# A local study on the KK-theory equivalence and more basics for C^*-algebras

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# A LOCAL STUDY ON THE KK-THEORY EQUIVALENCE AND MORE BASICS FOR $C^*$-ALGEBRAS

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**Abstract**

We review and study the KK-theory equivalence for $C^*$-algebras as the main subject. For this we review and study some more basics on the KK-theory for $C^*$-algebras. As a result as a collection we obtain a table on classification of some KK-equivalent $C^*$-algebras.

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## Contents



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# 1 Introduction

This paper is based on the reference textbook [1] of Blackadar on K-theory for operator algebras. We study basic elements of KK-theory of $C^*$-algebras, aimed at KK-equivalence of $C^*$-algebras mainly, to be contained and to be self-contained, but far from being, and some to be not included. With some considerable effort, we make and give some elementary and helpful, exact computations, proofs, or hints for some facts on the KK-theory to understand those completely or suitably at the basic level. Consequently, at the end we obtain a, perhaps or certainly, useful table on classification of some known KK-equivalent $C^*$-algebras, which would be a guiding map for further investigation. Since time and effort for publication are limited, we could not contain all the topics and their details in KK-theory story.

We also refer to the textbook [6] of Wegge-Olsen on K-theory and $C^*$-algebras, in particular, as for Hilbert modules over $C^*$-algebras. Also, especially, refer to [5] on Takai duality for crossed products of $C^*$-algebras. See also [3] of the author, containing a section on KK-theory basics as an appendix. As well, may refer to [4] on the UCT, based on [2].

This paper, viewed as a technical note, with some corrections or interpretations, possibly from misprints, is organized as Contents above.

Now, given a few:

*Notaions.* We denote by $\mathbb{K}$ the $C^*$-algebra of all compact operators on a separable, infinite dimensional, Hilbert space. Denote by $M_n(\mathbb{C})$ the $n \times n$ matrix $C^*$-algebra over the complex field $\mathbb{C}$.

We denote by $C(X)$ the $C^*$-algebra of all complex-valued, continuous functions on a compact Hausdorff space $X$. Denote by $C_0(X)$ the $C^*$-algebra of all $\mathbb{C}$-valued, continuous functions on a locally compact Hausdorff space $X$ vanishing at infinity.

A positive element $p$ of a $C^*$-algebra $\mathfrak{A}$ is said to be **strictly positive** if $\varphi(p) > 0$ for every state $\varphi : \mathfrak{A} \to \mathbb{C}$ a functional with norm one, which is equivalent to that $p\mathfrak{A}$ is dense in $\mathfrak{A}$. A $C^*$-algebra has a strictly positive element if and only if it has a countable approximate identity. In particular, every separable $C^*$-algebra contains a strictly positive element. A $C^*$-algebra is said to be **$\sigma$-unital** if it has a countable approximate identity.

Denote by $\oplus$ and $\otimes$ the usual direct sum and (mimimal) tensor product for $C^*$-algebras and some others. But the same symbols with suffix such as $\otimes_{\mathfrak{D}}$ are used frequently in the different sense in what follows.

## 2 Hilbert modules over $C^*$-algebras

Let $\mathfrak{B}$ be a $C^*$-algebra. A **Hilbert module** $E$ over $\mathfrak{B}$ is defined to be the completion of a right $\mathfrak{B}$-module $E_0$ with a $\mathfrak{B}$-valued inner product $\langle \cdot, \cdot \rangle : E_0 \times E_0 \to \mathfrak{B}$ such that the function is conjugate linear in the first variable and linear in the second; $\langle x, yb \rangle = \langle x, y \rangle b$ for all $x, y \in E$, $b \in \mathfrak{B}$; $\langle y, x \rangle = \langle x, y \rangle^*$ for all $x, y \in E$; $\langle x, x \rangle \geq 0$; if $\langle x, x \rangle = 0$, then $x = 0$, and with the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ for $x \in E$.

**Example 2.1.** • Any $C^*$-algebra $\mathfrak{B}$ is viewed as a Hilbert $\mathfrak{B}$-module with the $\mathfrak{B}$-valued inner product $\langle a, b \rangle = a^*b$ for $a, b \in \mathfrak{B}$.

Check it out: $\langle a, cb \rangle = a^*cb = \langle a, c \rangle b$; $\langle b, a \rangle = b^*a = (a^*b)^* = \langle a, b \rangle^*$; $\langle a, a \rangle = a^*a \geq 0$; $\|\langle a, a \rangle\|^{\frac{1}{2}} = \|a^*a\|^{\frac{1}{2}} = \|a\|$ (the $C^*$-norm condition).

• Let $\mathfrak{B}^n = \oplus^n \mathfrak{B}$ denote the direct sum of $n$ copies of a $C^*$-algebra $\mathfrak{B}$. Then $\mathfrak{B}^n$ is a Hilbert $\mathfrak{B}$-module (over $\mathfrak{B}$).

Indeed, $(b_1, \cdots, b_n)b = (b_1 b, \cdots, b_n b)$ and $\langle (a_j), (b_j) \rangle = \sum_{j=1}^n a_j^* b_j$ and then $\langle (a_j), (b_j)b \rangle = \langle (a_j), (b_j) \rangle b$; $\langle (b_j), (a_j) \rangle = \sum_{j=1}^n b_j^* a_j = (\sum_{j=1}^n a_j^* b_j)^* = \langle (a_j), (b_j) \rangle^*$; $\langle (a_j), (a_j) \rangle = \sum_{j=1}^n a_j^* a_j \geq 0$; $\|(a_j)\| = \| \sum_{j=1}^n a_j^* a_j \|^{\frac{1}{2}}$.

• Let $H_{\mathfrak{B}}$ be the **Hilbert space** over $\mathfrak{B}$, which is the completion of the direct sum of a countable number of copies of $\mathfrak{B}$, in the sense that $H_{\mathfrak{B}}$ consists of all sequences $(b_n)$ such that $\sum_{n=1}^\infty b_n^* b_n$ converges, with the inner product $\langle (a_n), (b_n) \rangle = \sum_{n=1}^\infty a_n^* b_n \in \mathfrak{B}$.

Check it out: $\langle (a_n), (b_n)c \rangle = \sum_{n=1}^\infty a_n^* b_n c = \langle (a_n), (b_n) \rangle c$; $\langle (b_n), (a_n) \rangle = \sum_{n=1}^\infty b_n^* a_n = (\sum_{n=1}^\infty a_n^* b_n)^* = \langle (a_n), (b_n) \rangle^*$; $\langle (a_n), (a_n) \rangle = \sum_{n=1}^\infty a_n^* a_n \geq 0$; $\|\langle (a_n), (a_n) \rangle\|^{\frac{1}{2}} = \| \sum_{n=1}^\infty a_n^* a_n \|^{\frac{1}{2}} \equiv \|(a_n)\|$.

Let $E$ be a Hilbert $\mathfrak{B}$-module. $\mathbb{B}(E)$ is the set of all module homomorphisms $T : E \to E$ for which there is an adjoint module homomorphism $T^* : E \to E$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in E$. $\mathbb{B}(E)$ is a $C^*$-algebra with respect to the operator norm, whose elements are bounded.

If $T$ is an adjointable map in that sense, then $T$ (and $T^*$) are modules maps and bounded. Indeed,

$$\begin{aligned} \langle T(x + \lambda y), z \rangle &= \langle x + \lambda y, T^*z \rangle = \langle x, T^*z \rangle + \lambda^* \langle y, T^*z \rangle \\ &= \langle Tx, z \rangle + \lambda^* \langle Ty, z \rangle = \langle Tx + \lambda Ty, z \rangle, \end{aligned}$$

so that $T(x + \lambda y) = T(x) + \lambda T(y)$, and also

$$\begin{aligned} \langle T(xb), y \rangle &= \langle xb, T^*y \rangle = \langle T^*y, xb \rangle^* = [\langle T^*y, x \rangle b]^* \\ &= [\langle y, T(x) \rangle b]^* = \langle y, T(x)b \rangle^* = \langle T(x)b, y \rangle, \end{aligned}$$

so that $T(xb) = T(x)b$.

Boundedness follows from that the graph of $T$ is closed. In fact, if $x_n \to x$ in $E$ and $Tx_n \to y$ in $E$, then for any $z \in E$,

$$\begin{aligned} 0 &= \langle x_n, T^*z\rangle - \langle x_n, T^*z\rangle \\ &= \langle Tx_n, z\rangle - \langle x_n, T^*z\rangle \\ &\to \langle y, z\rangle - \langle x, T^*z\rangle = \langle y - Tx, z\rangle = 0. \end{aligned}$$

Therefore, we get $Tx = y$.

The operator norm of $T$ is defined by

$$\|T\| = \sup\{\|Tx\| \mid x \in E, \|x\| \leq 1\}.$$

We see that the norm is submultiplicative and $\|T\| = \|T^*\|$, $\|T^*T\| = \|T\|^2$, and $\mathbb{B}(E)$ is complete.

$\mathbb{K}(E)$ is the closure of the linear spans of (rank one) operators on $E$ such as $\theta_{x,y}(z) = x\langle y, z\rangle$ for $x, y, z \in E$. $\mathbb{K}(E)$ is a closed ideal of $\mathbb{B}(E)$.

Check it out: $\theta_{x,y}(z_1 + z_2) = x\langle y, z_1\rangle + x\langle y, z_2\rangle = \theta_{x,y}(z_1) + \theta_{x,y}(z_2)$; $\theta_{x,y}(\alpha z) = xy^*(\alpha z) = \alpha x\langle y, z\rangle = \alpha\theta_{x,y}(z)$; $\theta_{x,y}(zb) = x\langle y, zb\rangle = x\langle y, z\rangle b = \theta_{x,y}(z)b$;

$$\begin{aligned} \langle \theta_{x,y}(z), w\rangle &= \langle x\langle y, z\rangle, w\rangle \\ &= \langle xy^*z, w\rangle = z^*yx^*w \\ &= \langle z, y\langle x, w\rangle\rangle = \langle z, \theta_{y,x}(w)\rangle, \end{aligned}$$

so that $\theta_{x,y}^* = \theta_{y,x} \in \mathbb{K}(E)$. Note that the maps $\theta_{x,y}$ are not projections, but $\theta_{x,x}^* = \theta_{x,x}$ self-adjoint and positive since

$$\langle \theta_{x,x}(z), z)\rangle = \langle xx^*z, z\rangle = z^*xx^*z = \langle z, x\rangle\langle z, x\rangle^* \geq 0.$$

Also,

$$\theta_{x,x} \circ \theta_{x,x}(z) = \theta_{x,x}(xx^*z) = xx^*xx^*z = \theta_{xx^*,xx^*}(z).$$

It certainly follows from this consideration that if $x$ is a projection, then the map $\theta_{x,x}$ is an idempotent, so that $\theta_{x,x}$ is also a projection.

With $T \in \mathbb{B}(E)$, we check that

$$\begin{aligned} T \circ \theta_{x,y}(z) &= T(x\langle y, z\rangle) = T(x)\langle y, z\rangle = \theta_{T(x),y}(z), \\ \theta_{x,y} \circ T(z) &= x\langle y, Tz\rangle = x\langle Tz, y\rangle^* = x\langle z, T^*y\rangle^* \\ &= x\langle T^*y, z\rangle = \theta_{x,T^*y}(z), \quad \text{and} \\ \theta_{x,y} \circ \theta_{u,v}(z) &= \theta_{x,y}(uv^*z) = xy^*uv^*z \\ &= \theta_{xy^*,vu^*}(z), \\ \|\theta_{x,y}(z)\| &= \|xy^*z\| \leq \|x\| \cdot \| y^* \| \cdot \| z\|, \end{aligned}$$

so that $\|\theta_{x,y}\| \leq \|x\| \cdot \|y\|$. It follows that $\mathbb{K}(E)$ is a two-sided ideal of $\mathbb{B}(E)$.

**Example 2.2.** • For any $C^*$-algebra $\mathfrak{B}$, we have $\mathbb{K}(\mathfrak{B}) \cong \mathfrak{B}$.

To show it we define a linear map $\Phi$ from finite sums of generators of $\mathbb{K}(\mathfrak{B})$ by

$$\Phi(\sum_k \lambda_k \theta_{a_k,b_k}) = \sum_k \lambda_k a_k b_k^* \in \mathfrak{A}.$$

This is well defined because if $\sum_k \lambda_k \theta_{a_k,b_k} = \sum_j \mu_j \theta_{c_j,d_j}$, then $\sum_k \lambda_k a_k b_k^* x = \sum_j \mu_j c_j d_j^* x$ for every $x \in \mathfrak{A}$. Thus,

$$\begin{aligned}\|\sum_k \lambda_k a_k b_k^* - \sum_j \mu_j c_j d_j^*\| &\le \|\sum_k \lambda_k a_k b_k^* - \sum_k \lambda_k a_k b_k^* u_s\| \\ &+ \|\sum_k \lambda_k a_k b_k^* u_s - \sum_j \mu_j c_j d_j^* u_s\| + \|\sum_j \mu_j c_j d_j^* u_s - \sum_j \mu_j c_j d_j^*\|\end{aligned}$$

goes to zero, with $(u_s)$ an approximate unit for $\mathfrak{B}$, a net of the positive part of the unit ball of $\mathfrak{B}$, with $\|b - bu_s\|$ and $\|b - u_s b\|$ going to zero for any $b \in \mathfrak{B}$. The map $\Phi$ is a $*$-homomorphism because

$$\theta_{a,b} \circ \theta_{c,d}(z) = \theta_{a,b}(cd^* z) = ab^* cd^* z = \theta_{ab^*,dc^*}(z)$$

and

$$\Phi(\theta_{a,b} \circ \theta_{c,d}) = ab^* cd^* = \Phi(\theta_{a,b})\Phi(\theta_{c,d}),$$

and

$$\Phi(\theta_{a,b}^*) = \Phi(\theta_{b,a}) = ba^* = (ab^*)^* = \Phi(\theta_{a,b})^*.$$

Surjectivity follows from the existence of a sort of polar decompositions in $\mathfrak{B}$: every $b \in \mathfrak{B}$ can be written as $b = u|b|^{\frac{1}{2}}$ for some $u \in \mathfrak{B}$. Hence, $b = \Phi(\theta_{u,|b|^{\frac{1}{2}}})$. The map $\Phi$ is an isometry. Indeed,

$$\begin{aligned}\|\sum_k \lambda_k \theta_{a_k,b_k}\| &= \sup_{\|x\|\le 1} \|\sum_k \lambda_k a_k b_k^* x\| \\ &\le \|\sum_k \lambda_k a_k b_k^*\| = \|\Phi(\sum_k \lambda_k \theta_{a_k,b_k})\|,\end{aligned}$$

and on the other hand,

$$\begin{aligned}\|\sum_k \lambda_k a_k b_k^*\| &\le \|\sum_k \lambda_k a_k b_k^* - \sum_k \lambda_k a_k b_k^* u_s\| + \|\sum_k \lambda_k a_k b_k^* u_s\| \\ &= \|\sum_k \lambda_k a_k b_k^* - \sum_k \lambda_k a_k b_k^* u_s\| + \|\sum_k \lambda_k \theta_{a_k,b_k}(u_s)\| \\ &\le \|\sum_k \lambda_k a_k b_k^* - \sum_k \lambda_k a_k b_k^* u_s\| + \|\sum_k \lambda_k \theta_{a_k,b_k}\|\end{aligned}$$

which converges to $\|\sum_k \lambda_k \theta_{a_k,b_k}\|$.

• If $\mathfrak{B}$ is a unital $C^*$-algebra, then $\mathbb{B}(\mathfrak{B}) \cong \mathfrak{B} \cong \mathbb{K}(\mathfrak{B})$ as well as $\mathfrak{B} \cong M(\mathfrak{B})$ the multiplier algebra of $\mathfrak{B}$.

Note that $\theta_{1,1}(z) = 1\langle 1, z\rangle = z = \mathrm{id}_{\mathfrak{B}}(z)$ the identitiy map on $\mathfrak{B}$ belongs to $\mathbb{K}(\mathfrak{B})$, so that $\mathbb{B}(\mathfrak{B}) = \mathbb{K}(\mathfrak{B})$. Also, the map from $\mathbb{B}(\mathfrak{B})$ to $\mathfrak{B}$ defined by $T \mapsto T(1)$ gives an isomorphism. Indeed, the map is injective because if $T(1) = S(1)$, then $T(z) = T(1)z = S(1)z = S(z)$. Also, $S \circ T(1) = S(T(1)) = S(1)T(1)$, and for any $b \in \mathfrak{B}$, we have $b = \theta_{b,1}(1)$ with $\theta_{b,1} \in \mathbb{B}(\mathfrak{B})$.

• As a remarkable theorem, for any Hilbert module $E$ over a $C^*$-algebra $\mathfrak{A}$, we have $\mathbb{B}(E)$ isomorphic to $M(\mathbb{K}(E))$. It follows that for any $C^*$-algebra $\mathfrak{A}$,

$$\mathbb{B}(\mathfrak{A}) \cong M(\mathbb{K}(\mathfrak{A})) \cong M(\mathfrak{A}).$$

Moreover,

$$\begin{aligned}\mathbb{B}(\oplus^n\mathfrak{A}) &\cong M(\mathbb{K}(\oplus^n\mathfrak{A})) \\ &\cong M(M_n(\mathfrak{A})) \cong M_n(M(\mathfrak{A})),\end{aligned}$$

where $M_n(\mathfrak{A})$ is the $n \times n$ matrix algebra over $\mathfrak{A}$.

Note that for $a = (a_j), b = (b_j), z = (z_j) \in \mathfrak{A}_n$,

$$\begin{aligned}\theta_{(a_j),(b_j)}(z_1, \cdots, z_n) &= a\langle b, z\rangle = a\sum_{k=1}^n b_k^* z_j = (a_j \sum_{k=1}^n b_k^* z_j)_{j=1}^n \\ &= \left[\begin{pmatrix} a_1 b_1^* & \cdots & a_1 b_n^* \\ \vdots & \ddots & \vdots \\ a_n b_1^* & \cdots & a_n b_n^* \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}\right]^t,\end{aligned}$$

where $[\cdot]^t$ means the transpose, and the equation implies an isomorphism between $\mathbb{K}(\mathfrak{A}^n)$ and $M_n(\mathfrak{A})$ by the argument as mentioned above.

• For the Hilbert module over a $C^*$-algebra $\mathfrak{A}$, we have

$$\mathbb{B}(H_{\mathfrak{A}}) \cong M(\mathbb{K}(H_{\mathfrak{A}})) \cong M(\mathfrak{A} \otimes \mathbb{K}).$$

Note that $\mathbb{K}(H_{\mathfrak{A}})$ is viewed as the closure of the union $\cup_{n=1}^\infty \mathbb{K}(\mathfrak{A}^n)$, and each $\mathbb{K}(\mathfrak{A}^n) \cong M_n(\mathfrak{A})$, and $\mathfrak{A} \otimes \mathbb{K}$ is viewed as the closure of the union $\cup_{n=1}^\infty M_n(\mathfrak{A})$. Thus, we have $\mathbb{K}(H_{\mathfrak{A}})$ isomorphic to $\mathfrak{A} \otimes \mathbb{K}$.

Let $E_1, E_2$ be Hilbert modules over $C^*$-algebras $\mathfrak{B}_1, \mathfrak{B}_2$ respectively, and $\varphi : \mathfrak{B}_1 \to \mathbb{B}(E_2)$ a $*$-homomorphism. The **tensor product** $E_1 \otimes_\varphi E_2$ is the completion of the algebraic tensor product $E_1 \odot_{\mathfrak{B}_1} E_2$ with $E_2$ as a left $\mathfrak{B}_1$-module via $\varphi$, with the product as a right $\mathfrak{B}_2$-module, with respect to the $\mathfrak{B}_2$-valued inner product

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_2, \varphi(\langle x_1, y_1 \rangle_1) y_2 \rangle_2 \in \mathfrak{B}_2,$$

where $\langle \cdot, \cdot \rangle_i$ is the $\mathfrak{B}_i$-valued inner product on $E_i$.

Check some out. For $\lambda \in \mathbb{C}$,

$$\begin{aligned}\langle \lambda(x_1 \otimes x_2), y_1 \otimes y_2 \rangle &= \langle (\lambda x_1) \otimes x_2, y_1 \otimes y_2 \rangle \\ &= \langle x_2, \varphi(\langle \lambda x_1, y_1 \rangle_1) y_2 \rangle_2 = \lambda^* \langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle;\end{aligned}$$

$$\begin{aligned}\langle x_1 \otimes x_2, \lambda(y_1 \otimes y_2) \rangle &= \langle (\lambda x_1) \otimes x_2, (\lambda y_1) \otimes y_2 \rangle \\ &= \langle x_2, \varphi(\langle x_1, \lambda y_1 \rangle_1) y_2 \rangle_2 = \lambda \langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle;\end{aligned}$$

and for $b \in \mathfrak{B}_2$,

$$\langle x_1 \otimes x_2, (y_1 \otimes y_2) b \rangle = \langle x_2, \varphi(\langle x_1, y_1 \rangle_1) y_2 b \rangle_2 = \langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle b;$$

and

$$\begin{aligned}\langle y_1 \otimes y_2, x_1 \otimes x_2 \rangle &= \langle y_2, \varphi(\langle y_1, x_1 \rangle_1) x_2 \rangle_2 = \langle y_2, \varphi(\langle x_1, y_1 \rangle_1^*) x_2 \rangle_2 \\ &= \langle y_2, \varphi(\langle x_1, y_1 \rangle_1)^* x_2 \rangle_2 = \langle \varphi(\langle x_1, y_1 \rangle_1) y_2, x_2 \rangle_2 \\ &= \langle x_2, \varphi(\langle x_1, y_1 \rangle_1) y_2 \rangle_2^* = \langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle^*;\end{aligned}$$

and

$$\begin{aligned}\langle x_1 \otimes x_2, x_1 \otimes x_2 \rangle &= \langle x_2, \varphi(\langle x_1, x_1 \rangle_1) x_2 \rangle_2 \\ &= \langle \varphi(\langle x_1, x_1 \rangle_1)^{\frac{1}{2}} x_2, \varphi(\langle x_1, x_1 \rangle_1)^{\frac{1}{2}} x_2 \rangle_2 \geq 0\end{aligned}$$

in $\mathfrak{B}_2$; and

$$\| x_1 \otimes x_2 \| = \| \langle \varphi(\langle x_1, x_1 \rangle_1)^{\frac{1}{2}} x_2, \varphi(\langle x_1, x_1 \rangle_1)^{\frac{1}{2}} x_2 \rangle_2 \|^{\frac{1}{2}}.$$

**Example 2.3.** • If $\varphi : \mathfrak{B}_1 \to \mathfrak{B}_2$ is a $*$-homomorphism of $C^*$-algebras, then $\mathfrak{B}_1 \otimes_\varphi \mathfrak{B}_2$ is isomorphic to the closed right ideal $\overline{\varphi(\mathfrak{B}_1)\mathfrak{B}_2}$ of $\mathfrak{B}_2$ generated by $\varphi(\mathfrak{B}_1)$, as a Hilbert module, where the overline means the norm closure.

Note that the simple tensor $xy \otimes z = x \otimes \varphi(y) z \in \mathfrak{B}_1 \otimes_\varphi \mathfrak{B}_2$, is mapped to $\varphi(xy)z = \varphi(x)\varphi(y)z$ in the closure, respectively. Also,

$$\begin{aligned}\| x \otimes y \| &= \| \langle x \otimes y, x \otimes y \rangle \|^{\frac{1}{2}} \\ &= \| y^* (\varphi(x^* x) y) \|^{\frac{1}{2}} = \| y^* \varphi(x)^* \varphi(x) y \|^{\frac{1}{2}} = \| \varphi(x) y \|.\end{aligned}$$

• If $\varphi : \mathfrak{B}_1 \to \mathfrak{B}_2$ is a unital $*$-homomorphism of unital $C^*$-algebras, of more generally, if $\varphi$ is **essential** in the sense that $\varphi(\mathfrak{B}_1)$ contains an approximate identity for $\mathfrak{B}_2$, then $\mathfrak{B}_1 \otimes_\varphi \mathfrak{B}_2$ may be identified with $\mathfrak{B}_2$, as a Hilbert module.

Because

$$\mathfrak{B}_1 \otimes_\varphi \mathfrak{B}_2 \cong \overline{\varphi(\mathfrak{B}_1)\mathfrak{B}_2} \cong \mathfrak{B}_2.$$

• If $\varphi : \mathbb{C} \to M(\mathfrak{B})$ is unital, then $H_\mathbb{C} \otimes_\varphi \mathfrak{B} \equiv H \otimes_\mathbb{C} \mathfrak{B} \cong H_\mathfrak{B}$.

Note that the element $(x_n)\lambda \otimes b = (x_n) \otimes \varphi(\lambda)b \in H_\mathbb{C} \otimes_\varphi \mathfrak{B}$ is mapped to $(x_n\lambda b) \in H_\mathfrak{B}$. Also.

$$\begin{aligned}
\|(\lambda_n) \otimes b\| &= \|\langle(\lambda_n) \otimes b, (\lambda_n) \otimes b\rangle\|^{\frac{1}{2}} \\
&= \|b^*\varphi(\langle(\lambda_n), (\lambda_n)\rangle)b\|^{\frac{1}{2}} \\
&= \|\sum_n \lambda_n^*\lambda_n b^*b\|^{\frac{1}{2}} \\
&= \|\sum_n (\lambda_n b)^*(\lambda_n b)\|^{\frac{1}{2}} \\
&= \|\langle(\lambda_n b), (\lambda_n b)\rangle\|^{\frac{1}{2}} = \|(\lambda_n b)\|.
\end{aligned}$$

• If $\varphi : \mathfrak{B}_1 \to \mathfrak{B}_2$ is essential, then we may identify $H_{\mathfrak{B}_1} \otimes_\varphi \mathfrak{B}_2$ with $H \otimes_\mathbb{C} \mathfrak{B}_2 \cong H_{\mathfrak{B}_2}$.

Because $H_{\mathfrak{B}_1} \otimes_\varphi \mathfrak{B}_2 \cong (H \otimes_\mathbb{C} \mathfrak{B}_1) \otimes_\varphi \mathfrak{B}_2 \cong H \otimes_\mathbb{C} (\mathfrak{B}_1 \otimes_\varphi \mathfrak{B}_2) \cong H \otimes_\mathbb{C} \mathfrak{B}_2$.

(**Stabilization or Absorption**). If $E$ is a countably generated Hilbert $\mathfrak{B}$-modules, then $E \oplus H_\mathfrak{B} \cong H_\mathfrak{B}$.

. As a corollary, if $E$ is a Hilbert $\mathfrak{B}$-module, then $E$ is countably generated if and only if $\mathbb{K}(E)$ has a strictly positive element.

# 3 Graded, $C^*$-algebras and Hilbert modules

Let $\mathfrak{A} = \mathfrak{A}^{(0)} \oplus \mathfrak{A}^{(1)}$ be a **graded** $C^*$-algebra such that each direct summand $\mathfrak{A}^{(j)}$ is a self-adjoint closed linear subspace and if $x \in \mathfrak{A}^{(j)}$, $y \in \mathfrak{A}^{(k)}$, then $xy \in \mathfrak{A}^{(j+k)}$ ($j+k$ mod 2). Set the degree $\partial x = j$ if $x \in \mathfrak{A}^{(j)}$. In particular,

$$\mathfrak{A}^{(0)}\mathfrak{A}^{(0)} \subset \mathfrak{A}^{(0)}, \quad \mathfrak{A}^{(1)}\mathfrak{A}^{(1)} \subset \mathfrak{A}^{(0)}$$

and $\mathfrak{A}^{(0)}\mathfrak{A}^{(1)} \subset \mathfrak{A}^{(1)}$ and $\mathfrak{A}^{(1)}\mathfrak{A}^{(0)} \subset \mathfrak{A}^{(1)}$.

If there is a self-adjoint unitary $g \in M(\mathfrak{A})$ such that $\mathfrak{A}^{(n)} = \{a \in \mathfrak{A} \mid gag = (-1)^n a\}$, then the grading is called **even** and $g$ is called a **grading operator** for the grading. If $\mathfrak{A}^{(1)} = \{0\}$, the grading is **trivial**. A trivial grading is even with the grading operator $1 \in M(\mathfrak{A})$.

A homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ of graded $C^*$-algebras is a graded homomorphism such that $\varphi(\mathfrak{A}^{(j)}) \subset \mathfrak{B}^{(j)}$ for $j = 0, 1$. Namely, as a diagonal map,

$$\varphi = \varphi^{(0)} \oplus \varphi^{(1)} : \mathfrak{A} = \mathfrak{A}^{(0)} \oplus \mathfrak{A}^{(1)} \to \mathfrak{B} = \mathfrak{B}^{(0)} \oplus \mathfrak{B}^{(1)}$$

A grading on a $C^*$-algebra $\mathfrak{A}$ is nothing but an action of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ on $\mathfrak{A}$, that is, an automorphism $\alpha$ of $\mathfrak{A}$ with period two $\alpha^2 = \mathrm{id}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$ the identity map on $\mathfrak{A}$. Then

$$\mathfrak{A} = \mathfrak{A}^{(0)} \oplus \mathfrak{A}^{(1)} = \{a \in \mathfrak{A} \mid \alpha(a) = a\} \oplus \{ a \in \mathfrak{A} \mid \alpha(a) = -a\}$$

with

$$a = (a^{(0)}, a^{(1)}) = \left( \frac{a + \alpha(a)}{2}, \frac{a - \alpha(a)}{2} \right).$$

Conversely, a grading on $\mathfrak{A}$ gives an $\mathbb{Z}_2$-action $\alpha$ defined by $\alpha(a^{(0)}, a^{(1)}) = (a^{(0)}, -a^{(1)})$. A grading is even if and only if the corresponding $\mathbb{Z}_2$-action is inner.

**Example 3.1.** • For $\mathfrak{A}$ any (ungraded) $C^*$-algebra, there is a grading on the $2 \times 2$ matrix algebra $M_2(\mathfrak{A})$ over $\mathfrak{A}$ such that

$$M_2(\mathfrak{A})^{(0)} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathfrak{A} \right\}, \quad M_2(\mathfrak{A})^{(1)} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \mid b, c \in \mathfrak{A} \right\}.$$

Note that

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix} = \begin{pmatrix} aa' & 0 \\ 0 & dd' \end{pmatrix} \in M_2(\mathfrak{A})^{(0)},$$
$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \begin{pmatrix} 0 & b' \\ c' & 0 \end{pmatrix} = \begin{pmatrix} bc' & 0 \\ 0 & cb' \end{pmatrix} \in M_2(\mathfrak{A})^{(0)},$$
$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & ab \\ dc & 0 \end{pmatrix} \in M_2(\mathfrak{A})^{(1)},$$
$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & bd \\ ca & 0 \end{pmatrix} \in M_2(\mathfrak{A})^{(1)}.$$

This is an even grading with grading operator $1 \oplus -1$ the diagonal sum, called the **standard even grading** on $M_2(\mathfrak{A})$. In fact,

$$(1 \oplus -1) \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} (1 \oplus -1) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad (1 \oplus -1) \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} (1 \oplus -1) = \begin{pmatrix} 0 & -b \\ -c & 0 \end{pmatrix}.$$

Since $\mathfrak{A} \otimes \mathbb{K} \cong M_2(\mathfrak{A} \otimes \mathbb{K})$, we obtan the standard even grading of $\mathfrak{A} \otimes \mathbb{K}$.

• For any (ungraded) $\mathfrak{A}$, there is the **standard odd grading** on the direct sum $\mathfrak{A} \oplus \mathfrak{A}$ such that $(\mathfrak{A} \oplus \mathfrak{A})^{(0)} = \{(a,a) \mid a \in \mathfrak{A}\}$ and $(\mathfrak{A} \oplus \mathfrak{A})^{(1)} = \{(a,-a) \mid a \in \mathfrak{A}\}$. If $\mathfrak{A} = \mathbb{C}$, there is the standard odd grading on $\mathbb{C}^2$, denoted by $\mathbb{C}_1$. Note that $\mathbb{C}_1$ is isomorphic to the group $C^*$-algebra $C^*(\mathbb{Z}_2)$ of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, and the grading is given by the dual action $\alpha^\wedge$ of the dual group $\mathbb{Z}_2^\wedge \cong \mathbb{Z}_2$ defined as $\alpha_\gamma^\wedge(g) = \langle g, \gamma \rangle g = \gamma(g)g$ for $\gamma \in \mathbb{Z}_2^\wedge$ and $g \in \mathbb{Z}_2$ viewed as a unitary in $C^*(\mathbb{Z}_2)$.

• A grading on a $C^*$-algebra $\mathfrak{A}$ induces a canonical grading on $M(\mathfrak{A})$ in the sense that

$$M(\mathfrak{A}) = M(\mathfrak{A})^{(0)} \oplus M(\mathfrak{A})^{(1)} = M(\mathfrak{A}^{(0)}) \oplus M(\mathfrak{A}^{(1)}).$$

For $\mathfrak{B}$ a graded $C^*$-algebra, a **graded Hilbert** $\mathfrak{B}$-module is a Hilbert $\mathfrak{B}$-module $E = E^{(0)} \oplus E^{(1)}$ such that $E^{(m)}\mathfrak{B}^{(n)} \subset E^{(m+n)}$ and $\langle E^{(m)}, E^{(n)} \rangle \subset \mathfrak{B}^{(m+n)}$. In particular, $E^{(m)}\mathfrak{B}^{(0)} \subset E^{(m)}$ stable under $\mathfrak{B}^{(0)}$, but $E^{(0)}\mathfrak{B}^{(1)} \subset E^{(1)}$ and $E^{(1)}\mathfrak{B}^{(1)} \subset E^{(0)}$.

**Example 3.2.** • Let $\mathfrak{B} = \mathfrak{B}^{(0)} \oplus \mathfrak{B}^{(1)}$ be a graded $C^*$-algebra. Then

$$\mathfrak{B}^n = [\mathfrak{B}^n]^{(0)} \oplus [\mathfrak{B}^n]^{(1)} = [\oplus^n \mathfrak{B}^{(0)}] \oplus [\oplus^n \mathfrak{B}^{(1)}].$$

Moreover,

$$H_{\mathfrak{B}} = H_{\mathfrak{B}}^{(0)} \oplus H_{\mathfrak{B}}^{(1)} = H_{\mathfrak{B}^{(0)}} \oplus H_{\mathfrak{B}^{(1)}}.$$

Note that

$$\begin{aligned}
&\langle (a_n^{(0)}, a_n^{(1)})_{n=1}^\infty, (b_n^{(0)}, b_n^{(1)})_{n=1}^\infty \rangle \\
&= \sum_{n=1}^\infty (a_n^{(0)}, a_n^{(1)})^* (b_n^{(0)}, b_n^{(1)}) \\
&= \sum_{n=1}^\infty ((a_n^{(0)})^* b_n^{(0)}, (a_n^{(1)})^* b_n^{(1)}) \\
&= \sum_{n=1}^\infty (a_n^{(0)})^* b_n^{(0)} \oplus \sum_{n=1}^\infty (a_n^{(1)})^* b_n^{(1)} \\
&= \langle (a_n^{(0)})_{n=1}^\infty, (b_n^{(0)})_{n=1}^\infty \rangle \oplus \langle (a_n^{(1)})_{n=1}^\infty, (b_n^{(1)})_{n=1}^\infty \rangle \in \mathfrak{B}^{(0)} \oplus \mathfrak{B}^{(1)}.
\end{aligned}$$

• A grading on $E$ induces a grading on $\mathbb{B}(E)$ and $\mathbb{K}(E)$ as:

$$\begin{aligned}
\mathbb{B}(E) &= \mathbb{B}(E)^{(0)} \oplus \mathbb{B}(E)^{(1)} \\
&= [\mathbb{B}(E^{(0)}) \oplus \mathbb{B}(E^{(1)})] \oplus [\mathbb{B}(E^{(1)}, E^{(0)}) \oplus \mathbb{B}(E^{(0)}, E^{(1)})] \\
&\cong \begin{pmatrix} \mathbb{B}(E^{(0)}) & \mathbb{B}(E^{(1)}, E^{(0)}) \\ \mathbb{B}(E^{(0)}, E^{(1)}) & \mathbb{B}(E^{(1)}) \end{pmatrix} \\
&\equiv \begin{pmatrix} \mathbb{B}(E)_{00} & \mathbb{B}(E)_{10} \\ \mathbb{B}(E)_{01} & \mathbb{B}(E)_{11} \end{pmatrix},
\end{aligned}$$

and for $T \in \mathbb{B}(E)$,

$$
\begin{aligned}
T &= \begin{pmatrix} T_{00} & T_{10} \\ T_{01} & T_{11} \end{pmatrix} : E = E^{(0)} \oplus E^{(1)} \to E = E^{(0)} \oplus E^{(1)}. \\
&= [(T_{00}, T_{11}] \oplus [(T_{10}, T_{01})].
\end{aligned}
$$

As for $T_{jk} \in \mathbb{B}(E^{(j)}, E^{(k)})$, note that

$$
\begin{aligned}
&\langle T(x_0, x_1), (y_0, y_1) \rangle = \langle (x_0, x_1), T^*(y_0, y_1) \rangle \\
&\Leftrightarrow \langle \begin{pmatrix} T_{00} & T_{10} \\ T_{01} & T_{11} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \rangle = \langle \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \begin{pmatrix} T_{00}^* & T_{01}^* \\ T_{10}^* & T_{11}^* \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \rangle \\
&\Leftrightarrow \langle \begin{pmatrix} T_{00}x_0 + T_{10}x_1 \\ T_{01}x_0 + T_{11}x_1 \end{pmatrix}, \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \rangle = \langle \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \begin{pmatrix} T_{00}^* y_0 + T_{01}^* y_1 \\ T_{10}^* y_0 + T_{11}^* y_1 \end{pmatrix} \rangle \\
&\Leftrightarrow \langle T_{00}x_0, y_0 \rangle = \langle x_0, T_{00}^* y_0 \rangle, \quad \langle T_{10}x_1, y_0 \rangle = \langle x_0, T_{01}^* y_1 \rangle, \\
&\quad \langle T_{01}x_0, y_1 \rangle = \langle x_1, T_{10}^* y_0 \rangle, \quad \langle T_{11}x_1, y_1 \rangle = \langle x_1, T_{11}^* y_1 \rangle,
\end{aligned}
$$

so that $T_{jk}^* \in \mathbb{B}(E^{(k)}, E^{(j)})$ in this sense and adjointableness for $\mathbb{B}(E^{(j)}, E^{(k)})$ should be defined together with $\mathbb{B}(E^{(k)}, E^{(j)})$. Similarly, $\mathbb{K}(E)$ is understood as well. Note also that for $x_j \in E^{(j)}$ and $y_{j+1} \in E^{(j+1)}$, $\theta_{y_{j+1}, x_j} \in \mathbb{K}(E^{(j)}, E^{(j+1)})$ is defined by $\theta_{y_{j+1}, x_j}(z_j) = y_{j+1}\langle x_j, z_j \rangle \in E^{(j+1)}$, and $\mathbb{K}(E^{(j)}, E^{(j+1)})$ should be defined as the closure of linear spans of such operators $\theta_{y,x}$.

• In particular, if $\mathfrak{A}$ is a graded $C^*$-algebra, then

$$
\begin{aligned}
\mathbb{K}(\mathfrak{A}) &= \begin{pmatrix} \mathbb{K}(\mathfrak{A}^{(0)}) & \mathbb{K}(\mathfrak{A}^{(1)}, \mathfrak{A}^{(0)}) \\ \mathbb{K}(\mathfrak{A}^{(0)}, \mathfrak{A}^{(1)}) & \mathbb{K}(\mathfrak{A}^{(1)}) \end{pmatrix} \\
&\cong \begin{pmatrix} \mathfrak{A}^{(0)} & \mathbb{K}(\mathfrak{A}^{(1)}, \mathfrak{A}^{(0)}) \\ \mathbb{K}(\mathfrak{A}^{(0)}, \mathfrak{A}^{(1)}) & \mathfrak{A}^{(1)} \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{B}(\mathfrak{A}) &= \begin{pmatrix} \mathbb{B}(\mathfrak{A}^{(0)}) & \mathbb{B}(\mathfrak{A}^{(1)}, \mathfrak{A}^{(0)}) \\ \mathbb{B}(\mathfrak{A}^{(0)}, \mathfrak{A}^{(1)}) & \mathbb{B}(\mathfrak{A}^{(1)}) \end{pmatrix} \\
&\cong \begin{pmatrix} M(\mathfrak{A}^{(0)}) & \mathbb{B}(\mathfrak{A}^{(1)}, \mathfrak{A}^{(0)}) \\ \mathbb{B}(\mathfrak{A}^{(0)}, \mathfrak{A}^{(1)}) & M(\mathfrak{A}^{(1)}) \end{pmatrix}.
\end{aligned}
$$

• For $E = E^{(0)} \oplus E^{(1)}$ a graded Hilbert $\mathfrak{B}$-module, we denote by $E^{op} = E^{(1)} \oplus E^{(0)}$ the **opposite** of $E$, also a graded Hilbert $\mathfrak{B}$-module.

Define $H_{\mathfrak{B}}^{\wedge} = H_{\mathfrak{B}} \oplus H_{\mathfrak{B}}^{op}$. $H_{\mathfrak{B}}^{\wedge}$ is isomorphic to $H_{\mathfrak{B}}$ as a Hilbert $\mathfrak{B}$-module, but not in general as a graded Hilbert $\mathfrak{B}$-module.

If $H_{\mathfrak{B}} = H_{\mathfrak{B}} \oplus \{0\}$ is trivially graded, then we say $H_{\mathfrak{B}}^{\wedge}$ has the standard even grading. Then $H_{\mathfrak{B}}^{\wedge} = (H_{\mathfrak{B}} \oplus \{0\}) \oplus (\{0\} \oplus H_{\mathfrak{B}}) \cong H_{\mathfrak{B}} \oplus H_{\mathfrak{B}}$ the usual direct sum, isomorphic to $H_{\mathfrak{B}}$.

We may write $E^{\wedge} = E \oplus E^{op}$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be graded $C^*$-algebras. Let $\mathfrak{A} \otimes \mathfrak{B}$ denote the (minimal) **graded tensor product** of $\mathfrak{A}$ and $\mathfrak{B}$, which is the same symbol as the usual (minimal) tensor product of $C^*$-algebras and is obtained as the (minimal) completion of the algebraic tensor product of $\mathfrak{A}$ and $\mathfrak{B}$ with the product and the involution given by

$$(a_1^{(i_1)} \otimes b_1^{(k_1)})(a_2^{(i_2)} \otimes b_2^{(k_2)}) = (-1)^{k_1 i_2}(a_1^{(i_1)} a_2^{(i_2)} \otimes b_1^{(k_1)} b_2^{(k_2)}),$$
$$(a^{(i)} \otimes b^{(k)})^* = (-1)^{i \cdot k}((a^{(i)})^* \otimes (b^{(k)})^*),$$

where $\partial(a^{(i)}) = i$ and $a^{(i)} \in \mathfrak{A}^{(i)}$, and with the degree $\partial(a^{(i)} \otimes b^{(k)}) = i + k$.

If $E_1$ and $E_2$ are graded Hilbert modules over $\mathfrak{A}$ and $\mathfrak{B}$ respectively, and $\varphi$ is a graded $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, we define the **graded tensor product** $E_1 \otimes_{\varphi} E_2$ to be the ordinary tensor product (of the same symbol) with grading $\partial(x^{(i)} \otimes y^{(k)}) = i + k$ for $x^{(i)} \in E^{(i)}$ and $y^{(k)} \in E^{(k)}$,

**Example 3.3.** • If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is an essential graded homomorphism of $C^*$-algebras, then $\mathfrak{A} \otimes_{\varphi} \mathfrak{B} \cong \mathfrak{B}$, $H_{\mathfrak{A}} \otimes_{\varphi} \mathfrak{B} \cong H_{\mathfrak{B}}$, and $H_{\mathfrak{A}}^{\wedge} \otimes_{\varphi} \mathfrak{B} \cong H_{\mathfrak{B}}^{\wedge}$ as graded Hilbert $\mathfrak{B}$-modules.

Because, especially,

$$\begin{aligned}
&\mathfrak{A} \otimes_{\varphi} \mathfrak{B} = (\mathfrak{A}^{(0)} \oplus \mathfrak{A}^{(1)}) \otimes_{\varphi} (\mathfrak{B}^{(0)} \oplus \mathfrak{B}^{(1)}) \\
&\cong \{[\mathfrak{A}^{(0)} \otimes_{\varphi} \mathfrak{B}^{(0)}] \oplus [\mathfrak{A}^{(0)} \otimes_{\varphi} \mathfrak{B}^{(1)}]\} \oplus \{[\mathfrak{A}^{(1)} \otimes_{\varphi} \mathfrak{B}^{(0)}] \oplus [\mathfrak{A}^{(1)} \otimes_{\varphi} \mathfrak{B}^{(1)}]\} \\
&\cong \{\mathfrak{B}^{(0)} \oplus \{0\}\} \oplus \{\{0\} \oplus \mathfrak{B}^{(1)}\} \cong \mathfrak{B}
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\mathfrak{A}^{op} \otimes_{\varphi} \mathfrak{B} = (\mathfrak{A}^{(1)} \oplus \mathfrak{A}^{(0)}) \otimes_{\varphi} (\mathfrak{B}^{(0)} \oplus \mathfrak{B}^{(1)}) \\
&\cong \{[\mathfrak{A}^{(1)} \otimes_{\varphi} \mathfrak{B}^{(0)}] \oplus [\mathfrak{A}^{(1)} \otimes_{\varphi} \mathfrak{B}^{(1)}]\} \oplus \{[\mathfrak{A}^{(0)} \otimes_{\varphi} \mathfrak{B}^{(0)}] \oplus [\mathfrak{A}^{(0)} \otimes_{\varphi} \mathfrak{B}^{(1)}]\} \\
&\cong \{\mathfrak{B}^{(1)} \oplus \{0\}\} \oplus \{\{0\} \oplus \mathfrak{B}^{(0)}\} \cong \mathfrak{B}^{op}
\end{aligned}$$

Also,

$$H_{\mathfrak{A}} \otimes_{\varphi} \mathfrak{B} \equiv \mathfrak{A}^{\infty} \otimes_{\varphi} \mathfrak{B} \cong \mathfrak{B}^{\infty} \equiv H_{\mathfrak{B}},$$
$$H_{\mathfrak{A}}^{op} \otimes_{\varphi} \mathfrak{B} \equiv (\mathfrak{A}^{op})^{\infty} \otimes_{\varphi} \mathfrak{B} \cong (\mathfrak{B}^{op})^{\infty} = H_{\mathfrak{B}^{op}} \cong H_{\mathfrak{B}}^{op},$$

so that $H_{\mathfrak{A}}^{\wedge} \otimes_{\varphi} \mathfrak{B} \cong H_{\mathfrak{B}}^{\wedge}$.

• If $E_1$ is a Hilbert $\mathfrak{A}$-module and $E_2$ is a Hilbert $\mathfrak{B}$-module, and if $\varphi : \mathfrak{A} \to \mathbb{B}(E_2)$ is a graded homomorphism, then $E_1^{op} \otimes_\varphi E_2 \cong E_1 \otimes_\varphi E_2^{op} \cong (E_1 \otimes_\varphi E_2)^{op}$ and $E_1^{op} \otimes_\varphi E_2^{op} \cong E_1 \otimes_\varphi E_2$ as graded Hilbert $\mathfrak{B}$-modules.

Indeed,

$$\begin{aligned}
&E_1^{op} \otimes_\varphi E_2 = (E_1^{(1)} \oplus E_1^{(0)}) \otimes_\varphi (E_2^{(0)} \oplus E_2^{(1)}) \\
&\cong \{[E_1^{(1)} \otimes_\varphi E_2^{(0)}] \oplus [E_1^{(1)} \otimes_\varphi E_2^{(1)}]\} \oplus \{[E_1^{(0)} \otimes_\varphi E_2^{(0)}] \oplus [E_1^{(0)} \otimes_\varphi E_2^{(1)}]\} \\
&\cong \{[E_1^{(1)} \otimes_\varphi E_2^{(1)}] \oplus [E_1^{(1)} \otimes_\varphi E_2^{(0)}]\} \oplus \{[E_1^{(0)} \otimes_\varphi E_2^{(1)}] \oplus [E_1^{(0)} \otimes_\varphi E_2^{(0)}]\} \\
&\cong \{[E_1^{(0)} \otimes_\varphi E_2^{(1)}] \oplus [E_1^{(0)} \otimes_\varphi E_2^{(0)}]\} \oplus \{[E_1^{(1)} \otimes_\varphi E_2^{(1)}] \oplus [E_1^{(1)} \otimes_\varphi E_2^{(0)}]\} \\
&\cong E_1 \otimes_\varphi E_2^{op},
\end{aligned}$$

where we use isomorphisms as Hilbert modules by interchanging direct sum components without breaking grading structure. Since we have

$$\begin{aligned}
&E_1 \otimes_\varphi E_2 = [E_1 \otimes_\varphi E_2]^{(0)} \oplus [E_1 \otimes_\varphi E_2]^{(1)} \\
&= [(E_1^{(0)} \otimes_\varphi E_2^{(0)}) \oplus (E_1^{(1)} \otimes_\varphi E_2^{(1)})] \oplus [(E_1^{(1)} \otimes_\varphi E_2^{(0)}) \oplus (E_1^{(0)} \otimes_\varphi E_2^{(1)})],
\end{aligned}$$

then we have

$$\begin{aligned}
&(E_1 \otimes_\varphi E_2)^{op} = [E_1 \otimes_\varphi E_2]^{(1)} \oplus [E_1 \otimes_\varphi E_2]^{(0)} \\
&= [(E_1^{(1)} \otimes_\varphi E_2^{(0)}) \oplus (E_1^{(0)} \otimes_\varphi E_2^{(1)})] \oplus [(E_1^{(0)} \otimes_\varphi E_2^{(0)}) \oplus (E_1^{(1)} \otimes_\varphi E_2^{(1)})] \\
&\cong [(E_1^{(1)} \otimes_\varphi E_2^{(0)}) \oplus (E_1^{(1)} \otimes_\varphi E_2^{(1)})] \oplus [(E_1^{(0)} \otimes_\varphi E_2^{(0)}) \oplus (E_1^{(0)} \otimes_\varphi E_2^{(1)})] \\
&\cong [E_1^{(1)} \otimes_\varphi E_2] \oplus [E_1^{(0)} \otimes_\varphi E_2] \cong E_1^{op} \otimes_\varphi E_2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
E_1 \otimes_\varphi E_2 &\cong ((E_1 \otimes_\varphi E_2)^{op})^{op} \\
&\cong (E_1^{op} \otimes_\varphi E_2)^{op} \cong (E_1 \otimes_\varphi E_2^{op})^{op} \cong E_1^{op} \otimes_\varphi E_2^{op}.
\end{aligned}$$

**Example 3.4.** • If $\mathfrak{A}$ is evenly graded and $\mathfrak{B}$ is trivially graded, then the graded $\mathfrak{A} \otimes \mathfrak{B}$ is the usual tensor product.

Because $\mathfrak{B}^{(1)} = 0$.

• If $\mathfrak{A}$ and $\mathfrak{B}$ are both evenly graded with grading operators $g$ and $h$ respectively, then $\mathfrak{A} \otimes \mathfrak{B}$ is evenly graded with grading operator $g \otimes h$.

• If $\mathfrak{A}$ is evenly graded and $M_2(\mathbb{C})$ has the standard even grading, then $\mathfrak{A} \otimes M_2(\mathbb{C}) \cong M_2(\mathfrak{A})$ with the standard even grading. In addition, if $\mathbb{K}$ has the standard even grading, then

$$\mathfrak{A} \otimes \mathbb{K} \cong \mathfrak{A} \otimes M_2(\mathbb{K}) \cong \mathfrak{A} \otimes (M_2(\mathbb{C}) \otimes \mathbb{K}) \cong (\mathfrak{A} \otimes \mathbb{K}) \otimes M_2(\mathbb{C}) \cong M_2(\mathfrak{A} \otimes \mathbb{K})$$

with the standard even grading.

• If $\mathfrak{A}$ is evenly graded, then $\mathfrak{A} \otimes \mathbb{C}_1 \cong \mathfrak{A} \oplus \mathfrak{A}$ with the standard odd grading.

Note that

$$\begin{aligned}
&(a^{(0)}, a^{(1)}) \otimes ((s, s) \oplus (t, -t)) \\
&= [(a^{(0)}, a^{(1)}) \otimes (s, s)] \oplus [(a^{(0)}, a^{(1)}) \otimes (t, -t)] \\
&= [(sa^{(0)}, sa^{(1)}) \oplus (sa^{(0)}, sa^{(1)})] \oplus [(ta^{(0)}, ta^{(1)}) \oplus (-ta^{(0)}, -ta^{(1)})] \\
&\in (\mathfrak{A} \oplus \mathfrak{A})^{(0)} \oplus (\mathfrak{A} \oplus \mathfrak{A})^{(1)}.
\end{aligned}$$

• If $\mathfrak{A}$ is a graded $C^*$-algebra with an action $\alpha$ of $\mathbb{Z}_2$, then $\mathfrak{A} \otimes \mathbb{C}_1 \cong \mathfrak{A} \rtimes_\alpha \mathbb{Z}_2$ the crossed product with the grading corresponding to $\alpha \otimes \alpha^\wedge$ (corrected) with $\alpha^\wedge$ the dual action of $\mathbb{Z}_2^\wedge \cong \mathbb{Z}_2$.

• $\mathbb{C}_1 \otimes \mathbb{C}_1 \cong M_2(\mathbb{C})$ with the standard even grading.

Takai duality implies that $\mathbb{C} \rtimes_\alpha \mathbb{Z}_2 \rtimes_{\alpha^\wedge} \mathbb{Z}_2 \cong \mathbb{C} \otimes M_2(\mathbb{C}) = M_2(\mathbb{C})$, with $\mathbb{C} \rtimes_\alpha \mathbb{Z}_2 \cong \mathbb{C} \otimes \mathbb{C}_1 \cong \mathbb{C}_1$ and $\mathbb{C}_1 \rtimes_{\alpha^\wedge} \mathbb{Z}_2 \cong \mathbb{C}_1 \otimes \mathbb{C}_1$.

(**Stabilization**). Let $\mathfrak{B}$ be a graded $C^*$-algebra and $E$ a countably generated graded Hilbert $\mathfrak{B}$-module. Then $E \oplus H_{\mathfrak{B}}^\wedge \cong H_{\mathfrak{B}}^\wedge$ as graded Hilbert $\mathfrak{B}$-modules with natural grading.

# 4 Kasparov modules and their KK-theory groups

Let $\mathfrak{A}, \mathfrak{B}$ be graded $C^*$-algebras. $\mathbb{E}(\mathfrak{A}, \mathfrak{B})$ is the set of all triples $(E, \varphi, F)$, where $E$ is a countably generated, graded Hilbert module over $\mathfrak{B}$, $\varphi$ is a graded $*$-homomorphism from $\mathfrak{A}$ to $\mathbb{B}(E)$, and $F$ is an operator in $\mathbb{B}(E)$ of degree 1, such that the (additive) commutators $[F, \varphi(a)]$, and $(F^2 - 1)\varphi(a)$, and $(F - F^*)\varphi(a)$ are all in $\mathbb{K}(E)$ for all $a \in \mathfrak{A}$. The elements of $\mathbb{E}(\mathfrak{A}, \mathfrak{B})$ are called **Kasparov modules** for $(\mathfrak{A}, \mathfrak{B})$. $\mathbb{D}(\mathfrak{A}, \mathfrak{B})$ is the set of triples in $\mathbb{E}(\mathfrak{A}, \mathfrak{B})$ such that $[F, \varphi(a)]$, $(F^2 - 1)\varphi(a)$, and $(F - F^*)\varphi(a)$ are zero for all $a \in \mathfrak{A}$. The elements of $\mathbb{D}(\mathfrak{A}, \mathfrak{B})$ are called **degenerate** Kasparov modules for $(\mathfrak{A}, \mathfrak{B})$.

Notes: $\varphi = \varphi^{(0)} \oplus \varphi^{(1)} : \mathfrak{A} = \mathfrak{A}^{(0)} \oplus \mathfrak{A}^{(1)} \to \mathbb{B}(E)^{(0)} \oplus \mathbb{B}(E)^{(1)}$ with

$\varphi^{(j)}\mathfrak{A}^{(j)} \subset \mathbb{B}(E)^{(j)}$ for $j = 0, 1$, and $F$ is in $\mathbb{B}(E)^{(1)}$. Also, in general,

$$\begin{aligned}
[F, \varphi(a)] &= \begin{pmatrix} 0 & F_{12} \\ F_{21} & 0 \end{pmatrix} \begin{pmatrix} \varphi^{(0)}(a^{(0)})_{11} & \varphi^{(1)}(a^{(1)})_{12} \\ \varphi^{(1)}(a^{(1)})_{21} & \varphi^{(0)}(a^{(0)})_{22} \end{pmatrix} \\
&- \begin{pmatrix} \varphi^{(0)}(a^{(0)})_{11} & \varphi^{(1)}(a^{(1)})_{12} \\ \varphi^{(1)}(a^{(1)})_{21} & \varphi^{(0)}(a^{(0)})_{22} \end{pmatrix} \begin{pmatrix} 0 & F_{12} \\ F_{21} & 0 \end{pmatrix} \\
&= \begin{pmatrix} F_{12}\varphi^{(1)}(a^{(1)})_{21} & F_{12}\varphi^{(0)}(a^{(0)})_{22} \\ F_{21}\varphi^{(0)}(a^{(0)})_{11} & F_{21}\varphi^{(1)}(a^{(1)})_{12} \end{pmatrix} \\
&- \begin{pmatrix} \varphi^{(1)}(a^{(1)})_{12}F_{21} & \varphi^{(0)}(a^{(0)})_{11}F_{12} \\ \varphi^{(0)}(a^{(0)})_{22}F_{21} & \varphi^{(1)}(a^{(1)})_{21}F_{12} \end{pmatrix} \\
&= \begin{pmatrix} F_{12}\varphi^{(1)}(a^{(1)})_{21} - \varphi^{(1)}(a^{(1)})_{12}F_{21} & F_{12}\varphi^{(0)}(a^{(0)})_{22} - \varphi^{(0)}(a^{(0)})_{11}F_{12} \\ F_{21}\varphi^{(0)}(a^{(0)})_{11} - \varphi^{(0)}(a^{(0)})_{22}F_{21} & F_{21}\varphi^{(1)}(a^{(1)})_{12} - \varphi^{(1)}(a^{(1)})_{21}F_{12} \end{pmatrix};
\end{aligned}$$

$$\begin{aligned}
(F^2 - 1)\varphi(a) &= \begin{pmatrix} F_{12}F_{21} - 1 & 0 \\ 0 & F_{21}F_{12} - 1 \end{pmatrix} \varphi(a) \\
&= \begin{pmatrix} (F_{12}F_{21} - 1)\varphi^{(0)}(a^{(0)})_{11} & (F_{12}F_{21} - 1)\varphi^{(1)}(a^{(1)})_{12} \\ (F_{21}F_{12} - 1)\varphi^{(1)}(a^{(1)})_{21} & (F_{21}F_{12} - 1)\varphi^{(0)}(a^{(0)})_{22} \end{pmatrix};
\end{aligned}$$

$$\begin{aligned}
(F - F^*)\varphi(a) &= \begin{pmatrix} 0 & F_{12} - F_{21}^* \\ F_{21} - F_{12}^* & 0 \end{pmatrix} \varphi(a) \\
&= \begin{pmatrix} (F_{12} - F_{21}^*)\varphi^{(1)}(a^{(1)})_{21} & (F_{12} - F_{21}^*)\varphi^{(0)}(a^{(0)})_{22} \\ (F_{21} - F_{12}^*)\varphi^{(0)}(a^{(0)})_{11} & (F_{21} - F_{12}^*)\varphi^{(1)}(a^{(1)})_{12} \end{pmatrix}.
\end{aligned}$$

Let $q : \mathbb{B}(E) \to \mathbb{B}(E)/\mathbb{K}(E) \equiv Q(E)$ be the canonical quotient homomorphism. Note that

$$\begin{aligned}
&\mathbb{B}(E)/\mathbb{K}(E) = [\mathbb{B}(E)/\mathbb{K}(E)]^{(0)} \oplus [\mathbb{B}(E)/\mathbb{K}(E)]^{(1)} \\
&\cong [Q(E^{(0)}) \oplus Q(E^{(1)})] \oplus [Q(E^{(1)}, E^{(0)}) \oplus Q(E^{(0)}, E^{(1)})] \\
&\cong \begin{pmatrix} Q(E^{(0)}) & Q(E^{(1)}, E^{(0)}) \\ Q(E^{(0)}, E^{(1)}) & Q(E^{(1)}) \end{pmatrix} \equiv \begin{pmatrix} Q(E)_{00} & Q(E)_{10} \\ Q(E)_{01} & Q(E)_{11} \end{pmatrix} = Q(E),
\end{aligned}$$

where we let $Q(E^{(j)}) = \mathbb{B}(E^{(j)})/\mathbb{K}(E^{(j)})$ for $j = 0, 1$ and $Q(E^{(j)}, E^{(k)}) = \mathbb{B}(E^{(j)}, E^{(k)})/\mathbb{B}(E^{(j)}, E^{(k)})$. Hence we may let

$$q = \begin{pmatrix} q_{00} & q_{10} \\ q_{01} & q_{11} \end{pmatrix} : \mathbb{B}(E) = \begin{pmatrix} \mathbb{B}(E)_{00} & \mathbb{B}(E)_{10} \\ \mathbb{B}(E)_{01} & \mathbb{B}(E)_{11} \end{pmatrix} \to Q(E).$$

This our indexing might be new and more useful than the usual one. Anyhow, we have

$$\begin{aligned} q(F)q(\varphi(a)) - q(\varphi(a))q(F) &= 0 \in Q(E), \\ (q(F)^2 - q(1))q(\varphi(a)) &= 0 \in Q(E), \\ (q(F) - q(F)^*)q(\varphi(a)) &= 0 \in Q(E), \end{aligned}$$

where we can write down their components by using $q_{ij}$ and the computation above, but better to be omitted. Importantly, those equations say that $F$ and each $\varphi(a)$ commute mod compact parts and if $\varphi$ is unital or essential, then $F$ is a self-adjoint unitary mod the same.

Two triples $(E_0, \varphi_0, F_0), (E_1, \varphi_1, F_1)$ of $\mathbb{E}(\mathfrak{A}, \mathfrak{B})$ are **unitarily equivalent** if there is a unitary, say $U$, in $\mathbb{B}(E_0, E_1)$, of degree zero (i.e., a diagonal sum of unitaries on graded $E_0$ to $E_1$), intertwining $\varphi_i$ and $F_i$;

$$\begin{array}{ccc} E_0 = E_0^{(0)} \oplus E_0^{(1)} & \xrightarrow{U} & E_1 = E_1^{(0)} \oplus E_1^{(1)} \\ {\scriptstyle \varphi_0(a)}\downarrow & & \downarrow{\scriptstyle \varphi_1(a)} \\ E_0 & \xrightarrow{U} & E_1 \\ E_0 & \xrightarrow{U} & E_1 \\ {\scriptstyle F_0}\downarrow & & \downarrow{\scriptstyle F_1} \\ E_0 & \xrightarrow{U} & E_1 \end{array}$$

for $a \in \mathfrak{A}$. Its unitary equivalence is denoted as $\approx_u$.

A **homotopy** connecting $(E_0, \varphi_0, F_0), (E_1, \varphi_1, F_1)$ is an element $(E, \varphi, F)$ of $\mathbb{E}(\mathfrak{A}, I\mathfrak{B})$ such that $(E \otimes_{f_i} \mathfrak{B}, f_i \circ \varphi, f_{i*}(F)) \approx_u (E_i, \varphi_i, F_i)$, where $f_i$ for $i = 0, 1$ is the evaluation homomorphism from $I\mathfrak{B} = C([0,1], \mathfrak{B}) \cong C([0,1]) \otimes \mathfrak{B}$ to $\mathfrak{B}$ (at $0, 1$ respectively):

$$\begin{array}{ccccc} E_{I\mathfrak{B}} & \xrightarrow{f_i} & E \otimes_{f_i} \mathfrak{B} & \xrightarrow{U_i} & E_i \\ {\scriptstyle \varphi(a)}\downarrow & & \downarrow{\scriptstyle f_i \circ \varphi(a)} & & \downarrow{\scriptstyle \varphi_i(a)} \\ E_{I\mathfrak{B}} & \xrightarrow{f_i} & E \otimes_{f_i} \mathfrak{B} & \xrightarrow{U_i} & E_i \\ E_{I\mathfrak{B}} & \xrightarrow{f_i} & E \otimes_{f_i} \mathfrak{B} & \xrightarrow{U_i} & E_i \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f_{i,*}(F)} & & \downarrow{\scriptstyle F_i} \\ E_{I\mathfrak{B}} & \xrightarrow{f_i} & E \otimes_{f_i} \mathfrak{B} & \xrightarrow{U_i} & E_i \end{array}$$

for $a \in \mathfrak{A}$, where the diagrams commute, and the middle down arrows are defined to be that the left squares are commutative, and $E_{I\mathfrak{B}} = E$ over $I\mathfrak{B}$, and $U_i$ are the implementing unitaries of the unitary equivalences $\approx_u$.

The homotopy respects direct sums. The **homotopy equivalence** on $\mathbb{E}(\mathfrak{A}, \mathfrak{B})$ is denoted by $\sim_h$.

If $E_0 = E_1$, a **standard homotopy** is a homotopy of the form $E = C([0,1], E_0)$, $\varphi = (\varphi_t)$, and $F = (F_t)$, where $[0,1] \ni t \mapsto F_t$ and $t \mapsto \varphi_t(a)$ are strong $*$-operator continuous for each $a \in \mathfrak{A}$. Any homotopy can be converted into one in standard form by using the stabilization theorem.

Note that

$$\varphi = (\varphi_t) : \mathfrak{A} \to \mathbb{B}(C([0,1]) \otimes E_0),$$

and $\varphi, F \in C([0,1], \mathbb{B}(E_0)) \cong C([0,1]) \otimes \mathbb{B}(E_0)$, which is strictly contained in $\mathbb{B}(C([0,1]) \otimes E_0)$ in general.

An **operator homotopy** is a standard homotopy where $\varphi_t$ is constant and $F_t$ is norm continuous.

• Degenerate Kasparov modules are homotopic to zero module.

The **operator homotopy** equivalence relation $\sim_{oh}$ on $\mathbb{E}(\mathfrak{A}, \mathfrak{B})$ is generated by operator homotopy and addition of degenerate elements. Namely, two Kasparov modules are operator homotopy equivalent if there is an operaptor homotopy (up to unitary equivalence) for their sums with some degenerate ones.

The **compact perturbation** equivalence relation $\sim_{cp}$ on $\mathbb{E}(\mathfrak{A}, \mathfrak{B})$ is generated by unitary equivalence, compact perturbation $F'$ of $F$ (via $\varphi$) in the sense that $(F - F')\varphi(a) \in \mathbb{K}(E)$ for $(E, \varphi, F), (E, \varphi, F') \in \mathbb{K}(\mathfrak{A}, \mathfrak{B})$, and addition of degenerate Kasparov modules.

The **stabilized compact perturbation** equivalence relation $\sim_{scp}$ on $\mathbb{E}(\mathfrak{A}, \mathfrak{B})$ is defined by that two Kasparov modules are stalilized compact perturbation equivalent if their sums with unitarily equivalent Kasparov modules are compact perturbation equivalent.

The equivalence relation $\sim_{scp}$ is also called homology or cobordism.

• When $\mathfrak{A}$ is separable and $\mathfrak{B}$ is $\sigma$-unital, the equivalence relations $\sim_h$, $\sim_{oh}$, and $\sim_{scp}$ on $\mathbb{E}(\mathfrak{A}, \mathfrak{B})$ all coincide.

• If $(E, \varphi, F)$ and $(E, \varphi, F')$ belong to $\mathbb{E}(\mathfrak{A}, \mathfrak{B})$, with $F'$ a compact perturbation of $F$ via $\varphi$, then $(E, \varphi, F)$ is operator homotopic to $(E, \varphi, F')$.

Indeed, define $F_t = (1-t)F + tF'$ for $t \in [0,1]$. Then $(C([0,1], E), (\varphi =$

$\varphi_t), F = (F_t))$ gives an operator homotopy. Check that

$$\begin{aligned}[F_t, \varphi(a)] &= \{(1-t)F + tF'\}\varphi(a) - \{(1-t)F + tF'\}\varphi(a) \\ &= (1-t)[F, \varphi(a)] + t[F', \varphi(a)] \in \mathbb{K}(E); \\ (F_t^2 - 1)\varphi(a) &= \{(1-t)^2(F^2-1) + t^2((F')^2 - 1)\}\varphi(a) \\ &\quad - \{(1-t)t(FF' - 1) - t(1-t)(F'F - 1)\}\varphi(a)\end{aligned}$$

with

$$\begin{aligned}FF' - 1 &= FF' - F^2 + F^2 - 1 = F(F' - F) + (F^2 - 1), \\ F'F - 1 &= F'F - (F')^2 + (F')^2 - 1 = F'(F - F') + ((F')^2 - 1),\end{aligned}$$

so that $(F_t^2 - 1)\varphi(a) \in \mathbb{K}(E)$, and

$$(F_t - F_t^*)\varphi(a) = (1-t)(F - F^*)\varphi(a) + t(F' - (F')^*)\varphi(a) \in \mathbb{K}(E).$$

• The equivalence relation $\sim_{cp}$ on $\mathbb{E}(\mathfrak{A}, \mathfrak{B})$ implies $\sim_{oh}$, and the equivalence relation $\sim_{oh}$ implies $\sim_h$.

• If $(E, \varphi, F)$ and $(E, \varphi, F')$ are Kasparov $(\mathfrak{A}, \mathfrak{B})$-modules, such that $\varphi(a)[F, F']\varphi(a)^* \geq \mod \mathbb{K}(E)$ for all $a \in \mathfrak{A}$, then $(E, \varphi, F) \sim_{oh} (E, \varphi, F')$.

**(KK-theory groups)**. $KK(\mathfrak{A}, \mathfrak{B}) \coloneqq KK_h(\mathfrak{A}, \mathfrak{B})$ is the set of all equivalence classes of $\mathbb{E}(\mathfrak{A}, \mathfrak{B})$ under $\sim_h$.

Set $KK^1(\mathfrak{A}, \mathfrak{B}) = KK(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{C}_1)$.

Similarly, define $KK_{oh}(\mathfrak{A}, \mathfrak{B})$, $KK_{cp}(\mathfrak{A}, \mathfrak{B})$, $KK_{scp}(\mathfrak{A}, \mathfrak{B})$ to be the sets of equivalence classes of $\mathbb{E}(\mathfrak{A}, \mathfrak{B})$ under $\sim_{op}$, $\sim_{cp}$, and $\sim_{scp}$, respectively.

Set $KK_{oh}^1(\mathfrak{A}, \mathfrak{B}) = KK_{oh}(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{C}_1)$, $KK_{cp}^1(\mathfrak{A}, \mathfrak{B}) = KK_{cp}(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{C}_1)$, and $KK_{scp}^1(\mathfrak{A}, \mathfrak{B}) = KK_{scp}(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{C}_1)$.

There are surjective maps:

$$KK_{cp}(\mathfrak{A}, \mathfrak{B}) \to KK_{oh}(\mathfrak{A}, \mathfrak{B}) \to KK(\mathfrak{A}, \mathfrak{B}).$$

• $KK(\mathfrak{A}, \mathfrak{B})$ and $KK_{oh}(\mathfrak{A}, \mathfrak{B})$ are abelian groups, and $KK_{cp}(\mathfrak{A}, \mathfrak{B})$ and $KK_{scp}(\mathfrak{A}, \mathfrak{B})$ are abelian semigroups with identity.

The proof is as follows. Two degenerate Kasparov modules are equivalent under $\sim_{cp}$, and the class of elements of $\mathbb{D}(\mathfrak{A}, \mathfrak{B})$ is the respective identity. If $(E, \varphi, F) \in \mathbb{E}(\mathfrak{A}, \mathfrak{B})$, then let $\varphi^\sim : \mathfrak{A} \to \mathbb{B}(E^{op})$ be defined by $\varphi^\sim(a^{(0)}, a^{(1)}) = \varphi(a^{(0)}, -a^{(1)})$ (perphaps, missed defined), and then

$$(E, \varphi, F) \oplus (E^{op}, \varphi^\sim, -F) \sim_{oh} (E \oplus E^{op}, \varphi \oplus \varphi^\sim, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$$

via the operator homotopy:

$$(C([0,1], E \oplus E^{op}), \varphi \oplus \varphi^{\sim}, F = \left(F_t = \begin{pmatrix} F\cos\frac{\pi t}{2} & 1\sin\frac{\pi t}{2} \\ 1\sin\frac{\pi t}{2} & -F\cos\frac{\pi t}{2} \end{pmatrix}\right))$$

(corrected). Note also that

$$\begin{aligned} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (\varphi \oplus \varphi^{\sim})(a)\right] &= \begin{pmatrix} 0 & \varphi^{\sim}(a) \\ \varphi(a) & 0 \end{pmatrix} - \begin{pmatrix} 0 & \varphi(a) \\ \varphi^{\sim}(a) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \varphi^{\sim}(a) - \varphi(a) \\ \varphi(a) - \varphi^{\sim}(a) & 0 \end{pmatrix} \end{aligned}$$

which should be zero (but not sure, even belong to $\mathbb{K}(E \oplus E^{op})$), however, is zero certainly when $\mathfrak{A}$ is trivially graded, and

$$\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\} (\varphi(a) \oplus \varphi^{\sim}(a)) = 0;$$

$$\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^*\right\} (\varphi(a) \oplus \varphi^{\sim}(a)) = 0.$$

(Possibly, if no misunderstanding, or as a case, $\varphi^{\sim}$ may be replaced and defined as $\varphi^{\sim}(a^{(0)}, a^{(1)}) = \varphi(0, a^{(1)}) \oplus \varphi(a^{(0)}, 0)$ in $\mathbb{B}(E^{op}) = \mathbb{B}(E^{(1)} \oplus E^{(0)})$, and then $\varphi^{\sim}$ is actually the same with $\varphi$ and is identified with $\varphi$, so that $\varphi^{\sim} - \varphi = 0$.)

As a note. If $\mathfrak{A}$ is separable, $KK_{scp}(\mathfrak{A}, \mathfrak{B})$ is also a group.

$KK_{cp}$ does not have cancellation in general. There is a surjective homomorphism $q$ from $KK_{scp}(\mathfrak{A}, \mathfrak{B})$ to the cancellation semigroup of $KK_{cp}(\mathfrak{A}, \mathfrak{B})$, temporarily denoted by $KK_{cp}(\mathfrak{A}, \mathfrak{B})_{can}$. There is also an induced surjective homomorphism $q'$ from $KK_{scp}(\mathfrak{A}, \mathfrak{B})$ to $KK_{oh}(\mathfrak{A}, \mathfrak{B})$.

As a summary, we obtain the following diagram:

$$\begin{array}{ccccc} & & 0 & & \\ & & \uparrow & & \\ KK_{cp}(\mathfrak{A}, \mathfrak{B}) & \xleftarrow{i} & KK_{cp}(\mathfrak{A}, \mathfrak{B})_{can} & \longleftarrow & 0 \\ \| & & q\uparrow & & \\ KK_{cp}(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{s} & KK_{scp}(\mathfrak{A}, \mathfrak{B}) & \longrightarrow & 0 \\ \| & & q'\downarrow & & \\ KK_{cp}(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{r} & KK_{oh}(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{r'} & KK(\mathfrak{A}, \mathfrak{B}) \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array}$$

where $q' = r \circ i \circ q$, with $r$, $r'$, and $s$ the canonical surjections by the definitions of the equivalence relations, and $i$ the canonical inclusion map.

**Example 4.1.** • There is a sequence of surjective maps:

$$\mathbb{Z} \to KK_{cp}(\mathbb{C},\mathbb{C}) \to KK_{sc}(\mathbb{C},\mathbb{C}) \to KK_{oh}(\mathbb{C},\mathbb{C}) \to \mathbb{Z},$$

so that all the maps are isomorphisms, and as well $KK(\mathbb{C},\mathbb{C}) \cong \mathbb{Z}$.

An element of $\mathbb{E}(\mathbb{C},\mathbb{C})$ is a module of the form $\alpha = (H_{\mathbb{C}}^{\wedge}, \varphi, F)$ or $((\mathbb{C}^n)^{\wedge}, \varphi, F)$, where we may let $H_{\mathbb{C}}^{\wedge} = H_0 \oplus H_1^{op}$. Then $\varphi(1)$ is a projection of degree zero, i.e., $\varphi(1) = P \oplus Q$ for some projections $P$ and $Q$, and $F$ is of the form $\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$ with degree one.

The Kasparov module $\alpha$ above is a compact perturbation via $\varphi$ of the module $(H_{\mathbb{C}}^{\wedge}, \varphi, \varphi(1)F\varphi(1))$. Thus, the equivalence class of $\alpha$ in $KK_{cp}(\mathbb{C},\mathbb{C})$ can be represented by a module of the form:

$$\beta = (H_0 \oplus H_1^{op}, \varphi = 1, \begin{pmatrix} 0 & S' \\ T' & 0 \end{pmatrix})$$

with $\varphi$ unital on $\mathbb{C}$, where $H_{\mathbb{C}}^{\wedge}$ may not be the same as the first one. Then $T'$ is essentially a unitary operator from $H_0$ to $H_1$ (mod $\mathbb{K}(H_0, H_1)$), and $S' = (T')^*$ essentially (mod $\mathbb{K}(H_1, H_0)$). By another compact perturbation, we may assume that $T'$ is either an isometry or coisometry, and that $S' = (T')^*$.

If $T'$ is unitary, then the module $\beta$ is degenerate.

If $T'$ is a proper coisometry, i.e., $T'(T')^* = 1$, $(T')^*T' \neq 1$, then set $P' = 1 - (T')^*T'$. Then $P'$ is a projection in $\mathbb{B}(H_0)$ of finite rank $n$, and the module $\beta$ is unitarily equivalent to

$$\{(P'H_0, 1, 0)\} \oplus \{ ((1 - P')H_0 \oplus H_1^{op}, 1, \begin{pmatrix} 0 & (T')^* \\ T' & 0 \end{pmatrix})\}.$$

The second module is degenerate, and the first module is isomorphic to $n$ times the module obtained from the identity map $\mathrm{id}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ as:

$$\oplus^n(\mathbb{C}, \mathrm{id}_{\mathbb{C}}, 0)$$

or equivalently, the first module is isomorphic to the module coming from the unital map $i : \mathbb{C} \to M_n(\mathbb{C})$ as:

$$(M_n(\mathbb{C}), i, 0).$$

Similarly, if $T'$ is a proper isometry, then set $Q' = 1-(T')(T')^* \in \mathbb{B}(H_1)$. Then the module $\beta$ is unitarily equivalent to

$$\{(Q'H_1, 1, 0)\} \oplus \{ (H_0 \oplus (1 - Q')H_1^{op}, 1, \begin{pmatrix} 0 & (T')^* \\ T' & 0 \end{pmatrix})\},$$

which is compact perturbation equivalent to the module (corrected) of a homomorphism.

Thus the map $\mathbb{Z} \to KK_{cp}(\mathbb{C}, \mathbb{C})$ sending 1 to the class of $\mathrm{id}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ is surjective, i.e., $KK_{cp}(\mathbb{C}, \mathbb{C})$ is a (cyclic) group generated by the class $[\mathrm{id}_{\mathbb{C}}]$.

There is an inverse map from $KK_{oh}(\mathbb{C}, \mathbb{C})$ to $\mathbb{Z}$ by sending the module in the first to the following Fredholm index:

$$(H_0 \oplus H_1^{op}, \varphi(1) = P \oplus Q, F = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}) \mapsto \mathrm{index}(QTP),$$

with

$$\varphi(1)F\varphi(1) = \begin{pmatrix} 0 & PSQ \\ QTP & 0 \end{pmatrix}.$$

and the Fredholm index equal to $\dim PH_0 - \dim QH_1$ if $PH_0$ and $QH_1$ are finite dimensional. This map is well defined because an operator homotopy preserves this index. Thus the map is surjective.

• If $\mathfrak{B}$ is $\sigma$-unital, then in the definition of $KK_{scp}(\mathfrak{A}, \mathfrak{B})$, hence also for $KK_{oh}(\mathfrak{A}, \mathfrak{B})$ and $KK(\mathfrak{A}, \mathfrak{B})$, it suffices to consider only the triples $(E, \varphi, F)$ with $E = H_{\mathfrak{B}}^{\wedge}$.

Since the triple $(H_{\mathfrak{B}}^{\wedge}, 0, 0)$ is in $\mathbb{D}(\mathfrak{A}, \mathfrak{B})$, then $(E, \varphi, F)$ has the same class in $KK_{scp}(\mathfrak{A}, \mathfrak{B})$ with $(E \oplus H_{\mathfrak{B}}^{\wedge}, \varphi \oplus 0, F \oplus 0)$, where $E \oplus H_{\mathfrak{B}}^{\wedge} \cong H_{\mathfrak{B}}^{\wedge}$ by the stabilization theorem.

• If $(E, \varphi, F) \in \mathbb{E}(\mathfrak{A}, \mathfrak{B})$, then there is its compact perturbation via $\varphi$: $(E, \varphi, G)$ with $G = G^*$. Thus, in the definition of $KK_{scp}(\mathfrak{A}, \mathfrak{B})$, hence also for $KK_{oh}(\mathfrak{A}, \mathfrak{B})$ and $KK(\mathfrak{A}, \mathfrak{B})$, it suffices to consider only the triples $(E, \varphi, F)$ with $F = F^*$.

If $(E, \varphi, F) \in \mathbb{E}(\mathfrak{A}, \mathfrak{B})$, then so are $(E, \varphi, F^*)$ and $(E, \varphi, \frac{1}{2}(F + F^*))$, which are compact perturbations of $(E, \varphi, F)$ via $\varphi$. Also, a homotopy (or operator homotopy) $(E_t, \varphi_t, F_t)$ from $(E_0, \varphi_0, F_0)$ to $(E_1, \varphi_1, F_1)$ yields a homotopy (or operator homotopy) $(E_t, \varphi_t, \frac{1}{2}(F_t + F_t^*))$ from $(E_0, \varphi_0, \frac{1}{2}(F_0 + F_0^*))$ to $(E_1, \varphi_1, \frac{1}{2}(F_1 + F_1^*))$.

• If $(E, \varphi, F) \in \mathbb{E}(\mathfrak{A}, \mathfrak{B})$, then there is its compact perturbation $(E, \varphi, G)$ with $G = G^*$ and $\|G\| \leq 1$. If $\mathfrak{A}$ is unital, we may in addition assume that $G^2 - 1 \in \mathbb{K}(E)$. Thus, in the definition of $KK_{scp}(\mathfrak{A}, \mathfrak{B})$, hence also for $KK_{oh}(\mathfrak{A}, \mathfrak{B})$ and $KK(\mathfrak{A}, \mathfrak{B})$, it suffices to consider only the triples $(E, \varphi, F)$ with $F = F^*$ and $\|F\| \leq 1$, and in addition $F^2 - 1 \in \mathbb{K}(E)$ if $\mathfrak{A}$ is unital.

**Example 4.2.** • If $\varphi = f : \mathfrak{A} \to \mathfrak{B}$ is a graded homomorphism, then its corresponding class $[f] \in KK(\mathfrak{A}, \mathfrak{B})$ is represented by the triple $(\mathfrak{B}, f, 0)$.

Note again that $\mathbb{B}(\mathfrak{B}) \cong M(\mathfrak{B})$ and $\mathbb{K}(\mathfrak{B}) \cong \mathfrak{B}$. Also, zero $0 \in \mathbb{B}(\mathfrak{B})$ has degree any, and all $[0, \varphi(a)] = 0$, $(0^2-1)\varphi(a) = -\varphi(a)$, and $(0-0^*)\varphi(a) = 0$ belong to $\mathbb{K}(\mathfrak{B})$.

One may also associate to $f : \mathfrak{A} \to \mathfrak{B}$ the Kasparov $(\mathfrak{A}, \mathfrak{B})$-module

$$(\mathfrak{B} \oplus \mathfrak{B}^{op}, \varphi = f \oplus 0, F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$$

to give the same KK-element. As a note,

$$\mathbb{B}(\mathfrak{B} \oplus \mathfrak{B}^{op}) = \begin{pmatrix} \mathbb{B}(\mathfrak{B}) & \mathbb{B}(\mathfrak{B}^{op}, \mathfrak{B}) \\ \mathbb{B}(\mathfrak{B}, \mathfrak{B}^{op}), & \mathbb{B}(\mathfrak{B}^{op}) \end{pmatrix} \cong \begin{pmatrix} M(\mathfrak{B}) & \mathbb{B}(\mathfrak{B}^{op}, \mathfrak{B}) \\ \mathbb{B}(\mathfrak{B}, \mathfrak{B}^{op}), & M(\mathfrak{B}^{op}) \end{pmatrix},$$

and

$$\mathbb{K}(\mathfrak{B} \oplus \mathfrak{B}^{op}) = \begin{pmatrix} \mathbb{K}(\mathfrak{B}) & \mathbb{K}(\mathfrak{B}^{op}, \mathfrak{B}) \\ \mathbb{K}(\mathfrak{B}, \mathfrak{B}^{op}), & \mathbb{K}(\mathfrak{B}^{op}) \end{pmatrix} \cong \begin{pmatrix} \mathfrak{B} & \mathbb{K}(\mathfrak{B}^{op}, \mathfrak{B}) \\ \mathbb{K}(\mathfrak{B}, \mathfrak{B}^{op}), & \mathfrak{B}^{op} \end{pmatrix}.$$

Hence $\varphi = f \oplus 0 : \mathfrak{A} \to \mathbb{K}(\mathfrak{B} \oplus \mathfrak{B}^{op})$. Also, $F \in \mathbb{B}(\mathfrak{B} \oplus \mathfrak{B}^{op})$ has degree one, and

$$\begin{aligned}
[F, \varphi(a)] &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f(a) & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} f(a) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -f(a) \\ f(a) & 0 \end{pmatrix} \in \mathbb{K}(\mathfrak{B} \oplus \mathfrak{B}^{op}), \\
(F^2 - 1)\varphi(a) &= (1-1)\varphi(a) = 0 \in \mathbb{K}(\mathfrak{B} \oplus \mathfrak{B}^{op}), \\
(F - F^*)\varphi(a) &= (F - F)\varphi(a) = 0 \in \mathbb{K}(\mathfrak{B} \oplus \mathfrak{B}^{op}).
\end{aligned}$$

• If $\varphi = f : \mathfrak{A} \to \mathfrak{B} \otimes \mathbb{K}$ is a graded homomorphism, then its corresponding class $[f] \in KK(\mathfrak{A}, \mathfrak{B})$ is represented by the triple $(H_{\mathfrak{B}}, f, 0)$, where $\mathbb{K}(H_{\mathfrak{B}})$ is identified with $\mathfrak{B} \otimes \mathbb{K}$.

Note again that $\mathbb{B}(H_{\mathfrak{B}}) \cong M(\mathbb{K}(H_{\mathfrak{B}})) \cong M(\mathfrak{B} \otimes \mathbb{K})$. Also, zero $0 \in \mathbb{B}(H_{\mathfrak{B}})$ has degree any, and all $[0, \varphi(a)] = 0$, $(0^2 - 1)\varphi(a) = -\varphi(a)$, and $(0 - 0^*)\varphi(a) = 0$ belong to $\mathbb{K}(H_{\mathfrak{B}})$.

One can also associate to $f : \mathfrak{A} \to \mathfrak{B} \otimes \mathbb{K}$ the Kasparov $(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{K})$-module

$$([\mathfrak{B} \otimes \mathbb{K}] \oplus [\mathfrak{B} \otimes \mathbb{K}]^{op}, \varphi = f \oplus 0, F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$$

(corrected) to give the same KK-element.

Note also that

$$\mathbb{B}([\mathfrak{B}\otimes\mathbb{K}]\oplus[\mathfrak{B}\otimes\mathbb{K}]^{op}) = \begin{pmatrix} \mathbb{B}(\mathfrak{B}\otimes\mathbb{K}) & \mathbb{B}([\mathfrak{B}\otimes\mathbb{K}]^{op},\mathfrak{B}\otimes\mathbb{K}) \\ \mathbb{B}(\mathfrak{B}\otimes\mathbb{K},[\mathfrak{B}\otimes\mathbb{K}]^{op}), & \mathbb{B}([\mathfrak{B}\otimes\mathbb{K}]^{op}) \end{pmatrix}$$
$$\cong \begin{pmatrix} M(\mathfrak{B}\otimes\mathbb{K}) \cong H_{\mathfrak{B}} & \mathbb{B}([\mathfrak{B}\otimes\mathbb{K}]^{op},\mathfrak{B}\otimes\mathbb{K}) \\ \mathbb{B}(\mathfrak{B}\otimes\mathbb{K},[\mathfrak{B}\otimes\mathbb{K}]^{op}), & M([\mathfrak{B}\otimes\mathbb{K}]^{op}) \end{pmatrix}$$

and similarly,

$$\mathbb{K}([\mathfrak{B}\otimes\mathbb{K}[\oplus[\mathfrak{B}\otimes\mathbb{K}]^{op}) \cong \begin{pmatrix} \mathfrak{B}\otimes\mathbb{K}\cong\mathbb{K}_{\mathfrak{B}} & \mathbb{K}([\mathfrak{B}\otimes\mathbb{K}]^{op},\mathfrak{B}\otimes\mathbb{K}) \\ \mathbb{K}(\mathfrak{B}\otimes\mathbb{K},[\mathfrak{B}\otimes\mathbb{K}]^{op}), & [\mathfrak{B}\otimes\mathbb{K}]^{op} \end{pmatrix}.$$

Hence $\varphi : f\oplus 0 : \mathfrak{A} \to \mathbb{K}([\mathfrak{B}\otimes\mathbb{K}[\oplus[\mathfrak{B}\otimes\mathbb{K}]^{op})$. Also, $F \in \mathbb{B}([\mathfrak{B}\otimes\mathbb{K}]\oplus[\mathfrak{B}\otimes\mathbb{K}]^{op})$ has degree one, and

$$[F,\varphi(a)] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} f(a) & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} f(a) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -f(a) \\ f(a) & 0 \end{pmatrix} \in \mathbb{K}([\mathfrak{B}\otimes\mathbb{K}]\oplus[\mathfrak{B}\otimes\mathbb{K}]^{op}),$$
$$(F^2-1)\varphi(a) = (1-1)\varphi(a) = 0 \in \mathbb{K}([\mathfrak{B}\otimes\mathbb{K}]\oplus[\mathfrak{B}\otimes\mathbb{K}]^{op}),$$
$$(F-F^*)\varphi(a) = (F-F)\varphi(a) = 0 \in \mathbb{K}([\mathfrak{B}\otimes\mathbb{K}]\oplus[\mathfrak{B}\otimes\mathbb{K}]^{op}).$$

- Given a split short exact sequence of graded $C^*$-algebras

$$0 \longrightarrow \mathfrak{B} \xrightarrow{i} \mathfrak{D} \xrightarrow{q} \mathfrak{A} \longrightarrow 0$$

with $s : \mathfrak{A} \to \mathfrak{D}$ a section, we associate a Kasparov $(\mathfrak{D},\mathfrak{B})$-module

$$(\mathfrak{B}\oplus\mathfrak{B}^{op}, w\oplus(w\circ s\circ q) \equiv w\oplus w_s, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}),$$

where $w$ is the canonical homomorphism from $\mathfrak{D}$ to $M(\mathfrak{B}) \cong \mathbb{B}(\mathfrak{B})$. This Kasparov module is called the **splitting morphism** of the split short exact sequence, denoted by $\pi_s$. When $\mathfrak{D} = \mathfrak{B}\oplus\mathfrak{A}$, then

$$\pi_s = (\mathfrak{B}\oplus\mathfrak{B}^{op}, p\oplus 0, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \in \mathbb{E}(\mathfrak{D},\mathfrak{B}),$$

with $p : \mathfrak{D} \to \mathfrak{B}$ the projection, so that $p\circ s\circ q = 0$ with $s : \mathfrak{A} \to \mathfrak{D}$ a section as the canonical injection.

Note that $\varphi = w \oplus w_s : \mathfrak{D} \to M(\mathfrak{B}) \oplus M(\mathfrak{B}^{op})$ in $\mathbb{B}(\mathfrak{B} \oplus \mathfrak{B}^{op})$. Also, $F \in \mathbb{B}(\mathfrak{B} \oplus \mathfrak{B}^{op})$ has degree one, and

$$\begin{aligned}
[F, \varphi(a)] &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w(d) & 0 \\ 0 & w_s(d) \end{pmatrix} - \begin{pmatrix} w(d) & 0 \\ 0 & w_s(d) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & w_s(d) - w(d) \\ w(d) - w_s(d) & 0 \end{pmatrix} \in \mathbb{K}(\mathfrak{B} \oplus \mathfrak{B}^{op}),
\end{aligned}$$

$$(F^2 - 1)\varphi(a) = (1 - 1)\varphi(a) = 0 \in \mathbb{K}(\mathfrak{B} \oplus \mathfrak{B}^{op}),$$

$$(F - F^*)\varphi(a) = (F - F)\varphi(a) = 0 \in \mathbb{K}(\mathfrak{B} \oplus \mathfrak{B}^{op}).$$

Indeed, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{B} & \xrightarrow{i} & \mathfrak{D} & \xrightarrow{q} & \mathfrak{A} & \longrightarrow & 0 \\
 & & \| & & \downarrow w & & \downarrow b & & \\
0 & \longrightarrow & \mathfrak{B} \cong \mathbb{K}(\mathfrak{B}) & \xrightarrow{i} & M(\mathfrak{B}) & \xrightarrow{q} & M(\mathfrak{B})/\mathfrak{B} & \longrightarrow & 0
\end{array}$$

with $b$ the Busby invariant, so that $w(d) - w_s(d)$ is mapped to zero under $q$ in the second line, and hence $w(d) - w_s(d) \in \mathbb{K}(\mathfrak{B})$.

• Similarly, for a split short exact sequence of graded $C^*$-algebras

$$0 \longrightarrow \mathfrak{B} \otimes \mathbb{K} \xrightarrow{i} \mathfrak{D} \xrightarrow{q} \mathfrak{A} \longrightarrow 0$$

with $s : \mathfrak{A} \to \mathfrak{D}$ a section, we associate a Kasparov $(\mathfrak{D}, \mathfrak{B} \otimes \mathbb{K})$-module

$$(H_{\mathfrak{B}} \oplus H_{\mathfrak{B}}^{op}, w \oplus (w \circ s \circ q), \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}),$$

also called the **splitting morphism**, where $w : \mathfrak{D} \to M(\mathfrak{B} \otimes \mathbb{K}) \cong \mathbb{B}(\mathfrak{B} \otimes \mathbb{K})$ is the canonical homomorphism, with $M(\mathfrak{B} \otimes \mathbb{K}) \cong M(\mathbb{K}_{\mathfrak{B}}) \cong H_{\mathfrak{B}}$.

• If $(E_j, \varphi_j, F_j)$ is a Kasparov $(\mathfrak{A}_j, \mathfrak{B})$-module, for $j = 1, 2$, then $(E_1 \oplus E_2, \varphi_1 \oplus \varphi_2, F_1 \oplus F_2)$ is a Kasparov $(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \mathfrak{B})$-module. Similarly, if $(E_j', \varphi_j', F_j')$ is a Kasparov $(\mathfrak{A}, \mathfrak{B}_j)$-module, for $j = 1, 2$, then $(E_1' \oplus E_2', \varphi_1' \oplus \varphi_2', F_1' \oplus F_2')$ is a Kasparov $(\mathfrak{A}, \mathfrak{B}_1 \oplus \mathfrak{B}_2)$, where each $E_j'$ is viewed as a Hilbert $\mathfrak{B}_1 \oplus \mathfrak{B}_2$-module by letting $\mathfrak{B}_k$ with $k \neq j$ act trivially on $E_j'$.

Note that $\varphi_1 \oplus \varphi_1 : \mathfrak{A}_1 \oplus \mathfrak{A}_2 \to \mathbb{B}(E_1) \oplus \mathbb{B}(E_2) \subset \mathbb{B}(E_1 \oplus E_2)$, and

$$\begin{aligned}
&[F_1 \oplus F_2, \varphi_1(a_1) \oplus \varphi_2(a_2)] \\
&= \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \begin{pmatrix} \varphi_1(a_1) & 0 \\ 0 & \varphi_2(a_2) \end{pmatrix} - \begin{pmatrix} \varphi_1(a_1) & 0 \\ 0 & \varphi_2(a_2) \end{pmatrix} \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \\
&= \begin{pmatrix} [F_1, \varphi_1(a_1)] & 0 \\ 0 & [F_2, \varphi_2(a_2)] \end{pmatrix} \in \mathbb{K}(E_1 \oplus E_2),
\end{aligned}$$

and

$$\begin{aligned}
&\{(F_1 \oplus F_2)^2 - (1 \oplus 1)\}(\varphi_1(a_1) \oplus \varphi_2(a_2)) \\
&= (F_1^2 - 1)\varphi_1(a_1) \oplus (F_2^2 - 1)\varphi_2(a_2) \in \mathbb{K}(E_1 \oplus E_2); \\
&\{(F_1 \oplus F_2) - (F_1 \oplus F_2)^*\}(\varphi_1(a_1) \oplus \varphi_2(a_2)) \\
&= (F_1 - F_1^*)\varphi_1(a_1) \oplus (F_2 - F_2^*)\varphi_2(a_2) \in \mathbb{K}(E_1 \oplus E_2).
\end{aligned}$$

The other case is omitted.

• If $0 \to \mathfrak{B} \to \mathfrak{D} \to \mathfrak{A} \to 0$ is an invertible extension, then the Busby invariant $\tau : \mathfrak{A} \to Q(\mathfrak{B}) = M(\mathfrak{B})/\mathfrak{B}$ dilates to a $*$-homomorphism:

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} : \mathfrak{A} \to M_2(M(\mathfrak{B}))$$

and $((\mathfrak{B} \oplus \mathfrak{B}) \otimes \mathbb{C}_1, \varphi \otimes 1, (1 \oplus -1) \otimes (1, -1))$ the Kasparov module in $E(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{C}_1)$ is associated to the invertible extension.

Note that $\varphi$ is identified via the inclusion $\mathfrak{A} \to \mathfrak{A} \otimes \mathbb{C}_1$, with $\varphi \otimes 1 : \mathfrak{A} \otimes \mathbb{C}_1 \to M_2(M(\mathfrak{B})) \otimes \mathbb{C}1$ in $\mathbb{B}(\mathfrak{B} \oplus \mathfrak{B}) \otimes \mathbb{B}(\mathbb{C}_1)$ contained in $\mathbb{B}((\mathfrak{B} \oplus \mathfrak{B}) \otimes \mathbb{C}_1)$, and $(1 \oplus -1) \otimes (1, -1) \in \mathbb{B}(\mathfrak{B} \oplus \mathfrak{B}) \otimes \mathbb{B}(\mathbb{C}_1)$, and

$$\begin{aligned}
&[(1 \oplus -1) \otimes (1, -1), \varphi(a) \otimes 1] \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \varphi_{11}(a) & \varphi_{12}(a) \\ \varphi_{21}(a) & \varphi_{22}(a) \end{pmatrix} \otimes (1, -1) \\
&\quad - \begin{pmatrix} \varphi_{11}(a) & \varphi_{12}(a) \\ \varphi_{21}(a) & \varphi_{22}(a) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes (1, -1) \\
&= \begin{pmatrix} \varphi_{11}(a) & \varphi_{12}(a) \\ -\varphi_{21}(a) & -\varphi_{22}(a) \end{pmatrix} \otimes (1, -1) - \begin{pmatrix} \varphi_{11}(a) & -\varphi_{12}(a) \\ \varphi_{21}(a) & -\varphi_{22}(a) \end{pmatrix} \otimes (1, -1) \\
&= \begin{pmatrix} 0 & 2\varphi_{12}(a) \\ -2\varphi_{21}(a) & 0 \end{pmatrix} \otimes (1, -1)
\end{aligned}$$

which should belong to $\mathbb{K}(\mathfrak{B} \oplus \mathfrak{B}) \otimes \mathbb{B}(\mathbb{C}_1) \cong M_2(\mathfrak{B}) \otimes \mathbb{C}_1$ (possibly in this sense), and

$$\begin{aligned}
&\{(1 \oplus -1)^2 \otimes (1, -1)^2 - (1 \otimes 1)\}(\varphi(a) \otimes 1) \\
&= \{(1 \otimes 1) - (1 \otimes 1)\}(\varphi(a) \otimes 1) = 0; \\
&\{(1 \oplus -1) \otimes (1, -1) - (1 \oplus -1)^* \otimes (1, -1)^*\}(\varphi(a) \otimes 1) = 0.
\end{aligned}$$

(**Additivity**). We have

$$KK(\mathfrak{A}_1 \oplus \mathfrak{A}_2, \mathfrak{B}) \cong KK(\mathfrak{A}_1, \mathfrak{B}) \oplus KK(\mathfrak{A}_2, \mathfrak{B})$$

and

$$KK(\mathfrak{A}, \mathfrak{B}_1 \oplus \mathfrak{B}_2) \cong KK(\mathfrak{A}, \mathfrak{B}_1) \oplus KK(\mathfrak{A}, \mathfrak{B}_2).$$

Additivity in the first variable also holds for countable direct sums, but does not in the second variable in general.

(**Functoriality**). If $f : \mathfrak{A}_1 \to \mathfrak{A}_2$ is a graded homomorphism, then the following homomorphism is induced:

$$f^* : KK(\mathfrak{A}_2, \mathfrak{B}) \to KK(\mathfrak{A}_1, \mathfrak{B})$$

for any $\mathfrak{B}$, with $f^*[(E, \varphi, F)] = [(E, \varphi \circ f, F)] \in KK(\mathfrak{A}_1, \mathfrak{B})$. Thus, $KK(\cdot, \mathfrak{B})$ with $\mathfrak{B}$ fiexd is a contravariant functor from $C^*$-algebras to abelian groups.

If $g : \mathfrak{B}_1 \to \mathfrak{B}_2$ is a graded homomorphism, then the following homomorphism is induced:

$$g_* : KK(\mathfrak{A}, \mathfrak{B}_1) \to KK(\mathfrak{A}, \mathfrak{B}_2)$$

for any $\mathfrak{A}$, with $g_*[(E, \varphi, F)] = [(E \otimes_g \mathfrak{B}_2, \varphi \otimes 1, F \otimes 1)] \in KK(\mathfrak{A}, \mathfrak{B}_2)$. Thus, $KK(\mathfrak{A}, \cdot)$ with $\mathfrak{A}$ fixed is a covariant functor from $C^*$-algebras to abelian groups.

Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{D}$ be $C^*$-algebras. There is a map from $\mathbb{E}(\mathfrak{A}, \mathfrak{B})$ to $\mathbb{E}(\mathfrak{A} \otimes \mathfrak{D}, \mathfrak{B} \otimes \mathfrak{D})$ given by $(E, \varphi, F) \mapsto (E \otimes \mathfrak{D}, \varphi \otimes 1, F \otimes 1)$. This map respects direct sums and the equivalence relation and thus induces a homomorphism $\tau_{\mathfrak{D}} : KK(\mathfrak{A}, \mathfrak{B}) \to KK(\mathfrak{A} \otimes \mathfrak{D}, \mathfrak{B} \otimes \mathfrak{D})$. The homomorphism $\tau_{\mathfrak{D}}$ is natural in each variable, which may be called by us the **tensor-inducing** homomorphism.

In particular, if $x = [f] \in KK(\mathfrak{A}, \mathfrak{B})$ with $f : \mathfrak{A} \to \mathfrak{B}$ a homomorphism, then $\tau_{\mathfrak{D}}(x) = [f \otimes \mathrm{id}_{\mathfrak{D}}] \in KK(\mathfrak{A} \otimes \mathfrak{D}, \mathfrak{B} \otimes \mathfrak{D})$ with $f \otimes \mathrm{id}_{\mathfrak{D}} : \mathfrak{A} \otimes \mathfrak{D} \to \mathfrak{B} \otimes \mathfrak{D}$. Indeed, $x = [(\mathfrak{B}, f, 0)]$ and

$$\tau_{\mathfrak{D}}(x) = [(\mathfrak{B} \otimes \mathfrak{D}, f \otimes \mathrm{id}_{\mathfrak{D}}, 0 \otimes 1 = 0)] = [f \otimes \mathrm{id}_{\mathfrak{D}}].$$

• If $h : \mathfrak{D}_1 \to \mathfrak{D}_2$ is a homomorphism, then we have the following composites, denoted by $(\otimes h)_*$ and $(\otimes h)^*$ respectively:

$$KK(\mathfrak{A}, \mathfrak{B}) \xrightarrow{\tau_{\mathfrak{D}_1}} KK(\mathfrak{A} \otimes \mathfrak{D}_1, \mathfrak{B} \otimes \mathfrak{D}_1) \xrightarrow{(\mathrm{id}_{\mathfrak{B}} \otimes h)_*} KK(\mathfrak{A} \otimes \mathfrak{D}_1, \mathfrak{B} \otimes \mathfrak{D}_2),$$

and

$$KK(\mathfrak{A}, \mathfrak{B}) \xrightarrow{\tau_{\mathfrak{D}_2}} KK(\mathfrak{A} \otimes \mathfrak{D}_2, \mathfrak{B} \otimes \mathfrak{D}_2) \xrightarrow{(\mathrm{id}_{\mathfrak{A}} \otimes h)^*} KK(\mathfrak{A} \otimes \mathfrak{D}_1, \mathfrak{B} \otimes \mathfrak{D}_2),$$

and moreover, the following diagram commutes:

$$\begin{array}{ccc} KK(\mathfrak{A},\mathfrak{B}) & \xrightarrow{\tau_{\mathfrak{D}_1}} & KK(\mathfrak{A}\otimes\mathfrak{D}_1,\mathfrak{B}\otimes\mathfrak{D}_1) \\ {\scriptstyle \tau_{\mathfrak{D}_2}}\downarrow & & \downarrow{\scriptstyle (\mathrm{id}_{\mathfrak{B}}\otimes h)_*} \\ KK(\mathfrak{A}\otimes\mathfrak{D}_2,\mathfrak{B}\otimes\mathfrak{D}_2) & \xrightarrow{(\mathrm{id}_{\mathfrak{A}}\otimes h)^*} & KK(\mathfrak{A}\otimes\mathfrak{D}_1,\mathfrak{B}\otimes\mathfrak{D}_2), \end{array}$$

so that $(\otimes h)_* = (\otimes h)^*$, which may be denoted by $\otimes h$.

- For any $\mathfrak{A}$ and $\mathfrak{B}$, the map

$$\tau_{\mathbb{K}} : KK(\mathfrak{A},\mathfrak{B}) \to KK(\mathfrak{A}\otimes\mathbb{K},\mathfrak{B}\otimes\mathbb{K})$$

is an isomorphism. Moreover, as a corollary,

$$\begin{aligned} KK(\mathfrak{A},\mathfrak{B}) &\cong KK(\mathfrak{A}\otimes M_n(\mathbb{C}),\mathfrak{B}\otimes M_m(\mathbb{C})) \\ &\cong KK(\mathfrak{A}\otimes\mathbb{K},\mathfrak{B}\otimes\mathbb{K}) \\ &\cong KK(\mathfrak{A}\otimes\mathbb{K},\mathfrak{B}) \cong KK(\mathfrak{A},\mathfrak{B}\otimes\mathbb{K}). \end{aligned}$$

The same holds for $KK_{oh}$ and $KK_{cp}$.

Indeed, the inverse map for $\tau_{\mathbb{K}}$ sends the Kasparov module $(E,\varphi,F) \in KK(\mathfrak{A}\otimes\mathbb{K},\mathfrak{B}\otimes\mathbb{K})$ to $(E\otimes_i H_{\mathfrak{B}},\varphi\circ h, F\otimes 1)$, where $i : \mathfrak{B}\otimes\mathbb{K} \to \mathfrak{B}\otimes\mathbb{K} \subset \mathbb{B}(H_{\mathfrak{B}}) \cong M(\mathfrak{B}\otimes\mathbb{K})$ the inclusion map and $h : \mathfrak{A} \to \mathfrak{A}\otimes\mathbb{K}$ is defined by sending $a$ to $a\otimes p$ for a rank one projection $p$ of degree zero.

For instance, $(\varphi\circ h)(a) = \varphi(a\otimes p) = \varphi(a)\otimes\varphi(p) \in \mathbb{B}(E)\otimes\mathbb{B}(H_{\mathfrak{B}}) \subset \mathbb{B}(E\otimes H_{\mathfrak{B}})$ and

$$\begin{aligned} [F\otimes 1, (\varphi\circ h)(a)] &= F\varphi(a)\otimes\varphi(p) - \varphi(a)F\otimes\varphi(p) \\ &= [F,\varphi(a)]\otimes\varphi(p) \in \mathbb{K}(E)\otimes\mathbb{K}(H_{\mathfrak{B}}), \end{aligned}$$

and

$$\begin{aligned} (F^2\otimes 1 - 1\otimes 1)(\varphi\circ h)(a) &= (F^2-1)\varphi(a)\otimes\varphi(p) \in \mathbb{K}(E)\otimes\mathbb{K}(H_{\mathfrak{B}}), \\ (F\otimes 1 - F^*\otimes 1)(\varphi\circ h)(a) &= (F-F^*)\varphi(a)\otimes\varphi(p) \in \mathbb{K}(E)\otimes\mathbb{K}(H_{\mathfrak{B}}). \end{aligned}$$

Note that for $(E,\varphi,F) \in \mathbb{E}(\mathfrak{A},\mathfrak{B})$,

$$\begin{aligned} (\tau_{\mathbb{K}}^{-1}\circ\tau_{\mathbb{K}})[(E,\varphi,F)] &= \tau_{\mathbb{K}}^{-1}[(E\otimes\mathbb{K},\varphi\otimes\mathrm{id}_{\mathbb{K}},F\otimes 1)] \\ &= [(E\otimes\mathbb{K}\otimes_i H_{\mathfrak{B}},(\varphi\otimes\mathrm{id}_{\mathbb{K}})\circ h, F\otimes 1\otimes 1)] \end{aligned}$$

with $\mathbb{K} = \mathbb{C}\otimes\mathbb{K} \cong \mathbb{K}(H_{\mathbb{C}})$.

Moreover,

$$\begin{aligned} KK(\mathfrak{A}\otimes M_n(\mathbb{C}),\mathfrak{B}\otimes M_m(\mathbb{C})) &\cong KK(\mathfrak{A}\otimes M_n(\mathbb{C})\otimes\mathbb{K},\mathfrak{B}\otimes M_m(\mathbb{C})\otimes\mathbb{K}) \\ &\cong KK(\mathfrak{A}\otimes\mathbb{K},\mathfrak{B}\otimes\mathbb{K}) \cong KK(\mathfrak{A},\mathfrak{B}), \end{aligned}$$

and

$$KK(\mathfrak{A}\otimes\mathbb{K},\mathfrak{B}) \cong KK(\mathfrak{A}\otimes\mathbb{K}\otimes\mathbb{K},\mathfrak{B}\otimes\mathbb{K})$$
$$\cong KK(\mathfrak{A}\otimes\mathbb{K},\mathfrak{B}\otimes\mathbb{K}) \cong KK(\mathfrak{A},\mathfrak{B}).$$

**(Formal Bott periodicity).** For any $\mathfrak{A}$ and $\mathfrak{B}$, the map

$$\tau_{\mathbb{C}_1} : KK(\mathfrak{A},\mathfrak{B}) \to KK(\mathfrak{A}\otimes\mathbb{C}_1,\mathfrak{B}\otimes\mathbb{C}_1)$$

is an isomorphism, and so there are isomorphisms:

$$KK^1(\mathfrak{A},\mathfrak{B}) \cong KK(\mathfrak{A}\otimes\mathbb{C}_1,\mathfrak{B})$$

and

$$KK(\mathfrak{A},\mathfrak{B}) \cong KK^1(\mathfrak{A},\mathfrak{B}\otimes\mathbb{C}_1) \cong KK^1(\mathfrak{A}\otimes\mathbb{C}_1,\mathfrak{B})$$
$$\cong KK(\mathfrak{A}\otimes\mathbb{C}_1,\mathfrak{B}\otimes\mathbb{C}_1).$$

The same holds for $KK_{oh}$ and $KK_{cp}$.

Note that the map $\tau_{\mathbb{C}_1}\circ\tau_{\mathbb{C}_1} = \tau_{M_2(\mathbb{C})}$ since $\mathbb{C}_1\otimes\mathbb{C}_1 \cong M_2(\mathbb{C})$, so that that the following diagram commutes:

$$\begin{array}{ccc} KK(\mathfrak{A},\mathfrak{B}) & \xrightarrow{\tau_{\mathbb{C}_1}} & KK(\mathfrak{A}\otimes\mathbb{C}_1,\mathfrak{B}\otimes\mathbb{C}_1) \\ {\scriptstyle \tau_{M_2(\mathbb{C})}}\downarrow & & \downarrow{\scriptstyle \tau_{\mathbb{C}_1}} \\ KK(\mathfrak{A}\otimes M_2(\mathbb{C}),\mathfrak{B}\otimes M_2(\mathbb{C})) & = & KK(\mathfrak{A}\otimes\mathbb{C}_1\otimes\mathbb{C}_1,\mathfrak{B}\otimes\mathbb{C}_1\otimes\mathbb{C}_1), \end{array}$$

with $\tau_{\mathbb{C}}^{-1} = \tau_{M_2(\mathbb{C})}^{-1}\circ\tau_{\mathbb{C}_1}$ as an inverse.

Moreover,

$$KK^1(\mathfrak{A},\mathfrak{B}) = KK(\mathfrak{A},\mathfrak{B}\otimes\mathbb{C}_1)$$
$$\cong KK(\mathfrak{A}\otimes\mathbb{C}_1,\mathfrak{B}\otimes\mathbb{C}_1\otimes\mathbb{C}_1)$$
$$\cong KK(\mathfrak{A}\otimes\mathbb{C}_1,\mathfrak{B}\otimes M_2(\mathbb{C})) \cong KK(\mathfrak{A}\otimes\mathbb{C}_1,\mathfrak{B}),$$

and also

$$KK^1(\mathfrak{A},\mathfrak{B}\otimes\mathbb{C}_1) = KK(\mathfrak{A},\mathfrak{B}\otimes\mathbb{C}_1\otimes\mathbb{C}_1)$$
$$\cong KK(\mathfrak{A}\otimes\mathbb{C}_1,\mathfrak{B}\otimes\mathbb{C}_1)$$
$$\cong KK(\mathfrak{A}\otimes\mathbb{C}_1\otimes\mathbb{C}_1,\mathfrak{B}\otimes\mathbb{C}_1\otimes\mathbb{C}_1)$$
$$\cong KK(\mathfrak{A}\otimes M_2(\mathbb{C}),\mathfrak{B}\otimes M_2(\mathbb{C})) \cong KK(\mathfrak{A},\mathfrak{B})$$

and similarly, for $KK^1(\mathfrak{A}\otimes\mathbb{C}_1,\mathfrak{B}) = KK(\mathfrak{A}\otimes\mathbb{C}_1,\mathfrak{B}\otimes\mathbb{C}_1)$.

• As for the case where $\mathfrak{A} = \mathbb{C}$, we have

$$\begin{aligned} KK(\mathbb{C}, \mathfrak{B}) &\cong KK(\mathbb{C}, \mathfrak{B} \otimes \mathbb{K}) \\ &\cong \{[T] : T \in M^s(\mathfrak{B}), T^*T - 1, TT^* - 1 \in \mathfrak{B} \otimes \mathbb{K}\}, \\ KK^1(\mathbb{C}, \mathfrak{B}) &\cong KK^1(\mathbb{C}, \mathfrak{B} \otimes \mathbb{K}) \\ &\cong \{[T] : T \in M^s(\mathfrak{B}), T = T^*, T^2 - 1 \in \mathfrak{B} \otimes \mathbb{K}\}, \end{aligned}$$

where $[T]$ means the homotopy class for $T$. Note that there is only one unital homomorphism $\varphi = 1_{\mathbb{C}}$ from $\mathbb{C}$ to $M^s(\mathfrak{B}) = M(\mathfrak{B} \otimes \mathbb{K}) = \mathbb{B}(\mathfrak{B} \otimes \mathbb{K})$. The elements $(E, 1_{\mathbb{C}}, F)$ of $\mathbb{E}(\mathbb{C}, \mathfrak{B} \otimes \mathbb{K})$ are identified (up to equivalence) with the preimages $T$ of unitaries in $Q^s(\mathfrak{B}) = M(\mathfrak{B} \otimes \mathbb{K})/\mathfrak{B} \otimes \mathbb{K}$. The equivalent relation in $KK_{oh}(\mathbb{C}, \mathfrak{B})$ is homotopy. Any homotopy in $Q^s(\mathfrak{B})$ can be lifted to a homotopy in $M^s(\mathfrak{B})$.

Similarly, the elements of $KK^1_{oh}(\mathbb{C}, \mathfrak{B})$ can be identified with self-adjoint elements in $M^s(\mathfrak{B})$ with unitary image in $Q^s(\mathfrak{B})$, and these may be identified with projections in $Q^s(\mathfrak{B})$.

It follows that

**(K-theory).** • If $\mathfrak{B}$ is a trivially graded $\sigma$-unital $C^*$-algebra, then

$$KK_{oh}(\mathbb{C}, \mathfrak{B}) \cong K_1(Q^s(\mathfrak{B})) \cong K_0(\mathfrak{B}).$$

and

$$KK^1_{oh}(\mathbb{C}, \mathfrak{B}) \cong K_0(Q^s(\mathfrak{B})) \cong K_1(\mathfrak{B}).$$

Note that the identification of $K_j(Q^s(\mathfrak{B}))$ with $K_{j+1}(\mathfrak{B})$ ($j = 0$) requires the Bott periodicity in K-theory of $C^*$-algebras and the triviality of the K-theory $K_*(M^s(\mathfrak{B}))$ for $* = 0, 1$.

• If $\mathfrak{B}$ is a trivially graded $\sigma$-unital $C^*$-algebra, then

$$KK^1_{oh}(C_0(\mathbb{R}), \mathfrak{B}) \cong K_1(Q^s(\mathfrak{B})) \cong K_0(\mathfrak{B}).$$

Any $*$-homomorphism $\psi$ from $C_0((0,1))$ into a unital $C^*$-algebra defines a unitary $u = \psi(f) + 1$, where $f(t) = e^{2\pi i t} - 1$, and conversely, any unitary $u$ of a unital $C^*$-algebra defines a homomorphism $\varphi$ of $C_0((0,1))$ by sending $f$ to $u - 1$.

Check that $(\psi(f) + 1)(\psi(f) + 1)^* = \psi(ff^* + f + f^*) + 1$ and

$$(ff^* + f + f^*)(t) = (e^{2\pi i t} - 1)(e^{-2\pi i t} - 1) + (e^{2\pi i t} - 1) + (e^{-2\pi i t} - 1) = 0.$$

Note that we extend $\varphi$ to $\varphi^+$ on $C_0((0,1))^+ \cong \mathbb{C}(\mathbb{T})$ by $\psi(f, 1) = (u - 1) + 1 = u$ with $(f, 1) = e^{2\pi i t}$ the generator for $C(\mathbb{T})$, with $\mathbb{T}$ the one-torus. Thus $\varphi$ on $C_0((0,1))$ is defined as the restriction of $\varphi^+$ from $C(\mathbb{T})$ to the unital $C^*$-algebra generated by $u$.

Two such homomorphisms are homotopic if and only if the corresponding unitaries are in the same connected component. Thus elements of $KK^1(C_0(\mathbb{R}), \mathfrak{B})$ with $\mathfrak{B}$ trivially graded are represented by triples $(H_{\mathfrak{B}}, U, T)$, where $U$ is a unitary in $\mathbb{B}(\mathfrak{B} \otimes \mathbb{K}) \cong M^s(\mathfrak{B})$ and $T$ is a self-adjoint, essentially unitary operator in $M^s(\mathfrak{B})$ essentially commuting with $U$. Set $P = \frac{1}{2}(T+1)$, a projection essentially. Set $u(U, P) = q(PUP + 1 - P)$ with $q : M^s(\mathfrak{B}) \to Q^s(\mathfrak{B})$ the quotient map. Then $u(U, P)$ is a unitary in $Q^s(\mathfrak{B})$, and $u(U, P)$ and a similar $u(V, Q)$ are in the same component if and only if $(U, P)$ and $(V, Q)$ are operator homotopic. If $P$ is a projection commuting with $U$, then $(U, P)$ is degenerate. If $v$ is a unitary in $Q^s(\mathfrak{B})$, let $V$ be a lift of $v \oplus v^{-1}$ in $M_2(M^s(\mathfrak{B}))$, and then $(V, 1 \oplus 0)$ has $v \oplus 1$ as an image.

# 5 Kasparov product

If $\mathfrak{A}$ is a separable $C^*$-algebra, $\mathfrak{B}$ is a $C^*$-algebra, and $\mathfrak{D}$ is a $\sigma$-unital $C^*$-algebra, then we have the **Kasparov (intersection) product** (map) defined as the following bilinear function denoted as:

$$\otimes_{\mathfrak{D}} : KK(\mathfrak{A}, D) \times KK(\mathfrak{D}, \mathfrak{B}) \to KK(\mathfrak{A}, \mathfrak{B}),$$

with $\otimes_{\mathfrak{D}}(x, y) = x \otimes_{\mathfrak{D}} y \in KK(\mathfrak{A}, \mathfrak{B})$.

Outline of the construction is as follows. Given $x \in KK(\mathfrak{A}, \mathfrak{D})$ and $y \in KK(\mathfrak{D}, \mathfrak{B})$, choose representatives $(E_1, \varphi_1, F_1) \in \mathbb{E}(\mathfrak{A}, \mathfrak{D})$ and $(E_2, \varphi_2, F_2) \in \mathbb{E}(\mathfrak{D}, \mathfrak{B})$. Then define $(E, \varphi, F) \in \mathbb{E}(\mathfrak{A}, \mathfrak{B})$ by $E = E_1 \otimes_{\varphi_2} E_2$, $\varphi = \varphi_1 \otimes_{\varphi_2} 1$, and $F$ a suitable combination of $F_1$ and $F_2$. Keep in mind the notations in the following examples below.

**Example 5.1.** • If $x = [f] \in KK(\mathfrak{A}, \mathfrak{D})$ with $f : \mathfrak{A} \to \mathfrak{D}$ a homomorphism, then we may let $(E_1, \varphi_1, F_1) = (\mathfrak{D}, f, 0)$. Assume that $\varphi_2$ is essential. Then $E_1 \otimes_{\varphi_2} E_2 = \mathfrak{D} \otimes_{\varphi_2} E_2 \cong E_2$ and we may take $F = F_2$. Namely, the Kasparov product is

$$[(\mathfrak{D}, f, 0)] \otimes_{\mathfrak{D}} [(E_2, \varphi_2, F_2)] = [(E_2, f \otimes_{\varphi_2} 1 = \varphi_2 \circ f, F_2)].$$

This class $x \otimes_{\mathfrak{D}} y = [f] \otimes_{\mathfrak{D}} y$ is denoted by $f^*(y)$. Note that $f(a) \otimes 1 = \varphi_2(f(a))$.

• If $x = [f] \in KK(\mathfrak{A}, \mathfrak{D} \otimes \mathbb{K}) \cong KK(\mathfrak{A}, \mathfrak{D})$ with $f : \mathfrak{A} \to \mathfrak{D} \otimes \mathbb{K}$ a homomorphism, then we may take $(E_1, \varphi_1, F_1) = (H_{\mathfrak{D}}, f, 0)$ and then $E = H_{\mathfrak{D}} \otimes_{\varphi_2} E_2 \cong H \otimes_{\mathbb{C}} E_2$ and we may take $F = 1 \otimes F_2$. Namely, the Kasparov product is

$$[(H_{\mathfrak{D}}, f, 0)] \otimes_{\mathfrak{D}} [(E_2, \varphi_2, F_2)] = [(H \otimes_{\mathbb{C}} E_2, f \otimes_{\varphi_2} 1 = 1 \otimes (\varphi_2 \circ f), 1 \otimes F_2)].$$

In general, $F$ is a combination of $F_1 \otimes 1$ and $1 \otimes F_2$. We may write $F = F(F_1, F_2)$ for a suitable combination (troublesome to make) of $F_1$ and $F_2$.

• Let $\mathfrak{A}$ and $\mathfrak{B}$ be graded $C^*$-algebras, with $\mathfrak{A}$ $\sigma$-unital. If $(E, \varphi, F) \in \mathbb{E}(\mathfrak{A}, \mathfrak{B})$, then there is $(E', \varphi', F') \in \mathbb{E}(\mathfrak{A}, \mathfrak{B})$ with $\varphi'$ essential and that $(E, \varphi, F) \sim_h (E', \varphi', F')$.

**Example 5.2.** • Consider $(E_1, \varphi_1, F_1) \in \mathbb{E}(\mathfrak{A}, \mathfrak{D})$ and $(\mathfrak{B}, g, 0) \in \mathbb{E}(\mathfrak{D}, \mathfrak{B})$ with $g : \mathfrak{D} \to \mathfrak{B}$ a homomorphism. Then the Kasparov product is

$$[(E_1, \varphi_1, F_1)] \otimes_{\mathfrak{D}} [(\mathfrak{B}, g, 0)] = [(E_1 \otimes_g \mathfrak{B}, \varphi_1 \otimes 1, F_1 \otimes 1)].$$

This class $x \otimes_{\mathfrak{D}} y = x \otimes_{\mathfrak{D}} [g]$ is denoted by $g_*(x)$.

• If $\mathfrak{A}$ is separable and $\mathfrak{D}$ is $\sigma$-unital, then there is a Kasparov product for $(E_1, \varphi_1, F_1) \in \mathbb{E}(\mathfrak{A}, \mathfrak{D})$ and $(E_2, \varphi_2, F_2) \in \mathbb{E}(\mathfrak{D}, \mathfrak{B})$, which is unique up to operator homotopy. If $F_1$ and $F_2$ are self-adjoint, then there is a self-adjoint $F = F(F_1, F_2)$.

(**Kasparov product map**). If $\mathfrak{A}$ is separable and $\mathfrak{D}$ is $\sigma$-unital, then the Kasparov product defines a bilinear function:

$$\begin{aligned} \otimes_{\mathfrak{D}} : KK(\mathfrak{A}, \mathfrak{D}) \times KK(\mathfrak{D}, \mathfrak{B}) &\to KK(\mathfrak{A}, \mathfrak{B}), \\ (x, y) &\mapsto x \otimes_{\mathfrak{D}} y. \end{aligned}$$

Note that $KK_{op} = KK$ under the assumption. The Kasparov product of two Kasparov modules is uniquely determined up to homotopy. It is shown that the map is well defined.

• If $\mathfrak{A}$ is separable and $\mathfrak{B}$ is $\sigma$-unital, then the equivalence relations $\sim_h$ and $\sim_{oh}$ coincide on $\mathbb{E}(\mathfrak{A}, \mathfrak{B})$.

• Consequently, if $\mathfrak{A}$ and $\mathfrak{B}$ are trivially graded, with $\mathfrak{A}$ separable and $\mathfrak{B}$ $\sigma$-unital, then

$$K_0(\mathfrak{B}) \cong KK(\mathbb{C}, \mathfrak{B}), \quad K_1(\mathfrak{B}) \cong KK^1(\mathbb{C}, \mathfrak{B}),$$

and $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B})^{-1} \cong KK^1(\mathfrak{A}, \mathfrak{B})$.

Note that $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B})^{-1}$ is the group of invertible elements of $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B})$ (recalled later below). If $\mathfrak{A}$ is separable, the Busby invariant $\tau : \mathfrak{A} \to Q^s(\mathfrak{B})$ corresponding to an extension: $0 \to \mathfrak{B} \otimes \mathbb{K} \to E \to \mathfrak{A} \to 0$ defines an invertible element of $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B})$ if and only if $\tau$ lifts to a completely positive contraction from $\mathfrak{A}$ to $M^s(\mathfrak{B})$.

If $\mathfrak{A}$ is a separable, nuclear $C^*$-algebra, then $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B})$ is a group for any $C^*$-algebra $\mathfrak{B}$. Hence then $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B})^{-1} = \mathrm{Ext}(\mathfrak{A}, \mathfrak{B})$.

(**Associativity**). Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ be graded $C^*$-algebras with $\mathfrak{A}$ and $\mathfrak{B}$ separable and $\mathfrak{C}$ $\sigma$-unital. Then the following diagram commutes:

$$\begin{array}{ccc} KK(\mathfrak{A},\mathfrak{B}) \times KK(\mathfrak{B},\mathfrak{C}) \times KK(\mathfrak{C},\mathfrak{D}) & \xrightarrow{\mathrm{id}\times\otimes_{\mathfrak{C}}} & KK(\mathfrak{A},\mathfrak{B}) \times KK(\mathfrak{B},\mathfrak{D}) \\ {\scriptstyle \otimes_{\mathfrak{B}}\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \otimes_{\mathfrak{B}}} \\ KK(\mathfrak{A},\mathfrak{C}) \times KK(\mathfrak{C},\mathfrak{D}) & \xrightarrow{\otimes_{\mathfrak{C}}} & KK(\mathfrak{A},\mathfrak{D}), \end{array}$$

with $x \otimes_{\mathfrak{B}} (y \otimes_{\mathfrak{C}} z) = (x \otimes_{\mathfrak{B}} y) \otimes_{\mathfrak{C}} z$.

(**Functoriality**). Let $\mathfrak{A}_1, \mathfrak{A}_2 = \mathfrak{A}, \mathfrak{B}_1 = \mathfrak{B}, \mathfrak{B}_2, \mathfrak{D}_1 = \mathfrak{D}, \mathfrak{D}_2$ be graded $C^*$-algebras, with $\mathfrak{A}_1, \mathfrak{A}_2$ separable and $\mathfrak{D}_1, \mathfrak{D}_2$ $\sigma$-unital, and let $f : \mathfrak{A}_1 \to \mathfrak{A}_2$, $g : \mathfrak{B}_1 \to \mathfrak{B}_2$, $h : \mathfrak{D}_1 \to \mathfrak{D}_2$ be graded $*$-homomorphisms. Then the following diagrams commute:

$$\begin{array}{ccc} KK(\mathfrak{A}_2,\mathfrak{D}) \times KK(\mathfrak{D},\mathfrak{B}) & \xrightarrow{\otimes_{\mathfrak{D}}} & KK(\mathfrak{A}_2,\mathfrak{B}) \\ {\scriptstyle f^*\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle f^*} \\ KK(\mathfrak{A}_1,\mathfrak{D}) \times KK(\mathfrak{D},\mathfrak{B}) & \xrightarrow{\otimes_{\mathfrak{D}}} & KK(\mathfrak{A}_1,\mathfrak{B}), \end{array}$$

with $f^*(x \otimes_{\mathfrak{D}} y) = f^*(x) \otimes_{\mathfrak{D}} y \in KK(\mathfrak{A}_1, \mathfrak{B})$, and

$$\begin{array}{ccc} KK(\mathfrak{A},\mathfrak{D}) \times KK(\mathfrak{D},\mathfrak{B}_1) & \xrightarrow{\otimes_{\mathfrak{D}}} & KK(\mathfrak{A},\mathfrak{B}_1) \\ {\scriptstyle \mathrm{id}\times g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ KK(\mathfrak{A},\mathfrak{D}) \times KK(\mathfrak{D},\mathfrak{B}_2) & \xrightarrow{\otimes_{\mathfrak{D}}} & KK(\mathfrak{A},\mathfrak{B}_2), \end{array}$$

with $g_*(x \otimes_{\mathfrak{D}} y) = x \otimes_{\mathfrak{D}} g_*(y) \in KK(\mathfrak{A}, \mathfrak{B}_2)$, and

$$\begin{array}{ccc} KK(\mathfrak{A},\mathfrak{D}_1) \times KK(\mathfrak{D}_2,\mathfrak{B}) & \xrightarrow{h_*\times\mathrm{id}} & KK(\mathfrak{A},\mathfrak{D}_2) \times KK(\mathfrak{D}_2,\mathfrak{B}) \\ {\scriptstyle \mathrm{id}\times h^*}\downarrow & & \downarrow{\scriptstyle \otimes_{\mathfrak{D}_2}} \\ KK(\mathfrak{A},\mathfrak{D}_1) \times KK(\mathfrak{D}_1,\mathfrak{B}) & \xrightarrow{\otimes_{\mathfrak{D}_1}} & KK(\mathfrak{A},\mathfrak{B}), \end{array}$$

with $h_*(x) \otimes_{\mathfrak{D}_2} y = x \otimes_{\mathfrak{D}_1} h^*(y) \in KK(\mathfrak{A}, \mathfrak{B})$, and

$$\begin{array}{ccc} KK(\mathfrak{A},\mathfrak{D}) & \xrightarrow{i_2} & KK(\mathfrak{A},\mathfrak{A}) \times KK(\mathfrak{A},\mathfrak{D}) \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle \otimes_{\mathfrak{A}}} \\ KK(\mathfrak{A},\mathfrak{D}) \times KK(\mathfrak{D},\mathfrak{D}) & \xrightarrow{\otimes_{\mathfrak{D}}} & KK(\mathfrak{A},\mathfrak{D}), \end{array}$$

with $i_1, i_2$ unital canonical injections and $[\mathrm{id}_{\mathfrak{A}}] \otimes_{\mathfrak{A}} x = x \otimes_{\mathfrak{D}} [\mathrm{id}_{\mathfrak{D}}] = x \in KK(\mathfrak{A}, \mathfrak{D})$.

• Let $\mathfrak{A}, \mathfrak{D}, \mathfrak{B}$ be graded $C^*$-algebras, with $\mathfrak{A}$ separable and $\mathfrak{D}$ $\sigma$-unital. Let $f : \mathfrak{A} \to \mathfrak{D}$ and $g : \mathfrak{D} \to \mathfrak{B}$ be graded homomorphisms. Then

$$\begin{aligned} f^* : KK(\mathfrak{D}, \mathfrak{B}) &\to KK(\mathfrak{A}, \mathfrak{B}), \\ y &\mapsto f^*(y), \\ \otimes_{\mathfrak{D}} : KK(\mathfrak{A}, \mathfrak{D}) \times KK(\mathfrak{D}, \mathfrak{B}) &\to KK(\mathfrak{A}, \mathfrak{B}), \\ ([(\mathfrak{D}, f, 0)], y) = ([f], y) &\mapsto [f] \otimes_{\mathfrak{D}} y = f^*(y) \end{aligned}$$

with the first $f^*(\cdot)$ equal to $[f] \otimes_{\mathfrak{D}} (\cdot) \equiv f^*(\cdot)$ the second, and

$$\begin{aligned} g_* : KK(\mathfrak{A}, \mathfrak{D}) &\to KK(\mathfrak{A}, \mathfrak{B}) \\ x &\mapsto g_*(x), \\ \otimes_{\mathfrak{D}} : KK(\mathfrak{A}, \mathfrak{D}) \times KK(\mathfrak{D}, \mathfrak{B}) &\to KK(\mathfrak{A}, \mathfrak{B}), \\ (x, [(\mathfrak{B}, g, 0)]) = (x, [g]) &\mapsto x \otimes_{\mathfrak{D}} [g] = g_*(x), \end{aligned}$$

with the first $g_*(\cdot)$ equal to $(\cdot) \otimes_{\mathfrak{D}} [g] \equiv g_*(\cdot)$ the second.

(**Ring structure**). If $\mathfrak{A}$ is a separable $C^*$-algebra, then $KK(\mathfrak{A}, \mathfrak{A})$ is a unital ring under the Kasparov intersection product, with the class $[\mathrm{id}_{\mathfrak{A}}]$ as the unit:

$$\otimes_{\mathfrak{A}} : KK(\mathfrak{A}, \mathfrak{A}) \times KK(\mathfrak{A}, \mathfrak{A}) \to KK(\mathfrak{A}, \mathfrak{A}),$$

with $x \otimes_{\mathfrak{A}} [\mathrm{id}_{\mathfrak{A}}] = [\mathrm{id}_{\mathfrak{A}}] \otimes_{\mathfrak{A}} x = x$.

**Example 5.3.** $KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$ as a ring, with the Kasparov product as multiplication:

$$\begin{aligned} \otimes_{\mathbb{C}} : KK(\mathbb{C}, \mathbb{C}) \times KK(\mathbb{C}, \mathbb{C}) &\to KK(\mathbb{C}, \mathbb{C}), \\ ([\mathbb{C}, \mathrm{id}_{\mathbb{C}}, 0)], [(\mathbb{C}, \mathrm{id}_{\mathbb{C}}, 0)]) = ([\mathrm{id}_{\mathbb{C}}], [\mathrm{id}_{\mathbb{C}}]) &\mapsto [\mathrm{id}_{\mathbb{C}}] \otimes_{\mathbb{C}} [\mathrm{id}_{\mathbb{C}}] \\ &= [(\mathbb{C} \otimes_{\mathrm{id}_{\mathbb{C}}} \mathbb{C}, \mathrm{id}_{\mathbb{C}} \circ \mathrm{id}_{\mathbb{C}}, 0)] \\ &= [(\mathbb{C}, \mathrm{id}_{\mathbb{C}}, 0)] = [\mathrm{id}_{\mathbb{C}}]. \end{aligned}$$

If $\mathfrak{A}$ is an (approximately finite dimensional) AF-algebra, that is an inductive limit of finite dimensional $C^*$-algebras, then

$$KK(\mathfrak{A}, \mathfrak{A}) \cong \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{A}))$$

the endomorphism ring of $K_0(\mathfrak{A})$. The isomorphism is deduced from the UCT (below later) and that $\mathfrak{A}$ is separable, belongs to the UCT class, $K_1(\mathfrak{A}) \cong 0$, and $K_0(\mathfrak{A})$ is free. If a $C^*$-algebra satisfies these four conditions, then the isomorphism also holds.

**(Generalized Kasparov product map).** Let $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{B}_1$, $\mathfrak{B}_2$, and $\mathfrak{D}$ be graded $C^*$-algebras, with $\mathfrak{A}_1$ and $\mathfrak{A}_2$ separable and $\mathfrak{B}_1$ and $\mathfrak{D}$ $\sigma$-unital. Then the (generalized) Kasparov (intersection) product:

$$\begin{array}{c} KK(\mathfrak{A}_1, \mathfrak{B}_1 \otimes \mathfrak{D}) \times KK(\mathfrak{D} \otimes \mathfrak{A}_2, \mathfrak{B}_2) \\ {\scriptstyle \otimes_{\mathfrak{D}}} \Big\downarrow \\ KK(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{B}_1 \otimes \mathfrak{B}_2) \end{array}$$

is bilinear, contravariantly functorial in $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and covariantly functorial in $\mathfrak{B}_1$ and $\mathfrak{B}_2$.

Let $x \in KK(\mathfrak{A}_1, \mathfrak{B}_1 \otimes \mathfrak{D})$, $y \in KK(\mathfrak{D} \otimes \mathfrak{A}_2, \mathfrak{B}_2)$. Define $x \otimes_{\mathfrak{D}} y$ to be the composite:

$$(x \otimes [\mathrm{id}_{\mathfrak{A}_2}]) \otimes_E ([\mathrm{id}_{\mathfrak{B}_1}] \otimes y) = \tau_{\mathfrak{A}_2}(x) \otimes_E \tau_{\mathfrak{B}_1}(y)$$

in $KK(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{B}_1 \otimes \mathfrak{B}_2)$, where $E = \mathfrak{B}_1 \otimes \mathfrak{D} \otimes \mathfrak{A}_2$.

Note that

$$x \otimes [\mathrm{id}_{\mathfrak{A}_2}] = \tau_{\mathfrak{A}_2}(x) \in KK(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{B}_1 \otimes \mathfrak{D} \otimes \mathfrak{A}_2),$$
$$[\mathrm{id}_{\mathfrak{B}_1}] \otimes y = \tau_{\mathfrak{B}_1}(y) \in KK(\mathfrak{B}_1 \otimes \mathfrak{D} \otimes \mathfrak{A}_2, \mathfrak{B}_1 \otimes \mathfrak{B}_2).$$

Hence, in fact we have

$$\begin{array}{c} KK(\mathfrak{A}_1, \mathfrak{B}_1 \otimes \mathfrak{D}) \times KK(\mathfrak{D} \otimes \mathfrak{A}_2, \mathfrak{B}_2) \\ {\scriptstyle (\tau_{\mathfrak{A}_2}, \tau_{\mathfrak{B}_1})} \Big\downarrow \\ KK(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{B}_1 \otimes \mathfrak{D} \otimes \mathfrak{A}_2) \times KK(\mathfrak{B}_1 \otimes \mathfrak{D} \otimes \mathfrak{A}_2, \mathfrak{B}_1 \otimes \mathfrak{B}_2) \\ {\scriptstyle \otimes_E = \otimes_{\mathfrak{B}_1 \otimes \mathfrak{D} \otimes \mathfrak{A}_2}} \Big\downarrow \\ KK(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{B}_1 \otimes \mathfrak{B}_2) \end{array}$$

and $\otimes_{\mathfrak{D}} = \otimes_{\mathfrak{B}_1 \otimes \mathfrak{D} \otimes \mathfrak{A}_2} \circ (\tau_{\mathfrak{A}_2}, \tau_{\mathfrak{B}_1})$.

• If $h : \mathfrak{D}_1 \to \mathfrak{D}_2$ is a homomorphism of $\sigma$-unital $C^*$-algebras, then we have the following composites:

$$\begin{array}{c} KK(\mathfrak{A}_1, \mathfrak{B}_1 \otimes \mathfrak{D}_1) \\ {\scriptstyle (\mathrm{id}_{\mathfrak{B}_1} \otimes h)_*} \Big\downarrow \\ KK(\mathfrak{A}_1, \mathfrak{B}_1 \otimes \mathfrak{D}_2) \times KK(\mathfrak{D}_2 \otimes \mathfrak{A}_2, \mathfrak{B}_2) \\ {\scriptstyle \otimes_{\mathfrak{D}_2}} \Big\downarrow \\ KK(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{B}_1 \otimes \mathfrak{B}_2), \end{array}$$

and

$$\begin{array}{c} KK(\mathfrak{D}_2 \otimes \mathfrak{A}_2, \mathfrak{B}_2) \\ \Big\downarrow{\scriptstyle (h\otimes \mathrm{id}_{\mathfrak{A}_2})^*} \\ KK(\mathfrak{A}_1, \mathfrak{B}_1 \otimes \mathfrak{D}_1) \times KK(\mathfrak{D}_1 \otimes \mathfrak{A}_2, \mathfrak{B}_2) \\ \Big\downarrow{\scriptstyle \otimes_{\mathfrak{D}_1}} \\ KK(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{B}_1 \otimes \mathfrak{B}_2), \end{array}$$

so that $\otimes_{\mathfrak{D}_2} \circ (\mathrm{id}_{\mathfrak{B}_1} \otimes h)_* = \otimes_{\mathfrak{D}_1} \circ (\mathrm{id}_{\mathfrak{A}_2} \otimes h)^*$ and

$$(\mathrm{id}_{\mathfrak{B}_1} \otimes h)_*(x) \otimes_{\mathfrak{D}_2} y = x \otimes_{\mathfrak{D}_1} (\mathrm{id}_{\mathfrak{A}_2} \otimes h)^*(y)$$

for $x \in KK(\mathfrak{A}_1, \mathfrak{B}_1 \otimes \mathfrak{D}_1)$ and $y \in KK(\mathfrak{D}_2 \otimes \mathfrak{A}_2, \mathfrak{B}_2)$.

- We have the following commuting composites:

$$\begin{array}{c} KK(\mathfrak{A}_1, \mathfrak{B}_1 \otimes \mathfrak{D}) \times KK(\mathfrak{D} \otimes \mathfrak{A}_2, \mathfrak{B}_2) \\ {\scriptstyle \otimes_{\mathfrak{D}}}\Big\downarrow \\ KK(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{B}_1 \otimes \mathfrak{B}_2) \\ {\scriptstyle \tau_{\mathfrak{D}_1}}\Big\downarrow \\ KK(\mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \mathfrak{D}_1, \mathfrak{B}_1 \otimes \mathfrak{B}_2 \otimes \mathfrak{D}_1) \end{array}$$

for any $\mathfrak{D}_1$, and

$$\begin{array}{c} KK(\mathfrak{A}_1, \mathfrak{B}_1 \otimes \mathfrak{D}) \times KK(\mathfrak{D} \otimes \mathfrak{A}_2, \mathfrak{B}_2) \\ {\scriptstyle \tau_{\mathfrak{D}_1}}\Big\downarrow{\scriptstyle \tau_{\mathfrak{D}_1}} \\ KK(\mathfrak{A}_1 \otimes \mathfrak{D}_1, \mathfrak{B}_1 \otimes \mathfrak{D} \otimes \mathfrak{D}_1) \times KK(\mathfrak{D} \otimes \mathfrak{D}_1 \otimes \mathfrak{A}_2, \mathfrak{B}_2 \otimes \mathfrak{D}_1) \\ \Big\downarrow{\scriptstyle \otimes_{\mathfrak{D}\otimes\mathfrak{D}_1}} \\ KK(\mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \mathfrak{D}_1, \mathfrak{B}_1 \otimes \mathfrak{B}_2 \otimes \mathfrak{D}_1) \end{array}$$

for any $\sigma$-unital $\mathfrak{D}_1$, so that

$$\tau_{\mathfrak{D}_1}(x \otimes_{\mathfrak{D}} y) = \tau_{\mathfrak{D}_1}(x) \otimes_{\mathfrak{D}\otimes\mathfrak{D}_1} \tau_{\mathfrak{D}_1}(y)$$

for $x \in KK(\mathfrak{A}_1, \mathfrak{B}_1 \otimes \mathfrak{D})$ and $y \in KK(\mathfrak{D} \otimes \mathfrak{A}_2, \mathfrak{B}_2)$.

- We have the following composites:

$$\begin{array}{c} KK(\mathfrak{A}_1, \mathfrak{B}_1 \otimes \mathfrak{D}_1 \otimes \mathfrak{D}) \times KK(\mathfrak{D} \otimes \mathfrak{D}_2 \otimes \mathfrak{A}_2, \mathfrak{B}_2) \\ {\scriptstyle \tau_{\mathfrak{D}_2}} \Big\downarrow {\scriptstyle \tau_{\mathfrak{D}_1}} \\ KK(\mathfrak{A}_1 \otimes \mathfrak{D}_2, \mathfrak{B}_1 \otimes \mathfrak{D}_1 \otimes \mathfrak{D} \otimes \mathfrak{D}_2) \times KK(\mathfrak{D}_1 \otimes \mathfrak{D} \otimes \mathfrak{D}_2 \otimes \mathfrak{A}_2, \mathfrak{D}_1 \otimes \mathfrak{B}_2) \\ \Big\downarrow {\scriptstyle \otimes_{\mathfrak{D}_1 \otimes \mathfrak{D} \otimes \mathfrak{D}_2}} \\ KK(\mathfrak{A}_1 \otimes \mathfrak{D}_2 \otimes \mathfrak{A}_2, \mathfrak{B}_1 \otimes \mathfrak{D}_1 \otimes \mathfrak{B}_2) \end{array}$$

and

$$\begin{array}{c} KK(\mathfrak{A}_1, \mathfrak{B}_1 \otimes \mathfrak{D}_1 \otimes \mathfrak{D}) \times KK(\mathfrak{D} \otimes \mathfrak{D}_2 \otimes \mathfrak{A}_2, \mathfrak{B}_2) \\ {\scriptstyle \otimes_{\mathfrak{D}}} \Big\downarrow \\ KK(\mathfrak{A}_1 \otimes \mathfrak{D}_2 \otimes \mathfrak{A}_2, \mathfrak{B}_1 \otimes \mathfrak{D}_1 \otimes \mathfrak{B}_2), \end{array}$$

so that $\otimes_{\mathfrak{D}_1 \otimes \mathfrak{D} \otimes \mathfrak{D}_2} \circ (\tau_{\mathfrak{D}_2}, \tau_{\mathfrak{D}_1}) = \otimes_{\mathfrak{D}}$ and

$$\tau_{\mathfrak{D}_2}(x) \otimes_{\mathfrak{D}_1 \otimes \mathfrak{D} \otimes \mathfrak{D}_2} \tau_{\mathfrak{D}_1}(y) = x \otimes_{\mathfrak{D}} y.$$

Note that by definition,

$$x \otimes_{\mathfrak{D}} y = (\tau_{\mathfrak{D}_2 \otimes \mathfrak{A}_2}(x)) \otimes_{\mathfrak{D}_2 \otimes \mathfrak{A}_2 \otimes \mathfrak{D} \otimes \mathfrak{B}_1 \otimes \mathfrak{D}_1} (\tau_{\mathfrak{B}_1 \otimes \mathfrak{D}_1}(y)).$$

**(Kasparov product on graded KK)**. We can define the following map:

$$\otimes_{\mathfrak{D}} : KK^i(\mathfrak{A}, \mathfrak{D}) \times KK^j(\mathfrak{D}, \mathfrak{B}) \to KK^{i+j}(\mathfrak{A}, \mathfrak{B}),$$

where $i, j, i+j$ (mod 2), and $KK^0 = KK$.

The map

$$\otimes_{\mathfrak{D}} : KK^1(\mathfrak{A}, \mathfrak{D}) \times KK(\mathfrak{D}, \mathfrak{B}) \to KK^1(\mathfrak{A}, \mathfrak{B})$$

with $KK^1(\mathfrak{A}, \mathfrak{D}) \cong KK(\mathfrak{A}, S\mathfrak{D})$ and $KK^1(\mathfrak{A}, \mathfrak{B}) \cong KK(S\mathfrak{A}, \mathfrak{B})$ is defined by $x \otimes_{\mathfrak{D}} y = \tau_S(x) \otimes_{\mathfrak{D}} y \in KK(S\mathfrak{A}, \mathfrak{B})$ or by $x \otimes_{\mathfrak{D}} y = x \otimes_{S\mathfrak{D}} \tau_S(y) \in KK(\mathfrak{A}, S\mathfrak{B})$, where $S = S\mathbb{C} = C_0(\mathbb{R})$.

The map

$$\otimes_{\mathfrak{D}} : KK(\mathfrak{A}, \mathfrak{D}) \times KK^1(\mathfrak{D}, \mathfrak{B}) \to KK^1(\mathfrak{A}, \mathfrak{B})$$

with $KK^1(\mathfrak{D}, \mathfrak{B}) \cong KK(S\mathfrak{D}, \mathfrak{B})$ and $KK^1(\mathfrak{A}, \mathfrak{B}) \cong KK(\mathfrak{A}, S\mathfrak{B})$ is defined by $x \otimes_{\mathfrak{D}} y = x \otimes_{\mathfrak{D}} \tau_S(y) \in KK(\mathfrak{A}, S\mathfrak{B})$ or $x \otimes_{\mathfrak{D}} y = \tau_S(x) \otimes_{S\mathfrak{D}} y \in KK(S\mathfrak{A}, \mathfrak{B})$.

The map

$$\otimes_{\mathfrak{D}} : KK^1(\mathfrak{A}, \mathfrak{D}) \times KK^1(\mathfrak{D}, \mathfrak{B}) \to KK(\mathfrak{A}, \mathfrak{B})$$

with $KK^1(\mathfrak{A}, \mathfrak{D}) \cong KK(S\mathfrak{A}, \mathfrak{D})$ and $KK^1(\mathfrak{D}, \mathfrak{B}) \cong KK(S\mathfrak{D}, \mathfrak{B})$ is defined by $x \otimes_{\mathfrak{D}} y = \tau_S(x) \otimes_{S\mathfrak{D}} (y) \in KK(\mathfrak{A}, \mathfrak{B})$ or $x \otimes_{\mathfrak{D}} y = x \otimes_{\mathfrak{D}} \tau_S(y) \in KK(S\mathfrak{A}, S\mathfrak{B})$.

# 6 KK-theory equivalence

An element $x \in KK(\mathfrak{A}, \mathfrak{B})$ is a **KK-equivalence** if there is $y \in KK(\mathfrak{B}, \mathfrak{A})$ such that $x \otimes_{\mathfrak{B}} y = [\mathrm{id}_{\mathfrak{A}}] \in KK(\mathfrak{A}, \mathfrak{A})$ and $y \otimes_{\mathfrak{A}} x = [\mathrm{id}_{\mathfrak{B}}] \in KK(\mathfrak{B}, \mathfrak{B})$. $C^*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are **KK-equivalent** if there exists a KK-equivalence in $KK(\mathfrak{A}, \mathfrak{B})$. Then we may denote its being by $\mathfrak{A} \sim_{KK} \mathfrak{B}$.

**Lemma 6.1.** *The KK-theory equivalence for $C^*$-algebras is an equivalence relation.*

*Proof.* The equivalence $\mathfrak{A} \sim_{KK} \mathfrak{A}$. For it there is the class $[\mathrm{id}_{\mathfrak{A}}] \in KK(\mathfrak{A}, \mathfrak{A})$ such that $[\mathrm{id}_{\mathfrak{A}}] \otimes_{\mathfrak{A}} [\mathrm{id}_{\mathfrak{A}}] = [\mathrm{id}_{\mathfrak{A}} \circ \mathrm{id}_{\mathfrak{A}}] = [\mathrm{id}_{\mathfrak{A}}] \in KK(\mathfrak{A}, \mathfrak{A})$.

If $\mathfrak{A} \sim_{KK} \mathfrak{B}$, then $\mathfrak{B} \sim_{KK} \mathfrak{A}$. For it, it is clear by definition.

If $\mathfrak{A} \sim_{KK} \mathfrak{B}$ and $\mathfrak{B} \sim_{KK} \mathfrak{C}$, then $\mathfrak{A} \sim_{KK} \mathfrak{C}$. For it, since there are $x \in KK(\mathfrak{A}, \mathfrak{B})$ and $y \in KK(\mathfrak{B}, \mathfrak{A})$ with $x \otimes_{\mathfrak{B}} y = [\mathrm{id}_{\mathfrak{A}}] \in KK(\mathfrak{A}, \mathfrak{A})$ and $y \otimes_{\mathfrak{A}} x = [\mathrm{id}_{\mathfrak{B}}] \in KK(\mathfrak{B}, \mathfrak{B})$, and there are $s \in KK(\mathfrak{B}, \mathfrak{C})$ and $t \in KK(\mathfrak{C}, \mathfrak{B})$ with $s \otimes_{\mathfrak{C}} t = [\mathrm{id}_{\mathfrak{B}}] \in KK(\mathfrak{B}, \mathfrak{B})$ and $t \otimes_{\mathfrak{B}} s = [\mathrm{id}_{\mathfrak{C}}] \in KK(\mathfrak{C}, \mathfrak{C})$, then $x \otimes_{\mathfrak{B}} s \in KK(\mathfrak{A}, \mathfrak{C})$ and $t \otimes_{\mathfrak{B}} y \in KK(\mathfrak{C}, \mathfrak{A})$ such that

$$\begin{aligned}
(x \otimes_{\mathfrak{B}} s) \otimes_{\mathfrak{C}} (t \otimes_{\mathfrak{B}} y) &= x \otimes_{\mathfrak{B}} ((s \otimes_{\mathfrak{C}} t) \otimes_{\mathfrak{B}} y) \\
&= x \otimes_{\mathfrak{B}} ([\mathrm{id}_{\mathfrak{B}}] \otimes_{\mathfrak{B}} y) = x \otimes_{\mathfrak{B}} y = [\mathrm{id}_{\mathfrak{A}}] \in KK(\mathfrak{A}, \mathfrak{A}), \\
(t \otimes_{\mathfrak{B}} y) \otimes_{\mathfrak{A}} (x \otimes_{\mathfrak{B}} s) &= t \otimes_{\mathfrak{B}} ((y \otimes_{\mathfrak{A}} x) \otimes_{\mathfrak{B}} s) \\
&= t \otimes_{\mathfrak{B}} ([\mathrm{id}_{\mathfrak{B}}] \otimes_{\mathfrak{B}} s) = t \otimes_{\mathfrak{B}} s = [\mathrm{id}_{\mathfrak{C}}] \in KK(\mathfrak{C}, \mathfrak{C})
\end{aligned}$$

by associativity of Kasparov product. □

If $x \in KK(\mathfrak{A}, \mathfrak{B})$ is a KK-equivalence with $y$ its inverse, then for any $C^*$-algebra $\mathfrak{D}$, the following maps:

$$x \otimes_{\mathfrak{B}} (\cdot) : KK(\mathfrak{B}, \mathfrak{D}) \to KK(\mathfrak{A}, \mathfrak{D}), \quad (\cdot) \otimes_{\mathfrak{A}} x : KK(\mathfrak{D}, \mathfrak{A}) \to KK(\mathfrak{D}, \mathfrak{B})$$

are isomorphisms. Indeed, if $x \otimes_{\mathfrak{B}} f = x \otimes_{\mathfrak{B}} g \in KK(\mathfrak{A}, \mathfrak{D})$, then

$$\begin{aligned}
y \otimes_{\mathfrak{A}} (x \otimes_{\mathfrak{B}} f) &= (y \otimes_{\mathfrak{A}} x) \otimes_{\mathfrak{B}} f = [\mathrm{id}_{\mathfrak{B}}] \otimes_{\mathfrak{B}} f = f, \\
y \otimes_{\mathfrak{A}} (x \otimes_{\mathfrak{B}} g) &= (y \otimes_{\mathfrak{A}} x) \otimes_{\mathfrak{B}} g = [\mathrm{id}_{\mathfrak{B}}] \otimes_{\mathfrak{B}} g = g,
\end{aligned}$$

so that $f = g \in KK(\mathfrak{B}, \mathfrak{D})$. Hence the map $x \otimes_{\mathfrak{B}} (\cdot)$ is injective. If $h \in KK(\mathfrak{A}, \mathfrak{D})$, then $y \otimes_{\mathfrak{A}} h \in KK(\mathfrak{B}, \mathfrak{D})$. Therefore,

$$x \otimes_{\mathfrak{B}} (y \otimes_{\mathfrak{A}} h) = (x \otimes_{\mathfrak{B}} y) \otimes_{\mathfrak{A}} h = [\mathrm{id}_{\mathfrak{A}}] \otimes_{\mathfrak{A}} h = h,$$

which shows that the map $x \otimes_{\mathfrak{B}} (\cdot)$ is surjective. The similar holds for $(\cdot) \otimes_{\mathfrak{A}} x$.

In particular, it follows that if $\mathfrak{A}$ and $\mathfrak{B}$ are $\sigma$-unital and $\mathfrak{A} \sim_{KK} \mathfrak{B}$, then $K_j(\mathfrak{A}) \cong K_j(\mathfrak{B})$ $(j = 0, 1)$, which may be viewed as an equivalence relation, because

$$K_j(\mathfrak{A}) \cong KK^j(\mathbb{C}, \mathfrak{A}) \cong KK^j(\mathbb{C}, \mathfrak{B}) \cong K_j(\mathfrak{B}).$$

Then we may say that KK-theory equivalence implies K-theory equivalence for $\sigma$-unital $C^*$-algebras.

If $x \in KK(\mathfrak{A}, \mathfrak{B})$ is a KK-equivalence with $y$ its inverse, then there is a ring-isomorphism from $KK(\mathfrak{A}, \mathfrak{A})$ to $KK(\mathfrak{B}, \mathfrak{B})$ by the map $y \otimes_{\mathfrak{A}} (\cdot) \otimes_{\mathfrak{A}} x$. Indeed, for $f, g \in KK(\mathfrak{A}, \mathfrak{A})$, if $y \otimes_{\mathfrak{A}} f \otimes_{\mathfrak{A}} x = y \otimes_{\mathfrak{A}} g \otimes_{\mathfrak{A}} x \in KK(\mathfrak{B}, \mathfrak{B})$, then

$$\begin{aligned} x \otimes_{\mathfrak{B}} (y \otimes_{\mathfrak{A}} f \otimes_{\mathfrak{A}} x) \otimes_{\mathfrak{B}} y &= (x \otimes_{\mathfrak{B}} y) \otimes_{\mathfrak{A}} f \otimes_{\mathfrak{A}} (x \otimes_{\mathfrak{B}} y) \\ &= [\mathrm{id}_{\mathfrak{A}}] \otimes_{\mathfrak{A}} f \otimes_{\mathfrak{A}} [\mathrm{id}_{\mathfrak{A}}] = f \end{aligned}$$

and similarly, $x \otimes_{\mathfrak{B}} (y \otimes_{\mathfrak{A}} g \otimes_{\mathfrak{A}} x) \otimes_{\mathfrak{B}} y = g$. Hence the map is injective. If $h \in KK(\mathfrak{B}, \mathfrak{B})$, then $x \otimes_{\mathfrak{B}} h \otimes_{\mathfrak{B}} y \in KK(\mathfrak{A}, \mathfrak{A})$ and

$$\begin{aligned} y \otimes_{\mathfrak{A}} (x \otimes_{\mathfrak{B}} h \otimes_{\mathfrak{B}} y) \otimes_{\mathfrak{A}} x &= (y \otimes_{\mathfrak{A}} x) \otimes_{\mathfrak{B}} h \otimes_{\mathfrak{B}} (y \otimes_{\mathfrak{A}} x) \\ &= [\mathrm{id}_{\mathfrak{B}}] \otimes_{\mathfrak{B}} h \otimes_{\mathfrak{B}} [\mathrm{id}_{\mathfrak{B}}] = h, \end{aligned}$$

which shows that the map is surjective.

**Example 6.2.** • Isomorphic $C^*$-algebras are KK-equivalent.

If $\alpha : \mathfrak{A} \to \mathfrak{B}$ and $\beta : \mathfrak{B} \to \mathfrak{A}$ are isomorphisms, then $[\alpha] \otimes_{\mathfrak{B}} [\beta] = [\beta \circ \alpha] = [\mathrm{id}_{\mathfrak{A}}] \in KK(\mathfrak{A}, \mathfrak{A})$ and $[\beta] \otimes_{\mathfrak{A}} [\alpha] = [\alpha \circ \beta] = [\mathrm{id}_{\mathfrak{B}}] \in KK(\mathfrak{B}, \mathfrak{B})$.

• Any $C^*$-algebra $\mathfrak{A}$, the $n \times n$ matrix algebra $M_n(\mathfrak{A})$ over $\mathfrak{A}$, and $\mathfrak{A} \otimes \mathbb{K}$ are all KK-equivalent. Stably isomorphic $C^*$-algebras are KK-equivalent.

Indeed, $KK(\mathfrak{A}, M_n(\mathfrak{A})) \cong KK(\mathfrak{A}, \mathfrak{A})$, $KK(\mathfrak{A}, \mathfrak{A} \otimes \mathbb{K}) \cong KK(\mathfrak{A}, \mathfrak{A})$, and $KK(M_n(\mathfrak{A}), \mathfrak{A} \otimes \mathbb{K}) \cong KK(\mathfrak{A}, \mathfrak{A})$ all contain KK-equivalences, because any $C^*$-algebra $\mathfrak{A}$ is KK-equivalent to itself by the KK-class of its identity map $\mathrm{id}_{\mathfrak{A}}$, i.e., $[\mathrm{id}_{\mathfrak{A}}] \in KK(\mathfrak{A}, \mathfrak{A})$ with the inverse itself and $[\mathrm{id}_{\mathfrak{A}}] \otimes_{\mathfrak{A}} [\mathrm{id}_{\mathfrak{A}}] = [\mathrm{id}_{\mathfrak{A}}] \in KK(\mathfrak{A}, \mathfrak{A})$. A $C^*$-algebra $\mathfrak{A}$ is stably isomorphic to $\mathfrak{B}$ if $\mathfrak{A} \otimes \mathbb{K} \cong \mathfrak{B} \otimes \mathbb{K}$.

• Homotopy equivalent $C^*$-algebras are KK-equivalent.

Indeed, two $C^*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are homotopy equivalent if there are homomorphisms $\alpha : \mathfrak{A} \to \mathfrak{B}$ and $\beta : \mathfrak{B} \to \mathfrak{A}$ such that $\beta \circ \alpha$ and $\alpha \circ \beta$ are homotopic to $\mathrm{id}_{\mathfrak{A}}$ and $\mathrm{id}_{\mathfrak{B}}$ respectively. Then for the corresponding classes $[\alpha] \in KK(\mathfrak{A}, \mathfrak{B})$ and $[\beta] \in KK(\mathfrak{B}, \mathfrak{A})$, we have

$$[\beta] \otimes_{\mathfrak{A}} [\alpha] = [\alpha \circ \beta] = [\mathrm{id}_{\mathfrak{B}}] \quad \text{and} \quad [\alpha] \otimes_{\mathfrak{B}} [\beta] = [\beta \circ \alpha] = [\mathrm{id}_{\mathfrak{A}}].$$

• Contractible $C^*$-algebras are KK-equivalent to zero.

If $\mathfrak{A}$ is a contractible $C^*$-algebra, then there are zero homomorphisms from $\mathfrak{A}$ to $\{0\}$ and from $\{0\}$ to $\mathfrak{A}$ and their compositions are zero homomorphisms that are homotopic to $\mathrm{id}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$ and the zero map $0 : \{0\} \to \{0\}$, respectively, by contractibility and triviality.

• If a compact Hausdorff space $X$ is contractible to a point, then $C(X)$ is KK-equivalent to $\mathbb{C}$.

Indeed, there are canonical unital $*$-homomorphisms $\alpha = 1 : C(X) \to \mathbb{C}$ and $\beta = 1 : \mathbb{C} \to C(X)$ and their compositions are the unit homomorphisms that are homotopic to $\mathrm{id}_{C(X)}$ and $\mathrm{id}_{\mathbb{C}}$ by contractibity of $X$ and triviality, respectively.

• If $\mathfrak{A}$ and $\mathfrak{B}$ are KK-equivalent via $x \in KK(\mathfrak{A}, \mathfrak{B})$, then for any $C^*$-algebra $\mathfrak{D}$, we have $\mathfrak{A} \otimes \mathfrak{D}$ and $\mathfrak{B} \otimes \mathfrak{D}$ KK-equivalent via $\tau_{\mathfrak{D}}$.

Check it out. Since $\tau_{\mathfrak{D}}(x) \in KK(\mathfrak{A} \otimes \mathfrak{D}, \mathfrak{B} \otimes \mathfrak{D})$ and $\tau_{\mathfrak{D}}(y) \in KK(\mathfrak{B} \otimes \mathfrak{D}, \mathfrak{A} \otimes \mathfrak{D})$, then we have

$$
\begin{aligned}
\tau_{\mathfrak{D}}(x) \otimes_{\mathfrak{B}\otimes\mathfrak{D}} \tau_{\mathfrak{D}}(y) &= \tau_{\mathfrak{D}}(x \otimes_{\mathfrak{B}} y) = \tau_{\mathfrak{D}}([\mathrm{id}_{\mathfrak{A}}]) \\
&= [\mathrm{id}_{\mathfrak{A}\otimes\mathfrak{D}}] \in KK(\mathfrak{A} \otimes \mathfrak{D}, \mathfrak{A} \otimes \mathfrak{D}), \\
\tau_{\mathfrak{D}}(y) \otimes_{\mathfrak{A}\otimes\mathfrak{D}} \tau_{\mathfrak{D}}(x) &= \tau_{\mathfrak{D}}(y \otimes_{\mathfrak{A}} x) = \tau_{\mathfrak{D}}([\mathrm{id}_{\mathfrak{B}}]) \\
&= [\mathrm{id}_{\mathfrak{B}\otimes\mathfrak{D}}] \in KK(\mathfrak{B} \otimes \mathfrak{D}, \mathfrak{B} \otimes \mathfrak{D})
\end{aligned}
$$

where the following diagrams commute:

$$
\begin{array}{ccc}
KK(\mathfrak{A}, \mathfrak{B}) \times KK(\mathfrak{B}, \mathfrak{A}) & \xrightarrow{\otimes_{\mathfrak{B}}} & KK(\mathfrak{A}, \mathfrak{A}) \\
{\scriptstyle \tau_{\mathfrak{D}} \times \tau_{\mathfrak{D}}} \downarrow & & \downarrow {\scriptstyle \tau_{\mathfrak{D}}} \\
KK(\mathfrak{A} \otimes \mathfrak{D}, \mathfrak{B} \otimes \mathfrak{D}) \times KK(\mathfrak{B} \otimes \mathfrak{D}, \mathfrak{A} \otimes \mathfrak{D}) & \xrightarrow{\otimes_{\mathfrak{B}\otimes\mathfrak{D}}} & KK(\mathfrak{A} \otimes \mathfrak{D}, \mathfrak{A} \otimes \mathfrak{D}),
\end{array}
$$

and

$$
\begin{array}{ccc}
KK(\mathfrak{B}, \mathfrak{A}) \times KK(\mathfrak{A}, \mathfrak{B}) & \xrightarrow{\otimes_{\mathfrak{A}}} & KK(\mathfrak{B}, \mathfrak{B}) \\
{\scriptstyle \tau_{\mathfrak{D}} \times \tau_{\mathfrak{D}}} \downarrow & & \downarrow {\scriptstyle \tau_{\mathfrak{D}}} \\
KK(\mathfrak{B} \otimes \mathfrak{D}, \mathfrak{A} \otimes \mathfrak{D}) \times KK(\mathfrak{A} \otimes \mathfrak{D}, \mathfrak{B} \otimes \mathfrak{D}) & \xrightarrow{\otimes_{\mathfrak{A}\otimes\mathfrak{D}}} & KK(\mathfrak{B} \otimes \mathfrak{D}, \mathfrak{B} \otimes \mathfrak{D}).
\end{array}
$$

• If $0 \to \mathfrak{A} \xrightarrow{i} \mathfrak{D} \xrightarrow{q} \mathfrak{B} \to 0$ is a split short exact sequence of $C^*$-algebras with $s : \mathfrak{B} \to \mathfrak{D}$ its section, then $\mathfrak{D}$ is KK-equivalent to the direct sum $\mathfrak{A} \oplus \mathfrak{B}$. Then we may write the split extension $\mathfrak{D} = \mathfrak{A} \rtimes \mathfrak{B} \sim_{KK} \mathfrak{A} \oplus \mathfrak{B}$.

Show it out. The element

$$
[i] \oplus [s] \in KK(\mathfrak{A}, \mathfrak{D}) \oplus KK(\mathfrak{B}, \mathfrak{D}) \cong KK(\mathfrak{A} \oplus \mathfrak{B}, \mathfrak{D})
$$

is a KK-equivalence with inverse

$$\pi_s \oplus [q] \in KK(\mathfrak{D},\mathfrak{A}) \oplus KK(\mathfrak{D},\mathfrak{B}) \cong KK(\mathfrak{D},\mathfrak{A}\oplus\mathfrak{B}),$$

where $\pi_s$ is the splitting morphism. Indeed,

$$\begin{aligned}
&([i]\oplus[s])\otimes_{\mathfrak{D}}(\pi_s\oplus[q])\\
&= ([i]\otimes_{\mathfrak{D}}\pi_s)\oplus([i]\otimes_{\mathfrak{D}}[q])\oplus([s]\otimes_{\mathfrak{D}}\pi_s)\oplus([s]\otimes_{\mathfrak{D}}[q])\\
&= (i^*(\pi_s))\oplus([q\circ i])\oplus([s]\otimes_{\mathfrak{D}}\pi_s)\oplus([q\circ s])\\
&= [\mathrm{id}_{\mathfrak{A}}]\oplus[0]\oplus[0]\oplus[\mathrm{id}_{\mathfrak{B}}] = [\mathrm{id}_{\mathfrak{A}}\oplus\mathrm{id}_{\mathfrak{B}}] \in KK(\mathfrak{A}\oplus\mathfrak{B},\mathfrak{A}\oplus\mathfrak{B}),
\end{aligned}$$

and

$$\begin{aligned}
&(\pi_s\oplus[q])\otimes_{\mathfrak{A}\oplus\mathfrak{B}}([i]\oplus[s])\\
&= (\pi_s\otimes_{\mathfrak{A}\oplus\mathfrak{B}}[i])\oplus(\pi_s\otimes_{\mathfrak{A}\oplus\mathfrak{B}}[s])\oplus([q]\otimes_{\mathfrak{A}\oplus\mathfrak{B}}[i])\oplus([q]\otimes_{\mathfrak{A}\oplus\mathfrak{B}}[s])\\
&= [i\circ\pi_s]\oplus[s\circ\pi_s]\oplus[i\circ q]\oplus[s\circ q]\\
&= [s\circ\pi_s]\oplus[0]\oplus[0]\oplus[s\circ q] = [\mathrm{id}_{\mathfrak{D}}] \in KK(\mathfrak{D},\mathfrak{D}).
\end{aligned}$$

• For $\mathbb{T}$ the one-torus or circle, there is $0 \to C_0(\mathbb{R}) \to C(\mathbb{T}) \to \mathbb{C} \to 0$ a split extension, so that $C(\mathbb{T}) = C_0(\mathbb{R}) \rtimes \mathbb{C} \sim_{KK} C_0(\mathbb{R}) \oplus \mathbb{C}$.

For $S^n$ the $n$-dimensional sphere, we have $0 \to C_0(\mathbb{R}^n) \to C(S^n) \to \mathbb{C} \to 0$ a split extension, and hence $C(S^n) = C_0(\mathbb{R}^n) \rtimes \mathbb{C} \sim_{KK} \mathbb{C} \oplus C_0(\mathbb{R}^n)$.

• If $\mathfrak{A}$ and $\mathfrak{B}$ are AF algebras, then $\mathfrak{A}$ and $\mathfrak{B}$ are KK-equivalent if and only their dimension groups are isomorphic as groups, ignoring the order structure.

Check it out. The UCT for KK-theory implies that

$$KK(\mathfrak{A},\mathfrak{B}) \cong \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{B}))$$

because the $K_0$-groups of $\mathfrak{A}$ and $\mathfrak{B}$, written as inductive limits of finite direct sums of $\mathbb{Z}$ as dimension groups, are torsion free, and their $K_1$-groups are zero. It follows that if $\mathfrak{A}$ and $\mathfrak{B}$ are KK-equivalent via $x \in KK(\mathfrak{A},\mathfrak{B})$ with inverse $y \in KK(\mathfrak{B},\mathfrak{A})$, then $x \otimes_{\mathfrak{B}} y = [\mathrm{id}_{\mathfrak{A}}]$ and $y \otimes_{\mathfrak{A}} x = [\mathrm{id}_{\mathfrak{B}}]$, so that the corresponding composites

$$\otimes_{\mathfrak{B}}y \circ \otimes_{\mathfrak{A}}x : K_0(\mathfrak{A}) \cong K_0(\mathbb{C},\mathfrak{A}) \xrightarrow{\otimes_{\mathfrak{A}} x} K_0(\mathbb{C},\mathfrak{B}) \cong K_0(\mathfrak{B}) \xrightarrow{\otimes_{\mathfrak{B}} y} K_0(\mathfrak{A}),$$

$$\otimes_{\mathfrak{A}}x \circ \otimes_{\mathfrak{B}}y : K_0(\mathfrak{B}) \cong K_0(\mathbb{C},\mathfrak{B}) \xrightarrow{\otimes_{\mathfrak{B}} y} K_0(\mathbb{C},\mathfrak{A}) \cong K_0(\mathfrak{A}) \xrightarrow{\otimes_{\mathfrak{A}} x} K_0(\mathfrak{B})$$

are $\mathrm{id}_{K_0(\mathfrak{A})}$ and $\mathrm{id}_{K_0(\mathfrak{B})}$, respectively. Hence $\otimes_{\mathfrak{A}}x$ and $\otimes_{\mathfrak{B}}y$ as maps are injective repectively, so that $K_0(\mathfrak{A}) \cong K_0(\mathfrak{B})$ as a group. The converse also holds via the UCT (given below).

• The Toeplitz algebra $\mathfrak{T}$ is KK-equivalent to $\mathbb{C}$.

Indeed, $\mathfrak{T}$ is generated by the unilateral shift $U$ on a Hilbert space with an orthogonal basis such as $H_{\mathbb{C}} \cong l^2(\mathbb{Z})$ of all square summable sequences of $\mathbb{C}$ on $\mathbb{Z}$. Then $1 - UU^*$ is a one-dimensional projection, and $\mathfrak{T}$ contains $\mathbb{K}$ as an essential ideal, i.e., which has non-zero intersection with any other non-zero closed ideal, and the quotient $\mathfrak{T}/\mathbb{K} \cong C(S^1)$ the $C^*$-algebra of all continous functions on the unit circle $S^1$, so that $0 \to \mathbb{K} \to \mathfrak{T} \xrightarrow{q} C(S^1) \to 0$ is exact, and the sequence dose not split, where the quotient map $q$ sends $U$ to the coordinate unitary function $u(z) = z \in S^1$. Let $ev_1 \circ q : \mathfrak{T} \to \mathbb{C}$ be the composition of the quotient map $q$ with the evaluation map $ev_1$ at $1 \in S^1$. Let $j : \mathbb{C} \to \mathfrak{T}$ be the unital embedding. Then $[ev_1 \circ q] \in KK(\mathfrak{T}, \mathbb{C})$ is a KK-equivalence with inverse $[j] \in KK(\mathbb{C}, \mathfrak{T})$. For this, $(ev_1 \circ q) \circ j = \mathrm{id}_{\mathbb{C}}$. Hence $[j] \otimes_{\mathfrak{T}} [ev_1 \circ q] = [\mathrm{id}_{\mathbb{C}}] \in KK(\mathbb{C}, \mathbb{C})$. Also, $[ev_1 \circ q] \otimes_{\mathbb{C}} [j] = [j \circ (ev_1 \circ q)] = [\mathrm{id}_{\mathfrak{T}}] \in KK(\mathfrak{T}, \mathfrak{T})$. For the last equality, a homotopy between the maps may be given by the maps $\varphi_t(u) = u^t$ for $t \in [0, 1]$.

For $\mathfrak{B}$ any $C^*$-algebra, $\mathfrak{B}$ and $\mathfrak{B} \otimes \mathfrak{T}$ are KK-equivalent. Indeed, since $\mathfrak{T}$ and $\mathbb{C}$ are KK-equivalent, then $\mathfrak{T} \otimes \mathfrak{B}$ and $\mathbb{C} \otimes \mathfrak{B} \cong \mathfrak{B}$ are KK-equivalent via the tensor-inducing homomorphism $\tau_{\mathfrak{B}}$.

• For $\mathfrak{A} *_{\mathfrak{D}} \mathfrak{B}$ an amalgamated free product (or amalgam) of $C^*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$ over $\mathfrak{D}$, if there are retractions $r_1, r_2$ of $\mathfrak{A}$ and $\mathfrak{B}$ onto $\mathfrak{D}$, then $\mathfrak{A} *_{\mathfrak{D}} \mathfrak{B}$ is KK-equivalent to the pullback $C^*$-algebra $P = \mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B} = \{(a, b) \in \mathfrak{A} \oplus \mathfrak{B} \mid r_1(a) = r_2(b)\}$.

In fact, $[k] \in KK(\mathfrak{A} *_{\mathfrak{D}} \mathfrak{B}, P)$ is a KK-equivalence with inverse $[f] - [g]$, where $k : \mathfrak{A} *_{\mathfrak{D}} \mathfrak{B} \to P$ is defined by $k(a) = (a, r_1(a))$ and $k(b) = (r_2(b), b)$ for $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$, $g = i \circ r : P \to \mathfrak{A} *_{\mathfrak{D}} \mathfrak{B}$ with $r : \mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B} \to \mathfrak{D}$ defined by $r(a, b) = r(a) = r(b)$ and with $i : \mathfrak{D} \to \mathfrak{A} *_{\mathfrak{D}} \mathfrak{B}$ the canonical inclusion map (corrected from [1]), and $f : P \to M_2(\mathfrak{A} *_{\mathfrak{D}} \mathfrak{B})$ is defined by $f(a, b) = a \oplus b$ the diagonal sum. It follows that $(1 \otimes k) \circ f : P \to M_2(P)$ sends $(a, b)$ to $(a, r_1(a)) \oplus (r_2(b), b)$, and this homomorphism is homotopic to $\mathrm{id}_P \oplus (k \circ g)$, and also the composition $f \circ k$ is homotopic to $\mathrm{id}_{\mathfrak{A} *_{\mathfrak{D}} \mathfrak{B}} \oplus (g \circ k)$. See [3] for more details for this. Thus, $[(1 \otimes k) \circ f] = [\mathrm{id}_P \oplus (k \circ g)] \in KK(P, M_2(P)) \cong KK(P, P)$ as an identification. Also, $[f \circ k] = [\mathrm{id}_{\mathfrak{A} *_{\mathfrak{D}} \mathfrak{B}} \oplus (g \circ k)] \in KK(\mathfrak{A} *_{\mathfrak{D}} \mathfrak{B}), M_2(\mathfrak{A} *_{\mathfrak{D}} \mathfrak{B})) \cong KK(\mathfrak{A} *_{\mathfrak{D}} \mathfrak{B}, \mathfrak{A} *_{\mathfrak{D}} \mathfrak{B})$. Therefore,

$$
\begin{aligned}
([f] - [g]) \otimes_{\mathfrak{A}_{\mathfrak{D}} \mathfrak{B}} [k] &= [f] \otimes_{M_2(\mathfrak{A}_{\mathfrak{D}} \mathfrak{B})} [1 \otimes k] - [g] \otimes_{\mathfrak{A}_{\mathfrak{D}} \mathfrak{B}} [k] \\
&= [(1 \otimes k) \circ f] - [k \circ g] = [\mathrm{id}_P] \in KK(P, P), \\
[k] \otimes_P ([f] - [g]) &= [k] \otimes_P [f] - [k] \otimes_P [g] \\
&= [f \circ k] - [g \circ k] = [\mathrm{id}_{\mathfrak{A} *_{\mathfrak{D}} \mathfrak{B}}],
\end{aligned}
$$

where those identifications are used implicitly.

In particular, let $F_2$ be the free group of two generators and $C^*(F_2)$ the full group $C^*$-algebra of $F_2$. Then $C^*(F_2) \cong C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T})$ is KK-equivalent to $C(\mathbb{T}) \oplus_{\mathbb{C}} C(\mathbb{T}) \cong C(\mathbf{8})$ with $\mathbf{8}$ the figure eight, homeomorphic to the one-point compactification of the disjoint union $\mathbb{R} \sqcup \mathbb{R}$, so that there is the following short exact sequence:

$$0 \to C_0(\mathbb{R}) \oplus C_0(\mathbb{R}) \to C(\mathbf{8}) \cong C(\mathbb{T}) \oplus_{\mathbb{C}} C(\mathbb{T}) \to \mathbb{C} \to 0.$$

More generally, let $F_n$ be the free group of $n$ generators. Then $C^*(F_n) \cong (\cdots(C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T}))\cdots) *_{\mathbb{C}} C(\mathbb{T})$ the unital successive amalgam by $\mathbb{C}$ is KK-equivalent to the successive pullback $\oplus_{\mathbb{C}}^n C(\mathbb{T}) \equiv (\cdots(C(\mathbb{T}) \oplus_{\mathbb{C}} C(\mathbb{T}))\cdots) \oplus_{\mathbb{C}} C(\mathbb{T}) \cong C((\sqcup^n \mathbb{R})^+)$ with $(\sqcup^n \mathbb{R})^+$ the one-point compactification of the disjoint union $\sqcup^n \mathbb{R}$, homeomorphic to the Hawaian ring $H_n$ of (disjoint) $n$ circles joined at a point, with $H_2 = \mathbf{8}$ the eight, so that

$$0 \to \oplus^n C_0(\mathbb{R}) \to C(H_n) \cong \oplus_{\mathbb{C}}^n C(\mathbb{T}) \to \mathbb{C} \to 0.$$

It also follows that

$$\begin{aligned} K_0(C(H_n)) &\cong [\oplus^n K_0(C_0(\mathbb{R}))] \oplus K_0(\mathbb{C}) \cong [\oplus^n 0] \oplus \mathbb{Z} \cong \mathbb{Z}, \\ K_1(C(H_n)) &\cong [\oplus^n K_1(C_0(\mathbb{R}))] \oplus K_1(\mathbb{C}) \cong [\oplus^n \mathbb{Z}] \oplus 0 \cong \mathbb{Z}^n. \end{aligned}$$

Similarly, one can define the Hawaian ring $H_\infty$ of countably inifinitely many (disjoint) circles joined at a point, so that

$$0 \to \oplus^\infty C_0(\mathbb{R}) \to C(H_\infty) \cong \oplus_{\mathbb{C}}^\infty C(\mathbb{T}) \to \mathbb{C} \to 0.$$

It also follows that

$$\begin{aligned} K_0(C(H_\infty)) &\cong [\oplus^\infty K_0(C_0(\mathbb{R}))] \oplus K_0(\mathbb{C}) \cong [\oplus^\infty 0] \oplus \mathbb{Z} \cong \mathbb{Z}, \\ K_1(C(H_\infty)) &\cong [\oplus^\infty K_1(C_0(\mathbb{R}))] \oplus K_1(\mathbb{C}) \cong [\oplus^\infty \mathbb{Z}] \oplus 0 \cong \mathbb{Z}^\infty. \end{aligned}$$

Furthermore, let $F_\infty$ the free group of countably infinitely many generators. Then the group $C^*$-algebra $C^*(F^\infty)$ is isomorphic to $*_{\mathbb{C}}^\infty C(\mathbb{T})$ the unital free product of countably infinite copies of $C(\mathbb{T})$ and has K-theory groups

$$\begin{aligned} K_0(C^*(F_\infty)) &\cong \varinjlim K_0(C^*(F_n)) \cong \varinjlim \mathbb{Z} \cong \mathbb{Z} \quad \text{and} \\ K_1(C^*(F_\infty)) &\cong \varinjlim K_0(C^*(F_n)) \cong \varinjlim \mathbb{Z}^n \cong \mathbb{Z}^\infty \end{aligned}$$

and is KK-equivalent to $C(H_\infty)$. In fact, the KK-equivalence should follow from the same argument as for $C^*(F_2) \sim_{KK} C(\mathbb{T}) \oplus_{\mathbb{C}} C(\mathbb{T})$, without using the UCT (below soon later).

• Let $\mathfrak{A}$ be a $C^*$-algebra and let $\mathfrak{A}_Q \equiv \mathfrak{A} * \mathfrak{A}$ the full free product. Then the identity maps $\mathrm{id}_{\mathfrak{A}}$ on $\mathfrak{A}$ as free product factors induce the quotient map $q(\mathrm{id}_{\mathfrak{A}}, \mathrm{id}_{\mathfrak{A}}) : \mathfrak{A}_Q \to \mathfrak{A}$, so that the following is exact:

$$0 \to \mathfrak{A}_q \to \mathfrak{A}_Q = \mathfrak{A} * \mathfrak{A} \to \mathfrak{A} \to 0,$$

where $\mathfrak{A}_q$ is the kernel of $q = q(\mathrm{id}_{\mathfrak{A}}, \mathrm{id}_{\mathfrak{A}})$. Then, similarly, define the quotient homomorphisms $\pi_0 = q(\mathrm{id}_{\mathfrak{A}}, 0), \pi_1 = q(0, \mathrm{id}_{\mathfrak{A}}) : \mathfrak{A}_Q \to \mathfrak{A}$ and its restrictions $\pi_0, \pi_1 : \mathfrak{A}_q \to \mathfrak{A}$. If follows that the KK-element $[\pi_0 \oplus \pi_1] \in KK(\mathfrak{A}_q, \mathfrak{A})$ is a KK-equivalence with inverse given below, where the KK-element is corrected from [1], and $\pi_0 \oplus \pi_1 : \mathfrak{A}_q \to \mathfrak{A} \oplus \mathfrak{A}$, but the image can be identified with $\mathfrak{A}$.

Indeed, let $\mathfrak{A} * \mathfrak{A} = \mathfrak{A}_1 * \mathfrak{A}_2$ and $i_j : \mathfrak{A} = \mathfrak{A}_j \to \mathfrak{A}_Q$ the canonical inclusion map. For $x \in \mathfrak{A}$, define $s(x) = i_1(x) - i_2(x) \in \mathfrak{A}_Q$. Then $q(s(x)) = 0 \in \mathfrak{A}$, and hence $s(x) \in \mathfrak{A}_q$. Then $\mathfrak{A}_q$ is the ideal of $\mathfrak{A}_Q$ generated by $s(x)$ for $x \in \mathfrak{A}$, but the map $s$ is not a homomorphism. Note that $\pi_0(s(x)) = x$ for $x \in \mathfrak{A}$. Also, as for $\pi_1 : q(0, \mathrm{id}_{\mathfrak{A}}) : \mathfrak{A}_Q \to \mathfrak{A}$, $\pi_1(s(x)) = -x$ for $x \in \mathfrak{A}$. Define the maps $k = \pi_0 \oplus \pi_1 : \mathfrak{A}_Q \to \mathfrak{A} \oplus \mathfrak{A}$ and $f = i_1 \oplus i_2 : \mathfrak{A} \oplus \mathfrak{A} \to M_2(\mathfrak{A}_Q)$. Then $[k] \in KK(\mathfrak{A}_Q, \mathfrak{A} \oplus \mathfrak{A})$ is a KK-equivalence with inverse $[f] \in KK(\mathfrak{A} \oplus \mathfrak{A}, \mathfrak{A}_Q)$, as shown above (with $[g] = [0]$), and hence, $\mathfrak{A}_Q$ is KK-equivalent to $\mathfrak{A} \oplus \mathfrak{A}$. Anyhow, one gets

$$[\pi_0 \oplus \pi_1] \otimes_{\mathfrak{A}\oplus\mathfrak{A}} [f] = [\mathrm{id}_{\mathfrak{A}_Q}] \in KK(\mathfrak{A}_Q, \mathfrak{A}_Q),$$
$$[f] \otimes_{\mathfrak{A}_Q} [\pi_0 \oplus \pi_1] = [\mathrm{id}_{\mathfrak{A}\oplus\mathfrak{A}}] \in KK(\mathfrak{A} \oplus \mathfrak{A}, \mathfrak{A} \oplus \mathfrak{A}).$$

By considering the restriction of those maps to $\mathfrak{A}_q$ and $D(\mathfrak{A} \oplus \mathfrak{A})$ the corresponding diagonal part of $\mathfrak{A} \oplus \mathfrak{A}$, it is obtained that

$$[(\pi_0 \oplus \pi_1)_{\mathfrak{A}_q}] \otimes_{\mathfrak{A}} [f_{D(\mathfrak{A}\oplus\mathfrak{A})}] = [\mathrm{id}_{\mathfrak{A}_q}] \in KK(\mathfrak{A}_q, \mathfrak{A}_q),$$
$$[f_{D(\mathfrak{A}\oplus\mathfrak{A})}] \otimes_{\mathfrak{A}_q} [(\pi_0 \oplus \pi_1)_{\mathfrak{A}_p}] = [\mathrm{id}_{D(\mathfrak{A}\oplus\mathfrak{A})}] \in KK(D(\mathfrak{A} \oplus \mathfrak{A}), D(\mathfrak{A} \oplus \mathfrak{A}))$$
$$= [\mathrm{id}_{\mathfrak{A}}] \in KK(\mathfrak{A}, \mathfrak{A})$$

(corrected from [1]).

A $C^*$-algebra $\mathfrak{A}$ is said to be **K-contractible** if $KK(\mathfrak{A}, \mathfrak{A}) = 0$. This implies that $KK(\mathfrak{A}, \mathfrak{B}) = KK(\mathfrak{B}, \mathfrak{A}) = 0$ for any $C^*$-algebra $\mathfrak{B}$.

Indeed, the classs $[\mathrm{id}_{\mathfrak{A}}] \in KK(\mathfrak{A}, \mathfrak{A})$ is a KK-equivalence. Because

$$[\mathrm{id}_{\mathfrak{A}}] \otimes_{\mathfrak{A}} [\mathrm{id}_{\mathfrak{A}}] = [\mathrm{id}_{\mathfrak{A}} \circ \mathrm{id}_{\mathfrak{A}}] = [\mathrm{id}_{\mathfrak{A}}] \in KK(\mathfrak{A}, \mathfrak{A}).$$

It then follows that the following map is an isomorphism and is zero:

$$[\mathrm{id}_{\mathfrak{A}}] \otimes_{\mathfrak{A}} (\cdot) : KK(\mathfrak{A}, \mathfrak{B}) \to KK(\mathfrak{A}, \mathfrak{B})$$

by $[\mathrm{id}_{\mathfrak{A}}] = 0$. Thus, $KK(\mathfrak{A}, \mathfrak{B}) = 0$. Similarly, $KK(\mathfrak{B}, \mathfrak{A}) = 0$ by that the map $(\cdot) \otimes_{\mathfrak{A}} [\mathrm{id}_{\mathfrak{A}}] : KK(\mathfrak{B}, \mathfrak{A}) \to KK(\mathfrak{B}, \mathfrak{A})$ is an isomorphism and is zero.

**Example 6.3.** • Any contractible $C^*$-algebra is K-contractible. In particular, the cone $C\mathfrak{B} = C_0([0,1),\mathfrak{B}) \cong C_0([0,1)) \otimes \mathfrak{B}$ for any $C^*$-algebra $\mathfrak{B}$ is K-contractible.

Indeed, the identity map $\mathrm{id}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$ is homotopic to the zero map on $\mathfrak{A}$, so that $[\mathrm{id}_{\mathfrak{A}}] = [0] \in KK(\mathfrak{A},\mathfrak{A})$. Then the following map is an isomorphism and is zero:

$$[\mathrm{id}_{\mathfrak{A}}] \otimes_{\mathfrak{A}} (\cdot) : KK(\mathfrak{A},\mathfrak{A}) \to KK(\mathfrak{A},\mathfrak{A}).$$

It follows that $KK(\mathfrak{A},\mathfrak{A}) = 0$. By the way, the cone $C\mathfrak{B}$ is contractible, and thus is K-contractible.

• For $0 \to \mathfrak{J} \xrightarrow{i} \mathfrak{A} \xrightarrow{q} \mathfrak{A}/\mathfrak{J} \to 0$ a short exact sequence of (graded) $C^*$-algebras, suppose that the following six-term exact sequences (given below) hold for a (graded) $C^*$-algebra $\mathfrak{D}$:

$$\begin{array}{ccccc}
KK(\mathfrak{D},\mathfrak{J}) & \xrightarrow{i_*} & KK(\mathfrak{D},\mathfrak{A}) & \xrightarrow{q_*} & KK(\mathfrak{D},\mathfrak{A}/\mathfrak{J}) \\
\partial \uparrow & & & & \downarrow \partial \\
KK^1(\mathfrak{D},\mathfrak{A}/\mathfrak{J}) & \xleftarrow{q_*} & KK^1(\mathfrak{D},\mathfrak{A}) & \xleftarrow{i_*} & KK^1(\mathfrak{D},\mathfrak{J})
\end{array}$$

and

$$\begin{array}{ccccc}
KK(\mathfrak{J},\mathfrak{D}) & \xleftarrow{i^*} & KK(\mathfrak{A},\mathfrak{D}) & \xleftarrow{q^*} & KK(\mathfrak{A}/\mathfrak{J},\mathfrak{D}) \\
\partial \downarrow & & & & \uparrow \partial \\
KK^1(\mathfrak{A}/\mathfrak{J},\mathfrak{D}) & \xrightarrow{q^*} & KK^1(\mathfrak{A},\mathfrak{D}) & \xrightarrow{i^*} & KK^1(\mathfrak{J},\mathfrak{D}).
\end{array}$$

If the class corresponding to the quotient map $q$ is a KK-equivalence, so that $[q] \in KK(\mathfrak{A},\mathfrak{A}/\mathfrak{J})$ is a KK-equivalence, then $\mathfrak{J}$ is K-contractible.

In particular, for the short exact sequence: $0 \to \mathfrak{T}_0 \to \mathfrak{T} \xrightarrow{ev_1 \circ q} \mathbb{C} \to 0$, with $\mathfrak{T}$ the Toeplitz algebra and $\mathfrak{T}_0$ the kernel of $ev_1 \circ q$, generated by $U-1$, the closed ideal $\mathfrak{T}_0$ and $\mathfrak{B} \otimes \mathfrak{T}_0$ for any $C^*$-algebra $\mathfrak{B}$ are K-contractible.

As a proof, it follows that the maps $q_*$ on $KK$ and $KK^1$ in the first diagram above are isomorphisms, so that the maps $i_*$ and $\partial$ on the left are zero. Hence $KK(\mathfrak{D},\mathfrak{J}) = 0$ for any $C^*$-algebra $\mathfrak{D}$. This is equivalent to that $\mathfrak{J}$ is K-contractible.

Similarly, the maps $q^*$ on $KK$ and $KK^1$ in the second are isomorphisms, so that the maps $i^*$ and $\partial$ on the left are zero. Hence $KK(\mathfrak{J},\mathfrak{D}) = 0$ for any $C^*$-algebra $\mathfrak{D}$. This is equivalent to that $\mathfrak{J}$ is K-contractible.

Since the short exact sequence for $\mathfrak{T}$ as an extension of $\mathbb{C}$ by $\mathfrak{T}_0$ is split, so that it is **semi-split**, i.e., there is a completely positive, norm-decreasing, grading preserving, cross section for $q$. Therefore, the six-term exact sequences for KK-theory groups hold. As well,

$$0 \to \mathfrak{B} \otimes \mathfrak{T}_0 \to \mathfrak{B} \otimes \mathfrak{T} \xrightarrow{\mathrm{id}_{\mathfrak{B}} \otimes (ev_1 \circ q)} \mathfrak{B} \otimes \mathbb{C} \to 0$$

is split.

Since

$$KK(\mathbb{C}_1, S\mathbb{C}) \cong KK(\mathbb{C}, S\mathbb{C} \otimes \mathbb{C}_1) = KK^1(\mathbb{C}, S\mathbb{C}) \cong \mathrm{Ext}(\mathbb{C}, S\mathbb{C}) \cong \mathbb{Z},$$

the generating class $x \in KK^1(\mathbb{C}, S\mathbb{C}) \cong KK(\mathbb{C}, \mathbb{C})$ is represented by the cone $C\mathbb{C} \cong C_0([0,1])$ as the extension of $\mathbb{C}$ by the suspension $S\mathbb{C} \cong C_0(\mathbb{R})$:

$$0 \to S\mathbb{C} \to C\mathbb{C} \to \mathbb{C} \to 0.$$

We may alternatively interpret the class $x$ as the element of $K_1(S\mathbb{C})$ corresponding to the unitary $u(t) = e^{2\pi i t}$ in $S\mathbb{C}^+ \cong C(\mathbb{T})$, restricted to $0 < t < 1$. We may call $x$ the **Bott class**.

The generating class $y \in KK^1(S\mathbb{C}, \mathbb{C}) \cong \mathrm{Ext}(S\mathbb{C}, \mathbb{C}) = \mathrm{Ext}(S\mathbb{C}) \cong \mathbb{Z}$ with $KK^1(S\mathbb{C}, \mathbb{C}) \cong KK^1(S\mathbb{C}, \mathbb{K}) \cong KK(\mathbb{C}, \mathbb{C})$ is represented by the extension

$$0 \to \mathbb{K} \to C^*(V-1) \to S\mathbb{C} = C_0(\mathbb{R}) = S \to 0$$

where $V$ is a coisometry of Fredholm index one, e.g., the adjoint $U^*$ of the unilateral shift $U$, and where the $C^*$-algebra $C^*(V)$ generated by $V$ is the Toeplitz algebra $\mathfrak{T}$ and the $C^*$-algebra $C^*(V-1)$ generated by $V-1$ is a $C^*$-subalgebra of $\mathfrak{T}$ and the quotient map $q : \mathfrak{T} \to C(\mathbb{T})$ sends $V-1$ to $u - 1 \in C(\mathbb{T})$, with the $C^*$-algebra $C^*(u-1)$ generated by $u-1$, isomorphic to $C_0(\mathbb{R})$.

It follows that $x \in KK(\mathbb{C}_1, S\mathbb{C})$ is a KK-equivalence with inverse $y \in KK(S\mathbb{C}, \mathbb{C}_1) = KK^1(S\mathbb{C}, \mathbb{C})$. We may call $y$ the **inverse Bott class**.

Indeed, $\mathrm{Ext}(\mathbb{C}, S\mathbb{C}) \cong \mathrm{Ext}(S\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$, and we need to check that

$$x \otimes_{S\mathbb{C}} y = [\mathrm{id}_{\mathbb{C}}] \in KK(\mathbb{C}_1, \mathbb{C}_1) \cong KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z},$$
$$y \otimes_{\mathbb{C}_1} x = [\mathrm{id}_{S\mathbb{C}}] \in KK(S\mathbb{C}, S\mathbb{C}) \cong KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$$

with $x \otimes_{S\mathbb{C}} y = x \otimes_{\mathbb{C}} y$ and $y \otimes_{\mathbb{C}_1} x = y \otimes_{\mathbb{C}} x$. Note that there is a pairing $\langle \cdot, \cdot \rangle$ between $\mathrm{Ext}(\mathfrak{A}) = \mathrm{Ext}(\mathfrak{A}, \mathbb{C}) \cong KK^1(\mathfrak{A}, \mathbb{C})$ and $K_1(\mathfrak{A}) \cong KK^1(\mathbb{C}, \mathfrak{A})$ for any $C^*$-algebra $\mathfrak{A}$ as

$$\langle [\tau], [u] \rangle = \mathrm{index}(\tau^{\sim}(u)) \in \mathbb{Z},$$

for $[\tau] \in \mathrm{Ext}(\mathfrak{A})$ and $[u] \in K_1(\mathfrak{A})$, where $\tau : \mathfrak{A} \to \mathbb{B}/\mathbb{K} = Q$ the Busby invariant, $u$ is a unitary in $M_n(\mathfrak{A})$ for some $n$, and $\tau^{\sim} : M_n(\mathfrak{A}) \to M_n(Q) \cong$

$Q$ the canonical extension of $\tau$, and where if $\mathfrak{A}$ is non-unital, $\mathfrak{A}$ is replaced by the unitization $\mathfrak{A}^+$. Thus each $[\tau] \in \mathrm{Ext}(\mathfrak{A})$ defines a homomorphism $\gamma([\tau])$ from $K_1(\mathfrak{A})$ to $\mathbb{Z}$ by $\gamma([\tau])([u]) = \langle [\tau], [u] \rangle \in \mathbb{Z}$. Also, for $\mathfrak{B}$ a trivially graded $\sigma$-unital $C^*$-algebra, we have

$$x \otimes_{\mathfrak{B}} y = \mathrm{index}(\tau^{\sim}(e^{2\pi i x})) = \langle [\tau], [e^{2\pi i x}] \rangle \in \mathbb{Z},$$

for $x \in KK^1(\mathbb{C}, \mathfrak{B}) \cong K_1(\mathfrak{B})$ and $y \in KK^1(\mathfrak{B}, \mathbb{C}) \cong \mathrm{Ext}(\mathfrak{B})$, where $y = [\tau]$ with $\tau : \mathfrak{B} \to Q = \mathbb{B}/\mathbb{K}$ the Busby invariant and $x \in M(\mathfrak{B})_+$ with $q(x) = p$ a projection of $Q(\mathfrak{B}) = M(\mathfrak{B})/\mathfrak{B}$ with $[p] = x$.

(**Bott Periodicity**). For any $C^*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$, we have

$$KK^1(\mathfrak{A}, \mathfrak{B}) \cong KK(\mathfrak{A}, S\mathfrak{B}) \cong KK(S\mathfrak{A}, \mathfrak{B})$$

and

$$\begin{aligned} KK(\mathfrak{A}, \mathfrak{B}) &\cong KK^1(\mathfrak{A}, S\mathfrak{B}) \cong KK^1(S\mathfrak{A}, \mathfrak{B}) \\ &\cong KK(S^2\mathfrak{A}, \mathfrak{B}) \cong KK(\mathfrak{A}, S^2\mathfrak{B}) \cong K(S\mathfrak{A}, S\mathfrak{B}). \end{aligned}$$

*Proof.* Indeed, the KK-equivalence $x \in KK(\mathbb{C}_1, S\mathbb{C})$ with inverse $y \in KK(S\mathbb{C}, \mathbb{C}_1)$ induces the KK-equivalence $\tau_{\mathfrak{B}}(x) \in KK(\mathbb{C}_1, \otimes\mathfrak{B}, S\mathfrak{B})$ with inverse $\tau_{\mathfrak{B}}(y) \in KK(S\mathfrak{B}, \mathbb{C}_1 \otimes \mathfrak{B})$. It then follows by the Kasparov product $\otimes_{\mathbb{C}_1 \otimes \mathfrak{B}} \tau_{\mathfrak{B}}(x)$ that

$$KK^1(\mathfrak{A}, \mathfrak{B}) = KK(\mathfrak{A}, \mathbb{C}_1 \otimes \mathfrak{B}) \cong KK(\mathfrak{A}, S\mathfrak{B})$$

and also by the Kasparov product $\otimes_{S\mathfrak{B}} \tau_{\mathfrak{B}}(y)$ that

$$KK(\mathfrak{A}, S\mathfrak{B}) \cong KK(\mathfrak{A}, \mathbb{C}_1 \otimes \mathfrak{B}) \cong KK^1(\mathfrak{A}, \mathfrak{B}).$$

Similarly, we have the KK-equivalence $\tau_{\mathfrak{A}}(x) \in KK(\mathbb{C}_1, \otimes\mathfrak{A}, S\mathfrak{A})$ with inverse $\tau_{\mathfrak{A}}(y) \in KK(S\mathfrak{A}, \mathbb{C}_1 \otimes \mathfrak{A})$. It then follows that by the formal Bott periodicity and the Kasparov product $\tau_{\mathfrak{A}}(y) \otimes_{\mathbb{C}_1 \otimes \mathfrak{A}}$ that

$$\begin{aligned} KK^1(\mathfrak{A}, \mathfrak{B}) = KK(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{C}_1) &\cong KK(\mathfrak{A} \otimes \mathbb{C}_1, \mathfrak{B}) \\ &\cong KK(S\mathfrak{A}, \mathfrak{B}) \end{aligned}$$

and also by the Kasparov product $\tau_{\mathfrak{A}}(x) \otimes_{S\mathfrak{A}}$ that

$$KK(S\mathfrak{A}, \mathfrak{B}) \cong KK(\mathbb{C}_1 \otimes \mathfrak{A}, \mathfrak{B}) \cong KK(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{C}_1) = KK^1(\mathfrak{A}, \mathfrak{B}).$$

Therefore, it is deduced that the commutativity for $S$ holds: $KK(\mathfrak{A}, S\mathfrak{B}) \cong KK(S\mathfrak{A}, \mathfrak{B}) \cong KK^1(\mathfrak{A}, \mathfrak{B})$, which may be viewed as the definition of $KK^1$.

Moreover, it follows by the formal Bott periodicity and the Kasparov product $\tau_{\mathfrak{A}}(y) \otimes_{\mathbb{C}_1 \otimes \mathfrak{A}}$ that

$$\begin{aligned} KK^1(\mathfrak{A}, S\mathfrak{B}) = KK(\mathfrak{A}, \mathbb{C}_1 \otimes S\mathfrak{B}) &\cong KK(\mathfrak{A} \otimes \mathbb{C}_1, S\mathfrak{B}) \\ &\cong KK(S\mathfrak{A}, S\mathfrak{B}) \end{aligned}$$

and also by the Kasparov product $\tau_{\mathfrak{A}}(x) \otimes_{S\mathfrak{A}}$ that

$$KK(S\mathfrak{A}, S\mathfrak{B}) \cong KK(\mathbb{C}_1 \otimes \mathfrak{A}, S\mathfrak{B}) \cong KK^1(\mathfrak{A}, S\mathfrak{B}).$$

By the commutativity for $S$,

$$KK(S\mathfrak{A}, S\mathfrak{B}) \cong KK(S^2\mathfrak{A}, \mathfrak{B}) \quad \text{and} \quad KK(S\mathfrak{A}, S\mathfrak{B}) = KK(\mathfrak{A}, S^2\mathfrak{B})$$

and as well

$$KK^1(\mathfrak{A}, S\mathfrak{B}) = KK_1(\mathfrak{A}, S\mathfrak{B} \otimes \mathbb{C}_1) \cong KK_1(S\mathfrak{A}, \mathfrak{B} \otimes \mathbb{C}_1) = KK^1(S\mathfrak{A}, \mathfrak{B}).$$

It also follows by the Kasparov product $\otimes_{S\mathfrak{B}} \tau_{\mathfrak{B}}(y)$ and the formal Bott periodicity that

$$\begin{aligned} KK^1(S\mathfrak{A}, \mathfrak{B}) = KK(S\mathfrak{A}, \mathfrak{B} \otimes \mathbb{C}_1) &\cong KK(\mathfrak{A} \otimes \mathbb{C}_1, S\mathfrak{B}) \\ &\cong KK(\mathfrak{A} \otimes \mathbb{C}_1, \mathfrak{B} \otimes \mathbb{C}_1) \\ &\cong KK(\mathfrak{A}, \mathfrak{B}). \end{aligned}$$

Since $\tau_{\mathfrak{A}}(y) \in KK(S\mathfrak{A}, \mathfrak{A} \otimes \mathbb{C}_1) \cong KK(S\mathfrak{A} \otimes \mathbb{C}_1, \mathfrak{A})$, we also have

$$KK(\mathfrak{A}, \mathfrak{B}) \cong KK(S\mathfrak{A} \otimes \mathbb{C}_1, \mathfrak{B}) \cong KK^1(S\mathfrak{A}, \mathfrak{B}).$$

□

**Example 6.4.** For any $C^*$-algebra $\mathfrak{A}$ given, $\mathfrak{A}$ and $S^2\mathfrak{A}$ are KK-equivalent. In particular, $\mathbb{C}$ and $C_0(\mathbb{R}^2)$ are KK-equivalent.

Because, the Bott periodicity implies that

$$KK(\mathfrak{A}, S^2\mathfrak{A}) \cong KK(\mathfrak{A}, \mathfrak{A}) \quad \text{and} \quad KK(S^2\mathfrak{A}, \mathfrak{A}) \cong KK(\mathfrak{A}, \mathfrak{A}),$$

both of which contain KK-equivalences. Indeed, it follows that the element $[\mathrm{id}_{\mathfrak{A}}] \in KK(\mathfrak{A}, \mathfrak{A})$ is viewed as in both $KK(\mathfrak{A}, S^2\mathfrak{A})$ and $KK(S^2\mathfrak{A}, \mathfrak{A})$, so that

$$[\mathrm{id}_{\mathfrak{A}}] \otimes_{\mathfrak{A}} [\mathrm{id}_{\mathfrak{A}}] = [\mathrm{id}_{\mathfrak{A}} \circ \mathrm{id}_{\mathfrak{A}}] = [\mathrm{id}_{\mathfrak{A}}]$$

in $KK(\mathfrak{A}, \mathfrak{A}) \cong KK(S^2\mathfrak{A}, S^2\mathfrak{A})$ both.

**(Thom isomorphism in KK-theory).** Let $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ be the crossed product $C^*$-algebra of a separable trivially graded $C^*$-algebra $\mathfrak{A}$ by an action $\alpha$ of $\mathbb{R}$ of reals on $\mathfrak{A}$. Then $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ is KK-equivalent to $S\mathfrak{A}$.

It follows that for any $\mathfrak{D}$,

$$\begin{array}{ccc} KK(\mathfrak{A} \rtimes_\alpha \mathbb{R}, \mathfrak{D}) & \xrightarrow{t_\alpha \otimes_{\mathfrak{A} \rtimes_\alpha \mathbb{R}} (\cdot)} & \cong KK(S\mathfrak{A}, \mathfrak{D}) \cong KK^1(\mathfrak{A}, \mathfrak{D}), \\ KK^1(\mathfrak{D}, \mathfrak{A}) \cong KK(\mathfrak{D}, S\mathfrak{A}) & \xrightarrow{(\cdot) \otimes_{S\mathfrak{A}} t_\alpha} & KK(\mathfrak{D}, \mathfrak{A} \rtimes_\alpha \mathbb{R}), \end{array}$$

with $t_\alpha \in KK(S\mathfrak{A}, \mathfrak{A} \rtimes_\alpha \mathbb{R}) \cong KK^1(\mathfrak{A}, \mathfrak{A} \rtimes_\alpha \mathbb{R})$ the KK-equivalence, called the **Thom class** for $\mathfrak{A}$ and $\alpha$, corresponding to the **Thom module** $(\mathfrak{A} \rtimes_\alpha \mathbb{R}, \varphi, F_f)$ for $\alpha$, a Kasparov $(\mathfrak{A}, \mathfrak{A} \rtimes_\alpha \mathbb{R})$-module, which is identified with a Kasparov $(\mathfrak{A}, (\mathfrak{A} \rtimes_\alpha \mathbb{R}) \otimes \mathbb{C}_1)$-module, such that $\varphi : \mathfrak{A} \to M(\mathfrak{A} \rtimes_\alpha \mathbb{R})$ is a canonical homomorphism and $F_f \in M(\mathfrak{A} \rtimes_\alpha \mathbb{R})$ is the **Thom operator** on $\mathfrak{A} \rtimes_\alpha \mathbb{R}$, corresponding to a continuous, complex-valued function $f$ on $\mathbb{R}$ with $\lim_{t\to\infty} f(t) = 1$ and $\lim_{t\to-\infty} f(t) = -1$.

**Example 6.5.** • As a corollary, let $G$ be a simply connected, solvable Lie group and $\mathfrak{A} \rtimes_\alpha G$ be the crossed product of a separable, trivially graded $C^*$-algebra $\mathfrak{A}$ by an action of $G$ on $\mathfrak{A}$. Then $\mathfrak{A} \rtimes_\alpha G$ is KK-equivalent to $\mathfrak{A}$ if $\dim G$ is even, and to $S\mathfrak{A}$ if $\dim G$ is odd.

Since $\mathfrak{A} \rtimes_\alpha G$ is obtained as a successive crossed product $\mathfrak{A} \rtimes_{\alpha_1} \mathbb{R} \cdots \rtimes_{\alpha_{\dim G}} \mathbb{R}$ of $\mathfrak{A}$ by actions $\alpha_j$ of $\mathbb{R}$ with $1 \le j \le \dim G$, we have $\mathfrak{A} \rtimes_\alpha G$ KK-equivalent to $S^{\dim G}\mathfrak{A}$.

It follows that for any $\mathfrak{D}$,

$$KK(\mathfrak{A} \rtimes G, \mathfrak{D}) \cong \begin{cases} KK(\mathfrak{A}, \mathfrak{D}) & \text{if } \dim G \text{ even}, \\ KK^1(\mathfrak{A}, \mathfrak{D}) & \text{if } \dim G \text{ odd}; \end{cases}$$

$$KK(\mathfrak{D}, \mathfrak{A} \rtimes G) \cong \begin{cases} KK(\mathfrak{D}, \mathfrak{A}) & \text{if } \dim G \text{ even}, \\ KK^1(\mathfrak{D}, \mathfrak{A}) & \text{if } \dim G \text{ odd}. \end{cases}$$

• Let $\mathfrak{A} = \mathbb{C}$ and $\alpha = \mathrm{id}$ the trivial action. Then $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \cong C_0(\mathbb{R})$. Then the Thom class $t_{\mathrm{id}} = x \in KK^1(\mathbb{C}, S\mathbb{C})$ the Bott class.

• Let $\mathfrak{A} = C_0(\mathbb{R})$ and $\mathrm{id}^\wedge$ the dual action of $\mathbb{R}^\wedge \cong \mathbb{R}$ on $C_0(\mathbb{R})$ by translation. Then $C_0(\mathbb{R}) \rtimes_{\mathrm{id}^\wedge} \mathbb{R} \cong \mathbb{K}$ by Takai duality. The Thom class $t_{\mathrm{id}^\wedge} \in KK^1(C_0(\mathbb{R}), \mathbb{K})$ corresponds to the inverse Bott class $y \in KK(S\mathbb{C}, \mathbb{C}_1) = KK^1(S\mathbb{C}, \mathbb{C})$.

As a key part of the proof for the Thom isomorphism in KK-theory, let

$\alpha^\wedge$ be the dual action of $\mathbb{R}$ on $\mathfrak{A} \rtimes_\alpha \mathbb{R}$. Then the Kasparov product:

$$\begin{array}{c} KK^1(\mathfrak{A}, \mathfrak{A} \rtimes_\alpha \mathbb{R}) \times KK^1(\mathfrak{A} \rtimes_\alpha \mathbb{R}, (\mathfrak{A} \rtimes_\alpha \mathbb{R}) \rtimes_{\alpha^\wedge} \mathbb{R}) \\ \otimes_{\mathfrak{A} \rtimes_\alpha \mathbb{R}} \Big\downarrow \\ KK(\mathfrak{A}, (\mathfrak{A} \rtimes_\alpha \mathbb{R}) \rtimes_{\alpha^\wedge} \mathbb{R}) \cong KK(\mathfrak{A}, \mathfrak{A} \otimes \mathbb{K}) \cong KK(\mathfrak{A}, \mathfrak{A}), \end{array}$$

by Takai duality, and that $t_\alpha \otimes_{\mathfrak{A} \rtimes_\alpha \mathbb{R}} t_{\alpha^\wedge} = [\mathrm{id}_{\mathfrak{A}}] \in KK(\mathfrak{A}, \mathfrak{A})$, with

$$KK^1(\mathfrak{A} \rtimes_\alpha \mathbb{R}, (\mathfrak{A} \rtimes_\alpha \mathbb{R}) \rtimes_{\alpha^\wedge} \mathbb{R}) \cong KK^1(\mathfrak{A} \rtimes_\alpha \mathbb{R}, \mathfrak{A}) \cong KK(\mathfrak{A} \rtimes_\alpha \mathbb{R}, S\mathfrak{A}).$$

If $\mathfrak{A}$ is a (graded) $C^*$-algebra, the cone $C\mathfrak{A}$ of $\mathfrak{A}$ is the (graded) $C^*$-algebra $\mathfrak{A} \otimes C\mathbb{C} \cong C_0([0,1), \mathfrak{A})$ (with the obvious grading).

If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a (graded) $*$-homomorphism of $C^*$-algebras, then the mapping cone $C_\varphi$ of $\varphi$ is the (graded) $C^*$-subalgebra of $\mathfrak{A} \oplus C\mathfrak{B}$:

$$C_\varphi = \{(x, f) \in \mathfrak{A} \oplus C\mathfrak{B} \,|\, \varphi(x) = f(0)\}.$$

There is a standard short exact sequences of $C^*$-algebras for $C_\varphi$:

$$0 \to S\mathfrak{B} \xrightarrow{\ i\ } C_\varphi \xrightarrow{\ p\ } \mathfrak{A} \to 0,$$

where $S\mathfrak{B}$ is identified with $C_0((0,1), \mathfrak{B}) \cong C_0((0,1)) \otimes \mathfrak{B}$ and $i(f) = (0, f)$ for $f \in S\mathfrak{B}$ with $\varphi(0) = 0 = f(0)$, and $p(x, f) = x \in \mathfrak{A}$.

The mapping cone of $\varphi : \mathfrak{A} \to \mathfrak{B}$ is an important example of the pullbacks of $C^*$-algebras, so as

$$\begin{array}{ccc} C_\varphi = \mathfrak{A} \oplus_{\mathfrak{B}} C\mathfrak{B} & \xrightarrow{\ p_2\ } & C\mathfrak{B} \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle ev_0} \\ \mathfrak{A} & \xrightarrow{\ \varphi\ } & \mathfrak{B}, \end{array}$$

where $p_1 = p$ and $p_2$ the projection defined by $p_2(x, f) = f$ and $ev_0$ the evaluation at zero defined by $ev_0(f) = f(0)$.

The mapping cone construction is natural in $\mathfrak{A}$ and $\mathfrak{B}$ in the sense that if we have a commutative diagram:

$$\begin{array}{ccc} \mathfrak{A}_1 & \xrightarrow{\ \varphi\ } & \mathfrak{B}_1 \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \mathfrak{A}_2 & \xrightarrow{\ \psi\ } & \mathfrak{B}_2 \end{array}$$

then there is a map $\omega : C_\varphi \to C_\psi$ making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S\mathfrak{B}_1 & \xrightarrow{i} & C_\varphi & \xrightarrow{p} & \mathfrak{A}_1 & \longrightarrow & 0 \\ & & {\scriptstyle Sg}\downarrow & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & S\mathfrak{B}_2 & \xrightarrow{i} & C_\psi & \xrightarrow{p} & \mathfrak{A}_2 & \longrightarrow & 0. \end{array}$$

**Example 6.6.** • $C_{\mathrm{id}_{\mathfrak{A}}} \cong C\mathfrak{A}$. Because,

$$C_{\mathrm{id}_{\mathfrak{A}}} = \{(x, f) \in \mathfrak{A} \oplus C\mathfrak{A} \mid \mathrm{id}_{\mathfrak{A}}(x) = x = f(0)\}$$

and

$$0 \to S\mathfrak{A} \xrightarrow{i} C_{\mathrm{id}_{\mathfrak{A}}} \xrightarrow{p} \mathfrak{A} \to 0,$$

and the isomorphism is given by the map $(f(0), f) \mapsto f \in C\mathfrak{A}$.

• $C_{S\varphi} \cong S(C_\varphi)$ for any $\varphi : \mathfrak{A} \to \mathfrak{B}$. Indeed, with $S\varphi : S\mathfrak{A} \to S\mathfrak{B}$,

$$C_{S\varphi} = \{(x, f) \in S\mathfrak{A} \oplus C(S\mathfrak{B}) \mid S\varphi(x) = f(0)\}.$$

For $(x, f) \in C_{S\varphi}$, if $x = y \otimes z$ for $y \in S\mathbb{C}$ and $z \in \mathfrak{A}$ and $f = f_1 \otimes f_2$ for $f_1 \in C\mathbb{C}$ and $f_2 \in S\mathfrak{B}$ with $f_2 = f_3 \otimes b$ for $f_3 \in S$ and $b \in \mathfrak{B}$, then

$$S\varphi(x) = S\varphi(y \otimes z) = y \otimes \varphi(z) = f(0) = f_1(0) \otimes f_2 = f_3 \otimes f_1(0)b \in S\mathfrak{B}.$$

Therefore, the following map defined as:

$$(x, f) = (y \otimes z, f_1 \otimes f_3 \otimes b) \mapsto y f_3 \otimes (z, f_1 \otimes b) \in S(C_\varphi)$$

induces the isomorphism.

(**Puppe sequences**). Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{D}$ be graded $C^*$-algebras and $\varphi : \mathfrak{A} \to \mathfrak{B}$ a graded $*$-homomorphism. Then the following sequences are exact:

$$\begin{array}{ccccc} KK(\mathfrak{D}, S\mathfrak{B}) & \xrightarrow{i_*} & KK(\mathfrak{D}, C_\varphi) & \xrightarrow{p_*} & KK(\mathfrak{D}, \mathfrak{A}) \\ {\scriptstyle S\varphi_*}\uparrow & & & & \downarrow{\scriptstyle \varphi_*} \\ KK(\mathfrak{D}, S\mathfrak{A}) & & & & KK(\mathfrak{D}, \mathfrak{B}) \end{array}$$

and

$$\begin{array}{ccccc} KK(S\mathfrak{B}, \mathfrak{D}) & \xleftarrow{i^*} & KK(C_\varphi, \mathfrak{D}) & \xleftarrow{p^*} & KK(\mathfrak{A}, \mathfrak{D}) \\ {\scriptstyle S\varphi^*}\downarrow & & & & \uparrow{\scriptstyle \varphi^*} \\ KK(S\mathfrak{A}, \mathfrak{D}) & & & & KK(\mathfrak{B}, \mathfrak{D}). \end{array}$$

*Proof.* Omitted, regrettably. □

A short exact sequence

$$0 \to \mathfrak{J} \xrightarrow{\;i\;} \mathfrak{A} \xrightarrow{\;q\;} \mathfrak{A}/\mathfrak{J} \to 0$$

of graded $C^*$-algebras is said to be **semi-split** if there is a completely positive, norm-decreasing, grading-preserving cross section for $q$, and then $\mathfrak{J}$ is called a **semi-split ideal** of $\mathfrak{A}$.

**Example 6.7.** • Semi-split ideals are exactly the ideals corresponding to invertible extensions.

• If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a graded homomorphism, then the mapping cone sequence:

$$0 \to S\mathfrak{B} \to C_\varphi \xrightarrow{\;p\;} \mathfrak{A} \to 0$$

is semi-split. A cross section for $p$ is given by the map defined by $\psi(a) = (a, (1-t)\varphi(a))$ for $a \in \mathfrak{A}$ and $(1-t)\varphi(a) \in C\mathfrak{B}$.

In fact, since $\psi$ is a $*$-homomorphism, $\psi$ is norm-decreasing and also that the entry-wise induced map $\psi^{(n)} : M_n(\mathfrak{A}) \to M_n(\mathfrak{B})$ is a $*$-homomorphism and hence, is positive, so that $\psi$ is completely positive.

As a theorem, *if $\mathfrak{A}$ is a separable $C^*$-algebra, then an extension $\tau : \mathfrak{A} \to Q^s(\mathfrak{B}) = M(\mathfrak{B} \otimes \mathbb{K})/\mathfrak{B} \otimes \mathbb{K}$ defines an invertible element of* $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B})$ *if and only if $\tau$ lifts to a completely positive contraction from $\mathfrak{A}$ to $M^s(\mathfrak{B}) = M(\mathfrak{B} \otimes \mathbb{K})$. If $\mathfrak{A}$ is unital and $\tau$ is unital, then $\tau$ defines an invertible element of* $\mathrm{Ext}_s^u(\mathfrak{A}, \mathfrak{B})$ *if and only if $\tau$ lifts to a (non necessarily unital) completely positive contraction from $\mathfrak{A}$ to $M^s(\mathfrak{B})$.* (Such an extension is called **semi-split**.)

As for the proof of this theorem, if $\tau$ is an invertible extension, then there is an extension $\tau^{-1}$ such that $\tau \oplus \tau^{-1}$ is trivial. Then $\tau \oplus \tau^{-1} : \mathfrak{A} \to M_2(Q^s(\mathfrak{B}))$ lifts to a $*$-homomorphism

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} : \mathfrak{A} \to M_2(M^s(\mathfrak{B}))$$

where $\varphi_{11}$ and also $\varphi_{22}$ being the comprerssions of a $*$-homomorphism must be completely positive contractions from $\mathfrak{A}$ to $M^s(\mathfrak{A})$, $\pi \circ \varphi_{11} = \tau$. The converse is true by the generalized **Stinespring Theorem**: *if $\mathfrak{A}$ is separable, and if $\tau$ has a completely positive contractive lifting $\varphi_{11}$ to $M^s(\mathfrak{B})$, then $\varphi_{11}$ can be dilated to a homomorphism*

$$\varphi = (\varphi_{ij})_{i,j=1}^2 : \mathfrak{A} \to M_2(M^s(\mathfrak{B})),$$

*which may be unital if* $\mathfrak{A}$ *is unital, and* $\pi \circ \varphi_{11} = \tau$ *is a homomorphism, so that* $\pi \circ \varphi_{22}$ *is also a homomorphism from* $\mathfrak{A}$ *to* $Q^s(\mathfrak{B})$ *and is an inverse for* $\tau$.

(**Basics in Extension theory**). Recall now that for an extension $E$ of $\mathfrak{A}$ by $\mathfrak{B}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{B} & \xrightarrow{i} & E & \xrightarrow{q} & \mathfrak{A} & \longrightarrow & 0 \\
 & & \| & & \sigma\downarrow & & \downarrow\tau & & \\
0 & \longrightarrow & \mathfrak{B} & \xrightarrow{i} & M(\mathfrak{B}) & \xrightarrow{\pi} & M(\mathfrak{B})/\mathfrak{B} = Q(\mathfrak{B}) & \longrightarrow & 0
\end{array}$$

with the canonical maps, the **Busby invariants** $\tau : \mathfrak{A} \cong E/\mathfrak{B} \to Q(\mathfrak{B})$ is defined and deduced from the composite $\pi \circ \sigma$ in the diagram.

The Busby invariants $\tau$ is injective if and only if $\mathfrak{B}$ is essential in $E$.

An extension $E$ of $\mathfrak{A}$ by $\mathfrak{B}$ is **trivial** if the Busby invariant $\tau : \mathfrak{A} \to Q(\mathfrak{B})$ lifts to a $*$-homomorphism from $\mathfrak{A}$ to $M(\mathfrak{B})$. This is the case where the short exact sequence splits, with a section $s : \mathfrak{A} \to E$.

In particular, $E$ is the direct sum $\mathfrak{A} \oplus \mathfrak{B}$ if and only if $\tau = 0$ the zero map. If $\mathfrak{B}$ is unital, this is the only extension since $\mathfrak{B} = M(\mathfrak{B})$ and $Q(\mathfrak{B}) = 0$. If $\mathfrak{B}$ is non-unital, then $\mathfrak{B} \neq M(\mathfrak{B})$ quite large.

Two extensions $E_j$ of $\mathfrak{A}$ by $\mathfrak{B}$ $(j = 1, 2)$ are **strongly (unitary) equivalent** if there is a unitary $u \in M(\mathfrak{B})$ such that $\tau_2(a) = \pi(u)\tau_1(a)\pi(u)^*$ for all $a \in \mathfrak{A}$.

Two extensions $E_j$ of $\mathfrak{A}$ by $\mathfrak{B}$ $(j = 1, 2)$ are **weakly (unitary) equivalent** if there is a unitary $v \in Q(\mathfrak{B})$ such that $\tau_2(a) = v\tau_1(a)v^*$ for all $a \in \mathfrak{A}$.

Two extensions $E_j$ of $\mathfrak{A}$ by $\mathfrak{B}$ $(j = 1, 2)$ are **homotopy equivalent** if the Busby homomorphisms $\tau_j : \mathfrak{A} \to Q(\mathfrak{B})$ are homotopic.

• The strong equivalence for extensions implies the weak equivalence. If the unitary group of $M(\mathfrak{B})$ is connected, in particular, if $\mathfrak{B}$ is a $\sigma$-unital, stable $C^*$-algebra, then the strong equivalence for extensions implies the homotopy equivalence.

Let $\mathbf{Ext}(\mathfrak{A}, \mathfrak{B}) = \mathbf{Ext}_s(\mathfrak{A}, \mathfrak{B})$ denote the set of strong equivalence classes of extensions of $\mathfrak{A}$ by $\mathfrak{B}$, which is a commutative semigroup.

Let $\mathbf{Ext}_w(\mathfrak{A}, \mathfrak{B})$ and $\mathbf{Ext}_h(\mathfrak{A}, \mathfrak{B})$ denote the sets of weak and homotopy equivalence classes of extensions of $\mathfrak{A}$ by $\mathfrak{B}$, respectively, which are quotients of $\mathbf{Ext}(\mathfrak{A}, \mathfrak{B})$.

Let $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B}) = \mathrm{Ext}_s(\mathfrak{A}, \mathfrak{B})$ denote the quotient of $\mathbf{Ext}(\mathfrak{A}, \mathfrak{B})$ by the subsemigroup of trivial extensions. Similarly, $\mathrm{Ext}_w(\mathfrak{A}, \mathfrak{B})$ and $\mathrm{Ext}_h(\mathfrak{A}, \mathfrak{B})$ are defined respectively as the quotients of $\mathbf{Ext}_w(\mathfrak{A}, \mathfrak{B})$ and $\mathbf{Ext}_h(\mathfrak{A}, \mathfrak{B})$ by the subsemigroup of trivial extensions.

For $* = s, w, h$, $\mathrm{Ext}_*^e(\mathfrak{A}, \mathfrak{B})$ is defined as the quotient of the subsemigroup of essential extensions by the subsemigroup of essential trivial extensions. If $\mathfrak{A}$ is unital, $\mathrm{Ext}_*^u(\mathfrak{A}, \mathfrak{B})$ is defined as the quotient of the subsemigroup of unital extensions by that of strongly unital trivial extensions.

**Example 6.8.** • When $\mathfrak{A} = \mathbb{C}$ and $\mathfrak{B} = C_0((0,1)) \cong C_0(\mathbb{R})$, the direct sum $C_0((0,1)) \oplus \mathbb{C}$ and $C(S^1)$ viewed as extensions of $\mathfrak{A}$ by $\mathfrak{B}$ are trivial. There are two other extensions $C_0([0,1)) \cong C\mathbb{C}$ and $C_0((0,1]) \cong C\mathbb{C}$. Moreover, the associated diagram becomes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(\mathbb{R}) & \longrightarrow & E & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & S\mathbb{C} & \longrightarrow & M(S\mathbb{C}) & \longrightarrow & Q(S\mathbb{C}) & \longrightarrow & 0 \end{array}$$

and $M(C_0(\mathbb{R})) \cong C(\beta\mathbb{R})$, where $\beta\mathbb{R}$ is the Stone-Čech compactification of $\mathbb{R}$.

• All extensions of $M_n(\mathbb{C})$ by $\mathbb{K}$ are trivial. In this case, the diagram is:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{K} & \longrightarrow & E & \longrightarrow & M_n(\mathbb{C}) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & \mathbb{K} & \longrightarrow & \mathbb{B} & \longrightarrow & \mathbb{B}/\mathbb{K} & \longrightarrow & 0. \end{array}$$

(We have no time to review and consider further, until the last minute.)

**Example 6.9.** • Let $\mathfrak{A}$ be a separable (graded) $C^*$-algebra, $\mathfrak{I}$ a semi-split (graded) ideal of $\mathfrak{A}$, and $q : \mathfrak{A} \to \mathfrak{A}/\mathfrak{I}$ the quotient map. Let $C_q$ be the mapping cone for $q$ and $e : \mathfrak{I} \to C_q$ defined by $e(x) = (x, 0)$. Then $[e] \in KK(\mathfrak{I}, C_q)$ is a KK-equivalence.

Note that $C_q = \mathfrak{A} \oplus_{\mathfrak{A}/\mathfrak{I}} C(\mathfrak{A}/\mathfrak{I})$ as a pull-back $C^*$-algebra, with $(x, 0)$ for $x \in \mathfrak{A}$ and $0 \in C(\mathfrak{A}/\mathfrak{I})$, so that $q(x) = 0 = 0(0) \in \mathfrak{A}/\mathfrak{I}$.

The inverse of $[e]$ is the element $u$ of $KK(C_q, \mathfrak{I}) \cong KK^1(C_q, S\mathfrak{I})$ represented by the extension:

$$0 \to S\mathfrak{I} \to C\mathfrak{A} \xrightarrow{\pi} C_q = \mathfrak{A} \oplus_{\mathfrak{A}/\mathfrak{I}} C(\mathfrak{A}/\mathfrak{I}) \to 0,$$

where $\pi(f \otimes a) = (f(0)a, f \otimes q(a))$ for $f \otimes a \in C\mathfrak{A} = C_0([0,1)) \otimes \mathfrak{A}$, so that $q(f(0)a) = f(0)q(a) = [f \otimes q(a)](0)$. More specifically, if $v \in KK^1(C_q, S\mathfrak{I}) \cong KK(C_q, \mathfrak{I})$ is the element represented by this extension, then $u = v \otimes_{\mathfrak{I}} ([\mathrm{id}_{\mathfrak{I}}] \otimes_{\mathbb{C}} y)$ (corrected), where $y \in KK^1(S\mathbb{C}, \mathbb{C}) \cong KK(S\mathbb{C}, S\mathbb{C})$, and

$$\otimes_{\mathbb{C}} : KK(\mathfrak{I}, \mathfrak{I} \otimes \mathbb{C}) \times KK(\mathbb{C} \otimes S\mathbb{C}, S\mathbb{C}) \to KK(S\mathfrak{I}, S\mathfrak{I}) \cong KK(\mathfrak{I}, \mathfrak{I}),$$
$$([\mathrm{id}_{\mathfrak{I}}], y) \mapsto [\mathrm{id}_{\mathfrak{I}}] \otimes_{\mathbb{C}} y,$$

and

$$\otimes_{\mathfrak{I}} : KK(C_q, \mathfrak{I}) \times KK(\mathfrak{I}, \mathfrak{I}) \to KK(C_q, \mathfrak{I}),$$
$$(v, [\mathrm{id}_{\mathfrak{I}}] \otimes_{\mathbb{C}} y) \mapsto v \otimes_{\mathfrak{I}} ([\mathrm{id}_{\mathfrak{I}}] \otimes_{\mathbb{C}} y) = u.$$

- We have $[e] \otimes_{C_q} u = e^*(u) = [\mathrm{id}_{\mathfrak{I}}] \in KK(\mathfrak{I}, \mathfrak{I})$.

It follows from the following diagram:

$$\begin{array}{ccccccccccc} KK^1(\mathfrak{I}, S\mathfrak{I}) \ni e^*(v) : 0 & \longrightarrow & S\mathfrak{I} & \longrightarrow & C\mathfrak{I} & \xrightarrow{ev_0} & \mathfrak{I} & \longrightarrow & 0 \\ e^* \uparrow & & \| & & \downarrow \mathrm{id}_C \otimes i & & \downarrow e & & \\ KK^1(C_q, S\mathfrak{I}) \ni v : 0 & \longrightarrow & S\mathfrak{I} & \longrightarrow & C\mathfrak{A} & \xrightarrow{\pi} & C_q & \longrightarrow & 0 \end{array}$$

that $[e] \otimes_{C_q} v = e^*(v) = [\mathrm{id}_{\mathfrak{I}}] \otimes_{\mathbb{C}} x$, where $x \in KK^1(\mathbb{C}, S\mathbb{C})$ is represented by the extension:

$$x : 0 \to S\mathbb{C} \to C\mathbb{C} \to \mathbb{C} \to 0.$$

Note that

$$\otimes_{\mathbb{C}} : KK(\mathfrak{I}, \mathfrak{I} \otimes \mathbb{C}) \times KK(\mathbb{C} \otimes S\mathbb{C}, S\mathbb{C}) \to KK(S\mathfrak{I}, S\mathfrak{I}) \cong KK^1(\mathfrak{I}, S\mathfrak{I}),$$
$$([\mathrm{id}_{\mathfrak{I}}], x) \mapsto [\mathrm{id}_{\mathfrak{I}}] \otimes_{\mathbb{C}} x,$$

Thus, $[e] \otimes_{C_q} u = e^*(u) = [\mathrm{id}_{\mathfrak{I}}] \in KK(\mathfrak{I}, \mathfrak{I})$. Because

$$\begin{aligned} [e] \otimes_{C_q} u &= [e] \otimes_{C_q} \{v \otimes_{\mathfrak{I}} ([\mathrm{id}_{\mathfrak{I}}] \otimes_{\mathbb{C}} y)\} \\ &= ([e] \otimes_{C_q} v) \otimes_{\mathfrak{I}} ([\mathrm{id}_{\mathfrak{I}}] \otimes_{\mathbb{C}} y) \\ &= ([\mathrm{id}_{\mathfrak{I}}] \otimes_{\mathbb{C}} x) \otimes_{\mathfrak{I}} ([\mathrm{id}_{\mathfrak{I}}] \otimes_{\mathbb{C}} y) \\ &= [\mathrm{id}_{\mathfrak{I}}] \otimes_{\mathbb{C}} (x \otimes_{\mathfrak{I}} [\mathrm{id}_{\mathfrak{I}}]) \otimes_{\mathbb{C}} y \\ &= [\mathrm{id}_{\mathfrak{I}}] \otimes_{\mathbb{C}} (x \otimes_{\mathbb{C}} y) \\ &= [\mathrm{id}_{\mathfrak{I}}] \otimes_{\mathbb{C}} [\mathrm{id}_{\mathbb{C}}] = [\mathrm{id}_{\mathfrak{I}}]. \end{aligned}$$

**(Six-term exact sequence)**. Let

$$0 \to \mathfrak{I} \xrightarrow{i} \mathfrak{A} \xrightarrow{q} \mathfrak{A}/\mathfrak{I} \to 0$$

be a semi-split short exact sequence of $\sigma$-unital graded $C^*$-algebras. Then, for any separable graded $C^*$-algebra $\mathfrak{D}$, the following sequence is exact:

$$\begin{array}{ccccc} KK(\mathfrak{D}, \mathfrak{I}) & \xrightarrow{i_*} & KK(\mathfrak{D}, \mathfrak{A}) & \xrightarrow{q_*} & KK(\mathfrak{D}, \mathfrak{A}/\mathfrak{I}) \\ \delta \uparrow & & & & \downarrow \delta \\ KK^1(\mathfrak{D}, \mathfrak{A}/\mathfrak{I}) & \xleftarrow{q_*} & KK^1(\mathfrak{D}, \mathfrak{A}) & \xleftarrow{i_*} & KK^1(\mathfrak{D}, \mathfrak{I}). \end{array}$$

If $\mathfrak{A}$ is separable, then for any $\sigma$-unital $\mathfrak{D}$, the following sequences is exact:

$$\begin{array}{ccccc} KK(\mathfrak{I},\mathfrak{D}) & \xleftarrow{i^*} & KK(\mathfrak{A},\mathfrak{D}) & \xleftarrow{q^*} & KK(\mathfrak{A}/\mathfrak{I},\mathfrak{D}) \\ \downarrow \delta & & & & \uparrow \delta \\ KK^1(\mathfrak{A}/\mathfrak{I},\mathfrak{D}) & \xrightarrow{q^*} & KK^1(\mathfrak{A},\mathfrak{D}) & \xrightarrow{i^*} & KK^1(\mathfrak{I},\mathfrak{D}). \end{array}$$

The maps $\delta$ in the first and second diagrams are given by the Kasparov product from the right or left by the class $\delta_q \in KK^1(\mathfrak{A}/\mathfrak{I},\mathfrak{I})$ corresponding to the extension. Under the identification of $KK^1(\mathfrak{A}/\mathfrak{I},\mathfrak{I})$ with $KK(S(\mathfrak{A}/\mathfrak{I},\mathfrak{I})$, $\delta_q$ corresponds to $j^*(u)$, where $j$ is the natural inclusion map of $S(\mathfrak{A}/\mathfrak{I})$ into $C_q = \mathfrak{A} \oplus_{\mathfrak{A}/\mathfrak{I}} C(\mathfrak{A}/\mathfrak{I})$ and $u \in KK(C_q,\mathfrak{I})$ is the inverse of $e : \mathfrak{I} \to C_q$.

Note that

$$KK^1(\mathfrak{D},\mathfrak{A}/\mathfrak{I}) \times KK^1(\mathfrak{A}/I,\mathfrak{I}) \xrightarrow{(\delta(\cdot),\delta_e)=(\cdot)\otimes_{\mathfrak{A}/\mathfrak{I}}\delta_q} KK(\mathfrak{D},\mathfrak{I}),$$

and similarly,

$$KK(\mathfrak{D},\mathfrak{A}/\mathfrak{I}) \times KK^1(\mathfrak{A}/I,\mathfrak{I}) \xrightarrow{(\delta(\cdot),\delta_e)=(\cdot)\otimes_{\mathfrak{A}/\mathfrak{I}}\delta_q} KK^1(\mathfrak{D},\mathfrak{I}),$$

and also,

$$KK^1(\mathfrak{A}/\mathfrak{I},\mathfrak{I}) \times KK(\mathfrak{I},\mathfrak{D}) \xrightarrow{(\delta_e,\delta(\cdot))=\delta_e\otimes_{\mathfrak{I}}(\cdot)} KK^1(\mathfrak{A}/\mathfrak{I},\mathfrak{D}),$$

and similarly,

$$KK^1(\mathfrak{A}/\mathfrak{I},\mathfrak{I}) \times KK^1(\mathfrak{I},\mathfrak{D}) \xrightarrow{(\delta_e,\delta(\cdot))=\delta_e\otimes_{\mathfrak{I}}(\cdot)} KK(\mathfrak{A}/\mathfrak{I},\mathfrak{D}).$$

Since $j : S(\mathfrak{A}/\mathfrak{I}) \to C_q$, we have

$$j^* : KK(C_q,\mathfrak{I}) \to KK(S(\mathfrak{A}/\mathfrak{I}),\mathfrak{I}) \cong KK^1(\mathfrak{A}/\mathfrak{I},\mathfrak{I}),$$

with $j^*(u) = \delta_e$.

Furthermore, the Puppe sequences for the extension:

$$0 \to \mathfrak{I} \xrightarrow{i} \mathfrak{A} \xrightarrow{q} \mathfrak{A}/\mathfrak{I} \to 0$$

become that the following sequences are exact:

$$\begin{array}{ccccc} KK(\mathfrak{D},S(\mathfrak{A}/\mathfrak{I})) \cong KK^1(\mathfrak{D},\mathfrak{A}/\mathfrak{I}) & \xrightarrow{j_*} & KK(\mathfrak{D},C_\varphi) & \xrightarrow{p_*} & KK(\mathfrak{D},\mathfrak{A}) \\ Sq_* \uparrow & & & & \downarrow q_* \\ KK(\mathfrak{D},S\mathfrak{A}) \cong KK^1(\mathfrak{D},\mathfrak{A}) & & & & KK(\mathfrak{D},\mathfrak{A}/\mathfrak{I}) \end{array}$$

and

$$\begin{array}{ccccc} KK(S(\mathfrak{A}/\mathfrak{J}),\mathfrak{D}) \cong KK^1(\mathfrak{A}/\mathfrak{J},\mathfrak{D}) & \xleftarrow{j^*} & KK(C_\varphi,\mathfrak{D}) & \xleftarrow{p^*} & KK(\mathfrak{A},\mathfrak{D}) \\ {\scriptstyle Sq^*}\downarrow & & & & \uparrow{\scriptstyle q^*} \\ KK(S\mathfrak{A},\mathfrak{D}) \cong KK^1(\mathfrak{A},\mathfrak{D}) & & & & KK(\mathfrak{A}/\mathfrak{J},\mathfrak{D}). \end{array}$$

In addition, the KK-equivalence $[e] \in KK(\mathfrak{J}, C_q)$ implies the isomorphisms:

$$KK(\mathfrak{D},\mathfrak{J}) \cong KK(\mathfrak{D},C_q) \quad \text{and} \quad KK(C_q,\mathfrak{D}) \cong KK(\mathfrak{J},\mathfrak{D}).$$

As well, similarly, we have the following Puppe exact sequences:

$$\begin{array}{ccccc} KK(S\mathfrak{D},S(\mathfrak{A}/\mathfrak{J})) \cong KK(\mathfrak{D},\mathfrak{A}/\mathfrak{J}) & \xrightarrow{j_*} & KK(S\mathfrak{D},C_\varphi) & \xrightarrow{p_*} & KK(S\mathfrak{D},\mathfrak{A}) \\ {\scriptstyle Sq_*}\uparrow & & & & \downarrow{\scriptstyle q_*} \\ KK(S\mathfrak{D},S\mathfrak{A}) \cong KK(\mathfrak{D},\mathfrak{A}) & & & & KK(S\mathfrak{D},\mathfrak{A}/\mathfrak{J}) \end{array}$$

and

$$\begin{array}{ccccc} KK(S(\mathfrak{A}/\mathfrak{J}),S\mathfrak{D}) \cong KK(\mathfrak{A}/\mathfrak{J},\mathfrak{D}) & \xleftarrow{j^*} & KK(C_\varphi,S\mathfrak{D}) & \xleftarrow{p^*} & KK(\mathfrak{A},S\mathfrak{D}) \\ {\scriptstyle Sq^*}\downarrow & & & & \uparrow{\scriptstyle q^*} \\ KK(S\mathfrak{A},S\mathfrak{D}) \cong KK(\mathfrak{A},\mathfrak{D}) & & & & KK(\mathfrak{A}/\mathfrak{J},S\mathfrak{D}), \end{array}$$

with

$$KK^1(\mathfrak{D},\mathfrak{J}) \cong KK(S\mathfrak{D},\mathfrak{J}) \cong KK(S\mathfrak{D},C_q) \quad \text{and}$$
$$KK(C_q,S\mathfrak{D}) \cong KK(\mathfrak{J},S\mathfrak{D}) \cong KK^1(\mathfrak{J},\mathfrak{D}).$$

The proof for the six-term exact sequence is done.

**(Pimsner-Voiculescu exact sequence for KK-theory).** Let $\mathfrak{A}$ be a trivially graded $\sigma$-unital $C^*$-algebra, and $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ the crossed product $C^*$-algebra of $\mathfrak{A}$, with $\alpha$ an action of $\mathbb{Z}$ of integers on $\mathfrak{A}$. Then, if $\mathfrak{D}$ is any separable graded $C^*$-algebra, then the following sequence is exact:

$$\begin{array}{ccccc} KK(\mathfrak{D},\mathfrak{A}) & \xrightarrow{\delta=(\mathrm{id}_{\mathfrak{A}}-\alpha)_*} & KK(\mathfrak{D},\mathfrak{A}) & \longrightarrow & KK(\mathfrak{D},\mathfrak{A}\rtimes_\alpha\mathbb{Z}) \\ \uparrow & & & & \downarrow \\ KK^1(\mathfrak{D},\mathfrak{A}\rtimes_\alpha\mathbb{Z}) & \longleftarrow & KK^1(\mathfrak{D},\mathfrak{A}) & \xleftarrow{\delta=(\mathrm{id}_{\mathfrak{A}}-\alpha)_*} & KK^1(\mathfrak{D},\mathfrak{A}). \end{array}$$

If $\mathfrak{A}$ is separable, then for any $\sigma$-unital graded $C^*$-algebra $\mathfrak{D}$, the following sequence is exact:

$$\begin{array}{ccccc} KK(\mathfrak{A},\mathfrak{D}) & \xleftarrow{\delta=(\mathrm{id}_{\mathfrak{A}}-\alpha)^*} & KK(\mathfrak{A},\mathfrak{D}) & \longleftarrow & KK(\mathfrak{A}\rtimes_\alpha \mathbb{Z},\mathfrak{D}) \\ \downarrow & & & & \uparrow \\ KK^1(\mathfrak{A}\rtimes_\alpha \mathbb{Z},\mathfrak{D}) & \longrightarrow & KK^1(\mathfrak{A},\mathfrak{D}) & \xrightarrow{\delta=(\mathrm{id}_{\mathfrak{A}}-\alpha)^*} & KK^1(\mathfrak{A},\mathfrak{D}). \end{array}$$

Indeed, there is the following short exact sequence:

$$0 \to S(\mathfrak{A}\otimes\mathbb{K}) \xrightarrow{i} (\mathfrak{A}\rtimes_\alpha \mathbb{Z})\rtimes_{\alpha^\wedge}\mathbb{R} \xrightarrow{q} \mathfrak{A}\otimes\mathbb{K} \to 0$$

corresponding to the mapping torus construction, with $\alpha^\wedge$ the dual action of $\mathbb{T}\cong\mathbb{Z}^\wedge$, extended to $\mathbb{R}$ periodically. This exact sequence is locally split, and hence semi-split. The maps $\delta$ in the diagrams above are given by the Kasparov product from the left or right by the class $\delta_q \in KK^1(\mathfrak{A}\otimes\mathbb{K}, S(\mathfrak{A}\otimes\mathbb{K}))\cong KK(\mathfrak{A},\mathfrak{A})$ corresponding to this extension, which also corresponds to the class $[\mathrm{id}_{\mathfrak{A}}-\alpha]\in KK(\mathfrak{A},\mathfrak{A})$.

Recall that the **mapping torus** for an action $\alpha$ of $\mathbb{Z}$ on $\mathfrak{A}$ is defined to be the $C^*$-algebra $M_\alpha$ of all $\mathfrak{A}$-valued continuous functions $f$ on $\mathbb{R}$ such that $f(x+1)=\alpha(f(x))$ for $x\in\mathbb{R}$. There is the following short exact sequence:

$$0\to S\mathfrak{A}\to M_\alpha\to\mathfrak{A}\to 0.$$

The dual action $\alpha^\wedge$ of $\mathbb{T}\cong\mathbb{Z}^\wedge$ on $\mathfrak{A}\rtimes_\alpha\mathbb{Z}$, extended to $\mathbb{R}$ periodically, induces the following isomorphism:

$$(\mathfrak{A}\rtimes_\alpha\mathbb{Z})\rtimes_{\alpha^\wedge}\mathbb{R}\cong M_{(\alpha^\wedge)^\wedge},$$

with the mapping torus $M_{(\alpha^\wedge)^\wedge}$ for the second dual action $(\alpha^\wedge)^\wedge$ of $\mathbb{Z}\cong\mathbb{T}^\wedge\cong(\mathbb{Z}^\wedge)^\wedge$ on $(\mathfrak{A}\rtimes_\alpha\mathbb{Z})\rtimes_{\alpha^\wedge}\mathbb{T}$ the crossed prodcut, which is isomorphic to $\mathfrak{A}\otimes\mathbb{K}$ by the Takai duality theorem.

The KK-theory six-term exact sequence and the Thom isomorphism imply that

$$\begin{array}{ccccc} KK(\mathfrak{D},S(\mathfrak{A}\otimes\mathbb{K})) & \xrightarrow{i_*} & KK^1(\mathfrak{D},\mathfrak{A}\rtimes_\alpha\mathbb{Z}) & \xrightarrow{q_*} & KK(\mathfrak{D},\mathfrak{A}\otimes\mathbb{K}) \\ \delta\uparrow & & & & \downarrow\delta \\ KK^1(\mathfrak{D},\mathfrak{A}\otimes\mathbb{K}) & \xleftarrow{q_*} & KK(\mathfrak{D},\mathfrak{A}\rtimes_\alpha\mathbb{Z}) & \xleftarrow{i_*} & KK^1(\mathfrak{D},S(\mathfrak{A}\otimes\mathbb{K})) \end{array}$$

and

$$\begin{array}{ccccc} KK(S(\mathfrak{A}\otimes\mathbb{K}),\mathfrak{D}) & \xleftarrow{i^*} & KK^1(\mathfrak{A}\rtimes_\alpha\mathbb{Z},\mathfrak{D}) & \xleftarrow{q^*} & KK(\mathfrak{A}\otimes\mathbb{K},\mathfrak{D}) \\ \delta\downarrow & & & & \uparrow\delta \\ KK^1(\mathfrak{A}\otimes\mathbb{K},\mathfrak{D}) & \xrightarrow{q^*} & KK(\mathfrak{A}\rtimes_\alpha\mathbb{Z},\mathfrak{D}) & \xrightarrow{i^*} & KK^1(\mathfrak{D},S(\mathfrak{A}\otimes\mathbb{K})). \end{array}$$

• As a byproduct of the PV sequence, we have that for $\mathfrak{A}$ a trivially graded $\sigma$-unital $C^*$-algebra, if $\mathfrak{D}$ is any separable graded $C^*$-algebra, then

$$\begin{array}{ccccc} KK(\mathfrak{D},\mathfrak{A}\times_\alpha\mathbb{T}) & \xrightarrow{\delta=(\mathrm{id}-\alpha^\wedge)_*} & KK(\mathfrak{D},\mathfrak{A}\rtimes_\alpha\mathbb{T}) & \longrightarrow & KK(\mathfrak{D},\mathfrak{A}) \\ q_*\uparrow & & & & \downarrow q_* \\ KK^1(\mathfrak{D},\mathfrak{A}) & \longleftarrow & KK^1(\mathfrak{D},\mathfrak{A}\rtimes_\alpha\mathbb{T}) & \xleftarrow{\delta=(\mathrm{id}-\alpha^\wedge)_*} & KK^1(\mathfrak{D},\mathfrak{A}\rtimes_\alpha\mathbb{T}) \end{array}$$

and if $\mathfrak{A}$ is separable, then for any $\sigma$-unital graded $C^*$-algebra $\mathfrak{D}$,

$$\begin{array}{ccccc} KK(\mathfrak{A}\rtimes_\alpha\mathbb{T},\mathfrak{D}) & \xleftarrow{\delta=(\mathrm{id}-\alpha^\wedge)^*} & KK(\mathfrak{A}\rtimes_\alpha\mathbb{T},\mathfrak{D}) & \longleftarrow & KK(\mathfrak{A},\mathfrak{D}) \\ q^*\downarrow & & & & \uparrow q^* \\ KK^1(\mathfrak{A},\mathfrak{D}) & \longrightarrow & KK^1(\mathfrak{A}\rtimes_\alpha\mathbb{T},\mathfrak{D}) & \xrightarrow{\delta=(\mathrm{id}-\alpha^\wedge)^*} & KK^1(\mathfrak{A}\rtimes_\alpha\mathbb{T},\mathfrak{D}), \end{array}$$

where $\mathfrak{A}\cong(\mathfrak{A}\rtimes_\alpha\mathbb{T})\rtimes_{\alpha^\wedge}\mathbb{Z}$ with $\alpha^\wedge$ the dual action of $\mathbb{Z}\cong\mathbb{T}^\wedge$ for an action $\alpha$ of the one torus $\mathbb{T}$ on $\mathfrak{A}$, by Takai duality, and the maps $\delta$ in the diagrams above are given by the Kasparov product from the left or right by the class $\delta_q\in KK^1(\mathfrak{A}\rtimes_\alpha\mathbb{T},S((\mathfrak{A}\rtimes_\alpha\mathbb{T})\otimes\mathbb{K}))\cong KK(\mathfrak{A}\rtimes_\alpha\mathbb{T},\mathfrak{A}\rtimes_\alpha\mathbb{T})$ corresponding to the following extension:

$$\begin{aligned} 0\to S((\mathfrak{A}\rtimes_\alpha\mathbb{T})\otimes\mathbb{K}) &\xrightarrow{i} [(\mathfrak{A}\rtimes_\alpha\mathbb{T})\rtimes_{\alpha^\wedge}\mathbb{Z}]\rtimes_{(\alpha^\wedge)^\wedge}\mathbb{R} \\ &\xrightarrow{q} (\mathfrak{A}\rtimes_\alpha\mathbb{T})\otimes\mathbb{K}\to 0, \end{aligned}$$

where this extension is viewed as the mapping torus $M_{((\alpha^\wedge)^\wedge)^\wedge}$ for the third dual action $((\alpha^\wedge)^\wedge)^\wedge$ of $\mathbb{Z}$ on $[(\mathfrak{A}\rtimes_\alpha\mathbb{T})\rtimes_{\alpha^\wedge}\mathbb{Z}]\rtimes_{(\alpha^\wedge)^\wedge}\mathbb{T}$, which is isomorphic to $(\mathfrak{A}\rtimes_\alpha\mathbb{T})\otimes\mathbb{K}$ by Takai duality.

• As another byproduct of the PV sequence, in the same manner we have that for $\mathfrak{A}$ a trivially graded $\sigma$-unital $C^*$-algebra, if $\mathfrak{D}$ is any separable graded $C^*$-algebra, then

$$\begin{array}{ccccc} KK(\mathfrak{D},\mathfrak{A}\times_\alpha\mathbb{Z}_n) & \xrightarrow{\delta=(\mathrm{id}-\alpha^\wedge)_*} & KK(\mathfrak{D},\mathfrak{A}\rtimes_\alpha\mathbb{Z}_n) & \longrightarrow & KK^1(\mathfrak{D},\mathfrak{A}\rtimes_\alpha\mathbb{Z}) \\ q_*\uparrow & & & & \downarrow q_* \\ KK(\mathfrak{D},\mathfrak{A}\rtimes_\alpha\mathbb{Z}) & \longleftarrow & KK^1(\mathfrak{D},\mathfrak{A}\rtimes_\alpha\mathbb{Z}_n) & \xleftarrow{\delta=(\mathrm{id}-\alpha^\wedge)_*} & KK^1(\mathfrak{D},\mathfrak{A}\rtimes_\alpha\mathbb{Z}_n) \end{array}$$

and if $\mathfrak{A}$ is separable, then for any $\sigma$-unital graded $C^*$-algebra $\mathfrak{D}$,

$$\begin{array}{ccccc} KK(\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n, \mathfrak{D}) & \xleftarrow{\delta=(\mathrm{id}-\alpha^\wedge)^*} & KK(\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n, \mathfrak{D}) & \longleftarrow & KK^1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}, \mathfrak{D}) \\ {\scriptstyle q^*}\downarrow & & & & \uparrow{\scriptstyle q^*} \\ KK(\mathfrak{A} \rtimes_\alpha \mathbb{Z}, \mathfrak{D}) & \longrightarrow & KK^1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n, \mathfrak{D}) & \xrightarrow{\delta=(\mathrm{id}-\alpha^\wedge)^*} & KK^1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n, \mathfrak{D}), \end{array}$$

where $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ with the action $\alpha$ extended to $\mathbb{Z}$ from an action $\alpha$ of a cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ on $\mathfrak{A}$ is isomorphic to the mapping torus $M_{\alpha^\wedge}$ for the dual action $\alpha^\wedge$ of $\mathbb{Z}_n \cong \mathbb{Z}_n^\wedge$, extended to $\mathbb{Z}$ and to $\mathbb{R}$ periodically, on $\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n$ with

$$0 \to S(\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n) \xrightarrow{i} M_{\alpha^\wedge} \cong \mathfrak{A} \rtimes_\alpha \mathbb{Z} \xrightarrow{q} \mathfrak{A} \rtimes_\alpha \mathbb{Z}_n \to 0,$$

and the maps $\delta$ in the diagrams above are given by the Kasparov product from the left or right by the class $\delta_q \in KK^1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n, S(\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n)) \cong KK(\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n, \mathfrak{A} \rtimes_\alpha \mathbb{Z}_n)$ corresponding to this extension. Note that

$$(\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n) \rtimes_{\alpha^\wedge} \mathbb{R} \cong M_{\alpha^\wedge}.$$

As well, we have another one:

$$0 \to S(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \to (\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \rtimes_{\alpha^\wedge} \mathbb{R} \cong M'_{\alpha^\wedge} \to \mathfrak{A} \rtimes_\alpha \mathbb{Z} \to 0.$$

This quotient should give a quotient map from $M'_{\alpha^\wedge}$ to $M_{\alpha^\wedge}$.

# 7 UCT and KK-equivalence

We denote by $N$ the smallest class of separable nuclear $C^*$-algebras such that:

(N1) $N$ contains $\mathbb{C}$;

(N2) $N$ is closed under countable inductive limits of $C^*$-algebras;

(N3) For a short exact sequence of $C^*$-algebras, if non-zero two terms are in $N$, then so is the nonzero third;

(N4) $N$ is closed under KK-equivalence.

Let $N_0$ denote the smallest class of separable nuclear $C^*$-algebras closed under

($N_0$1) $N_0$ contains $\mathbb{C}$ and $C_0(\mathbb{R})$; ($N_0$2)= (N2);

($N_0$3) For a split short exact sequence of $C^*$-algebras, if nonzero two terms are in $N_0$, then so is the nonzero third; ($N_0$4)=(N4).

The class $N$ is called the **bootstrap category** in this sense.

The class $N$ is also the smallest category of all separable nuclear $C^*$-algebras such that the UCT below holds for their pairs of every $C^*$-algebra. In this sense, the class $N$ is also called the **UCT class**.

Set $K_*(\mathfrak{A}) = K_0(\mathfrak{A}) \oplus K_1(\mathfrak{A})$, $KK(\mathfrak{A}, \mathfrak{B}) = KK^0(\mathfrak{A}, \mathfrak{B})$, and $KK^*(\mathfrak{A}, \mathfrak{B}) = KK^0(\mathfrak{A}, \mathfrak{B}) \oplus KK^1(\mathfrak{A}, \mathfrak{B})$.

**(Universal Coefficient Theorem (UCT)).** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable $C^*$-algebras, with $\mathfrak{A}$ in the bootstrap category or the UCT class $N$. Then we have the following short exact sequence:*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \xrightarrow{\delta} KK^*(\mathfrak{A}, \mathfrak{B}) \xrightarrow{\gamma} \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to 0$$

*and so that, more exactly,*

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_0(\mathfrak{B})) \\ &\xrightarrow{\delta} KK(\mathfrak{A}, \mathfrak{B}) \\ &\xrightarrow{\gamma} \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{B})) \oplus \mathrm{Hom}(K_1(\mathfrak{A}), K_1(\mathfrak{B})) \to 0 \end{aligned}$$

*and*

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_0(\mathfrak{B})) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_1(\mathfrak{B})) \\ &\xrightarrow{\delta} KK^1(\mathfrak{A}, \mathfrak{B}) \\ &\xrightarrow{\gamma} \mathrm{Hom}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \oplus \mathrm{Hom}(K_1(\mathfrak{A}), K_0(\mathfrak{B})) \to 0. \end{aligned}$$

*The sequence is natural in each variable, and splits unnaturally. If $K_*(\mathfrak{A})$ is free or $K_*(\mathfrak{B})$ is divisible, then $\gamma$ is an isomorphism.*

*In particular,*

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{A})) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_0(\mathfrak{A})) \\ &\xrightarrow{\delta} KK(\mathfrak{A}, \mathfrak{A}) \\ &\xrightarrow{\gamma} \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{A})) \oplus \mathrm{Hom}(K_1(\mathfrak{A}), K_1(\mathfrak{A})) \to 0 \end{aligned}$$

*and*

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_0(\mathfrak{A})) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_1(\mathfrak{A})) \\ &\xrightarrow{\delta} KK^1(\mathfrak{A}, \mathfrak{A}) \\ &\xrightarrow{\gamma} \mathrm{Hom}(K_0(\mathfrak{A}), K_1(\mathfrak{A})) \oplus \mathrm{Hom}(K_1(\mathfrak{A}), K_0(\mathfrak{A})) \to 0, \end{aligned}$$

*and $KK(\mathfrak{A}, \mathfrak{A})$ as well as $KK^*(\mathfrak{A}, \mathfrak{A}) = KK^0(\mathfrak{A}, \mathfrak{A}) \oplus KK^1(\mathfrak{A}, \mathfrak{A})$ a graded ring have the following ring structure: the product of any two $\mathrm{Ext}^1_{\mathbb{Z}}$ is zero, and $\mathrm{Hom} = \mathrm{Ext}^0_{\mathbb{Z}}$ acts on $\mathrm{Hom}$ and $\mathrm{Ext}^1_{\mathbb{Z}}$ as usual, so the respective $\mathrm{Ext}^1_{\mathbb{Z}}$*

*terms in $KK(\mathfrak{A},\mathfrak{A})$ and $KK^*(\mathfrak{A},\mathfrak{A})$ form ideals with square zero. Also, $KK(\mathfrak{A},\mathfrak{A})$ is viewed as a subring of $KK^*(\mathfrak{A},\mathfrak{A})$, but $KK^1(\mathfrak{A},\mathfrak{A})$ is not, without ring structure.*

We denote by $N'$ the class of $C^*$-algebras such that the UCT holds for their pairs with every $C^*$-algebra, i.e., if $\mathfrak{A} \in N'$, then the UCT holds for $(\mathfrak{A},\mathfrak{B})$ with $\mathfrak{B}$ any. Certainly, we have $N \subset N'$. That the inclusion is strict has already shown above. The reason is that there are some non-nuclear, separable $C^*$-algebras for which the UCT holds. We may call the class $N'$ the **general UCT** class, to distinguish from the UCT class $N$.

**Example 7.1.** • Let $\mathfrak{A}$ and $\mathfrak{B}$ be $C^*$-algebras in $N'$. If they have their K-theory groups isomorphic, then $\mathfrak{A}$ and $\mathfrak{B}$ are KK-equivalent. Namely, in the general UCT class $N'$, the K-theory equivalence implies the KK-theory equivalence. In particular, the same holds for the UCT class $N$.

This is deduced as a corollary from that if $\mathfrak{A}$ and $\mathfrak{B}$ in $N'$ and there is $x \in KK(\mathfrak{A},\mathfrak{B})$ such that $\gamma(x) \in \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B}))$ is an isomorphism, then $x$ is a KK-equivalence. Hence, if $K_*(\mathfrak{A}) \cong K_*(\mathfrak{A})$, then there is the canonical isomorphism in $\mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{B}))$, and it is viewed as an element of $KK(\mathfrak{A},\mathfrak{B})$ by splitting of the UCT, which gives a KK-equivalence between $\mathfrak{A}$ and $\mathfrak{B}$.

Indeed, by naturality of the UCT we have the following commutative diagram:

$$\begin{array}{ccc}
0 & = & 0 \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{B}), K_*(\mathfrak{D})) & \xrightarrow{\theta} & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{D})) \\
{\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta} \\
KK^*(\mathfrak{B},\mathfrak{D}) & \xrightarrow{x\otimes_{\mathfrak{B}}(\cdot)} & KK^*(\mathfrak{A},\mathfrak{D}) \\
{\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\
\mathrm{Hom}(K_*(\mathfrak{B}), K_*(\mathfrak{D})) & \xrightarrow{\eta} & \mathrm{Hom}(K_*(\mathfrak{A}), K_*(\mathfrak{D})) \\
\downarrow & & \downarrow \\
0 & = & 0
\end{array}$$

with $\theta$ and $\eta$ the isomorphisms induced by $K_*(\mathfrak{B}) \cong K_*(\mathfrak{A})$. Apply the Five Lemma to obtain that the middle map $x \otimes_{\mathfrak{B}} (\cdot)$ by Kasparov product is an isomorphism. Taking $\mathfrak{D} = \mathfrak{A}$, we have that there is $y \in KK(\mathfrak{B},\mathfrak{A})$ such that $x \otimes_{\mathfrak{B}} y = [\mathrm{id}_{\mathfrak{A}}] \in KK(\mathfrak{A},\mathfrak{A})$. Similarly, by naturality of the UCT we have

$$
\begin{array}{ccc}
0 & = & 0 \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{D}), K_*(\mathfrak{A})) & \xrightarrow{\theta} & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{D}), K_*(\mathfrak{B})) \\
{\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta} \\
KK^*(\mathfrak{D}, \mathfrak{A}) & \xrightarrow{(\cdot)\otimes_{\mathfrak{A}} x} & KK^*(\mathfrak{D}, \mathfrak{B}) \\
{\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\
\mathrm{Hom}(K_*(\mathfrak{D}), K_*(\mathfrak{A})) & \xrightarrow{\eta} & \mathrm{Hom}(K_*(\mathfrak{D}), K_*(\mathfrak{B})) \\
\downarrow & & \downarrow \\
0 & = & 0
\end{array}
$$

with $\theta$ and $\eta$ the isomorphisms induced by $K_*(\mathfrak{A}) \cong K_*(\mathfrak{B})$. Apply the Five Lemma to obtain that the middle map $(\cdot) \otimes_{\mathfrak{A}} x$ by Kasparov product is an isomorphism. Taking $\mathfrak{D} = \mathfrak{B}$, we have that there is $y \in KK(\mathfrak{B}, \mathfrak{A})$ such that $y \otimes_{\mathfrak{A}} x = [\mathrm{id}_{\mathfrak{B}}] \in KK(\mathfrak{B}, \mathfrak{B})$.

• Consequently, the K-theory equivalence is the same as the KK-theory equivalence, for $\sigma$-unital $C^*$-algebras in the general UCT class $N'$. In particular, the same holds for $C^*$-algebras in the class $N$.

• *Let $\mathfrak{A}$ be any (separable) $C^*$-algebra. Then there is a (separable) commutative $C^*$-algebra $\mathfrak{C}$, whose spectrum has dimension at most three, and there is an element $x \in KK(\mathfrak{C}, \mathfrak{B})$ such that $\gamma(x) : K_*(\mathfrak{C}) \to K_*(\mathfrak{B})$ is an isomorphism. For all $\mathfrak{A} \in N'$, we have that $(\cdot) \otimes_{\mathfrak{C}} x : KK^*(\mathfrak{A}, \mathfrak{C}) \to KK^*(\mathfrak{A}, \mathfrak{B})$ is an isomorphism. If $\mathfrak{B} \in N'$, then $x$ is a KK-equivalence. We may choose $\mathfrak{C}$ to be the direct sum $\mathfrak{C}_0 \oplus \mathfrak{C}_1$ such that $K_1(\mathfrak{C}_0) = 0$ and $K_0(\mathfrak{C}_1) = 0$. If $K_*(\mathfrak{B})$ is finitely generated, then we may also choose $\mathfrak{C}$ whose spectrum is a finite complex of dimension at most three.*

The proof is as follows. Since the map $\gamma : KK^*(\mathfrak{C}, \mathfrak{B}) \to \mathrm{Hom}(K_*(\mathfrak{C}), K_*(\mathfrak{B}))$ is surjective, it is only necessary to find a commutative $C^*$-algebra with its spectrum specified and its K-groups isomorphic to those of $\mathfrak{B}$. There is a standard way to construct a commutative $\mathfrak{C}_0$ with $K_0$ specified and $K_1$ trivial: choose a free resolution:

$$0 \to F_1 \xrightarrow{f} F_2 \longrightarrow K_0(\mathfrak{B}) \to 0,$$

and let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ each be a $c_0$-direct sum of copies of $S\mathbb{C} = C_0(\mathbb{R})$ and $\varphi : \mathfrak{D}_1 \to \mathfrak{D}_2$ with $\varphi_* = f$ on $K^*(\mathfrak{D}_1)$, and let $\mathfrak{C}_0$ be the mapping cone of $\varphi$:

$$\mathfrak{C}_0 = \{(x, f) \in \mathfrak{D}_1 \oplus C\mathfrak{D}_2 \,|\, \varphi(x) = f(0)\}$$

and then let $\mathfrak{C}_1 = S\mathfrak{C}_0$, to obtain $\mathfrak{C} = \mathfrak{C}_0 \oplus \mathfrak{C}_1$. Other things follow as before or from a moment of thought.

• It follows that the class $N$ is equal to the smallest class $N^\sim$ of separable nuclear $C^*$-algebras such that:

$N^\sim$ contains commutative $C^*$-algebras and

(N4) $N^\sim$ is closed under KK-equivalence.

We may call the class $N^\sim$ the **KK-commutative** class.

• *Let $\mathfrak{A}$ be a $C^*$-algebra in the class $N$ with torsion free K-theory. Then there are simple AF $C^*$-algebras $\mathfrak{A}_0$ and $\mathfrak{A}_1$ such that $\mathfrak{A} \sim_{KK} \mathfrak{A}_0 \oplus S\mathfrak{A}_1$. Thus $\mathfrak{A}$ is in fact in the class $N_0$.*

The proof is as follows. Any countable torsion-free abelian group $G$ with $G \neq \mathbb{Z}$ can be embedded as an additive subgroup of $\mathbb{R}$, so that the image is dense in $\mathbb{R}$. For instance, $\mathbb{Z}^2 \cong \mathbb{Z} \oplus \theta\mathbb{Z} \subset \mathbb{R}^2$ with $\theta$ an irrational number. If $K_i(\mathfrak{A}) = \mathbb{Z}$, then set $\mathfrak{A}_i = \mathbb{C}$; otherwise embed $K_i(\mathfrak{A})$ into $\mathbb{R}$ and let $\mathfrak{A}_i$ be an AF $C^*$-algebra with the image of $K_i(\mathfrak{A})$ as dimension group.

• It follows that the class $N_0$ consists of $C^*$-algebras in $N$ with torsion free K-theroy.

We may call the class $N_0$ the **torsion free** UCT class or the **torsion free** bootstrap category.

• *Let $\mathfrak{B}$ be a $C^*$-algebra. The following conditions are equivalent:* (i) $\mathfrak{B} \in N'$*;* (ii) $\mathfrak{B}$ *is KK-equivalent to a $C^*$-algebra in $N$;* (iii) $\mathfrak{B}$ *is KK-equivalent to a (separable) commutative $C^*$-algebra;* (iv) *If $\mathfrak{D}$ is any $C^*$-algebra with $K_*(\mathfrak{D}) = 0$, then $KK^*(\mathfrak{B}, \mathfrak{D}) = 0$.*

The proof is as in the following.

(i) $\Rightarrow$ (ii). The UCT implies that if $K_*(\mathfrak{D}) = 0$, then both $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{B}), K_*(\mathfrak{D}))$ and $\mathrm{Hom}(K_*(\mathfrak{B}), K_*(\mathfrak{D}))$ are zero and hence $KK^*(\mathfrak{B}, \mathfrak{D}) = 0$.

(iii) $\Rightarrow$ (ii). The class $N$ contains any $C(X)$ for every finite simplicial complex $X$. Since every compact space is an inverse limit of simplicial complexes, then $C(X) \in N$ for every compact space $X$. Also, $N$ contains $C_0(X)$ for every locally compact $X$ because we have $0 \to C_0(X) \to C(X^+) \to \mathbb{C} \to 0$ and use (N3). Thus $N$ contains all separable commutative $C^*$-algebra. Thus if $\mathfrak{B} \sim_{KK} \mathfrak{A} = C(X)$ or $C_0(X)$, then $\mathfrak{B} \sim_{KK} \mathfrak{A} \in N$.

(ii) $\Rightarrow$ (i). If $\mathfrak{B} \sim_{KK} \mathfrak{A} \in N$ with $x \in KK(\mathfrak{B}, \mathfrak{A})$ a KK-equivalence, then $x \otimes_{\mathfrak{A}} (\cdot) : KK(\mathfrak{A}, \mathfrak{D}) \to KK(\mathfrak{B}, \mathfrak{D})$ and $(\cdot) \otimes_{\mathfrak{B}} x : KK(\mathfrak{D}, \mathfrak{B}) \to KK(\mathfrak{D}, \mathfrak{A})$ are isomorphism for any $C^*$-algebra $\mathfrak{D}$, so that in particular,

$$K_j(\mathfrak{B}) \cong KK^j(\mathbb{C}, \mathfrak{B}) \cong KK^j(\mathbb{C}, \mathfrak{A}) \cong K_j(\mathfrak{A})$$

for $\mathfrak{A}$ and $\mathfrak{B}$ both $\sigma$-unital. Therefore, for a separable $C^*$-algebra $\mathfrak{D}$, the UCT for $(\mathfrak{A}, \mathfrak{D})$ implies that the UCT for $(\mathfrak{B}, \mathfrak{D})$ by $KK(\mathfrak{B}, \mathfrak{D}) \cong KK(\mathfrak{A}, \mathfrak{D})$ and $K_j(\mathfrak{B}) \cong K_j(\mathfrak{A})$.

(iv) $\Rightarrow$ (iii). There is a (separable) commutative $C^*$-algebra $\mathfrak{A}$ and $x \in KK(\mathfrak{A}, \mathfrak{B})$ with $\gamma(x) : K_*(\mathfrak{A}) \to K_*(\mathfrak{B})$ is an isomorphism. Represent $x$ by a semi-split extension:

$$0 \to S\mathfrak{B} \otimes \mathbb{K} \xrightarrow{\ i\ } \mathfrak{D} \xrightarrow{\ q\ } \mathfrak{A} \to 0.$$

Then the six-term exact sequence of K-groups becomes:

$$\begin{array}{ccccc} K_1(\mathfrak{B}) & \xrightarrow{i_*} & K_0(\mathfrak{D}) & \xrightarrow{q_*} & K_0(\mathfrak{A}) \\ \partial \uparrow \cong & & & & \cong \downarrow \partial \\ K_1(\mathfrak{A}) & \xleftarrow{q_*} & K_1(\mathfrak{D}) & \xleftarrow{i_*} & K_0(\mathfrak{B}) \end{array}$$

and hence $K_*(\mathfrak{D}) = 0$ Thus, $KK^*(\mathfrak{B}, \mathfrak{D}) = 0$. And then the KK-theory six-term exact sequence in the second variable becomes:

$$\begin{array}{ccccc} KK^1(\mathfrak{B}, \mathfrak{B}) & \xrightarrow{i_*} & KK(\mathfrak{B}, \mathfrak{D}) = 0 & \xrightarrow{q_*} & KK(\mathfrak{B}, \mathfrak{A}) \\ \uparrow (\cdot)\otimes_{\mathfrak{A}} x & & & & \downarrow (\cdot)\otimes_{\mathfrak{A}} x \\ KK^1(\mathfrak{B}, \mathfrak{A}) & \xleftarrow{q_*} & KK^1(\mathfrak{B}, \mathfrak{D}) = 0 & \xleftarrow{i_*} & KK(\mathfrak{B}, \mathfrak{B}) \end{array}$$

and thus $(\cdot) \otimes_{\mathfrak{A}} x : KK(\mathfrak{B}, \mathfrak{A}) \to KK(\mathfrak{B}, \mathfrak{B})$ is an isomorphism, and hence there is $y \in KK(\mathfrak{B}, \mathfrak{A})$ such that $y \otimes_{\mathfrak{A}} x = [\mathrm{id}_{\mathfrak{B}}] \in KK(\mathfrak{B}, \mathfrak{B})$. Moreover, similarly, represent $y$ by a semi-split extension:

$$0 \to S\mathfrak{A} \otimes \mathbb{K} \xrightarrow{\ i\ } \mathfrak{D} \xrightarrow{\ q\ } \mathfrak{B} \to 0,$$

so that if follows that

$$\begin{array}{ccccc} KK^1(\mathfrak{A}, \mathfrak{A}) & \xrightarrow{i_*} & KK(\mathfrak{A}, \mathfrak{D}) = 0 & \xrightarrow{q_*} & KK(\mathfrak{A}, \mathfrak{B}) \\ \uparrow (\cdot)\otimes_{\mathfrak{B}} y & & & & \downarrow (\cdot)\otimes_{\mathfrak{B}} y \\ KK^1(\mathfrak{A}, \mathfrak{B}) & \xleftarrow{q_*} & KK^1(\mathfrak{A}, \mathfrak{D}) = 0 & \xleftarrow{i_*} & KK(\mathfrak{A}, \mathfrak{A}) \end{array}$$

by using (iii) $\Rightarrow$ (iv) and thus $(\cdot) \otimes_{\mathfrak{B}} y : KK(\mathfrak{A}, \mathfrak{B}) \to KK(\mathfrak{A}, \mathfrak{A})$ is an isomorphism, and hence there is $x' \in KK(\mathfrak{A}, \mathfrak{B})$ such that $x' \otimes_{\mathfrak{B}} y = [\mathrm{id}_{\mathfrak{A}}] \in KK(\mathfrak{A}, \mathfrak{A})$. It then follows that $\mathfrak{B} \sim_{KK} \mathfrak{A}$.

(iii) $\Rightarrow$ (iv). If $\mathfrak{B}$ is KK-equivalent to a commutative $C^*$-algebra $\mathfrak{C}$, then $KK^*(\mathfrak{B}, \mathfrak{D}) \cong KK^*(\mathfrak{C}, \mathfrak{D})$ for any $C^*$-algebra. The UCT implies that if $K_*(\mathfrak{D}) = 0$, then $KK^*(\mathfrak{C}, \mathfrak{D}) = 0$. Therefore, $KK^*(\mathfrak{B}, \mathfrak{D}) = 0$.

# 8 Classification of $C^*$-algebras by KK-equivalence

Obtained as a collection is the following table in the next page:

Table 1: Classification for some KK-equivalent $C^*$-algebras

| Representatives<br>K-theory groups | Classes | Examples |
|---|---|---|
| Zero 0<br>$K_0 = 0$, $K_1 = 0$ | Contractible to 0 | $C\mathbb{C} = C_0((0,1])$,<br>$C\mathfrak{A} = C_0(0,1]) \otimes \mathfrak{A}$ |
| Point $\mathbb{C}$<br><br>$K_0 = \mathbb{Z}$, $K_1 = 0$ | Commutative<br>Contracitible to $\mathbb{C}$<br>Elementary<br>Type I, non-split ext<br>Crossed product by $\mathbb{R}$ | $C_0(\mathbb{R}^{2n})$<br>$C([0,1]^n)$<br>$M_n(\mathbb{C})$, $\mathbb{K}$<br>Toeplitz $\mathfrak{T}$<br>Even solvable Lie $C^*(G)$ |
| Line $C_0(\mathbb{R})$<br>$K_0 = 0$, $K_1 = \mathbb{Z}$ | Commutative<br>Suspended<br>Crossed product by $\mathbb{R}$ | $C_0(\mathbb{R}^{2n+1})$<br>$S\mathfrak{A} \sim_{KK} S\mathbb{C} = C_0(\mathbb{R})$<br>Odd solvable Lie $C^*(G)$ |
| Circle $C(\mathbb{T})$<br>$K_0 = \mathbb{Z}$, $K_1 = \mathbb{Z}$ | Direct sum<br>Split extension<br><br>Free product (=FP) | $\mathfrak{A} \oplus \mathfrak{B} \sim_{KK} \mathbb{C} \oplus C_0(\mathbb{R})$<br>$C(S^{2n+1}) \cong C_0(\mathbb{R}^{2n+1}) \rtimes \mathbb{C}$<br>$\mathfrak{A} \rtimes \mathfrak{B} \sim_{KK} C_0(\mathbb{R}) \oplus \mathbb{C}$<br>$\mathfrak{A} * \mathfrak{B} \sim_{KK} \mathbb{C} \oplus C_0(\mathbb{R})$ |
| Two points $\mathbb{C}^2$<br>$K_0 = \mathbb{Z}^2$, $K_1 = 0$ | Direct sum<br>Split, extension (=ext)<br><br>Free product | $\mathfrak{A} \oplus \mathfrak{B} \sim_{KK} \mathbb{C} \oplus \mathbb{C}$<br>$C(S^{2n}) = C_0(\mathbb{R}^{2n}) \rtimes \mathbb{C}$<br>$\mathfrak{A} \rtimes \mathfrak{B} \sim_{KK} \mathbb{C} \oplus \mathbb{C}$<br>$\mathfrak{A} * \mathfrak{B} \sim_{KK} \mathbb{C}^2$ |
| Eight $C(\mathbf{8})$<br>$K_0 = \mathbb{Z}$, $K_1 = \mathbb{Z}^2$ | Split extension<br>Pull back<br>Unital free product | $C(\mathbf{8}) \cong C_0(\mathbb{R} \sqcup \mathbb{R}) \rtimes \mathbb{C}$<br>$C(\mathbf{8}) \cong C(\mathbb{T}) \oplus_{\mathbb{C}} C(\mathbb{T})$<br>$C^*(F_2) \cong C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T})$ |
| $n$ points $\mathbb{C}^n$<br>$K_0 = \mathbb{Z}^n$, $K_1 = 0$ | Direct sum<br>Successive split ext<br>Successive FP | $\oplus^n \mathfrak{A}_i \sim_{KK} \mathbb{C}^n$<br>$\mathfrak{A}_1 \rtimes \cdots \rtimes \mathfrak{A}_n \sim_{KK} \mathbb{C} \rtimes \cdots \rtimes \mathbb{C}$<br>$\mathfrak{A}_1 * \cdots * \mathfrak{A}_n \sim_{KK} \mathbb{C} * \cdots * \mathbb{C}$ |
| Hawaian $C(H_n)$<br>$K_0 = \mathbb{Z}$, $K_1 = \mathbb{Z}^n$ | Split extension<br>Pull back<br>Unital FP | $C(H_n) \cong C_0(\sqcup^n \mathbb{R}) \rtimes \mathbb{C}$<br>$C(H_n) \cong C(\mathbb{T}) \oplus_{\mathbb{C}} \cdots \oplus_{\mathbb{C}} C(\mathbb{T})$<br>$C^*(F_n) \cong C(\mathbb{T}) *_{\mathbb{C}} \cdots *_{\mathbb{C}} C(\mathbb{T})$ |
| $\infty$ points $\oplus^\infty \mathbb{C}$<br>$K_0 = \mathbb{Z}^\infty$, $K_1 = 0$ | Direct sum<br>Successive split ext<br>Successive FP | $\oplus^\infty \mathfrak{A}_j \sim_{KK} \oplus^\infty \mathbb{C}$<br>$\mathfrak{A}_1 \rtimes \cdots \rtimes \mathfrak{A}_n \cdots \sim_{KK} \mathbb{C} \rtimes \cdots \rtimes \mathbb{C} \cdots$<br>$\mathfrak{A}_1 * \cdots * \mathfrak{A}_n \cdots \sim_{KK} \mathbb{C} * \cdots * \mathbb{C} \cdots$ |
| Hawaian $C(H_\infty)$<br>$K_0 = \mathbb{Z}$, $K_1 = \mathbb{Z}^\infty$ | Split extension<br>Pull back<br>Unital FP | $C(H_\infty) \cong C_0(\sqcup^\infty \mathbb{R}) \rtimes \mathbb{C}$<br>$C(H_\infty) \cong \oplus_{\mathbb{C}}^\infty C(\mathbb{T})$<br>$C^*(F_\infty) \cong *_{\mathbb{C}}^\infty C(\mathbb{T})$ |

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