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THE CYCLIC COHOMOLOGY THEORY FOR SMOOTH ALGEBRA CROSSED PRODUCTS BY THE GROUP OF REALS — AN OVERVIEW

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Abstract

We review and study the cyclic cohomology theory for smooth algebra crossed products by actions of the group of reals, which is obtained by Elliott, Natsume, and Nest.

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1 Introduction

We review and study the cyclic cohomology theory for smooth algebra crossed products by actions of the group of reals, which is obtained by Elliott, Natsume, and Nest [3]. This is a continuation of the paper contained in this volume of Ryukyu Mathematical Journal. This paper is exactly based on the paper [3]. We made some extended effort to understand the contents by reading and writing on the straight way, and to become some more detailed and interpreted and slightly be corrected from, possibly, misprints, but in a limited number of places because of the time extended limited. It is hoped that our effort here would not be in vain.

Without mentioning we refer to [2] of Connes for some details in the texts. Refer also to [4] of Natsume and Moriyoshi.

The texts are started as below.

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It is proved by Connes in [1] that for the crossed product $A \rtimes_\alpha \mathbb{R}$ of a C^*-algebra A by an action α of $\mathbb{R}$ of reals, associated to a C^*-dynamical system $(A, \alpha, \mathbb{R})$, there is a K-theory isomorphism:

$$\Phi_\alpha^j : K_j(A) \to K_{j+1}(A \rtimes_\alpha \mathbb{R}),$$

$j+1$ (mod 2), called the Connes' Thom isomorphism. It is also shown by him that, given an α-invariant trace τ on A, with the dual trace $\tau^\wedge$ on $A \rtimes_\alpha \mathbb{R}$, the equality

$$\tau^\wedge(\Phi_\alpha^1([u])) = \frac{1}{2\pi i}\tau(u^*\delta(u))$$

holds for any unitary u in the domain of the infinitesimal generator δ of α, where $\delta(u) = \lim_{t\to 0} \frac{1}{t}(\alpha_t(u) - u)$, and $\tau^\wedge(f) = \tau(f(0))$ for $f \in A \rtimes_\alpha \mathbb{R}$. The right hand side of the above equality is viewed as the pairing between a unitary and a cyclic one-cocycle, and $\tau^\wedge$ is a zero-cocycle. Therefore, the above equality reveals a certain relation between cyclic cocycles on an algebra and those on its crossed product by $\mathbb{R}$.

In this paper we consider the relation between the cyclic cohomology theory of a smooth algebra and that of its crossed product by $\mathbb{R}$.

Given a Fréchet algebra $\mathfrak{A}$ and automorphisms α_t of $\mathfrak{A}$ for $t \in \mathbb{R}$ to make a Fréchet (or smooth) dynamical system satisfying certain smoothness conditions, we define the smooth (or Fréchet) algebra crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{R}$. The main result is as follows.

Theorem 1.1. *There is a map in cyclic cohomology* (*denoted by double notations*)

$$\#_\alpha : H_\lambda^*(\mathfrak{A}) = HC^*(\mathfrak{A}) \to H_\lambda^{*+1}(\mathfrak{A} \rtimes_\alpha \mathbb{R})$$

which commutes with the operator S and implies the isomorphisms

$$HC^{\mathrm{ev}}(\mathfrak{A}) \cong HC^{\mathrm{odd}}(\mathfrak{A} \rtimes_\alpha \mathbb{R}),$$
$$HC^{\mathrm{odd}}(\mathfrak{A}) \cong HC^{\mathrm{ev}}(\mathfrak{A} \rtimes_\alpha \mathbb{R})$$

in even and odd parts in cyclic cohomology.

Note that, by definition

$$\begin{aligned} HC^*(\mathfrak{A}) &= HC^{\mathrm{ev}}(\mathfrak{A}) \oplus HC^{\mathrm{odd}}(\mathfrak{A}), \\ HC^{\mathrm{ev}}(\mathfrak{A}) &= \varinjlim HC^{2n}(\mathfrak{A}), \quad \text{and} \\ HC^{\mathrm{odd}}(\mathfrak{A}) &= \varinjlim HC^{2n+1}(\mathfrak{A}), \end{aligned}$$

where the inductive limits are defined via the S map of Connes.

It is shown in the way of the proof that

Theorem 1.2. (Stability). *Let $\mathbb{K}^\infty$ denote the algebra of smooth compact operators defined in the section* 2. *Then there is an isomorphism*

$$HC^*(\mathfrak{A}) \cong HC^*(\mathfrak{A} \otimes \mathbb{K}^\infty).$$

As a corollary of Theorem 1.1 we have

Theorem 1.3. (Bott periodicity). *There is an isomorphism*

$$HC^*(\mathfrak{A} \otimes S^\infty(\mathbb{R})) \cong HC^{*+1}(\mathfrak{A}),$$

and hence we have an isomorphism.

$$HC^*(\mathfrak{A} \otimes S^\infty(\mathbb{R}^2)) \cong HC^*(\mathfrak{A}),$$

where we may call the tensor product of $\mathfrak{A}$ with the smooth algebra $S^\infty(\mathbb{R})$ defined in the section 2 *the smooth suspension of $\mathfrak{A}$, and $S^\infty(\mathbb{R}^2) \cong S^\infty(\mathbb{R}) \otimes S^\infty(\mathbb{R})$.*

This paper is organized as follows. In Section 2, we construct the smooth crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ of a smooth algebra $\mathfrak{A}$ by a smooth action α of $\mathbb{R}$. We also define the algebra $\mathbb{K}^\infty$ of smooth compact operators and prove a smooth version of the Takesaki-Takai duality theorem for smooth crossed products by $\mathbb{R}$. In Section 3 we construct a map $\#_\alpha$ in cyclic cohomology associated to smooth crossed products by $\mathbb{R}$ and derive its basic properties. In Section 4 we prove the stability theorem, which is used in the proof of the main result given in Section 5. In Section 6 we consider the comparison of the map $\#_\alpha$ with the map Φ_α^j in K-theory defined by Connes. We obtain the following as a generalization of the equality of Connes first given above:

Theorem 1.4. *For any cyclic cocycle φ on $\mathfrak{A}$ (which may be identified with its cyclic cohomology class $[\varphi]$) and a K_j-theory class x of $\mathfrak{A}$, the following equality holds:*

$$\langle \varphi, x \rangle = \langle \#_\alpha \varphi, \Phi_\alpha^j(x) \rangle$$

as a pairing in cyclic cohomology and K-theory for $\mathfrak{A}$ and $\mathfrak{A} \rtimes_\alpha \mathbb{R}$.

Finally, in Section 7 we consider a variant of the main result in the case that the action α is not smooth in the sense of Section 2.

All the cochains considered in this paper are assumed (or can be proved) to be continuous. By the tensor product of locally convex spaces which may not be C^*-algebras we mean the complete projective tensor product.

2 Smooth crossed products by $\mathbb{R}$

Let $\mathfrak{A}$ be a Fréchet algebra with a topology given by an increasing sequence of seminorms $\|\cdot\|_n$ for $n \in \mathbb{N}$.

Definition 2.1. A homomorphism α from $\mathbb{R}$ the group of reals to the automorphism group $\mathrm{Aut}(\mathfrak{A})$ of $\mathfrak{A}$ is said to be a smooth action if the following two conditions are satisfied.

(1) For each $a \in \mathfrak{A}$, the function on $\mathbb{R}$: $t \mapsto \alpha_t(a) \in \mathfrak{A}$ is strongly infinitely differentiable, where

$$\frac{d}{dt}\alpha_t(a) = \lim_{h\to 0} \frac{\alpha_{t+h}(a) - \alpha_t(a)}{h}, \quad \frac{d^k}{dt^k}\alpha_t(a) = \frac{d}{dt}\left(\frac{d^{k-1}}{dt^{k-1}}\alpha_t(a)\right).$$

(2) For arbitrary $m, k \in \mathbb{N}$, there exist $n, j \in \mathbb{N}$ and a positive constant C such that for any $a \in \mathfrak{A}$, we have

$$\left\|\frac{d^k}{dt^k}\alpha_t(a)\right\|_m \leq C(1+t^2)^{j/2}\|a\|_n.$$

A typical example of a smooth action is given by a smooth flow on a closed C^∞-manifold X, i.e. we may let $\mathfrak{A} = C^\infty(X)$ of all smooth functions on X. An especially pertinent example is the translation on $S^\infty(\mathbb{R})$ defined below.

Notation. We denote by $S^\infty(\mathbb{R})$ the Fréchet algebra of all rapidly decreasing smooth functions on $\mathbb{R}$ with pointwise multiplication. We also denote by the same symbol $S^\infty(\mathbb{R})$ the Fréchet algebra of all rapidly decreasing smooth functions with convolution as a product, just for convenience.

It is well known that both algebras can be identified with by Fourier transform, but we should distinguish both algabras in the texts below, and keep it in mind.

Remark. Since $S^\infty(\mathbb{R})$ is nuclear, the tensor product $S^\infty(\mathbb{R}) \otimes \mathfrak{A}$ can be viewed as an $\mathfrak{A}$-valued function space $S^\infty(\mathbb{R}, \mathfrak{A})$ on $\mathbb{R}$ for $\mathfrak{A}$ any complete Hausdorff locally convex space. If, in particular, $\mathfrak{A}$ is a Fréchet algebra as given above, then the topology on $S^\infty(\mathbb{R}, \mathfrak{A})$ is given by the seminorms defined as:

$$\|f\|_{k,m} = \sup_{t\in\mathbb{R}}(1+t^2)^{k/2}\left\|\frac{d^m}{dt^m}f(t)\right\|_k$$

for $f \in S^\infty(\mathbb{R}, \mathfrak{A})$.

Suppose that α is a smooth action of $\mathbb{R}$ on $\mathfrak{A}$. Define a jointly continuous product on $S^\infty(\mathbb{R}, \mathfrak{A})$, the α-convolution, by

$$(f * g)(t) = \int_{\mathbb{R}} f(s)\alpha_s(g(t-s))ds$$

for $f, g \in S^\infty(\mathbb{R}, \mathfrak{A})$.

Check that

$$\begin{aligned}
\frac{d}{dt}(f * g)(t) &= \lim_{h\to 0} \frac{(f * g)(t+h) - (f * g)(t)}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \int_{\mathbb{R}} f(s)\alpha_s(g(t+h-s) - g(t-s))ds \\
&= \int_{\mathbb{R}} f(s)\alpha_s \left(\lim_{h\to 0} \frac{g(t+h-s) - g(t-s)}{h} \right) ds \\
&= \int_{\mathbb{R}} f(s)\alpha_s \left(\frac{dg}{dt}(t-s) \right) ds = (f * \frac{dg}{dt})(t),
\end{aligned}$$

and thus, that

$$\begin{aligned}
\|f * g\|_{k,m} &= \sup_{t\in\mathbb{R}} (1+t^2)^{k/2} \| \frac{d^m}{dt^m}(f * g)(t) \|_k \\
&= \sup_{t\in\mathbb{R}} (1+t^2)^{k/2} \| (f * \frac{d^m}{dt^m} g)(t) \|_k
\end{aligned}$$

and moreover,

$$\begin{aligned}
\|(f * \frac{d^m}{dt^m} g)(t)\|_k &= \| \int_{\mathbb{R}} f(s)\alpha_s \left(\frac{d^m g}{dt^m}(t-s) \right) ds \|_k \\
&\le \int_{\mathbb{R}} \| f(s)\alpha_s \left(\frac{d^m g}{dt^m}(t-s) \right) \|_k ds \\
&\le \int_{\mathbb{R}} \|f(s)\|_k \cdot \|\alpha_s \left(\frac{d^m g}{dt^m}(t-s) \right) \|_k ds
\end{aligned}$$

and furthermore,

$$\|\alpha_s \left(\frac{d^m g}{dt^m}(t-s) \right) \|_k \le C(1+s^2)^{j/2} \left\| \frac{d^m g}{dt^m}(t-s) \right\|_n$$

for some $j, n \in \mathbb{N}$ and $C > 0$, and thus, the joint continuity of the α-convolution product is now ready to be proved.

Definition 2.2. Denote by $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ the Fréchet algebra $S^\infty(\mathbb{R}, \mathfrak{A})$ with the product defined above. We say that $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ is the smooth crossed prdouct of $\mathfrak{A}$ by an action α of $\mathbb{R}$.

Example 2.3. Let us consider the action γ of $\mathbb{R}$ on $S^\infty(\mathbb{R})$ by translation. This action extends to the translation action of $\mathbb{R}$ on the C^*-algebra $C_0(\mathbb{R})$ of continuous functions on $\mathbb{R}$ vanishing at infinity. The smooth crossed product $S^\infty(\mathbb{R}) \rtimes_\gamma \mathbb{R}$ is embedded as a dense subalgebra of the C^*-algebra crossed product $C_0(\mathbb{R}) \rtimes_\gamma \mathbb{R}$. Via the isomorphism between $C_0(\mathbb{R}) \rtimes_\gamma \mathbb{R}$ and the C^*-algebra $\mathbb{K}$ of all compact operators on the Hilbert space $L^2(\mathbb{R})$ of all square integrable, measurable functions on $\mathbb{R}$ (up to null sets), the smooth algebra $S^\infty(\mathbb{R}) \rtimes_\alpha \mathbb{R}$ is identified with the subalgebra $\mathbb{K}^\infty$ of $\mathbb{K}$, of Hilbert-Schmidt operators whose integral kernels belong to $S^\infty(\mathbb{R}^2)$. Furthermore, this subalgebra consists of compact operators of trace class.

Let α be a smooth action of a smooth algebra $\mathfrak{A}$. The dual action $\alpha^\wedge$ of $\mathbb{R}$ on $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ is given by

$$\alpha_s^\wedge(f)(t) = e^{2\pi i s t} f(t)$$

for $f \in \mathfrak{A} \rtimes_\alpha \mathbb{R}$, $s, t \in \mathbb{R} \cong \mathbb{R}^\wedge$. Then the action $\alpha^\wedge$ is a smooth action on $\mathfrak{A} \rtimes_\alpha \mathbb{R}$. Thus, we can define the iterated (or dual) smooth crossed product:

$$\mathfrak{A} \rtimes_\alpha \mathbb{R} \rtimes_{\alpha^\wedge} \mathbb{R}.$$

The following lemma plays a crucial role in the proof of the main result:

Lemma 2.4. (A Fréchet algebra version of Takesaki-Takai duality). *The dual smooth crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{R} \rtimes_{\alpha^\wedge} \mathbb{R}$ is isomorphic to the tensor product $\mathfrak{A} \otimes \mathbb{K}^\infty$ as a Fréchet algebra.*

Proof. The proof follows as that of the C^*-algebra case, i.e. Takai duality.

Let γ be the action of $\mathbb{R}$ on $\mathfrak{A} \otimes S^\infty(\mathbb{R}) = S^\infty(\mathbb{R}, \mathfrak{A})$ with pointwise multiplication, given by

$$(\gamma_t f)(s) = \alpha_s(f(s-t))$$

for $t, s \in \mathbb{R}$ and $f \in S^\infty(\mathbb{R}, \mathfrak{A})$. Let β be the action of $\mathbb{R}$ on $\mathfrak{A} \rtimes_{\mathrm{id}} \mathbb{R}$ with id the action associated to the identity automorphism of $\mathfrak{A}$ and with convolution, given by

$$(\beta_t f)(u) = e^{-2\pi i t u} \alpha_t(f(u)).$$

Define the following maps:

$$\begin{aligned}
&\pi : \mathfrak{A} \otimes \mathbb{K}^\infty = \mathfrak{A} \otimes (S^\infty(\mathbb{R}) \rtimes_\gamma \mathbb{R}) \to (\mathfrak{A} \otimes S^\infty(\mathbb{R})) \rtimes_\gamma \mathbb{R}, \\
\text{by}\quad &\pi(f)(s,t) = \alpha_s(f(s, s-t)), \quad \textit{and} \\
&\rho : (\mathfrak{A} \otimes S^\infty(\mathbb{R})) \rtimes_\gamma \mathbb{R} \to (\mathfrak{A} \rtimes_{\mathrm{id}} \mathbb{R}) \rtimes_\beta \mathbb{R} \\
\text{by}\quad &\rho(f)(u,t) = \int_{\mathbb{R}} f(s,t) e^{-2\pi i s u} ds, \quad \text{and} \\
&\psi : (\mathfrak{A} \rtimes_{\mathrm{id}} \mathbb{R}) \rtimes_\beta \mathbb{R} \to \mathfrak{A} \rtimes_\alpha \mathbb{R} \rtimes_{\alpha^\wedge} \mathbb{R} \\
\text{by}\quad &\psi(f)(t,u) = e^{2\pi i t u} f(u,t).
\end{aligned}$$

It is straightforward to see that the maps π, ρ, and ψ are isomorphisms as topological algebras, and hence the composed map

$$T = \psi \circ \rho \circ \pi : \mathfrak{A} \otimes \mathbb{K}^\infty \to \mathfrak{A} \rtimes_\alpha \mathbb{R} \rtimes_{\alpha^\wedge} \mathbb{R}$$

gives the required isomorphism. □

Note that for $f \in \mathfrak{A} \otimes \mathbb{K}^\infty = \mathfrak{A} \otimes (S^\infty(\mathbb{R}) \rtimes_\gamma \mathbb{R})$, we have

$$\begin{aligned}
(Tf)(t,u) &= (\psi \circ \rho \circ \pi) f(s,t) = \psi((\rho \circ \pi) f)(t,u) \\
&= e^{2\pi i t u} (\rho \circ \pi) f(u,t) \\
&= e^{2\pi i t u} \int_{\mathbb{R}} \pi(f)(s,t) e^{-2\pi i s u} ds \\
&= e^{2\pi i t u} \int_{\mathbb{R}} \alpha_s(f(s, s-t)) e^{-2\pi i s u} ds.
\end{aligned}$$

3 Construction of a map in cyclic cohomology

Let $\mathfrak{A}$ be a locally convex topological algebra. The construction of the universal differential graded algebra given by Connes in noncommutative differential topology extends to the topological case if we set

$$\begin{aligned}
\Omega_0(\mathfrak{A}) &= \mathfrak{A}, \\
\Omega_n(\mathfrak{A}) &= \mathfrak{A}^+ \otimes (\otimes^n \mathfrak{A}), \quad n \geq 1, \\
\Omega(\mathfrak{A}) &= \oplus_{n \geq 0} \Omega_n(\mathfrak{A}),
\end{aligned}$$

where $\mathfrak{A}^+$ denotes the algebra $\mathfrak{A} \oplus \mathbb{C}$ with unit adjoined, and the graded multiplication and differential are extended by continuity to the projective completions of the algebraic tensor products. Furthermore, any automorphism α of $\mathfrak{A}$ has a natural extension to an automorphism of $\Omega(\mathfrak{A})$ commuting with the differential d. In particular, when $\mathfrak{A}$ is a Fréchet algebra and α is a smooth action on $\mathfrak{A}$, it is immediate to see that for each $n \geq 0$, the extension of α acts smoothly on $\Omega_n(\mathfrak{A})$.

Let $S^\infty(\mathbb{R})$ be with convolution. Set

$$E = \Omega(S^\infty(\mathbb{R}))/ \oplus_{n\geq 2} \Omega_n(S^\infty(\mathbb{R}))$$

and then E has a differential graded algebra structure induced by the quotient map

$$\Omega(S^\infty(\mathbb{R})) \to E.$$

In what follows we assume that $\mathfrak{A}$ is a Fréchet algebra and that α is a smooth action of $\mathbb{R}$ on $\mathfrak{A}$.

Endow the space $\Omega(\mathfrak{A}) \otimes E$ with the structure of a locally convex differential graded algebra as follows.

(1) Define the differential

$$\begin{aligned} & d : \Omega(\mathfrak{A}) \otimes E \to \Omega(\mathfrak{A}) \otimes E \\ \text{by} \quad & d(\omega \otimes x) = d\omega \otimes x + (-1)^{\deg \omega} \omega \otimes dx. \end{aligned}$$

(2) Define a left E_0-module structure on $\Omega(\mathfrak{A}) \otimes E_0$ as the one induced from the product structure of

$$\Omega(\mathfrak{A})^+ \rtimes_\alpha \mathbb{R}$$

and the inclusion of $1 \otimes E_0$ into the algebra $\Omega(\mathfrak{A})^+ \rtimes_\alpha \mathbb{R}$, where E_0 in E corresponds to $\Omega_0(S^\infty(\mathbb{R})) = S^\infty(\mathbb{R})$, to make sense.

(3) Define a left E-module structure on $\Omega(\mathfrak{A}) \otimes E$ by

$$\begin{aligned} f(\omega \otimes gdh) &= (f\omega \otimes g)dh, \\ df(\omega \otimes g) &= d(f\omega \otimes g)) - fd(\omega \otimes g), \\ df(\omega \otimes gdh) &= 0. \end{aligned}$$

(4) Define the product in $\Omega(\mathfrak{A}) \otimes E$ by

$$(\omega \otimes x)(\omega_1 \otimes x_1) = \omega(x(\omega_1 \otimes x_1)).$$

The above formulae extend by continuity to $\Omega(\mathfrak{A}) \otimes E$ to have the required structure, with the differential d.

Definition 3.1. We denote by $\Omega(\mathfrak{A}) \otimes_\alpha E$ the differential graded algebra constructed above.

Following the construction above applied to $\mathfrak{A}^+ \otimes C$ and the action $\alpha \otimes \mathrm{id}$, we get a differential graded algebra

$$\Omega(\mathfrak{A}^+ \otimes C) \otimes_{\alpha\otimes\mathrm{id}} E^+$$

and a closed graded trace $\#_{\alpha\otimes\mathrm{id}}\Psi$ of degree $n+1$. Since we have a natural homomorphism

$$(\mathfrak{A} \rtimes_\alpha \mathbb{R})^+ \otimes C \to (\Omega(\mathfrak{A}^+ \otimes C) \otimes_{\alpha\otimes\mathrm{id}} E^+)^0$$

this leads to a cyclic $(n+1)$-cocycle Ψ^+ on $(\mathfrak{A} \rtimes_\alpha \mathbb{R})^+ \otimes C$ such that

$$\rho^*\Psi^+ = \#_\alpha\psi,$$

where ρ is the map from $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ into $(\mathfrak{A} \rtimes_\alpha \mathbb{R})^+ \otimes C$ defined by $x \mapsto x \otimes e_{11}$ with e_{11} the matrix of $a_{11} = 1$ and $a_{ij} = 0$ otherwise, and an application of Connes' noncommutative geomerty finishes the proof.

Note that

$$\begin{array}{ccc} B^n_\lambda(\mathfrak{A}) \subset Z^n_\lambda(\mathfrak{A}) & \xrightarrow{\#_\alpha} & Z^{n+1}_\lambda(\mathfrak{A} \rtimes_\alpha \mathbb{R}) \\ & & \| \\ Z^{n+1}_\lambda((\mathfrak{A} \rtimes_\alpha \mathbb{R})^+ \otimes C) & \xrightarrow{\rho^*} & Z^{n+1}_\lambda(\mathfrak{A} \rtimes_\alpha \mathbb{R}) \end{array}$$

and possibly, the proof above says that there is an extension in cyclic coboundaries from the top left down to the bottom left to make the diagram commute. □

According to the lemmas above the linear map $\#_\alpha$ in cyclic cocycles desends to cyclic cohomology, i.e. the map in cyclic cohomology

$$\#_\alpha : HC^n(\mathfrak{A}) = H^n_\lambda(\mathfrak{A}) \to H^{n+1}_\lambda(\mathfrak{A} \rtimes_\alpha \mathbb{R})$$

is well defined.

Lemma 3.4. *We have $\#_\alpha S = S\#_\alpha$, i.e. the map $\#_\alpha$ commutes with the S map of Connes in cyclic cohomology.*

Proof. This is obtained by an argument analogous to that in [5]. See also [6]. □

Let us denote by id the trivial action of $\mathbb{R}$ on $\mathbb{C}$. Then we have

$$\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} = S^\infty(\mathbb{R})$$

Suppose that φ is a closed graded trace of degree n on $\Omega(\mathfrak{A})$. Set, for $f \in \Omega(\mathfrak{A}) \otimes E$,

$$\begin{aligned} \#_\alpha\varphi(f) &= 2\pi i \int_{-\infty}^{\infty} dt \int_0^t ds\, \varphi(\alpha_s(f(-t,t))) \\ &\qquad\qquad \text{for } f \in \Omega_n(\mathfrak{A}) \otimes S^\infty(\mathbb{R}^2) \subset \Omega_n(\mathfrak{A}) \otimes E_1, \\ \#_\alpha\varphi(f) &= 0 \quad \text{otherwise.} \end{aligned}$$

Lemma 3.2. *$\#_\alpha\varphi$ is a continuous closed graded trace of degree $n+1$ on $\Omega(\mathfrak{A}) \otimes_\alpha E$.*

We may reformulate this lemma by a linear map of Connes

$$\#_\alpha : Z^n_\lambda(\mathfrak{A}) \to Z^{n+1}_\lambda(\mathfrak{A} \rtimes_\alpha \mathbb{R})$$

in cyclic cocycles on $\mathfrak{A}$ and $\mathfrak{A} \rtimes_\alpha \mathbb{R}$.

Remark. Note that the map $\#_\alpha$ is natural with respect to smooth actions in the sense that given two smooth actions α on $\mathfrak{A}$ and β on $\mathfrak{B}$, and an equivariant homomorphism of Fréchet algebras $\rho : \mathfrak{A} \to \mathfrak{B}$, then

$$(\rho^\wedge)^* \#_\beta = \#_\alpha (\rho^\wedge)^*$$

where $\rho^\wedge : \mathfrak{A} \rtimes_\alpha \mathbb{R} \to \mathfrak{B} \rtimes_\beta \mathbb{R}$ the homomorphism induced by ρ, and hence the following diagram commutes:

$$\begin{array}{ccc} Z^n_\lambda(\mathfrak{B}) & \xrightarrow{\#_\beta} & Z^{n+1}_\lambda(\mathfrak{B} \rtimes_\alpha \mathbb{R}) \\ {\scriptstyle (\rho^\wedge)^*}\downarrow & & \downarrow{\scriptstyle (\rho^\wedge)^*} \\ Z^n_\lambda(\mathfrak{A}) & \xrightarrow{\#_\alpha} & Z^{n+1}_\lambda(\mathfrak{A} \rtimes_\alpha \mathbb{R}). \end{array}$$

Lemma 3.3. *We have $\#_\alpha(B^n_\lambda(\mathfrak{A})) \subset B^{n+1}_\lambda(\mathfrak{A} \rtimes_\alpha \mathbb{R})$ an inclusion of cyclic coboundaries on $\mathfrak{A}$ and $\mathfrak{A} \rtimes_\alpha \mathbb{R}$.*

Proof. Let $\mathbb{B}(l^2(\mathbb{N}))$ be the C^*-algebra of all bounded operators on the Hilbert space $l^2(\mathbb{N})$. Let C denote the Banach subalgebra of $\mathbb{B}(l^2(\mathbb{N}))$ generated by the infinite matrices $(a_{ij})_{i,j\in\mathbb{N}}$ with $a_{ij} \in \mathbb{C}$ such that

(I) the set $\{a_{ij} \mid i, j \in \mathbb{N}\}$ of complex numbers a_{ij} is finite,

(II) the numbers of non zero a_{ij} per row or column are bounded.

Let $\psi \in B^n_\lambda(\mathfrak{A})$. We can extend ψ to an element $\psi^+ \in B^n_\lambda(\mathfrak{A}^+)$ canonically. Using the argument of Connes we get a cyclic n-cocycle Ψ on $\mathfrak{A}^+ \otimes C$.

with convolution, and

$$\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\mathrm{id}^\wedge} \mathbb{R} \cong \mathbb{C} \otimes \mathbb{K}^\infty \cong \mathbb{K}^\infty.$$

Let τ be the normalized trace on $\mathbb{C}$, also viewed as zero cycle on $\Omega_0(\mathbb{C}) = \mathbb{C}$ and set

$$\begin{aligned} \varepsilon &= \#_{\mathrm{id}}\tau, \\ \omega &= \#_{\mathrm{id}^\wedge}\varepsilon \end{aligned}$$

in $Z^1_\lambda(\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R})$ and $Z^2_\lambda(\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\mathrm{id}^\wedge} \mathbb{R}) = Z^2_\lambda(\mathbb{K}^\infty)$ respectively.

Proposition 3.5. *The following equalities hold:*

$$(1) \quad \varepsilon(f,g) = 2\pi i \int_{\mathbb{R}} t f(-t) g(t) dt$$

for $f, g \in S^\infty(\mathbb{R})$, *and*

$$(2) \quad \omega(f,g,h) = -2\pi\{\mathrm{Tr}(f[D,g][M,h]) - \mathrm{Tr}(f[M,g][D,h])\}$$

for $f, g, h \in \mathbb{K}^\infty$, *where* D *and* M *are the unbounded operators on* $L^2(\mathbb{R})$ *given by*

$$(D\xi)(x) = -i\frac{d\xi}{dx}(x), \quad (M\xi)(x) = x\xi(x)$$

for $\xi \in L^2(\mathbb{R})$ *respectively.*

Note that D and M can be defined on the dense differential subalgebra $S^\infty(\mathbb{R})$ of $L^2(\mathbb{R})$, and be with self-adjoint closures, such that

$$DM - MD = [D, M] = -iI$$

with I the identity operator on $S^\infty(\mathbb{R})$, the canonical commutation relation CCR, because

$$(DM - MD)\xi(x) = -i(x\xi(x))' - x(-i\xi'(x)) = -i\xi(x).$$

Proof. (1) By definition, we have

$$\begin{aligned} \varepsilon(f,g) &= (\#_{\mathrm{id}}\tau)(fdg) \\ &= 2\pi i \int_{\mathbb{R}} dt \int_0^t ds\, \tau(f(-t)g(t)) \\ &= 2\pi i \int_{\mathbb{R}} t f(-t) g(t) dt. \end{aligned}$$

(2) By the identification of $S^\infty(\mathbb{R})$ with convolution, with itself with pointwise multiplication via the Fourier transform, ε becomes the cyclic cocycle given by

$$\varepsilon(f,g) = \int f dg.$$

The dual action id$^\wedge$ also becomes the translation action γ of $\mathbb{R}$ on $S^\infty(\mathbb{R})$. The rest of the computation for the proof consists of a straightforwad chasing of the definition of $\#_\gamma \varepsilon$ and an application of the fact that $S^\infty(\mathbb{R}) \rtimes_\gamma \mathbb{R}$ acts on $L^2(\mathbb{R})$ as integral operators with kernels

$$f(s, s-t), \quad f \in S^\infty(\mathbb{R}) \rtimes_\gamma \mathbb{R} \cong S^\infty(\mathbb{R}^2).$$

□

Note that

$$\begin{aligned}\varepsilon(g,f) &= 2\pi i \int_{-\infty}^{\infty} tg(-t)f(t)dt, \quad \text{and by setting } -t = s, \\ &= 2\pi i \int_{\infty}^{-\infty} (-s)g(s)f(-s)(-ds) = -\varepsilon(f,g),\end{aligned}$$

which shows that $\varepsilon(\cdot,\cdot)$ is a cyclic one-cocycle on $S^\infty(\mathbb{R})$.

Let p be a rank one projection in $\mathbb{K}^\infty$. Then, using the equality $[D, M] = \frac{1}{i} I$, we get

$$\omega(p,p,p) = -2\pi i.$$

Check that

$$\begin{aligned}
&p[D,p][M,p] = p(Dp - pD)(Mp - pM) \\
&= p(DpMp - DpM - pDMp + pDpM) \\
&= (pDp)Mp - (pDp)M - pDMp + (pDp)M \\
&= -pDMp; \\
&p[M,p][D,p] = p(Mp - pM)(Dp - pD) \\
&= p(MpDp - MpD - pMDp + pMpD \\
&= pM(pDp) - pMpD - pMDp + pMpD \\
&= -pMDp,
\end{aligned}$$

where $Dp = Dp^2 = Dpp + pDp$ and hence $pDp = 0$, so that the subtraction of the lower from the upper becomes

$$p[D,p][M,p] - p[M,p][D,p] = -p(DM - MD)p = -p[D,M]p = -\frac{1}{i}p,$$

Note that in the case of the trivial action, we have $\mathfrak{A} \rtimes_{\mathrm{id}} \mathbb{R} \cong \mathfrak{A} \otimes S^\infty(\mathbb{R})$ and

$$\#_{\mathrm{id}}\, \rho = \varphi \# \varepsilon, \quad \varphi \in H^n_\lambda(\mathfrak{A})$$

where $\#$ in the right hand side means the cup product in cyclic cohomology.

Suppose that α is a smooth action of $\mathbb{R}$ on $\mathfrak{A}$ and that τ is an α-invariant trace on $\mathfrak{A}$. Then we define a trace $\tau^\wedge$ on $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ by

$$\tau^\wedge(f) = \tau(f(0)), \quad f \in \mathfrak{A} \rtimes_\alpha \mathbb{R}.$$

Let e be the projection of $L^2(\mathbb{R})$ onto the one-dimensional subspace spanned by a certain vector h_0 in $L^2(\mathbb{R})$ to have that $e \in \mathbb{K}^\infty$.

Define a homomorphism

$$r : \mathfrak{A} \to \mathfrak{A} \otimes \mathbb{K}^\infty, \quad a \mapsto a \otimes e.$$

Proposition 3.6. *Let δ be the derivation of $\mathfrak{A}$ associated to a smooth action α of $\mathbb{R}$ on $\mathfrak{A}$ and $T : \mathfrak{A} \otimes \mathbb{K}^\infty \to \mathfrak{A} \rtimes_\alpha \mathbb{R} \rtimes_{\alpha^\wedge} \mathbb{R}$ the isomorphism obtained above. Then*

$$(r^* T^* \#_{\alpha^\wedge} \tau^\wedge)(a_0, a_1) = \tau(a_0 \delta(a_1)).$$

Proof. Recall that

$$\begin{aligned}(T \circ r(a))(s,t) &= T(a \otimes e)(s,t)\\ &= e^{2\pi i s t} \int_{\mathbb{R}} \alpha_u(a \otimes e(u, u-s)) e^{-2\pi i u t} du\\ &= \int_{\mathbb{R}} e^{2\pi i s t} e^{-2\pi i u t} \alpha_u(a) h_0(u-s) h_0(u) du.\end{aligned}$$

Given $y_0, y_1 \in (\mathfrak{A} \rtimes_\alpha \mathbb{R}) \rtimes_{\alpha^\wedge} \mathbb{R}$ we have, by definition,

$$\begin{aligned}\#_{\alpha^\wedge} \tau^\wedge(y_0, y_1) &= 2\pi i \int_{\mathbb{R}} dt \int_0^t ds\, \tau^\wedge(\alpha_s^\wedge(y_0(-t) y_1(t)))\\ &= 2\pi i \int_{\mathbb{R}} dt \int_0^t ds\, \tau(e^{2\pi i s 0}(y_0(-t) y_1(t))(0))\\ &= 2\pi i \int_{\mathbb{R}} t \cdot \tau^\wedge(y_0(-t) y_1(t)) dt.\end{aligned}$$

Inserting that equation in this equation we get, by a routine computation,

$$\begin{aligned}(r^* T^* \#_{\alpha^\wedge} \tau^\wedge)(a_0, a_1) &= \#_{\alpha^\wedge} \tau^\wedge(T \circ r(a_0), T \circ r(a_1))\\ &= -\tau(\delta(a_0) a_1) = \tau(a_0 \delta(a_1)).\end{aligned}$$

Note that

$$
\begin{aligned}
&\tau^{\wedge}((T\circ r(a_0))(-t)(T\circ r(a_1))(t))\\
&=\tau\left(\int_{\mathbb{R}} e^{2\pi i u t}\alpha_u(a_0\otimes e(u,u))du\cdot\int_{\mathbb{R}} e^{-2\pi i u t}\alpha_u(a_1\otimes e(u,u))du\right)
\end{aligned}
$$

and then, for the first multiple in $\tau(\cdot)$, by partial integration

$$
\begin{aligned}
&2\pi i\int_{\mathbb{R}} t\,dt\int_{\mathbb{R}} e^{2\pi i u t}\alpha_u(a_0\otimes e(u,u))du\\
&=\int_{\mathbb{R}} dt\int_{\mathbb{R}}\frac{d}{du}e^{2\pi i u t}\alpha_u(a_0\otimes e(u,u))du\\
&=\int_{\mathbb{R}} dt\left([e^{2\pi i u t}\alpha_u(a_0\otimes e(u,u))]_{u=-\infty}^{\infty}-\int_{\mathbb{R}} e^{2\pi i u t}\frac{d}{du}\alpha_u(\alpha_0\otimes e(u,u))du\right)\\
&=-\int_{\mathbb{R}} dt\int_{\mathbb{R}} e^{2\pi i u t}\frac{d}{du}\alpha_u(\alpha_0\otimes e(u,u))du
\end{aligned}
$$

where the last integral by u could be related to $\delta(a_0)$, and similarly, to $\delta(a_1)$ for the second multiple instead. □

Remark. It comes from differential topology. Let X be a closed C^∞-manifold and α an action of $\mathbb{R}$ on $C^\infty(X)$ generated by a smooth vector field ξ. For any α-invariant measure μ on X, the Ruelle-Sullivan one-dimensional current C is defined by

$$C(\omega)=\int\omega(\xi)d\mu.$$

This current is, in a natural way, a cyclic one-cocycle on $C^\infty(X)$. The proposition above says that

$$\#_{\alpha^\wedge}\mu^\wedge = C.$$

Note that for $f,g\in C^\infty(X)$, with $\omega=(f,g)$,

$$
\begin{aligned}
\#_{\alpha^\wedge}\mu^\wedge(f,g)&=\mu(f\delta(g))\\
&=\int_X d\mu f(x)\lim_{h\to 0}\frac{\alpha_h(g(x))-g(x)}{h}\\
&=\int_X f(x)\xi_x(g)d\mu=\int\omega(\xi)d\mu
\end{aligned}
$$

possibly, in this sense.

4 The theorem of stability

Given an n-cycle $(\Omega = (\Omega, d), \rho, \varphi)$ over $\mathfrak{A}$, such that $\Omega = \sum_{j=0}^{n} \Omega^j$ a graded algebra over $\mathbb{C}$ with d a graded derivation such that $d^2 = 0$, $\varphi : \Omega^n \to \mathbb{C}$ a closed graded trace, and $\rho : \mathfrak{A} \to \Omega_0$ a homomorphism, the canonical extension $(\Omega^+, \rho^+, \varphi^+)$ is an n-cycle over $\mathfrak{A}^+$. Moreover, if two n-cycles $(\Omega_1, \rho_1, \varphi_1)$ and $(\Omega_2, \rho_2, \varphi_2)$ over $\mathfrak{A}$ are cobordant over $\mathfrak{A}$, then their extensions over $\mathfrak{A}^+$ cobordant over $\mathfrak{A}^+$.

Note that those cycles are cobordant over $\mathfrak{A}$ if there is a chain Ω_3 with boundary $\Omega_1 \oplus \Omega_2$ and a homomorphism $\rho_3 : \mathfrak{A} \to \Omega_3$ such that $r \circ \rho_3 = (\rho_1, \rho_2)$, where

$$\mathfrak{A} \xrightarrow{\rho_3} \Omega_3 \xrightarrow{r} \partial\Omega_3 = \Omega_1 \oplus \Omega_2 \to 0$$

The character of an n-cycle over $\mathfrak{A}$ is the $(n+1)$-linear function τ defined by

$$\tau(a_0, \cdots, a_n) = \varphi[\rho(a_0) d(\rho(a_1)) \cdots d(\rho(a_n))]$$

for $a_j \in \mathfrak{A}$.

The characters τ_1 and τ_2 of two cobordant n-cycles over $\mathfrak{A}$ satisfy

$$\tau_1^+ - \tau_2^+ = B\psi$$

for some Hochschild cocycle $\psi \in Z^{n+1}(\mathfrak{A}^+, (\mathfrak{A}^+)^*)$, and hence

$$[S\tau_1^+] = [S\tau_2^+] \in H_\lambda^{n+3}(\mathfrak{A}^+).$$

Since the operator S of Connes commutes with the restriction map

$$H_\lambda^n(\mathfrak{A}^+) \to H_\lambda^n(\mathfrak{A}),$$

it follows that

$$[S\tau_1] = [S\tau_2] \in HC^{n+3}(\mathfrak{A}).$$

This observation in particular extends the homotopy invariance of $HC^*(\mathfrak{A})$ with $\mathfrak{A}$ unital to the case of a non-unital algebra $\mathfrak{A}$.

Theorem 4.1. *Let $\mathfrak{K}$ be a locally convex topological algebra. Suppose that*

(1) *there exists an idempotent $e \in \mathfrak{K}$ and a cyclic cocycle $\omega \in Z_\lambda^{2k}(\mathfrak{K})$ such that*

$$\omega(e, \cdots, e) = -k!(2\pi i)^k,$$

(2) *the flip automorphism $\sigma \in \mathrm{Aut}(\mathfrak{K} \otimes \mathfrak{K})$ defined by*

$$\sigma(a \otimes b) = b \otimes a$$

is connected to the identity automorphism by a C^1-path of endomorphims of $\mathfrak{K} \otimes \mathfrak{K}$.

Then, for any locally convex topological algebra $\mathfrak{A}$, the map

$$HC^*(\mathfrak{A}) \to HC^*(\mathfrak{A} \otimes \mathfrak{K}), \quad [\varphi] \mapsto [\varphi \# \omega]$$

is an isomorphism in cyclic cohomology.

Proof. Define two homomorphisms

$$\begin{aligned} r_1 : \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{K} \quad &\text{by} \quad r_1(a) = a \otimes e; \\ r_2 : \mathfrak{A} \otimes \mathfrak{K} \to \mathfrak{A} \otimes \mathfrak{K} \otimes \mathfrak{K} \quad &\text{by} \quad r_2(a \otimes b) = a \otimes b \otimes e. \end{aligned}$$

We have immediately

$$\begin{aligned} (r_2)^*(\varphi \# \omega) &= S^k \varphi, \\ (r_2)^*(\mathrm{id}_{\mathfrak{A}} \otimes \sigma)^*(\varphi \# \omega) &= (r_1^* \varphi) \# \omega, \end{aligned}$$

for any $\varphi \in Z^*_\lambda(\mathfrak{A} \otimes \mathfrak{K})$. Recall that in that case

$$S : HC^n(\mathfrak{A} \otimes \mathfrak{K}) \to HC^{n+2}(\mathfrak{A} \otimes \mathfrak{K}), \quad [\varphi] \mapsto [A_s(\varphi \# \mu)],$$

with $[\mu]$ a generator $HC^2(\mathbb{C})$ such that $\mu(1,1,1) = 1$ and with A_s the antisymmetrization operator.

By the assumption and the observation above, we have

$$(\mathrm{id}_{\mathfrak{A}} \otimes \sigma)^* = \mathrm{id} \quad \text{on} \quad HC^*(\mathfrak{A} \otimes \mathfrak{K} \otimes \mathfrak{K})$$

and hence

$$[\varphi] = [(r_1^* \varphi) \# \omega] \in HC^*(\mathfrak{A} \otimes \mathfrak{K}).$$

If we denote by $\sharp\omega$ the map defined by $\varphi \mapsto \varphi \# \omega$, the last equality says that

$$\sharp\omega \circ r_1^* = \mathrm{id} \quad \text{on} \quad HC^*(\mathfrak{A} \otimes \mathbb{K}).$$

On the other hand, for any cyclic cocycle φ on $\mathfrak{A}$,

$$(r_1^* \circ \sharp\omega)\varphi = S^k \varphi$$

as well. It follows that the maps $\sharp\omega$ and r_1^* are inverses of each other at the level of periodic cyclic cohomology, in particular, they are isomorphisms. $\square$

Theorem 4.2. (Stability). *The Fréchet algebra $\mathbb{K}^\infty$ satisfies the assumptions in the theorem above. Therefore, we have*

$$HC^*(\mathbb{K}^\infty) \cong HC^*(\mathbb{C}),$$

and

$$HC^*(\mathfrak{A} \otimes \mathbb{K}^\infty) \cong HC^*(\mathfrak{A})$$

for any locally convex topological algebra $\mathfrak{A}$.

Proof. As noted before, there is a rank one projection p in $\mathbb{K}^\infty$ and an two (or even) cocycles $\omega = \#_{\mathrm{id}^\wedge}(\#_{\mathrm{id}}\tau)$ on $\mathbb{K}^\infty$ such that

$$\omega(p,p,p) = -2\pi i.$$

As for the second assumption on the flip automorphism, note that $\mathbb{K}^\infty$ consists of the integral operators with smooth rapidly decreasing kernels, and that we have the isomorphisms

$$\mathbb{K}^\infty \otimes \mathbb{K}^\infty \cong S^\infty(\mathbb{R}^2) \otimes S^\infty(\mathbb{R}^2) \cong S^\infty(\mathbb{R}^4)$$

as topological vector spaces. Since any one-parameter subgroup of rotations in the coordinate space asts smoothly on $S^\infty(\mathbb{R}^4)$ and gives a required path of automorphisms of $\mathbb{K}^\infty \otimes \mathbb{K}^\infty$, unitarily implemented in $L^2(\mathbb{R}^2)$, so that the result follows.

Note that $\mathbb{K}^\infty \otimes \mathbb{C} \cong \mathbb{K}^\infty$ and thus,

$$HC^*(\mathbb{K}^\infty) \cong HC^*(\mathbb{C} \otimes \mathbb{K}^\infty) \cong HC^*(\mathbb{C}).$$

□

5 The main theorem

Let $\mathfrak{A}$ be a Fréchet algebra and α a smooth action of $\mathbb{R}$ on $\mathfrak{A}$.

Proposition 5.1. *Given a cocycle $\varphi \in Z^n_\lambda(\mathfrak{A})$, we have the class identity of cocycles:*

$$[S(\varphi\#\omega)] = [S(T^*_\alpha \#_{\alpha^\wedge}\#_\alpha)\varphi]$$

in $H^{n+4}_\lambda(\mathfrak{A} \otimes \mathbb{K}^\infty)$, where T_α is the isomorphism associated to the dual crossed product of $\mathfrak{A}$ by a smooth action α and its dual $\alpha^\wedge$.

Proof. We use the notation in the proof of the duality theorem for the dual crossed product of a Fréchet algebra. Extending the homomorphism

$$\psi \circ \rho : (\mathfrak{A} \otimes S^{\infty}(\mathbb{R})) \rtimes_{\gamma} \mathbb{R} \to \mathfrak{A} \rtimes_{\alpha} \mathbb{R} \rtimes_{\alpha^{\wedge}} \mathbb{R}$$

we get a homomorphism

$$\psi \circ \rho : (\Omega(\mathfrak{A}) \otimes E) \otimes_{\alpha} E \to (\Omega(\mathfrak{A}) \otimes_{\alpha} E) \otimes_{\alpha^{\wedge}} E$$

and hence an equality

$$(\psi \circ \rho)^{*} \#_{\alpha^{\wedge}} \#_{\alpha} \varphi = \#_{\gamma}(\varphi \# \varepsilon),$$

where $\varepsilon = \#_{\mathrm{id}} \tau$ is the canonical one-cocycle on $S^{\infty}(\mathbb{R}) = \mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R}$. Note that both sides belong to the bottom right corner in the following commutative diagram:

$$\begin{tikzcd}
H_{\lambda}^{n}(\mathfrak{A}) \arrow[r, "\#_{\alpha}"] \arrow[d, equal] & H_{\lambda}^{n+1}(\mathfrak{A} \rtimes_{\alpha} \mathbb{R}) \arrow[r, "\#_{\alpha^{\wedge}}"] & H_{\lambda}^{n+2}(\mathfrak{A} \times_{\alpha} \mathbb{R} \rtimes_{\alpha^{\wedge}} \mathbb{R}) \arrow[d, "(\psi \circ \rho)^{*}"] \\
H_{\lambda}^{n}(\mathfrak{A}) \arrow[r, "\sharp \varepsilon"] & H_{\lambda}^{n+1}(\mathfrak{A} \otimes S^{\infty}(\mathbb{R})) \arrow[r, "\#_{\gamma}"] & H_{\lambda}^{n+2}((\mathfrak{A} \otimes S^{\infty}(\mathbb{R})) \rtimes_{\gamma} \mathbb{R}).
\end{tikzcd}$$

Let $t \in [0,1]$ and γ^{t} be the action of $\mathbb{R}$ on

$$\mathfrak{A} \otimes S^{\infty}(\mathbb{R}) \cong S^{\infty}(\mathbb{R}, \mathfrak{A})$$

given by

$$((\gamma^{t})_{u} f)(s) = \alpha_{tu}(f(s-u))$$

for $u, s \in \mathbb{R}$ and $f \in S^{\infty}(\mathbb{R}, \mathfrak{A})$.

Define an action β of $\mathbb{R}$ on

$$S^{\infty}(\mathbb{R}, \mathfrak{A}) \otimes C^{\infty}([0,1])$$

by

$$(\beta_{u} f)_{t}(s) = ((\gamma^{t})_{y} f_{t})(s) = \alpha_{tu}(f_{t}(s-u))$$

for $t \in [0,1]$, $u, s \in \mathbb{R}$ and $f \in S^{\infty}(\mathbb{R}, \mathfrak{A}) \otimes C^{\infty}([0,1])$. Note that β is smooth and therefore we can define the smooth crossed product:

$$(S^{\infty}(\mathbb{R}, \mathfrak{A}) \otimes C^{\infty}([0,1])) \rtimes_{\beta} \mathbb{R}.$$

The evaluation maps

$$g_{i} : S^{\infty}(\mathbb{R}, \mathfrak{A}) \otimes C^{\infty}([0,1]) \to S^{\infty}(\mathbb{R}, \mathfrak{A}), \quad i = 0, 1$$

defined by $(g_0 f)(s) = f_0(s)$ and $(g_1 f)(s) = f_1(s)$ are equivariant homomorphisms and hence give rise to homomorphisms:

$$g_0 : (S^\infty(\mathbb{R}, \mathfrak{A}) \otimes C^\infty([0,1])) \rtimes_\beta \mathbb{R} \to S^\infty(\mathbb{R}, \mathfrak{A}) \rtimes_{\gamma^0} \mathbb{R},$$
$$g_1 : (S^\infty(\mathbb{R}, \mathfrak{A}) \otimes C^\infty([0,1])) \rtimes_\beta \mathbb{R} \to S^\infty(\mathbb{R}, \mathfrak{A}) \rtimes_{\gamma^1} \mathbb{R}$$

denoted by the same symbols as the evalutaion maps.

Define a homomorphism:

$$\mu : \mathfrak{A} \otimes \mathbb{K}^\infty \to (S^\infty(\mathbb{R}, \mathfrak{A}) \otimes C^\infty([0,1])) \rtimes_\beta \mathbb{R}$$

by

$$\mu(f)_t(s, r) = \alpha_{st}(f(s, r))$$

for $t \in [0,1]$, $s, r \in \mathbb{R}$ and t the variable corresponding to the crossed product by the action β of $\mathbb{R}$. It is an easy observation that

$$\mu^* g_0^*(\#_{\gamma^0}(\varphi \# \varepsilon)) = T_{\mathrm{id}}^* \#_{\mathrm{id}^\wedge} \#_{\mathrm{id}} \varphi,$$
$$\mu^* g_1^*(\#_{\gamma^1}(\varphi \# \varepsilon)) = T_\alpha^* \#_{\alpha^\wedge} \#_\alpha \varphi.$$

Note that both sides in the first equality belong to the top or bottom right corners in the following commutative diagram:

$$\begin{array}{ccccc}
H_\lambda^n(\mathfrak{A}) & \xrightarrow{\#_{\mathrm{id}^\wedge} \#_{\mathrm{id}}} & H_\lambda^{n+2}(\mathfrak{A} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\mathrm{id}^\wedge} \mathbb{R}) & \xrightarrow{T_{\mathrm{id}}^*} & H_\lambda^{n+2}(\mathfrak{A} \otimes \mathbb{K}^\infty) \\
{\scriptstyle \sharp\varepsilon}\downarrow & & & & \| \\
H_\lambda^{n+1}(S^\infty(\mathbb{R}, \mathfrak{A})) & \xrightarrow{\#_{\gamma^0}} & H_\lambda^{n+2}(S^\infty(\mathbb{R}, \mathfrak{A}) \rtimes_{\gamma^0} \mathbb{R}) & \xrightarrow{\mu^* g_0^*} & H_\lambda^{n+2}(\mathfrak{A} \otimes \mathbb{K}^\infty)
\end{array}$$

and both sides in the second equality do to the top or bottom right corners in the following diagram:

$$\begin{array}{ccccc}
H_\lambda^n(\mathfrak{A}) & \xrightarrow{\#_{\alpha^\wedge} \#_\alpha} & H_\lambda^{n+2}(\mathfrak{A} \rtimes_\alpha \mathbb{R} \rtimes_{\alpha^\wedge} \mathbb{R}) & \xrightarrow{T_\alpha^*} & H_\lambda^{n+2}(\mathfrak{A} \otimes \mathbb{K}^\infty) \\
{\scriptstyle \sharp\varepsilon}\downarrow & & & & \| \\
H_\lambda^{n+1}(S^\infty(\mathbb{R}, \mathfrak{A})) & \xrightarrow{\#_{\gamma^1}} & H_\lambda^{n+2}(S^\infty(\mathbb{R}, \mathfrak{A}) \rtimes_{\gamma^1} \mathbb{R}) & \xrightarrow{\mu^* g_1^*} & H_\lambda^{n+2}(\mathfrak{A} \otimes \mathbb{K}^\infty).
\end{array}$$

Let us consider the two $(n+2)$-cycles over $\mathfrak{A} \otimes \mathbb{K}^\infty$ given by

$$(\Omega' = (\Omega(\mathfrak{A}) \otimes E) \otimes_{\gamma_0} E, g_0\mu, \#_{\gamma_0}(\varphi \# \varepsilon)),$$
$$(\Omega' = (\Omega(\mathfrak{A}) \otimes E) \otimes_{\gamma_1} E, g_1\mu, \#_{\gamma_1}(\varphi \# \varepsilon)).$$

Note that

$$\mathfrak{A} \otimes \mathbb{K}^\infty \xrightarrow{\mu} (S^\infty(\mathbb{R}, \mathfrak{A}) \otimes C^\infty([0,1])) \rtimes_\beta \mathbb{R} \xrightarrow{g_i} S^\infty(\mathbb{R}, \mathfrak{A}) \rtimes_{\gamma^i} \mathbb{R} = \Omega'_0$$

for $i = 0, 1$ and

$$[\#_{\gamma_i}(\varphi \# \varepsilon)] \in H_\lambda^{n+2}(S^\infty(\mathbb{R}, \mathfrak{A}) \rtimes_{\gamma^i} \mathbb{R}),$$

whose representative is a closed graded trace from Ω'_{n+2} to $\mathbb{C}$.

We claim that those two cycles are cobordant. In fact, let $\psi^\wedge$ be the canonical graded trace of degree one on the differential graded algebra $\Omega^*([0,1])$ of smooth differential forms on $[0,1]$. The graded trace

$$\#_\beta((\varphi \# \varepsilon) \# \psi^\wedge)$$

over

$$(\Omega(\mathfrak{A}) \otimes E \otimes \Omega^*(([0,1])) \otimes_\beta E$$

gives us the required cobordism. As a result, we obtain

$$ST^*_{\mathrm{id}} \#_{\mathrm{id}^\wedge} \#_{\mathrm{id}} \varphi = ST^*_\alpha \#_{\alpha^\wedge} \#_\alpha \varphi \quad \text{in} \quad H_\lambda^{n+4}(\mathfrak{A} \otimes \mathbb{K}^\infty).$$

Note also that

$$\begin{tikzcd}
H_\lambda^n(\mathfrak{A}) \arrow[r, "\#_{\mathrm{id}^\wedge}\#_{\mathrm{id}}\varphi"] \arrow[d, equal] & H_\lambda^{n+2}(\mathfrak{A} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\mathrm{id}^\wedge} \mathbb{R}) \arrow[r, "T^*_{\mathrm{id}}"] & H_\lambda^{n+2}(\mathfrak{A} \otimes \mathbb{K}^\infty) \arrow[d, equal] \\
H_\lambda^n(\mathfrak{A}) \arrow[r, "\sharp\varepsilon"] & H_\lambda^{n+1}(\mathfrak{A} \otimes S^\infty(\mathbb{R})) \arrow[r, "\#_{\mathrm{id}^\wedge}"] & H_\lambda^{n+2}(\mathfrak{A} \otimes \mathbb{K}^\infty)
\end{tikzcd}$$

and hence $T^*_{\mathrm{id}} \#_{\mathrm{id}^\wedge} \#_{\mathrm{id}} \varphi = \#_{\mathrm{id}^\wedge}(\varphi \# \varepsilon) = \varphi \# (\#_{\mathrm{id}^\wedge} \varepsilon) = \varphi \# \omega = \sharp\omega(\varphi)$. □

Theorem 5.2. *For a smooth action α of $\mathbb{R}$ on a Fréchet algebra $\mathfrak{A}$, the map*

$$\#_\alpha : Z_\lambda^n(\mathfrak{A}) \to Z_\lambda^{n+1}(\mathfrak{A} \rtimes_\alpha \mathbb{R})$$

induces the following isomorphism in the cyclic cohomology groups of $\mathfrak{A}$ and $\mathfrak{A} \rtimes_\alpha \mathbb{R}$:

$$H_\lambda^n(\mathfrak{A}) \cong H_\lambda^{n+1}(\mathfrak{A} \rtimes_\alpha \mathbb{R})$$

and hence, does the isomorphisms of even and odd parts in the cyclic cohomology groups:

$$HC^{\mathrm{ev}}(\mathfrak{A}) \cong HC^{\mathrm{odd}}(\mathfrak{A} \rtimes_\alpha \mathbb{R}),$$
$$HC^{\mathrm{odd}}(\mathfrak{A}) \cong HC^{\mathrm{ev}}(\mathfrak{A} \rtimes_\alpha \mathbb{R}).$$

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
HC^n(\mathfrak{A}) & \xrightarrow{\natural\omega} & HC^{n+2}(\mathfrak{A}\otimes\mathbb{K}^\infty) & \xrightarrow{r^*} & HC^{n+2}(\mathfrak{A}) \\
\#_\alpha\big\downarrow & & \big\uparrow T^*_\alpha & & \\
HC^{n+1}(\mathfrak{A}\rtimes_\alpha\mathbb{R}) & \xrightarrow{\#_{\alpha^\wedge}} & HC^{n+2}(\mathfrak{A}\rtimes_\alpha\mathbb{R}\rtimes_{\alpha^\wedge}\mathbb{R}). & &
\end{array}$$

According to the proposition above and the stability theorem, this diagram is commutative and the map $\natural\omega$ is an isomorphism. In particular, the map $\#_\alpha$ is injective and the map $\#_{\alpha^\wedge}$ is surjective since $r^*\circ T^*_\alpha$ is an isomorphism. Applying the same argument to dual action $\alpha^\wedge$, as in the commutative diagram below:

$$\begin{array}{ccccc}
H^n_\lambda(\mathfrak{A}\rtimes_\alpha\mathbb{R}) & \xrightarrow{\natural\omega} & H^{n+2}_\lambda((\mathfrak{A}\rtimes_\alpha\mathbb{R})\otimes\mathbb{K}^\infty) & \xrightarrow{r^*} & H^{n+2}_\lambda(\mathfrak{A}\rtimes_\alpha\mathbb{R}) \\
\#_{\alpha^\wedge}\big\downarrow & & \big\uparrow T^*_{\alpha^\wedge} & & \\
H^{n+1}_\lambda(\mathfrak{A}\rtimes_\alpha\mathbb{R}\rtimes_{\alpha^\wedge}\mathbb{R}) & \xrightarrow{\#_{(\alpha^\wedge)^\wedge}} & H^{n+2}_\lambda(\mathfrak{A}\rtimes_\alpha\mathbb{R}\rtimes_{\alpha^\wedge}\mathbb{R}\rtimes_{(\alpha^\wedge)^\wedge}\mathbb{R}). & &
\end{array}$$

we conclude that the map $\#_{\alpha^\wedge}$ is injective, and hence that the map $\#_\alpha$ is surjective. □

Corollary 5.3. *We have*

$$(1): \quad H^*_\lambda(S^\infty(\mathbb{R}^k)) \cong \mathbb{C}$$

with a generator given by the n-cocycle

$$(f_0, f_1, \cdots, f_n) \mapsto \int f_0 df_1 \cdots df_n.$$

We also have

$$(2) \quad H^*_\lambda(\mathfrak{A}\otimes S^\infty(\mathbb{R})) \cong H^{*+1}_\lambda(\mathfrak{A})$$

and

$$(3) \quad H^*_\lambda(\mathfrak{A}\otimes S^\infty(\mathbb{R}^2)) \cong H^*_\lambda(\mathfrak{A}).$$

Proof. Note that

$$S^\infty(\mathbb{R}^k) \cong S^\infty(\mathbb{R}^{k-1}) \rtimes_{\mathrm{id}} \mathbb{R},$$

and thus, if $n \geq k$, which we may assume, then

$$\begin{aligned}
H^n_\lambda(S^\infty(\mathbb{R}^k)) &\cong H^n_\lambda(S^\infty(\mathbb{R}^{k-1})\rtimes_{\mathrm{id}}\mathbb{R}) \\
&\cong H^{n-1}_\lambda(S^\infty(\mathbb{R}^{k-1})) \cong \cdots \cong H^{n-k}_\lambda(\mathbb{C}) \\
&\cong \begin{cases} \mathbb{C} & \text{if } n-k \text{ even}, \\ 0 & \text{if } n-k \text{ odd}. \end{cases}
\end{aligned}$$

Hence, if k is even,

$$H_\lambda^{\mathrm{ev}}(S^\infty(\mathbb{R}^k)) = \varinjlim H_\lambda^{2n}(S^\infty(\mathbb{R}^k)) \cong \mathbb{C},$$
$$H_\lambda^{\mathrm{odd}}(S^\infty(\mathbb{R}^k)) = \varinjlim H_\lambda^{2n+1}(S^\infty(\mathbb{R}^k)) \cong 0$$

and thus

$$H_\lambda^*(S^\infty(\mathbb{R}^k)) = H_\lambda^{\mathrm{ev}}(S^\infty(\mathbb{R}^k)) \oplus H_\lambda^{\mathrm{odd}}(S^\infty(\mathbb{R}^k)) \cong \mathbb{C}.$$

Similarly, if k is odd, we obtain $H_\lambda^*(S^\infty(\mathbb{R}^k)) \cong \mathbb{C}$.

We have

$$\begin{aligned} H_\lambda^n(\mathfrak{A} \otimes S^\infty(\mathbb{R})) &\cong H_\lambda^n((\mathfrak{A} \otimes \mathbb{C}) \rtimes_{\mathrm{id}} \mathbb{R}) \\ &\cong H_\lambda^{n+1}(\mathfrak{A}), \end{aligned}$$

so that

$$\begin{aligned} H_\lambda^*(\mathfrak{A} \otimes S^\infty(\mathbb{R})) &= H_\lambda^{\mathrm{ev}}(\mathfrak{A} \otimes S^\infty(\mathbb{R})) \oplus H_\lambda^{\mathrm{odd}}(\mathfrak{A} \otimes S^\infty(\mathbb{R})) \\ &\cong H_\lambda^{\mathrm{odd}}(\mathfrak{A}) \oplus H_\lambda^{\mathrm{ev}}(\mathfrak{A}) = H_\lambda^{*+1}(\mathfrak{A}). \end{aligned}$$

Since

$$\begin{aligned} H_\lambda^{n+2}(\mathfrak{A} \otimes S^\infty(\mathbb{R}^2)) &\cong H_\lambda^{n+2}(\mathfrak{A} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\mathrm{id}} \mathbb{R}) \\ &\cong H_\lambda^n(\mathfrak{A}), \end{aligned}$$

we get

$$\begin{aligned} H_\lambda^*(\mathfrak{A} \otimes S^\infty(\mathbb{R}^2)) &= H_\lambda^{\mathrm{ev}}(\mathfrak{A} \otimes S^\infty(\mathbb{R}^2)) \oplus H_\lambda^{\mathrm{odd}}(\mathfrak{A} \otimes S^\infty(\mathbb{R}^2)) \\ &\cong H_\lambda^{\mathrm{ev}}(\mathfrak{A}) \oplus H_\lambda^{\mathrm{odd}}(\mathfrak{A}) = H_\lambda^*(\mathfrak{A}). \end{aligned}$$

□

Let G be any connected, simply connected nilpotent Lie group, with dimension m. Since G can be written as an iterated semi-direct product by $\mathbb{R}$:

$$G \cong \mathbb{R} \rtimes \mathbb{R} \rtimes \cdots \rtimes \mathbb{R}$$

as a result in Lie group theory and since the nilpotency of G implies the smoothness of the successive actions of $\mathbb{R}$, we can apply the theorem above and get

$$\begin{aligned} HC^*(S^\infty(G)) &\cong HC^*(S^\infty(\mathbb{R}) \rtimes \mathbb{R} \rtimes \mathbb{R} \cdots \rtimes \mathbb{R})) \\ &\cong H_\lambda^{*+m}(\mathbb{C}) \cong \mathbb{C}, \end{aligned}$$

where the Fréchet algebra $S^\infty(G)$ on G is equal to $S^\infty(\mathbb{R}^m)$ as a topological vector space, with convolution over G as a product, and $* + m$ means the exchanging of even and odd parts of $H^*_\lambda(\mathbb{C})$ m-times.

In the case of the Heisenberg Lie group H, written as $H \cong \mathbb{R}^2 \rtimes \mathbb{R}$, the corresponding generator of $HC^*(S^\infty(H))$ is given by the cyclic 3-cocycle:

$$\tau(f_0, f_1, f_2, f_3) =$$
$$(2\pi i)^3 \iiint_{g_0 g_1 g_2 g_3 = 1} f_0(g_0) f_1(g_1) f_2(g_2) f_3(g_3) c(g_1, g_2, g_3) dg_1 dg_2 dg_3$$

where $c(g_1, g_2, g_3)$ is a continuous normalized group 3-cocycle generating $H^3_c(H, \mathbb{R}) \cong \mathbb{R}$.

6 Comparison with the Connes' Thom isomorphism in K-theory

Let α be an action of $\mathbb{R}$ on a C^*-algebra A. In [1], Connes constructed a map

$$\Phi^j_\alpha : K_j(A) \to K_{j+1}(A \rtimes_\alpha \mathbb{R})$$

satisfying certain natural axioms, and proved that it is unique up to a choice of orientation, and is an isomorphism. We briefly review the construction of Φ^0_α in the following.

The subalgebra $\mathfrak{A}$ of all smooth elements of A with respect to the action α has, in a natural way, the structure of a Fréchet algebra, and the inclusion map of $\mathfrak{A}$ into A induces an group isomorphism:

$$K_0(\mathfrak{A}) \to K_0(A).$$

Working instead, if necessary, in a matrix algebra over $\mathfrak{A}^+$, we may assume that we are given a projection $e \in \mathfrak{A}$

With $h = \delta(e)e - e\delta(e)$ we get

$$(\delta - \mathrm{ad}(h))(e) = 0,$$

and hence $\delta - \mathrm{ad}(h)$ generates an action β of $\mathbb{R}$ on $\mathfrak{A}$ such that $\beta_t(e) = e$.

We check that

$$\begin{aligned}(\delta - \mathrm{ad}(h))(e) &= \delta(e) - [h, e] \\ &= \delta(e) - (he - eh) \\ &= \delta(e) - ((\delta(e)e - e\delta(e))e - e(\delta(e)e - e\delta(e)) \\ &= \delta(e) - (\delta(e)e + e\delta(e)) = 0\end{aligned}$$

since $\delta(e) = \delta(e^2) = \delta(e)e + e\delta(e)$, and so that $e\delta(e)e = 0$. Note also that

$$\beta_t(e) = \exp(t(\delta - \mathrm{ad}(h))(e)) = \mathrm{id}(e) = e.$$

Note that both actions α and β are smooth on $\mathfrak{A}$. Therefore, by Connes [1], there exists an isomorphism:

$$i_e : A \rtimes_\alpha \mathbb{R} \to A \rtimes_\beta \mathbb{R},$$

and it also defines an isomorphism:

$$i_e : \mathfrak{A} \rtimes_\alpha \mathbb{R} \to \mathfrak{A} \rtimes_\beta \mathbb{R}.$$

Let u be a unitary in the C^*-algebra $C^*(\mathbb{R})^+$ such that the K_1-class of u is the positive generator of $K_1(C^*(\mathbb{R})) \cong K_1(C_0(\mathbb{R})) \cong \mathbb{Z}$, and $u - 1 \in S^\infty(\mathbb{R})$. Then

$$i_e^{-1}(1 - e + eu)$$

is a unitary element of $(\mathfrak{A} \rtimes_\alpha \mathbb{R})^+$ and represents the K_1-class $\Phi_\alpha^0([e])$ in $K_1(A \rtimes_\alpha \mathbb{R})$.

Check that

$$\begin{aligned}(1 - e + eu)(1 - e + eu)^* &= 1 - e + (1 - e)u^*e + eu(1 - e) + e,\\ (1 - e + eu)^*(1 - e + eu) &= 1 - e + (1 - e)eu + u^*e(1 - e) + u^*eu\\ &= 1 - e + u^*eu,\end{aligned}$$

both of which can be the identity if $u^*eu = e$, i.e. $u^*e = eu^*$, or $eu = ue$.

According to Nest [5], the K_1-class $\Phi_\alpha^0([e])$ viewed as in $K_1(\mathfrak{A} \rtimes_\alpha \mathbb{R})$ can be evaluated by a pairing $\langle \cdot, \cdot \rangle$ with any odd-dimensional cyclic cocycle on $\mathfrak{A} \rtimes_\alpha \mathbb{R}$.

Theorem 6.1. *Given $\varphi \in H_\lambda^{2n}(\mathfrak{A})$, we have the equality:*

$$\langle \varphi, [e] \rangle = \langle \#_\alpha \varphi, \Phi_\alpha^0([e]) \rangle$$

between the pairing of $2n$-cyclic cohomology classes and K_0-group classes of $\mathfrak{A}$ and the pairing of $(2n + 1)$-cyclic cohomology classes and K_1-group classes of $\mathfrak{A} \rtimes_\alpha \mathbb{R}$.

Proof. Assume first that a projection e of $\mathfrak{A}$ is α-invariant. Then the homomorphism defined as

$$\rho : \mathbb{C} \to \mathfrak{A}, \quad \rho(c) = c \cdot e$$

induces a homomorphism:

$$\rho^\wedge : \mathbb{C} \rtimes_{\text{id}} \mathbb{R} \longrightarrow \mathfrak{A} \rtimes_\alpha \mathbb{R}.$$

By naturality of both Φ^0_α and $\#_\alpha$, we can pull everything back to $\mathbb{C} \rtimes_{\text{id}} \mathbb{R}$ and thus reduce the computation to the case where $\mathfrak{A} = \mathbb{C}$ and $\alpha = \text{id}$. We have thus to show that

$$\langle \omega, 1\rangle = \langle \#_{\text{id}}\omega, \Phi^0_{\text{id}}(1)\rangle.$$

By construction, the K_1-group class $\Phi^0_{\text{id}}(1)$ is the positively oriented generator of $K_1(C^*(\mathbb{R})) \cong \mathbb{Z}$. Using the Fourier transform we identity $S^\infty(\mathbb{R})$ with convolution, with that with pointwise multiplication, and then

$$\Phi^0_{\text{id}}(1) = \exp(2\pi i h)$$

where h is any C^∞ real-valued function on $\mathbb{R}$ such that

$$h(t) = \begin{cases} 0 & \text{for } t \le 0, \\ 1 & \text{for } t \ge 1, \end{cases}$$

and otherwise, for instance, we may define as:

$$h(t) = \frac{1}{2}\left(\frac{2}{\pi}\text{Arctan}\left(-\frac{1}{x} + \frac{1}{1-x}\right) + 1\right), \quad 0 < t < 1.$$

Since the map $\#_{\text{id}}$ commutes with the map S, we may assume that ω is equal to τ the normalized trace on $\mathbb{C}$. Then

$$\langle \tau, 1\rangle = 1,$$

and

$$\begin{aligned} \langle \#_{\text{id}}\tau, \Phi^0_{\text{id}}(1)\rangle &= \langle \varepsilon, \exp(2\pi i h)\rangle \\ &= \frac{1}{2\pi i}\varepsilon(\exp(-2\pi i h) - 1, \exp(2\pi i h) - 1) \\ &= \frac{1}{2\pi i}\int e^{-2\pi i h} d(e^{2\pi i h}) = 1. \end{aligned}$$

Note that

$$\begin{aligned} \int e^{2\pi i h} d(e^{2\pi i h}) &= \int e^{-2\pi i h}(e^{2\pi i h})' dh \\ &= 2\pi i \int dh \end{aligned}$$

and $\int dh = \int_0^1 dh = 1$ to make sense.

The lemma below finishies the proof. ∎

Lemma 6.2. *The following diagram is commutative:*

$$\begin{array}{ccc} HC^*(\mathfrak{A}) & \xrightarrow{\#_\beta} & HC^{*+1}(\mathfrak{A} \rtimes_\beta \mathbb{R}) \\ \| & & \downarrow i_e^* \\ HC^*(\mathfrak{A}) & \xrightarrow{\#_\alpha} & HC^{*+1}(\mathfrak{A} \rtimes_\alpha \mathbb{R}), \end{array}$$

where the action β is induced by α and a projection $e \in \mathfrak{A}$.

Proof. Let us recall first that the construction of the isomorphism i_e in the following. Given a smooth action α of $\mathbb{R}$ on $\mathfrak{A}$, for the action β there is a smooth unitary one-cocycle u_t in the multiplier algebra of the C^*-algebra A such that $\mathbb{R} \ni t \mapsto u_t(a)$ is norm continuous for any $a \in \mathfrak{A}$ and that

$$\beta_t(a) = u_t \alpha_t(a) u_t^*$$

in other words, β is exterior equivalent to α (see [1]). One can then define an action γ of $\mathbb{R}$ on

$$M_2(\mathfrak{A}) \cong \mathfrak{A} \otimes M_2(\mathbb{C})$$

by

$$\begin{aligned} \gamma_t(a_{11} \otimes e_{11}) &= \alpha_t(a_{11}) \otimes e_{11}, \\ \gamma_t(a_{12} \otimes e_{12}) &= \alpha_t(a_{12}) u_t^* \otimes e_{12}, \\ \gamma_t(a_{21} \otimes e_{21}) &= u_t \alpha_t(a_{21}) \otimes e_{21}, \\ \gamma_t(a_{22} \otimes e_{22}) &= \beta_t(a_{22}) \otimes e_{22}, \end{aligned}$$

for $t \in \mathbb{R}$, $a_{ij} \in \mathfrak{A}$, and (e_{ij}) the matrix unit for $M_2(\mathbb{C})$.

From the smoothness of the actions α and u, it follows that γ is a smooth action of $\mathbb{R}$ on $M_2(\mathfrak{A})$, and so we can construct the smooth crossed product:

$$M_2(\mathfrak{A}) \rtimes_\gamma \mathbb{R}.$$

Then there are two imbeddings:

$$\begin{aligned} \rho_1^\wedge &: \mathfrak{A} \rtimes_\alpha \mathbb{R} \to M_2(\mathfrak{A}) \rtimes_\gamma \mathbb{R}, \\ \rho_2^\wedge &: \mathfrak{A} \rtimes_\alpha \quad \to M_2(\mathfrak{A}) \rtimes_\gamma \mathbb{R}, \end{aligned}$$

where the homomorphisms $\rho_j^\wedge$ are induced by the homomorphisms $\rho_j : \mathfrak{A} \to M_2(\mathfrak{A})$ defined by $\rho_j(a) = a \otimes e_{jj}$ for $a \in \mathfrak{A}$. The isomorphism i_e is now given by

$$\mathrm{Ad} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since we have

$$\#_\alpha\varphi = (\rho_1^\wedge)^*(\#_\gamma\varphi),$$
$$\#_\beta\varphi = (\rho_2^\wedge)^*(\#_\gamma\varphi),$$

where, possibly, more precisely,

$$\begin{tikzcd}
H_\lambda^{2n}(\mathfrak{A}) \arrow[r, "\sharp\mathrm{Tr}"] \arrow[d, equal] & H_\lambda^{2n}(M_2(\mathfrak{A})) \arrow[r, "\#_\gamma"] & H_\lambda^{2n+1}(M_2(\mathfrak{A}) \rtimes_\gamma \mathbb{R}) \arrow[d, "(\rho_j^\wedge)^*"] \\
H_\lambda^{2n}(\mathfrak{A}) \arrow[r, "\#_{\alpha_j}"] & H_\lambda^{2n+1}(\mathfrak{A} \rtimes_{\alpha_j} \mathbb{R}) \arrow[r, equal] & H_\lambda^{2n+1}(\mathfrak{A} \rtimes_{\alpha_j} \mathbb{R})
\end{tikzcd}$$

with $\alpha_1 = \alpha$ and $\alpha_2 = \beta$ and Tr the trace on $M_2(\mathbb{C})$, and since

$$\mathrm{Ad}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is connected to the identity by a smooth path in $\mathrm{Aut}(M_2(\mathfrak{A}) \rtimes_\gamma \mathbb{R})$, an application of homotopy invariance of HC^* gives the required formula:

$$i_e^*\#_\beta = \#_\alpha.$$

Namely,

$$\begin{tikzcd}
H_\lambda^{2n}(\mathfrak{A}) \arrow[r, "\#_\beta"] \arrow[d, equal] & H_\lambda^{2n+1}(\mathfrak{A} \rtimes_\beta \mathbb{R}) \arrow[d, "i_e^*"] \\
H_\lambda^{2n}(\mathfrak{A}) \arrow[r, "\#_\alpha"] & H_\lambda^{2n+1}(\mathfrak{A} \rtimes_\alpha \mathbb{R})
\end{tikzcd}$$

with the arrows isomorphisms. □

Let us consider the map $\Phi_\alpha^1 : K_1(A) \to K_0(A \rtimes_\alpha \mathbb{R})$. Note that Φ_α^1 is the inverse of $\Phi_{\alpha^\wedge}^0$:

$$\Phi_{\alpha^\wedge}^0 : K_0(A \rtimes_\alpha \mathbb{R}) \to K_1(A \rtimes_\alpha \mathbb{R} \rtimes_{\alpha^\wedge} \mathbb{R}) \cong K_1(A \otimes \mathbb{K}) \cong K_1(A),$$

where $\mathbb{K}$ is the C^*-algebra of all compact operators on the Hilbert space $L^2(\mathbb{R})$, via the Takai C^*-algebra duality theorem. Since $\#_\alpha$ is the inverse of $\#_{\alpha^\wedge}$:

$$\begin{aligned}
\#_{\alpha^\wedge} : H_\lambda^n(\mathfrak{A} \rtimes_\alpha \mathbb{R}) &\to H_\lambda^{n+1}(\mathfrak{A} \rtimes_\alpha \mathbb{R} \rtimes_{\alpha^\wedge} \mathbb{R}) \\
&\xrightarrow{T_\alpha^*} H_\lambda^{n+1}(\mathfrak{A} \otimes \mathbb{K}^\infty) \\
&\xrightarrow{(\sharp\omega)^{-1}} H_\lambda^{n-1}(\mathfrak{A}),
\end{aligned}$$

with the maps isomorphisms, we have the following result:

Proposition 6.3. *With the same assumptions above, let u be a unitary and suppose that $\Phi_\alpha^1([u])$ is represented by an element of $K_0(\mathfrak{A} \rtimes_\alpha \mathbb{R})$. Then, for any odd-dimensional cyclic cocycle ψ on $\mathfrak{A}$, we obtain the equality*

$$\langle \psi, [u] \rangle = \langle \#_\alpha \psi, \Phi_\alpha^1([u]) \rangle$$

between the pairing of $(2n-1)$-cyclic cohomology classes and K_1-group classes of $\mathfrak{A}$ and the pairing of $2n$-cyclic cohomology classes and K_0-group classes of $\mathfrak{A} \rtimes_\alpha \mathbb{R}$.

Proof. Because, by the theorem above we have

$$\begin{aligned} \langle \#_\alpha \psi, \Phi_\alpha^1([u]) \rangle &= \langle \#_{\alpha^\wedge}(\#_\alpha \psi), \Phi_{\alpha^\wedge}^0(\Phi_\alpha^1([u])) \rangle \\ &= \langle (\#_{\alpha^\wedge} \circ \#_\alpha)\psi, (\Phi_{\alpha^\wedge}^0 \circ \Phi_\alpha^1)([u]) \rangle \\ &= \langle \psi, [u] \rangle. \end{aligned}$$

□

7 Further remarks

Let us consider an action α of $\mathbb{R}$ on $\mathbb{R}$ given by

$$\alpha_t(s) = e^{2t}s, \quad t, s \in \mathbb{R}.$$

The corresponding semi-direct product group $\mathbb{R} \rtimes_\alpha \mathbb{R}$ is a minimal parabolic subgroup of $PSL_2(\mathbb{R})$. We then have an action α of $\mathbb{R}$ on $S^\infty(\mathbb{R})$ with convolution given by

$$(\alpha_t f)(s) = e^{-2t} f(e^{-2t}s), \quad f \in S^\infty(\mathbb{R}), t, s, \in \mathbb{R},$$

which is not smooth in the terminology as before. This is because the exponetial multiple e^{-2t} never belong to $O((1+t^2)^{j/2}) (t \to -\infty)$ for any $j \in \mathbb{N}$. We would like to sketch below how to extend the results above to this situation.

Suppose that we are given a Fréchet algebra $\mathfrak{A}$ and a strongly infinitely differentiable action α of $\mathbb{R}$ on $\mathfrak{A}$. Suppose, moreover, that there exists an increasing sequence of functions:

$$\rho_n : \mathbb{R} \to \mathbb{R}_+,$$

satisfying the following conditions:

(1) $(1+t^2)^{1/2} \rho_{n-1}(t) \leq \rho_n(t), \quad t \in \mathbb{R};$

(2) $\rho_n(t) \leq \rho_n(s) \rho_n(t-s), \quad s, t \in \mathbb{R};$

(3) $\|D^k \alpha_t(a)\|_n \leq \rho_m(t) \|a\|_n$, uniformly for $t \in \mathbb{R}$ and $a \in \mathfrak{A}$,

where m depends on n and k.

Denote by $S_\rho^\infty(\mathbb{R})$ the convolution algebra of rapidly decreasing smooth functions f on $\mathbb{R}$ satisfying

$$\|f\|_{k,n} = \sup_{t\in\mathbb{R}} \rho_n(t)|D^{k-1}f(t)| < \infty.$$

One checks that $S_\rho^\infty(\mathbb{R})$ with the topology defined by semi-norms $\|\cdot\|_{k,n}$ for $k, n \in \mathbb{N}$, becomes a Fréchet algebra, and be nuclear as a locally convex space.

Define

$$\mathfrak{A} \rtimes_\alpha^\rho \mathbb{R} = S_\rho^\infty(\mathbb{R}, \mathfrak{A}) = S_\rho^\infty(\mathbb{R}) \otimes \mathfrak{A}$$

with the multiplication as the α-convolution defined by

$$(f * g)(t) = \int_{\mathbb{R}} f(s)\alpha_s(g(t-s))ds.$$

It is straightforward to see that the construction of the map $\#_\alpha$ goes through and we can define a map

$$\#_\alpha : HC^n(\mathfrak{A}) \to HC^{n+1}(\mathfrak{A} \rtimes_\alpha^\rho \mathbb{R}).$$

The all-important, though completely trivial, fact is that the dual action $\alpha^\wedge$ on $\mathfrak{A} \rtimes_\alpha^\rho \mathbb{R}$ is smooth, and so we can define the dual crossed product $(\mathfrak{A} \rtimes_\alpha^\rho \mathbb{R}) \rtimes_{\alpha^\wedge} \mathbb{R}$. It is easy to see that the proof of the duality theorem for dual smooth crossed products still goes through and we have

$$(\mathfrak{A} \rtimes_\alpha^\rho \mathbb{R}) \rtimes_{\alpha^\wedge} \mathbb{R} \cong \mathfrak{A} \otimes \mathbb{K}_\rho^\infty,$$

where $\mathbb{K}_\rho^\infty = S_\rho^\infty(\mathbb{R}) \rtimes_{\mathrm{id}^\wedge} \mathbb{R}$.

Furthermore, since the required conditions can be satisfied with in this situation, we can obtain the following isomorphisms:

$$\begin{aligned} HC^{n+1}(\mathfrak{A} \rtimes_\alpha^\rho \mathbb{R}) &\cong HC^{n+2}(\mathfrak{A} \rtimes_\alpha^\rho \mathbb{R} \rtimes_{\alpha^\wedge} \mathbb{R}) \\ &\cong HC^{n+2}(\mathfrak{A} \otimes \mathbb{K}_\rho^\infty) \cong HC^n(\mathfrak{A}). \end{aligned}$$

It follows from the proof sketched above that

Proposition 7.1. *Assume that we have $\mathfrak{A}$, α, ρ_n as given above. Then the map $\#_\alpha$ implies an isomorphism*

$$\#_\alpha : HC^*(\mathfrak{A}) \to HC^{*+1}(\mathfrak{A} \rtimes_\alpha^\rho \mathbb{R}).$$

The preceding proposition applies to the non-smooth example in the first of this section if we set

$$\rho_n(t) = e^{2n|t|}.$$

Check the following conditions required:

$$(1) \quad (1+t^2)^{1/2} e^{2(n-1)|t|} \leq e^{2n|t|}, \quad t \in \mathbb{R},$$

which is equivalent to $(1+t^2)^{1/2} \leq e^{2|t|}$, and to $(1+t^2) \leq e^{4|t|}$, and, indeed, $e^{4|t|} \geq 1 + \frac{1}{2!}(4|t|)^2$.

$$(2) \quad \rho_n(t) = e^{2n|t|} = e^{2n|s+(t-s)|} \leq e^{2n|s|} e^{2n|t-s|} = \rho_n(s)\rho_n(t-s), \quad s, t \in \mathbb{R}.$$

As for the condition (3), for instance, note that

$$\begin{aligned} |D\alpha_t(f)(s)| &= |D(e^{-2t} f(e^{-2t}s))| \\ &= |-2e^{-2t} f(e^{-2t}s) + e^{-2t} f'(e^{-2t}s)(-2se^{-2t})| \\ &\leq 2e^{2|t|} |f(e^{-2t}s)| + 2e^{4|t|} |sf'(e^{2t}s)| \\ &\leq e^{4|t|} \cdot 2(\sup_s |f(e^{-2t}s)| + \sup_s |sf'(e^{2t}s)|) \end{aligned}$$

and thus, some of the given semi-norms for f should dominate the last multiple.

In general, given an infinitely differentiable action α of $\mathbb{R}$ on a Fréchet algebra $\mathfrak{A}$ such that α_t is continuous with respect to each of the semi-norms defining the topology of $\mathfrak{A}$, then we can use the weight functions ρ_n given by

$$\rho_n(s) = \sum_{k=0}^{n} \sum_{i=0}^{n} \left(\int_{-s}^{s} \left\| \frac{d^k}{dt^k} \alpha_t \right\|_i dt \right)^n.$$

Incidentally, it is only in this last construction that $\mathfrak{A}$ need be a Fréchet algebra, — everywhere else in this paper $\mathfrak{A}$ may be any locally convex algebra.

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