The universal coefficient thorem and the Künneth theorem in Kasparov KK-theory : a review

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THE UNIVERSAL COEFFICIENT THOREM AND THE KÜNNETH THEOREM IN KASPAROV KK-THEORY — A REVIEW

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Abstract

We review and study the universal coefficient theorem (UCT) and the Künneth theorem (KT) for Kasparov KK-theory groups, both of which are obtained by Jonathan Rosenberg and Claude Schochet.

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Contents

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1 Introduction

This is nothing but a review on a paper by Jonathan Rosenberg and Claude Schochet [8]. But we made some considerable effort to read the paper carefully to make some additional and helpful explanations for some proofs to become more accessible to the readers for convenience. Some notations are changed by our taste. The contents start from below almost along the story of [8], but in this section we briefly recall several notations, definitions, and basic properties without proofs and with possibly incomplete explanations.

The BDF theory of Brown, Douglas, and Fillmore classifies the C^*-algebra extensions of the form

$$0 \to \mathbb{K} \to \mathfrak{E} \to \mathfrak{A} \to 0$$

to classify essentially normal operators, where $\mathbb{K}$ is the C^*-algebra of all compact operators on an infinite dimensional Hilbert space

For a separable nuclear C^*-algebra $\mathfrak{A}$, the abelian extension group $\mathrm{Ext}(\mathfrak{A})$ consists of unitary equivalence classes of essential extensions $\mathfrak{E}$ of the form above, where the addition in $\mathrm{Ext}(\mathfrak{A})$ comes from the corresponding Busby invariant $\tau_{\mathfrak{E}} : \mathfrak{A} \to Q = \mathbb{B}/\mathbb{K}$ a homomorphism from $\mathfrak{A}$ to the Calkin algebra Q and composing with an injection from $Q \oplus Q$ to Q.

Let $S^n\mathfrak{A} = C_0(\mathbb{R}^n) \otimes \mathfrak{A}$ be the n-th suspension of a C^*-algebra $\mathfrak{A}$. It is shown by BDF that the Bott periodicity holds for Ext with definitions:

$$\mathrm{Ext}(\mathfrak{A}) \equiv \mathrm{Ext}_0(\mathfrak{A}) \cong \mathrm{Ext}(S^2\mathfrak{A}) \equiv \mathrm{Ext}_2(\mathfrak{A})$$

and hence $\mathrm{Ext}_*(\star) = \{\mathrm{Ext}_0(\star), \mathrm{Ext}_1(\star)\}$ with $\mathrm{Ext}_1(\star) = \mathrm{Ext}(S\star)$ is a periodic cohomology theory on separable nuclear C^*-algebras $(\star)$ as variables.

When $\mathfrak{A} = C(X)$ with X a compact metric space, then $\mathrm{Ext}_*(C(\star)) = \mathrm{Ext}_*(\star)$ with $(\star)$ spaces generates a $\mathbb{Z}_2$-graded Steenrod homology theory. If X is finite dimensional, then by Kahn-Kaminker-Schochet,

$$\mathrm{Ext}_*(C(X)) \cong K^*(F(X)) \equiv K_*^s(X),$$

where $F(X)$ is the fundamental dual of the space X and $K^*(\star)$ is the (representable) topological K-theory for spaces $(\star)$ (and both of sides above may as well be viewed as the K-homology theory $K_*(X)$ for X).

There is a natural index map

$$\gamma : \mathrm{Ext}(\mathfrak{A}) \to \mathrm{Hom}(K_1(\mathfrak{A}), K_0(\mathbb{K}))$$

with $K_0(\mathbb{K}) \cong \mathbb{Z}$, where $\gamma([\mathfrak{E}]) = \partial_{\mathfrak{E}}$ the index map in the six-term exact sequence of K-groups for the extension $\mathfrak{E}$ of $\mathfrak{A}$ by $\mathbb{K}$.

The Universal Coefficient Theorem (UCT) for unital commutative C^*-algebras $C(X)$ of all continuous functions on compact Hausdorff spaces X, proved by L. G. Brown, is the following natural short exact sequence:

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(K^0(X), \mathbb{Z}) \xrightarrow{\delta} \operatorname{Ext}(C(X)) \xrightarrow{\gamma} \operatorname{Hom}(K^1(X), \mathbb{Z}) \to 0,$$

where the map γ preserves degrees and the map δ reverses degrees, the first term as well as the third one come from homology theory (and also $K^*(X)$ is the topological K-theory for X).

As a generalization of the above UCT to non-commutative C^*-algebras, the Universal Coefficient Theorem (UCT) for inductive limits of type I C^*-algebras, obtained by L. G. Brown, is the following natural short exact sequence:

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), \mathbb{Z}) \xrightarrow{\delta} \operatorname{Ext}(\mathfrak{A}) \xrightarrow{\gamma} \operatorname{Hom}(K_1(\mathfrak{A}), \mathbb{Z}) \to 0.$$

In the meantime, M. Pimsner, S. Popa, and D. Voiculescu classify C^*-algebra extenstions of the form

$$0 \to C(Y) \otimes \mathbb{K} \to \mathfrak{E} \to \mathfrak{A} \to 0$$

with Y compact metric spaces, by the functors $\operatorname{Ext}_*(Y; \mathfrak{A})$. It is shown by them that their functors are homotopy invariant, periodic, and satisfy exactness properties in each variable. In the commutative case $\mathfrak{A} = C(X)$, it is calculated by Schochet as

$$\operatorname{Ext}_*(Y; C(X)) \cong K^*(Y \wedge F(X)).$$

This equation and the Künneth formula for toplogical K-theory imply a Künneth formula of the form

$$0 \to K^*(Y) \otimes K^s_*(X) \to \operatorname{Ext}_*(Y; C(X)) \to \operatorname{Tor}^{\mathbb{Z}}_1(K^*(Y), K^s_*(X)) \to 0$$

with $K^s_*(X) = K^*(F(X))$, provided that $K^*(Y)$ is finitely generated.

Let $\mathfrak{A}$ be a separable nuclear C^*-algebra and $\mathfrak{B}$ a C^*-algebra with countable approximate units. We **assume** this throughout this paper if not mentioned otherwise.

The Kasparov (KK_1-)group $KK_1(\mathfrak{A}, \mathfrak{B})$ consists of stable equivalence classes of extensions of the form:

$$0 \to \mathfrak{B} \otimes \mathbb{K} \to \mathfrak{E} \to \mathfrak{A} \to 0,$$

the addition comes from the corresponding Busby invariant $\tau : \mathfrak{A} \to Q(\mathfrak{B}) = M(\mathfrak{B} \otimes \mathbb{K})/\mathfrak{B} \otimes \mathbb{K}$ the quotient of the multiplier algebra $M(\mathfrak{B} \otimes \mathbb{K})$

(the outer multiplier algebra of $\mathfrak{B} \otimes \mathbb{K}$) and the composition with an injection from $Q(\mathfrak{B}) \oplus Q(\mathfrak{B}) \to Q(\mathfrak{B})$, and the identity of the group corresponds to those extensions τ which are stably split, i.e. $\tau \oplus \tau'$ is split for some split extension τ'.

There are natural isomorphisms between Kasparov KK-theory groups and Karoubi K-theory groups as well as BDF extension theory groups:

$$KK_*(\mathbb{C}, \mathfrak{B}) \cong K_*(\mathfrak{B}) \quad \text{and} \quad KK_*(\mathfrak{A}, \mathbb{C}) \cong \mathrm{Ext}_*(\mathfrak{A})$$

with $\mathfrak{A}$ unital.

There is also the isomorphism obtained by Rosenberg and Schochet:

$$KK_*(\mathfrak{A}, C_0(Y)) \cong \mathrm{Ext}_*(Y^+, +; \mathfrak{A}^+),$$

where Y is a locally compact subset of the Euclidean space $\mathbb{R}^n$ and Y^+ is the one-point compactification of Y, and the right hand side in the isomorphism is the reduced Pimsner-Popa-Voiculescu group with $\mathfrak{A}^+$ the unitization of a C^*-algebra $\mathfrak{A}$. See [7].

A homology theory is a sequence $\{H_n\}$ of covariant functors from an admissible category $\mathcal{C}$ of C^*-algebras to abelian groups satisfying the following two axioms:

- *Homotopy Axom*: If $f, g : \mathfrak{A} \to \mathfrak{B}$ are maps of C^*-algebras in $\mathcal{C}$ and f is homotopic to g, then

$$f_* = g_* : H_n(\mathfrak{A}) \to H_n(\mathfrak{B}).$$

- *Exactness Axiom*: Let

$$0 \longrightarrow \mathfrak{I} \xrightarrow{i} \mathfrak{A} \xrightarrow{q} \mathfrak{B} \longrightarrow 0$$

be a short exact sequence of C^*-algebras in $\mathcal{C}$ with i an inclusion map and q a quotient map. Then there is a boundary map $\partial : H_n(\mathfrak{B}) \to H_{n-1}(\mathfrak{I})$ and a long exact sequence:

$$\cdots \xrightarrow{\partial} H_n(\mathfrak{I}) \xrightarrow{i_*} H_n(\mathfrak{A}) \xrightarrow{q_*} H_n(\mathfrak{B}) \xrightarrow{\partial} H_{n-1}(\mathfrak{I}) \xrightarrow{i_*} \cdots.$$

The map ∂ is natural with respect to morphisms of short exact sequences.

The homology theory is said to be additive if the following axiom holds:

- *Additivity Axiom*: Let $\mathfrak{A} = \oplus_i \mathfrak{A}_i$ a direct sum in $\mathcal{C}$. Then the natural maps $H_n(\mathfrak{A}_i) \to H_n(\mathfrak{A})$ induce an isomorphism:

$$\oplus_i H_n(\mathfrak{A}_i) \to H_n(\mathfrak{A})$$

where $\oplus_i$ means the direct sum.

Similarly, a cohomology theory is a sequence $\{H^n\}$ of contravariant functors from an admissible category of C^*-algebras to abelian groups satisfying the analogous homotopy and exactness axioms.

A cohomology theory is said to be additive if the natural maps $H^n(\mathfrak{A}) \to H^n(\mathfrak{A}_i)$ induce an isomorphism:

$$H^n(\mathfrak{A}) = H^n(\oplus_i \mathfrak{A}_i) \to \Pi_i H^n(\mathfrak{A}_i)$$

where Π_i means the direct product.

We now recall some about the basics of KK-theory groups.

Theorem 1.1. **(Kasparov).**

(1) *For each* $\mathfrak{A}$, *the functors* $KK_j(\mathfrak{A}, \star)$ $(j = 0, 1)$ *with respect to the second variables* $\star$ *form a homology theory.*

(2) *For each* $\mathfrak{B}$, *the functors* $KK_j(\star, \mathfrak{B})$ *form a cohomology theory.*

(3) *Bott periodicity is satisfied in each variable:*

$$KK_j(\mathfrak{A}, \mathfrak{B}) \cong KK_j(\mathfrak{A}, S^2\mathfrak{B}) \cong KK_j(S^2\mathfrak{A}, \mathfrak{B}),$$

where $S\mathfrak{A} = C_0(\mathbb{R}) \otimes \mathfrak{A}$ *and* $S^2\mathfrak{A} = S(S\mathfrak{A})$. *In fact, we also have*

$$KK_1(\mathfrak{A}, \mathfrak{B}) \cong KK_0(\mathfrak{A}, S\mathfrak{B}) \cong KK_0(S\mathfrak{A}, \mathfrak{B}),$$
$$KK_0(\mathfrak{A}, \mathfrak{B}) \cong KK_1(\mathfrak{A}, S\mathfrak{B}) \cong KK_1(S\mathfrak{A}, \mathfrak{B}).$$

(4) *There is a natural Kasparov intersection product*

$$KK_i(\mathfrak{A}_1, \mathfrak{B}_1 \otimes \mathfrak{D}) \otimes KK_j(\mathfrak{D} \otimes \mathfrak{A}_2, \mathfrak{B}_2) \xrightarrow{\otimes_{\mathfrak{D}}} KK_{i+j}(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{B}_1 \otimes \mathfrak{B}_2)$$

where $i + j$ *(mod 2). In particular, if* $\mathfrak{B}_1 = \mathbb{C}$ *and* $\mathfrak{A}_2 = \mathbb{C}$, *then*

$$KK_i(\mathfrak{A}_1, \mathfrak{D}) \otimes KK_j(\mathfrak{D}, \mathfrak{B}_2) \xrightarrow{\otimes_{\mathfrak{D}}} KK_{i+j}(\mathfrak{A}_1, \mathfrak{B}_2).$$

(5) *The inclusion maps* $\mathfrak{A} \to \mathfrak{A} \otimes \mathbb{K}$ *and* $\mathfrak{B} \to \mathfrak{B} \otimes \mathbb{K}$ *induced by a rank-one projection in* $\mathbb{K}$ *give rise to natural isomorphisms*

$$KK_j(\mathfrak{A} \otimes \mathbb{K}, \mathfrak{B}) \cong KK_j(\mathfrak{A}, \mathfrak{B})$$

and

$$KK_j(\mathfrak{A}, \mathfrak{B}) \cong KK_j(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{K}).$$

Theorem 1.2. **(Rosenberg).** *For each* $\mathfrak{B}$, $KK_j(\star, \mathfrak{B})$ *form an additive cohomology theory. That is, if* I *is a countable index set, then the projections* $p_j : \oplus_{i \in I} \mathfrak{A}_i \to \mathfrak{A}_j$ *induce an isomorphism*

$$KK_k(\oplus_{i \in I} \mathfrak{A}_i, \mathfrak{B}) \cong \Pi_{i \in I} KK_k(\mathfrak{A}_i, \mathfrak{B}).$$

Proof. It suffices to deal with KK_0-groups. Indeed, if we have the isomorphism for KK_0, then

$$\begin{aligned} KK_1(\oplus_i \mathfrak{A}_i, \mathfrak{B}) &\cong KK_0(\oplus_i \mathfrak{A}_i, S\mathfrak{B}) \\ &\cong \Pi_i KK_0(\mathfrak{A}_i, S\mathfrak{B}) \cong \Pi_i KK_1(\mathfrak{A}_i, \mathfrak{B}). \end{aligned}$$

The maps p_j induce a homomorphism $\Theta : KK_0(\oplus_i \mathfrak{A}_i, \mathfrak{B}) \to \Pi_i KK_0(\mathfrak{A}_i, \mathfrak{B})$.

In fact, there are injections $s_j : \mathfrak{A}_j \to \oplus_i \mathfrak{A}_i$, so that there are induced maps $(s_j)_* : KK_*(\oplus_i \mathfrak{A}_i, \mathfrak{B}) \to KK_*(\mathfrak{A}_j, \mathfrak{B})$. Hence there is $\Pi_j(s_j)_* : KK_*(\oplus_i \mathfrak{A}_i, \mathfrak{B}) \to \Pi_j KK_*(\mathfrak{A}_j, \mathfrak{B})$. This should be the reason for Θ.

Surjectiviety of the map Θ is proved as follows. Given graded Hilbert $\mathfrak{B}$-module E_i, bounded operators $T_i \in \mathbb{B}(E_i)$ of degree one, and representations $\varphi_i : \mathfrak{A}_i \to \mathbb{B}(E_i)$ of degree zero, so that the triple (E_i, T_i, φ_i) called Kasparov module defines an element of $KK_0(\mathfrak{A}_i, \mathfrak{B})$. Then form the direct sum $(E, T, \varphi) = (\oplus_i E_i, \oplus_i T_i, \oplus_i \varphi_i)$. This defines an element of $KK_0(\oplus_i \mathfrak{A}_i, \mathfrak{B})$ which is mapped to $\Pi_i(E_i, T_i, \varphi_i)$ under Θ, since the following relations hold:

$$[\varphi(a), T] \in \mathbb{K}(E), \quad \varphi(a)(T^2 - 1) \in \mathbb{K}(E), \quad \varphi(a)(T - T^*) \in \mathbb{K}(E)$$

for $a = (a_i) \in \oplus_i \mathfrak{A}_i$, where $[x, y] = xy - yx$ the commutator for x, y.

Note that for $a_i \in \mathfrak{A}_i$,

$$[\varphi_i(a_i), T_i] \in \mathbb{K}(E_i), \quad \varphi_i(a_i)(T_i^2 - 1) \in \mathbb{K}(E_i), \quad \varphi_i(a_i)(T_i - T_i^*) \in \mathbb{K}(E_i).$$

Also, the KK_0-group is defined to be set of homotopy equivalence classes of Kasparov modules such as above, and to be an abelian group with respect to direct sum. And also, for E a (right) Hilbert $\mathfrak{B}$-module, $\mathbb{B}(E)$ is the set of all module homomorphisms T on E with adjoint T with respect to $\mathfrak{B}$-valued inner product $\langle \cdot, \cdot \rangle$ for E: $\langle Tx, y \rangle = \langle x, T^* y \rangle \in \mathfrak{B}$ for $x, y \in E$, and $\mathbb{K}(E)$ is defined to be the closure of linear space of rank one operators $\theta_{x,y}$ for $x, y \in E$ defined as $\theta_{x,y}(z) = x\langle y, z \rangle$ for $z \in E$, with adjoint $\theta_{x,y}^* = \theta_{y,x}$.

Injectivity of the map Θ is proved as follows. Suppose that one is given an element of $KK_0(\oplus_i \mathfrak{A}_i, \mathfrak{B})$ which is trivial in each $KK_0(\mathfrak{A}_i, \mathfrak{B})$. There is the following split short exact sequence:

$$0 \to \oplus_i \mathfrak{A}_i \to \oplus_i \mathfrak{A}_i^+ \to c_0(I) \to 0$$

where $c_0(I)$ is the C^*-algebra of all sequences of $\mathbb{C}$ on I vanishing at infinity. Thus we may suppose that we have an element of $KK_0(\oplus_i \mathfrak{A}_i^+, \mathfrak{B})$ which is trivial in each $KK_0(\mathfrak{A}_i^+, \mathfrak{B})$, say (E, T, φ) a corresponding Kasparov module. Let e_i be the unit element of $\mathfrak{A}_i^+$. Then the image of the

KK-element in $KK_0(\mathfrak{A}_i^+, \mathfrak{B})$ is defined by the $\mathfrak{B}$-module e_iE together with the compression of T and φ, say $T|_{e_iE}$ and the compression of φ restricted to $\mathfrak{A}_i$, say ψ_i. Then we may assume that $T|_{e_iE}$ is homotopic to some T_i on e_iE satisfying the following relations:

$$[\psi_i(a_i), T_i] = 0, \quad \psi_i(a_i)(T_i^2 - 1) = 0, \quad \psi_i(a_i)(T_i - T_i^*) = 0$$

for $a_i \in \mathfrak{A}_i$, realized in $\mathbb{B}(e_iE)$. It follows that the homotopies can be added to obtain a homotopy from T to $\oplus_i T_i$ which defines the zero element in KK_0. Thus the kernel of Θ is trivial. □

Theorem 1.3. (1) *Any (countable) additive homology theory commutes with (countably) inductive limits of C^*-algebras.*

(2) *If H^* is a (countably) additive cohomology theory and $\varinjlim \mathfrak{A}_j$ is a direct sequence of C^*-algebras $\mathfrak{A}_j$, then there is a natural Milnor* $\lim^1$ *sequence of the form:*

$$0 \to \varprojlim{}^1 H^{n-1}(\mathfrak{A}_j) \to H^n(\varinjlim \mathfrak{A}_j) \to \varprojlim H^n(\mathfrak{A}_j) \to 0.$$

Let

$$K_*(\star) = K_0(\star) \oplus K_1(\star) \quad \text{and} \quad KK_*(\star, \star') = KK_0(\star, \star') \oplus KK_1(\star, \star')$$

denote the direct sums of K-theory groups and KK-theory groups respectively, but these notations in the left sides are also used to mean respective unions of K-groups and KK-groups to mean the respective theories.

Let $K_{*\pm 1}(\star) = K_{0+1}(\star) \oplus K_{1+1}(\star) = K_1(\star) \oplus K_0(\star)$. Let $KK_{*\pm 1}(\star, \star') = KK_1(\star, \star') \oplus KK_0(\star, \star')$. These notations are used to match up to the degree of maps between K-theory or KK-theory groups.

Note that

$$KK_*(\mathbb{C}, \mathfrak{B}) \cong K_*(\mathfrak{B})$$

and also

$$KK_*(\mathfrak{A}, \mathbb{C}) \equiv K^*(\mathfrak{A}) \cong \mathrm{Ext}_*(\mathfrak{A}),$$

where $\mathrm{Ext}_*(\mathfrak{A}) = \mathrm{Ext}_0(\mathfrak{A}) \oplus \mathrm{Ext}_1(\mathfrak{A})$, with $\mathrm{Ext}_1(\mathfrak{A}) = \mathrm{Ext}(\mathfrak{A})$ and $\mathrm{Ext}_0(\mathfrak{A}) = \mathrm{Ext}(S\mathfrak{A})$.

There are two natural maps via the Kasparov product:

$$\begin{aligned} \alpha : & K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \\ & \cong KK_*(\mathfrak{A}, \mathbb{C}) \otimes KK_*(\mathbb{C}, \mathfrak{B}) \to KK_*(\mathfrak{A}, \mathfrak{B}) \end{aligned}$$

with degree zero, and

$$\gamma : KK_*(\mathfrak{A}, \mathfrak{B}) \to \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B}))$$

with degree zero, which comes from the pairing:

$$KK_*(\mathbb{C},\mathfrak{A}) \otimes KK_*(\mathfrak{A},\mathfrak{B}) \to KK_*(\mathbb{C},\mathfrak{B}) = K_*(\mathfrak{B}).$$

Moreover, the map γ has the following interpretation as well. Let $\tau \in KK_1(\mathfrak{A},\mathfrak{B})$ be represented by an extension

$$0 \to \mathfrak{B} \otimes \mathbb{K} \to \mathfrak{E} \to \mathfrak{A} \to 0.$$

The associated six-term exact sequence of K-groups has the form

$$\begin{array}{ccccc} K_0(\mathfrak{B}) & \longrightarrow & K_0(\mathfrak{E}) & \longrightarrow & K_0(\mathfrak{A}) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\partial'} \\ K_1(\mathfrak{A}) & \longleftarrow & K_1(\mathfrak{E}) & \longleftarrow & K_1(\mathfrak{B}) \end{array}$$

with ∂, ∂' index maps and then we have

$$\begin{aligned}\gamma(\tau) = (\partial,\partial') &\in \mathrm{Hom}(K_1(\mathfrak{A}),K_0(\mathfrak{B})) \oplus \mathrm{Hom}(K_0(\mathfrak{A}),K_1(\mathfrak{B})) \\ &\subset \mathrm{Hom}(K_*(\mathfrak{A}),K_*(\mathfrak{B})).\end{aligned}$$

Furthermore, if $\rho \in KK_0(\mathfrak{A},\mathfrak{B}) = KK_1(S\mathfrak{A},\mathfrak{B})$, then similarly,

$$\begin{aligned}\gamma(\rho) = (\partial_S,\partial'_S) &\in \mathrm{Hom}(K_1(S\mathfrak{A}),K_0(\mathfrak{B})) \oplus \mathrm{Hom}(K_0(S\mathfrak{A}),K_1(\mathfrak{B})) \\ &= \mathrm{Hom}(K_0(\mathfrak{A}),K_0(\mathfrak{B})) \oplus \mathrm{Hom}(K_1(\mathfrak{A}),K_1(\mathfrak{B})) \\ &\subset \mathrm{Hom}(K_*(\mathfrak{A}),K_*(\mathfrak{B})),\end{aligned}$$

where $\partial_S : K_1(S\mathfrak{A}) \to K_0(\mathfrak{B})$, $\partial'_S : K_0(S\mathfrak{S}) \to K_1(\mathfrak{B})$ are corresponding index maps. Therefore, the map γ is determined by the index maps as

$$\gamma(\rho,\tau) = (\gamma(\rho),\gamma(\tau)) = (\partial_S,\partial'_S,\partial,\partial')$$

Suppose that $\gamma(\tau) = 0$, i.e., the index maps ∂ and ∂' vanish, so that the six-term exact sequence splits into two short exact sequences

$$0 \to K_j(\mathfrak{B}) \to K_j(\mathfrak{E}) \to K_j(\mathfrak{A}) \to 0$$

for $j = 0,1$, which determine an element

$$\kappa(\tau) \in \mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}),K_0(\mathfrak{B}))\oplus\mathrm{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}),K_1(\mathfrak{B})) \subset \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}),K_*(\mathfrak{B}))$$

with degree zero, while τ has degree one. Similarly, if $\tau(\rho) = 0$, then we have two short exact sequences

$$0 \to K_j(\mathfrak{B}) \to K_j(\mathfrak{E}) \to K_j(S\mathfrak{A}) \to 0$$

for $j = 0, 1$, which determine an element

$\kappa(\rho) \in \mathrm{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_0(\mathfrak{B})) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \subset \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B}))$

with degree one, while ρ has degree zero.

Let N be the smallest full subcategory of separable nuclear C^*-algebras which contains separable type I C^*-algebras and is closed under strong Morita equivalence (that is the same as stable isomorphism), inductive limits, extensions (and if two terms of an extension are in N, then so is the other), and crossed products by $\mathbb{R}$ and by $\mathbb{Z}$.

We are going to check the following two theorems UCT and KT of Rosenberg-Schochet:

Universal Coefficient Theorem (UCT). (Rosenberg-Schochet). *Let $\mathfrak{A} \in N$. Then there is a short exact sequence (denoted as a diagram):*

$$\begin{tikzcd} \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) \arrow[r, "\delta"] & KK_*(\mathfrak{A}, \mathfrak{B}) \arrow[r, "\gamma"] & \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \arrow[d] \\ 0 \arrow[u] & & 0 \end{tikzcd}$$

which is natural in each variable. The map γ has degree 0 and the map δ has degree 1.

The UCT sequence splits unnaturally.

Taking $\mathbb{C}$ as $\mathfrak{B}$, we have

Corollary 1.4. *Suppose that $\mathfrak{A} \in N$. Then there is a short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), \mathbb{Z}) \xrightarrow{\delta} K^1(\mathfrak{A}) \xrightarrow{\gamma} \mathrm{Hom}(K_1(\mathfrak{A}), \mathbb{Z}) \to 0,$$

which is a generalization of the UCT of Brown for inductive limits of type I C^*-algebras, where $K^1(\mathfrak{A}) = KK_1(\mathfrak{A}, \mathbb{C}) = \mathrm{Ext}(\mathfrak{A})$.

In particular, taking $C(X)$ as $\mathfrak{A}$ we get the UCT of Brown:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K^0(X), \mathbb{Z}) \xrightarrow{\delta} \mathrm{Ext}(C(X)) \xrightarrow{\gamma} \mathrm{Hom}(K^1(X), \mathbb{Z}) \to 0.$$

Separable C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are said to be KK-equivalent if there exist $\lambda \in KK_0(\mathfrak{A}, \mathfrak{B})$ and $\lambda^{-1} \in KK_0(\mathfrak{B}, \mathfrak{A})$ such that $\lambda \otimes_{\mathfrak{B}} \lambda^{-1} = \mathrm{id}_{\mathfrak{A}}$ and $\lambda^{-1} \otimes_{\mathfrak{A}} \lambda = \mathrm{id}_{\mathfrak{B}}$ the KK-theory classes corresponding to the identity maps on $\mathfrak{A}$ and $\mathfrak{B}$ respectively.

Künneth Theorem (KT). (Rosenberg-Schochet). *Let $\mathfrak{A} \in N$ and suppose that $K_*(\mathfrak{B})$ is finitely generated. Then there is a short exact sequence (denoted as a diagram):*

$$\begin{tikzcd} K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \arrow[r, "\alpha"] & KK_*(\mathfrak{A}, \mathfrak{B}) \arrow[r, "\beta"] & \mathrm{Tor}^{\mathbb{Z}}_1(K^*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) \arrow[d] \\ 0 \arrow[u] & & 0 \end{tikzcd}$$

which is natural in each variable. The map α has degree 0 *and the map β has degree* 1.

The KT holds if either $K_*(\mathfrak{A})$ or $K_*(\mathfrak{B})$ is finitely generated. The KT sequence splits unnaturally.

If neither $K_*(\mathfrak{A})$ nor $K_*(\mathfrak{B})$ is finitely generated, then there are counterexamples to the KT as noted by George Elliott.

2 Special cases of the UCT

This section is devoted to prove the following:

Theorem 2.1. *Let $\mathfrak{A} \in N$ and let $\mathfrak{B}$ be a C^*-algebra with countable approximate units such that $K_*(\mathfrak{B})$ is an injective (i.e., divisible) $\mathbb{Z}$-module. Then the map*

$$\gamma(\mathfrak{A}, \mathfrak{B}) : KK_*(\mathfrak{A}, \mathfrak{B}) \to \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B}))$$

is an isomorphism.

This is equivalent to the assertion that the UCT holds for all pair $(\mathfrak{A}, \mathfrak{B})$ with $\mathfrak{A} \in N$ and with $K_*(\mathfrak{B})$ injective.

Remark. Recall a fact from ([5]) in the following. An additive abelian group is viewed as a $\mathbb{Z}$-module. A $\mathbb{Z}$-module G is said to be divisible (or complete) if for any $g \in G$ and any $n \in \mathbb{Z}$, there is $x_n \in G$ such that $g = nx_n = x_n + \cdots + x_n$ ($n-1$-times sum). A torsion free, divisible $\mathbb{Z}$-module coinsides with a direct product of additive $\mathbb{Q}$ of all rationals.

Proposition 2.2. *Assume that $K_*(\mathfrak{B})$ is an injective $\mathbb{Z}$-module. Then $KK_*(\star, \mathfrak{B})$ and $\mathrm{Hom}(K_*(\star), K_*(\mathfrak{B}))$ are additive cohomology theories on the category of separable nuclear C^*-algebras $\star$, and $\gamma(\star, \mathfrak{B})$ is a natural transformation of cohomology theories.*

Proof. The assertion to $KK_*(\star, \mathfrak{B})$ is always true without any hypothesis on $\mathfrak{B}$ by Theorem 1.2.

The Karoubi K-theory $K_*(\star)$ is an additive homology theory. If $K_*(\mathfrak{B})$ is injective, then $\mathrm{Hom}(*, K_*(\mathfrak{B}))$ becomes an exact functor, and hence $\mathrm{Hom}(K_*(\star), K_*(\mathfrak{B}))$ satisfies the exactness axiom and so is a cohomology theory. It is additive since $\mathrm{Hom}(*, *)$ transforms direct sums in the first variable into direct products, that is,

$$\mathrm{Hom}(\oplus_i X_i, *) = \Pi_i \mathrm{Hom}(X_i, *).$$

Finally, the naturality of $\gamma(\star, \mathfrak{B})$ follows from that of the Kasparov product. Indeed, there is the following commutative diagram:

$$\begin{array}{ccc}
KK_*(\mathfrak{A}, \mathfrak{B}) & \xrightarrow[\cong]{\gamma(\mathfrak{A},\mathfrak{B})} & \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \\
\uparrow{\scriptstyle \otimes_{\mathfrak{A}'}\lambda} & & \uparrow{\scriptstyle (\otimes_{\mathfrak{A}'}\lambda)^*} \\
KK_*(\mathfrak{A}', \mathfrak{B}) & \xrightarrow[\cong]{\gamma(\mathfrak{A}',\mathfrak{B})} & \mathrm{Hom}(K_*(\mathfrak{A}'), K_*(\mathfrak{B}))
\end{array}$$

via $\lambda \in KK_0(\mathfrak{A}, \mathfrak{A}')$, where the map $(\otimes_{\mathfrak{A}'}\lambda)^*$ is defined to be the composite of the maps in the clock-wise, with $\gamma(\mathfrak{A}', \mathfrak{B})$ replaced with its inverse. □

Proposition 2.3. *Suppose that $K_*(\mathfrak{B})$ is injective. Let $\mathfrak{J}$ be a closed ideal of a separable nuclear C^*-algebra $\mathfrak{A}$. If two of the following maps:*

$$\begin{aligned}
\gamma(\mathfrak{J}, \mathfrak{B}) &: KK_*(\mathfrak{J}, \mathfrak{B}) \to \mathrm{Hom}(K_*(\mathfrak{J}), K_*(\mathfrak{B})), \\
\gamma(\mathfrak{A}, \mathfrak{B}) &: KK_*(\mathfrak{A}, \mathfrak{B}) \to \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B})), \\
\gamma(\mathfrak{A}/\mathfrak{J}, \mathfrak{B}) &: KK_*(\mathfrak{A}/\mathfrak{J}, \mathfrak{B}) \to \mathrm{Hom}(K_*(\mathfrak{A}/\mathfrak{J}), K_*(\mathfrak{B}))
\end{aligned}$$

are isomorphisms, then so is the third map.

Proof. We have the following commutative diagram:

$$\begin{array}{ccc}
KK_{*+1}(\mathfrak{J}, \mathfrak{B}) & \xrightarrow{\gamma(\mathfrak{J},\mathfrak{B})} & \mathrm{Hom}(K_{*+1}(\mathfrak{J}), K_*(\mathfrak{B})) \\
\uparrow & & \uparrow \\
KK_{*+1}(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{\gamma(\mathfrak{A},\mathfrak{B})} & \mathrm{Hom}(K_{*+1}(\mathfrak{A}), K_*(\mathfrak{B})) \\
\uparrow & & \uparrow \\
KK_{*+1}(\mathfrak{A}/\mathfrak{J}, \mathfrak{B}) & \xrightarrow{\gamma(\mathfrak{A}/\mathfrak{J},\mathfrak{B})} & \mathrm{Hom}(K_{*+1}(\mathfrak{A}/\mathfrak{J}), K_*(\mathfrak{B})) \\
\uparrow & & \uparrow \\
KK_*(\mathfrak{J}, \mathfrak{B}) & \xrightarrow{\gamma(\mathfrak{J},\mathfrak{B})} & \mathrm{Hom}(K_*(\mathfrak{J}), K_*(\mathfrak{B})) \\
\uparrow & & \uparrow \\
KK_*(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{\gamma(\mathfrak{A},\mathfrak{B})} & \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \\
\uparrow & & \uparrow \\
KK_*(\mathfrak{A}/\mathfrak{J}, \mathfrak{B}) & \xrightarrow{\gamma(\mathfrak{A}/\mathfrak{J},\mathfrak{B})} & \mathrm{Hom}(K_*(\mathfrak{A}/\mathfrak{J}), K_*(\mathfrak{B}))
\end{array}$$

Apply the Five-Lemma to the diagram above. □

Proposition 2.4. *Suppose that $K_*(\mathfrak{B})$ is injective. Let $\varinjlim \mathfrak{A}_j$ be an inductive limit of countable separable nuclear C^*-algebras $\mathfrak{A}_j$. If each map:*

$$\gamma(\mathfrak{A}_j, \mathfrak{B}) : KK_*(\mathfrak{A}_j, \mathfrak{B}) \to \mathrm{Hom}(K_*(\mathfrak{A}_j), K_*(\mathfrak{B}))$$

is an isomorphism, then

$$\gamma(\varinjlim \mathfrak{A}_j, \mathfrak{B}) : KK_*(\varinjlim \mathfrak{A}_j, \mathfrak{B}) \to \mathrm{Hom}(K_*(\varinjlim \mathfrak{A}_j), K_*(\mathfrak{B}))$$

is an isomorphism.

Proof. We have the following commutative diagram:

$$\begin{array}{ccc}
0 & = & 0 \\
\uparrow & & \uparrow \\
\varprojlim KK_*(\mathfrak{A}_j, \mathfrak{B}) & \xrightarrow{\cong} & \varprojlim \mathrm{Hom}(K_*(\mathfrak{A}_j), K_*(\mathfrak{B})) \\
\uparrow & & \uparrow \\
KK_*(\varinjlim \mathfrak{A}_j, \mathfrak{B}) & \longrightarrow & \mathrm{Hom}(K_*(\varinjlim \mathfrak{A}_j), K_*(\mathfrak{B})) \\
\uparrow & & \uparrow \\
\varprojlim^1 KK_{*-1}(\mathfrak{A}_j, \mathfrak{B}) & \xrightarrow{\cong} & \varprojlim^1 \mathrm{Hom}(K_{*-1}(\mathfrak{A}_j), K_*(\mathfrak{B})) \\
\uparrow & & \uparrow \\
0 & = & 0
\end{array}$$

where $\varprojlim$ means projective limit and $\varprojlim^1$ means Milnor limit, and the vertical sequences are Milnor $\lim^1$-sequences. Apply the Five-Lemma to the diagram above. □

Proposition 2.5. *If $\mathfrak{A}$ is strongly Morita equivalent to a commutative C^*-algebra and if $K_*(\mathfrak{B})$ is injective, then $\gamma(\mathfrak{A}, \mathfrak{B})$ is an isomorphism.*

Proof. First note that both $K_*(\star, \mathfrak{B})$ and $\mathrm{Hom}(K_*(\star), K_*(\mathfrak{B}))$ are invariant under strong Morita equivalence. Thus we may assume that $\mathfrak{A} = C_0(X)$ the C^*-algebra of all continuous functions on a locally compact Hausdorff space X vanishing at infinity. If $X = \mathbb{R}^k$ with k even, then by Bott periodicity,

$$\begin{aligned}
KK_*(C_0(\mathbb{R}^k), \mathfrak{B}) &\cong KK_*(\mathbb{C}, \mathfrak{B}) \cong K_*(\mathfrak{B}), \\
\mathrm{Hom}(K_*(C_0(\mathbb{R}^k)), K_*(\mathfrak{B})) &\cong \mathrm{Hom}(K_*(\mathbb{C}), K_*(\mathfrak{B})) \\
&\cong \mathrm{Hom}(\mathbb{Z}, K_*(\mathfrak{B})) \cong K_*(\mathfrak{B})
\end{aligned}$$

and if k is odd, then by Bott periodicity,

$$\begin{aligned} KK_*(C_0(\mathbb{R}^k), \mathfrak{B}) &\cong KK_*(S\mathbb{C}, \mathfrak{B}) \\ &\cong KK_{*+1}(\mathbb{C}, \mathfrak{B}) \cong K_{*+1}(\mathfrak{B}) \cong K_*(\mathfrak{B}), \\ \mathrm{Hom}(K_*(C_0(\mathbb{R}^k)), K_*(\mathfrak{B})) &\cong \mathrm{Hom}(K_*(S\mathbb{C}), K_*(\mathfrak{B})) \\ &\cong \mathrm{Hom}(K_{*+1}(\mathbb{C}), K_*(\mathfrak{B})) \\ &\cong \mathrm{Hom}(\mathbb{Z}, K_*(\mathfrak{B})) \cong K_*(\mathfrak{B}). \end{aligned}$$

For X a finite cell complex, we use induction on the number of cells and the above Proposition 2.3 for short exact sequences of C^*-algebras such as $0 \to C_0(U) \to C_0(V) \to C_0(W) \to 0$, where U is an open subset of a locally compact Hausdorff space V and W is closed in V, repeatedly. If X is compact and metrizable, then X is a countable inverse (or projective) limit $\varprojlim X_i$ of finite complexes X_i, so that $C(X) = \varinjlim C(X_i)$ an inductive limit of $C(X_i)$ of all continuous functions on X_i and thus $\gamma(C(X), \mathfrak{B})$ is an isomorphism for the above Proposition 2.4 for inductive limits. Finally, if $\mathfrak{A} = C_0(X)$ with X non-compact, we use the result for the unitization $\mathfrak{A}^+ = C(X^+)$, where X^+ is the one-point compactification and the short exact sequence: $0 \to C_0(X) \to C(X^+) \to \mathbb{C} \to 0$. □

Proposition 2.6. *If $K_*(\mathfrak{B})$ is injective and $\mathfrak{A}$ is a separable type I C^*-algebra, then $\gamma(\mathfrak{A}, \mathfrak{B})$ is an isomorphism.*

Proof. We prove this as follows, not omitted as in [8].

Since $\mathfrak{A}$ is separable and of type I, it has a countable composition series of closed ideals $\mathfrak{I}_j$ such that the union of $\mathfrak{I}_j$ is dense in $\mathfrak{A}$ and subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ have continuous trace.

(1) If such a subquotient is finite homogeneous, with respect to dimension of irreducible representation, then it can be obtained by taking finitely extensions by tensor products of commutative C^*-algebras with matrix algebras over $\mathbb{C}$.

(2) If such a subquotient has a composition series of finite homogeneous C^*-algebras, then we use the case of (1) and the Propositions 2.3 and 2.4 for extensions and inductive limits.

(3) If such a subquotient is infinite homogeneous, with respect to dimension of irreducible representation, and if its spectrum has dimension finite, then by local triviality, it has a composition series of closed ideals such that subquotients are tensor products of commutative C^*-algebra with $\mathbb{K}$. If the spectrum has dimension infinite, we decompose the spectrum into an inverse limit of spaces with dimension finite and the subquotient can be

an inductive limit of infinite homogeneous C^*-algebras with spectrums of dimension finite.

(4) The general case can be treated by taking a composition series of closed ideals such that subquotients are contained in those three cases. $\square$

Proposition 2.7. *If $K_*(\mathfrak{B})$ is injective and if $\gamma(\mathfrak{A}, \mathfrak{B})$ is an isomorphism, then $\gamma(\mathfrak{A} \rtimes_\alpha \mathbb{R}, \mathfrak{B})$ is an isomorphism for $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ the crossed product of $\mathfrak{A}$ by an action α of the real group $\mathbb{R}$ on $\mathfrak{A}$.*

Proof. The Thom isomorphisms of Connes and of Fack-Skandalis for K-theory and KK-theory respectively yield the natural isomorphisms:

$$\mathrm{Hom}(K_i(\mathfrak{A} \rtimes_\alpha \mathbb{R}), K_j(\mathfrak{B})) \cong \mathrm{Hom}(K_{i+1}(\mathfrak{A}), K_j(\mathfrak{B}))$$

and

$$KK_i(\mathfrak{A} \rtimes_\alpha \mathbb{R}, \mathfrak{B}) \cong KK_{i+1}(\mathfrak{A}, \mathfrak{B}).$$

Hence, we have

$$\begin{aligned} KK_*(\mathfrak{A} \rtimes_\alpha \mathbb{R}, \mathfrak{B}) &\cong KK_{*+1}(\mathfrak{A}, \mathfrak{B}) \\ &\cong \mathrm{Hom}(K_{*+1}(\mathfrak{A}), K_*(\mathfrak{B})) \\ &\cong \mathrm{Hom}(K_*(\mathfrak{A} \rtimes_\alpha \mathbb{R}), K_*(\mathfrak{B})). \end{aligned}$$

$\square$

Proposition 2.8. *If $K_*(\mathfrak{B})$ is injective and if $\gamma(\mathfrak{A}, \mathfrak{B})$ is an isomorphism, then $\gamma(\mathfrak{A} \rtimes_\alpha \mathbb{Z}, \mathfrak{B})$ is an isomorphism for $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ the crossed product of $\mathfrak{A}$ by an action α of the integer group $\mathbb{Z}$ on $\mathfrak{A}$.*

Proof. Consider the following diagram:

$$\begin{array}{ccc}
KK_{*+1}(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{\gamma(\mathfrak{A},\mathfrak{B})} & \mathrm{Hom}(K_{*+1}(\mathfrak{A}), K_*(\mathfrak{B})) \\
{\scriptstyle (\mathrm{id}-\alpha)^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}((\mathrm{id}-\alpha)_*, \mathrm{id}_*)} \\
KK_{*+1}(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{\gamma(\mathfrak{A},\mathfrak{B})} & \mathrm{Hom}(K_{*+1}(\mathfrak{A}), K_*(\mathfrak{B})) \\
{\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(\partial, \mathrm{id}_*)} \\
KK_*(\mathfrak{A} \rtimes_\alpha \mathbb{Z}, \mathfrak{B}) & \xrightarrow{\gamma(\mathfrak{A}\rtimes_\alpha\mathbb{Z},\mathfrak{B})} & \mathrm{Hom}(K_*(\mathfrak{A} \rtimes_\alpha \mathbb{Z}), K_*(\mathfrak{B})) \\
{\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(i_*, \mathrm{id}_*)} \\
KK_*(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{\gamma(\mathfrak{A},\mathfrak{B})} & \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \\
{\scriptstyle (\mathrm{id}-\alpha)^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}((\mathrm{id}-\alpha)_*, \mathrm{id}_*)} \\
KK_*(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{\gamma(\mathfrak{A},\mathfrak{B})} & \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B}))
\end{array}$$

where $i : \mathfrak{A} \to \mathfrak{A} \rtimes_\alpha \mathbb{Z}$ is the inclusion map, and the left column is exact by Fack-Skandalis, and the right column is exact by Pimsner-Voiculescu and the fact that $K_*(\mathfrak{B})$ is injective. The diagram commutes and hence the map $\gamma(\mathfrak{A} \rtimes_\alpha \mathbb{Z}, \mathfrak{B})$ in the middle is an isomorphism by the Five-Lemma. □

Proof of Theorem 2.1. The propositons proved so far in this section complete the proof of Theorem 2.1. □

3 Geometric injective resolutions

This section is devoted to show the construction of a geometric injective resolution for a C^*-algebra.

Recall that an injective resolution of an abelian group G is a short exact sequence of abelian groups:

$$0 \to G \to I_0 \to I_1 \to 0$$

such that I_0 and I_1 are injective (i.e., divisible) groups. For instance, the following short exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

is an injective resolution of $\mathbb{Z}$.

Every abelian group G has an injective resolution. We may construct such a resolution as follows. Let $f : F_0 \to G$ be a homomorphism from a free abelian group F_0 onto G, and let $F_1 = \ker(f)$ the kernel. Then F_1 is free and $F_0/F_1 \cong G$. Let $g : G \to I_0$ be the composition of homomorphisms:

$$G \to F_0/F_1 \to (F_0 \otimes \mathbb{Q})/F_1 \equiv I_0$$

and then g is a monomorphism, and I_0 is injective as it is a quotient of $F_0 \otimes \mathbb{Q}$ injective. Then the short exact sequence:

$$0 \to G \to I_0 \to I_0/G \equiv I_1 \to 0$$

is an injective resolution of G.

Let $f : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism of C^*-algebras. The mapping cone Cf for f is defined by

$$Cf = \{(\xi, a) \in I\mathfrak{B} \oplus \mathfrak{A} \,|\, \xi(0) = 0, \xi(1) = f(a)\},$$

where $I\mathfrak{B} = C(I, \mathfrak{B})$ the C^*-algebra of all $\mathfrak{B}$-valued, continuous functions on the interval $I = [0, 1]$. There is a natural map $Cf \to \mathfrak{A}$ given by sending (ξ, a) to a, and the resulting short exact sequence:

$$0 \to S\mathfrak{B} \to Cf \to \mathfrak{A} \to 0$$

called the mapping cone sequence for f, which splits.

Theorem 3.1. *Let $\mathfrak{B}$ be a C^*-algebra. Then there exists a C^*-algebra $\mathfrak{D}$ whose K-theory groups $K_*(\mathfrak{D})$ are injective and a homomorphism $f : S\mathfrak{B} \to \mathfrak{D}$ such that the induced map*

$$K_{*+1}(\mathfrak{B}) = K_*(S\mathfrak{B}) \xrightarrow{f_*} K_*(\mathfrak{D})$$

is a monomorphism.

Note that this theorem implies the existence of a geometric injective resolution for $K_*(\mathfrak{B})$. Indeed, the mapping cone sequence for f has the form:

$$0 \to S\mathfrak{D} \to Cf \to S\mathfrak{B} \to 0.$$

It follows from the associated six-term exact K-theory group sequence and the theorem above that

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & K_j(S\mathfrak{B}) \arrow[r, "f_*"] \arrow[d, "\cong"'] & K_j(\mathfrak{D}) \arrow[d, "\cong"] & & \\
0 \arrow[r] & K_j(S\mathfrak{B}) \arrow[r, "\partial"] & K_{j-1}(S\mathfrak{D}) \arrow[r] & K_{j-1}(Cf) \arrow[r] & 0
\end{tikzcd}$$

where the last right zero also means the kernel of the map ∂. Therefore, $K_j(Cf)$ become quotients of injective $K_{j+1}(\mathfrak{D})$, and hence be injective. Thus, the last short exact sequence on the bottom line is a geometric injective resolution for $K_*(\mathfrak{B})$.

Proof. It suffices to assume that $\mathfrak{B}$ is unital, for the non-unital case follows from the unital case and the fact that the map from $\mathfrak{B}$ to its unitization $\mathfrak{B}^+$ induces an inclusion from $K_*(\mathfrak{B})$ to $K_*(\mathfrak{B}^+)$. Namely, the maps $f^+ : S(\mathfrak{B}^+) \to \mathfrak{D}$ and $f_* : K_*(S(\mathfrak{B}^+)) = K_{*+1}(\mathfrak{B}^+) \to K_*(\mathfrak{D})$ as in the last theorem above induce the same maps for $\mathfrak{B}$ by restriction.

Let $\mathfrak{B}$ be a unital C^*-algebra. Let $r : \mathfrak{F} \to \mathfrak{B} \otimes \mathbb{K}$ be a geometric projective resolution for $\mathfrak{B}$ obtained by Schochet. That is, $\mathfrak{F}$ is a C^*-algebra with $K_*(\mathfrak{F})$ a free abelian group and the map

$$r_* : K_*(\mathfrak{F}) \to K_*(\mathfrak{B} \otimes \mathbb{K}) \cong K_*(\mathfrak{B})$$

is onto. The resulting mapping cone sequence for the map r:

$$0 \to S\mathfrak{B} \otimes \mathbb{K} \xrightarrow{i} Cr \xrightarrow{s} \mathfrak{F} \to 0$$

yields the following K-theory group sequence:

$$0 \to K_*(Cr) \xrightarrow{s_*} K_*(\mathfrak{F}) \xrightarrow{r_*} K_*(\mathfrak{B} \otimes \mathbb{K}) \cong K_*(\mathfrak{B}) \to 0$$

since r_* is onto, and via the part of the six-term K-theory group diagram:

$$\begin{array}{ccccccc}
K_j(\mathfrak{F}) & \xrightarrow{\partial} & K_{j-1}(S\mathfrak{B}\otimes\mathbb{K}) & \xrightarrow{i_*} & K_{j-1}(Cr) & \xrightarrow{s_*} & K_{j-1}(\mathfrak{F}) \\
\| & & \| & & & & \\
K_j(\mathfrak{F}) & \xrightarrow{r_*} & K_j(\mathfrak{B}\otimes\mathbb{K}) & \longrightarrow & 0 & &
\end{array}$$

from which, i_* is the zero map, and hence, s_* is injective by the exactness of the diagram. Then $K_*(Cr)$ is a subgroup of $K_*(\mathfrak{F})$ free, and is hence free.

Let $\mathfrak{N}$ be a unital UHF or AF-algebra with $K_0(\mathfrak{N}) = \mathbb{Q}$. For instance, let $\mathfrak{N}$ be an inductive limit of tensor products of matrix algebras over $\mathbb{C}$:

$$\mathfrak{N} = \varinjlim M_2(\mathbb{C}) \otimes M_3(\mathbb{C}) \otimes \cdots \otimes M_{n!}(\mathbb{C})$$

with the canonical inclusion maps, for instance,

$$M_2(\mathbb{C}) \ni A \mapsto \begin{pmatrix} A & 0_2 & 0_2 \\ 0_2 & A & 0_2 \\ 0_2 & 0_2 & A \end{pmatrix} \in M_2(\mathbb{C}) \otimes M_3(\mathbb{C}) \cong M_6(\mathbb{C})$$

with 0_2 the 2×2 zero matrix in $M_2(\mathbb{C})$..

Define a map $t : \mathfrak{F} \to \mathfrak{F}\otimes\mathfrak{N}$ by $t(x) = x\otimes 1$ for $x \in \mathfrak{F}$. Then the induced map

$$t_* : K_*(\mathfrak{F}) \to K_*(\mathfrak{F}\otimes\mathfrak{N})$$

is a monomorphism, and

$$K_*(\mathfrak{F}\otimes\mathfrak{N}) \cong K_*(\mathfrak{F}) \otimes K_*(\mathfrak{N}) = K_*(\mathfrak{F}) \otimes \mathbb{Q}$$

via the Künneth theorem for K-theory groups by Schochet since $K_1(\mathfrak{N}) = 0$ and (additive) $\mathbb{Q}$ torsion free, and also, as a note, $K_*(F) \otimes \mathbb{Q}$ is injective since $\mathbb{Q}$ is divisible.

The mapping cone sequence for $t \circ s : Cr \to \mathfrak{F}\otimes\mathfrak{N}$:

$$0 \to S\mathfrak{F}\otimes\mathfrak{N} \to Ct\circ s \to Cr \to 0$$

implies the following K-theory group sequence:

$$0 \to K_*(Cr) \xrightarrow{(t\circ s)_*} K_*(\mathfrak{F}\otimes\mathfrak{N}) \longrightarrow K_*(Ct\circ s) \to 0$$

since $(t\circ s)_* = t_* \circ s_*$ is injective, and via the part of the six-term K-theory group exact sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_j(Cr) & \xrightarrow{(t\circ r)_*} & K_j(\mathfrak{F}\otimes\mathfrak{N}) & & & & \\
\| & & \| & & \| & & & & \\
0 & \longrightarrow & K_j(Cr) & \xrightarrow{\partial} & K_{j-1}(S\mathfrak{F}\otimes\mathfrak{N}) & \longrightarrow & K_{j-1}(Ct\circ r) & \longrightarrow & 0
\end{array}$$

where the last right zero comes from being zero of the kernel of the map ∂ which is the image of $K_{j-1}(Ct \circ r)$ by exactness of the diagram. It follows that $K_*(Ct \circ s)$ is divisible since it is a quotient of a divisible $\mathbb{Z}$-module by a free one. Possibly, if the free module is finitely generated, then the quotient is certainly divisible, and if the free module is (countably) infinitely generated, the quotient is still divisible or may be zero, but divisible.

The naturality of the cone construction by Schochet implies that there is a map of mapping cone sequences:

$$\begin{tikzcd}
SCr \arrow[r] \arrow[d] & S\mathfrak{F} \arrow[r] \arrow[d] & Cs \arrow[r] \arrow[d, "u"] & Cr \arrow[r, "s"] \arrow[d] & \mathfrak{F} \arrow[d, "t"] \\
SCr \arrow[r] & S\mathfrak{F} \otimes \mathfrak{N} \arrow[r] & Ct \circ s \arrow[r] & Cr \arrow[r] & \mathfrak{F} \otimes \mathfrak{N}
\end{tikzcd}$$

where note that the mapping cone sequence for $s : Cr \to \mathfrak{F}$ is

$$0 \to S\mathfrak{F} \to Cs \to Cr \to 0$$

and the mapping cone sequence for $t \circ s : Cr \to \mathfrak{F} \otimes \mathfrak{N}$ is

$$0 \to S\mathfrak{F} \otimes \mathfrak{N} \to Ct \circ s \to Cr \to 0.$$

Hence, there is the following commuting diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & K_*(Cr) \arrow[r, "s_*"] \arrow[d, equal] & K_*(\mathfrak{F}) \arrow[r] \arrow[d, "t_*"] & K_{*+1}(Cs) \arrow[r] \arrow[d, "u_*"] & 0 \arrow[d, equal] \\
0 \arrow[r] & K_*(Cr) \arrow[r] & K_*(\mathfrak{F} \otimes \mathfrak{N}) \arrow[r] & K_{*+1}(Ct \circ s) \arrow[r] & 0
\end{tikzcd}$$

with t_* injective. The Snake Lemma implies that the map $u_* : K_*(Cs) \to K_*(Ct \circ s)$ is a monomorphism.

Indeed, check that u_* is mono. Suppose that x is in the kernel of u_*. Then there is $y \in K_*(\mathfrak{F})$ which is mapped to x from the left by exactness of the diagram. Then $t_*(y)$ is mapped to zero from the left by commutativeness of the diagram. And hence there is $z \in K_*(Cr)$ which is mapped to $t_*(y)$ from the left by exactness of the diagram, and z is also mapped to y under s_* by commutativeness of the diagram. Therefore, z is mapped to x under the composition with s_*, and hence x is zero by exactness of the diagram, which shows the claim.

Finally, since $S\mathfrak{B} \otimes \mathbb{K}$ is the kernel of the map $s : Cr \to \mathfrak{F}$, a homotopy argument of Schochet yields the short exact sequence:

$$0 \to S\mathfrak{B} \otimes \mathbb{K} \xrightarrow{v} Cs \xrightarrow{q} C\mathfrak{F} \to 0.$$

where, indeed, without looking at the another item of Schochet we check that

$$Cs = \{(\xi, a) \in I\mathfrak{F} \oplus Cr \mid \xi(0) = 0, \xi(1) = s(a)\}$$

and thus the quotient map q is defined by $(\xi, a) \mapsto \xi \in C\mathfrak{F}$ and if $\xi = 0$, then $\xi(1) = 0 = s(a)$, and hence $(0, a) = a$ is in the kernel of the map s. Thus $\ker(q) \subset \ker(s)$. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S\mathfrak{B} \otimes \mathbb{K} & \longrightarrow & Cr & \xrightarrow{s} & \mathfrak{F} & \longrightarrow & 0 \\ \| & & \uparrow & & \uparrow & & \uparrow & & \| \\ 0 & \longrightarrow & \ker(q) & \longrightarrow & Cs & \xrightarrow{q} & C\mathfrak{F} & \longrightarrow & 0. \end{array}$$

If $s(a) = 0$, then there is $(\xi, a) \in Cs$ with $\xi(1) = 0$. And this ξ can be deformed to zero, and hence $a = (0, a)$ is in the kernel of q.

Note that $C\mathfrak{F}$ means the cone over $\mathfrak{A}$, that is, it is the C^*-algebra of all $\mathfrak{F}$-valued continuous functions on the closed interval $[0, 1]$ vanishing at zero. Since $C\mathfrak{F}$ is contractible, so that $K_0(C\mathfrak{F}) = 0$, and then the suspention $SC\mathfrak{F}$ is contractible, so that $K_1(C\mathfrak{F}) = K_0(SC\mathfrak{F}) = 0$.

It follows that $v_* : K_*(S\mathfrak{B} \otimes \mathbb{K}) \to K_*(Cs)$ is an isomorphism.

Let $w : S\mathfrak{B} \to S\mathfrak{B} \otimes \mathbb{K}$ be the inclusion map induced by the choice of a rank-one projection in $\mathbb{K}$, which induces an isomorphism $K_0(S\mathfrak{B}) \cong K_0(S\mathfrak{B} \otimes \mathbb{Z})$, so that also $K_1(S\mathfrak{B}) \cong K_0(S^2\mathfrak{B}) \cong K_0(S^2\mathfrak{B} \otimes \mathbb{K}) \cong K_1(S\mathfrak{B} \otimes \mathbb{K})$, and hence, $w_* : K_*(S\mathfrak{B}) \to K_*(S\mathfrak{B} \otimes \mathbb{K})$ is an isomorphism.

Define f to be the following composite:

$$S\mathfrak{B} \xrightarrow{w} S\mathfrak{B} \otimes \mathbb{K} \xrightarrow{v} Cs \xrightarrow{u} Ct \circ s$$

and set $\mathfrak{D} = Ct \circ s$. Then $f_* = u_* \circ v_* \circ w_* : K_*(S\mathfrak{B}) \to K_*(\mathfrak{D})$ is a monomorphism with $K_*(\mathfrak{D})$ injective, as desired. □

4 The general UCT

Theorem 4.1. *Let $\mathfrak{A}, \mathfrak{B}$ be C^*-algebras. Suppose that the UCT holds for all pairs $(\mathfrak{A}, \mathfrak{B})$ with $K_*(\mathfrak{B})$ injective. Then the UCT holds for all pairs $(\mathfrak{A}, \mathfrak{B})$ (with $K_*(\mathfrak{B})$ arbitrary).*

Proof. Let $f : S\mathfrak{B} \to \mathfrak{D}_0$ be a geometric injective resolution such that $K_*(\mathfrak{D}_0)$ is injective and $f_* : K_*(S\mathfrak{B}) \to K_*(\mathfrak{D}_0)$ is mono. Let $\mathfrak{D}_1$ be the mapping cone Cf:

$$0 \to S\mathfrak{D}_0 \xrightarrow{g} Cf = \mathfrak{D}_1 \to S\mathfrak{B} \to 0.$$

This mapping cone sequence yields the following K-theory sequence:

$$0 \to K_j(S\mathfrak{B}) \xrightarrow{\partial} K_{j-1}(S\mathfrak{D}_0) \xrightarrow{g_*} K_{j-1}(\mathfrak{D}_1) \to 0$$

which gives a geometric injective resolution of $K_*(S\mathfrak{B})$, where note that the boundary map(s) ∂ is mono because f_* is mono.

The long exact sequence for the KK-theory groups associated to the mapping cone sequence above has the form

$$\begin{aligned} \to KK_j(\mathfrak{A}, S\mathfrak{D}_0) &\xrightarrow{(\mathrm{id},g_j)_*} KK_j(\mathfrak{A}, \mathfrak{D}_1) \to KK_j(\mathfrak{A}, S\mathfrak{B}) \\ &\xrightarrow{(\mathrm{id}_*,\partial)} KK_{j-1}(\mathfrak{A}, S\mathfrak{D}_0) \xrightarrow{(\mathrm{id},g_{j-1})_*} KK_{j-1}(\mathfrak{A}, \mathfrak{D}_1) \to \end{aligned}$$

with $g_j = g = g_{j-1}$, from which it follows that

$$\begin{aligned} 0 &\to \mathrm{coker}(\mathrm{id}, g_j)_* = KK_j(\mathfrak{A}, \mathfrak{D}_1)/\mathrm{im}(\mathrm{id}, g_j)_* \\ &\to KK_{j+1}(\mathfrak{A}, \mathfrak{B}) \cong KK_j(\mathfrak{A}, S\mathfrak{B}) \\ &\to \ker(\mathrm{id}, g_{j-1})_* \to 0. \end{aligned}$$

Since the following diagram:

$$\begin{array}{ccc} KK_*(\mathfrak{A}, S\mathfrak{D}_0) & \xrightarrow{(\mathrm{id},g)_*} & KK_*(\mathfrak{A}, \mathfrak{D}_1) \\ \big\downarrow \gamma(\mathfrak{A}, S\mathfrak{D}_0) & & \big\downarrow \gamma(\mathfrak{A}, \mathfrak{D}_1) \\ \mathrm{Hom}(K_*(\mathfrak{A}), K_*(S\mathfrak{D}_0)) & \xrightarrow{\mathrm{Hom}(\mathrm{id},g)_*} & \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{D}_1)) \end{array}$$

commutes and the vertical maps are isomorphism by the assumption, we see that

$$\begin{aligned} \ker(\mathrm{id}, g)_* &\cong \ker[\mathrm{Hom}(\mathrm{id}, g)_*], \\ \mathrm{coker}(\mathrm{id}, g)_* &\cong \mathrm{coker}[\mathrm{Hom}(\mathrm{id}, g)_*]. \end{aligned}$$

And also, similarly, the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & KK_{*+1}(\mathfrak{A}, S\mathfrak{B}) & \xrightarrow{(\mathrm{id}_*,\partial)} & KK_*(\mathfrak{A}, S\mathfrak{D}_0) \\ \| & & \big\downarrow \gamma(\mathfrak{A}, S\mathfrak{B}) & & \big\downarrow \gamma(\mathfrak{A}, S\mathfrak{D}_0) \\ 0 & \longrightarrow & \mathrm{Hom}(K_*(\mathfrak{A}), K_{*+1}(S\mathfrak{B})) & \xrightarrow{\mathrm{Hom}(\mathrm{id},\partial)} & \mathrm{Hom}(K_*(\mathfrak{A}), K_*(S\mathfrak{D}_0)) \end{array}$$

commutes and the vertical maps are isomorphism by the assumption, we get

$$\begin{aligned} &\mathrm{Hom}(K_*(\mathfrak{A}), K_{*+1}(S\mathfrak{B})) \cong \ker[\mathrm{Hom}(\mathrm{id}, g)_*], \\ &\mathrm{Hom}(K_*(\mathfrak{A}), K_*(S\mathfrak{D}_0))/\mathrm{Hom}(K_*(\mathfrak{A}), K_{*+1}(S\mathfrak{B})) \cong \mathrm{im}[\mathrm{Hom}(\mathrm{id}, g)_*] \end{aligned}$$

and hence, we obtain

$$\begin{aligned}
&\operatorname{coker}[\operatorname{Hom}(\mathrm{id}, g)_*] \\
&\cong \frac{\operatorname{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{D}_1))}{\operatorname{Hom}(K_*(\mathfrak{A}), K_*(S\mathfrak{D}_0))/\operatorname{Hom}(K_*(\mathfrak{A}), K_{*+1}(S\mathfrak{B}))} \\
&\cong \operatorname{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(S\mathfrak{B})).
\end{aligned}$$

Indeed, we have the following exact sequence as a fact of homology theory:

$$\begin{aligned}
0 &\to \operatorname{Hom}(K_*(\mathfrak{A}), K_{*+1}(S\mathfrak{B})) \\
&\to \operatorname{Hom}(K_*(\mathfrak{A}), K_*(S\mathfrak{D}_0)) \\
&\to \operatorname{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{D}_1)) \\
&\to \operatorname{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(S\mathfrak{B})) \\
&\to \operatorname{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(S\mathfrak{D}_0)) \\
&\cong 0 \to \cdots
\end{aligned}$$

since

$$0 \to K_{*+1}(S\mathfrak{B}) \to K_*(S\mathfrak{D}_0) \to K_*(\mathfrak{D}_1) \to 0$$

is exact, and since $K_*(S\mathfrak{D}_0)$ is injective.

We have verified that

$$\begin{aligned}
0 &\to \operatorname{coker}(\mathrm{id}, g)_* \cong \operatorname{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(S\mathfrak{B})) \\
&\to KK_*(\mathfrak{A}, S\mathfrak{B}) \\
&\xrightarrow{\gamma(\mathfrak{A}, S\mathfrak{B})} \ker(\mathrm{id}, g)_* \cong \operatorname{Hom}(K_*(\mathfrak{A}), K_*(S\mathfrak{B})) \to 0
\end{aligned}$$

so that we obtain the UCT for the pair $(\mathfrak{A}, \mathfrak{B})$:

$$\begin{aligned}
0 \to \operatorname{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) \to KK_*(\mathfrak{A}, \mathfrak{B}) \\
\xrightarrow{\gamma(\mathfrak{A}, \mathfrak{B})} \operatorname{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to 0
\end{aligned}$$

with the injection of degree one and the quotient map of degree zero. □

Proof for the UCT except naturality. Theorem 4.1 and Theorem 2.1 prove the UCT in the introduction. The naturality of the UCT follows from the next theorem. □

Theorem 4.2. *The UCT is natural in both variables. More precisely, if* $\mathfrak{A}$, $\mathfrak{A}'$ *in* N, $\mathfrak{B}$ *and* $\mathfrak{B}'$ *are* C^**-algebras with countable approximate units, and*

$\lambda \in KK_0(\mathfrak{A}, \mathfrak{A}')$, $\mu \in KK_0(\mathfrak{B}, \mathfrak{B}')$, *then the following diagrams commute:*

$$\begin{array}{ccc}
0 & = & 0 \\
\downarrow & & \downarrow \\
\operatorname{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}'), K_{*+1}(\mathfrak{B})) & \xrightarrow{(\lambda \otimes_{\mathfrak{A}'})^*} & \operatorname{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) \\
{\scriptstyle\delta}\downarrow & & {\scriptstyle\delta}\downarrow \\
KK_*(\mathfrak{A}', \mathfrak{B}) & \xrightarrow{\lambda \otimes_{\mathfrak{A}'}} & KK_*(\mathfrak{A}, \mathfrak{B}) \\
{\scriptstyle\gamma}\downarrow & & {\scriptstyle\gamma}\downarrow \\
\operatorname{Hom}(K_*(\mathfrak{A}'), K_*(\mathfrak{B})) & \xrightarrow{(\lambda \otimes_{\mathfrak{A}'})^*} & \operatorname{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \\
\downarrow & & \downarrow \\
0 & = & 0
\end{array}$$

where

$$KK_0(\mathfrak{A}, \mathfrak{A}') \otimes KK_*(\mathfrak{A}', \mathfrak{B}) \xrightarrow{\otimes_{\mathfrak{A}'}} KK_*(\mathfrak{A}, \mathfrak{B}),$$

and also

$$\begin{array}{ccc}
0 & = & 0 \\
\downarrow & & \downarrow \\
\operatorname{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) & \xrightarrow{(\otimes_{\mathfrak{B}} \mu)_*} & \operatorname{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B}')) \\
{\scriptstyle\delta}\downarrow & & {\scriptstyle\delta}\downarrow \\
KK_*(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{\otimes_{\mathfrak{B}} \mu} & KK_*(\mathfrak{A}, \mathfrak{B}') \\
{\scriptstyle\gamma}\downarrow & & {\scriptstyle\gamma}\downarrow \\
\operatorname{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) & \xrightarrow{(\otimes_{\mathfrak{B}} \mu)_*} & \operatorname{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B}')) \\
\downarrow & & \downarrow \\
0 & = & 0
\end{array}$$

where

$$KK_0(\mathfrak{A}, \mathfrak{B}) \otimes KK_*(\mathfrak{B}, \mathfrak{B}') \xrightarrow{\otimes_{\mathfrak{B}}} KK_*(\mathfrak{A}, \mathfrak{B}').$$

Proof. The naturality of γ follows from functoriality of the Kasparov product. Note that, for $\rho \in KK_*(\mathfrak{A}', \mathfrak{B})$ we have

$$\begin{array}{ccc} \rho & \xrightarrow{\gamma} & \gamma(\rho) \\ \Big\downarrow{\scriptstyle \lambda\otimes_{\mathfrak{A}'}} & & \Big\downarrow{\scriptstyle (\lambda\otimes_{\mathfrak{A}'})^*} \\ \lambda \otimes_{\mathfrak{A}'} \rho & \xrightarrow{\gamma} & \gamma(\lambda \otimes_{\mathfrak{A}'} \rho) = (\lambda\otimes_{\mathfrak{A}'})^* \gamma(\rho) \end{array}$$

where the equation in the lower right corner is the definition for the map $(\lambda\otimes_{\mathfrak{A}'})^*$, so that the resulting diagram

$$\begin{array}{ccc} KK_*(\mathfrak{A}', \mathfrak{B}) & \xrightarrow{\gamma} & \mathrm{Hom}(K_*(\mathfrak{A}'), K_*(\mathfrak{B})) \\ \Big\downarrow{\scriptstyle \lambda\otimes_{\mathfrak{A}'}} & & \Big\downarrow{\scriptstyle (\lambda\otimes_{\mathfrak{A}'})^*} \\ KK_*(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{\gamma} & \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \end{array}$$

commutes.

We show naturality of the map

$$\delta : \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) \to KK_*(\mathfrak{A}, \mathfrak{B})$$

as in the following 4 steps, which is equivalent to the following map:

$$\delta : \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to KK_{*+1}(\mathfrak{A}, \mathfrak{B}).$$

Step 1. We show that the map δ is independent of the choice of the geometric injective resolution. Indeed, given two such resolutions $f : S\mathfrak{B} \to \mathfrak{D}_0$ and $f' : S\mathfrak{B} \to \mathfrak{D}_0'$ with $K_*(\mathfrak{D}_0)$, $K_*(\mathfrak{D}_0')$ injective and f_*, f'_* monomorphisms, such that

$$0 \to S\mathfrak{D}_0 \xrightarrow{g} \mathfrak{D}_1 = Cf \xrightarrow{p} S\mathfrak{B} \to 0$$

$$0 \to S\mathfrak{D}_0' \xrightarrow{h} \mathfrak{D}_1' = Cf' \xrightarrow{p} S\mathfrak{B} \to 0$$

(corrected), where p is the canonical projection (to the second coordinate). Let

$$\mathfrak{D}_1'' = \{(x, y) \in \mathfrak{D}_1 \oplus \mathfrak{D}_1' \,|\, p(x) = p(y)\}.$$

Then we obtain another geometric injective resolution: $f'' = f \oplus f' : S\mathfrak{B} \to \mathfrak{D}_0'' = \mathfrak{D}_0 \oplus \mathfrak{D}_0'$, such that

$$0 \to S\mathfrak{D}_0'' \xrightarrow{g\oplus h} \mathfrak{D}_1'' = Cf'' \xrightarrow{p} S\mathfrak{B} \to 0.$$

Indeed, an element $(\xi_1, a) \oplus (\xi_2, a) \in \mathfrak{D}_1''$ with $\xi_1(0) = 0, \xi_2(0) = 0$ and $\xi_1(1) = f(a)$, $\xi_2(1) = f'(a)$ corresponds to the element $(\xi, a) = (\xi_1 \oplus$

$\xi_2, a) \in Cf''$ with $\xi(0) = \xi_1(0) \oplus \xi_2(0) = 0 \oplus 0$ and $f''(a) = f(a) \oplus f'(a) = \xi_1(1) \oplus \xi_2(1)$.

The projection to the first coordinate: $\mathfrak{D}_1'' \to \mathfrak{D}_1$ gives the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S\mathfrak{D}_0'' & \xrightarrow{g\oplus h} & \mathfrak{D}_1'' & \xrightarrow{p} & S\mathfrak{B} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & S\mathfrak{D}_0 & \xrightarrow{g} & \mathfrak{D}_1 & \xrightarrow{p} & S\mathfrak{B} & \longrightarrow & 0.
\end{array}$$

It follows from the induced maps in K-theory and KK-theory that we obtain the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{H}(K_*(\mathfrak{A}), K_*(S\mathfrak{D}_0'')) & \longrightarrow & \mathrm{H}(K_*(\mathfrak{A}), K_*(\mathfrak{D}_1'')) & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(S\mathfrak{B})) \\
\| & & \| & & \\
KK_*(\mathfrak{A}, S\mathfrak{D}_0'') & \xrightarrow{(\mathrm{id}, g\oplus h)_*} & KK_*(\mathfrak{A}, \mathfrak{D}_1'') & \xrightarrow{(\mathrm{id}, p)_*} & KK_*(\mathfrak{A}, S\mathfrak{B}) \\
\downarrow & & \downarrow & & \| \\
KK_*(\mathfrak{A}, S\mathfrak{D}_0) & \xrightarrow{(\mathrm{id}, g)_*} & KK_*(\mathfrak{A}, \mathfrak{D}_1) & \xrightarrow{(\mathrm{id}, p)_*} & KK_*(\mathfrak{A}, S\mathfrak{B}) \\
\| & & \| & & \\
\mathrm{H}(K_*(\mathfrak{A}), K_*(S\mathfrak{D}_0)) & \longrightarrow & \mathrm{H}(K_*(\mathfrak{A}), K_*(\mathfrak{D}_1)) & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(S\mathfrak{B}))
\end{array}$$

where $\mathrm{H}(*, *) = \mathrm{Hom}(*, *)$ just for short. It follows from the diagram above that

$$\begin{array}{ccc}
0 & = & 0 \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(S\mathfrak{B})) & = & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(S\mathfrak{B})) \\
\downarrow & & \downarrow \\
KK_*(\mathfrak{A}, \mathfrak{D}_1'')/(\mathrm{id}, g \oplus h)_* KK_*(\mathfrak{A}, S\mathfrak{D}_0'') & & KK_*(\mathfrak{A}, \mathfrak{D}_1)/(\mathrm{id}, g)_* KK_*(\mathfrak{A}, S\mathfrak{D}_0) \\
\| & & \| \\
KK_*(\mathfrak{A}, \mathfrak{D}_1'')/\ker(\mathrm{id}, p)_* & & KK_*(\mathfrak{A}, \mathfrak{D}_1)/\ker(\mathrm{id}, p)_* \\
\downarrow & & \downarrow \\
(\mathrm{id}, p)_* KK_*(\mathfrak{A}, \mathfrak{D}_1'') & = & (\mathrm{id}, p)_* KK_*(\mathfrak{A}, \mathfrak{D}_1) \\
\downarrow & & \downarrow \\
KK_*(\mathfrak{A}, S\mathfrak{B}) & = & KK_*(\mathfrak{A}, S\mathfrak{B})
\end{array}$$

and both of the composites give the same map:

$$\delta : \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to KK_{*+1}(\mathfrak{A}, \mathfrak{B}).$$

Replacing $\mathfrak{D}_1$ with $\mathfrak{D}_1'$ from the beginning, one can show exactly in the same way that the resulting map is the same as the above δ, as wanted.

Step 2. We prove naturality with respect to $\lambda \in KK_0(\mathfrak{A}, \mathfrak{A}')$. Functoriality of the Kasparov product gives the following commutative diagram:

$$\begin{tikzcd}
\mathrm{H}(K_*(\mathfrak{A}'), K_*(S\mathfrak{D}_0)) \arrow[r] \arrow[d, equal] & \mathrm{H}(K_*(\mathfrak{A}'), K_*(\mathfrak{D}_1)) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}'), K_{*+1}(S\mathfrak{B})) \\
KK_*(\mathfrak{A}', S\mathfrak{D}_0) \arrow[r, "(\mathrm{id}, g)_*"] \arrow[d, "\lambda\otimes_{\mathfrak{A}'}"] & KK_*(\mathfrak{A}', \mathfrak{D}_1) \arrow[r, "(\mathrm{id}, p)_*"] \arrow[d, "\lambda\otimes_{\mathfrak{A}'}"] & KK_*(\mathfrak{A}', S\mathfrak{B}) \arrow[d, "\lambda\otimes_{\mathfrak{A}'}"] \\
KK_*(\mathfrak{A}, S\mathfrak{D}_0) \arrow[r, "(\mathrm{id}, g)_*"] \arrow[d, equal] & KK_*(\mathfrak{A}, \mathfrak{D}_1) \arrow[r, "(\mathrm{id}, p)_*"] \arrow[d, equal] & KK_*(\mathfrak{A}, S\mathfrak{B}) \\
\mathrm{H}(K_*(\mathfrak{A}), K_*(S\mathfrak{D}_0)) \arrow[r] & \mathrm{H}(K_*(\mathfrak{A}), K_*(\mathfrak{D}_1)) \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(S\mathfrak{B}))
\end{tikzcd}$$

where $\mathrm{H}(*, *) = \mathrm{Hom}(*, *)$ just for short. It follows that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] \\
\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}'), K_{*+1}(S\mathfrak{B})) \arrow[r, "(\lambda\otimes_{\mathfrak{A}'})^*"] \arrow[d] & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(S\mathfrak{B})) \arrow[d] \\
KK_*(\mathfrak{A}', \mathfrak{D}_1)/(\mathrm{id}, g)_*KK_*(\mathfrak{A}', S\mathfrak{D}_0) \arrow[d, equal] & KK_*(\mathfrak{A}, \mathfrak{D}_1)/(\mathrm{id}, g)_*KK_*(\mathfrak{A}, S\mathfrak{D}_0) \arrow[d, equal] \\
KK_*(\mathfrak{A}', \mathfrak{D}_1)/\ker(\mathrm{id}, p)_* \arrow[d] & KK_*(\mathfrak{A}, \mathfrak{D}_1)/\ker(\mathrm{id}, p)_* \arrow[d] \\
(\mathrm{id}, p)_*KK_*(\mathfrak{A}', \mathfrak{D}_1) \arrow[r, "\lambda\otimes_{\mathfrak{A}'}"] \arrow[d] & (\mathrm{id}, p)_*KK_*(\mathfrak{A}, \mathfrak{D}_1) \arrow[d] \\
KK_*(\mathfrak{A}, S\mathfrak{B}) \arrow[r, "\lambda\otimes_{\mathfrak{A}'}"] & KK_*(\mathfrak{A}, S\mathfrak{B})
\end{tikzcd}$$

where the map $(\lambda\otimes_{\mathfrak{A}'})^*$ is defined to be the composite of the two down arrows in the left with the map $\lambda\otimes_{\mathfrak{A}'}$ in the line up from the bottom line and with the reverses of the two down arrows in the right, so that

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}'), K_{*+1}(S\mathfrak{B})) & \xrightarrow{(\lambda\otimes_{\mathfrak{A}'})^*} & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(S\mathfrak{B})) \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ KK_*(\mathfrak{A}, S\mathfrak{B}) & \xrightarrow{\lambda\otimes_{\mathfrak{A}'}} & KK_*(\mathfrak{A}, S\mathfrak{B}) \end{array}$$

commutes, and $S\mathfrak{B}$ may be replaced with $\mathfrak{B}$ in the diagram above to obtain the corresponding diagram in the statement.

Step 3. We next show that the map δ may be computed instead using a geometric projective resolution of $\mathfrak{A}$. Suppose that we are given a short exact sequence of C^*-algebras:

$$0 \to S\mathfrak{A} \to \mathfrak{F}_1 \to \mathfrak{F}_0 \to 0$$

with $\mathfrak{F}_0, \mathfrak{F}_1 \in N$, $K_*(\mathfrak{F}_0)$ and $K_*(\mathfrak{F}_1)$ free abelian, and the boundary map $K_*(\mathfrak{F}_0) \to K_{*-1}(S\mathfrak{A}) = K_*(\mathfrak{A})$ surjective, such that

$$0 \to K_{*+1}(\mathfrak{F}_1) \to K_{*+1}(\mathfrak{F}_0) \to K_*(S\mathfrak{A}) \to 0,$$

where if necessary, we replace $\mathfrak{A}$ with $S^2\mathfrak{A}\otimes\mathbb{K}$ which has the same K-groups as those of $\mathfrak{A}$. Such sequence does exist by Schochet [9] (see also another paper in this volume of RMJ). Suppose also that we are given the mapping cone sequence:

$$0 \to S\mathfrak{D}_0 \to \mathfrak{D}_1 \to S\mathfrak{B} \to 0$$

which comes from a geometric injective resolution as above. The short exact sequences give rise to the following double complex of KK-groups with exact rows and columns:

$$\begin{array}{ccccc} KK_*(\mathfrak{F}_0, S\mathfrak{D}_0) & \longrightarrow & KK_*(\mathfrak{F}_1, S\mathfrak{D}_0) & \longrightarrow & KK_{*+1}(\mathfrak{A}, S\mathfrak{D}_0) \\ \downarrow & & \downarrow & & \downarrow \\ KK_*(\mathfrak{F}_0, \mathfrak{D}_1) & \longrightarrow & KK_*(\mathfrak{F}_1, \mathfrak{D}_1) & \longrightarrow & KK_{*+1}(\mathfrak{A}, \mathfrak{D}_1) \\ \downarrow & & \downarrow & & \downarrow \\ KK_{*+1}(\mathfrak{F}_0, \mathfrak{B}) & \longrightarrow & KK_{*+1}(\mathfrak{F}_1, \mathfrak{B}) & \longrightarrow & KK_*(\mathfrak{A}, \mathfrak{B}). \end{array}$$

On the other hand, since $K_*(\mathfrak{F}_j)$ are free, so that $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{F}_j), K_*(S\mathfrak{D}_0))$ and $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{F}_j), K_*(\mathfrak{B}))$ are trivial, and since $K_*(\mathfrak{D}_j)$ are injective, the UCT tells us that the above double complex can be written in the following form:

$$
\begin{tikzcd}
\mathrm{H}(K_*(\mathfrak{F}_0), K_*(S\mathfrak{D}_0)) \arrow[r] \arrow[d] & \mathrm{H}(K_*(\mathfrak{F}_1), K_*(S\mathfrak{D}_0)) \arrow[r] \arrow[d] & \mathrm{H}(K_{*+1}(\mathfrak{A}), K_*(S\mathfrak{D}_0)) \arrow[d] \\
\mathrm{H}(K_*(\mathfrak{F}_0), K_*(\mathfrak{D}_1)) \arrow[r] \arrow[d] & \mathrm{H}(K_*(\mathfrak{F}_1), K_*(\mathfrak{D}_1)) \arrow[r] \arrow[d] & \mathrm{H}(K_{*+1}(\mathfrak{A}), K_*(\mathfrak{D}_1)) \arrow[d, "\rho"] \\
\mathrm{H}(K_{*+1}(\mathfrak{F}_0), K_*(\mathfrak{B})) \arrow[r] & \mathrm{H}(K_{*+1}(\mathfrak{F}_1), K_*(\mathfrak{B})) \arrow[r, "\sigma"] & KK_*(\mathfrak{A}, \mathfrak{B})
\end{tikzcd}
$$

where $\mathrm{H}(*,*) = \mathrm{Hom}(*,*)$ just for short. The images of the maps σ and ρ can each be identified with $\mathrm{Ext}^1_{\mathbb{Z}}(K_{*+1}(\mathfrak{A}), K_*(\mathfrak{B}))$. Indeed, we have the following exact sequences as a fact of homology theory:

$$
\begin{aligned}
0 &\to \mathrm{Hom}(K_*(S\mathfrak{A}), K_*(\mathfrak{B})) \\
&\to \mathrm{Hom}(K_{*+1}(\mathfrak{F}_0), K_*(\mathfrak{B})) \\
&\to \mathrm{Hom}(K_{*+1}(\mathfrak{F}_1), K_*(\mathfrak{B})) \\
&\xrightarrow{\sigma} \mathrm{Ext}^1_{\mathbb{Z}}(K_*(S\mathfrak{A}), K_*(\mathfrak{B})) \\
&\to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{F}_0), K_*(\mathfrak{B})) \cong 0 \to \cdots
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\to \mathrm{Hom}(K_{*+1}(\mathfrak{A}), K_{*+1}(S\mathfrak{B})) \\
&\to \mathrm{Hom}(K_{*+1}(\mathfrak{A}), K_*(S\mathfrak{D}_0)) \\
&\to \mathrm{Hom}(K_{*+1}(\mathfrak{A}), K_*(\mathfrak{D}_1)) \\
&\xrightarrow{\rho} \mathrm{Ext}^1_{\mathbb{Z}}(K_{*+1}(\mathfrak{A}), K_{*+1}(S\mathfrak{B})) \\
&\to \mathrm{Ext}^1_{\mathbb{Z}}(K_{*+1}(\mathfrak{A}), K_*(S\mathfrak{D}_0)) \cong 0 \to \cdots
\end{aligned}
$$

so that both of $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(S\mathfrak{A}), K_*(\mathfrak{B}))$ and $\mathrm{Ext}^1_{\mathbb{Z}}(K_{*+1}(\mathfrak{A}), K_{*+1}(S\mathfrak{B}))$ the images under the respective maps σ and ρ above can be identified with $\mathrm{Ext}^1_{\mathbb{Z}}(K_{*+1}(\mathfrak{A}), K_*(\mathfrak{B}))$. Hence the diagram commutes at the right and bottom corner, so that the map δ induced by σ coincides with that by ρ.

Step 4. Finally, we show that the map δ is natural with respect to an element $\mu \in KK_0(\mathfrak{B}, \mathfrak{B}')$. Take a geometric projective resolution for $\mathfrak{A}$:

$$0 \to S\mathfrak{A} \xrightarrow{i} \mathfrak{F}_1 \xrightarrow{q} \mathfrak{F}_0 \to 0$$

with $\mathfrak{F}_0, \mathfrak{F}_1 \in N$, $K_*(\mathfrak{F}_0)$ and $K_*(\mathfrak{F}_1)$ free abelian, and the boundary map $K_*(\mathfrak{F}_0) \to K_{*-1}(S\mathfrak{A}) = K_*(\mathfrak{A})$ surjective, where i means the inclusion map and q does the quotient map. It then follows the following diagram:

$$\begin{tikzcd}
\mathrm{H}(K_*(\mathfrak{F}_0), K_*(\mathfrak{B})) \arrow[r] & \mathrm{H}(K_*(\mathfrak{F}_1), K_*(\mathfrak{B})) \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(K_{*+1}(S\mathfrak{A}), K_*(\mathfrak{B})) \\
KK_*(\mathfrak{F}_0, \mathfrak{B}) \arrow[u, "\cong"'] \arrow[r, "(q,\mathrm{id})^*"] \arrow[d, "\otimes_{\mathfrak{B}} \mu"] & KK_*(\mathfrak{F}_1, \mathfrak{B}) \arrow[u, "\cong"'] \arrow[r, "(i,\mathrm{id})^*"] \arrow[d, "\otimes_{\mathfrak{B}} \mu"] & KK_{*+1}(\mathfrak{A}, \mathfrak{B}) \arrow[d, "\otimes_{\mathfrak{B}} \mu"] \\
KK_*(\mathfrak{F}_0, \mathfrak{B}') \arrow[r, "(q,\mathrm{id})^*"] \arrow[d, "\cong"] & KK_*(\mathfrak{F}_1, \mathfrak{B}') \arrow[r, "(i,\mathrm{id})^*"] \arrow[d, "\cong"] & KK_{*+1}(\mathfrak{A}, \mathfrak{B}') \\
\mathrm{H}(K_*(\mathfrak{F}_0), K_*(\mathfrak{B}')) \arrow[r] & \mathrm{H}(K_*(\mathfrak{F}_1), K_*(\mathfrak{B}')) \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(K_{*+1}(S\mathfrak{A}), K_*(\mathfrak{B}'))
\end{tikzcd}$$

where $\mathrm{H}(*,*) = \mathrm{Hom}(*,*)$ just for short, since $K_*(\mathfrak{F}_0)$ and $K_*(\mathfrak{F}_1)$ are free, so that for $j = 0, 1$, both $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{F}_j), K_*(\mathfrak{B}))$ and $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{F}_j), K_*(\mathfrak{B}'))$ are trivial. It follows that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] \\
\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \arrow[r, "(\otimes_{\mathfrak{B}} \mu)_*"] \arrow[d] & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B}')) \arrow[d] \\
KK_*(\mathfrak{F}_1, \mathfrak{B})/(q,\mathrm{id})^* KK_*(\mathfrak{F}_0, \mathfrak{B}) \arrow[d, equal] & KK_*(\mathfrak{F}_0, \mathfrak{B}')/(q,\mathrm{id})^* KK_*(\mathfrak{F}_0, \mathfrak{B}') \arrow[d, equal] \\
KK_*(\mathfrak{F}_1, \mathfrak{B})/\ker(i,\mathrm{id})^* \arrow[d] & KK_*(\mathfrak{F}_1, \mathfrak{B}')/\ker(i,\mathrm{id})^* \arrow[d] \\
(i,\mathrm{id})^* KK_*(\mathfrak{F}_1, \mathfrak{B}) \arrow[r, "\otimes_{\mathfrak{B}} \mu"] \arrow[d] & (i,\mathrm{id})^* KK_*(\mathfrak{F}_1, \mathfrak{B}') \arrow[d] \\
KK_{*+1}(\mathfrak{A}, \mathfrak{B}) \arrow[r, "\otimes_{\mathfrak{B}} \mu"] & KK_{*+1}(\mathfrak{A}, \mathfrak{B}')
\end{tikzcd}$$

where the map $(\otimes_{\mathfrak{B}})_*$ is defined to be the composite of the two down arrows in the left with the map $\otimes_{\mathfrak{B}} \mu$ in the line up from the bottom line and with the reverses of the two down arrows in the right, so that

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \arrow[r, "(\otimes_{\mathfrak{B}} \mu)_*"] \arrow[d, "\delta"] & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B}')) \arrow[d, "\delta"] \\
KK_{*+1}(\mathfrak{A}, \mathfrak{B}) \arrow[r, "\otimes_{\mathfrak{B}} \mu"] & KK_{*+1}(\mathfrak{A}, \mathfrak{B}')
\end{tikzcd}$$

commutes, and $S\mathfrak{B}$ may be replaced with $\mathfrak{B}$ in the diagram above to obtain the corresponding diagram in the statement. □

5 Special cases of the KT

This section is devoted to prove the following:

Theorem 5.1. *Let $\mathfrak{A} \in N$ and $\mathfrak{B}$ a C^*-algebra such that $K_*(\mathfrak{B})$ is finitely generated and free. Then the map*

$$\alpha(\mathfrak{A}, \mathfrak{B}) : K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to KK_*(\mathfrak{A}, \mathfrak{B})$$

is an isomorphism.

This theorem is equivalent to say that the KT holds for all pair $(\mathfrak{A}, \mathfrak{B})$ wih $\mathfrak{A} \in N$ and $K_*(\mathfrak{B})$ finitely generated and free.

In fact, note that $\mathrm{Tor}_1^{\mathbb{Z}}(K^*(\mathfrak{A}), K_*(\mathfrak{B}))$ is trivial in such a case.

The theorem above is proved via several propositions below.

Proposition 5.2. *Assume that $K_*(\mathfrak{B})$ is a free $\mathbb{Z}$-module. Then $K^j(\star) \otimes K_*(\mathfrak{B})$ $(j = 0, 1)$ form a cohomology theory on the category of separable nuclear C^*-algebras $\star$, which is invariant under strong Morita equivalence, and $\alpha(\star, \mathfrak{B})$ is a natural transformation of cohomology theories. If $K_*(\mathfrak{B})$ is finitely generated and free, then $K^j(\star) \otimes K_*(\mathfrak{B})$ $(j = 0, 1)$ form an additive cohomology theory.*

Proof. With no restriction on $\mathfrak{B}$ there is an isomorphism

$$K^*(\oplus_i \mathfrak{A}_i) \otimes K_*(\mathfrak{B}) \cong (\Pi_i K^*(\mathfrak{A}_i)) \otimes K_*(\mathfrak{B}).$$

Note that $K^*(\mathfrak{A}) = KK_*(\mathfrak{A}, \mathbb{C})$, and $KK_*(\star, \mathbb{C})$ forms an additive cohomology theory.

If $K_*(\mathfrak{B})$ is finitely generated and free, then

$$(\Pi_i K^*(\mathfrak{A}_i)) \otimes K_*(\mathfrak{B}) \cong \Pi_i (K^*(\mathfrak{A}_i) \otimes K_*(\mathfrak{B})).$$

□

Proposition 5.3. *Let $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$ be a short exact sequence of C^*-algebras. Suppose that $K_*(\mathfrak{B})$ is torsion free and that two of the three maps $\alpha(\mathfrak{I}, \mathfrak{B})$, $\alpha(\mathfrak{A}, \mathfrak{B})$, and $\alpha(\mathfrak{A}/\mathfrak{I}, \mathfrak{B})$ are isomorphisms. Then so is the third map.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc}
K^{n+1}(\mathfrak{A}/\mathfrak{J}) \otimes K_*(\mathfrak{B}) & \xrightarrow{\alpha(\mathfrak{A}/\mathfrak{J},\mathfrak{B})} & KK_{n+1}(\mathfrak{A}/\mathfrak{J},\mathfrak{B}) \\
\uparrow & & \uparrow \\
K^n(\mathfrak{J}) \otimes K_*(\mathfrak{B}) & \xrightarrow{\alpha(\mathfrak{J},\mathfrak{B})} & KK_n(\mathfrak{J},\mathfrak{B}) \\
\uparrow & & \uparrow \\
K^n(\mathfrak{A}) \otimes K_*(\mathfrak{B}) & \xrightarrow{\alpha(\mathfrak{A},\mathfrak{B})} & KK_n(\mathfrak{A},\mathfrak{B}) \\
\uparrow & & \uparrow \\
K^n(\mathfrak{A}/\mathfrak{J}) \otimes K_*(\mathfrak{B}) & \xrightarrow{\alpha(\mathfrak{A}/\mathfrak{J},\mathfrak{B})} & KK_n(\mathfrak{A}/\mathfrak{J},\mathfrak{B}) \\
\uparrow & & \uparrow \\
K^{n-1}(\mathfrak{J}) \otimes K_*(\mathfrak{B}) & \xrightarrow{\alpha(\mathfrak{J},\mathfrak{B})} & KK_{n-1}(\mathfrak{J},\mathfrak{B}).
\end{array}$$

The Five-Lemma argument completes the proof for the middle map to be an isomorphism, and other cases are proved by using the similar diagrams shifted (or added) one line up or down. □

Proposition 5.4. *Let $\{\mathfrak{A}_j\}$ be a countable set of C^*-algebras. Suppose that $K_*(\mathfrak{B})$ is finitely generated and free and the map $\alpha(\mathfrak{A}_j,\mathfrak{B})$ is an isomorphism for all i. Then the map $\alpha(\oplus_j\mathfrak{A}_j,\mathfrak{B})$ is an isomorphism.*

Proof. There is the following commuting diagram:

$$\begin{array}{ccc}
KK_*(\oplus_j\mathfrak{A}_j,\mathfrak{B}) & \xrightarrow{\cong} & \Pi_j KK_*(\mathfrak{A}_j,\mathfrak{B}) \\
\uparrow{\scriptstyle \alpha(\oplus_j\mathfrak{A}_j,\mathfrak{B})} & & {\scriptstyle\cong}\uparrow{\scriptstyle \Pi_j\alpha(\mathfrak{A}_j,\mathfrak{B})} \\
K^*(\oplus_j\mathfrak{A}_j) \otimes K_*(\mathfrak{B}) & \xrightarrow{\cong} & \Pi_j[K^*(\mathfrak{A}_j) \otimes K_*(\mathfrak{B})]
\end{array}$$

which implies that the map $\alpha(\oplus_j\mathfrak{A}_j,\mathfrak{B})$ is an isomorphism. □

Proposition 5.5. *Suppose that $\mathfrak{A} = \varinjlim \mathfrak{A}_k$, $K_*(\mathfrak{B})$ is finitely generated and free, and that $\alpha(\mathfrak{A}_k,\mathfrak{B})$ is an isomorphism for all k. Then the map $\alpha(\mathfrak{A},\mathfrak{B})$ is an isomorphism.*

Proof. The KK-groups $KK_j(\star,\mathfrak{B})$ always form an additive cohomology theory and the $K^j(\star) \otimes K_*(\mathfrak{B})$ also do the same theory by the assumption on $K_*(\mathfrak{B})$. Thus both theories satisfy the Milnor $\lim^1$ sequence, so that the following commuting diagram is induced:

$$\begin{array}{ccc}
0 & = & 0 \\
\downarrow & & \downarrow \\
\varprojlim^1 K^{j-1}(\mathfrak{A}_k) \otimes K_*(\mathfrak{B}) & \xrightarrow{\varprojlim^1 \alpha(\mathfrak{A}_k, \mathfrak{B})} & \varprojlim^1 KK_{*+j-1}(\mathfrak{A}_k, \mathfrak{B}) \\
\downarrow & & \downarrow \\
K^j(\varinjlim \mathfrak{A}_k) \otimes K_*(\mathfrak{B}) & \xrightarrow{\alpha(\mathfrak{A}, \mathfrak{B})} & KK_{*+j}(\varinjlim \mathfrak{A}_k, \mathfrak{B}) \\
\downarrow & & \downarrow \\
\varprojlim K^j(\mathfrak{A}_k) \otimes K_*(\mathfrak{B}) & \xrightarrow{\varprojlim \alpha(\mathfrak{A}_k, \mathfrak{B})} & \varprojlim KK_{*+j}(\mathfrak{A}_k, \mathfrak{B}) \\
\downarrow & & \downarrow \\
0 & = & 0.
\end{array}$$

Since both of the maps $\varprojlim^1 \alpha(\mathfrak{A}_k, \mathfrak{B})$ and $\varprojlim \alpha(\mathfrak{A}_k, \mathfrak{B})$ are isomorphisms, the Five-Lemma implies that the map $\alpha(\mathfrak{A}, \mathfrak{B})$ is an isomorphism. □

Proposition 5.6. *Suppose that $\mathfrak{A}$ is a separable C^*-algebra of type I and that $K_*(\mathfrak{A})$ is finitely generated and free. Then the map $\alpha(\mathfrak{A}, \mathfrak{B})$ is an isomorphism.*

Proof. This follows from the general structure theory for separable C^*-algebras of type I as considered in the proof of Proposition 2.6 and from combining Propositions 5.3 to 5.5 for the respective operations taking extensions, direct sums, and inductive limits. Indeed, a type I C^*-algebra has a composition series of closed ideals with subqutients of continuous trace. □

Proposition 5.7. *If $K_*(\mathfrak{B})$ is finitely generated and free and the map $\alpha(\mathfrak{A}, \mathfrak{B})$ is an isomorphism, then the map $\alpha(\mathfrak{A} \rtimes_\rho \mathbb{R}, \mathfrak{B})$ is an isomorphism for any crossed product $\mathfrak{A} \rtimes_\rho \mathbb{R}$ by an action of $\mathbb{R}$ on a C^*-algebra $\mathfrak{A}$.*

Proof. By the Thom isomorphism of Fack-Skandalis for KK-theory used in Proposition 2.7, we have

$$\begin{aligned}
K^*(\mathfrak{A} \rtimes_\rho \mathbb{R}) \otimes K_*(\mathfrak{B}) &= KK_*(\mathfrak{A} \rtimes_\rho \mathbb{R}, \mathbb{C}) \otimes K_*(\mathfrak{B}) \\
&\cong KK_{*+1}(\mathfrak{A}, \mathbb{C}) \otimes K_*(\mathfrak{B}) = K^{*+1}(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \\
&\cong KK_{*+1}(\mathfrak{A}, \mathfrak{B}) \quad \text{(by the assumption)} \\
&\cong KK_*(\mathfrak{A} \rtimes_\rho \mathbb{R}, \mathfrak{B}).
\end{aligned}$$

□

Proposition 5.8. *Suppose that $K_*(\mathfrak{B})$ is free and the map $\alpha(\mathfrak{A},\mathfrak{B})$ is an isomorphism. Then the map $\alpha(\mathfrak{A}\rtimes_\rho\mathbb{Z},\mathfrak{B})$ is an isomorphism, where $\mathfrak{A}\rtimes_\rho\mathbb{Z}$ is the crossed product C^*-algebra by an action ρ of $\mathbb{Z}$ on $\mathfrak{A}$.*

Proof. Consider the following diagram (corrected):

$$\begin{array}{ccc}
\mathrm{Ext}_{*+1}(\mathfrak{A})\otimes K_*(\mathfrak{B}) = K^*(S\mathfrak{A})\otimes K_*(\mathfrak{B}) & \xrightarrow{\alpha(S\mathfrak{A},\mathfrak{B})} & KK_{*+1}(S\mathfrak{A},\mathfrak{B}) \\
\downarrow{\scriptstyle (\mathrm{id}-\rho)^*\otimes\mathrm{id}_*} & & \downarrow{\scriptstyle (\mathrm{id}-\rho)^*} \\
\mathrm{Ext}_{*+1}(\mathfrak{A})\otimes K_*(\mathfrak{B}) = K^*(S\mathfrak{A})\otimes K_*(\mathfrak{B}) & \xrightarrow{\alpha(S\mathfrak{A},\mathfrak{B})} & KK_{*+1}(S\mathfrak{A},\mathfrak{B}) \\
\downarrow{\scriptstyle \partial\otimes\mathrm{id}_*} & & \downarrow{\scriptstyle \partial} \\
\mathrm{Ext}_*(\mathfrak{A}\rtimes_\rho\mathbb{Z})\otimes K_*(\mathfrak{B}) = K^*(\mathfrak{A}\rtimes_\rho\mathbb{Z})\otimes K_*(\mathfrak{B}) & \xrightarrow{\alpha(\mathfrak{A}\rtimes_\rho\mathbb{Z},\mathfrak{B})} & KK_{*+1}(\mathfrak{A},\mathfrak{B}) \\
\downarrow{\scriptstyle i^*\otimes\mathrm{id}_*} & & \downarrow{\scriptstyle i^*} \\
\mathrm{Ext}_*(\mathfrak{A})\otimes K_*(\mathfrak{B}) = K^*(\mathfrak{A})\otimes K_*(\mathfrak{B}) & \xrightarrow{\alpha(\mathfrak{A},\mathfrak{B})} & KK_{*+1}(\mathfrak{A},\mathfrak{B}) \\
\downarrow{\scriptstyle (\mathrm{id}-\rho)^*\otimes\mathrm{id}_*} & & \downarrow{\scriptstyle (\mathrm{id}-\rho)^*} \\
\mathrm{Ext}_*(\mathfrak{A})\otimes K_*(\mathfrak{B}) = K^*(\mathfrak{A})\otimes K_*(\mathfrak{B}) & \xrightarrow{\alpha(\mathfrak{A},\mathfrak{B})} & KK_{*+1}(\mathfrak{A},\mathfrak{B}).
\end{array}$$

The left column is exact by the six-term exact sequence of C^*-algebra extension theory $\mathrm{Ext}_*(\star)$ by Pimsner-Voicuclescu and the assumption that $K_*(\mathfrak{B})$ is free. The right column is exact by the Thom isomorphism of Fack-Skandalis. See also [2, Sections 16,4 and 19.6]. The diagram commutes by examination of the maps involved, and then the Five-Lemma implies that the map $\alpha(\mathfrak{A}\rtimes_\rho\mathbb{Z},\mathfrak{B})$ is an isomorphism. □

Proof of Theorem 5.1. The proof is completed by the propositions proved above in this section. □

6 The general KT

Theorem 6.1. *Let $\mathfrak{A},\mathfrak{B}$ be C^*-algebras. Suppose that the KT holds for all pairs $(\mathfrak{A},\mathfrak{B})$ with $K_*(\mathfrak{B})$ finitely generated and free. Then the KT holds for all pairs $(\mathfrak{A},\mathfrak{B})$ with $K_*(\mathfrak{B})$ finitely generated.*

Proof. Construct a geometric projective resolution for $\mathfrak{B}$ as follows. If necessary, replacing $\mathfrak{B}$ by $S^2\mathfrak{B}$, there is a C^*-algebra $\mathfrak{F}_0$ and a $*$-homomorphism $f:\mathfrak{F}_0\to\mathfrak{B}\otimes\mathbb{K}$ such that $K_*(\mathfrak{F}_0)$ is free and finitely generated (since $K_*(\mathfrak{B})$ is finitely generated), and

$$f_*: K_*(\mathfrak{F}_0)\to K_*(\mathfrak{B}\otimes\mathbb{K})\cong K_*(\mathfrak{B})$$

is surjective (where we use the construction method for $\mathfrak{F}_0$). Let $\mathfrak{F}_1 = Cf$ be the mapping cone of f. The mapping cone sequence:

$$0 \to S\mathfrak{B} \otimes \mathbb{K} \xrightarrow{i} \mathfrak{F}_1 \xrightarrow{p} \mathfrak{F}_0 \to 0$$

has the associated K-theory sequence as

$$0 \to K_j(\mathfrak{F}_1) \xrightarrow{p_*} K_j(\mathfrak{F}_0) \xrightarrow{\partial = f_*} K_{j-1}(S\mathfrak{B} \otimes \mathbb{K}) \cong K_j(\mathfrak{B}) \to 0.$$

This is a geometric projective resolution of $K_*(\mathfrak{B})$ by finitely generated, free abelian groups. The associated sequence of KK-groups has the following form:

$$\begin{aligned} &\to KK_{j+1}(\mathfrak{A}, \mathfrak{F}_1) \xrightarrow{(\mathrm{id}, p_{j+1})_*} KK_{j+1}(\mathfrak{A}, \mathfrak{F}_0) \xrightarrow{(\mathrm{id}_*, \partial)} KK_{j+1}(\mathfrak{A}, \mathfrak{B}) \\ &\xrightarrow{(\mathrm{id}, i)_*} KK_j(\mathfrak{A}, \mathfrak{F}_1) \xrightarrow{(\mathrm{id}, p_j)_*} KK_j(\mathfrak{A}, \mathfrak{F}_0) \xrightarrow{(\mathrm{id}_*, \partial)} KK_j(\mathfrak{A}, \mathfrak{B}) \longrightarrow \end{aligned}$$

with $KK_{j+1}(\mathfrak{A}, \mathfrak{B}) \cong KK_j(\mathfrak{A}, S\mathfrak{B} \otimes \mathbb{K})$ and with $p_{j+1} = p = p_j$. It yields the following short exact sequence:

$$\begin{aligned} 0 &\to \mathrm{coker}(\mathrm{id}, p_{j+1})_* = KK_{j+1}(\mathfrak{A}, \mathfrak{F}_0)/\mathrm{im}(\mathrm{id}, p_{j+1})_* \\ &= KK_{j+1}(\mathfrak{A}, \mathfrak{F}_0)/\ker(\mathrm{id}_*, \partial) \\ &\to KK_{j+1}(\mathfrak{A}, \mathfrak{B}) \to \mathrm{im}(\mathrm{id}, i)_* = \ker(\mathrm{id}, p_j)_* \to 0. \end{aligned}$$

Check below that this is the Künneth theorem.

Since $K_*(\mathfrak{F}_i)$ are finitely generated and free, the maps $\alpha(\mathfrak{A}, \mathfrak{F}_i)$ are isomorphisms in the following diagram:

$$\begin{array}{ccc} K^*(\mathfrak{A}) \otimes K_*(\mathfrak{F}_1) & \xrightarrow{\mathrm{id}_* \otimes p_*} & K^*(\mathfrak{A}) \otimes K_*(\mathfrak{F}_0) \\ \downarrow{\scriptstyle \alpha(\mathfrak{A}, \mathfrak{F}_1)} & & \downarrow{\scriptstyle \alpha(\mathfrak{A}, \mathfrak{F}_0)} \\ KK_*(\mathfrak{A}, \mathfrak{F}_1) & \xrightarrow{(\mathrm{id}, p)_*} & KK_*(\mathfrak{A}, \mathfrak{F}_0). \end{array}$$

Thus,

$$\ker(\mathrm{id}, p)_* \cong \ker(\mathrm{id}_* \otimes p_*) \cong \mathrm{Tor}_1^{\mathbb{Z}}(K^*(\mathfrak{A}), K_{*+1}(\mathfrak{B})),$$

where the long exact sequence for torsion product Tor is

$$\begin{aligned} &\mathrm{Tor}_1^{\mathbb{Z}}(K^*(\mathfrak{A}), K_j(\mathfrak{F}_0)) \cong 0 \xrightarrow{(\mathrm{id}_*, \partial)} \mathrm{Tor}_1^{\mathbb{Z}}(K^*(\mathfrak{A}), K_j(\mathfrak{B})) \xrightarrow{(\mathrm{id}_*, i_*)} \\ &K^*(\mathfrak{A}) \otimes K_{j-1}(\mathfrak{F}_1) \xrightarrow{\mathrm{id}_* \otimes (p_{j-1})_*} K^*(\mathfrak{A}) \otimes K_{j-1}(\mathfrak{F}_0) \xrightarrow{\mathrm{id}_* \otimes \partial} \\ &K^*(\mathfrak{A}) \otimes K_{j-1}(\mathfrak{B}) \longrightarrow 0 \end{aligned}$$

and hence,

$$\ker(\mathrm{id}, p_j)_* = \mathrm{Tor}_1^{\mathbb{Z}}(K^*(\mathfrak{A}), K_{j+1}(\mathfrak{B}))$$

and the quotient map β in the KT has degree one. It also follows from the diagram above that

$$\mathrm{coker}(\mathrm{id}, p)_* \cong \mathrm{coker}(\mathrm{id} \otimes p_*) \cong K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B})$$

and indeed, from the long exact sequence,

$$\begin{aligned} \mathrm{coker}(\mathrm{id}_* \otimes (p_j)_*) &= K^*(\mathfrak{A}) \otimes K_j(\mathfrak{F}_0)/\mathrm{im}(\mathrm{id}_* \otimes (p_j)_*) \\ &= K^*(\mathfrak{A}) \otimes K_j(\mathfrak{F}_0)/\ker(\mathrm{id}_* \otimes \partial) \\ &\cong K^*(\mathfrak{A}) \otimes K_j(\mathfrak{B}) \end{aligned}$$

and the inclusion map α in the KT has degree zero. □

Proof for the KT except naturality. Theorems 5.1 and 6.1 prove the KT in the introduction. The naturality of the KT may be proved as that of the UCT in Theorem 4.2. □

But we state the following:

Theorem 6.2. *The KT is natural in both variables in the sense that if $\mathfrak{A}, \mathfrak{A}' \in N$, $\mathfrak{B}$ and $\mathfrak{B}'$ are C^*-algebras with countable approximate units with $K_*(\mathfrak{B})$ and $K_*(\mathfrak{B}')$ finitely generated, and $\lambda \in KK_0(\mathfrak{A}, \mathfrak{A}')$, $\mu \in KK_0(\mathfrak{B}, \mathfrak{B}')$, then the following diagrams commute:*

$$\begin{array}{ccccccccc} 0 \to & K^*(\mathfrak{A}') \otimes K_*(\mathfrak{B}) & \xrightarrow{\alpha} & KK_*(\mathfrak{A}', \mathfrak{B}) & \xrightarrow{\beta} & \mathrm{Tor}_1^{\mathbb{Z}}(K^*(\mathfrak{A}'), K_{*+1}(\mathfrak{B})) & \to 0 \\ & \big\downarrow (\lambda \otimes_{\mathfrak{A}'})^* & & \big\downarrow \lambda \otimes_{\mathfrak{A}'} & & \big\downarrow (\lambda \otimes_{\mathfrak{A}'})^* & \\ 0 \to & K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) & \xrightarrow{\alpha} & KK_*(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{\beta} & \mathrm{Tor}_1^{\mathbb{Z}}(K^*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) & \to 0 \end{array}$$

with $\otimes_{\mathfrak{A}'} : KK_0(\mathfrak{A}, \mathfrak{A}') \otimes KK_*(\mathfrak{A}', \mathfrak{B}) \to KK_*(\mathfrak{A}, \mathfrak{B})$, *and also*

$$\begin{array}{ccccccccc} 0 \to & K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) & \xrightarrow{\alpha} & KK_*(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{\beta} & \mathrm{Tor}_1^{\mathbb{Z}}(K^*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) & \to 0 \\ & \big\downarrow (\otimes_{\mathfrak{B}} \mu)^* & & \big\downarrow \otimes_{\mathfrak{B}} \mu & & \big\downarrow (\otimes_{\mathfrak{B}} \mu)^* & \\ 0 \to & K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B}') & \xrightarrow{\alpha} & KK_*(\mathfrak{A}, \mathfrak{B}') & \xrightarrow{\beta} & \mathrm{Tor}_1^{\mathbb{Z}}(K^*(\mathfrak{A}), K_{*+1}(\mathfrak{B}')) & \to 0 \end{array}$$

with $\otimes_{\mathfrak{B}} : KK_*(\mathfrak{A}, \mathfrak{B}) \otimes KK_0(\mathfrak{B}, \mathfrak{B}') \to KK_*(\mathfrak{A}, \mathfrak{B}')$.

Sketch of the proof. As in the proof of Theorem 6.1, take a geometric projective resolution for $\mathfrak{B}$ which implies a geometric projective resolution for $K_*(\mathfrak{B})$ by finitely generated, free abelian groups such that

$$0 \to K_j(\mathfrak{F}_1) \to K_j(\mathfrak{F}_0) \to K_j(\mathfrak{B}) \to 0$$

$(j = 0, 1)$. Then we have the following commutative diagram:

$$\begin{tikzcd}
K^*(\mathfrak{A}') \otimes K_*(\mathfrak{J}_1) \arrow[r] \arrow[d, "\alpha"', "\cong"] & K^*(\mathfrak{A}') \otimes K_*(\mathfrak{J}_0) \arrow[r, "q"] \arrow[d, "\alpha"', "\cong"] & K^*(\mathfrak{A}') \otimes K_*(\mathfrak{B}) \arrow[r] \arrow[d, "\alpha"] & 0 \\
KK_*(\mathfrak{A}', \mathfrak{J}_1) \arrow[r] \arrow[d, "\lambda\otimes_{\mathfrak{A}'}"] & KK_*(\mathfrak{A}', \mathfrak{J}_0) \arrow[r] \arrow[d, "\lambda\otimes_{\mathfrak{A}'}"] & KK_*(\mathfrak{A}', \mathfrak{B}) \arrow[d, "\lambda\otimes_{\mathfrak{A}'}"] & \\
KK_*(\mathfrak{A}, \mathfrak{J}_1) \arrow[r] & KK_*(\mathfrak{A}, \mathfrak{J}_0) \arrow[r] & KK_*(\mathfrak{A}, \mathfrak{B}) & \\
K^*(\mathfrak{A}) \otimes K_*(\mathfrak{J}_1) \arrow[r] \arrow[u, "\alpha", "\cong"'] & K^*(\mathfrak{A}) \otimes K_*(\mathfrak{J}_0) \arrow[r, "q"] \arrow[u, "\alpha", "\cong"'] & K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \arrow[r] \arrow[u, "\alpha"'] & 0
\end{tikzcd}$$

and one can construct the map $(\lambda\otimes_{\mathfrak{A}'})^*$ from the right upper corner $K^*(\mathfrak{A}')\otimes K_*(\mathfrak{B})$ to the right bottom corner $K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B})$ as the composite of the reverse of the quotient map q, the three maps α, $\lambda\otimes_{\mathfrak{A}'}$, and the reverse of α in the middle column, and as well the quotient map q.

We also have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1^{\mathbf{Z}}(K^*(\mathfrak{A}'), K_{*+1}(\mathfrak{B})) \arrow[r, "i"] & K^*(\mathfrak{A}') \otimes K_*(\mathfrak{J}_1) \arrow[d, "\alpha"', "\cong"] \\
 & KK_*(\mathfrak{A}', \mathfrak{B}) \arrow[u, "\beta"'] \arrow[d, "\lambda\otimes_{\mathfrak{A}'}"] & KK_*(\mathfrak{A}', \mathfrak{J}_1) \arrow[d, "\lambda\otimes_{\mathfrak{A}'}"] \\
 & KK_*(\mathfrak{A}, \mathfrak{B}) \arrow[d, "\beta"] & KK_*(\mathfrak{A}, \mathfrak{J}_1) \\
0 \arrow[r] & \mathrm{Tor}_1^{\mathbf{Z}}(K^*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) \arrow[r, "i"] & K^*(\mathfrak{A}) \otimes K_*(\mathfrak{J}_1) \arrow[u, "\alpha", "\cong"']
\end{tikzcd}$$

and one can construct the map $(\lambda\otimes_{\mathfrak{A}'})^*$ from $\mathrm{Tor}_1^{\mathbf{Z}}(K^*(\mathfrak{A}'), K_{*+1}(\mathfrak{B}))$ at the left upper corner to $\mathrm{Tor}_1^{\mathbf{Z}}(K^*(\mathfrak{A}), K_{*+1}(\mathfrak{B}))$ at the left bottom corner as the composite of the inclusion map i, the three maps α, $\lambda\otimes_{\mathfrak{A}'}$, and the reverse of α in the right column, and as well the reverse of the inclusion map i. □

7 Further consequences and generalizations

Proposition 7.1. *Suppose that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are separable C^*-algebras, $\mathfrak{B}$ is a C^*-algebra with a countable approximate unit, and the UCT or KT holds* *for the pair* $(\mathfrak{A}_1, \mathfrak{B})$. *If there is an invertible KK-element* λ *in* $KK_0(\mathfrak{A}_1, \mathfrak{A}_2)$, *then the theorem also holds for* $(\mathfrak{A}_2, \mathfrak{B})$.

Proof. The Kasparov product $\lambda^{-1} \otimes_{\mathfrak{A}_1}$ with $\lambda^{-1} \in KK_0(\mathfrak{A}_2, \mathfrak{A}_1)$ on the left side induces the following isomorphisms:

$$KK_*(\mathfrak{A}_1, \mathfrak{B}) \to KK_*(\mathfrak{A}_2, \mathfrak{B}) \quad \text{and} \quad K^*(\mathfrak{A}_1) = KK_*(\mathfrak{A}_1, \mathbb{C}) \to K^*(\mathfrak{A}_2).$$

The product $\otimes_{\mathfrak{A}_2} \lambda^{-1}$ on the right side induces the following isomorphism:

$$K_*(\mathfrak{A}_2) = KK_*(\mathbb{C}, \mathfrak{A}_2) \to K_*(\mathfrak{A}_1).$$

It follows that the following diagram is induced:

$$\begin{array}{ccc}
0 & = & 0 \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}_1), K_{*+1}(\mathfrak{B})) & \xleftarrow{\mathrm{Ext}^1_{\mathbb{Z}}(\lambda^{-1}\otimes_{\mathfrak{A}_1}, \mathrm{id}_*)} & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}_2), K_{*+1}(\mathfrak{B})) \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\
KK_*(\mathfrak{A}_1, \mathfrak{B}) & \xrightarrow{\lambda^{-1}\otimes_{\mathfrak{A}_1}} & KK_*(\mathfrak{A}_2, \mathfrak{B}) \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\
\mathrm{Hom}(K_*(\mathfrak{A}_1), K_*(\mathfrak{B})) & \xleftarrow{\mathrm{Hom}(\otimes_{\mathfrak{A}_2}\lambda^{-1}, \mathrm{id}_*)} & \mathrm{Hom}(K_*(\mathfrak{A}_2), K_*(\mathfrak{B})) \\
\downarrow & & \downarrow \\
0 & = & 0
\end{array}$$

and commutes to have the right column exact, and also the following diagram is induced:

$$\begin{array}{ccc}
0 & = & 0 \\
\downarrow & & \downarrow \\
K^*(\mathfrak{A}_1) \otimes K_*(\mathfrak{B}) & \xrightarrow{\lambda^{-1}\otimes_{\mathfrak{A}_1}} & K^*(\mathfrak{A}_2) \otimes K_*(\mathfrak{B}) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\
KK_*(\mathfrak{A}_1, \mathfrak{B}) & \xrightarrow{\lambda^{-1}\otimes_{\mathfrak{A}_1}} & KK_*(\mathfrak{A}_2, \mathfrak{B}) \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} \\
\mathrm{Tor}^{\mathbb{Z}}_1(K_*(\mathfrak{A}_1), K_{*+1}(\mathfrak{B})) & \xleftarrow{\mathrm{Tor}^{\mathbb{Z}}_1(\otimes_{\mathfrak{A}_2}\lambda^{-1}, \mathrm{id}_*)} & \mathrm{Tor}^{\mathbb{Z}}_1(K_*(\mathfrak{A}_2), K_{*+1}(\mathfrak{B})) \\
\downarrow & & \downarrow \\
0 & = & 0
\end{array}$$

and commutes to have the right column exact. □

Corollary 7.2. *The UCT holds for $(\mathfrak{A} \rtimes_\alpha F_n, \mathfrak{B})$ or for $(\mathfrak{A} \rtimes_{\alpha,r} F_n, \mathfrak{B})$, where $\mathfrak{A} \rtimes_\alpha F_n$ and $\mathfrak{A} \rtimes_{\alpha,r} F_n$ are the full and reduced crossed product of a C^*-algebra $\mathfrak{A}$ in N by an action α of a free group F_n respectively (in fact, of any torsion free, discrete subgroup of $SO(n,1)$ for some n).*

Sketch of the proof. Note that the canonical quotient map from $\mathfrak{A} \rtimes_\alpha F_n$ to $\mathfrak{A} \rtimes_{\alpha,r} F_n$ gives an invertible KK-element, i.e., a KK-equivalence (see [2, Section 20.9]). Assuming that $K_*(\mathfrak{B})$ is injective, we have the following commutative diagram:

$$\begin{array}{ccc}
KK_{*+1}(\mathfrak{A},\mathfrak{B}) & \xrightarrow{\gamma} & \mathrm{Hom}(K_{*+1}(\mathfrak{A}), K_*(\mathfrak{B})) \\
\downarrow{\scriptstyle (\sigma^*,\mathrm{id}^*)} & & \downarrow{\scriptstyle \mathrm{Hom}(\sigma_*,\mathrm{id}_*)} \\
KK_{*+1}(\oplus^n\mathfrak{A},\mathfrak{B}) & \xrightarrow{\gamma} & \mathrm{Hom}(\oplus^n K_*(\mathfrak{A}), K_*(\mathfrak{B})) \\
\downarrow{\scriptstyle (\partial,\mathrm{id}^*)} & & \downarrow{\scriptstyle \mathrm{Hom}(\partial,\mathrm{id}_*)} \\
KK_*(\mathfrak{A} \rtimes_{\alpha,r} F_n,\mathfrak{B}) & \xrightarrow{\gamma} & \mathrm{Hom}(K_*(\mathfrak{A} \rtimes_{\alpha,r} F_n), K_*(\mathfrak{B})) \\
\downarrow{\scriptstyle (i^*,\mathrm{id}^*)} & & \downarrow{\scriptstyle \mathrm{Hom}(i_*,\mathrm{id}_*)} \\
KK_*(\mathfrak{A},\mathfrak{B}) & \xrightarrow{\gamma} & \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \\
\downarrow{\scriptstyle (\sigma^*,\mathrm{id}^*)} & & \downarrow{\scriptstyle \mathrm{Hom}(\sigma_*,\mathrm{id}_*)} \\
KK_*(\oplus^n\mathfrak{A},\mathfrak{B}) & \xrightarrow{\gamma} & \mathrm{Hom}(\oplus^n K_*(\mathfrak{A}), K_*(\mathfrak{B}))
\end{array}$$

where $\sigma^* = \sum_{j=1}^n (\mathrm{id} - \alpha_j)^*$ and $\sigma_* = \sum_{j=1}^n (\mathrm{id} - \alpha_j)_*$ with α_j automorphisms of $\mathfrak{A}$ corresponding to generators of F_n, and the maps γ except the middle one are isomorphisms by the assumption, and the right column is exact by the K-theory six-term exact sequence of Pimsner-Voiculescu (see [2, Section 10.8]), and as well is the left column (but not checked). The Five-Lemma implies the special UCT holds for $(\mathfrak{A} \rtimes_{\alpha,r} F_n, \mathfrak{B})$, and hence that the general UCT holds for the pair. The Proposition 7.1 above implies that the UCT holds for the pair $(\mathfrak{A} \rtimes_\alpha F_n, \mathfrak{B})$. □

Proposition 7.3. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be C^*-algebras in N. Suppose that there is an element $\lambda \in KK_0(\mathfrak{A}_1,\mathfrak{A}_2)$ such that $\gamma(\lambda) = \lambda_* \in \mathrm{Hom}(K_*(\mathfrak{A}_1), K_*(\mathfrak{A}_2))$ is an isomorphism. Then λ is a KK-equivalence, i.e., there exists $\lambda^{-1} \in KK_0(\mathfrak{A}_2,\mathfrak{A}_1)$ such that*

$$\lambda \otimes_{\mathfrak{A}_2} \lambda^{-1} = \mathrm{id}_{\mathfrak{A}_1} \quad \textit{and} \quad \lambda^{-1} \otimes_{\mathfrak{A}_1} \lambda = \mathrm{id}_{\mathfrak{A}_2}.$$

Proof. Let $\mathfrak{B}$ be any separeble C^*-algebra. The UCT for $(\mathfrak{A}_1, \mathfrak{B})$ and $(\mathfrak{A}_2, \mathfrak{B})$ implies the following commutative diagram:

$$\begin{array}{ccc}
0 & = & 0 \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}_2), K_*(\mathfrak{B})) & \xrightarrow{\gamma(\lambda)^*} & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}_1), K_*(\mathfrak{B})) \\
\downarrow & & \downarrow \\
KK_*(\mathfrak{A}_2, \mathfrak{B}) & \xrightarrow{\lambda \otimes_{\mathfrak{A}_2}} & KK_*(\mathfrak{A}_1, \mathfrak{B}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(K_*(\mathfrak{A}_2), K_*(\mathfrak{B})) & \xrightarrow{\gamma(\lambda)^*} & \mathrm{Hom}(K_*(\mathfrak{A}_1), K_*(\mathfrak{B})) \\
\downarrow & & \downarrow \\
0 & = & 0.
\end{array}$$

The Five-Lemma together with the hypothesis of $\gamma(\lambda)$ implies that the map $\lambda \otimes_{\mathfrak{A}_2} : KK_*(\mathfrak{A}_2, \mathfrak{B}) \to KK_*(\mathfrak{A}_1, \mathfrak{B})$ is an isomorphism. Similarly, the UCT for $(\mathfrak{N}, \mathfrak{A}_1)$ and $(\mathfrak{N}, \mathfrak{A}_2)$ with $\mathfrak{N} \in N$ implies that the map

$$\otimes_{\mathfrak{A}_1} \lambda : KK_*(\mathfrak{N}, \mathfrak{A}_1) \to KK_*(\mathfrak{N}, \mathfrak{A}_2)$$

is an isomorphism. Choosing $\mathfrak{B} = \mathfrak{A}_1$ and $\mathfrak{N} = \mathfrak{A}_2$, we see that

$$\begin{aligned}
&\lambda \otimes_{\mathfrak{A}_2} : KK_*(\mathfrak{A}_2, \mathfrak{A}_1) \to KK_*(\mathfrak{A}_1, \mathfrak{A}_1) \quad \text{and} \\
&\otimes_{\mathfrak{A}_1} \lambda : KK_*(\mathfrak{A}_2, \mathfrak{A}_1) \to KK_*(\mathfrak{A}_2, \mathfrak{A}_2)
\end{aligned}$$

are isomorphisms, so that there are elements μ and $\mu' \in KK_0(\mathfrak{A}_2, \mathfrak{A}_1)$ such that $\lambda \otimes_{\mathfrak{A}_2} \mu = \mathrm{id}_{\mathfrak{A}_1} \in KK_0(\mathfrak{A}_1, \mathfrak{A}_1)$ and $\mu' \otimes_{\mathfrak{A}_1} \lambda = \mathrm{id}_{\mathfrak{A}_2} \in KK_0(\mathfrak{A}_2, \mathfrak{A}_2)$. Moreover, one can check that

$$\begin{aligned}
\mu &= \mathrm{id}_{\mathfrak{A}_2} \otimes_{\mathfrak{A}_2} \mu \\
&= (\mu' \otimes_{\mathfrak{A}_1} \lambda) \otimes_{\mathfrak{A}_2} \mu \\
&= \mu' \otimes_{\mathfrak{A}_1} (\lambda \otimes_{\mathfrak{A}_2} \mu) \quad \text{(by associativity)} \\
&= \mu' \otimes_{\mathfrak{A}_1} \mathrm{id}_{\mathfrak{A}_1} = \mu'
\end{aligned}$$

and thus, $\mu = \mu' = \lambda^{-1} \in KK_0(\mathfrak{A}_2, \mathfrak{A}_1)$. □

Proposition 7.4. *Let $\mathfrak{A}$ be any separable C^*-algebra, not necessarily nuclear. Then there is a separable commutative C^*-algebra $\mathfrak{C}$ (which we* *can choose with the spectrum $\mathfrak{C}^\wedge$ of $\mathfrak{C}$ finite-dimensional), and an element $\lambda \in KK_0(\mathfrak{C}, \mathfrak{A})$ for which*

$$\gamma(\lambda) = \lambda_* : K_*(\mathfrak{C}) \to K_*(\mathfrak{A})$$

is an isomorphism. Furthermore, we may take $\mathfrak{C} = \mathfrak{C}^0 \oplus \mathfrak{C}^1$ with $K_0(\mathfrak{C}^1) = 0$ and $K_1(\mathfrak{C}^0) = 0$. If $K_*(\mathfrak{A})$ is finitely generated, then we can take the one-point compactification $(\mathfrak{C}^\wedge)^+$ of $\mathfrak{C}^\wedge$ as a finite complex.

Sketch of the proof. One can construct commutative C^*-algebras $\mathfrak{C}^0$ and $\mathfrak{C}^1$ such that $K_0(\mathfrak{C}^1) = 0$, $K_1(\mathfrak{C}^0) = 0$, and

$$K_j(\mathfrak{C}^j) \cong K_j(\mathfrak{A}) \quad (j = 0, 1)$$

as a group. For instance, to construct $\mathfrak{C}^0$, choose a free resolution of the $\mathbb{Z}$-module $K_0(\mathfrak{A})$:

$$0 \to F_1 \xrightarrow{f} F_2 \to K_0(\mathfrak{A}) \to 0$$

with each F_j a direct sum of $\mathbb{Z}$. We may choose commutative C^*-algebras $\mathfrak{D}_1$ and $\mathfrak{D}_2$ each as a c_0-direct sum of copies of $C_0(\mathbb{R})$, and a $*$-homomorphism $\varphi : \mathfrak{D}_1 \to \mathfrak{D}_2$, such that $K_1(\mathfrak{D}_j) \cong F_j$ $(j = 1, 2)$ and $\varphi_* = f$. Then let $\mathfrak{C}^0$ be the mapping cone $C\varphi$:

$$0 \to S\mathfrak{D}_2 \to \mathfrak{C}^0 = C\varphi \to \mathfrak{D}_1 \to 0.$$

Note that the associated six-term exact K-theory sequence becomes:

$$\begin{array}{ccccc} F_2 & \longrightarrow & K_0(\mathfrak{C}^0) & \longrightarrow & 0 \\ \Big\uparrow{\scriptstyle \partial = \varphi_* = f} & & & & \Big\downarrow{\scriptstyle \partial} \\ F_1 & \longleftarrow & K_1(\mathfrak{C}^0) & \longleftarrow & 0 \end{array}$$

so that $K_0(\mathfrak{C}^0) \cong K_0(\mathfrak{A})$ by the Five-Lemma, and hence, $K_1(\mathfrak{C}^0) = 0$ by the diagram.

Similarly, one can define $\mathfrak{C}^1$ as required. Indeed, choose a free resolution of the $\mathbb{Z}$-module $K_1(\mathfrak{A})$:

$$0 \to G_1 \xrightarrow{g} G_2 \to K_1(\mathfrak{A}) \to 0$$

with each G_j a direct sum of $\mathbb{Z}$. We may choose commutative C^*-algebras $\mathfrak{H}_1$ and $\mathfrak{H}_2$ each as a c_0-direct sum of copies of $C_0(\mathbb{R}^2)$, and a $*$-homomorphism $\psi : \mathfrak{H}_1 \to \mathfrak{H}_2$, such that $K_0(\mathfrak{H}_j) \cong G_j$ $(j = 1, 2)$ and $\psi_* = g$. Then let $\mathfrak{C}^1$ be the mapping cone $C\psi$:

$$0 \to S\mathfrak{H}_2 \to \mathfrak{C}^1 = C\psi \to \mathfrak{H}_1 \to 0.$$

Note that the associated six-term exact K-theory sequence becomes:

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(\mathfrak{C}^1) & \longrightarrow & G_1 \\ \uparrow{\scriptstyle \partial} & & & & \downarrow{\scriptstyle \partial=\psi_*=g} \\ 0 & \longleftarrow & K_1(\mathfrak{C}^1) & \longleftarrow & G_2 \end{array}$$

so that $K_1(\mathfrak{C}^1) \cong K_1(\mathfrak{A})$ by the Five-Lemma, and hence, $K_0(\mathfrak{C}^1) = 0$ by the diagram.

Let $\mathfrak{C} = \mathfrak{C}^0 \oplus \mathfrak{C}^1$. By construction, there is an isomorphism of graded groups in $\mathrm{Hom}(K_*(\mathfrak{C}), K_*(\mathfrak{A}))$. Surjectivity of the map γ of the UCT for the pair $(\mathfrak{C}, \mathfrak{A})$ implies the existence of the KK-theory class λ with $\gamma(\lambda)$ equal to the isomorphism. □

Remark. By construction, we see that the spectrums $(\mathfrak{C}^0)^\wedge$ and $(\mathfrak{C}^1)^\wedge$ (as well as their one-point compactifications) have dimension at most two and three, respectively.

Corollary 7.5. *Any C^*-algebra $\mathfrak{A}$ in N is KK-equivalent to a separable commutative C^*-algebra $\mathfrak{C}$ (with $(\mathfrak{C}^\wedge)^+$ finite-dimensional). In fact, we can choose $\mathfrak{C}$ of the form $\mathfrak{C}^0 \oplus \mathfrak{C}^1$ with $K_1(\mathfrak{C}^0) = 0$ and $K_0(\mathfrak{C}^1) = 0$. If $K_*(\mathfrak{A})$ is finitely generated, one may take $(\mathfrak{C}^\wedge)^+$ to be a finite (3-dimensional) cell complex.*

Any two C^*-algebras in N with the same K-groups are KK-equivalent.

Proof. Just combine Proposition 7.4 with Proposition 7.3. □

Remark. Let $\mathfrak{A}$ be any separable C^*-algebra and $\mathfrak{C}$ be a commutative C^*-algebra with the same K-groups. Then there is an element $\lambda \in KK_0(\mathfrak{C}, \mathfrak{A})$ such that $\gamma(\lambda)$ is an isomorphism on K-theory groups. By Kasparov theory, this corresponds to an extension:

$$0 \to \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{B} \to S\mathfrak{C} \to 0,$$

and the six-term K-theory group exact sequence:

$$\begin{array}{ccccc} K_0(\mathfrak{A}) & \longrightarrow & K_0(\mathfrak{B}) & \longrightarrow & K_1(\mathfrak{C}) \\ \cong\uparrow{\scriptstyle \partial} & & & & \cong\downarrow{\scriptstyle \partial} \\ K_0(\mathfrak{C}) & \longleftarrow & K_1(\mathfrak{B}) & \longleftarrow & K_1(\mathfrak{A}) \end{array}$$

implies that $K_*(\mathfrak{B}) = 0$.

Theorem 7.6. *Suppose that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are in N. Then the UCT holds for $(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{B})$ for all $\mathfrak{B}$.*

Proof. Fix a C^*-algebra $\mathfrak{A}_1$ in N and fix a C^*-algebra $\mathfrak{B}$ with $K_*(\mathfrak{B})$ injective. Define the functors h^* and k^* by

$$\begin{aligned} h^j(\mathfrak{A}_2) &= KK_j(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{B}) \quad \text{and} \\ k^j(\mathfrak{A}_2) &= \mathrm{Hom}(K_j(\mathfrak{A}_1 \otimes \mathfrak{A}_2), K_*(\mathfrak{B})). \end{aligned}$$

Both h^* and k^* are additive cohomology theories with respect to $\mathfrak{A}_2$. The following map:

$$\gamma_{\mathfrak{A}_2} \equiv \gamma(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{B}) : h^*(\mathfrak{A}_2) \to k^*(\mathfrak{A}_2)$$

is a natural transformation of additive cohomology theories. The following map:

$$\gamma_{\mathbb{C}} : KK_*(\mathfrak{A}_1 \otimes \mathbb{C}, \mathfrak{B}) \to \mathrm{Hom}(K_*(\mathfrak{A}_1 \otimes \mathbb{C}), K_*(\mathfrak{B}))$$

is an isomorphism since $\mathfrak{A}_1 \otimes \mathbb{C} \cong \mathfrak{A}_1 \in N$.

The same arguments as for the special UCT in Theorem 2.1 imply that the map $\gamma_{\mathfrak{A}_2}$ is an isomorphism for all $\mathfrak{A}_2 \in N$ with $K_*(\mathfrak{B})$ injective. This establishes the UCT for all $(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{B})$ with both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in N and $K_*(\mathfrak{B})$ injective.

The general case with $\mathfrak{B}$ arbitrary follows by the geometric injective resolution argument, just as for the general UCT in Theorem 4.1. □

Theorem 7.7. *Suppose that $\mathfrak{A} \in N$, and $\mathfrak{B}_1$ or $\mathfrak{B}_2 \in N$, and the groups $K_*(\mathfrak{B}_j)$ are finitely generated. Then there is a natural short exact sequence:*

$$\begin{aligned} 0 \longrightarrow\ & KK_*(\mathfrak{A}, \mathfrak{B}_1) \otimes K_*(\mathfrak{B}_2) \\ \xrightarrow{\ \alpha\ }\ & KK_*(\mathfrak{A}, \mathfrak{B}_1 \otimes \mathfrak{B}_2) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(KK_*(\mathfrak{A}, \mathfrak{B}_1), K_*(\mathfrak{B}_2)) \to 0. \end{aligned}$$

Proof. Suppose first that $K_*(\mathfrak{B}_2)$ is finitely generated and free. Fix $\mathfrak{B}_1$. Define the functors h^* and k^* by

$$\begin{aligned} h^j(\mathfrak{A}) &= KK_j(\mathfrak{A}, \mathfrak{B}_1) \otimes K_*(\mathfrak{B}_2) \quad \text{and} \\ k^j(\mathfrak{A}) &= KK_j(\mathfrak{A}, \mathfrak{B}_1 \otimes \mathfrak{B}_2). \end{aligned}$$

These are additive cohomology theories with respect to $\mathfrak{A}$. The Kasparov product:

$$\otimes_{\mathbb{C}} : KK_j(\mathfrak{A}, \mathfrak{B}_1 \otimes \mathbb{C}) \otimes KK_*(\mathbb{C} \otimes \mathbb{C}, \mathfrak{B}_2) \to KK_{j+*}(\mathfrak{A} \otimes \mathbb{C}, \mathfrak{B}_1 \otimes \mathfrak{B}_2)$$

together with $KK_*(\mathbb{C}, \mathfrak{B}_2) = K_*(\mathfrak{B}_2)$ induces the following map:

$$\alpha_{\mathfrak{A}} : h^*(\mathfrak{A}) \to k^*(\mathfrak{A})$$

as a natural transformation of the theories. The map $\alpha_{\mathbb{C}}$ is:

$$\begin{aligned}\alpha_{\mathbb{C}} : KK_*(\mathbb{C}, \mathfrak{B}_1) \otimes K_*(\mathfrak{B}_2) &= K_*(\mathfrak{B}_2) \otimes K_*(\mathfrak{B}_2) \\ &\to KK_*(\mathbb{C}, \mathfrak{B}_1 \otimes \mathfrak{B}_2) = K_*(\mathfrak{B}_1 \otimes \mathfrak{B}_2)\end{aligned}$$

an isomorphism since one of $\mathfrak{B}_j$ is in N. This is the Künneth theorem for K-theory groups by Schochet.

The same arguments as for the special KT in Theorem 5.1 imply that $\alpha_{\mathfrak{A}}$ is an isomorphism for all $\mathfrak{A} \in N$ with $K_*(\mathfrak{B}_2)$ free.

The general case with $K_*(\mathfrak{B}_i)$ finitely generated follows by the geometric projective resolution argument as for $\mathfrak{B}_2$, just as for the general KT in Theorem 6.1. □

As it contains a generalization, it holds that

Theorem 7.8. *The Künneth Theorem KT for KK-theroy groups holds if either $K_*(\mathfrak{A})$ or $K_*(\mathfrak{B})$ is finitely generated, where we assume that $\mathfrak{A} \in N$ and $\mathfrak{B}$ is separable.*

Furthermore, the KT exact sequence splits (unnaturally).

Proof. If we assume that $\mathfrak{A} \in N$, then we may assume by Corollary 7.5 that $\mathfrak{A}$ is commutative, with the same K-theory groups and with the spectrum $(\mathfrak{A}^\wedge)^+$ finite-dimensional (even a finite complex if $K_*(\mathfrak{A})$ is finitely generated). Furthermore, by Proposition 7.4, there is a commutative C^*-algebra $\mathfrak{B}'$ with similar properties and a KK-element in $KK_0(\mathfrak{B}', \mathfrak{B})$ inducing an isomorphism of K-theory groups of $\mathfrak{B}'$ and $\mathfrak{B}$. Using the UCT for $(\mathfrak{A}, \mathfrak{B}')$ and $(\mathfrak{A}, \mathfrak{B})$ and their UCT diagrams together with the Five Lemma, we see that there is a natural isomorphism:

$$KK_*(\mathfrak{A}, \mathfrak{B}') \xrightarrow{\cong} KK_*(\mathfrak{A}, \mathfrak{B}).$$

Hence there is no loss of generality in assuming that $\mathfrak{B}$ is commutative.

Now set $\mathfrak{A} = C_0(X)$ and $\mathfrak{B} = C_0(Y)$. Then

$$KK_*(C_0(X), C_0(Y)) \cong K^{*-1}(F(X^+) \wedge Y^+).$$

If $K_*(\mathfrak{B})$ is finitely generated, then we may assume that Y^+ is a finite complex and apply the Künneth theorem for representable K-theory of spectra (not checked).

If $K_*(\mathfrak{A})$ is finitely generated, we may assume that X^+ is a finite complex and replace the functional Spanier-Whitehead dual spectrum $F(X^+)$ by a finite complex $D(X^+)$, namely, the classical Spanier-Whitehead dual, and then use the Künneth theorem for topological K-theory of compact spaces. (Not checked).

In either case, the KT exact sequence splits (unnaturally). □

Theorem 7.9. *Suppose that $\mathfrak{A} \in N$ and $\mathfrak{B}$ is any separable C^*-algebra. Then the UCT exact sequence splits (unnaturally). If $\mathfrak{B} = \mathfrak{A}$, then the splitting is even a splitting as a ring, and the graded ring $KK_*(\mathfrak{A}, \mathfrak{A})$ is (anti)-isomorphic to*

$$\begin{aligned} &\oplus_{i,j,k} \operatorname{Ext}^i_{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{A})) \quad (i, j, k \in \mathbb{Z}_2) \\ \equiv &[\oplus_{j,k} \operatorname{Hom}(K_j(\mathfrak{A}), K_k(\mathfrak{A}))] \oplus [\oplus_{j,k} \operatorname{Ext}^1_{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{A}))] \end{aligned}$$

with the following ring structure: the product of any two $\operatorname{Ext}^1_{\mathbb{Z}}$*-terms is zero, and* $\operatorname{Ext}^0_{\mathbb{Z}} = \operatorname{Hom}$ *operates as usual on* Hom *and* $\operatorname{Ext}^1_{\mathbb{Z}}$*. Thus, for instance, if* $x \in \operatorname{Hom}(K_0(\mathfrak{A}), K_1(\mathfrak{A})) = \operatorname{Ext}^0_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{A}))$ *and* $y \in \operatorname{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_0(\mathfrak{A}))$*, then* $x^2 = y^2 = 0$*,*

$$\begin{aligned} xy &= x_*(y) \in \operatorname{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_0(\mathfrak{A})) \quad \textit{and} \\ yx &= x^*(y) \in \operatorname{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_1(\mathfrak{A})). \end{aligned}$$

Proof. As in the proof of the Theorem 7.8 above, we may reduce to the case where both $\mathfrak{A}$ and $\mathfrak{B}$ are in N and are commutative. Furthermore, we may assume that $\mathfrak{A} = \mathfrak{A}^0 \oplus \mathfrak{A}^1$ and $\mathfrak{B} = \mathfrak{B}^0 \oplus \mathfrak{B}^1$ with $K_i(\mathfrak{A}^j) = 0 = K_i(\mathfrak{B}^j)$ for $i \neq j$. And then

$$KK_0(\mathfrak{A}, \mathfrak{B}) \cong [\oplus_{i=0,1} KK_0(\mathfrak{A}^0, \mathfrak{B}^i)] \oplus [\oplus_{j=0,1} KK_0(\mathfrak{A}^1, \mathfrak{B}^j)].$$

Applying the UCT to each of the four terms separately, we obtain

$$\begin{aligned} KK_0(\mathfrak{A}^0, \mathfrak{B}^0) &\cong \operatorname{Hom}(K_0(\mathfrak{A}^0), K_0(\mathfrak{B}^0)) = \operatorname{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{B})), \\ KK_0(\mathfrak{A}^0, \mathfrak{B}^1) &\cong \operatorname{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}^0), K_1(\mathfrak{B}^1)) = \operatorname{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{B})), \\ KK_0(\mathfrak{A}^1, \mathfrak{B}^0) &\cong \operatorname{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}^1), K_0(\mathfrak{B}^0)) = \operatorname{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_0(\mathfrak{B})), \\ KK_0(\mathfrak{A}^1, \mathfrak{B}^1) &\cong \operatorname{Hom}(K_1(\mathfrak{A}^1), K_1(\mathfrak{B}^1)) = \operatorname{Hom}(K_1(\mathfrak{A}), K_1(\mathfrak{B})). \end{aligned}$$

Also,

$$\begin{aligned} KK_1(\mathfrak{A}^0, \mathfrak{B}^0) &\cong \operatorname{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}^0), K_0(\mathfrak{B}^0)) = \operatorname{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_0(\mathfrak{B})), \\ KK_1(\mathfrak{A}^0, \mathfrak{B}^1) &\cong \operatorname{Hom}(K_0(\mathfrak{A}^0), K_1(\mathfrak{B}^1)) = \operatorname{Hom}(K_0(\mathfrak{A}), K_1(\mathfrak{B})), \\ KK_1(\mathfrak{A}^1, \mathfrak{B}^0) &\cong \operatorname{Hom}(K_1(\mathfrak{A}^1), K_0(\mathfrak{B}^0)) = \operatorname{Hom}(K_1(\mathfrak{A}), K_0(\mathfrak{B})), \\ KK_1(\mathfrak{A}^1, \mathfrak{B}^1) &\cong \operatorname{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}^1), K_1(\mathfrak{B}^1)) = \operatorname{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_1(\mathfrak{B})). \end{aligned}$$

And it then follows that

$$KK_*(\mathfrak{A}, \mathfrak{B}) \cong \operatorname{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \oplus \operatorname{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B})).$$

This gives us the desired splitting of the UCT.

When $\mathfrak{A} = \mathfrak{B}$ we take $\mathfrak{A}^0 = \mathfrak{B}^0$ and $\mathfrak{A}^1 = \mathfrak{B}^1$. Note that $KK_*(\mathfrak{A}^0, \mathfrak{A}^0)$ and $KK_*(\mathfrak{A}^1, \mathfrak{A}^1)$ are graded subrings of $KK_*(\mathfrak{A}, \mathfrak{A})$, and that

$$\begin{aligned} &KK_*(\star, \mathfrak{A}^0) \otimes_{\mathfrak{A}} KK_*(\mathfrak{A}^1, \star) = 0 \quad \text{and} \\ &KK_*(\star, \mathfrak{A}^1) \otimes_{\mathfrak{A}} KK_*(\mathfrak{A}^0, \star) = 0. \end{aligned}$$

Check the ring structure on $KK_*(\mathfrak{A}, \mathfrak{A})$ as follows. For instance, given

$$\begin{aligned} &x \in KK_0(\mathfrak{A}^0, \mathfrak{A}^1) \cong \mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{A})) \quad \text{and} \\ &y \in KK_0(\mathfrak{A}^1, \mathfrak{A}^0) \cong \mathrm{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_0(\mathfrak{A})), \end{aligned}$$

we have

$$x \otimes_{\mathfrak{A}^1} y \in KK_0(\mathfrak{A}^0, \mathfrak{A}^0) \cong \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{A})) = \mathrm{Ext}^0_{\mathbb{Z}}(K_0(\mathfrak{A}), K_0(\mathfrak{A})).$$

But $\gamma(x) = 0$ and $\gamma(y) = 0$ and thus $\gamma(x \otimes_{\mathfrak{A}^1} y) = \gamma(y)\gamma(x) = 0$, and hence $x \otimes_{\mathfrak{A}^1} y = 0$. Similarly, $y \otimes_{\mathfrak{A}^0} x = 0$. Thus, the $\mathrm{Ext}^1_{\mathbb{Z}}$-terms in the UCT give rise to an ideal in $KK_*(\mathfrak{A}, \mathfrak{A})$ with square product zero.

Also, given

$$\begin{aligned} &x \in KK_1(\mathfrak{A}^0, \mathfrak{A}^1) \cong \mathrm{Hom}(K_0(\mathfrak{A}), K_1(\mathfrak{A})) = \mathrm{Ext}^0_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{A})), \\ &y \in KK_0(\mathfrak{A}^1, \mathfrak{A}^0) \cong \mathrm{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_0(\mathfrak{A})), \end{aligned}$$

we have

$$x \otimes_{\mathfrak{A}^1} y \in KK_1(\mathfrak{A}^0, \mathfrak{A}^0) \cong \mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_0(\mathfrak{A}))$$

and

$$y \otimes_{\mathfrak{A}^0} x \in KK_1(\mathfrak{A}^1, \mathfrak{A}^1) \cong \mathrm{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_1(\mathfrak{A}))$$

(corrected). □

Remark. Note that the above argument can also be adapted to prove splitting of the Künneth exact sequence for K-theory groups of tensor products of C^*-algebras. Thus, if $\mathfrak{A}, \mathfrak{B} \in N$, then the following KT short exact sequence:

$$0 \to K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{A} \otimes \mathfrak{B}) \to \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to 0$$

of Schochet splits (unnaturally). As a proof, choose KK-equivalence $\lambda \in KK_0(\mathfrak{A}, \mathfrak{A}')$ and $\mu \in KK_0(\mathfrak{B}, \mathfrak{B}')$, where $\mathfrak{A}' = \mathfrak{A}^0 \oplus \mathfrak{A}^1$ and $\mathfrak{B}' = \mathfrak{B}^0 \oplus \mathfrak{B}^1$ are commutative with $K_i(\mathfrak{A}^j) = 0 = K_i(\mathfrak{B}^j)$ for $i \neq j$. Then the Kasparov product by μ induces an isomorphism

$$K_*(\mathfrak{A} \otimes \mathfrak{B}) \to K_*(\mathfrak{A} \otimes \mathfrak{B}')$$

and applying the Kasparov product by λ in turn gives

$$\begin{aligned}
&K_*(\mathfrak{A} \otimes \mathfrak{B}) \cong K_*(\mathfrak{A}' \otimes \mathfrak{B}') \\
&\cong [\oplus_{i=0,1} K_*(\mathfrak{A}^0 \otimes \mathfrak{B}^i)] \oplus [\oplus_{j=0,1} K_*(\mathfrak{A}^1 \otimes \mathfrak{B}^j)]
\end{aligned}$$

(corrected). Now argue as in the proof above.

Specializing to the case where $\mathfrak{A}$ and $\mathfrak{B}$ are commutative, one recover the splittings of the K-theory Künneth sequence of Atiyah [1] for compact spaces X and Y:

$$0 \to K^*(X) \otimes K^*(Y) \to K^*(X \times Y) \to \mathrm{Tor}_1^{\mathbb{Z}}(K^*(X), K^*(Y)) \to 0,$$

a result due to Bödigheimer [3] and [4].

Remark. The KK-theory groups are not additive in the second variable. For instance, let $\mathfrak{A} = \oplus_i \mathfrak{A}_i$ with $\mathfrak{A}_i = \mathbb{C}$, so that $\mathfrak{A} = C_0(\mathbb{N})$. Then

$$KK_0(\mathfrak{A}, \mathfrak{A}) \cong \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{A}))$$

and its identity element corresponds to the identity map on $K_0(\mathfrak{A})$, which dose not belong to the group:

$$\oplus_i \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{A}_i)) \cong \oplus_i KK_0(\mathfrak{A}, \mathfrak{A}_i).$$

On the other hand, suppose that $\mathfrak{A} \in N$ and $K_0(\mathfrak{A}) = \mathbb{Q}$ and $K_1(\mathfrak{A}) = 0$, for instance, $\mathfrak{A}$ a UHF-algebra (an inductive limit of tensor products of a matrix algebra over $\mathbb{C}$) with $K_*(\mathfrak{A})$, not finitely generated. Since we have

$$\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}) = 0, \quad \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}, \quad \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) = 0$$

it follows from the UCT that

$$K^0(\mathfrak{A}) = KK_0(\mathfrak{A}, \mathbb{C}) = 0, \quad KK_0(\mathfrak{A}, \mathfrak{A}) = \mathbb{Q}, \quad KK_1(\mathfrak{A}, \mathfrak{A}) = 0.$$

Therefore,

$$[K^0(\mathfrak{A}) \otimes K_0(\mathfrak{A})] \oplus [K^1(\mathfrak{A}) \otimes K_1(\mathfrak{A})] = 0 \oplus 0 \not\cong KK_0(\mathfrak{A}, \mathfrak{A})$$

and hence, the Künneth theorem for KK-groups does fail in general.

Nevertheless, one has additivity of KK-theory groups under an assumption as in:

Proposition 7.10. *Suppose that $\mathfrak{A} \in N$ and that $K_*(\mathfrak{A})$ or $K_*(\mathfrak{B})$ is finitely generated, with $\mathfrak{B} = \oplus_i \mathfrak{B}_i$. Then the natural map*

$$\oplus_i KK_*(\mathfrak{A}, \mathfrak{B}_i) \to KK_*(\mathfrak{A}, \mathfrak{B})$$

is an isomorphism.

Proof. Suppose that $K_*(\mathfrak{A})$ is finitely generated. Then

$$\mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) = \oplus_i \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B}_i))$$

and

$$\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) = \oplus_i \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B}_i)).$$

The result follows from the UCT for $(\mathfrak{A}, \mathfrak{B})$ and $(\mathfrak{A}, \mathfrak{B}_i)$ and the Five-Lemma. Indeed, the following diagram is obtained:

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] \\
\oplus_i \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B}_i)) \arrow[r, equal] \arrow[d, "\oplus_i \delta"] & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \arrow[d, "\delta"] \\
\oplus_i KK_*(\mathfrak{A}, \mathfrak{B}_i) \arrow[r] \arrow[d, "\oplus_i \gamma"] & KK_*(\mathfrak{A}, \mathfrak{B}) \arrow[d, "\gamma"] \\
\oplus_i \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B}_i)) \arrow[r, equal] \arrow[d] & \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \arrow[d] \\
0 \arrow[r, equal] & 0.
\end{tikzcd}$$

Suppose that $K_*(\mathfrak{B})$ is finitely generated. Then

$$K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) = \oplus_i (K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B}_i))$$

and

$$\mathrm{Tor}^{\mathbb{Z}}_1(K^*(\mathfrak{A}), K_*(\mathfrak{B})) = \oplus_i \mathrm{Tor}^{\mathbb{Z}}_1(K^*(\mathfrak{A}), K_*(\mathfrak{B}_i)).$$

The result follows from the KT for $(\mathfrak{A}, \mathfrak{B})$ and $(\mathfrak{A}, \mathfrak{B}_i)$ and the Five-Lemma. Indeed, obtained is the following diagram:

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] \\
\oplus_i (K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B}_i)) \arrow[r, equal] \arrow[d, "\oplus_i \alpha"] & K^*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \arrow[d, "\alpha"] \\
\oplus_i KK_*(\mathfrak{A}, \mathfrak{B}_i) \arrow[r] \arrow[d, "\oplus_i \beta"] & KK_*(\mathfrak{A}, \mathfrak{B}) \arrow[d, "\beta"] \\
\oplus_i \mathrm{Tor}^{\mathbb{Z}}_1(K^*(\mathfrak{A}), K_*(\mathfrak{B}_i)) \arrow[r, equal] \arrow[d] & \mathrm{Tor}^{\mathbb{Z}}_1(K^*(\mathfrak{A}), K_*(\mathfrak{B})) \arrow[d] \\
0 \arrow[r, equal] & 0.
\end{tikzcd}$$

□

Now, let us assume given a cohomology theory H^* and an associated additive homology theory H_*. We say that the UCT holds for an algebra $\mathfrak{A}$ if there are abelian groups G_n and for each n, a natural short exact sequence:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(\mathfrak{A}), G_{n-1}) \to H^n(\mathfrak{A}) \to \mathrm{Hom}(H_n(\mathfrak{A}), G_n) \to 0.$$

Proposition 7.11. *Suppose that $\mathfrak{A} = \mathfrak{A}_i$ is a direct limit of a directed sequence of C^*-algebras $\mathfrak{A}_i$ and that the UCT holds for each $\mathfrak{A}_i$:*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(\mathfrak{A}_i), G_{n-1}) \to H^n(\mathfrak{A}_i) \to \mathrm{Hom}(H_n(\mathfrak{A}_i), G_n) \to 0.$$

Then the following are equivalent:
(a) The UCT holds for $\mathfrak{A}$.
(b) The Milnor $\lim^1$ *sequence holds for $\mathfrak{A}$:*

$$0 \to \varprojlim{}^1 H^{n-1}(\mathfrak{A}_i) \to H^n(\mathfrak{A}) \to \varprojlim H^n(\mathfrak{A}_i) \to 0.$$

Observe that

$$\varprojlim{}^1 \mathrm{Ext}^1_{\mathbb{Z}}(L_i, G) = 0$$

for any directed sequence of abelian groups L_i, and that there is an exact sequence

$$0 \to \varprojlim{}^1 \mathrm{Hom}(L_i, G) \to \mathrm{Ext}^1_{\mathbb{Z}}(\varinjlim L_i, G) \to \varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(L_i, G) \to 0.$$

Note that

$$\mathrm{Hom}(L_i, G) = \mathrm{Ext}^0_{\mathbb{Z}}(L_i, G) = \mathrm{Ext}^{1-1}_{\mathbb{Z}}(L_i, G).$$

8 Applications to mod p K-theory

Let us fix an integer $n \geq 2$ (in almost all applications this will be a prime p). Let C_n be the mapping cone of a $*$-homomorphism $f : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ of degree n. That is, the following sequence:

$$0 \to SC_0(\mathbb{R}) \to C_n = Cf \to C_0(\mathbb{R}) \to 0$$

is exact, where $(\xi, a) \in Cf \subset IC_0(\mathbb{R}) \oplus C_0(\mathbb{R})$ with $\xi(0) = 0$ and $\xi(1) = f(a)$. The spectrum $C_n^\wedge$ is homeomorphic to the locally compact space obtained by removing the basepoint from a 2-dimensional Moore space. For example, $C_2 = C_0(\mathbb{R}P^2 \setminus \{\mathrm{pt}\})$ where pt means a point and $\mathbb{R}P^2 = (\mathbb{R}^3 \setminus \{0\})/\sim$ (or $= S^2/\sim$) the real projective plane with the equivalence relation $\sim$ defined by $x \sim y$ if there is $t \in \mathbb{R}$ non-zero (or $t = \pm 1$) such that $x = ty$. Note that C_n belongs to the class N since it is abelian and has K-theory groups $K_0(C_n) \cong \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $K_1(C_n) \cong 0$. Indeed, deduced is the following six-term exact sequence:

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & K_0(C_n) & \longrightarrow & 0 \\ {\scriptstyle\partial}\uparrow{\scriptstyle\times n} & & & & \downarrow{\scriptstyle\partial} \\ \mathbb{Z} & \longleftarrow & K_1(C_n) & \longleftarrow & 0 \end{array}$$

where the map $\times n$ means the multiplication by n, so that we have the required isomorphisms by the diagram.

In fact, recall a fact from [6, 12.3] in the following. Let n be a natural number. The Moore space X_n is defined to be the quotient space $\mathbb{D}/\sim_n$ of the 2-dimensional, closed unit disc $\mathbb{D}$ with the equivalence relation $\sim_n$ defined by: for $z, w \in \mathbb{D}$, $z \sim_n w$ if $|z| = |w|$ and $z^n = w^n$, or if $z = w$. For instance, $z \sim_n ze^{2\pi i/n}$ for any $z \in \mathbb{T}$ the one-torus. Then

$$C(X_n) \cong \{g \in C(\mathbb{D}) \mid g(z) = g(ze^{2\pi i/n}), z \in \mathbb{T}\}.$$

Define a $*$-homomorphism $f^+ : C(\mathbb{T}) \to C(\mathbb{T})$ by $f^+(g)(z) = g(z^n)$, whose restriction to $C_0(\mathbb{R})$ as a closed ideal of $C(\mathbb{T})$ gives the map f of degree n above. There is the following short exact sequence and the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(\mathbb{R}^2) & \longrightarrow & C(X_n) & \longrightarrow & C(\mathbb{T}) & \longrightarrow & 0 \\ \| & & \| & & \uparrow & & \uparrow & & \| \\ 0 & \longrightarrow & SC_0(\mathbb{R}) & \longrightarrow & C_n = Cf & \longrightarrow & C_0(\mathbb{R}) & \longrightarrow & 0 \end{array}$$

and it is shown that the boundary map in the six-term exact K-theory sequence of the short exact sequence at the first line is the multiplication by n. Note that the unitization C_n^+ of C_n by $\mathbb{C}$ is isomorphic to $C(X_n)$, and in particular, $C(X_2) \cong C(\mathbb{R}P^2) \cong C_2^+$.

Let $\mathfrak{D}$ be any C^*-algebra. Define the K-theory groups with coefficients in $\mathbb{Z}_n$ (the mod n K-theory groups) for $\mathfrak{D}$ as

$$K_j(\mathfrak{D}; \mathbb{Z}_n) \equiv K_j(\mathfrak{D} \otimes C_n).$$

Theorem 8.1. (Schochet [10]). *For any C^*-algebra $\mathfrak{N} \in N$ with $K_0(\mathfrak{N}) = \mathbb{Z}_n$ and $K_1(\mathfrak{N}) = 0$, there is a natural equivalence of homology theories:*

$$K_*(\star; \mathbb{Z}_n) \cong K_*(\star \otimes \mathfrak{N}).$$

The Cuntz algebra $\mathcal{O}_{n+1}$ generated by $n+1$ orthogonal isometries with the sum of their range projections equal to the identity satisfies $K_0(\mathcal{O}_{n+1}) = \mathbb{Z}_n$ and $K_1(\mathcal{O}_{n+1}) = 0$ (see [11, 12.2]).

Proposition 8.2. *Let $\mathfrak{A} \in N$ with $K_0(\mathfrak{A}) \cong \mathbb{Z}_n$ and $K_1(\mathfrak{A}) = 0$. Then the $\mathbb{Z}_2$-graded ring $KK_*(\mathfrak{A}, \mathfrak{A})$ is a free $\mathbb{Z}_n$-module of rank two with generators $\mathrm{id}_{\mathfrak{A}}$ of degree 0 and $\beta_{\mathfrak{A}}$ the Bockstein element of degree 1, with multiplication determined by the relation $\beta_{\mathfrak{A}}^2 = 0$.*

Proof. By the UCT, we have

$$\begin{aligned} KK_0(\mathfrak{A}, \mathfrak{A}) &\cong \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{A})) \oplus \mathrm{Hom}(K_1(\mathfrak{A}), K_1(\mathfrak{A})) \\ &= \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_n) \oplus 0 \cong \mathbb{Z}_n \end{aligned}$$

even as a ring, where the last isomorphism is obtained by the group homomorphisms φ_k with $\varphi_k(1) = 1 + \cdots + 1 = k \cdot 1 \in \mathbb{Z}_n$ for $k \in \mathbb{Z}_n$. Note that φ_1 corresponds to $\mathrm{id}_{\mathfrak{A}}$ in KK_0. And also, the UCT implies that

$$\begin{aligned} KK_1(\mathfrak{A}, \mathfrak{A}) &\cong \mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_0(\mathfrak{A})) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_1(\mathfrak{A})) \\ &= \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_n) \oplus 0 \cong \mathbb{Z}_n \end{aligned}$$

(additively). The Bockstein element $\beta_{\mathfrak{A}}$ is the generator of $KK_1(\mathfrak{A}, \mathfrak{A})$ corresponding to the extension of abelian groups:

$$0 \to K_0(\mathfrak{A}) = \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to K_0(\mathfrak{A}) = \mathbb{Z}_n \to 0.$$

However, $\beta_{\mathfrak{A}}$ induces the zero map $\gamma(\beta_{\mathfrak{A}}) = 0 : K_{*+1}(\mathfrak{A}) \to K_*(\mathfrak{A})$. Thus, the element $\beta_{\mathfrak{A}}^2 \in KK_1(\mathfrak{A}, \mathfrak{A}) \otimes_{\mathfrak{A}} KK_1(\mathfrak{A}, \mathfrak{A}) \cong KK_0(\mathfrak{A}, \mathfrak{A})$ induces the zero map

$$\gamma(\beta_{\mathfrak{A}}^2) = \gamma(\beta_{\mathfrak{A}}) \circ \gamma(\beta_{\mathfrak{A}}) = 0 : K_*(\mathfrak{A}) \to K_*(\mathfrak{A})$$

in $\mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{A}))$. Hence, the UCT implies that $\beta_{\mathfrak{A}}^2 = 0$ in $KK_0(\mathfrak{A}, \mathfrak{A})$.

Note that other group extension classes of $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_n)$ are, possibly, given by the direct product $\mathbb{Z}_n \times \mathbb{Z}_n$ and (non-abelian) semi-direct products $\mathbb{Z}_n \rtimes_{\rho(k)} \mathbb{Z}_n$ with the action $\rho_1^{(k)}$ on $\mathbb{Z}_n$ given by multiplication by k mod n, where the case of $k = 1$ corresponds to the direct product with the trivial action $\square$

Corollary 8.3. *Any two C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in the class N with*

$$K_0(\mathfrak{A}_j) = \mathbb{Z}_n \quad \textit{and} \quad K_1(\mathfrak{A}_j) = 0 \quad (j = 0, 1)$$

are KK-equivalent.

Proof. This is a special case of Corollary 7.5, which says in particular that the same K-theory groups of two C^*-algebras in N imply KK-equivalence, i.e., the equivalence between the identity elements of their KK-theory rings.

The UCT implies that there is an element:

$$\lambda \in KK_0(\mathfrak{A}_1, \mathfrak{A}_2) \cong \mathrm{Hom}(K_0(\mathfrak{A}_1), K_0(\mathfrak{A}_2)) \cong \mathbb{Z}_n$$

which induces an isomorphism on K_0, and there is a unique element $\lambda^{-1} \in KK_0(\mathfrak{A}_2, \mathfrak{A}_1)$ which induces the inverse isomorphism. Then the Kasparov products:

$$\lambda \otimes_{\mathfrak{A}_2} \lambda^{-1} \in KK_0(\mathfrak{A}_1, \mathfrak{A}_1) \quad \text{and} \quad \lambda^{-1} \otimes_{\mathfrak{A}_1} \lambda \in KK_0(\mathfrak{A}_2, \mathfrak{A}_2)$$

induce respectively the identity maps on $K_0(\mathfrak{A}_1)$ and $K_0(\mathfrak{A}_2)$, again by the UCT. Hence, the Kasparov products coincide with $\mathrm{id}_{\mathfrak{A}_1} \in KK_0(\mathfrak{A}_1, \mathfrak{A}_1)$ and $\mathrm{id}_{\mathfrak{A}_2} \in KK_0(\mathfrak{A}_2, \mathfrak{A}_2)$, by Proposition 8.2 above. □

Remark. Invertible KK-elements induce natural equivalences of $K_*(\star \otimes \mathfrak{A}_1)$ with $K_*(\star \otimes \mathfrak{A}_2)$.

Proposition 8.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras in N. Then the natural map $\gamma = \gamma(\mathfrak{A}, \mathfrak{B})$ in the UCT factors as $\gamma = \psi \circ \varphi$:*

$$\begin{array}{ccc} KK_0(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{\gamma} & \mathrm{H}(K_0(\mathfrak{A}), K_0(\mathfrak{A})) \oplus \mathrm{H}(K_1(\mathfrak{A}), K_1(\mathfrak{A})) \\ {\scriptstyle\varphi}\big\downarrow & & \big\| \\ \mathrm{Nat}(K_0(\star \otimes \mathfrak{A}), K_0(\star \otimes \mathfrak{B})) & \xrightarrow{\psi} & \mathrm{H}(K_0(\mathfrak{A}), K_0(\mathfrak{A})) \oplus \mathrm{H}(K_1(\mathfrak{A}), K_1(\mathfrak{A})) \end{array}$$

where $\mathrm{H}(\cdot,\cdot) = \mathrm{Hom}(\cdot,\cdot)$, and $\mathrm{Nat}(\cdot,\cdot)$ means the group of natural transformations (possibly),and where the map φ is determined by

$$\varphi(x)(y) = y \otimes_{\mathfrak{A}} x \in KK_0(\mathbb{C}, \mathfrak{D} \otimes \mathfrak{B}) = K_0(\mathfrak{D} \otimes \mathfrak{B})$$

for $x \in KK_0(\mathfrak{A}, \mathfrak{B})$ and $y \in K_0(\mathfrak{D} \otimes \mathfrak{A}) = KK_0(\mathbb{C}, \mathfrak{D} \otimes \mathfrak{A})$ with $\mathfrak{D}$ a C^*-algebra, and where the map ψ is determined by the restriction of the variables $\star$ to the C^*-algebras $\mathbb{C}$ and $C_0(\mathbb{R})$.

If we have

$$\mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) = 0 = \mathrm{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_0(\mathfrak{B})),$$

then φ is injective.

The map ψ is always surjective.

If $K_*(\mathfrak{A})$ is torsion free, then ψ is injective.

Proof. The first statement is obvious. The second follows immediately from the UCT. The third does also.

Assume now that $K_*(\mathfrak{A})$ is torsion free. The Künneth theorem of K-theory for C^*-algebras by Schochet implies that

$$K_0(\mathfrak{D} \otimes \mathfrak{A}) = [K_0(\mathfrak{D}) \otimes K_0(\mathfrak{A})] \oplus [K_1(\mathfrak{D}) \otimes K_1(\mathfrak{A})]$$

for any C^*-algebra $\mathfrak{D}$. If $x \in K_1(\mathfrak{D})$ and $y \in K_1(\mathfrak{A})$, then there is some map $f : C_0(\mathbb{R}) \to \mathfrak{D} \otimes \mathbb{K}$ such that $x = f_*(\lambda)$ by Rosenberg, where λ is the canonical generator for $K_1(C_0(\mathbb{R}))$, and so $x \otimes y = (f \otimes \mathrm{id})_*(\lambda \otimes y)$ with $\lambda \otimes y \in K_1(C_0(\mathbb{R})) \otimes K_1(\mathfrak{A})$, so that the diagram for a natural transformation θ:

$$\begin{array}{ccc} K_1(C_0(\mathbb{R})) \otimes K_1(\mathfrak{A}) & \xrightarrow{\theta} & K_1(C_0(\mathbb{R})) \otimes K_1(\mathfrak{B}) \\ \downarrow{\scriptstyle (f\otimes\mathrm{id})_*} & & \downarrow{\scriptstyle (f\otimes\mathrm{id})_*} \\ K_1(\mathfrak{D}) \otimes K_1(\mathfrak{A}) & \xrightarrow{\theta} & K_1(\mathfrak{D}) \otimes K_1(\mathfrak{B}) \end{array}$$

commutes, and the natural transformation θ is determined by its image in $\mathrm{Hom}(K_1(\mathfrak{A}), K_1(\mathfrak{B}))$.

A similar argument applies to the case of $K_0(\mathfrak{D}) \otimes K_0(\mathfrak{A})$. Indeed, $K_0(\mathfrak{D}) \cong K_1(S\mathfrak{D})$. Let $x \in K_0(\mathfrak{D})$ and $y \in K_0(\mathfrak{A})$. There is a map $g : C_0(\mathbb{R}) \to S\mathfrak{D} \otimes \mathbb{K}$ such that $x = g_*(\lambda)$. Then we have

$$\begin{array}{ccc} K_0(\mathbb{C}) \otimes K_0(\mathfrak{A}) & \xrightarrow{\theta} & K_0(\mathbb{C}) \otimes K_0(\mathfrak{B}) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ K_1(C_0(\mathbb{R})) \otimes K_0(\mathfrak{A}) & \xrightarrow{\theta} & K_1(C_0(\mathbb{R})) \otimes K_0(\mathfrak{B}) \\ \downarrow{\scriptstyle (g\otimes\mathrm{id})_*} & & \downarrow{\scriptstyle (g\otimes\mathrm{id})_*} \\ K_1(S\mathfrak{D}) \otimes K_0(\mathfrak{A}) & \xrightarrow{\theta} & K_1(S\mathfrak{D}) \otimes K_0(\mathfrak{B}) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ K_0(\mathfrak{D}) \otimes K_0(\mathfrak{A}) & \xrightarrow{\theta} & K_0(\mathfrak{D}) \otimes K_0(\mathfrak{B}). \end{array}$$

$\square$

Corollary 8.5. *If $K_*(\mathfrak{A})$ is torsion free and*

$$\mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{A})) = 0,$$
$$\mathrm{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_0(\mathfrak{A})) = 0,$$

then the ring of degree-preserving, (self-) homology operations (or natural transformations) for $K_*(\star \otimes \mathfrak{A})$ is naturally (anti-) isomorphic to the KK-ring $KK_*(\mathfrak{A}, \mathfrak{A})$.

Proof. The proposition above with the assumptions implies that

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] \\
0 \to KK_0(\mathfrak{A},\mathfrak{A}) \arrow[r, "\varphi"] \arrow[d, "\cong"', "\gamma"] & \mathrm{Nat}(K_0(\star\otimes\mathfrak{A}), K_0(\star\otimes\mathfrak{A})) \arrow[d, "\cong"', "\psi"] \\
\oplus_{j=0,1}\mathrm{Hom}(K_j(\mathfrak{A}), K_j(\mathfrak{A})) \arrow[r, equal] \arrow[d] & \oplus_{j=0,1}\mathrm{Hom}(K_j(\mathfrak{A}), K_j(\mathfrak{A})) \arrow[d] \\
0 \arrow[r, equal] & 0.
\end{tikzcd}$$

It follows that the map φ is an isomorphism.

Note also that $KK_1(\mathfrak{A},\mathfrak{B}) \cong KK_0(S\mathfrak{A},\mathfrak{B}) \cong KK_0(\mathfrak{A},S\mathfrak{B})$ and $K_0(\star\otimes S\mathfrak{A}) = K_1(\star\otimes\mathfrak{A})$, $K_0(\star\otimes S\mathfrak{B}) = K_1(\star\otimes\mathfrak{B})$. □

Remark. Homology operators need not be degree-preserving. We are mostly interested in the case where $K_*(\mathfrak{A})$ has torsion.

The Bockstein homology operation $\beta = \beta_n$ of degree 1 is the connecting map in the long exact sequence: for any C^*-algebra $\mathfrak{D}$,

$$\cdots \to K_{*+1}(\mathfrak{D};\mathbb{Z}_n) \xrightarrow{\beta} K_*(\mathfrak{D};\mathbb{Z}_n) \to K_*(\mathfrak{D};\mathbb{Z}_{n^2}) \to K_*(\mathfrak{D};\mathbb{Z}_n) \xrightarrow{\beta} K_{*-1}(\mathfrak{D};\mathbb{Z}_n) \to K_{*-1}(\mathfrak{D};\mathbb{Z}_{n^2}) \to \cdots\cdots$$

with $K_*(\mathfrak{D};\mathbb{Z}_n) = K_*(\mathfrak{D}\otimes C_n)$. Indeed, the KT implies that

$$0 \to K_*(\mathfrak{D})\otimes\mathbb{Z}_n \to K_*(\mathfrak{D}\otimes C_n) \to \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{D}),\mathbb{Z}_n) \to 0$$

and also, the following diagram is induced by splitting of the KT:

$$\begin{tikzcd}
K_*(\mathfrak{D})\otimes\mathbb{Z}_n \arrow[r] \arrow[d] & K_*(\mathfrak{D})\otimes\mathbb{Z}_{n^2} \arrow[r] \arrow[d] & K_*(\mathfrak{D})\otimes\mathbb{Z}_n \arrow[d] \\
K_*(\mathfrak{D}\otimes C_n) \arrow[r] \arrow[d] & K_*(\mathfrak{D}\otimes C_{n^2}) \arrow[r] \arrow[d] & K_*(\mathfrak{D}\otimes C_n) \arrow[d] \\
\mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{D}),\mathbb{Z}_n) \arrow[r] & \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{D}),\mathbb{Z}_{n^2}) \arrow[r] & \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{D}),\mathbb{Z}_n)
\end{tikzcd}$$

which corresponds to the short exact sequences:

$$0 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 0$$

and, moreover (it should be),

$$\begin{array}{ccc}
K_{*+1}(\mathfrak{D}) \otimes \mathbb{Z}_n & \xrightarrow{\partial} & K_*(\mathfrak{D}) \otimes \mathbb{Z}_n \\
\downarrow & & \downarrow \\
K_{*+1}(\mathfrak{D}; \mathbb{Z}_n) & \xrightarrow{\beta=\partial} & K_*(\mathfrak{D}; \mathbb{Z}_n) \\
\downarrow & & \downarrow \\
\mathrm{Tor}_1^{\mathbb{Z}}(K_{*+1}(\mathfrak{D}), \mathbb{Z}_n) & & \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{D}), \mathbb{Z}_n)
\end{array}$$

(but note that there are no arrows at the bottom line to make the extended diagram to be commutative in general).

Theorem 8.6. *The $\mathbb{Z}_2$-graded ring of (self-) homology operations for $K_*(\star; \mathbb{Z}_n)$ (on the category of separable C^*-algebras $\star$) is a free $\mathbb{Z}_n$-module of rank 2 with generators the identity map of degree 0 and the Bockstein operation β_n of degree 1.*

As a $\mathbb{Z}_2$-graded ring over $\mathbb{Z}_n$, the ring of homology operations for $K_*(\star; \mathbb{Z}_n)$ is the exterior algebra over $\mathbb{Z}_n$ on β.

Proof. As computed in Proposition 8.2, the $\mathbb{Z}_2$-graded ring $KK_*(C_n, C_n)$ is a free $\mathbb{Z}_n$-module of rank 2 with generators id_{C_n} and β_{C_n}. This ring operators via the Kasparov product $\otimes_{C_n}$ on

$$K_*(\mathfrak{D}; \mathbb{Z}_n) \cong KK_*(\mathbb{C}, \mathfrak{D} \otimes C_n)$$

as

$$KK_*(\mathbb{C}, \mathfrak{D} \otimes C_n) \otimes KK_j(C_n, C_n) \xrightarrow{\otimes_{C_n}} KK_{*+j}(\mathbb{C}, \mathfrak{D} \otimes C_n)$$

with $\otimes_{C_n} \mathrm{id}_{C_n}$ the identity and $\otimes_{C_n} \beta_{C_n} = \beta$. We must show that there are no other homology operations other than those via the Kasparov product of $KK_*(C_n, C_n)$.

Suppose that θ is a homology operation for $K_*(\star; \mathbb{Z}_n)$ and that $\mathfrak{D}$ is any separable C^*-algebra. It suffices to show that the operation θ on $K_*(\mathfrak{D}; \mathbb{Z}_n)$ is determined by that on $K_*(\mathcal{O}_{n+1}; \mathbb{Z}_n)$, where $\mathcal{O}_{n+1}$ is the Cuntz algebra generated by $n+1$ orthogonal isometries with the sum of their range projections the identity, with $K_0(\mathcal{O}_{n+1}) = \mathbb{Z}_n$ and $K_1(\mathcal{O}_{n+1}) = 0$. The KT for K-theory implies that

$$\begin{aligned}
0 &\to K_*(\mathcal{O}_{n+1}) \otimes \mathbb{Z}_n \to K_*(\mathcal{O}_{n+1} \otimes C_n) = K_*(\mathcal{O}_{n+1}; \mathbb{Z}_n) \\
&\to \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathcal{O}_{n+1}), \mathbb{Z}_n) \to 0
\end{aligned}$$

so that

$$K_0(\mathcal{O}_{n+1} \otimes C_n) \cong K_0(\mathcal{O}_{n+1}) \otimes \mathbb{Z}_n = \mathbb{Z}_n \otimes \mathbb{Z}_n \cong \mathbb{Z}_n,$$
$$K_1(\mathcal{O}_{n+1} \otimes C_n) \cong \mathrm{Tor}_1^{\mathbb{Z}}(K_0(\mathcal{O}_{n+1}), \mathbb{Z}_n) \cong \mathbb{Z}_n,$$

that is, $K_*(\mathcal{O}_{n+1} \otimes C_n)$ is a free $\mathbb{Z}_n$-module $\mathbb{Z}_n \oplus \mathbb{Z}_n$ of rank 2, with generators $[1]_0$ and $[1]_1$ corresponding to generators of the K-theory groups K_0 and K_1 respectively. Since the boundary map $\partial : K_{*+1}(\mathcal{O}_{n+1}) \otimes \mathbb{Z}_n \to K_*(\mathcal{O}_{n+1}) \otimes \mathbb{Z}_n$ is zero, we have $\beta([1]_0) = 0$ and $\beta([1]_1) = [1]_0$, because the K_1-group $K_1(\mathcal{O}_{n+1} \otimes C_n)$ is mapped to $K_0(\mathcal{O}_{n+1} \otimes C_n)$ under $\partial = \beta$. Thus, if θ on $K_*(\mathcal{O}_{n+1}; \mathbb{Z}_n)$ is degree-preserving, it must be given by mulitiplication by some $x_0 \in \mathbb{Z}_n$ in degree 0 and by some $x_1 \in \mathbb{Z}_n$ in degree 1. It follows from the above K-theory isomorphisms $K_j(\mathcal{O}_{n+1} \otimes C_n) \cong \mathbb{Z}_n$ via $K_0(\mathcal{O}_{n+1})$ that θ becomes multplication by a constant. If θ is degree-reversing, then it becomes a multiple of β by the same reasoning (slightly different from that in the text).

It remains to show that θ is determined by its operation on $K_*(\mathcal{O}_{n+1}; \mathbb{Z}_n)$. Since θ is compatible with taking suspentions in the sense as: the following diagram:

$$\begin{array}{ccc} K_*(\mathcal{O}_{n+1}; \mathbb{Z}_n) & \xrightarrow{\theta} & K_*(\mathcal{O}_{n+1}; \mathbb{Z}_n) \\ \cong \big\uparrow S & & \cong \big\uparrow S \\ K_*(S\mathcal{O}_{n+1}; \mathbb{Z}_n) & \xrightarrow{\theta} & K_*(S\mathcal{O}_{n+1}; \mathbb{Z}_n) \end{array}$$

commetes (i.e., for θ to be stable), it is enough to consider elements $x \in K_0(\mathfrak{D}; \mathbb{Z}_n)$ and to compute $\theta(x)$ in terms of θ for $\mathcal{O}_{n+1}$. Note that

$$\begin{array}{ccc} K_0(\mathfrak{D}) \otimes \mathbb{Z}_n & \xrightarrow{\partial} & K_1(\mathfrak{D}) \otimes \mathbb{Z}_n \\ \big\downarrow & & \big\downarrow \\ K_0(\mathfrak{D}; \mathbb{Z}_n) & \xrightarrow{\beta=\partial} & K_1(\mathfrak{D}; \mathbb{Z}_n) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Tor}_1^{\mathbb{Z}}(K_1(\mathfrak{D}), \mathbb{Z}_n) & & \mathrm{Tor}_1^{\mathbb{Z}}(K_0(\mathfrak{D}), \mathbb{Z}_n). \end{array}$$

Thus, if $\beta(x) = 0$, then there is a class $u \in K_0(\mathfrak{D})$ that corresponds to x. Then the class u corresponds to a projection p in $\mathfrak{D}^+ \otimes \mathbb{K}$. We view u as $u = f_*([1_{\mathbb{C}}])$, where $f : \mathbb{C} \to \mathfrak{D}^+ \otimes \mathbb{K}$ with $f(1_{\mathbb{C}}) = p$ and $[1_{\mathbb{C}}]$ is the standard generator for $K_0(\mathbb{C})$. We then have

$$x = (f \otimes \mathrm{id})_*([1_{\mathbb{C}}] \otimes [1_{\mathcal{O}_{n+1}}]),$$

where $f \otimes \mathrm{id} : \mathbb{C} \otimes \mathcal{O}_{n+1} \to \mathfrak{D}^+ \otimes \mathbb{K} \otimes \mathcal{O}_{n+1}$, where we identify $K_0(\mathfrak{D}; \mathbb{Z}_n)$ with a summand in $K_0(\mathfrak{D}^+ \otimes \mathbb{K}; \mathbb{Z}_n) \cong K_0(\mathfrak{D}^+ \otimes \mathbb{K} \otimes \mathcal{O}_{n+1})$. Note that

$$\begin{array}{ccc}
0 & = & 0 \\
\downarrow & & \downarrow \\
K_0(\mathbb{C}) \otimes K_0(\mathcal{O}_{n+1}) & \xrightarrow{(f\otimes \mathrm{id})_*} & K_0(\mathfrak{D}^+ \otimes \mathbb{K}) \otimes K_0(\mathcal{O}_{n+1}) \\
\downarrow & & \downarrow \\
K_0(\mathbb{C}; \mathbb{Z}_n) & \xrightarrow{(f\otimes \mathrm{id})_*} & K_0(\mathfrak{D}^+ \otimes \mathbb{K}; \mathbb{Z}_n) \\
\downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta} \\
K_0(\mathbb{C}; \mathbb{Z}_n) & \xrightarrow{(f\otimes \mathrm{id})_*} & K_0(\mathfrak{D}^+ \otimes \mathbb{K}; \mathbb{Z}_n)
\end{array}$$

where the last square commutes by naturality of θ. In particular,

$$\begin{aligned}
\theta(x) &= \theta(f \otimes \mathrm{id})_*([1_{\mathbb{C}}] \otimes [1_{\mathcal{O}_{n+1}}]) \\
&= (f \otimes \mathrm{id})_* \theta([1])
\end{aligned}$$

with $[1]$ the standard generator of $K_0(\mathbb{C}; \mathbb{Z}_n)$, which is identified with $[1_{\mathbb{C}}] \otimes [1_{\mathcal{O}_{n+1}}]$. Since the unital inclusion of $\mathbb{C}$ in $\mathcal{O}_{n+1}$ induces an isomorphism on $K_0(\star; \mathbb{Z}_n)$, we see that $\theta(x)$ is determined by the restriction of θ to $K_0(\mathcal{O}_{n+1}; \mathbb{Z}_n)$. Note that

$$\begin{aligned}
K_0(\mathcal{O}_{n+1}; \mathbb{Z}_n) = K_0(\mathcal{O}_{n+1} \otimes \mathcal{O}_{n+1}) &\cong K_0(\mathcal{O}_{n+1}) \otimes K_0(\mathcal{O}_{n+1}) \\
&\cong \mathbb{Z}_n \otimes \mathbb{Z}_n \cong \mathbb{Z}_n, \\
K_0(\mathbb{C}; \mathbb{Z}_n) = K_0(\mathbb{C} \otimes \mathcal{O}_{n+1}) &\cong K_0(\mathcal{O}_{n+1}) \cong \mathbb{Z}_n
\end{aligned}$$

by the KT.

Now even if $\beta(x) \neq 0$ in $K_1(\mathfrak{D}; \mathbb{Z}_n)$, then $\beta(\beta(x)) = \partial^2(x) = 0$ with

$$\begin{array}{ccc}
K_1(\mathfrak{D}) \otimes \mathbb{Z}_n & \xrightarrow{\partial} & K_0(\mathfrak{D}) \otimes \mathbb{Z}_n \\
\downarrow & & \downarrow \\
K_1(\mathfrak{D}; \mathbb{Z}_n) & \xrightarrow{\beta = \partial} & K_0(\mathfrak{D}; \mathbb{Z}_n) \\
\downarrow & & \downarrow \\
\mathrm{Tor}_1^{\mathbb{Z}}(K_0(\mathfrak{D}), \mathbb{Z}_n) & & \mathrm{Tor}_1^{\mathbb{Z}}(K_1(\mathfrak{D}), \mathbb{Z}_n)
\end{array}$$

so that there is a class $w \in K_1(\mathfrak{D})$ that corresponds to $\beta(x)$. Since

$$K_1(\mathfrak{D}) \cong K_1(\mathfrak{D}) \otimes \mathbb{Z} \cong K_0(S\mathfrak{D}) \otimes K_0(\mathcal{O}_\infty)$$

where $\mathcal{O}_\infty$ is the Cuntz algebra generated by countably infinite, orthogonal projections with the finite sums of their range projections not equal to the identity. and has K_0 equal to $\mathbb{Z}$ and K_1 trivial. And the KT implies

$$\begin{aligned}0 \to K_1(\mathfrak{D}) &\cong K_0(S\mathfrak{D}) \otimes K_0(\mathcal{O}_\infty) \\ &\to K_0(S\mathfrak{D} \otimes \mathcal{O}_\infty) \\ &\to K_0(S\mathfrak{D}^+ \otimes \mathcal{O}_\infty)\end{aligned}$$

where the last map is injective because the following sequence:

$$K_0(S\mathfrak{D} \otimes \mathcal{O}_\infty) \to K_0(S\mathfrak{D}^+ \otimes \mathcal{O}_\infty) \to K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$$

splits. Hence w may be identified with a class of $K_0(S\mathfrak{D}^+ \otimes \mathcal{O}_\infty)$. By a lemma of Cuntz (see [10]), there is a $*$-homomorphism:

$$\varphi : \mathcal{O}_{n+1} \to (C_0(\mathbb{R}) \otimes \mathfrak{D})^+ \otimes C_n \otimes \mathcal{O}_\infty$$

(slightly corrected in our sense) such that $\varphi_*([1]) = w$, where we view $w \in K_1(\mathfrak{D})$ as with torsion in

$$K_1(\mathfrak{D}) \otimes \mathbb{Z}_n \cong K_0(S\mathfrak{D}) \otimes \mathbb{Z}_n \to K_0((S\mathfrak{D})^+ \otimes \mathcal{O}_\infty) \otimes \mathbb{Z}_n$$

which has the following inclusion by the KT:

$$\begin{aligned}K_0((S\mathfrak{D})^+ \otimes \mathcal{O}_\infty) \otimes \mathbb{Z}_n &= K_0((S\mathfrak{D})^+ \otimes \mathcal{O}_\infty) \otimes K_0(C_n) \\ &\to K_0((S\mathfrak{D})^+ \otimes C_n \otimes \mathcal{O}_\infty).\end{aligned}$$

Recall that $[1]_0 = \beta([1]_1)$ in $K_*(\mathcal{O}_{n+1}; \mathbb{Z}_n)$. Consider now the following commutative diagram with exact two row and added with the inclusion via a lemma of Cuntz:

$$\begin{array}{ccccc}
K_0(\mathcal{O}_{n+1}; \mathbb{Z}_n) & \xrightarrow{\beta} & K_1(\mathcal{O}_{n+1}; \mathbb{Z}_n) & \xrightarrow{\beta} & K_0(\mathcal{O}_{n+1}; \mathbb{Z}_n) \\
\downarrow{\scriptstyle (\varphi\otimes \mathrm{id})_*} & & \downarrow{\scriptstyle (\varphi\otimes \mathrm{id})_*} & & \downarrow{\scriptstyle (\varphi\otimes \mathrm{id})_*} \\
K_0(\mathfrak{D}; \mathbb{Z}_n) & \xrightarrow{\beta} & K_1(\mathfrak{D}; \mathbb{Z}_n) \supset K_1(\mathfrak{D}) \otimes \mathbb{Z}_n & \xrightarrow{\beta} & K_0(\mathfrak{D}; \mathbb{Z}_n) \\
 & & \downarrow & & \\
K_0(\mathcal{O}_{n+1}; \mathbb{Z}_n) & \xrightarrow{(\varphi\otimes \mathrm{id})_*} & K_0((S\mathfrak{D})^+ \otimes \mathcal{O}_\infty; \mathbb{Z}_n) & &
\end{array}$$

Then we have

$$\begin{aligned}\beta(x) = w &= (\varphi \otimes \mathrm{id})_*([1]_0) \\ &= (\varphi \otimes \mathrm{id})_*(\beta([1]_1)) \\ &= \beta((\varphi \otimes \mathrm{id})_*([1]_1)).\end{aligned}$$

Therefore, $\beta(x - (\varphi \otimes \mathrm{id})_*([1]_1)) = 0$, which is in the case already dealt with above. Thus,

$$\begin{aligned}\theta(x) &= \theta(x - (\varphi \otimes \mathrm{id})_*([1]_1) + (\varphi \otimes \mathrm{id})_*([1]_1)) \\ &= \theta(x - (\varphi \otimes \mathrm{id})_*([1]_1)) + \theta((\varphi \otimes \mathrm{id})_*([1]_1))) \\ &= \theta(x - (\varphi \otimes \mathrm{id})_*([1]_1)) + (\varphi \otimes \mathrm{id})_*(\theta([1]_1)))\end{aligned}$$

both of the first and second terms of which are determined by the operation θ restricted to $K_*(\mathcal{O}_{n+1}; \mathbb{Z}_n)$.

We have shown in the process that the Cuntz algebra $\mathcal{O}_{n+1}$ is a sort of universal object for mod n K-theory. □

An admissible multiplication on $K_*(\star; \mathbb{Z}_n)$ is a bilinear natural transformation:

$$\mu : K_i(\star; \mathbb{Z}_n) \times K_j(\star'; \mathbb{Z}_n) \to K_{i+j}(\star \otimes \star'; \mathbb{Z}_n)$$

satisfying certain reasonable axioms. We may as well assume that μ is to be associative (for triples of separable nuclear C^*-algebras). The other key axioms are that μ should commute up to sign with suspension in either variable, and that μ should be the multiplication when one or the other of the K-theroy classes is in the image of the composite:

$$K_*(\star) \to K_*(\star) \otimes \mathbb{Z}_n \to K_*(\star; \mathbb{Z}_n),$$

and that the Bockstein operation β should be a (graded) derivation.

Possibly, those things mean as that the following two maps are the same:

$$\begin{aligned}&\mu \times \mu : [K_i(\star; \mathbb{Z}_n) \times K_j(\star'; \mathbb{Z}_n)] \times K_k(\star''; \mathbb{Z}_n) \to K_{(i+j)+k}(\star \otimes \star' \otimes \star''; \mathbb{Z}_n), \\ &\mu \times \mu : K_i(\star; \mathbb{Z}_n) \times [K_j(\star'; \mathbb{Z}_n) \times K_k(\star''; \mathbb{Z}_n)] \to K_{i+(j+k)}(\star \otimes \star' \otimes \star''; \mathbb{Z}_n),\end{aligned}$$

and that the following two maps are the same up to sign:

$$\begin{aligned}&\mu : K_i(S\star; \mathbb{Z}_n) \times K_j(\star'; \mathbb{Z}_n) \to K_{i+j}(S \star \otimes \star'; \mathbb{Z}_n), \\ &\mu : K_i(\star; \mathbb{Z}_n) \times K_j(S\star'; \mathbb{Z}_n) \to K_{i+j}(\star \otimes S\star'; \mathbb{Z}_n),\end{aligned}$$

and that the restriction of μ to $K_i(\star) \times K_j(\star')$ is a map as:

$$\mu : K_i(\star) \times K_j(\star') \to K_{i+j}(\star \otimes \star'),$$

and that the following composite:

$$K_i(\star; \mathbb{Z}_n) \times K_j(\star'; \mathbb{Z}_n) \xrightarrow{\mu} K_{i+j}(\star \otimes \star'; \mathbb{Z}_n) \xrightarrow{\beta} K_{i+j-1}(\star \otimes \star'; \mathbb{Z}_n)$$

should be equal to the sum of the following composites:

$$\begin{array}{ccc} K_i(\star;\mathbb{Z}_n)\times K_j(\star';\mathbb{Z}_n) & \xrightarrow{\beta\times\mathrm{id}} & K_{i-1}(\star;\mathbb{Z}_n)\times K_j(\star';\mathbb{Z}_n) \\ \downarrow{\scriptstyle \mathrm{id}\times\beta} & & \downarrow{\scriptstyle \mu} \\ K_i(\star;\mathbb{Z}_n)\times K_{j-1}(\star';\mathbb{Z}_n) & \xrightarrow{\mu} & K_{i+j-1}(\star\otimes\star';\mathbb{Z}_n), \end{array}$$

so that

$$\beta\circ\mu = \mu\circ(\beta\times\mathrm{id}) + \mu\circ(\mathrm{id}\times\beta).$$

Theorem 8.7. *The admissible multiplications on $K_*(\star;\mathbb{Z}_n)$ are in natural one-to-one correspondence with the elements of $KK_0(C_n\otimes C_n, C_n)$ whose image by the UCT in the group*

$$\mathrm{Hom}(K_0(C_n)\otimes K_0(C_n), K_0(C_n))$$

with

$$K_0(C_n)\otimes K_0(C_n)\cong K_0(C_n\otimes C_n)$$

by the K-theory KT and with $K_0(C_n)\cong\mathbb{Z}_n$ is exactly the usual multiplication map from $\mathbb{Z}_n\otimes\mathbb{Z}_n$ to $\mathbb{Z}_n$.

There are exactly n such elements. When n is odd, exactly one of the admissible multilications is commutative.

When $n=2$, neither multiplication is commutative and the two multiplications are essentially equivalent.

The multiplication μ_λ corresponding to a KK-element λ in $KK_0(C_n\otimes C_n, C_n)$ is given by the counter-clock-wise composition involving the K-theroy KT and Kasparov product:

$$\begin{array}{ccc} K_i(\mathfrak{D}\otimes C_n)\otimes K_j(\mathfrak{E}\otimes C_n) & \xrightarrow{\mu_\lambda} & K_{i+j}(\mathfrak{D}\otimes\mathfrak{E}\otimes C_n) \\ \downarrow & & \uparrow{\scriptstyle \otimes_{C_n\otimes C_n}\lambda} \\ K_{i+j}(\mathfrak{D}\otimes C_n\otimes\mathfrak{E}\otimes C_n) & \underset{\cong}{\xrightarrow{\sigma_{2,3}}} & K_{i+j}(\mathfrak{D}\otimes\mathfrak{E}\otimes C_n\otimes C_n), \end{array}$$

for any C^*-algebras $\mathfrak{D}$ and $\mathfrak{E}$, where $\sigma_{2,3}$ is induced by the filp from $C_n\otimes\mathfrak{E}$ to $\mathfrak{E}\otimes C_n$.

Proof. By naturality and associativity of the Kasparov product, any element $\lambda\in KK_0(C_n\otimes C_n, C_n)$ gives rise to a natural bilinear associative multiplication μ_λ on $K_*(\star;\mathbb{Z}_n)$ for which degrees add correctly.

Check the associativity for μ_λ as in the following computation: for $x\otimes a\in K_i(\mathfrak{D};\mathbb{Z}_n)=K_i(\mathfrak{D}\otimes C_n)$, $y\otimes b\in K_j(\mathfrak{E};\mathbb{Z}_n)$, and $z\otimes c\in K_k(\mathfrak{F};\mathbb{Z}_n)$, we have

$$\begin{aligned}
&\mu_\lambda(\mu_\lambda((x\otimes a)\otimes(y\otimes b))\otimes(z\otimes c))\\
&=\otimes_{C_n\otimes C_n}\lambda[\sigma_{2,3}(\otimes_{C_n\otimes C_n}\lambda(\sigma_{2,3}((x\otimes a)\otimes(y\otimes b)))\otimes(z\otimes c))]\\
&=\otimes_{C_n\otimes C_n}\lambda[\sigma_{2,3}(\otimes_{C_n\otimes C_n}\lambda((x\otimes y)\otimes(a\otimes b))\otimes(z\otimes c))]\\
&=\otimes_{C_n\otimes C_n}\lambda[\sigma_{2,3}(((x\otimes y)\otimes(a\otimes b)\otimes_{C_n\otimes C_n}\lambda)\otimes(z\otimes c))]\\
&=\otimes_{C_n\otimes C_n}\lambda[(x\otimes y\otimes z)\otimes((a\otimes b)\otimes_{C_n\otimes C_n}\lambda)\otimes c)]\\
&=(x\otimes y\otimes z)\otimes[((a\otimes b)\otimes_{C_n\otimes C_n}\lambda)\otimes c)\otimes_{C_n\otimes C_n}\lambda]
\end{aligned}$$

and

$$\begin{aligned}
&\mu_\lambda((x\otimes a)\otimes\mu_\lambda((y\otimes b)\otimes(z\otimes c)))\\
&=\otimes_{C_n\otimes C_n}\lambda[\sigma_{2,3}((x\otimes a)\otimes(\otimes_{C_n\otimes C_n}\lambda(\sigma_{2,3}((y\otimes b)\otimes(z\otimes c)))))]\\
&=\otimes_{C_n\otimes C_n}\lambda[\sigma_{2,3}((x\otimes a)\otimes(\otimes_{C_n\otimes C_n}\lambda((y\otimes z)\otimes(b\otimes c))))]\\
&=\otimes_{C_n\otimes C_n}\lambda[\sigma_{2,3}((x\otimes a)\otimes(y\otimes z)\otimes((b\otimes c)\otimes_{C_n\otimes C_n}\lambda))]\\
&=\otimes_{C_n\otimes C_n}\lambda[(x\otimes y\otimes z)\otimes a\otimes((b\otimes c)\otimes_{C_n\otimes C_n}\lambda)]\\
&=(x\otimes y\otimes z)\otimes[(a\otimes((b\otimes c)\otimes_{C_n\otimes C_n}\lambda))\otimes_{C_n\otimes C_n}\lambda]
\end{aligned}$$

with

$$((a\otimes b)\otimes_{C_n\otimes C_n}\lambda)\otimes c)\otimes_{C_n\otimes C_n}\lambda=(a\otimes((b\otimes c)\otimes_{C_n\otimes C_n}\lambda))\otimes_{C_n\otimes C_n}\lambda$$

by the associativity of the Kasparov product.

It is also clear that if μ_λ is the usual multiplication in the case where $\mathfrak{D}=\mathbb{C}=\mathfrak{C}$, then it must project to the usual multiplication in $\mathrm{Hom}(K_0(C_n)\otimes K_0(C_n),K_0(C_n))$.

Indeed, with $K_i(\mathbb{C};\mathbb{Z}_n)=K_i(\mathbb{C}\otimes C_n)\cong K_i(C_n)$,

$$\begin{tikzcd}
K_i(\mathbb{C}\otimes C_n)\otimes K_j(\mathbb{C}\otimes C_n) \arrow[r, "\mu_\lambda"] \arrow[d] & K_{i+j}(\mathbb{C}\otimes\mathbb{C}\otimes C_n) \\
K_{i+j}(\mathbb{C}\otimes C_n\otimes\mathbb{C}\otimes C_n) \arrow[r, "\sigma_{2,3}", "\cong"'] & K_{i+j}(\mathbb{C}\otimes\mathbb{C}\otimes C_n\otimes C_n) \arrow[u, "\otimes_{C_n\otimes C_n}\lambda"']
\end{tikzcd}$$

so that μ_λ maps $K_0(C_n)\otimes K_0(C_n)$ to $K_0(C_n)$ and is zero otherwise.

Now by the K-theory Künneth theorem, we have

$$\begin{aligned}
K_0(C_n\otimes C_n)&\cong K_0(C_n)\otimes K_0(C_n)\cong\mathbb{Z}_n,\\
K_1(C_n\otimes C_n)&\cong\mathrm{Tor}_1^{\mathbb{Z}}(K_0(C_n),K_0(C_n))\cong\mathbb{Z}_n.
\end{aligned}$$

We then have the following short exact sequence by the UCT:

$$\begin{aligned}&0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_1(C_n \otimes C_n), K_0(C_n)) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_n)\\&\to KK_0(C_n \otimes C_n, C_n)\\&\xrightarrow{\gamma} \mathrm{Hom}(K_0(C_n) \otimes K_0(C_n), K_0(C_n)) \cong \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_n) \to 0.\end{aligned}$$

This, together with $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_n) \cong \mathbb{Z}_n$, confirms that there are exactly n KK-elements λ of the desired type (which are mapped by γ to the usual multiplication map for $\mathbb{Z}_n$).

The naturality of the map μ_λ in both variables and the representability theorem proved in the course of the proof of Theorem 8.6 show that it is enough to check the axioms only for the various combinations of two special elements: $[1] \in K_0(\mathbb{C}; \mathbb{Z}_n)$ and $[1]_1 \in K_1(\mathcal{O}_{n+1}; \mathbb{Z}_n) \cong K_1(C_n; \mathbb{Z}_n)$ with $\beta([1]_1) = [1]_0$ in $K_*(\mathcal{O}_{n+1}, \mathbb{Z}_n) \cong K_*(C_n, \mathbb{Z}_n)$. There are only three equations to check (as corrected):

$$\begin{aligned}\mu_\lambda(\beta([1]_1), [1]_1) &= \mu_\lambda([1]_0, [1]_1) = 0,\\ \mu_\lambda([1]_1, \beta([1]_1)) &= \mu_\lambda([1]_1, [1]_0) = [1]_1 \otimes [1],\\ \beta \circ \mu_\lambda([1]_1, [1]_1) &= \mu_\lambda(\beta([1]_1), [1]_1) = 0,\end{aligned}$$

where the first and second equations take place in $K_1(\mathcal{O}_{n+1}; \mathbb{Z}_n)$ or $K_1(C_n; \mathbb{Z}_n)$, and the third equation takes place in $K_1(\mathcal{O}_{n+1} \otimes \mathcal{O}_{n+1}; \mathbb{Z}_n)$ or $K_1(C_n \otimes C_n; \mathbb{Z}_n)$. And also, deduced is

$$\mu_\lambda(\beta([1]_1), \beta([1]_1)) = \mu_\lambda([1]_0, [1]_0) = [1]_0 \otimes [1] = \beta(\mu_\lambda([1]_1, [1]_0))$$

By definition of μ_λ,

$$\begin{aligned}\mu_\lambda([1]_0, [1]_1) &= \sigma_{2,3}([1]_0 \otimes [1]_1) \otimes_{C_n \otimes C_n} \lambda;\\ \mu_\lambda([1]_1, [1]_0) &= \sigma_{2,3}([1]_1 \otimes [1]_0) \otimes_{C_n \otimes C_n} \lambda.\end{aligned}$$

Moreover, in $K_1(C_n \otimes C_n; \mathbb{Z}_n) = K_1(C_n \otimes C_n \otimes C_n)$, we compute

$$\begin{aligned}\sigma_{2,3}([1]_0 \otimes [1]_1) &= \sigma_{2,3}([1_{C_n}]_0 \otimes [1_{C_n}]_0 \otimes [1]_1)\\ &= \sigma_{2,3}([1_{C_n}]_0 \otimes [1_{C_n}]_0 \otimes \beta^{-1}([1_{C_n}]_0, [1_{C_n}]_0))\\ &= [1_{C_n}]_0 \otimes [1_{C_n}]_0 \otimes \beta^{-1}([1_{C_n}]_0, [1_{C_n}]_0);\\ \sigma_{2,3}([1]_1 \otimes [1]_0) &= \sigma_{2,3}([1]_1 \otimes [1_{\mathbb{C}}]_0 \otimes [1_{C_n}]_0)\\ &= \sigma_{2,3}(\beta^{-1}([1_{C_n}]_0, [1_{C_n}]_0) \otimes [1_{C_n}]_0 \otimes [1_{C_n}]_0)\\ &= \beta^{-1}([1_{C_n}]_0, [1_{C_n}]_0) \otimes [1_{C_n}]_0 \otimes [1_{C_n}]_0,\end{aligned}$$

where β^{-1} is the splitting map in the K-theory KT, so that

$$\sigma_{2,3}([1]_0 \otimes [1]_1) \otimes_{C_n \otimes C_n} \lambda = [1]_0 \otimes ([1]_1 \otimes_{C_n \otimes C_n} \lambda) = 0;$$
$$\sigma_{2,3}([1]_1 \otimes [1]_0) \otimes_{C_n \otimes C_n} \lambda = [1]_1 \otimes ([1]_0 \otimes_{C_n \otimes C_n} \lambda) = [1]_1.$$

Note that by the Kasparov product,

$$K_1(C_n \otimes C_n) \otimes K_0(C_n \otimes C_n, C_n) \xrightarrow{\otimes_{C_n \otimes C_n} \lambda} K_1(C_n) = 0;$$
$$K_0(C_n \otimes C_n) \otimes K_0(C_n \otimes C_n, C_n) \xrightarrow{\otimes_{C_n \otimes C_n} \lambda} K_0(C_n) \cong \mathbb{Z}_n,$$

and hence $[1]_1 \otimes_{C_n \otimes C_n} \lambda = 0$ and $[1]_0 \otimes_{C_n \otimes C_n} \lambda = [1]$.

We next compute

$$\begin{aligned}
\mu_\lambda([1]_1, [1]_1) &= \sigma_{2,3}(\beta^{-1}([1_{C_n}]_0, [1_{C_n}]) \otimes \beta^{-1}([1_{C_n}], [1_{C_n}])) \otimes_{C_n \otimes C_n} \lambda \\
&= (\beta^{-1}([1_{C_n}]_0, [1_{C_n}]) \otimes \beta^{-1}([1_{C_n}], [1_{C_n}])) \otimes_{C_n \otimes C_n} \lambda \\
&= [1]_1 \otimes ([1]_1 \otimes_{C_n \otimes C_n} \lambda) = 0
\end{aligned}$$

and hence $\beta(\mu_\lambda([1]_1, [1]_1)) = 0$. (Possibly, the plausible derivation equation would fail.)

We next check that the map $\lambda \mapsto \mu_\lambda$ is injective. Suppose that this is false. Considering the difference of two distinct KK-elements λ_1, λ_2 inducing the same multiplication $\mu_{\lambda_1} = \mu_{\lambda_2}$, we obtain an element $\tau = \lambda_1 - \lambda_2 \in KK_0(C_n \otimes C_n, C_n)$ with

$$\begin{aligned}
&\gamma(\tau) = 0 \in \mathrm{Hom}(K_0(C_n \otimes C_n), K_0(C_n)), \\
\text{and} \quad &\kappa(\tau) \neq 0 \in \mathrm{Ext}^1_{\mathbb{Z}}(K_1(C_n \otimes C_n), K_0(C_n))
\end{aligned}$$

but such that for all C^*-algebras $\mathfrak{D}$,

$$x \otimes_{C_n \otimes C_n} \tau = 0 \in K_*(\mathfrak{D} \otimes C_n)$$

for all $x \in K_*(\mathfrak{D} \otimes C_n \otimes C_n)$ since $\mu_\tau = \mu_{\lambda_1} - \mu_{\lambda_2} = 0$. This is impossible. Because, if

$$0 \to \mathbb{Z}_n \to E \to \mathbb{Z}_n \to 0$$

is an abelian group extension realizing the element $\kappa(\tau)$, then the following sequence:

$$\mathbb{Z}_n \cong \mathbb{Z}_n \otimes \mathbb{Z}_n \to \mathbb{Z}_n \otimes E \to \mathbb{Z}_n \otimes \mathbb{Z}_n \cong \mathbb{Z}_n$$

is no longer exact, hence when extended to a long exact sequence, it gives a non-trivial connecting map

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_n) \to \mathbb{Z}_n \otimes \mathbb{Z}_n$$

(not checked, but it always exists since $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_n) \cong \mathbb{Z}_n$). This says that if $\mathfrak{D} = C_n$ and x is suitably chosen, then $x \otimes_{C_n \otimes C_n} \tau$ can be non-zero, a contradiction.

Note, indeed, that

$$K_0(C_n \otimes C_n) \cong K_0(C_n) \otimes K_0(C_n) \cong \mathbb{Z}_n \otimes \mathbb{Z}_n,$$
$$\mathrm{Tor}_1^{\mathbb{Z}}(K_0(C_n \otimes C_n), K_0(C_n)) \cong K_1(C_n \otimes C_n; \mathbb{Z}_n)$$

which contains $K_0(C_n) \otimes K_1(C_n \otimes C_n)$.

We now consider the part of the theorem about commutativity. A multiplication μ_λ is commutative in the graded sense if and only if the corresponding λ is invariant under the automorphism of $KK_0(C_n \otimes C_n, C_n)$ induced by the flip σ interchanging the two tensor factors in $C_n \otimes C_n$. The UCT gives the following sequence:

$$\begin{aligned} 0 &\to \mathrm{Ext}_{\mathbb{Z}}^1(K_1(C_n \otimes C_n), \mathbb{Z}_n) \to KK_0(C_n \otimes C_n, C_n) \\ &\to \mathrm{Hom}(K_0(C_n \otimes C_n), \mathbb{Z}_n) \to 0. \end{aligned}$$

Clearly, σ is trivial on $K_0(C_n \otimes C_n)$, however, it acts by -1 on $K_1(C_n \otimes C_n)$, because

$$K_0(C_n \otimes C_n) \cong K_0(C_n) \otimes K_0(C_n), \quad K_1(C_n \otimes C_n) \cong \mathrm{Tor}_1^{\mathbb{Z}}(K_0(C_n), K_0(C_n)).$$

We now distinguish two cases: n odd and n even.

If n is odd, then σ has two two distinct eigenvalues ± 1 on $KK_0(C_n \otimes C_n)$, we have a direct sum splitting:

$$KK_0(C_n \otimes C_n, C_n) \cong \mathrm{Ext}_{\mathbb{Z}}^1(K_1(C_n \otimes C_n), \mathbb{Z}_n) \oplus KK_0(C_n \otimes C_n, C_n)^\sigma$$

with $KK_0(C_n \otimes C_n, C_n)^\sigma$ the subgroup of elements fixed under σ. The admissible multiplications all have the same component in the fixed-point set, so μ_λ is commutative if and only if λ has projection zero in $\mathrm{Ext}_{\mathbb{Z}}^1(K_1(C_n \otimes C_n), \mathbb{Z}_n)$, which happens for exactly one λ.

Note that the commutativity for μ_λ means that $\mu_\lambda(a \otimes c, b \otimes d) = \mu_\lambda(b \otimes d, a \otimes c)$, so that

$$\begin{aligned} a \otimes b \otimes ((c \otimes d) \otimes_{C_n \otimes C_n} \lambda) &= b \otimes a \otimes ((d \otimes c) \otimes_{C_n \otimes C_n} \lambda) \\ &= b \otimes a \otimes ((\sigma(c \otimes d)) \otimes_{C_n \otimes C_n} \lambda) \end{aligned}$$

(where it probably should be that $a = b$).

On the other hand, consider the case where $n = 2$. Then σ acts trivially on both $\mathrm{Ext}_{\mathbb{Z}}^1(K_1(C_2 \otimes C_2), \mathbb{Z}_2)$ and $\mathrm{Hom}(K_0(C_2 \otimes C_2), \mathbb{Z}_2)$ (note that it seems like that it is non-trivial on the first summand, but it is trivial because $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2$, so that $1 = -1$ in $\mathbb{Z}_2$). However, σ is not diagonalizable as an operator on $KK_0(C_2 \otimes C_2, C_2)$. Instead, it acts by the unipotent 2×2 matrix over $\mathbb{Z}_2$ the field of two elements:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{conjugate over } \mathbb{Z}_2 \text{ to} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

i.e., σ interchanges the two admissible multiplications and neither is commutative. Note that $1 + 1 = 1 + (-1) = 0$ in $\mathbb{Z}_2$ and also

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For an amusing proof, one may use the following identification:

$$\begin{aligned} KK_0(C_2 \otimes C_2, C_2) &\cong K^1(F(\mathbb{R}P^2 \wedge \mathbb{R}P^2) \wedge \mathbb{R}P^2) \\ &\cong K^1(Y \wedge Y \wedge \mathbb{R}P^2), \end{aligned}$$

where Y is a suitable retract of the complement of an embedded copy of $\mathbb{R}P^2$ in S^4. In effect, we may take $Y = \mathbb{R}P^2$.

Note that $C(\mathbb{R}P^2) \cong C_2^+$ as stated before. And also, with $C_2 \cong C_0((\mathbb{R}P^2)^-)$ with $(\mathbb{R}P^2)^-$ the non-compactification of $\mathbb{R}P^2$ by removing one point, we have

$$\begin{aligned} KK_0(C_2 \otimes C_2, C_2) &\cong KK_0(C_0((\mathbb{R}P^2)^- \times (\mathbb{R}P^2)^-), C_0((\mathbb{R}P^2)^-)) \\ &\cong KK_1(C_0((\mathbb{R}P^2)^- \times (\mathbb{R}P^2)^-), SC_0((\mathbb{R}P^2)^-)) \\ &\cong K^0(F([(\mathbb{R}P^2)^- \times (\mathbb{R}P^2)^-]^+) \wedge [S(\mathbb{R}P^2)^-]^+) \\ &\cong K^1(F(\mathbb{R}P^2 \wedge \mathbb{R}P^2) \wedge \mathbb{R}P^2), \end{aligned}$$

where we have

$$KK_1(C_0(X), C_0(Y)) \cong K^0(F(X^+) \wedge Y^+)$$

by Rosenberg-Schochet [7].

Should be computed the action of the flip of the first two factors in

$$\begin{aligned} K^1(\mathbb{R}P^2 \wedge \mathbb{R}P^2 \wedge \mathbb{R}P^2) &\cong K^1(\mathbb{R}P^2 \wedge \mathbb{R}P^2; \mathbb{Z}_2) \\ &\cong H^{\mathrm{odd}}(\mathbb{R}P^2 \wedge \mathbb{R}P^2; \mathbb{Z}_2). \end{aligned}$$

By the Künneth Theorem in cohomology over the field $\mathbb{Z}_2$,

$$\begin{aligned} H^*(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}_2) &\cong H^*(\mathbb{R}P^2; \mathbb{Z}_2) \otimes H^*(\mathbb{R}P^2; \mathbb{Z}_2) \\ &\cong \{\mathbb{Z}_2[\omega]/(\omega^3)\} \otimes \{\mathbb{Z}_2[\omega]/(\omega^3)\}, \end{aligned}$$

with ω the generator of $H^1(\mathbb{R}P^2, \mathbb{Z}_2) \cong \mathbb{Z}_2$, so that the odd cohomology (ring) is spanned by $\omega \otimes \omega^2$ and $\omega^2 \otimes \omega$, which are exchanged by the flip.

Note that there are isomorphisms called Chern character as:

$$K^0(X) \otimes \mathbb{Q} \cong \bigoplus_{n \text{ even}} H^n(X; \mathbb{Q}), \quad K^1(X) \otimes \mathbb{Q} \cong \bigoplus_{n \text{ odd}} H^n(X; \mathbb{Q}),$$

with X a compact space, where $H^n(X;\mathbb{Q})$ denotes the n-th (Alexander or Čech) cohomology group of X with coefficients in $\mathbb{Q}$. Note also that

$$H_0(\mathbb{R}P^2) \cong \mathbb{Z}, \quad H_1(\mathbb{R}P^2) \cong \mathbb{Z}_2, \quad H_2(\mathbb{R}P^2) \cong 0$$

and that

$$H^0(\mathbb{R}P^2, \mathbb{Q}) \cong H_2(\mathbb{R}P^2, \mathbb{Q}) \cong 0$$

and also

$$H^1(\mathbb{R}P^2, \mathbb{Q}) \cong H_1(\mathbb{R}P^2, \mathbb{Q}) \cong \mathbb{Z}_2 \otimes \mathbb{Q} \cong \mathbb{Z}^\infty$$

and

$$H^2(\mathbb{R}P^2, \mathbb{Q}) \cong H_0(\mathbb{R}P^2, \mathbb{Q}) \cong \mathbb{Z}^\infty.$$

Moreover, since $\mathbb{R}P^2 \cong C_2$, we have

$$K^0(\mathbb{R}P^2) \cong K_0(C(\mathbb{R}P^2)) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \quad \text{and} \quad K^1(\mathbb{R}P^2) \cong K_1(C(\mathbb{R}P^2)) \cong 0$$

(see [6], but it looks like that the Chern character would fail to be an isomorphism in that sense, or possibly, something is wrong).

Note that the Künneth theorem in homology implies that

$$\begin{aligned}
&H^0(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}_2) \cong H_4(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}_2) \\
&\cong \oplus_{p+q=4}[H_p(\mathbb{R}P^2; \mathbb{Z}_2) \otimes H_q(\mathbb{R}P^2; \mathbb{Z}_2)] \cong 0; \\
&H^1(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}_2) \cong H_3(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}_2) \\
&\cong \oplus_{p+q=3}[H_p(\mathbb{R}P^2; \mathbb{Z}_2) \otimes H_q(\mathbb{R}P^2; \mathbb{Z}_2)] \cong 0
\end{aligned}$$

and also

$$\begin{aligned}
&H^2(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}_2) \cong H_2(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}_2) \\
&\cong H_1(\mathbb{R}P^2; \mathbb{Z}_2) \otimes H_1(\mathbb{R}P^2; \mathbb{Z}_2) \\
&\cong H^1(\mathbb{R}P^2; \mathbb{Z}_2) \otimes H^1(\mathbb{R}P^2; \mathbb{Z}_2).
\end{aligned}$$

and

$$\begin{aligned}
&H^3(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}_2) \cong H_1(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}_2) \\
&\cong [H_1(\mathbb{R}P^2; \mathbb{Z}_2) \otimes H_0(\mathbb{R}P^2; \mathbb{Z}_2)] \oplus [H_0(\mathbb{R}P^2; \mathbb{Z}_2) \otimes H_1(\mathbb{R}P^2; \mathbb{Z}_2)] \\
&\cong [H^1(\mathbb{R}P^2; \mathbb{Z}_2) \otimes H^2(\mathbb{R}P^2; \mathbb{Z}_2)] \oplus [H^1(\mathbb{R}P^2; \mathbb{Z}_2) \otimes H^2(\mathbb{R}P^2; \mathbb{Z}_2)],
\end{aligned}$$

and

$$
\begin{aligned}
&H^4(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}_2) \cong H_0(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}_2) \\
&\cong H_0(\mathbb{R}P^2; \mathbb{Z}_2) \otimes H_0(\mathbb{R}P^2; \mathbb{Z}_2) \\
&\cong H^2(\mathbb{R}P^2; \mathbb{Z}_2) \otimes H^2(\mathbb{R}P^2; \mathbb{Z}_2).
\end{aligned}
$$

□

Remark. The non-commutativity of multiplications on $K_*(\star; \mathbb{Z}_2)$ and its proof give an illustration that the UCT and the KT cannot have natural splittings. We have seen that for the case of $KK_0(C_2 \otimes C_2, C_2)$, there are no splittings equivariant for the flip automorphism σ.

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