The Künneth theorem for tensor products of C^*-algebras : a review as a prelude

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THE KÜNNETH THEOREM FOR TENSOR PRODUCTS OF C^*-ALGEBRAS — A REVIEW AS A PRELUDE

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Abstract

We review and study the Künneth theorem for tensor products of C^*-algebras, which is obtained by Claude Schochet.

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Contents

1 Introduction

This is nothing but a review on the paper by Claude Schochet [7]. But not merely, we made some considerable effort to understand the contents

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perfectly and interpret and explain them plainly in our sense, so that some elementary but helpful computations or proofs are added by us in some places. Several notations are changed from the original ones in [7] by our taste. This paper may be viewed as a prelude to the next one in this volume of RMJ and would be to more in the future.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. There is a $\mathbb{Z}_2$-graded pairing (defined below)

$$\alpha : K_p(\mathfrak{A}) \otimes K_q(\mathfrak{B}) \to K_{p+q}(\mathfrak{A} \otimes \mathfrak{B}), \quad p, q \in \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$$

where $K_j(\mathfrak{A})$ and $K_j(\mathfrak{B})$ are the K-theory groups of $\mathfrak{A}$ and $\mathfrak{B}$ ($j = 0, 1$), and $\otimes$ is the minimal (or injective) tensor product defined as: for an element $x \in \mathfrak{A} \otimes \mathfrak{B}$ the tensor product C^*-algebra of $\mathfrak{A}$ and $\mathfrak{B}$ defined as the C^*-algebra completion of their algebraic tensor product by the norm:

$$\|x\| = \sup_{\pi, \rho} \|(\pi \otimes \rho)(x)\|$$

where the supremum of the operator norms is taken over π and ρ all representations of $\mathfrak{A}$ and $\mathfrak{B}$ on Hilbert spaces, respectively, with $\pi \otimes \rho$ the tensor product representation of π and ρ (cf. Takesaki [8, Page 207]).

We denote by $\mathfrak{N}$ the smallest subcategory of the category of separable nuclear C^*-algebras, which contains the separable type I C^*-algebras and is closed under the operations of taking closed ideals, quotients, extensions, inductive limits, stable isomorphism, and crossed products by $\mathbb{Z}$ of integers and by $\mathbb{R}$ of reals.

The **Künneth theorem** in K-theory for C^*-algebras, established by Schochet, is:

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A} \in \mathfrak{N}$. Then there is a natural short exact sequence:

$$0 \to K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \xrightarrow{\alpha} K_*(\mathfrak{A} \otimes \mathfrak{B}) \xrightarrow{\beta} \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to 0$$

where $K_*(\cdot) = K_0(\cdot) \oplus K_1(\cdot)$, and the sequence is $\mathbb{Z}_2$-graded with the degree $\deg(\alpha) = 0$ and $\deg(\beta) = 1$, where $K_p(\cdot) \otimes K_q(\cdot)$, $K_p(\cdot \otimes \cdot) \oplus K_q(\cdot \otimes \cdot)$, and the torsion product $\mathrm{Tor}_1^{\mathbb{Z}}(K_p(\cdot), K_q(\cdot))$ all have degree given by $p + q \in \mathbb{Z}_2$.

If $\mathfrak{A} = C(X)$ and $\mathfrak{B} = C(Y)$ the C^*-algebras of all continuous functions on finite CW-complexes (or more generally compact spaces) X and Y, then the hypothesis is satisfied and the theorem becomes:

The **classical Künneth theorem** in topological K-theory for spaces, due to Atiyah [1]:

$$0 \to K^*(X) \otimes K^*(Y) \xrightarrow{\alpha} K^*(X \times Y) \xrightarrow{\beta} \mathrm{Tor}_1^{\mathbb{Z}}(K^*(X), K^*(Y)) \to 0.$$

Note that $K^*(\cdot) = K^0(\cdot) \oplus K^1(\cdot)$, $K_0(C(X)) \cong K^0(X)$ the topological K-theory group of X, and $K^{\pm 1}(X) = K^0(X \times \mathbb{R}) \cong K_1(C(X))$ (cf. [2]).

The proof of the theorem above of Atiyah is as follows. Let Y be a compact space. Then there is a compact space Y_1 and a continuous function $f_1 : Y \to Y_1$ such that $K^*(Y_1)$ is torsion free and $f_1^* : K^*(Y_1) \to K^*(Y)$ is surjective. A homotopy argument then yields a cofibration:

$$Y \xrightarrow{f_1} Y_1 \xrightarrow{f_2} Y_2$$

such that the associated long exact sequence degenerates to a free presentation:

$$0 \to K^*(Y_2) \xrightarrow{f_2^*} K^*(Y_1) \xrightarrow{f_1^*} K^*(Y) \to 0$$

of the $\mathbb{Z}_2$-graded group $K^*(Y)$. The space Y_1 is given by a product of Grassmann manifolds and their suspensions. □

The proof for the Künneth theorem by Schochet parallels the argument of Atiyah. The key step is the following theorem:

Theorem (*Geometric Realization*). *Let $\mathfrak{B}$ be a unital C^*-algebra. Then there exists a commutative C^*-algebra $\mathfrak{F} = C_0(Y)$ of all continuous functions on Y a disjoint union of points and lines, vanishing at infinity, and an inclusion map $\mu : \mathfrak{F} \to \mathfrak{B} \otimes \mathbb{K}$ of C^*-algebras such that $K_*(\mathfrak{F})$ is free abelian and the induced map by μ on K-theory groups:*

$$\mu_* : K_*(\mathfrak{F}) \to K_*(\mathfrak{B} \otimes \mathbb{K}) \cong K_*(\mathfrak{B})$$

is surjective, where $\mathbb{K} = \mathbb{K}(H)$ denote the C^*-algebra of all compact operators on an infinite dimensional Hilbert space H. If $K_*(\mathfrak{B})$ is free abelian, then μ_* is an isomorphism.

The theorem above and a homopoty argument imply that there is a short exact sequence of C^*-algebras:

$$0 \to \mathfrak{B} \otimes \mathbb{K} \otimes C_0(\mathbb{R}) \to \mathfrak{C} \xrightarrow{\nu} \mathfrak{F} \to 0$$

whose associated exact sequence in K-theory becomes a free resolution of $K_*(\mathfrak{B})$:

$$0 \to K_*(\mathfrak{C}) \xrightarrow{\nu_*} K_*(\mathfrak{F}) \xrightarrow{\mu_*} K_*(\mathfrak{B}) \to 0.$$

Note that not every map $f : C(X) \to C(X') \otimes M_n(\mathbb{C})$ or $f : C(X) \to C(X') \otimes \mathbb{K}$ arises via a map of spaces $X' \to X$, where $M_n(\mathbb{C})$ denotes the C^*-algebra of all $n \times n$ matrices over $\mathbb{C}$. It follows from this fact that the geometric realization theorem does not imply Atiyah's geometric realization theorem.

The Künneth theorem in C^*-algebras follows from the geometric realization theorem in the same way as the Künneth theorem in spaces follows from Atiyah's geometric realization theorem.

Note that if $Y^+ \subset \mathbb{C}$, then $K^*(Y)$ has a trivial ring structure and so $\mu_* : K^*(Y) \to K_*(\mathfrak{B})$ has no chance of being a ring map when $\mathfrak{B}$ is commutative. Maps $X \to X'$ induce ring maps $K^*(X') \to K^*(X)$, and maps $C(X') \to C(X) \otimes \mathbb{K}$ do not in general.

Künneth theorems are a necessary prelude to product structures and to the introduction of coefficients into homology and cohomology theories for C^*-algebras.

In Section 2, it is shown that the map $\alpha : K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{A} \otimes \mathfrak{B})$ is an isomorphism when $\mathfrak{A} \in \mathfrak{N}$ and $K_*(\mathfrak{B})$ is torsion free. In Section 3, the geometric realization theorem is proved. In Section 4, the results in the previous sections are combined to prove the Künneth theorem.

2 The Künneth formula in special cases

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. The following map:

$$\alpha : K_p(\mathfrak{A}) \otimes K_q(\mathfrak{B}) \to K_{p+q}(\mathfrak{A} \otimes \mathfrak{B}), \quad p, q \in \mathbb{Z}_2$$

is a $\mathbb{Z}_2$-graded pairing defined as follows. Let us check it.

Denote by $M_n(\mathfrak{A})$ the C^*-algebra of all $n \times n$ matrices over a C^*-algebra $\mathfrak{A}$. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are unital C^*-algebras. The natural mappings:

$$i_{r,s} : M_r(\mathfrak{A}) \otimes M_s(\mathfrak{B}) \to M_{r+s}(\mathfrak{A} \otimes \mathfrak{B}),$$

for both r and s positive integers are defined as

$$i_{r,s}(a \otimes b) = \begin{pmatrix} a \otimes 1_r & 0 \\ 0 & 1_s \otimes b \end{pmatrix} \equiv (a \otimes 1_r) \oplus (1_s \otimes b)$$

for $a \in M_r(\mathfrak{A})$ and $b \in M_s(\mathfrak{B})$ with 1_r and 1_s the identity matrices of $M_r(\mathfrak{B})$ and $M_s(\mathfrak{A})$ respectively. The mappings induce a paring:

$$i_{0,0} : K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B}) \to K_0(\mathfrak{A} \otimes \mathfrak{B})$$

defined as $i_{0,0}([a] \otimes [b]) = [a \otimes 1_r] \oplus [1_s \otimes b]$ for $[a] \in K_0(\mathfrak{A})$ and $[b] \in K_0(\mathfrak{B})$, which is natural with respect to pairs of $*$-homomorphisms $f : \mathfrak{A} \to \mathfrak{A}'$ and $g : \mathfrak{B} \to \mathfrak{B}'$ of C^*-algebras.

Check that

$$\begin{aligned} i_{r,s}((a\otimes b)(c\otimes d)) &= i_{r,s}(ac\otimes bd) \\ &= \begin{pmatrix} ac\otimes 1_r & 0 \\ 0 & bd\otimes 1_s \end{pmatrix} = \begin{pmatrix} a\otimes 1_r & 0 \\ 0 & b\otimes 1_s \end{pmatrix}\begin{pmatrix} c\otimes 1_r & 0 \\ 0 & d\otimes 1_s \end{pmatrix} \\ &= i_{r,s}(a\otimes b)i_{r,s}(c\otimes d) \end{aligned}$$

for $a, c \in M_r(\mathfrak{A})$ and $b, d \in M_s(\mathfrak{B})$, it follows from which that $i_{r,s}$ is a $*$-homomorphism. Also,

$$\begin{aligned} i_{0,0}(([a]+[a'])\otimes [b]) &= i_{0,0}([a\otimes a']\otimes [b]) \\ &= [(a\oplus a')\otimes 1_{2r}]\oplus [1_s\otimes b] \\ &= [a\otimes 1_r]\oplus [1_s\otimes b] + [a'\otimes 1_r]\oplus [1_s\otimes b] \\ &= i_{0,0}([a]\otimes [b]) + i_{0,0}([a']\otimes [b]) \end{aligned}$$

and similarly,

$$i_{0,0}([a]\otimes ([b]+[b'])) = i_{0,0}([a]\otimes [b]) + i_{0,0}([a]\otimes [b']),$$

from which the map $i_{0,0}$ is a group homomorphism between abelian groups. The naturality of $i_{0,0}$ with respect to f and g above should mean that the following diagram is commutative:

$$\begin{tikzcd} K_0(\mathfrak{A})\otimes K_0(\mathfrak{B}) \arrow[r, "f_*\otimes g_*"] \arrow[d, "i_{0,0}"'] & K_0(\mathfrak{A}')\otimes K_0(\mathfrak{B}') \arrow[d, "i_{0,0}"] \\ K_0(\mathfrak{A}\otimes \mathfrak{B}) \arrow[r, "f_*\otimes g_*"] & K_0(\mathfrak{A}'\otimes \mathfrak{B}') \end{tikzcd}$$

where $f_*\otimes g_*([a]\otimes [b]) = [f(a)]\otimes [g(b)]$ for $[a]\in K_0(\mathfrak{A})$ and $[b]\in K_0(\mathfrak{B})$, and

$$\begin{aligned} (f_*\otimes g_*)\circ i_{0,0}([a]\otimes [b]) &= f_*\otimes g_*([a\otimes 1_r]\oplus [1_s\otimes b]) \\ &= [f(a)\otimes 1_r]\oplus [1_s\otimes g(b)] \\ &= i_{0,0}\circ (f_*\otimes g_*)([a]\otimes [b]). \end{aligned}$$

When $\mathfrak{A}$ is a nonunital C^*-algebra and $\mathfrak{B}$ is a unital C^*-algebra, we consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & K_0(\mathfrak{A})\otimes K_0(\mathfrak{B}) \arrow[r, "i_*\otimes \mathrm{id}"] \arrow[d] & K_0(\mathfrak{A}^+)\otimes K_0(\mathfrak{B}) \arrow[r, "q_*\otimes \mathrm{id}"] \arrow[d, "i_{0,0}"'] & K_0(\mathbb{C})\otimes K_0(\mathfrak{B}) \arrow[r] \arrow[d, "i_{0,0}"] & 0 \\ 0 \arrow[r] & K_0(\mathfrak{A}\otimes \mathfrak{B}) \arrow[r, "i_*\otimes \mathrm{id}"] & K_0(\mathfrak{A}^+\otimes \mathfrak{B}) \arrow[r, "q_*\otimes \mathrm{id}"] & K_0(\mathbb{C}\otimes \mathfrak{B}) \arrow[r] & 0 \end{tikzcd}$$

where $\mathfrak{A}^+$ is the unitization of $\mathfrak{A}$ and

$$0 \to \mathfrak{A} \xrightarrow{i} \mathfrak{A}^+ \xrightarrow{q} \mathbb{C}$$

is the short exact sequence associated to $\mathfrak{A}^+$, and one can show by chasing the left square in the diagram that there is a map from $K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})$ to $K_0(\mathfrak{A} \otimes \mathfrak{B})$ defined that becomes the left arrow from up to down in the diagram. Moreover, one can extend this case to show that the same thing holds when both $\mathfrak{A}$ and $\mathfrak{B}$ are non-unital.

On the other hand, note that as a fact in K-theory for C^*-algebras, $K_1(\mathfrak{A}) \cong K_0(S\mathfrak{A})$ for a C^*-algebra $\mathfrak{A}$, where $S\mathfrak{A} = C_0(\mathbb{R}) \otimes \mathfrak{A}$ is the suspension of $\mathfrak{A}$. Hence we get the map $i_{1,0} : K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B}) \to K_1(\mathfrak{A} \otimes \mathfrak{B})$ defined as

$$\begin{aligned} i_{1,0}: \quad K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B}) &\cong K_0(S\mathfrak{A}) \otimes K_0(\mathfrak{B}) \\ &\xrightarrow{i_{0,0}} K_0(S\mathfrak{A} \otimes \mathfrak{B}) = K_0(S(\mathfrak{A} \otimes \mathfrak{B})) \\ &\cong K_1(\mathfrak{A} \otimes \mathfrak{B}) \end{aligned}$$

and similarly, the map $i_{0,1} : K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B}) \to K_1(\mathfrak{A} \otimes \mathfrak{B})$ is defined, and

$$\begin{aligned} i_{1,1}: \quad K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B}) &\cong K_0(S\mathfrak{A}) \otimes K_0(S\mathfrak{B}) \\ &\xrightarrow{i_{0,0}} K_0(S\mathfrak{A} \otimes S\mathfrak{B}) = K_0(S(S(\mathfrak{A} \otimes \mathfrak{B}))) \\ &\cong K_0(\mathfrak{A} \otimes \mathfrak{B}) \end{aligned}$$

where the last isomorphism is obtained by the Bott periodicity.

We now obtain the map $\alpha : K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{A} \otimes \mathfrak{B})$ defined by

$$\alpha = i_{0,0} \oplus i_{0,1} \oplus i_{1,0} \oplus i_{1,1}.$$

That the map $\alpha : K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{A} \otimes \mathfrak{B})$ is a $\mathbb{Z}_2$ graded isomorphism means that both of the following maps:

$$\begin{aligned} [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})] &\xrightarrow{\quad\alpha\quad} K_0(\mathfrak{A} \otimes \mathfrak{B}), \\ [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})] &\xrightarrow{\quad\alpha\quad} K_1(\mathfrak{A} \otimes \mathfrak{B}) \end{aligned}$$

are isomorphisms.

Proposition 2.1. *Let $\mathfrak{A} = \varinjlim \mathfrak{A}_n$ be an inductive (or direct) limit of nuclear C^*-algebras $\mathfrak{A}_n$ ($n \in \mathbb{N}$ natural numbers). Suppose that $\alpha : K_*(\mathfrak{A}_n) \otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{A}_n \otimes \mathfrak{B})$ for all n are isomorphisms. Then $\alpha : K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{A} \otimes \mathfrak{B})$ is an isomorphism.*

Proof. By the nuclearity, we have

$$\mathfrak{A} \otimes \mathfrak{B} = (\varinjlim \mathfrak{A}_n) \otimes \mathfrak{B} \cong \varinjlim(\mathfrak{A}_n \otimes \mathfrak{B}).$$

Indeed, note that $(\varinjlim \mathfrak{A}_n) \otimes \mathfrak{B}$ is equal to $\overline{\cup_{n=1}^{\infty} \mathfrak{A}_n \otimes \mathfrak{B}}$ and $\varinjlim(\mathfrak{A}_n \otimes \mathfrak{B})$ is $\overline{\cup_{n=1}^{\infty}(\mathfrak{A}_n \otimes \mathfrak{B})}$, where the overlines mean the respective norm closures of the unions. Since $(\cup_{n=1}^{m} \mathfrak{A}_n) \otimes \mathfrak{B} = \cup_{n=1}^{m}(\mathfrak{A}_n \otimes \mathfrak{B})$ as a finite union for any m, the usual argument on proofs of an inclusion and its reverse inclusion about the closures shows the equality. And then we obtain the following diagram by continuity of K-theory groups:

$$\begin{array}{ccc}
\varinjlim[K_*(\mathfrak{A}_n) \otimes K_*(\mathfrak{B})] & \xrightarrow{\cong} & \varinjlim K_*(\mathfrak{A}_n \otimes \mathfrak{B}) \\
\cong\downarrow & & \downarrow\cong \\
[\varinjlim K_*(\mathfrak{A}_n)] \otimes K_*(\mathfrak{B}) & & K_*(\varinjlim(\mathfrak{A}_n \otimes \mathfrak{B})) \\
\cong\downarrow & & \downarrow\cong \\
K_*(\varinjlim \mathfrak{A}_n) \otimes K_*(\mathfrak{B}) & \longrightarrow & K_*((\varinjlim \mathfrak{A}_n) \otimes \mathfrak{B})
\end{array}$$

where the last arrow at the bottom line is obtained as the clockwise composition of the isomorphisms in the diagram, and $\varinjlim K_*(\cdot)$ means the inductive limit as a group with respect to n. □

Proposition 2.2. *Suppose that $\alpha : K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{A} \otimes \mathfrak{B})$ is an isomorphism. Then for the crossed product $\mathfrak{A} \rtimes_\gamma \mathbb{R}$ by an action γ of $\mathbb{R}$ on $\mathfrak{A}$, the map $\alpha : K_*(\mathfrak{A} \rtimes_\gamma \mathbb{R}) \otimes K_*(\mathfrak{B}) \to K_*((\mathfrak{A} \rtimes_\gamma \mathbb{R}) \otimes \mathfrak{B})$ is an isomorphism.*

Proof. The Connes' Thom isomorphism for K-theory groups of crossed product C^*-algebras by $\mathbb{R}$ says that

$$K_j(\mathfrak{A} \rtimes_\gamma \mathbb{R}) \cong K_{j+1}(\mathfrak{A})$$

for $j = 0, 1$. Note that $(\mathfrak{A} \rtimes_\gamma \mathbb{R}) \otimes \mathfrak{B} \cong (\mathfrak{A} \otimes \mathfrak{B}) \rtimes_{\gamma \otimes \mathrm{id}} \mathbb{R}$. Therefore,

$$\begin{aligned}
&[K_0(\mathfrak{A} \rtimes_\gamma \mathbb{R}) \oplus K_1(\mathfrak{A} \rtimes_\gamma \mathbb{R})] \otimes K_*(\mathfrak{B}) \\
&\cong [K_1(\mathfrak{A}) \oplus K_0(\mathfrak{A})] \otimes K_*(\mathfrak{B}) \cong K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \\
&\xrightarrow{\alpha} K_*(\mathfrak{A} \otimes \mathfrak{B}) \\
&\cong K_1((\mathfrak{A} \otimes \mathfrak{B}) \rtimes_{\gamma \otimes \mathrm{id}} \mathbb{R}) \oplus K_0((\mathfrak{A} \otimes \mathfrak{B}) \rtimes_{\gamma \otimes \mathrm{id}} \mathbb{R}) \\
&\cong K_*((\mathfrak{A} \rtimes_\gamma \mathbb{R}) \otimes \mathfrak{B}).
\end{aligned}$$

□

If we assume that $K_*(\mathfrak{B})$ is torsion free, then the torsion product $\mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B}))$ is always trivial for any C^*-algebra $\mathfrak{A}$, so that the Künneth formula is just the statement that the map $\alpha : K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{A} \otimes \mathfrak{B})$ is an isomorphism.

Proposition 2.3. *Let*

$$0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{J} \to 0$$

be a short exact sequence of nuclear C^*-algebras. Suppose that $K_*(\mathfrak{B})$ is torsion free. If any two among the three maps:

$$\begin{aligned} &\alpha : K_*(\mathfrak{J}) \otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{J} \otimes \mathfrak{B}), \\ &\alpha : K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{A} \otimes \mathfrak{B}), \\ &\alpha : K_*(\mathfrak{A}/\mathfrak{J}) \otimes K_*(\mathfrak{B}) \to K_*((\mathfrak{A}/\mathfrak{J}) \otimes \mathfrak{B}) \end{aligned}$$

are isomorphisms, then so is the other map.

Proof. There is the six-term exact sequence for the short exact sequence of C^*-algebras:

$$\begin{tikzcd} K_0(\mathfrak{J}) \arrow[r] & K_0(\mathfrak{A}) \arrow[r] & K_0(\mathfrak{A}/\mathfrak{J}) \arrow[d, "\partial"] \\ K_1(\mathfrak{A}/\mathfrak{J}) \arrow[u, "\partial"] & K_1(\mathfrak{A}) \arrow[l] & K_1(\mathfrak{J}) \arrow[l] \end{tikzcd}$$

This implies the following diagram:

$$\begin{tikzcd} K_*(\mathfrak{J}) \arrow[r] & K_*(\mathfrak{A}) \arrow[r] & K_*(\mathfrak{A}/\mathfrak{J}) \arrow[d, "\partial\oplus\partial"] \\ K_{*+1}(\mathfrak{A}/\mathfrak{J}) \arrow[u, "\partial\oplus\partial"] & K_{*+1}(\mathfrak{A}) \arrow[l] & K_{*+1}(\mathfrak{J}) \arrow[l] \end{tikzcd}$$

where $K_*(\cdot) = K_0(\cdot) \oplus K_1(\cdot)$. Since the torsion-free groups are flat, tensoring the diagram with $K_*(\mathfrak{B}) = K_0(\mathfrak{B}) \oplus K_1(\mathfrak{B})$ we have

$$\begin{tikzcd} K_*(\mathfrak{J}) \otimes K_*(\mathfrak{B}) \arrow[r] & K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \arrow[r] & K_*(\mathfrak{A}/\mathfrak{J}) \otimes K_*(\mathfrak{B}) \arrow[d, "(\partial\oplus\partial)\otimes\mathrm{id}"] \\ K_{*+1}(\mathfrak{A}/\mathfrak{J}) \otimes K_*(\mathfrak{B}) \arrow[u, "(\partial\oplus\partial)\otimes\mathrm{id}"] & K_{*+1}(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \arrow[l] & K_{*+1}(\mathfrak{J}) \otimes K_*(\mathfrak{B}) \arrow[l] \end{tikzcd}$$

where $K_{*+1} = K_1(\cdot) \oplus K_0(\cdot)$.

If we assume that $K_*(\mathfrak{I})\otimes K_*(\mathfrak{B}) \cong K_*(\mathfrak{I}\otimes\mathfrak{B})$ and $K_*(\mathfrak{A}/\mathfrak{I})\otimes K_*(\mathfrak{B}) \cong K_*((\mathfrak{A}/\mathfrak{I}) \otimes \mathfrak{B})$ by α, then we have the following commutative diagram by naturality for α:

$$\begin{array}{ccc}
K_{*+1}(\mathfrak{A}/\mathfrak{I}) \otimes K_*(\mathfrak{B}) & \xrightarrow[\cong]{\alpha} & K_{*+1}((\mathfrak{A}/\mathfrak{I}) \otimes \mathfrak{B}) \\
\partial\downarrow & & \partial\downarrow \\
K_*(\mathfrak{I}) \otimes K_*(\mathfrak{B}) & \xrightarrow[\cong]{\alpha} & K_*(\mathfrak{I} \otimes \mathfrak{B}) \\
\downarrow & & \downarrow \\
K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) & \xrightarrow{\alpha} & K_*(\mathfrak{A} \otimes \mathfrak{B}) \\
\downarrow & & \downarrow \\
K_*(\mathfrak{A}/\mathfrak{I}) \otimes K_*(\mathfrak{B}) & \xrightarrow[\cong]{\alpha} & K_*((\mathfrak{A}/\mathfrak{I}) \otimes \mathfrak{B}) \\
\partial\downarrow & & \partial\downarrow \\
K_{*+1}(\mathfrak{I}) \otimes K_*(\mathfrak{B}) & \xrightarrow[\cong]{\alpha} & K_{*+1}(\mathfrak{I} \otimes \mathfrak{B}).
\end{array}$$

Therefore, the Five Lemma implies that the map α in the middle in the diagram is an isomorphism.

The other two cases are proved similarly. $\square$

Proposition 2.4. *Suppose that $K_*(\mathfrak{B})$ is torsion free, and $\alpha : K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{A} \otimes \mathfrak{B})$ is an isomorphism, with $\mathfrak{A}$ nuclear. Then for the crossed product $\mathfrak{A} \rtimes_\gamma \mathbb{Z}$ by an action γ of $\mathbb{Z}$ on $\mathfrak{A}$, the map $\alpha : K_*(\mathfrak{A} \rtimes_\gamma \mathbb{Z}) \otimes K_*(\mathfrak{B}) \to K_*((\mathfrak{A} \rtimes_\gamma \mathbb{Z}) \otimes \mathfrak{B})$ is an isomorphism.*

Proof. The Pimsner-Voiculescu six-term exact sequence for K-theory groups of crossed product C^*-algebras by $\mathbb{Z}$ is the following diagram:

$$\begin{array}{ccccc}
K_0(\mathfrak{A}) & \xrightarrow{(\mathrm{id}-\gamma)_*} & K_0(\mathfrak{A}) & \xrightarrow{i_*} & K_0(\mathfrak{A} \rtimes_\gamma \mathbb{Z}) \\
\partial\uparrow & & & & \downarrow\partial \\
K_1(\mathfrak{A} \rtimes_\gamma \mathbb{Z}) & \xleftarrow{i_*} & K_1(\mathfrak{A}) & \xleftarrow{(\mathrm{id}-\gamma)_*} & K_1(\mathfrak{A})
\end{array}$$

where $i : \mathfrak{A} \to \mathfrak{A} \rtimes_\gamma \mathbb{Z}$ is the canonical inclusion map. This diagram implies that

$$\begin{array}{ccccc}
K_*(\mathfrak{A}) & \xrightarrow{(\mathrm{id}-\gamma)_*} & K_*(\mathfrak{A}) & \xrightarrow{i_*} & K_*(\mathfrak{A} \rtimes_\gamma \mathbb{Z}) \\
\partial\oplus\partial\uparrow & & & & \downarrow\partial\oplus\partial \\
K_{*+1}(\mathfrak{A} \rtimes_\gamma \mathbb{Z}) & \xleftarrow{i_*} & K_{*+1}(\mathfrak{A}) & \xleftarrow{(\mathrm{id}-\gamma)_*} & K_{*+1}(\mathfrak{A})
\end{array}$$

Since $K_*(\mathfrak{B})$ is torsion free, we have

$$\begin{tikzcd} K_{*+1}(\mathfrak{A} \rtimes_\gamma \mathbb{Z}) \otimes K_*(\mathfrak{B}) \arrow[r, "\partial\oplus\partial"] & K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \arrow[d, "(\mathrm{id}-\gamma)_*\otimes \mathrm{id}_*"] \\ K_{*+1}(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \arrow[u, "i_*\otimes \mathrm{id}_*"] & K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \arrow[d, "i_*\otimes \mathrm{id}_*"] \\ K_{*+1}(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \arrow[u, "(\mathrm{id}-\gamma)_*\otimes \mathrm{id}_*"] & K_*(\mathfrak{A} \rtimes_\gamma \mathbb{Z}) \otimes K_*(\mathfrak{B}). \arrow[l, "\partial\oplus\partial"'] \end{tikzcd}$$

We now note that $(\mathfrak{A} \otimes \mathfrak{B}) \rtimes_{\gamma\otimes\mathrm{id}} \mathbb{Z} \cong (\mathfrak{A} \rtimes_\gamma \mathbb{Z}) \otimes \mathfrak{B}$. Thus, the diagram before the last one becomes by replacing $\mathfrak{A}$ with $\mathfrak{A} \otimes \mathfrak{B}$ as:

$$\begin{tikzcd} K_{*+1}((\mathfrak{A} \rtimes_\gamma \mathbb{Z}) \otimes \mathfrak{B}) \arrow[r, "\partial\oplus\partial"] & K_*(\mathfrak{A} \otimes \mathfrak{B}) \arrow[d, "(\mathrm{id}\otimes\mathrm{id}-\gamma\otimes\mathrm{id})_*"] \\ K_{*+1}(\mathfrak{A} \otimes \mathfrak{B}) \arrow[u, "i_*"] & K_*(\mathfrak{A} \otimes \mathfrak{B}) \arrow[d, "i_*"] \\ K_{*+1}(\mathfrak{A} \otimes \mathfrak{B}) \arrow[u, "(\mathrm{id}\otimes\mathrm{id}-\gamma\otimes\mathrm{id})_*"] & K_*((\mathfrak{A} \rtimes_\gamma \mathbb{Z}) \otimes \mathfrak{B}). \arrow[l, "\partial\oplus\partial"'] \end{tikzcd}$$

The hypotheses imply that the four terms among six terms in the diagram before the last one are isomorphic by $\alpha : K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{A} \otimes \mathfrak{B})$ respectively to the four terms among six terms in the last diagram. Therefore, the Five Lemma implies that the map $\alpha : K_*(\mathfrak{A} \rtimes_\gamma \mathbb{Z}) \otimes K_*(\mathfrak{B}) \to K_*((\mathfrak{A} \rtimes_\gamma \mathbb{Z}) \otimes \mathfrak{B})$ is an isomorphism. $\square$

Proposition 2.5. *Suppose that $K_*(\mathfrak{B})$ is torsion free. Then for $C_0(Y)$ the C^*-algebra of all continuous functions on a locally compact Hausdorff space Y vanishing at infinity, the map $\alpha : K_*(C_0(Y)) \otimes K_*(\mathfrak{B}) \to K_*(C_0(Y) \otimes \mathfrak{B})$ is an isomorphism.*

Proof. If $Y = \mathbb{R}^n$, then

$$K_*(C_0(Y)) \cong K_{*+n}(\mathbb{C}),$$
$$K_*(C_0(Y) \otimes \mathfrak{B}) \cong K_{*+n}(\mathbb{C} \otimes \mathfrak{B})$$

by the Bott periodicity, and $K_{*+n}(\cdot) \cong K_*(\cdot)$, and

$$K_*(\mathbb{C}) \otimes K_*(\mathfrak{B}) \cong \mathbb{Z} \otimes K_*(\mathfrak{B}) \cong K_*(\mathfrak{B}) \cong K_*(\mathbb{C} \otimes \mathfrak{B}).$$

If $Y = S^n$ the n-dimensional sphere, then there is the following short exact sequence of C^*-algebras:

$$0 \to C_0(\mathbb{R}^n) \to C(S^n) \to \mathbb{C} \to 0.$$

Since the maps α for pairs $(C_0(\mathbb{R}^n), \mathfrak{B})$ and $(\mathbb{C}, \mathfrak{B})$ are isomorphisms, the map α for $(C(S^n), \mathfrak{B})$ is also an isomorphism by applying the corresponding proposition above.

If Y is a finite complex, then the map α for $(C_0(Y), \mathfrak{B})$ is an isomorphism by induction on the number of cells of Y.

If Y is a compact (Hausdorff) space, then it can be written as an inverse limit of finite complexes Y_j (by Eilenberg-Steenrod), so that $C(Y) \cong \varinjlim C(Y_j)$ an inductive (or directed) limit of C^*-algebras. Hence the corresponding proposition above implies that the map α is an isomorphism.

If Y is not compact in general, then there is the following short exact sequence:

$$0 \to C_0(Y) \to C(Y^+) \to \mathbb{C} \to 0$$

where Y^+ is the one-point compactification of Y. Since the maps α for pairs $(C(Y^+), \mathfrak{B})$ and $(\mathbb{C}, \mathfrak{B})$ are isomorphisms, the map α for $(C_0(Y), \mathfrak{B})$ is also an isomorphism. $\square$

Recall that a C^*-algebra $\mathfrak{A}$ is said to be solvable if there is an ascending sequence of closed ideals $\mathfrak{A}_n$ of $\mathfrak{A}$ such that $\mathfrak{A}$ is the closure of the union $\cup_n \mathfrak{A}_n$ and subquotients $\mathfrak{A}_n/\mathfrak{A}_{n-1} \cong C_0(Y_n) \otimes \mathbb{K}(H_n)$ for some locally compact Hausdorff space Y_n and $\mathbb{K}(H_n)$ the C^*-algebra of all compact operators on some finite or infinite dimensional Hilbert space H_n.

Proposition 2.6. *If $K_*(\mathfrak{B})$ is torsion free and $\mathfrak{A}$ is solvable, then the map α for $(\mathfrak{A}, \mathfrak{B})$ is an isomorphism.*

Proof. This follows from the propositions above concerning inductive limits and extensions of C^*-algebras.

There is another proof as follows. Define functors L_q and M_q by

$$L_q(\cdot) = (K_*(\cdot) \otimes K_*(\mathfrak{B}))_q, \quad M_q(\cdot) = K_q((\cdot) \otimes \mathfrak{B}).$$

Each of these functors satisfy the exactness axiom and α induces a natural transformation $\alpha : L_* \to M_*$ which is an isomorphism for C^*-algebras that are tensor products of commutative C^*-algebras and $\mathbb{K}(H)$. A spectral sequence comparison theorem of Schochet implies that $\alpha : L_*(\mathfrak{A}) \to M_*(\mathfrak{A})$ is an isomorphism for any solvable C^*-algebra. (Not checked.) $\square$

Theorem 2.7. *If $K_*(\mathfrak{B})$ is torsion free and $\mathfrak{A}$ is separable C^*-algebra of type I, then the map α for $(\mathfrak{A}, \mathfrak{B})$ is an isomorphism.*

Proof. Since $\mathfrak{A}$ is a separable type I C^*-algebra, $\mathfrak{A}$ has a countable composition series of closed ideals $\mathfrak{I}_j$ such that each subquotient $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ has continuous trace. Using the propositions above concerning inductive limits and extensions of C^*-algebras repeatedly if necessary, we may assume that $\mathfrak{A}$ has continuous trace and $\mathbb{K}$ as a tensor factor. Such a continuous trace C^*-algebra is infinite homogeneous and hence the associated continuous field of elementary C^*-algebras over the spectrum $\mathfrak{A}^\wedge$ of $\mathfrak{A}$ (or the primitive ideal space of $\mathfrak{A}$) is locally trivial. Let $\{U_j\}$ be a countable open cover for the spectrum $\mathfrak{A}^\wedge$ such that each continuous field restricted to U_j is trivial. There is an increasing sequence of closed ideals $\mathfrak{L}_i$ of $\mathfrak{A}$ corresponding to the sequence of open sets $\cup_{j=1}^{i} U_j$ in $\mathfrak{A}^\wedge$ such that $\mathfrak{L}_i/\mathfrak{L}_{i-1} \cong C_0(Y_i) \otimes \mathbb{K}$ for some locally compact Hausdorff space Y_i. Hence $\mathfrak{A}$ is solvable in that sense. □

Remark. But, possibly, in each step, finite dimensionality of the spectrum should be necessary. Otherwise, such a splitting with $\mathbb{K}$ is not always clear, so that the argument may collapse, probably. However, one could avoid such a difficulty from infinite dimensionality by taking a suitable inverse limit of spaces with finite dimensions, to reduce to the cases of finite dimensions.

The results obtained in this section deduce that

Theorem 2.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. If $\mathfrak{A} \in \mathfrak{N}$ and $K_*(\mathfrak{B})$ is torsion free, then there is a natural isomorphism:*

$$\alpha : K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{A} \otimes \mathfrak{B}).$$

3 Geometric realization

Let $\mathfrak{B}$ be a unital C^*-algebra. Then for $q = 0, 1 \in \mathbb{Z}_2$, there are free ablian groups G'_q and G''_q such that

$$0 \to G'_q \to G''_q \to K_q(\mathfrak{B}) \to 0$$

is a free resolution of $K_q(\mathfrak{B})$. Moreover, one can find C^*-algebras $\mathfrak{B}'$ and $\mathfrak{B}''$ such that

$$K_q(\mathfrak{B}') \cong G'_q \quad \text{and} \quad K_q(\mathfrak{B}'') = G''_q$$

and hence

$$0 \to K_q(\mathfrak{B}') \xrightarrow{\ \nu\ } K_q(\mathfrak{B}'') \xrightarrow{\ \eta\ } K_q(\mathfrak{B}) \to 0$$

for $q = 0, 1 \in \mathbb{Z}_2$. However, the maps ν and η do not arise from maps at the level of C^*-algebras, so that such resolutions are said to be not geometric. In this section, given is the construction of geometric resolutions as:

$$0 \to K_q(\mathfrak{C}) \to K_q(\mathfrak{F}) \to K_q(\mathfrak{B}) \to 0$$

some C^*-algebras $\mathfrak{C}$ and $\mathfrak{F}$ given below, with useful properties, enough to prove the Künneth theorem in the next section.

Lemma 3.1. *Let $\mathfrak{B}$ be a unital C^*-algebra. Then there is a commutative C^*-algebra $\mathfrak{F} = C_0(Y)$ and an inclusion map $\mu : \mathfrak{F} \to \mathfrak{B} \otimes \mathbb{K}(H)$ such that the induced map*

$$\mu_* : K_*(\mathfrak{F}) \to K_*(\mathfrak{B})$$

is surjective, where the space Y is given by a disjoint union of points and copies of $\mathbb{R}$.

If $K_*(\mathfrak{B})$ is free abelian, then μ_* is an isomorphism.

If $K_*(\mathfrak{B})$ is countably generated (for example, if $\mathfrak{B}$ is separable), then H is separable and Y has countably many path components, so Y can be embedded into the plane.

Proof. Select a minimal family of generators for $K_*(\mathfrak{B})$. Each generator of $K_0(\mathfrak{B})$ is of the form $[p_s] - r(s)[1]$ with $r(s) \in \mathbb{Z}$, where $[1]$ represents the class of the identity of $\mathfrak{B}$ and $p_s \in \mathfrak{B} \otimes \mathbb{K}(H_s)$ a projection, with H_s a finite-dimensional Hilbert space, so that $\mathbb{K}(H_s) = M_{n(s)}(\mathbb{C})$ with $n(s) = \dim H_s$. Each generator of $K_1(\mathfrak{B})$ can be represented by some unitary $u_t \in \mathfrak{B} \otimes \mathbb{K}(H_t)$ with H_t of finite dimension. Let H be the Hilbert space direct sum of Hilbert spaces $\{H_s\}_{s \in S}$ and $\{H_t\}_{t \in T}$ with S, T index sets. Define elements in $\mathfrak{B} \otimes \mathbb{K}(H)$ by $p_s \equiv p_s \oplus 0$ (an identification) with respect to the inclusion:

$$[\mathfrak{B} \otimes \mathbb{K}(H_s)] \oplus [\mathfrak{B} \otimes \mathbb{B}(H_s^{\perp})] \subset \mathfrak{B} \otimes \mathbb{B}(H)$$

where $\mathbb{B}(H)$ and $\mathbb{B}(H_s^{\perp})$ are the C^*-algebras of all bounded operators on Hilbert spaces H and $H_s^{\perp}$ the orthogonal complement to H_s in H. Also define $w_t = (u_t - 1) \oplus 0 \in \mathfrak{B} \otimes \mathbb{K}(H)$ with respect to the inclusion:

$$[\mathfrak{B} \otimes \mathbb{K}(H_t)] \oplus [\mathfrak{B} \otimes \mathbb{B}(H_t^{\perp})] \subset \mathfrak{B} \otimes \mathbb{B}(H).$$

Then we have

$$\begin{aligned} p_s p_{s'} &= (p_s \oplus 0)(p_{s'} \oplus 0) = 0, \\ w_t w_{t'} &= ((u_t - 1) \oplus 0)((u_{t'} - 1) \oplus 0) = 0, \\ p_s w_t &= (p_s \oplus 0)((u_t - 1) \oplus 0) = 0 \end{aligned}$$

for $s \neq s'$, $t \neq t'$ and any s, t, because each $H_s \perp H_{s'}$, $H_t \perp H_{t'}$, and $H_s \perp H_t$ orthogonal in H. Further, we have the following identifications $[p_s] \in K_0(\mathfrak{B} \otimes \mathbb{K}) \cong K_0(\mathfrak{B}) \ni [p_s]$ and $[w_t + 1] \in K_1((\mathfrak{B} \otimes \mathbb{K})^+) \cong K_1(\mathfrak{B}) \ni [u_t]$. Check that $[u_t]$ corresponds to the connected component of the diagonal sum $u_t \oplus 1_\infty$, which is equal to $((u_t - 1) + 1) \oplus 1_\infty = w_t + (1 \oplus 1_\infty)$, which may be identified with $(w_t, 1) \in (\mathfrak{B} \otimes \mathbb{K})^+$, and its K_1-class is $[w_t + 1]$.

Let $\mathfrak{F}$ be the C^*-subalgebra of $\mathfrak{B} \otimes \mathbb{K}(H)$ generated by $\{p_s\}_{s \in S}$ and $\{w_t\}_{t \in T}$ with S, T index sets. Then $\mathfrak{F}$ is commutative. Indeed, $\mathfrak{F}$ is the direct sum of the C^*-algebras $C^*(p_s)$ ($s \in S$) generated by p_s and the C^*-algebras $C^*(w_t)$ ($t \in T$) generated by $w_t = u_t - 1$. Since each p_s is a projection, $C^*(p_s) \cong \mathbb{C}$, whose spectrum contributes a discrete point in the spectrum Y of $\mathfrak{F}$. Since w_t+1 identified with $u_t \oplus 1_\infty$ is unitary, the spectrum X_t of $C^*(w_t)^+$ is a closed subset of the unit circle S^1, and the class $[u_t]$ of u_t is non-zero in $K_1(\mathfrak{B})$, so that $C^*(w_t)^+ \cong C(S^1)$. Because since the class $[u_t]$ is one generator in $K_1(\mathfrak{B})$, the closed set X_t is connected, and if X_t is not equal to S^1, then u_t is connected to the unit within unitaries, so that the class $[u_s]$ is zero, and thus, X_t must be S^1. It follows that $C^*(w_t) \cong C_0(\mathbb{R})$. Therefore, $\mathfrak{F}$ is isomorphic to the direct sum $(\oplus_{s \in S}\mathbb{C}) \oplus (\oplus_{t \in T} C_0(\mathbb{R}))$. Hence the spectrum (or the maximal ideal space) Y of $\mathfrak{F}$ is the disjoint union $(\sqcup_{s \in S}\{\text{point}\}) \sqcup (\sqcup_{t \in T}\mathbb{R})$.

Let $\mu : \mathfrak{F} \to \mathfrak{B} \otimes \mathbb{K}(H)$ be the inclusion map. Then $\mu^+ : \mathfrak{F}^+ \to (\mathfrak{B} \otimes \mathbb{K}(H))^+$ is unital, and so

$$\begin{aligned}
\mu_*^+([p_s] - r(s)[1]) &= [\mu(p_s)] - r(s)[\mu^+(1)] = [p_s] - r(s)[1] \in K_0(\mathfrak{B}), \\
\mu_*^+([w_t + 1]) &= \mu_*^+([u_t \oplus 1_\infty]) = [\mu^+(u_t, 1)] \\
&= [(u_t, 1)] = [u_t \oplus 1_\infty] = [u_t] \in K_1(\mathfrak{B}).
\end{aligned}$$

Thus, $\mu_*^+ : K_*(\mathfrak{F}^+) \to K_*((\mathfrak{B} \otimes \mathbb{K}(H))^+)$ is surjective. Since $K_0(A^+) \cong K_0(A) \oplus \mathbb{Z}$ and $K_1(A^+) \cong K_1(A)$ for any C^*-algebra A, it follows that the induced map $\mu_* : K_*(\mathfrak{F}) \to K_*(\mathfrak{B} \otimes \mathbb{K}(H)) \cong K_*(\mathfrak{B})$ is also surjective.

If $K_*(\mathfrak{B})$ is free abelian, then the map μ_*^+ and also μ_* are isomorphisms.

Finally, if $K_*(\mathfrak{B})$ is countably generated, then H is a separable Hilbert space and Y has countably many path components, by the construction above. In this case, if $Y = (\sqcup_{s \in \mathbb{N}}\{\text{point}\}) \sqcup (\sqcup_{t \in \mathbb{N}}\mathbb{R})$ with $\mathbb{N}$ of positive integers, then the points may be identified with positive integers in the real line in the place, and the lines $\mathbb{R}$ may be identified with the vertical lines which go through negative integers in the real line in the plane, so that Y can be embedded into the plane. $\square$

Suppose given a unital C^*-algebra $\mathfrak{B}$. The lemma above implies that there is a commutative C^*-algebra $\mathfrak{F} = C_0(Y)$ and an inclusion map $\mu : \mathfrak{F} \to \mathfrak{B} \otimes \mathbb{K}$ such that the induced map

$$\mu_* : K_*(\mathfrak{F}) \to K_*(\mathfrak{B} \otimes \mathbb{K}) \cong K_*(\mathfrak{B})$$

is surjective. Let $\mathfrak{C}$ be the mapping cone of μ defined as:

$$\mathfrak{C} = \{(g, x) \mid g \in C([0,1], \mathfrak{B} \otimes \mathbb{K}), x \in \mathfrak{F}, g(0) = 0, g(1) = \mu(x)\},$$

where $C([0,1], \mathfrak{B} \otimes \mathbb{K})$ is the C^*-algebra of all $\mathfrak{B} \otimes \mathbb{K}$-valued continuous functions on the interval $[0,1]$. This is a C^*-algebra. Because for $(g, x), (h, y) \in \mathfrak{C}$, we have, with $\|\cdot\|$ the norm,

$$\begin{aligned}
(g,x) + (h,y) &= (g+h, x+y) \quad \text{and} \quad (g+h)(0) = 0, (g+h)(1) = \mu(x+y), \\
(g,x)(h,y) &= (gh, xy) \quad \text{and} \quad (gh)(0) = 0, (gh)(1) = \mu(xy), \\
(g,x)^* &= (g^*, x^*) \quad \text{and} \quad g^*(0) = 0, g^*(1) = \mu(x^*), \\
\|(g,x)^*(g,x)\| &= \|(g^*g, x^*x)\| = \max\{\|g^*g\|, \|x^*x\|\} \\
&= \max\{\|g\|^2, \|x\|^2\} = (\max\{\|g\|, \|x\|\})^2 = \|(g,x)\|^2.
\end{aligned}$$

Let $\nu : \mathfrak{C} \to \mathfrak{F}$ be the $*$-homomorphism defined by $\nu(g, x) = x \in \mathfrak{F}$. Then ν is surjective, and

$$\begin{aligned}
\ker(\mu) &= \{(g,x) \in \mathfrak{C} \mid x = 0\} \\
&= \{g \in C([0,1], \mathfrak{B} \otimes \mathbb{K}) \mid g(0) = g(1) = 0\} \\
&\cong \mathfrak{B} \otimes \mathbb{K} \otimes C_0(\mathbb{R}).
\end{aligned}$$

So there is the following exact sequence:

$$0 \longrightarrow \mathfrak{B} \otimes \mathbb{K} \otimes C_0(\mathbb{R}) \longrightarrow \mathfrak{C} \xrightarrow{\nu} \mathfrak{F} \longrightarrow 0.$$

And the associated six-term exact sequence is the following:

$$\begin{array}{ccccc}
K_0(\mathfrak{B} \otimes \mathbb{K} \otimes C_0(\mathbb{R})) & \longrightarrow & K_0(\mathfrak{C}) & \xrightarrow{\nu_*} & K_0(\mathfrak{F}) \\
\partial \big\uparrow & & & & \big\downarrow \partial \\
K_1(\mathfrak{F}) & \xleftarrow{\nu_*} & K_1(\mathfrak{C}) & \longleftarrow & K_1(\mathfrak{B} \otimes \mathbb{K} \otimes C_0(\mathbb{R})).
\end{array}$$

Moreover, the boundary maps:

$$\partial : K_q(\mathfrak{F}) \to K_{q+1}(\mathfrak{B} \otimes \mathbb{K} \otimes C_0(\mathbb{R})) \cong K_q(\mathfrak{B} \otimes \mathbb{K})$$

for $q = 0, 1$ correspond to the maps $\mu_* : K_q(\mathfrak{F}) \to K_q(\mathfrak{B} \otimes \mathbb{K})$. Indeed, $\partial([u]) = [w1_n w^*] - n[1]$ for $[u] \in K_1(\mathfrak{F})$ with u a $n \times n$ unitary matrix over $\mathfrak{F}$ and w is a unitary lift of $u \oplus u^*$ over $\mathfrak{C}$ by definition, but this class is converted to $[u]$ by the isomorphism from $K_0(\mathfrak{B} \otimes \mathbb{K} \otimes C_0(\mathbb{R}))$ to $K_1(\mathfrak{B} \otimes \mathbb{K})$. See [9] for details on these. The same also holds for

$$\partial : K_0(\mathfrak{A}) \cong K_1(\mathfrak{A} \otimes C_0(\mathbb{R})) \to K_1(\mathfrak{B} \otimes \mathbb{K} \otimes C_0(\mathbb{R})) \cong K_0(\mathfrak{B} \otimes \mathbb{K} \otimes C_0(\mathbb{R}^2)).$$

Thus the maps ∂ are surjective. It follows that the six-term exact sequence above splits into the following short exact sequences:

$$0 \to K_q(\mathfrak{C}) \xrightarrow{\nu_*} K_q(\mathfrak{F}) \xrightarrow{\partial} K_{q+1}(\mathfrak{B} \otimes \mathbb{K} \otimes C_0(\mathbb{R})) \to 0$$

for $q = 0, 1$, and $K_{q+1}(\mathfrak{B} \otimes \mathbb{K} \otimes C_0(\mathbb{R})) \cong K_q(\mathfrak{B})$. This implies that $K_q(\mathfrak{C})$ for $q = 0, 1$ are free abelian groups since so are $K_q(\mathfrak{F})$. This also becomes a geometric resolution for $K_*(\mathfrak{B})$.

Remark. It is also shown that there is an injective resolution as

$$0 \to K_*(\mathfrak{B}) \to K_*(\mathfrak{J}_1) \to K_*(\mathfrak{J}_2) \to 0$$

with each $K_*(\mathfrak{J}_j)$ $(j = 1, 2)$ injective (i.e., divisible) abelian groups.

Recall from (a math dictionary [5]) that an additive abelian group is viewed as a $\mathbb{Z}$-module. A $\mathbb{Z}$-module G is said to be divisible (or complete) if for any $g \in G$ and any $n \in \mathbb{Z}$, there is $x_n \in G$ such that $g = nx_n = x_n + \cdots + x_n$ ($n-1$-times sum). A torsion free, divisible $\mathbb{Z}$-module coinsides with a direct product of additive $\mathbb{Q}$ of all rationals.

Also established are the Künneth theorems and the universal coefficient theorems (UCT) for the Kasparov groups $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B})$ for C^*-algebras $\mathfrak{A}, \mathfrak{B}$ with $\mathfrak{A} \in \mathfrak{N}$, which classify extensions of closed ideals $\mathfrak{B} \otimes \mathbb{K}$ by the quotients $\mathfrak{A}$, up to stable equivalence, using geometric realization techniques. See the next article in this volume of RMJ.

4 The Künneth formula in the general case

Theorem 4.1. (Künneth formula). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A} \in \mathfrak{N}$. Then there is the following short exact sequence:*

$$0 \to K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \xrightarrow{\alpha} K_*(\mathfrak{A} \otimes \mathfrak{B}) \xrightarrow{\beta} \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to 0$$

with α of degree 0 and β of degree 1. The sequence is natural for maps of pairs $(\mathfrak{A}, \mathfrak{B}) \to (\mathfrak{A}, \mathfrak{B}')$.

The following proof uses only a result obtained above that $\alpha : K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{A} \otimes \mathfrak{B})$ is an isomorphism if $K_*(\mathfrak{B})$ is torsion free.

Proof. (The unital case). Suppose first that $\mathfrak{B}$ is unital. Then there is a geometric resolution of the form:

$$0 \to \mathfrak{B} \otimes \mathbb{K} \otimes C_0(\mathbb{R}) \to \mathfrak{C} \xrightarrow{\nu} \mathfrak{F} \to 0$$

with the inclusion map $\mu : \mathfrak{F} \to \mathfrak{B} \otimes \mathbb{K}$ and

$$\mathfrak{C} = C_0((0,1], \{\mathfrak{B} \otimes \mathbb{K}\}_{t\in(0,1)} \cup \{\mu(\mathfrak{F})\}_{t=1})$$

the mapping cone of ν. Tensoring the sequence above with $\mathfrak{A}$ a nuclear C^*-algebra since $\mathfrak{A} \in \mathfrak{N}$ yields the following:

$$0 \to \mathfrak{A} \otimes \mathfrak{B} \otimes \mathbb{K} \otimes C_0(\mathbb{R}) \to \mathfrak{A} \otimes \mathfrak{C} \xrightarrow{\mathrm{id}\otimes\nu} \mathfrak{A} \otimes \mathfrak{F} \to 0.$$

Then there is the following six-term exact sequence:

$$\begin{array}{ccc} K_1(\mathfrak{A}\otimes\mathfrak{F}) & \xrightarrow{\partial} & K_0(\mathfrak{A}\otimes\mathfrak{B}\otimes\mathbb{K}\otimes C_0(\mathbb{R})) \\ {\scriptstyle(\mathrm{id}\otimes\nu)_*}\uparrow & & \downarrow \\ K_1(\mathfrak{A}\otimes\mathfrak{C}) & & K_0(\mathfrak{A}\otimes\mathfrak{C}) \\ \uparrow & & \downarrow{\scriptstyle(\mathrm{id}\otimes\nu)_*} \\ K_1(\mathfrak{A}\otimes\mathfrak{B}\otimes\mathbb{K}\otimes C_0(\mathbb{R})) & \xleftarrow{\partial} & K_0(\mathfrak{A}\otimes\mathfrak{F}) \end{array}$$

with $K_j(\mathfrak{A}\otimes\mathfrak{B}\otimes\mathbb{K}\otimes C_0(\mathbb{R})) \cong K_{j+1}(\mathfrak{A}\otimes\mathfrak{B})$. Then it follows from exactness that

$$0 \to \mathrm{coker}((\mathrm{id}\otimes\nu)_*)_q \xrightarrow{\partial} K_q(\mathfrak{A}\otimes\mathfrak{B}) \longrightarrow \ker((\mathrm{id}\otimes\nu)_*)_{q+1} \to 0$$

for $q = 0, 1 \in \mathbb{Z}_2$, where note that the kerel $\ker(\partial)$ is equal to the image $\mathrm{im}((\mathrm{id}\otimes\nu)_*)$ and the cokernel $\mathrm{coker}((\mathrm{id}\otimes\nu)_*)_q$ is equal to the quotient $K_q(\mathfrak{A}\otimes\mathfrak{F})/\mathrm{im}((\mathrm{id}\otimes\nu)_*)_q$ by the image.

Since there is the following free resolution for $K_*(\mathfrak{B})$:

$$0 \to K_*(\mathfrak{C}) \xrightarrow{\nu_*} K_*(\mathfrak{F}) \xrightarrow{\partial} K_*(\mathfrak{B}) \to 0,$$

tensoring this sequence with $K_*(\mathfrak{A})$ yields

$$0 \to K_*(\mathfrak{A}) \otimes K_*(\mathfrak{C}) \xrightarrow{\mathrm{id}_*\otimes\nu_*} K_*(\mathfrak{A}) \otimes K_*(\mathfrak{F}) \xrightarrow{\mathrm{id}_*\otimes\partial} K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to 0.$$

with

$$\begin{aligned} 0 = \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{F})) \to \ker(\mathrm{id}_* \otimes \nu_*) & \\ = \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to K_*(\mathfrak{A}) \otimes K_*(\mathfrak{C}). & \end{aligned}$$

Since $\mathfrak{A} \in \mathfrak{N}$, and $K_*(\mathfrak{C})$ and $K_*(\mathfrak{F})$ are free abelian groups, the maps

$$\alpha : K_*(\mathfrak{A}) \otimes K_*(\mathfrak{C}) \to K_*(\mathfrak{A} \otimes \mathfrak{C}),$$
$$\alpha : K_*(\mathfrak{A}) \otimes K_*(\mathfrak{F}) \to K_*(\mathfrak{A} \otimes \mathfrak{F})$$

are isomorphisms, with $(\mathrm{id} \otimes \nu)_* : K_*(\mathfrak{A} \otimes \mathfrak{C}) \to K_*(\mathfrak{A} \otimes \mathfrak{J})$. Thus,

$$\begin{aligned}\operatorname{coker}((\mathrm{id} \otimes \nu)_*) &= \operatorname{coker}(\mathrm{id}_* \otimes \nu_*) \\ &= K_*(\mathfrak{A}) \otimes K_*(\mathfrak{J})/\operatorname{im}(\mathrm{id}_* \otimes \nu_*) = K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}),\end{aligned}$$

and

$$\ker((\mathrm{id} \otimes \nu)_*) = \ker(\mathrm{id}_* \otimes \nu_*) = \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})).$$

The composition of the index map ∂ from $\operatorname{coker}((\mathrm{id}\otimes\nu)_*)$ viewed as $K_*(\mathfrak{A})\otimes K_*(\mathfrak{B})$ with the isomorphism from $K_*(\mathfrak{A}\otimes\mathfrak{B}\otimes\mathbb{K}\otimes C_0(\mathbb{R}))$ to $K_{*+1}(\mathfrak{A}\otimes\mathfrak{B}\otimes\mathbb{K}) \cong K_*(\mathfrak{A}\otimes\mathfrak{B})$ is to become the map $\alpha : K_*(\mathfrak{A})\otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{A}\otimes\mathfrak{B})$. □

For the proof in the non-unital case,

Lemma 4.2. *Suppose that the Künneth theorem holds for the pairs* $(\mathfrak{A}, \mathfrak{B}^+)$, $(\mathfrak{A}, \mathbb{C})$ *with* $\mathfrak{A}$ *a nuclear* C^**-algebra. Then it holds for* $(\mathfrak{A}, \mathfrak{B})$ *(with* $\mathfrak{B}$ *a non-unital* C^**-algebra).*

Proof. Let

$$0 \to \mathfrak{B} \xrightarrow{\sigma} \mathfrak{B}^+ \xrightarrow{\tau} \mathbb{C} \to 0$$

be the unital extension of $\mathfrak{B}$ by $\mathbb{C}$. And then the following is exact:

$$0 \to \mathfrak{A} \otimes \mathfrak{B} \xrightarrow{\mathrm{id}\otimes\sigma} \mathfrak{A} \otimes \mathfrak{B}^+ \xrightarrow{\mathrm{id}\otimes\tau} \mathfrak{A} \otimes \mathbb{C} \to 0.$$

Contemlate the following commutative diagram with exact rows and columns induced by the above short exact sequences and the assumption:

$$\begin{tikzcd}
0 \arrow[r] & K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}^+) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_*\otimes\tau_*"] & K_*(\mathfrak{A} \otimes \mathfrak{B}^+) \arrow[r, "\beta"] \arrow[d, "(\mathrm{id}\otimes\tau)_*"] & \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B}^+)) \arrow[d, "\curvearrowright"'] \\
0 \arrow[r] & K_*(\mathfrak{A}) \otimes K_*(\mathbb{C}) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_*\otimes\partial"] & K_*(\mathfrak{A} \otimes \mathbb{C}) \arrow[r, "\beta"] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_*\otimes\sigma_*"] & K_*(\mathfrak{A} \otimes \mathfrak{B}) \arrow[d, "(\mathrm{id}\otimes\sigma)_*"] & \\
0 \arrow[r] & K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}^+) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_*\otimes\tau_*"] & K_*(\mathfrak{A} \otimes \mathfrak{B}^+) \arrow[r, "\beta"] \arrow[d, "(\mathrm{id}\otimes\tau)_*"] & \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B}^+)) \arrow[d, "\curvearrowright"'] \\
0 \arrow[r] & K_*(\mathfrak{A}) \otimes K_*(\mathbb{C}) \arrow[r, "\alpha"] & K_*(\mathfrak{A} \otimes \mathbb{C}) \arrow[r, "\beta"] & 0
\end{tikzcd}$$

where $\partial : K_*(\mathbb{C}) \to K_*(\mathfrak{B})$ and $\partial : K_*(\mathfrak{A} \otimes \mathbb{C}) \to K_*(\mathfrak{A} \otimes \mathfrak{B})$ are the boundary maps in the six-term exact sequences of K-groups for the first and second, short exact sequences of C^*-algebras in the first of the proof, and the symbols $\curvearrowright$ mean that the maps β in the first and fourth horizontal lines in the diagram are onto. Moreover, the map $\partial : K_*(\mathbb{C}) \to K_*(\mathfrak{B})$ is the zero map because $\tau_* : K_0(\mathfrak{B}^+) \to K_0(\mathbb{C})$ is onto and $K_1(\mathbb{C}) = 0$, so that the six-term exact sequence of K-groups for the first short exact sequence splits into two short exact sequences of K_0 and K_1-groups. Hence, the left column in the diagram above is indeed exact. Furthermore, $\mathrm{id}_* \otimes \tau_*$ is surjective, which implies that $(\mathrm{id} \otimes \tau)_*$ is surjective by the commutativity of the diagram. Therefore, it follows that the boundary map $\partial : K_*(\mathfrak{A} \otimes \mathbb{C}) \to K_*(\mathfrak{A} \otimes \mathbb{K})$ is the zero map (not injective!), from which the map $(\mathrm{id} \otimes \sigma)_* : K_*(\mathfrak{A} \otimes \mathfrak{B}) \to K_*(\mathfrak{A} \otimes \mathfrak{B}^+)$ is in fact injective (this part is corrected from the original part and it should be true). And hence we obtain

$$K_*(\mathfrak{A} \otimes \mathfrak{B}) \cong \ker((\mathrm{id} \otimes \tau)_*).$$

The commutative diagram above can then be rewritten as:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_* \otimes \sigma_*"] & K_*(\mathfrak{A} \otimes \mathfrak{B}) \arrow[r, "\beta"] \arrow[d, "(\mathrm{id} \otimes \sigma)_*"] & \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \arrow[d, "\cong"] \\
0 \arrow[r] & K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}^+) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_* \otimes \tau_*"] & K_*(\mathfrak{A} \otimes \mathfrak{B}^+) \arrow[r, "\beta"] \arrow[d, "(\mathrm{id} \otimes \tau)_*"] & \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B}^+)) \arrow[d, "\curvearrowright"'] \\
0 \arrow[r] & K_*(\mathfrak{A}) \otimes K_*(\mathbb{C}) \arrow[r, "\alpha"] \arrow[d] & K_*(\mathfrak{A} \otimes \mathbb{C}) \arrow[r, "\beta"] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

since we have

$$\begin{aligned}
&\mathrm{Tor}_2^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathbb{C})) \cong 0 \to \\
&\mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B}^+)) \to \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathbb{C})) = 0.
\end{aligned}$$

Therefore, the diagram above yields the following:

$$0 \to K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \xrightarrow{\alpha} K_*(\mathfrak{A} \otimes \mathfrak{B}) \xrightarrow{\beta} \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to 0,$$

which is the Künneth formula for the pair $(\mathfrak{A}, \mathfrak{B})$. $\square$

Remark. As mentioned by Schochet, it might be possible to drop the assumption that $\mathfrak{A}$ belongs to $\mathfrak{N}$, and to replace it by another one such as $\mathfrak{A}$ either nuclear or perhaps separable. For this, one needs to know that if $K_*(\mathfrak{B})$ is torsion free and if a $*$-homomorphism $f : \mathfrak{A}' \to \mathfrak{A}''$ of C^*-algebras induces an isomorphism $f_* : K_*(\mathfrak{A}') \to K_*(\mathfrak{A}'')$, then

$$(f \otimes \mathrm{id})_* : K_*(\mathfrak{A}' \otimes \mathfrak{B}) \to K_*(\mathfrak{A}'' \otimes \mathfrak{B})$$

is an isomorphism. The difficulty is equivalent to the following conjecture:

Conjecture. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras (separable or nuclear if that is necessary) with $K_*(\mathfrak{A}) = 0$ and $K_*(\mathfrak{B})$ torsion free. Then is it true that $K_*(\mathfrak{A} \otimes \mathfrak{B}) = 0$?

It is also conjectured at that time that the Künneth formula splits (unnaturally), at least if $\mathfrak{A}$ and $\mathfrak{B}$ are separable, so that $\mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B}))$ is countable. Some generalization of Bödigheimer's technique [3] should suffice. See also [4] and the next article in this volume of RMJ.

Also, as another remark, suppose that $\mathfrak{C} \in \mathfrak{N}$ is some fixed C^*-algebra with $K_0(\mathfrak{C}) = G$ an abelian group and $K_1(\mathfrak{C}) = 0$. Define the K-theory groups with coefficients in G for a C^*-algebra $\mathfrak{B}$ as

$$K_q(\mathfrak{B}; G) = K_q(\mathfrak{B} \otimes \mathfrak{C}), \quad q \in \mathbb{Z}_2.$$

The Künneth theorem implies that for any C^*-algebra $\mathfrak{B}$ there is the following short exact sequence:

$$\begin{array}{ccccc} 0 \to K_q(\mathfrak{B}) \otimes G & \longrightarrow & K_q(\mathfrak{B}; G) & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}(K_{q+1}(\mathfrak{B}), G) \to 0 \\ \| & & \| & & \| \\ 0 \to K_q(\mathfrak{B}) \otimes K_0(\mathfrak{C}) & \longrightarrow & K_q(\mathfrak{B} \otimes \mathfrak{C}) & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}(K_{q+1}(\mathfrak{B}), K_0(\mathfrak{C})) \to 0 \end{array}$$

which shows that, at least up to group extension, the groups $K_q(\mathfrak{B}, G)$ are independent of choice of $\mathfrak{C}$, because the extension is determined by both $K_q(\mathfrak{B}) \otimes G$ and $\mathrm{Tor}_1^{\mathbb{Z}}(K_{q+1}(\mathfrak{B}), G)$ only, without information about $\mathfrak{C}$, up to group extension.

Suppose further that $\mathfrak{C} \otimes \mathfrak{C} \otimes \mathbb{K} \cong \mathfrak{C} \otimes \mathbb{K}$, so that $G \otimes G \cong G$ (such as $G = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ or $\mathbb{Q}$). Check that

$$\begin{array}{ccccc} 0 \to K_0(\mathfrak{C}) \otimes K_0(\mathfrak{C}) & \longrightarrow & K_0(\mathfrak{C} \otimes \mathfrak{C}) & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}(K_1(\mathfrak{C}), K_0(\mathfrak{C})) \to 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 \to G \otimes G & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

Then there is the following Künneth formula for the K-theory $K_*(\cdots;G)$ with coefficient in G of the form:

$$
\begin{aligned}
0 &\to \sum_p [K_p(\mathfrak{A};G)\otimes K_{q-p}(\mathfrak{B};G)] \to K_q(\mathfrak{A}\otimes\mathfrak{B};G) \\
&\to \sum_p \mathrm{Tor}_1^{\mathbb{Z}}(K_p(\mathfrak{A};G),K_{q-p-1}(\mathfrak{B};G)) \to 0
\end{aligned}
$$

for $p,q\in\mathbb{Z}_2$. Indeed, check that

$$
K_q(\mathfrak{A}\otimes\mathfrak{B};G)=K_q(\mathfrak{A}\otimes\mathfrak{B}\otimes\mathfrak{C})\cong K_q((\mathfrak{A}\otimes\mathfrak{C})\otimes(\mathfrak{B}\otimes\mathfrak{C}))
$$

and hence the Künneth theorem implies the following:

$$
\begin{aligned}
0 &\to K_q(\mathfrak{A}\otimes\mathfrak{C})\otimes K_0(\mathfrak{B}\otimes\mathfrak{C}) \to K_q((\mathfrak{A}\otimes\mathfrak{C})\otimes(\mathfrak{B}\otimes\mathfrak{C})) \\
&\to \mathrm{Tor}_1^{\mathbb{Z}}(K_{q+1}(\mathfrak{A}\otimes\mathfrak{C}),K_0(\mathfrak{B}\otimes\mathfrak{C})) \to 0
\end{aligned}
$$

which is equivalent to the following:

$$
\begin{aligned}
0 &\to K_q(\mathfrak{A};G)\otimes K_0(\mathfrak{B};G) \to K_q(\mathfrak{A}\otimes\mathfrak{B};G) \\
&\to \mathrm{Tor}_1^{\mathbb{Z}}(K_{q+1}(\mathfrak{A};G),K_0(\mathfrak{B};G)) \to 0.
\end{aligned}
$$

Moreover, we get

$$
\begin{aligned}
K_{q+1}(\mathfrak{A};G)\otimes K_1(\mathfrak{B};G) &= K_{q+1}(\mathfrak{A}\otimes\mathfrak{C})\otimes K_1(\mathfrak{B}\otimes\mathfrak{C})\cong 0,\\
\mathrm{Tor}_1^{\mathbb{Z}}(K_q(\mathfrak{A};G),K_1(\mathfrak{B};G)) &= \mathrm{Tor}_1^{\mathbb{Z}}(K_q(\mathfrak{A}\otimes\mathfrak{C}),K_1(\mathfrak{B}\otimes\mathfrak{C}))\cong 0
\end{aligned}
$$

both of which follows from $K_1(\mathfrak{B}\otimes\mathfrak{C})\cong 0$, that follows by the Künneth formula and $K_1(\mathfrak{C})=0$. Furthermore, one can write that

$$
\begin{aligned}
&K_q(\mathfrak{A};G)\otimes K_0(\mathfrak{B};G)\\
&=[K_q(\mathfrak{A};G)\otimes K_0(\mathfrak{B};G)]\oplus[K_{q+1}(\mathfrak{A};G)\otimes K_1(\mathfrak{B};G)]\\
&=\sum_p [K_p(\mathfrak{A};G)\otimes K_{q-p}(\mathfrak{B};G)],\\
&\mathrm{Tor}_1^{\mathbb{Z}}(K_{q+1}(\mathfrak{A};G),K_0(\mathfrak{B};G))\\
&=[\mathrm{Tor}_1^{\mathbb{Z}}(K_{q+1}(\mathfrak{A};G),K_0(\mathfrak{B};G))]\oplus[\mathrm{Tor}_1^{\mathbb{Z}}(K_q(\mathfrak{A};G),K_1(\mathfrak{B};G))]\\
&=\sum_p [\mathrm{Tor}_1^{\mathbb{Z}}(K_p(\mathfrak{A};G),K_{q+1-p}(\mathfrak{B};G))]
\end{aligned}
$$

and the last index $q+1-p$ may be replaced with $q-p-1$.

For example as $\mathfrak{C}$, the Cuntz algebras O_{n+1} (for n prime) with K_0 isomorphic to $\mathbb{Z}/n\mathbb{Z}=\mathbb{Z}_n$ and K_1 trivial (see [9]) yield coefficients in the group $\mathbb{Z}_n$ and the UHF algebra with K_0 isomorphic to $\mathbb{Q}$ and K_1 trivial yields rational coefficients.

In fact, both of O_2 and O_∞ have the self-absorbing property for taking tensor product (and have K-theory groups, both zero and both trivial as $\mathbb{C}$, respectively), but both of O_n ($n \geq 3$) and the above UHF algebra do not have the self-absoring property and even the stably self-absorbing property for taking tensor product, which follows from the classification theorem of C^*-algebras by K-theory (see [6]), via the Künneth theorem (without coefficients), because their self-tensor products have K_1 non-trivial.

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