes e_{11}}(x_0,\cdots,x_n)$$

$$= \int_0^{\pi/2} h_J b((\varphi\#\mathrm{Tr})\circ\alpha_\theta)d\theta|_{\mathfrak{C}\otimes e_{11}}(x_0,\cdots,x_n)$$

$$+ (-1)^{n-1}\sum_{i=0}^{n}\int_0^{\pi/2}((\varphi\#\mathrm{Tr})\circ\alpha_\theta)d\theta|_{\mathfrak{C}\otimes e_{11}}(x_0,\cdots,[J,x_i],\cdots,x_n)$$

$$= \bar{\eta}b\varphi + (-1)^{n-1}\delta\varphi,$$

where note that $\alpha_0 = \mathrm{Ad}(U)$ and $\alpha_{\pi/2} = \mathrm{Ad}(UJ)$, and

$$\mathrm{Ad}(U)\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} d & cu^* \\ ub & uau^*\end{pmatrix}, \quad \mathrm{Ad}(UJ)\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a & -bu^* \\ -uc & udu^*\end{pmatrix}.$$

We also have $(1-T)\bar{\eta}\varphi(x_0,x_1,\cdots,x_{n-1})$

$$= \int_0^{\pi/2}(1-T)h_J((\varphi\#\mathrm{Tr})\circ\alpha_\theta)d\theta|_{\mathfrak{C}\otimes e_{11}}(x_0,x_1,\cdots,x_{n-1})$$

$$= \int_0^{\pi/2} h_J(1-T)((\varphi\#\mathrm{Tr})\circ\alpha_\theta)d\theta|_{\mathfrak{C}\otimes e_{11}}(x_0,x_1,\cdots,x_{n-1})$$

$$+ \int_0^{\pi/2}(-1)^n(1-T)((\varphi\#\mathrm{Tr})\circ\alpha_\theta)d\theta|_{\mathfrak{C}\otimes e_{11}}(J,x_0,x_1,\cdots,x_{n-1})$$

$$= \bar{\eta}(1-T)\varphi(x_0,x_1,\cdots,x_{n-1})$$

$$+ \int_0^{\pi/2}(-1)^n(1-T)((\varphi\#\mathrm{Tr})\circ\alpha_\theta)d\theta(J,\bar{x}_0,\bar{x}_1,\cdots,\bar{x}_{n-1}),$$

where $\bar{x}_i = x_i\otimes e_{11}$.

(D) If we define $\eta\varphi = \bar{\eta}\varphi|_{\mathfrak{A}}$, then it follows from those identities in (C) that

$$\varphi|_{\mathfrak{A}} = 0 \Rightarrow b\eta\varphi = \eta b\varphi \quad \text{and} \quad (1-T)\varphi = 0 \Rightarrow (1-T)\eta\varphi = 0.$$

Thus we can consider η as the map $\eta : (Q_n^*)_{\mathrm{hom}} \to (M_{n-1})^*$, which commutes with the coboundary operator b and sends cyclic cochains to cyclic cochains, since $L_n = \mathfrak{D}M_n \oplus Q_n$ as obtained before.

(E) Note that, given any cochain φ on $\mathfrak{A}$, the formula $\varphi((\lambda+a_0)da_1\cdots da_n) = \varphi(a_0,a_1,\cdots,a_n)$ defines a linear functional on $\Omega(\mathfrak{A})$ (by ignoring the scalar λ). Moreover, the construction of $\#_\alpha\varphi$ extends to this more general case and gives us the map $\#_\alpha; M_{n-1}^* \to (L_n^*)_{\mathrm{hom}}$

Lemma 1.10.1. *Let* $\omega_0, \omega_1, \omega_2 \in \Omega(\mathfrak{A})$, $a \in \mathfrak{A}$, *and suppose that* φ *is a cochain on* $\mathfrak{A}$. *Set*

$$\bar{\omega}_i = \omega_i \otimes e_{11} \in \Omega(\mathfrak{A}) \otimes \Omega(M_2(\mathbb{C})) \quad \textit{and} \quad \bar{a} = a \otimes e_{11} \in \Omega(\mathfrak{A}) \otimes \Omega(M_2(\mathbb{C})).$$

Then the following identities hold:

$$(1) \quad \int_0^{\pi/2} d\theta((\#_\alpha\varphi)\#\mathrm{Tr}) \circ \alpha_\theta(\bar{\omega}_0(DJ)\bar{\omega}_1) = (-1)^{\deg\omega_0+1}\varphi(\omega_0\omega_1),$$

$$(2) \quad \int_0^{\pi/2} d\theta((\#_\alpha\varphi)\#\mathrm{Tr}) \circ \alpha_\theta(\bar{\omega}_0(dJ)\bar{\omega}_1(D\bar{a})\bar{\omega}_2) = 0,$$

$$(3) \quad \int_0^{\pi/2} d\theta((\#_\alpha\varphi)\#\mathrm{Tr}) \circ \alpha_\theta(\bar{\omega}_0(D\bar{a})\bar{\omega}_1(dJ)\bar{\omega}_2) = 0,$$

$$(4) \quad \int_0^{\pi/2} d\theta((\#_\alpha\varphi)\#\mathrm{Tr}) \circ \alpha_\theta(\bar{\omega}_0(dJ)(D\bar{a})\bar{\omega}_1) = \frac{(-1)^{\deg\omega_0+1}}{2}\varphi(\omega_0(d1)a\omega_1),$$

$$(5) \quad \int_0^{\pi/2} d\theta((\#_\alpha\varphi)\#\mathrm{Tr}) \circ \alpha_\theta(\bar{\omega}_0(D\bar{a})(dJ)\bar{\omega}_1) = \frac{(-1)^{\deg\omega_0}}{2}\varphi(\omega_0 a(d1)\omega_1)$$

Proof. (1) Set $\sin\theta = s$ and $\cos\theta = c$. We compute the integrand I_θ

$$\begin{aligned}
&= ((\#_\alpha\varphi)\#\mathrm{Tr})\left(\alpha_\theta\begin{pmatrix}\omega_0 & 0\\ 0 & 0\end{pmatrix} D(\alpha_\theta(J))\alpha_\theta\begin{pmatrix}\omega_1 & 0\\ 0 & 0\end{pmatrix}\right)\\
&= ((\#_\alpha\varphi)\#\mathrm{Tr})\left(\begin{pmatrix}\omega_0 s^2 & \omega_0 u^* sc\\ u\omega_0 sc & \alpha(\omega_0)c^2\end{pmatrix}\begin{pmatrix}0 & Du^*\\ -Du & 0\end{pmatrix}\begin{pmatrix}\omega_1 s^2 & \omega_1 u^* sc\\ u\omega_1 sc & \alpha(\omega_1)c^2\end{pmatrix}\right)\\
&= ((\#_\alpha\varphi)\#\mathrm{Tr})\\
&\left(\begin{pmatrix}\omega_0(-u^*(Du)+(Du^*)u)\omega_1 s^3 c & \omega_0(-u^*(Du)\omega_1 u^* + (Du^*)\alpha(\omega_1))s^2c^2\\ (-\alpha(\omega_0)(Du)+u\omega_0(Du^*)u)\omega_1 s^2c^2 & (-\alpha(\omega_0)(Du)\omega_1 u^* + u\omega_0(Du^*)\alpha(\omega_1))sc^3\end{pmatrix}\right)\\
&= \#_\alpha\varphi(\omega_0(-u^*(Du)+(Du^*)u)\omega_1 s^3 c + (-\alpha(\omega_0)(Du)\omega_1 u^* + u\omega_0(Du^*)\alpha(\omega_1))sc^3)\\
&= \#_\alpha\varphi((-1)^{\deg\omega_0}(-u^*(Du)+(Du^*)u)\omega_0\omega_1 s^3 c\\
&+ (-1)^{\deg\omega_0}(-(Du)u^*\alpha(\omega_0)\alpha(\omega_1) + u(Du^*)\alpha(\omega_0)\alpha(\omega_1))sc^3)\\
&= \#_\alpha\varphi((-1)^{\deg\omega_0}(-2u^{-1}du)\omega_0\omega_1 s^3 c + (-1)^{\deg\omega_0}(-2u(u^{-1}Du)u^{-1}\alpha(\omega_0\omega_1))sc^3)\\
&= 2(-1)^{\deg\omega_0+1}(\varphi(\omega_0\omega_1)s^3c + \varphi(\alpha^{-1}(\alpha(\omega_0\omega_1)))sc^3)\\
&= 2(-1)^{\deg\omega_0+1}\varphi(\omega_0\omega_1)\sin\theta\cos\theta = (-1)^{\deg\omega_0+1}\varphi(\omega_0\omega_1)\sin 2\theta
\end{aligned}$$

where we used the graded trace property of $\#_\alpha$ and note that $Du = du$

and $Du^* = du^{-1} = -(u^{-1}du)u^{-1}$ by definition. Therefore, we get

$$\int_0^{\pi/2} I_\theta d\theta = (-1)^{\deg \omega_0 + 1}\varphi(\omega_0\omega_1)\int_0^{\pi/2}\sin 2\theta d\theta$$
$$= (-1)^{\deg \omega_0 + 1}\varphi(\omega_0\omega_1)\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2} = (-1)^{\deg \omega_0 + 1}\varphi(\omega_0\omega_1).$$

The other integrals are computed in the same way. As for (2), check that

$$\alpha_\theta(\bar\omega_0)\alpha_\theta\begin{pmatrix}0 & -d1\\ d1 & 0\end{pmatrix}\alpha_\theta(\bar\omega_1)D\alpha_\theta\begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix}\alpha_\theta(\bar\omega_2)$$
$$= \alpha_\theta(\bar\omega_0)\alpha_\theta\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\alpha_\theta(\bar\omega_1)D\alpha_\theta\begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix}\alpha_\theta(\bar\omega_2) = 0$$

(possibly, this computation is allowed), where note that $d1 = d(1\cdot 1) = (d1)1 + 1(d1)$, and hence $d1 = 0$. □

Proposition 1.10.2. (1) *For a cochain $\psi \in M^*_{n-1}$, we have*

$$\eta\pi\psi = \psi + \frac{1}{2}b(\psi|_1) + \frac{1}{2}(b\psi)|_1.$$

(2) *For a cyclic cocycle $\varphi \in C^{n-1}_\lambda(\mathfrak{A}, \mathfrak{A}^*)$, we have*

$$\eta\#_\alpha\varphi = (-1)^n n\varphi.$$

Proof. (1) Since $\pi\psi = (-1)^n\pi_1\psi$, and

$$\pi_1\psi(a_0, a_1, \cdots, a_{n-1}) = \#_\alpha\psi(a_0 Da_1 da_2\cdots da_{n-1})$$

we have $\eta\pi\psi(a_0, a_1, \cdots, a_{n-1})$

$$= \int_0^{\pi/2} h_J((\pi_1\psi)\#\mathrm{Tr})\circ\alpha_\theta d\theta|_{\mathfrak{A}\otimes e_{11}}(a_0, a_1, \cdots, a_{n-1})$$
$$= \int_0^{\pi/2}\sum_{i=0}^{n-i-1}((\pi_1\psi)\#\mathrm{Tr})\circ\alpha_\theta d\theta|_{\mathfrak{A}\otimes e_{11}}(a_0, a_1, \cdots, a_i, J, a_{i+1}, \cdots, a_{n-1})$$
$$= (-1)^n(-1)^{n-1}[\int_0^{\pi/2} d\theta((\#_\alpha\psi)\#\mathrm{Tr})\circ\alpha_\theta(\bar a_0 DJd\bar a_1, \cdots d\bar a_{n-1})$$
$$- \int_0^{\pi/2} d\theta((\#_\alpha\psi)\#\mathrm{Tr})\circ\alpha_\theta(\bar a_0 D\bar a_1 dJd\bar a_2\cdots d\bar a_{n-1})]$$

according to the lemma above, and moreover,

$$= (-1)[-\psi(a_0 da_1 \cdots da_{n-1}) - \frac{1}{2}\psi((a_0a_1)d1da_2 \cdots da_{n-1})]$$
$$= \psi(a_0, a_1, \cdots, a_{n-1}) + \frac{1}{2}\psi(a_0a_1, 1, a_2, \cdots, a_{n-1}).$$

On the other hand, check the following:

$$((b\psi)|_1 + b(\psi|_1))(a_0, \cdots, a_{n-1}) = b\psi(a_0, 1, a_1, \cdots, a_{n-1})$$
$$+ \sum_{i=0}^{n-2}(-1)^i \psi|_1(a_0, a_1, \cdots, a_i a_{i+1}, \cdots, a_{n-1}) + (-1)^{n-1}\psi|_1(a_0 a_{n-1}^{\circ}, a_1, \cdots, a_{n-2})$$
$$= \psi(a_0, a_1, \cdots, a_{n-1}) - \psi(a_0, a_1, \cdots, a_{n-1})$$
$$\sum_{j=1}^{n-2}(-1)^{j+1}\psi(a_0, 1, a_1, \cdots, a_j a_{j+1}, \cdots, a_{n-1}) + (-1)^n \psi(a_0 a_{n-1}^{\circ}, 1, a_1, \cdots, a_{n-2})$$
$$+ \psi(a_0a_1, 1, a_2, \cdots, a_{n-1})$$
$$+ \sum_{i=1}^{n-2}(-1)^i \psi(a_0, 1, a_1, \cdots, a_i a_{i+1}, \cdots, a_{n-1}) + (-1)^{n-1}\psi(a_0 a_{n-1}^{\circ}, 1, a_1, \cdots, a_{n-2})$$
$$= \psi(a_0a_1, 1, a_2, \cdots, a_{n-1}).$$

(2) Since $(\#_\alpha \varphi)\#\mathrm{Tr}$ is a cyclic cocycle on $\mathfrak{C} \otimes M_2(\mathbb{C})$, we have the equality

$$\eta\#_\alpha\varphi(a_0, a_1, \cdots, a_{n-1}) = \int_0^{\pi/2} h_J((\#_\alpha\varphi)\#\mathrm{Tr}) \circ \alpha_\theta) d\theta|_{\mathfrak{C}\otimes e_{11}}(a_0, a_1, \cdots, a_{n-1})$$
$$= nN_a \int_0^{\pi/2} d\theta((\#_\alpha\varphi)\#\mathrm{Tr}) \circ \alpha_\theta(\bar{a}_0, \cdots, \bar{a}_{n-1}, J),$$

where we denote by N_a the cyclic antisymmetrization operator in the variables a_j and J. Using (1) and (5) of the lemma above, we get

$$\int_0^{\pi/2} d\theta((\#_\alpha\varphi)\#\mathrm{Tr}) \circ \alpha_\theta(\bar{a}_0 d\bar{a}_1 \cdots d\bar{a}_{n-1} DJ + \bar{a}_0 d\bar{a}_1 \cdots d\bar{a}_{n-2} D\bar{a}_{n-1} dJ)$$
$$= (-1)^n \varphi(a_0 da_1 \cdots da_{n-1}) + \frac{1}{2}(-1)^{n-2}\varphi((a_0 da_1 \cdots da_{n-2}) a_{n-1} d1).$$

But we have $\varphi(a_0 da_1 \cdots da_{n-2}) a_{n-1} d1)$

$$= \varphi(a_0 da_1 \cdots da_{n-2} d(a_{n-1}1)) - \varphi(a_0 da_1 \cdots da_{n-1}1) = 0$$

since $d(a_{n-1}1) = da_{n-1}1 + a_{n-1}d1$, with $d(a_{n-1}1) = d(a_{n-1}) = d(a_{n-1})1$. (But the second term in the first expression could be unnecessary.) Hence

$$\eta\#_\alpha\varphi = (-1)^n N\varphi = (-1)^n n\varphi.$$

□

1.11 The long exact sequence

We have the four (induced) cochain maps: $k : M^*_{n-1} \to (Q^*_n)_{\text{hom}}$ and $h : (Q^*_n)_{\text{hom}} \to M^*_{n-1}$, where defined in the subsection 1.4 are $h : \mathfrak{D}M_n \to Q_{n+1}$ and $k : Q_m \to \mathfrak{D}M_{m-1}$ (not π), and $\#^\sim : M^*_{n-1} \to (L^*_n)_{\text{hom}}$ and $\eta : (Q^*_n)_{\text{hom}} \to M^*_{n-1}$, which are defined in the subsections 1.7 and 1.10, respectively, and $\pi : M^*_{n-1} \to (L^*_n)_{\text{hom}}$ as in the subsection 1.7. Moreover, in cohomology, the induced maps satisfy $h\pi = \eta\pi = \text{id}$ and $\pi = \#^\sim$, while $h\pi = \text{id}$ on the cochain level.

Now let us look at

$$(\pi h)\#^\sim\eta(\pi h) = \pi(h\#^\sim)(\eta\pi)h = \pi h,$$

which holds on the cohomology level. Since πh is the identity on the cohomology level, and all maps considered commute with b as cochain maps, we can find a cochain homotopy $\bar\rho : (Q^*_n)_{\text{hom}} \to (Q^*_{n-1})_{\text{hom}}$, $(n > 1)$ such that $\#^\sim\eta = \text{id} - b\bar\rho - \bar\rho b$. Define $\bar B : (Q^*_n)_{\text{hom}} \to (Q^*_{n-1})_{\text{hom}}$ by $\bar B\varphi = \frac{1}{2\pi i n(n+1)}B\varphi$.

Lemma 1.11.1. *Suppose that φ is a homogeneous cyclic cocycle on $\mathfrak{D}$ which is zero when restricted to $\mathfrak{A}$. Then $\varphi = \sum_{j=0}^{\infty} S^k\#^\sim\eta(\bar B\bar\rho)^j\varphi$ on the cochain level.*

Proof. We have $\varphi = (\#^\sim\eta + b\bar\rho + \bar\rho b)\varphi = \#^\sim\eta\varphi + b\bar\rho\varphi$. In particular, $b\bar\rho\varphi$ is cyclic, and hence, $b\bar\rho\varphi = S\bar B\bar\rho\varphi$ and $\varphi = \#^\sim\eta\varphi + S\bar B\bar\rho\varphi$ (where $SB = n(n+1)b$ by Connes [1, Lemma 23 at p. 201], so that $S\bar B = \frac{1}{2\pi i}b$). Note that $\bar B\bar\rho\varphi$ again is a homogeneous cyclic cocycle vanishing on $\mathfrak{A}$. By induction on j we get

$$(\bar B\bar\rho)^j\varphi = \#^\sim\eta(\bar B\bar\rho)^j\varphi + S(\bar B\bar\rho)^{j+1}\varphi.$$

Indeed, check that

$$\begin{aligned}(\bar B\bar\rho)^{j+1}\varphi = (\bar B\bar\rho)^j(\bar B\bar\rho\varphi) &= \#^\sim\eta(\bar B\bar\rho)^j(\bar B\bar\rho\varphi) + S(\bar B\bar\rho)^{j+1}(\bar B\bar\rho\varphi)\\ &= \#^\sim(\bar B\bar\rho)^{j+1}\varphi + S(\bar B\bar\rho)^{j+2}\varphi.\end{aligned}$$

Acting on both sides by S^j and summing over $j \geq 0$ we get

$$\sum_{j\geq 0} S^j(\bar B\bar\rho)^j\varphi = \sum_{j\geq 0} S^j\#^\sim\eta(\bar B\bar\rho)^j\varphi + \sum_{j\geq 0} S^{j+1}(\bar B\bar\rho)^{j+1}\varphi,$$

where the sums are finite for dimensional reasons, and hence we obtain $\varphi = \sum_{j\geq 0} S^j \#^\sim \eta(\bar{B}\bar{\rho})^k\varphi$. □

Theorem 1.11.2. *The following long cohomology sequence is exact:*

$$\cdots \xrightarrow{\delta} H_\lambda^{n-1}(\mathfrak{A}) \xrightarrow{\#} H_\lambda^n(\mathfrak{C})_{\text{hom}} \xrightarrow{i} H_\lambda^n(\mathfrak{A}) \xrightarrow{\delta} \cdots .$$

Proof. Step 1. $i \circ \# = 0$. This follows directly from the definition of $\#$, because $\# = (-1)^n n\#^\sim$, and $\#^\sim = \pi$ in cohomology, and $i \circ \pi = 0$. Hence we get $\mathrm{Im}(\#) \subset \ker(i)$.

Step 2. $\ker(i) \subset \mathrm{Im}(\#)$. Suppose that φ is a homogeneous cyclic n-cocycle on $\mathfrak{D}$ and $i[\varphi] = 0$ in $H_\lambda^n(\mathfrak{A})$. Then there is a cyclic $(n-1)$-cochain λ on $\mathfrak{A}$ such that $\varphi|_{\mathfrak{A}} = b\lambda$. Set $\lambda^\sim(a_0u^{m_0}, \cdots, a_{n-1}u^{m_{n-1}}) = \lambda(a_0, \cdots, a_{n-1})\delta_{m_0,0}\cdots\delta_{m_{n-1},0}$ for $a_i \in \mathfrak{A}$. Then $\lambda^\sim$ defines a cyclic element of $(Q_{n-1}^*)_{\text{hom}}$ and $(\varphi - b\lambda^\sim)|_{\mathfrak{A}} = 0$. By the lemma above, we get

$$\varphi - b\lambda^\sim = \sum_{k\geq 0} S^k \#^\sim \eta(\bar{B}\bar{\rho})^k\varphi.$$

But $\mathrm{Im}(S^k\#^\sim) = \mathrm{Im}(S^k\#) \subset \mathrm{Im}(\#)$ for $k \geq 0$. Indeed, by definition, $\#\psi = (-1)^n n\#^\sim\psi$ for an $(n-1)$-cochain ψ, and $(-1)^n n\#^\sim = \#_\alpha$ in cyclic cohomology level and $\#_\alpha$ commutes with S, as obtained before. Thus we obtain $\varphi - b\lambda^\sim \in \mathrm{Im}(\#)$. Hence the class of φ is contained in the image under $\#$ in cyclic cohomology.

Step 3. We have $\#\delta = 0$, because $\#\delta = (-1)^n n\#^\sim\delta = \#_\alpha\delta = 0$ in cyclic cohomology.

Step 4. $\ker(\#) \subset \mathrm{Im}(\delta)$. Suppose that φ is a cyclic cocycle on $\mathfrak{A}$ such that $\#\varphi = b\lambda$ for λ a cyclic cochain. Since $\#\varphi$ is homogeneous, we can assume that λ is homogeneous as well. But then we have

$$\varphi = \eta\#\varphi = \eta b\lambda = b\eta\lambda \pm \delta(\lambda|_{\mathfrak{A}}).$$

Step 5. $\delta i = 0$. This can be deduced from the fact that inner automorphisms act trivially on the level of cyclic cohomology, by Connes. Alternatively, given a cyclic cocycle φ on $\mathfrak{D}$, we have the equality $\delta(\varphi|_{\mathfrak{A}}) = \pm b\eta\varphi$.

Step 6. $\ker(\delta) \subset \mathrm{Im}(i)$. Suppose that φ is a cyclic cocycle on $\mathfrak{A}$ such that $\delta\varphi = b\lambda$ for λ a cyclic cochain on $\mathfrak{A}$. Set $\varphi^\sim = \frac{1}{n}(\sum_i (-1)^i \rho_i\varphi - \#\lambda)$. Then $\varphi^\sim$ is cyclic on $\mathfrak{A}$, $\varphi^\sim|_{\mathfrak{A}} = \varphi$, and

$$\begin{aligned} nb\varphi^\sim &= b(\sum_i (-1)^i \rho_i\varphi) - b\#\lambda \\ &= \#_\alpha\delta\varphi - \#b\lambda = 0, \end{aligned}$$

where note that $\# = \#_\alpha$ on cyclic cocycles and the identity $b(\sum_i (-1)^i\rho_i\varphi) = \#_\alpha\delta\varphi$ is obtained in the subsection 1.9. □

1.12 Periodic cyclic cohomology of the smooth crossed product

Theorem 1.12.1. *The following sequence is exact:*

$$\begin{array}{ccccc} HC^{\mathrm{ev}}(\mathfrak{A}) & \xrightarrow{\#} & HC^{\mathrm{odd}}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) & \xrightarrow{i} & HC^{\mathrm{odd}}(\mathfrak{A}) \\ {\scriptstyle 1-\alpha}\uparrow & & & & \downarrow{\scriptstyle 1-\alpha} \\ HC^{\mathrm{ev}}(\mathfrak{A}) & \xleftarrow{i} & HC^{\mathrm{ev}}(\mathfrak{D}) & \xleftarrow{\#} & HC^{\mathrm{odd}}(\mathfrak{A}), \end{array}$$

where $HC^{\mathrm{ev}}(\cdot)$ and $HC^{\mathrm{odd}}(\cdot)$ are the even and odd parts of $HC(\cdot)$ respectively, with $HC(\cdot) = HC^{\mathrm{ev}}(\cdot) \oplus HC^{\mathrm{odd}}(\cdot) = \varinjlim HC^{2n}(\cdot) \oplus \varinjlim HC^{2n+1}(\cdot)$.

Proof. We have

$$HC^{\mathrm{ev}}(\mathfrak{D}) \text{ or } HC^{\mathrm{odd}}(\mathfrak{D}) = \varinjlim_{n=n(k)} S^k H^n_\lambda(\mathfrak{D}) = \varinjlim_{n=n(k)} S^k H^n_\lambda(\mathfrak{D})_{\mathrm{hom}},$$

since $H^n_\lambda(\mathfrak{D})_e \subset \ker(S)$ as shown before. Hence it suffices to look at the homogeneous cyclic cohomology of the crossed product $\mathfrak{D}$. Let us look at the diagram:

$$\begin{array}{ccccccccc} \xrightarrow{\delta} & H^{n-1+2k}_\lambda(\mathfrak{A}) & \xrightarrow{\#} & H^{n+2k}_\lambda(\mathfrak{D})_{\mathrm{hom}} & \xrightarrow{i} & H^{n+2k}_\lambda(\mathfrak{A}) & \xrightarrow{\delta} \\ & {\scriptstyle S^k}\uparrow & & {\scriptstyle S^k}\uparrow & & {\scriptstyle S^k}\uparrow & \\ \xrightarrow{\delta} & H^{n-1}_\lambda(\mathfrak{A}) & \xrightarrow{\#} & H^n_\lambda(\mathfrak{D})_{\mathrm{hom}} & \xrightarrow{i} & H^n_\lambda(\mathfrak{A}) & \xrightarrow{\delta} \end{array}$$

where this is commutative because S commutes with $\#$, and the rows are exact by the long exact sequence in the last subsection, and $\delta = 1 - \alpha$ in cohomology. Now a straightforward diagram chase proves the desired result. □

1.13 Coupling with K-theory

Let $\mathfrak{A}$ be a Fréchet algebra, nuclear as a topological vector space. Denote by $\mathfrak{A}^+$ the unitization of $\mathfrak{A}$.

When $\mathfrak{A}$ has no unit, its K_0-group $K_0(\mathfrak{A})$ is defined to be the Grothendieck group of stable equivalence classes of projections in matrix algebras over $\mathfrak{A}^+$. When $\mathfrak{A}$ has the unit, we take $K_0(\mathfrak{A})$ as the kernel of the K-theory homomorphism induced by the injection from $\mathfrak{A}$ to $\mathfrak{A}^+$.

We define $K_1(\mathfrak{A})$ as a quotient of

$$GL_\infty(\mathfrak{A}) = \{v \in GL_\infty(\mathfrak{A}^+) = \cup_n GL_n(\mathfrak{A}^+) \mid v \equiv 1 \bmod \mathfrak{A}\}$$

by the continuous, piecewise C^1 equivalence relation $\sim_{C^1}$, given by that $w_1 \sim_{C^1} w_2$ if and only if there is a continuous, piecewise C^1 path $[0,1] \ni t \mapsto p_t \in GL_\infty(\mathfrak{A})$ such that $p_0 = w_1$ and $p_1 = w_2$.

Denote by $\langle \cdot, \cdot \rangle$ the pairing between $K_0(\mathfrak{A})$ and $HC^{\mathrm{ev}}(\mathfrak{A})$ and between $GL_\infty(\mathfrak{A})$ and $HC^{\mathrm{odd}}(\mathfrak{A})$ constructed by Connes, where we extend cocycles on $\mathfrak{A}$ to those on $\mathfrak{A}^+$ by setting $\varphi(1, a_1, \cdots, a_n) = 0$.

Lemma 1.13.1. *That pairing $\langle \cdot, \cdot \rangle$ desends to a pairing between $K_1(\mathfrak{A})$ and $HC^{\mathrm{odd}}(\mathfrak{A})$.*

Proof. Suppose that φ is an odd-dimensional cyclic cocycle on $\mathfrak{A}$ and that $t \mapsto v_t$ is a continuous piecewise C^1 path of elements of $GL_\infty(\mathfrak{A})$. By passing to a matrix algebra over $\mathfrak{A}^+$ we can assume that $v_t \in \mathfrak{A}^+$. It is enough to show the equality: $\frac{d}{dt}\varphi(v_t^{-1}, v_t, v_t^{-1}, \dots, v_t) = 0$. Let $\varphi^\wedge$ denote the closed graded trace on $\Omega(\mathfrak{A}^+)$ corresponding to φ. Then the left-hand side is equal to

$$\frac{d}{dt}\varphi^\wedge(v_t^{-1}dv_t d(v_t^{-1})\cdots dv_t) = \varphi^\wedge((\dot{v_t^{-1}})dv_t d(v_t^{-1})\cdots dv_t)$$
$$+ \varphi^\wedge(v_t^{-1}d\dot{v}_t d(v_t^{-1})\cdots dv_t) + \cdots + \varphi^\wedge(v_t^{-1}dv_t d(v_t^{-1})\cdots d\dot{v}_t).$$

Since φ is cyclic, it is enough to show that the sum of the first two terms is zero. Indeed, check that

$$\varphi^\wedge((\dot{v_t^{-1}})dv_t\cdots dv_t) + \varphi^\wedge(v_t^{-1}d\dot{v}_t\cdots dv_t)$$
$$= -\varphi^\wedge(v_t^{-1}\dot{v}_t v_t^{-1}dv_t\cdots dv_t)$$
$$- \varphi^\wedge((d(v_t^{-1})v_t + (\dot{v_t^{-1}})dv_t + d(v_t^{-1})\dot{v}_t)d(v_t^{-1})\cdots dv_t)$$
$$= -\varphi^\wedge(d(\dot{v_t^{-1}})v_t d(v_t^{-1})\cdots dv_t) - \varphi^\wedge(d(v_t^{-1})\dot{v}_t)d(v_t^{-1})\cdots dv_t) = 0,$$

where $1 = v_t v_t^{-1}$ implies $0 = \dot{v}_t v_t^{-1} + v_t(\dot{v_t^{-1}})$, so that $(\dot{v_t^{-1}}) = -v_t^{-1}\dot{v}_t v_t^{-1}$, and also $1 = v_t^{-1}v_t$ implies $0 = (\dot{v_t^{-1}})v_t + v_t^{-1}\dot{v}_t$, so that

$$v_t d\dot{v}_t = -d(\dot{v_t^{-1}})v_t - (\dot{v_t^{-1}})dv_t - d(v_t^{-1})\dot{v}_t,$$

and we use of $\varphi^\wedge$ being a trace.

Hence $\varphi(v_t^{-1}, v_t, v_t^{-1}, \cdots, v_t)$ is a constant with respect to t, and one can also check that $b\varphi(v_t^{-1}, v_t, v_t^{-1}, \cdots, v_t) = 0$, so that the bilinear map is defined by such a constant for a pair of classes in K_1 and HC^{odd}. □

Lemma 1.13.2. *Let φ be a cyclic cocycle on $\mathfrak{A}$ and ω a cyclic cocycle on $\mathfrak{B}$ either a matrix algebra over $\mathbb{C}$ or $C(\mathbb{T})$. Then $\#_{\alpha\otimes\mathrm{id}}(\varphi\#\omega) = (\#_\alpha\varphi)\#\omega$.*

Proof. It is enough to note that both sides of the equality stated are computed by columns of the commutative diagram:

$$\begin{array}{ccc}
\Omega((\mathfrak{A}\otimes\mathfrak{B})\rtimes_{\alpha\otimes\mathrm{id}}[u,u^{-1}]) & = & \Omega((\mathfrak{A}\rtimes_{\alpha}[u,u^{-1}])\otimes\mathfrak{B}) \\
\downarrow & & \downarrow \\
E\otimes_{\alpha\otimes\mathrm{id}}\Omega(\mathfrak{A}\otimes\mathfrak{B}) & & \Omega(\mathfrak{A}\rtimes_{\alpha}[u,u^{-1}])\otimes\Omega(\mathfrak{B}) \\
\downarrow & & \downarrow \\
E\otimes_{\alpha\otimes\mathrm{id}}(\Omega(\mathfrak{A})\otimes\Omega(\mathfrak{B})) & \longrightarrow & (E\otimes_{\alpha}\Omega(\mathfrak{A}))\otimes\Omega(\mathfrak{B}),
\end{array}$$

where the bottom arrow is an isomorphism. The extension of both sides of the equality to continuous cocycles on respective algebras is handled as in Proposition 1.9.3. □

Let us now introduce the Bott map Bt as follows: For a projection $p \in M_k(\mathfrak{A}^+)$, define $Bt(p) = e^{2\pi i t}p + (1-p)$ an invertible-matrix valued, smooth function on $\mathbb{T}$ in $GL_k(\mathfrak{A}^+ \otimes C_0^\infty(\mathbb{T}))$, where $C_0^\infty(\mathbb{T})$ is the ideal of $C^\infty(\mathbb{T})$ of smooth functions on $\mathbb{T}$ vanishing at zero, and for an element $v \in GL_k(\mathfrak{A}^+)$, define

$$Bt(v) = v_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v_t^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

a projection-valued smooth function on $\mathbb{T}$ in $M_{2k}(\mathfrak{A}^+ \otimes C_0^\infty(\mathbb{T}))$, where the map $t \mapsto v_t$ is any continuous piecewise C^∞ path inside $M_2(\mathbb{C}[v, v^{-1}])$ connecting the diagonal matrix $v \oplus v^{-1}$ to the 2×2 identity matrix $1 \oplus 1$. Note that the Bott map Bt descends to K-theory homomorphisms: $Bt : K_j(\mathfrak{A}) \to K_{j+1}(S^\infty\mathfrak{A})$ for $j = 0, 1$, where $S^\infty\mathfrak{A} = \mathfrak{A} \otimes C_0^\infty(\mathbb{T})$, which we may call the smooth suspension of $\mathfrak{A}$. Set $(S^\infty)^{k+1}\mathfrak{A} = S^\infty((S^\infty)^k\mathfrak{A})$ inductively.

Definition 1.13.3. Define K^p-groups of $\mathfrak{A}$ by $K_j^p(\mathfrak{A}) = \varinjlim K_j((S^\infty)^{2n}\mathfrak{A})$ $(n \to \infty)$ for an inductive system $\{K_j((S^\infty)^{2n}(\mathfrak{A}))\}_{n\in\mathbb{N}}$ of abelian groups connected by even powers of Bt.

Proposition 1.13.4. *The pairing* $\langle\cdot,\cdot\rangle$ *extends to a bilinear pairing between* K^p*-groups of* $\mathfrak{A}$ *and* $HC(\mathfrak{A})$.

Proof. Applying the six-term exact sequence of HC for the smooth crossed product of $\mathfrak{A}$ by an action α of $\mathbb{Z}$ in the subsection 1.12 to $\mathfrak{A}^+ \rtimes_{\mathrm{id}} \mathbb{Z} \cong \mathfrak{A}^+ \otimes C^\infty(\mathbb{T})$, we get short exact sequences:

$$0 \to HC^i(\mathfrak{A}^+) \xrightarrow{\#} HC^{i+1}(\mathfrak{A}^+ \otimes C_0^\infty(\mathbb{T})) \xrightarrow{i} HC^{i+1}(\mathfrak{A}^+) \to 0.$$

These give us the maps $\#_{\rm id} : HC^i(\mathfrak{A}) \to HC^{i+1}(S^\infty\mathfrak{A})$. It is easily seen that $\#_{\rm id}$ is given by the shuffle product with a generator of $H^1_\lambda(C_0^\infty(\mathbb{T}))$, and we have the equality $\langle\varphi, x\rangle = \langle\#_{\rm id}\varphi, Bt(x)\rangle$ by Pimsner. Now an application of the lemmas above gives the desired result, where we have $\#^2_{\rm id}(\varphi\#{\rm Tr}) = (\#^2_{\rm id}\varphi)\#{\rm Tr}$ and hence that $\#^2_{\rm id}$ is compatible with the identifications involved in the construction of $GL_\infty(\mathfrak{A})$ and $M_\infty(\mathfrak{A})$. □

As before, $\mathfrak{A}$ is a dense unital subalgebra of unital C^*-algeba A, α is an automorphism of $\mathfrak{A}$ with $\alpha(\mathfrak{A}) = \mathfrak{A}$, and the imbedding $\mathfrak{A} \to A$ is continuous.

Note that the map $Bt^2 : K_i(A) \to K_i((S^\infty)^2A)$ is an isomorhism since $K_i(A) \cong K_i(S^2A)$ the Bott periodisity in the category of C^*-algebras, where $SA = C_0(\mathbb{R}) \otimes A$ is the usual suspension of A. Hence we have the natural maps:

$$K_i^p(\mathfrak{A}) = \varinjlim K_i((S^\infty)^{2n}\mathfrak{A}) \to \varinjlim K_i((S^\infty)^{2n}A) = K_i(A),$$
$$K_i^p(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) = \varinjlim K_i((S^\infty)^{2n}(\mathfrak{A} \rtimes_\alpha \mathbb{Z})) \to K_i(A \rtimes_\alpha \mathbb{Z}).$$

Theorem 1.13.5. *Suppose that the maps $K_j^p(\mathfrak{A}) \to K_j(A)$ $(j = 0, 1)$ are isomorphisms. Then the above maps $I_{p,j} : K_j^p(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \to K_j(A \rtimes_\alpha \mathbb{Z})$ $(j = 0, 1)$ are surjective and the pairing $\langle\cdot,\cdot\rangle$ between K^p and HC of $\mathfrak{A}$ descends to a pairing between the K-groups of the C^*-crossed product $A\rtimes_\alpha\mathbb{Z}$ and $HC(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$.*

Proof. Start with the following diagram:

$$\begin{array}{ccccccccc}
K_1^p(\mathfrak{A}) & \xrightarrow{1-\alpha} & K_1^p(\mathfrak{A}) & \longrightarrow & K_1^p(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) & \longrightarrow & K_0^p(\mathfrak{A}) & \xrightarrow{1-\alpha} & K_0^p(\mathfrak{A}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_1(A) & \xrightarrow{1-\alpha} & K_1(A) & \longrightarrow & K_1(A \rtimes_\alpha \mathbb{Z}) & \longrightarrow & K_0(A) & \xrightarrow{1-\alpha} & K_0(A)
\end{array}$$

where the bottom sequence, being a part of the six-term exact sequence of Pimsner-Voiculescu for C^*-crossed products by $\mathbb{Z}$, is exact. Apply the following result of G. Elliott and T. Natsume: the map f from the set of pairs (e, v) with e a projection of A and v a unitary of A such that $vev^{-1} = \alpha(e)$ to the unitary group of $A \rtimes_\alpha \mathbb{Z}$, defined by $f(e, v) = ue + v(1-e)$ is a right inverse for the boundary map ∂ and its range, after passing to matrix algebras over A, generates $K_1(A \rtimes_\alpha \mathbb{Z})$ as an abelian group. Check that

$$\begin{aligned}
(ue + v(1-e))^*(ue + v(1-e)) &= (eu^* + (1-e)v^*)(ue + v(1-e)) \\
&= e + eu^*v(1-e) + (1-e)v^*ue + (1-e) \\
&= 1 + ((1-e)v^*ue)^* + (1-e)v^*ue
\end{aligned}$$

and moreover, since $ev^* = v^*\alpha(e)$ we have

$$\begin{aligned}(1-e)v^*ue &= v^*ue - ev^*ue \\ &= v^*ue - v^*\alpha(e)ue \\ &= v^*\alpha(1-e)ue \\ &= v^*u(1-e)u^*ue = v^*u(1-e)e = 0.\end{aligned}$$

Since we can choose e, v in a matrix algebra over $(S^\infty)^{2n}\mathfrak{A}$ for some n by the assumption, the surjectivity of the maps $I_{p,j}$ follows.

Suppose now that φ is an odd-dimensional cyclic cocycle on $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ and that pairs $(e, v), (\bar{e}, \bar{v})$ of projections and unitaries of $\mathfrak{A}$ are such that $f(e, v) \sim f(\bar{e}, \bar{v})$ in $A \rtimes_\alpha \mathbb{Z}$. Since then $[e] = \partial[f(e, v)] = \partial[f(\bar{e}, \bar{v})] = [\bar{e}]$ in $K_0(A)$, we can, after passing to a matrix algebra over some smooth suspension $(S^\infty)^{2n}\mathfrak{A}$, suppose that there exists an invertible element $w \in \mathfrak{A}$ such that $wew^{-1} = \bar{e}$. We set

$$X = \alpha(e) + \alpha(w^{-1})\bar{v}wv^{-1}(1 - \alpha(e)),$$

then a straightforward calculation gives

$$f(\bar{e}, \bar{v}) = \alpha(w)Xf(e, v)w^{-1},$$

where in the both equations above, $\alpha(w^{-1})$ and $\alpha(w)$ are corrected from $\alpha^{-1}(w^{-1})$ and $\alpha^{-1}(w)$ in the text, respectively. Indeed, we check that $\alpha(w)Xf(e, v)w^{-1}$

$$\begin{aligned}&= \alpha(w)(\alpha(e) + \alpha(w^{-1})\bar{v}wv^{-1}(1 - \alpha(e))(ue + v(1-e))w^{-1} \\ &= \alpha(w)\alpha(e)uew^{-1} + \alpha(w)\alpha(e)v(1-e)w^{-1} \\ &+ \bar{v}wv^{-1}(1 - \alpha(e))uew^{-1} + \bar{v}wv^{-1}(1 - \alpha(e))v(1-e)w^{-1} \\ &= (uwu^*)ueu^*uew^{-1} + \alpha(w)vev^*v(1-e)w^{-1} \\ &+ \bar{v}wv^{-1}(1 - ueu^*)uew^{-1} + \bar{v}wv^{-1}(1 - vev^*)v(1-e)w^{-1} \\ &= uwew^{-1} + \alpha(w)ve(1-e)w^{-1} \\ &+ \bar{v}wv^{-1}u(1-e)ew^{-1} + \bar{v}wv^{-1}v(1-e)(1-e)w^{-1} \\ &= u\bar{e} + \bar{v}(1-\bar{e}) = f(\bar{e}, \bar{v}).\end{aligned}$$

Moreover, $X \sim 1$ in (matrix algebras over, corrected) $A \rtimes_\alpha \mathbb{Z}$, because

$$\begin{aligned}f(e, v) \sim f(\bar{e}, \bar{v}) &= \alpha(w)Xf(e, v)w^{-1} \\ &\sim wXf(e, v)w^{-1} \sim Xf(e, v)\end{aligned}$$

so that there is a unitary y such that $yf(e,v)y^* = Xf(e,v)$, and thus $X = yf(e,v)y^*f(e,v)^* \sim f(e,v)f(e,v)^* = 1$ in the corrected case, where we use a fact that unitary equivalence is equivalent to homotopy equivalence in matrix algebras over $A \rtimes_\alpha \mathbb{Z}$ (but is not equivalent in general in $A \rtimes_\alpha \mathbb{Z}$) and we need to assume that w is a unitary in this case. If we use unitary equivalence only, we just get

$$\begin{aligned} f(e,v) \sim f(\bar{e},\bar{v}) &= \alpha(w)Xf(e,v)w^{-1} \\ &= uwu^*Xf(e,v)w^{-1} \sim wu^*Xf(e,v)w^{-1}u \end{aligned}$$

so that there is a unitary y such that $yf(e,v)y^* = wu^*Xf(e,v)w^{-1}u$, and thus $X = uw^{-1}yf(e,v)y^*u^*wf(e,v)^* \sim 1$ (not yet checked).

And hence $[X] \in \mathrm{Im}(1-\alpha)|_{K_1(A)} = \mathrm{Im}(1-\alpha)|_{K_1^p(\mathfrak{A})}$, because check that

$$\begin{aligned} X &= \alpha^{-1}(w)f(\bar{e},\bar{v})ws(e,v) \sim \alpha^{-1}(w)f(\bar{e},\bar{v})wf(\bar{e},\bar{v}) \\ &\sim \alpha^{-1}(w)w \end{aligned}$$

so that we have $[X] = [\alpha^{-1}(w)w] = [w\alpha^{-1}(w)] = [w][(\alpha(w))^{-1}] = (1-\alpha)[w]$. Since $\varphi|_{\mathfrak{A}}$ is α-invariant in cyclic cohomology, we get the equality $\langle\varphi, f(e,v)\rangle = \langle\varphi, f(\bar{e},\bar{v})\rangle$. This implies that $\langle\cdot,\cdot\rangle$ descends to $K_1(A \rtimes_\alpha \mathbb{Z})$. Note that $f(\bar{e},\bar{v}) = \alpha(w)Xf(e,v)w^{-1} \sim \alpha(w)\alpha(w)^{-1}wf(e,v)w^{-1} = wf(e,v)w^{-1} \sim f(e,v)$.

To deal with the K_0-case, note that the K^p-groups satisfy the Bott isomorphism $Bt : K_j^p(\mathfrak{A}) \xrightarrow{\cong} K_{j+1}^p(S^\infty\mathfrak{A})$, and hence it suffices to apply the K_1-case dealt with above to $S^\infty\mathfrak{A}$ in the diagram

$$\begin{array}{ccccc} K_1^p(S^\infty(\mathfrak{A} \rtimes_\alpha \mathbb{Z})) & \xrightarrow{\cong} & K_0^p(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) & \xrightarrow{\cong} & K_1^p(S^\infty(\mathfrak{A} \rtimes_\alpha \mathbb{Z})) \\ \downarrow & & \downarrow & & \downarrow \\ K_1(S(A \rtimes_\alpha \mathbb{Z})) & \xrightarrow{\cong} & K_0(A \rtimes_\alpha \mathbb{Z}) & \xrightarrow{\cong} & K_1(S(A \rtimes_\alpha \mathbb{Z})) \end{array}$$

and note that the pairing $\langle\cdot,\cdot\rangle$ commutes with the Bott map and that $\langle\varphi, p\rangle = \langle\#_{\mathrm{id}}\varphi, Bt(p)\rangle$. □

Proposition 1.13.6. *Suppose that the maps $K_j^p(\mathfrak{A}) \to K_j(A)$ $(j = 0,1)$ are isomorphisms. Then the maps $\partial : K_j(A \rtimes_\alpha \mathbb{Z}) \to K_{j+1}(A)$ $(i = 0,1)$ and $\# : HC^j(\mathfrak{A}) \to HC^{j+1}(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$ are dual to each other.*

Proof. Since the pairing $\langle\cdot,\cdot\rangle$ is compatible with the maps Bt and $\#_{\mathrm{id}}$ and since $\#\#_{\mathrm{id}} = \#_{\mathrm{id}}\#$, which means that $\#_\alpha$ is a shuffle product, it is enough to show that

$$\frac{1}{2\pi i}\langle\#\varphi, [f(e,v)]\rangle = \langle\varphi, [e]\rangle$$

holds for $\varphi \in H_\lambda^{2n+1}(\mathfrak{A})$ and e a projection of A and v a unitary of A.

Suppose first that $v = 1$, i.e., $ueu^* = e$, because

$$1 = f(e,v)f(e,v)^* = (ue + (1-e))(ue + (1-e))^* = ueu^* + (1-e).$$

Set $Y = ue + 1 - e$. Then we have $\#\varphi(Y^{-1}-1, Y-1, Y^{-1}-1, \cdots, Y-1) =$

$$\sum_{k=0}^{n-1} \#_\alpha\varphi((u^*-1)^{k+1}(u-1)^k(du)(u^*-1)^{n-k}(u-1)^{n-k}e(de)^{2k}e(de)^{2(n-k)}) +$$
$$\sum_{k=1}^{n} \#_\alpha\varphi((u^*-1)^k(u-1)^{k+1}(du^{-1})(u-1)^{n-k}(u-1)^{n-k}e(de)^{2k-1}e(de)^{2(n-k)+1})$$

where note that $\#\varphi = \#_\alpha\varphi$ and also $Y^{-1}-1 = e(u-1)$ and $Y-1 = (u-1)e$. Check indeed that when $n = 0$, we have

$$\begin{aligned}&\#\varphi(Y^{-1}-1, Y-1) = \#_\alpha\varphi((u^*-1)e(du)e)\\ &= \#\varphi_\alpha((u^*-1)(u-1)^0(du)(u^*-1)^0(u-1)^0e(de)^0e(de)^0),\end{aligned}$$

where $e(du) = (du)u^*eu = (du)e$, and when $n = 1$ consider the first term of $\#_\alpha\varphi(Y^{-1}-1, Y-1, Y^{-1}-1, Y-1)$ as follows:

$$\begin{aligned}&\#_\alpha\varphi((u^*-1)e(du)ed((u^*-1)e)d((u-1)e))\\ &= \#_\alpha\varphi((u^*-1)e(du)e((du^*)e - (u^*-1)de)((du)e - (u-1)de))\\ &= \#_\alpha\varphi((u^*-1)e(du)e(u^*-1)(de)(u-1)(de))\\ &= \#_\alpha\varphi((u^*-1)(du)(u^*-1)(u-1)e(de)(de))\\ &= \#_\alpha\varphi((u^*-1)^{0+1}(u-1)^0(du)(u^*-1)^{1-0}(u-1)^{1-0}e(de)^0e(de)^{2-0}).\end{aligned}$$

Using the identities $e(de)^{2k}e = e(de)^{2k}$ and $e(de)^{2k+1}e = 0$ and the α-invariance of α we get $\#\varphi(Y^{-1}-1, Y-1, \cdots, Y-1)$

$$= (n+1)\#_\alpha\varphi((u^*-1)^{n+1}(u-1)^n(du)e(de)^{2n}) = (n+1)\binom{2n+1}{n}\varphi(e, \cdots, e),$$

and the result follows in this case.

Note that since $e = e^2$, we have $de = (de)e + e(de)$, which implies $e(de)e = 0$. Also,

$$\begin{aligned}(de)^2 &= ((de)e + e(de))^2\\ &= (de)e(de)e + (de)e(de) + e(de)^2e + e(de)e(de)\\ &= (de)e(de) + e(de)^2e,\end{aligned}$$

and hence $e(de)^2 = e(de)^2e$. Moreover, if we assume that $e(de)^{2k}e = e(de)^{2k}$, then

$$\begin{aligned} e(de)^{2k+2}e &= e(de)^{2k}(de)^2e \\ &= e(de)^{2k}e(de)^2e = e(de)^{2k}e(de)^2 \\ &= e(de)^{2k}(de)^2 = e(de)^{2k+2}. \end{aligned}$$

Thus, $e(de)^{2k+1}e = e(de)^{2k}(de)e = e(de)^{2k}e(de)e = 0$. And the combination corresponds to the number of the terms with $u^{-1}(du)e(de)^{2n}$, since the other terms are mapped to zero by $\#_\alpha\varphi$ by its definition. For instance, if $n = 1$, then $\binom{3}{1} = 3$ and we have

$$(u^{-1} - 1)^2(u - 1) = (u^{-2} - 2u^{-1} + 1)(u - 1) = -u^{-2} + 3u^{-1} - 3 + u.$$

In general, since $\#\varphi|_{\mathfrak{A}} = 0$ as a cochain, we can suppose that Y has the form $Y = uwe + 1 - e$ with $uwe = euw$ and that there is a C^∞-path of invertibles $w_t \in \mathfrak{A}$ such that $w_0 = w$ and $w_1 = 1$. Note that $ue + v(1 - e) \sim v^*(ue + v(1 - e)) = v^*ue + (1 - e)$ since $v^* \sim 1$, and we may exchange the roles of u and v^*. Applying the homotopy invariance of cyclic cohomology proved by Connes to the family of homomorphisms $\rho_t : \mathfrak{A} \rtimes_\alpha [u, u^*] \to \mathfrak{A} \rtimes_\alpha \mathbb{Z}$ defined by $\rho_t(u) = uw_t$ and $\rho_t(a) = w_t^{-1}aw_t$, we get, by the above case, that

$$\frac{1}{2\pi i}\langle \#\varphi, Y\rangle = \frac{1}{2\pi i}\langle \#\varphi, [w^{-1}ew]\rangle = \langle \varphi, [e]\rangle$$

as well. Check that

$$\begin{aligned} ue + (1 - e) &= uw_1w_1^{-1}ew_1 + 1 - w_1^{-1}ew_1 \\ &\sim uw_t(w_t^{-1}ew_t) + 1 - w_t^{-1}ew_t = \rho_t(u)\rho_t(e) + \rho_t(1 - e) \\ &\sim \rho_0(u)\rho_0(e) + \rho_0(1 - e) = uw(w^{-1}ew) + 1 - (w^{-1}ew), \end{aligned}$$

and the unitary equivalence class is the same as the homotopy equivalence class in K-theory. □

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