The K-theory and the E-theory for C^*-algebras and the Baum-Connes conjecture for discrete groups : a commentative local study

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THE K-THEORY AND THE E-THEORY FOR C^*-ALGEBRAS AND THE BAUM-CONNES CONJECTURE FOR DISCRETE GROUPS — a commentative local study

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Dedicated to the memory of Takayuki Furuta

Abstract

We study the K-theory and the E-theory for C^*-algebras and consider as an application the Baum-Connes conjecture for discrete groups and their group C^*-algebras and crossed product C^*-algebras, mostly based on the lecture notes by Nigel Higson and Erik Guentner.

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1 Introduction as preface

This is nothing but a running commentary on their lecture notes as the section titled: Group C^*-algebras and K-theory, in the book: Noncommutative Geometry (LNM 1831), by Nigel Higson and Erik Guentner [24]. As a back to the past for a return to the future, we would like to study thoroughly almost all the details of the lecture notes by them, at the basic level. With some considerable effort within the time and the (page) space limited for publication as well as revising the text, we made some additional, helpful, and technical notes or computations, (minor or more) corrections (from

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misprints, miscitations, or not), and proofs or elementary lemmas added in details, as well as even solutions for exercises, for convenience to the readers and for our self-containedness, to somewhat extent (but not completed in some cases). Some notations are changed from the original ones by our taste. We go through their lecture notes along the same way, but numberings or titles of sections or subsections are a bit changed. Several items in the original references are not at hand, but cited, and a few or more are not cited by running short of preparation, but with some effort for such to be as less as possible. This paper after this introduction is organized as follows.

Contents

2 Brief reviewing K-theory groups for C^*-algebras

Definition 2.1. Let $\mathfrak{A}$ be a ring with a multiplicative unit. The K-theory group $K_0(\mathfrak{A})$ of $\mathfrak{A}$ is defined to be the abelian group generated by (stably) isomorphism classes $[E], [F]$ of finitely generated and projective (right) $\mathfrak{A}$-modules E, F, with the addition by the direct sum $\oplus$ and the formal difference as

$$[E] + [F] = [E \oplus F] \quad \text{and} \quad [E] - [F]$$

with $[0]$ the class of the zero $\mathfrak{A}$-module as the unit.

Remark. Let $M_n(\mathfrak{A})$ be the ring (or C^*-algebra) of all $n \times n$ matrices over a ring (or C^*-algebra) $\mathfrak{A}$. An idempotent $p \in M_n(\mathfrak{A})$ with $p^2 = p$ corresponds to a finitely generated and projective $\mathfrak{A}$-module $E = p\mathfrak{A}^n$. In particular, the $n \times n$ identity matrix $1_n \in M_n(\mathfrak{A})$ for $\mathfrak{A}$ unital corresponds to $\mathfrak{A}^n$ a free $\mathfrak{A}$-module.

Note also that $[E] = [F] \in K_0(\mathfrak{A})$ if and only if E and F are stably isomorphic as $E \oplus \mathfrak{A}^n \cong F \oplus \mathfrak{A}^n$ for some $n \geq 0$. This may say that E and F are in fact isomorphic, but its isomorphism may be constructed in a bigger matrix algebra over $\mathfrak{A}$. Hence, $[E] - [F] = [G] - [H]$ if and only if $E \oplus H$ and $F \oplus G$ are stably isomorphic. Refer to Milnor [44].

Remark. A ring homomorphism φ from $\mathfrak{A}$ to $\mathfrak{B}$ extends to a homomorphism from $M_n(\mathfrak{A})$ to $M_n(\mathfrak{B})$ as $\varphi((a_{ij})) = (\varphi(a_{ij}))$ or $\varphi(E) = E \otimes_{\mathfrak{A}} \mathfrak{B}$, with $(a_{ij})a \otimes b = (a_{ij}) \otimes \varphi(a)b$. Thus, as a functor there is an induced group homomoprhism φ_* from $K_0(\mathfrak{A})$ to $K_0(\mathfrak{B})$ as $\varphi_*[p] = [\varphi(p)]$ or $\varphi_*[p\mathfrak{A}^n] = [\varphi(p\mathfrak{A}^n)] = [\varphi(p)\mathfrak{B}^n]$. In the end, $E \otimes_{\mathfrak{A}} \mathfrak{B}$ may be defined by replacing involved elements in $\mathfrak{A}$ by corresponding elements in $\mathfrak{B}$ via φ.

Theorem 2.2. *For the pull back commutative diagram of rings (or C^*-algebras) with coordinate projections p_1, p_2 and homomorphisms q_1, q_2 :*

$$\begin{tikzcd}
\mathfrak{A} = \mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2 \arrow[r, "p_1"] \arrow[d, "p_2"'] & \mathfrak{A}_1 \arrow[d, "q_1"] \\
\mathfrak{A}_2 \arrow[r, "q_2"] & \mathfrak{B}
\end{tikzcd}$$

with $\mathfrak{A} \subset \mathfrak{A}_1 \oplus \mathfrak{A}_2$ the direct sum, assume that either q_1 or q_2 is surjective. If E_1 and E_2 are finitely generated, projective modules over $\mathfrak{A}_1$ and $\mathfrak{A}_2$, and if $f : (p_1)_* E_1 \to (p_2)_* E_2$ is an isomorphism of $\mathfrak{B}$-modules, then the $\mathfrak{A}$-module

$$E = \{(e_1, e_2) \in E_1 \times E_2 \mid f(e_1 \otimes 1) = e_2 \otimes 1\}$$

is finitely generated and projective.

Moreover, up to isomorphism, every finitely generated, projective module over $\mathfrak{A}$ has this form.

Remark. Note that $(p_j)_* E_j = \mathfrak{B} \otimes_{\mathfrak{A}_j} E_j$ or $E_j \otimes_{\mathfrak{A}_j} \mathfrak{B}$ with $(p_j)_* e_j = 1 \otimes_{\mathfrak{A}_j} e_j$ or $e_j \otimes_{\mathfrak{A}_j} 1$.

Definition 2.3. Let $\mathfrak{A}$ be a C^*-algebra and $I = [0,1]$ the closed interval in $\mathbb{R}$. Denote by $\mathfrak{A}[0,1] = C([0,1], \mathfrak{A})$ or $\mathfrak{A}(I) = C(I, \mathfrak{A})$ the C^*-algebra of all continuous functions from $[0,1]$ to $\mathfrak{A}$, with the supremum norm.

Similarly, define $\mathfrak{A}X = C(X, \mathfrak{A})$ for X a compact Hausdorff space X. If X is non-compact, let $\mathfrak{A}X = C_0(X, \mathfrak{A})$ the C^*-algebra of all continuous functions from X to $\mathfrak{A}$ vanishing at infinity.

Theorem 2.4. *Let $\mathfrak{A}$ be a C^*-algebra with unit. If E is a finitely generated, projective module over $\mathfrak{A}[0,1]$, then the induced modules over $\mathfrak{A}$ obtained by evaluation at points $0, 1 \in [0,1]$ are isomorphic.*

Proof. (Added). Let $E = p(\mathfrak{A}[0,1])^n$ for some $p \in M_n(\mathfrak{A}[0,1])$. By evaluation, $E(0) = p(0)\mathfrak{A}^n$ and $E(1) = p(1)\mathfrak{A}^n$, and also $E(t) = p(t)\mathfrak{A}^n$ for any $t \in [0,1]$. Then $p(t)$ is a continuous path of projections of $M_n(\mathfrak{A})$ between $p(0)$ and $p(1)$. By considering some finite partition of $[0,1]$, $p(0)$ and $p(1)$ are homotopic by some homotopes with respect to the partition, so that they are unitarily equivalent in $M_n(\mathfrak{A})$. □

Definition 2.5. A homotopy of $*$-homomorphisms of C^*-algebras is a family of homomorphisms $\varphi_t : \mathfrak{A} \to \mathfrak{B}$ for $t \in [0,1]$ such that the functions $\varphi_t(a)$ on $[0,1]$ are continuous for all $a \in \mathfrak{A}$.

A functor F on the category of C^*-algebras is a homotopy functor in the sense that all the homomorphisms φ_t in any such homotopy induce $F(\varphi_t) = F(\varphi_s) : F(\mathfrak{A}) \to F(\mathfrak{B})$ for any $t, s \in [0,1]$.

It follows from Theorem 2.4 that the K-theory $K_0(\mathfrak{A})$ for C^*-algebras $\mathfrak{A}$ is a homotopy functor.

Definition 2.6. Let $\mathfrak{A}$ be a C^*-algebra with unit. Denote by $GL_n(\mathfrak{A})$ the group of invertible matrices of $M_n(\mathfrak{A})$. Define a diagonal embedding of $GL_n(\mathfrak{A})$ as a subgroup of $GL_{n+k}(\mathfrak{A})$ by

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1_k \end{pmatrix}$$

with 1_k the $k \times k$ identity matrix. Define the K-theory group $K_1(\mathfrak{A})$ to be the direct limit of the path-connected component groups $\pi_0(GL_n(\mathfrak{A}))$:

$$K_1(\mathfrak{A}) = \varinjlim \pi_0(GL_n(\mathfrak{A})).$$

Remark. This is an abelian group. Because

$$\begin{pmatrix} gh & 0 \\ 0 & 1_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \quad \text{and hecce} \quad \begin{pmatrix} hg & 0 \\ 0 & 1_n \end{pmatrix}$$

are homotopic repectively. For instance, see Blackadar [7] or Wegge-Olsen [57]. As a statement, C^*-algebras are noncommutative in general, but their K-theroy groups are always commutative.

For $\mathfrak{A} = \mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2$ a (surjective) pull back of C^*-algebras, there is the following six-term Mayer-Vietoris exact sequence of K-theory groups:

$$\begin{array}{ccccc} K_0(\mathfrak{A}) & \longrightarrow & K_0(\mathfrak{A}_1) \oplus K_0(\mathfrak{A}_2) & \longrightarrow & K_0(\mathfrak{B}) \\ \uparrow & & & & \downarrow \partial \\ K_1(\mathfrak{B}) & \longleftarrow & K_1(\mathfrak{A}_1) \oplus K_1(\mathfrak{A}_2) & \longleftarrow & K_1(\mathfrak{A}). \end{array}$$

Proof. (Edited). It is straightforward to derive all but the down arrow $\partial : K_0(\mathfrak{B}) \to K_1(\mathfrak{A})$ on the right in the diagram. To complete the diagram, consider the pull back diagram: with S^1 the 1-dimensional circle and $\mathfrak{A}S^1 = C(S^1, \mathfrak{A})$,

$$\begin{array}{ccc} \mathfrak{A}S^1 & \xrightarrow{p_1} & \mathfrak{A}_1 S^1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle q_1} \\ \mathfrak{A}_2 S^1 & \xrightarrow{q_2} & \mathfrak{B}S^1. \end{array}$$

Then

$$\begin{array}{ccccc} K_0(\mathfrak{A}S^1) & \longrightarrow & K_0(\mathfrak{A}_1 S^1) \oplus K_0(\mathfrak{A}_2 S^1) & \longrightarrow & K_0(\mathfrak{B}S^1) \\ \partial \uparrow & & & & \\ K_1(\mathfrak{B}S^1) & \longleftarrow & K_1(\mathfrak{A}_1 S^1) \oplus K_1(\mathfrak{A}_2 S^1) & \longleftarrow & K_1(\mathfrak{A}S^1) \end{array}$$

which is mapped to the Mayer-Vietoris sequence before the proof by the operation $(\mathrm{id} \otimes \varepsilon_1)_*$ of the evaluation ε_1 at $1 \in S^1$. This map on each is a projection onto a direct summand since ε_1 splits. The complementary summands are computed as in Theorem 2.7 below. In fact, the map $\partial : K_0(\mathfrak{B}) \to K_1(\mathfrak{A})$ is obtained as a direct summand of the connecting map $\partial : K_1(\mathfrak{B}S^1) \to K_0(\mathfrak{A}S^1)$ with $K_1(S\mathfrak{B}) \cong K_0(\mathfrak{B})$ and $K_0(S\mathfrak{A}) \cong K_1(\mathfrak{A})$, where $S\mathfrak{A} = C_0(\mathbb{R}) \otimes \mathfrak{A}$ the tensor product C^*-algebra with $C_0(\mathbb{R})$ the C^*-algebra of all continuous, complex-valued functions on the real line $\mathbb{R}$ vanishing at infinity, and is isomorphic to $C_0(\mathbb{R}, \mathfrak{A}) = \mathfrak{A}\mathbb{R}$ (see Murphy [45]). □

Theorem 2.7. *Let $\mathfrak{A}$ be a C^*-algebra. The K-theory homomorphism induced by the evaluation $\varepsilon_1 = \mathrm{ev}_1$ at $1 \in S^1$*

$$(\mathrm{id} \otimes \varepsilon_1)_* : K_0(\mathfrak{A}S^1) \to K_0(\mathfrak{A})$$

has the kernel isomorphic to $K_1(\mathfrak{A})$. Also, the homomorphism induced by the evaluation ε_1

$$(\mathrm{id} \otimes \varepsilon_1)_* : K_1(\mathfrak{A}S^1) \to K_1(\mathfrak{A})$$

has the kernel isomorphic to $K_0(\mathfrak{A})$.

Proof. (Added). There is a split short exact sequence of C^*-algebras:

$$0 \to S = C_0(\mathbb{R}) \to C(S^1) \xrightarrow{\varepsilon_1} \mathbb{C} \to 0$$

(for instance, see [57]), which implies that

$$0 \to S\mathfrak{A} = S \otimes \mathfrak{A} \to \mathfrak{A}S^1 \cong C(S^1) \otimes \mathfrak{A} \xrightarrow{\mathrm{id}\otimes\varepsilon_1} \mathfrak{A} \to 0,$$

which also splits. It then follows that

$$K_j(\mathfrak{A}S^1) \cong K_j(S\mathfrak{A}) \oplus K_j(\mathfrak{A})$$

for $j = 0, 1$. Note that $C(S^1) \cong S \oplus \mathbb{C}$ and $C(S^1) \cong S \oplus_0 \mathbb{C}$ a pull back C^*-algebra on the zero C^*-algebra. As well, $\mathfrak{A}S^1 \cong S\mathfrak{A} \oplus_0 \mathfrak{A}$ (corrected). The Mayer-Vietoris sequence in this case does become:

$$\begin{tikzcd}
K_0(\mathfrak{A}S^1) \arrow[r] & K_0(S\mathfrak{A}) \oplus K_0(\mathfrak{A}) \arrow[r] & 0 \arrow[d] \\
0 \arrow[u] & K_1(S\mathfrak{A}) \oplus K_1(\mathfrak{A}) \arrow[l] & K_1(\mathfrak{A}S^1). \arrow[l]
\end{tikzcd}$$

It is known that $K_0(S\mathfrak{A}) \cong K_1(SS\mathfrak{A}) \cong K_1(\mathfrak{A})$ by the Bott peridocity in K-theory of C^*-algebras. As well, we have $K_1(S\mathfrak{A}) \cong K_0(\mathfrak{A})$. See [57]. □

The Mayer-Vietoris sequence implies that the functors $X \mapsto K_j(\mathfrak{A}X)$ on compact spaces X are cohomology theories. Especially, with $\mathfrak{A} = \mathbb{C}$, $X \mapsto K_j(C(X))$.

With S^2 the 2-dimensional sphere, we have

$$K_j(\mathfrak{A}S^2) \cong K_j(\mathfrak{A}) \oplus K_j(\mathfrak{A}), \quad j = 0, 1.$$

Proof. (Added). There is a split short exact sequence of C^*-algebras for $C(S^2)$ as

$$0 \to C_0(\mathbb{R}^2) \cong S \otimes S = SS \to C(S^2) \to \mathbb{C} \to 0.$$

It then follows that

$$0 \to SS\mathfrak{A} \to \mathfrak{A}S^2 \cong C(S^2) \otimes \mathfrak{A} \to \mathfrak{A} \to 0,$$

which splits. Therefore, for $j = 0, 1$,

$$\begin{aligned} K_j(\mathfrak{A}S^2) &\cong K_j(SS\mathfrak{A}) \oplus K_j(\mathfrak{A}) \\ &\cong K_j(\mathfrak{A}) \oplus K_j(\mathfrak{A}). \end{aligned}$$

□

The Bott homomorphism $\mathrm{B}_{p,u}$ from $K_0(\mathfrak{A})$ to $K_1(\mathfrak{A}S^1)$ is defined by associating to the class $[p]$ of a projection $p \in M_n(\mathfrak{A})$ the class $[u_p]$ of the unitary element

$$u_p(z) = zp + (1 - p) \in GL_n(\mathfrak{A}S^1).$$

It then follows that this map is an isomorphism onto the kernel of the map $(\mathrm{id} \otimes \varepsilon_1)_* : K_1(\mathfrak{A}S^1) \to K_1(\mathfrak{A})$. Refer to Atiyah and Bott [3] and [57].

Proof. (Edited). The key step is to show that the map $\mathrm{B}_{p,u}$ is surjective.

Reduce to show that a polynomial loop of invertible matrices

$$u(z) = b_0 + zb_1 + \cdots + z^m b_m \in GL_n(\mathfrak{A}), b_j \in M_n(\mathfrak{A})$$

which defines an element of the kernel of $(\mathrm{id} \otimes \varepsilon_1)_*$ belongs to the image of $\mathrm{B}_{p,u}$.

By elementary row operations, the loop $u(z)$ is equivalent to the linear loop $v(z)$:

$$v(z) = az + b \equiv \begin{pmatrix} 0 & & & 0 \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & -1 & 0 \end{pmatrix} z + \begin{pmatrix} b_0 & b_1 & \cdots & b_m \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

Evaluating at $z = 1$ and from that v is in the kernel of the evaluation map we see that the matrix $a + b$ is path connected to the identity matrix 1_k in some suitable $GL_k(\mathfrak{A})$, so that v is equivalent to

$$\begin{aligned} w(z) &= (a + b)^{-1}(az + b) = (a + b)^{-1}az + (a + b)^{-1}(a + b - a) \\ &\equiv cz + (1_k - c). \end{aligned}$$

Since $w(z)$ is invertible for all $z \in S^1$, the spectrum of c contains no element on the vertical line $\mathrm{Re}(z) = \frac{1}{2}$ in $\mathbb{C}$. If p is the projection associated to the part of the spectrum of c to the right of the line, then $w(z)$ is homotopic to the unitary:

$$u_p(z) = pz + (1 - p),$$

the K-theory class of which belongs to the image of $\mathrm{B}_{p,u}$. $\square$

3 Graded C^*-algebras $\times$ amplification and stabilization

Definition 3.1. Let $\mathfrak{A}$ be a C^*-algebra. A grading on $\mathfrak{A}$ is a $*$-automorphism α of $\mathfrak{A}$ such that $\alpha^2 = \mathrm{id}_{\mathfrak{A}}$ the identity map on $\mathfrak{A}$. Giving a grading on $\mathfrak{A}$ is equivalent to doing a direct sum of $*$-linear subspaces as $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ such that $\mathfrak{A}_i\mathfrak{A}_j \subset \mathfrak{A}_{i+j}$ for $i, j \in \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. The subspace $\mathfrak{A}_0$ and its elements x are said to be even, while $\mathfrak{A}_1$ and its elements y are done to be odd, for which $\alpha(x) = x$ and $\alpha(y) = -y$.

Indeed, as the direct sum of even and odd parts we have

$$\mathfrak{A} = \frac{1}{2}(\mathfrak{A} + \alpha(\mathfrak{A})) \oplus \frac{1}{2}(\mathfrak{A} - \alpha(\mathfrak{A})) \equiv \mathfrak{A}^{\mathrm{ev}} \oplus \mathfrak{A}^{\mathrm{od}},$$

which may be viewed as a column vector. Namely, we have

$$\alpha = \begin{pmatrix} \mathrm{id}_{\mathfrak{A}_0} & 0 \\ 0 & -\mathrm{id}_{\mathfrak{A}_1} \end{pmatrix} \quad \text{on} \quad \begin{pmatrix} \mathfrak{A}^{\mathrm{ev}} \\ \mathfrak{A}^{\mathrm{od}} \end{pmatrix}$$

with $\mathrm{id}_{\mathfrak{A}} + \alpha = 2\,\mathrm{id}_{\mathfrak{A}_0} \oplus 0$ and $\mathrm{id}_{\mathfrak{A}} - \alpha = 0 \oplus 2\,\mathrm{id}_{\mathfrak{A}_1}$ as diagonal sums.

Example 3.2. Any C^*-algebra $\mathfrak{A}$ is trivially graded in the sense that $\mathfrak{A} = \mathfrak{A}_0$ and $\mathfrak{A}_1 = \{0\}$ and $\alpha = \mathrm{id}_{\mathfrak{A}}$ or $\alpha = \mathrm{id}_{\mathfrak{A}} \oplus 0$ as a diagonal sum.

Example 3.3. A Hilbert space H is said to be graded if equipped with an orthogonal decomposition $H = H_0 \oplus H_1$. The C^*-algebra $\mathbb{K}(H)$ of all compact operators on H and the C^*-algebra $\mathbb{B}(H)$ of all bounded operators on H are graded in the sense that for an element $t \in \mathbb{B}(H_0 \oplus H_1)$ viewed as a 2×2 matrix of operators on H_0 and H_1 to H_0 or H_1, we define

$$t = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} t_{11} & 0 \\ 0 & t_{22} \end{pmatrix} \oplus \begin{pmatrix} 0 & t_{12} \\ t_{21} & 0 \end{pmatrix} \equiv t^{\mathrm{ev}} \oplus t^{\mathrm{od}} \quad \text{on} \quad \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$$

with $\alpha(t^{\mathrm{ev}}) = t^{\mathrm{ev}}$ and $\alpha(t^{\mathrm{od}}) = -t^{\mathrm{od}}$.

Namely,

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \equiv 1_2 \oplus -1_2 \equiv I \oplus -I \quad \text{on} \quad \begin{pmatrix} \mathbb{K}(H)^{\mathrm{ev}} \\ \mathbb{K}(H)^{\mathrm{od}} \end{pmatrix}, \begin{pmatrix} \mathbb{B}(H)^{\mathrm{ev}} \\ \mathbb{B}(H)^{\mathrm{od}} \end{pmatrix}.$$

Example 3.4. Let $S = C_0(\mathbb{R})$ be the suspensiòn C^*-algebra. Define a grading on S by the decompostion

$$S = C_0(\mathbb{R}) = C_0(\mathbb{R})^{\mathrm{ev}} \oplus C_0(\mathbb{R})^{\mathrm{od}} \equiv \{\text{even functions}\} \oplus \{\text{ odd functions}\}.$$

The grading operator α is given by the automorphism which sends $f(x)$ to $f(-x)$. Indeed,

$$f(x) = \frac{1}{2}(f(x) + f(-x)) \oplus \frac{1}{2}(f(x) - f(-x)) \equiv f^{\mathrm{ev}}(x) \oplus f^{\mathrm{od}}(x),$$

with $\alpha(f^{\mathrm{ev}}) = f^{\mathrm{ev}}$ and $\alpha(f^{\mathrm{od}}) = -f^{\mathrm{od}}$.

Definition 3.5. A graded C^*-algebra $\mathfrak{A}$ is inner-graded if there exists a self-adjoint unitary ε in the multiplier algebra $M(\mathfrak{A})$ of $\mathfrak{A}$ such that

$$\alpha(a) = \varepsilon a \varepsilon = \varepsilon a \varepsilon^* \equiv \mathrm{Ad}(\varepsilon)(a), \quad a \in \mathfrak{A}.$$

Example 3.6. The trivial grading on a C^*-algebra $\mathfrak{A}$ is inner with $\varepsilon = 1_{M(\mathfrak{A})} \in M(\mathfrak{A})$ as

$$\alpha(a) = \mathrm{id}_{\mathfrak{A}}(a) = a = 1_{M(\mathfrak{A})} a 1_{M(\mathfrak{A})} = \varepsilon a \varepsilon \in M(\mathfrak{A}),$$

where the unit $1_{M(\mathfrak{A})}$ is identified with $\mathrm{id}_{\mathfrak{A}}$.

The gradings on $\mathbb{K}(H)$ and $\mathbb{B}(H)$ are inner with $\varepsilon = 1_2 \oplus (-1_2) = I \oplus -I$ on $H = H_0 \oplus H_1$ as

$$\alpha(t) = t^{\mathrm{ev}} \oplus (-t^{\mathrm{od}}) = (1_2 \oplus -1_2)(t^{\mathrm{ev}} \oplus (-t^{\mathrm{od}}))(1_2 \oplus -1_2) = \varepsilon t \varepsilon$$

for $t \in \mathbb{B}(H)$, so that $\alpha = \mathrm{Ad}(\varepsilon)$ the adjoint map by ε.

The grading on S is not inner.

Note that $M(S) \cong C(\beta\mathbb{R}) \cong C^b(\mathbb{R})$, where βX is the Stone-Čech compactification of X and $C^b(\mathbb{R})$ the C^*-algebra of all bounded continuous $\mathbb{C}$-valued functions on X (see [57]). Suppose that $\alpha(a) = \varepsilon a \varepsilon$ for some self-adjoint unitary $\varepsilon \in M(\mathfrak{A})$. But then $\varepsilon a \varepsilon = \varepsilon^2 a = a$. □

Let $\mathfrak{A}$ be a graded C^*-algebra. Let us introduce the symbol of degree defined as for $a \in \mathfrak{A}_0$ or $a \in \mathfrak{A}_1$,

$$\partial a = \begin{cases} 0 & \text{if } a \in \mathfrak{A}_0, \\ 1 & \text{if } a \in \mathfrak{A}_1 \end{cases}$$

where $\{0,1\} = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Elements of $\mathfrak{A}_0$ or $\mathfrak{A}_1$ are said to be homogeneous. Note that if a, b are homogeneous, then $\partial(ab) = \partial a + \partial b$ and $\partial a^* = \partial a$.

Definition 3.7. Let $\mathfrak{A}$ and $\mathfrak{B}$ be graded C^*-algebras. Let $\mathfrak{A} \otimes \mathfrak{B}$ be the algebraic tensor product of $\mathfrak{A}$ and $\mathfrak{B}$ as linear spaces. Define multiplication, involution, and grading on $\mathfrak{A} \otimes \mathfrak{B}$ by

$$\begin{aligned} (a_1 \otimes b_1)(a_2 \otimes b_2) &= (-1)^{\partial b_1 \partial a_2} a_1 a_2 \otimes b_1 b_2, \\ (a \otimes b)^* &= (-1)^{\partial a \partial b} a^* \otimes b^*, \\ \partial(a \otimes b) &= \partial a + \partial b, \quad (\text{mod } 2), \end{aligned}$$

for homogeneous elements $a, a_1, a_2 \in \mathfrak{A}$ and $b, b_1, b_2 \in \mathfrak{B}$. These are extended by linearity to all tensors $a \otimes b$ for $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$.

For example, there is an isomorphism Φ from $\mathfrak{A} \otimes \mathfrak{B}$ to $\mathfrak{B} \otimes \mathfrak{A}$ defined by sending $a \otimes b \mapsto (-1)^{\partial a \partial b} b \otimes a$ for homogeneous $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$.

Proof. (Added). Indeed,

$$\begin{aligned} \Phi((a_1 \otimes b_1)(a_2 \otimes b_2)) &= \Phi((-1)^{\partial b_1 \partial a_2} a_1 a_2 \otimes b_1 b_2) \\ &= (-1)^{\partial b_1 \partial a_2} (-1)^{\partial(a_1 a_2)\partial(b_1 b_2)} b_1 b_2 \otimes a_1 a_2 \\ &= (-1)^{\partial a_1 \partial b_1 + \partial a_1 \partial b_2 + \partial a_2 \partial b_2} b_1 b_2 \otimes a_1 a_2, \\ \Phi(a_1 \otimes b_1)\Phi(a_2 \otimes b_2) &= (-1)^{\partial a_1 \partial b_1} (-1)^{\partial a_2 \partial b_2} (b_1 \otimes a_1)(b_2 \otimes a_2) \\ &= (-1)^{\partial a_1 \partial b_1} (-1)^{\partial a_2 \partial b_2} (-1)^{\partial a_1 \partial b_2} b_1 b_2 \otimes a_1 a_2 \end{aligned}$$

and hence, these coincide, and

$$\begin{aligned} \Phi((a \otimes b)^*) &= \Phi((-1)^{\partial a \partial b} a^* \otimes b^*) = (-1)^{\partial a \partial b} (-1)^{\partial a^* \partial b^*} b^* \otimes a^* = b^* \otimes a^*, \\ \Phi(a \otimes b)^* &= ((-1)^{\partial a \partial b} b \otimes a)^* = (-1)^{\partial a \partial b} (-1)^{\partial b \partial a} b^* \otimes a^* = b^* \otimes a^*. \end{aligned}$$

Note as well that

$$\begin{aligned} (\mathfrak{A} \otimes \mathfrak{B})_0 &= [\mathfrak{A}_0 \otimes \mathfrak{B}_0] \oplus [\mathfrak{A}_1 \otimes \mathfrak{B}_1], \\ (\mathfrak{A} \otimes \mathfrak{B})_1 &= [\mathfrak{A}_0 \otimes \mathfrak{B}_1] \oplus [\mathfrak{A}_1 \otimes \mathfrak{B}_0]. \end{aligned}$$

□

Definition 3.8. The graded commutator of elements of a graded C^*-algebra $\mathfrak{A}$ is given by

$$[a, b] = ab - (-1)^{\partial a \partial b} ba$$

for homogeneous elements a and b of $\mathfrak{A}$. This is extended by linearity to all elements a and b of $\mathfrak{A}$.

For example, if $a_0, b_0 \in \mathfrak{A}_0$ and $a_1, b_1 \in \mathfrak{A}_1$, then $[a_0, b_0] = ab - ba$, $[a_1, b_1] = ab + ba$, and $[a_j, b_{j+1}] = ab - ba$ for $j, j+1$ (mod 2).

Lemma 3.9. *Let $\mathfrak{C}$ be a graded C^*-algebra and $\varphi : \mathfrak{A} \to \mathfrak{C}$ and $\psi : \mathfrak{B} \to \mathfrak{C}$ be graded $*$-homomorphisms. If their images commute in the sense that all graded commutators $[\varphi(a), \psi(b)]$ are zero, then there is a unique graded $*$-homomorphism $\varphi \otimes \psi$ from $\mathfrak{A} \otimes \mathfrak{B}$ to $\mathfrak{C}$, which sends $a \otimes b$ to $\varphi(a)\psi(b)$:*

$$\begin{array}{ccc} \mathfrak{A} \otimes \mathfrak{B} & \xrightarrow{\varphi\otimes\psi} & \mathfrak{C} \\ \uparrow & & \| \\ \mathfrak{A}, \mathfrak{B} & \xrightarrow{\varphi,\psi} & \mathfrak{C}. \end{array}$$

Proof. (Added). Define $(\varphi\otimes\psi)(a\otimes b) = \varphi(a)\psi(b)$ and extend it by linearity. Check that

$$\begin{aligned} &(\varphi \otimes \psi)((a_1 \otimes b_1)(a_2 \otimes b_2)) = (\varphi \otimes \psi)((-1)^{\partial b_1 \partial a_2} a_1 a_2 \otimes b_1 b_2) \\ &= (-1)^{\partial b_1 \partial a_2} \varphi(a_1 a_2)\psi(b_1 b_2) = (-1)^{\partial b_1 \partial a_2} \varphi(a_1)\varphi(a_2)\psi(b_1)\psi(b_2), \\ &(\varphi \otimes \psi)(a_1 \otimes b_1)(\varphi \otimes \psi)(a_2 \otimes b_2) = \varphi(a_1)\psi(b_1)\varphi(a_2)\psi(b_2) \\ &= \varphi(a_1)\{[\psi(b_1), \varphi(a_2)] + (-1)^{\partial b_1 \partial a_2}\varphi(a_2)\psi(b_1)\}\psi(b_2) \end{aligned}$$

with $[\psi(b_1), \varphi(a_2)] = 0$, and thus those coincide. Also,

$$\begin{aligned} (\varphi \otimes \psi)((a \otimes b)^*) &= (-1)^{\partial a \partial b}\varphi(a)^*\psi(b)^*, \\ (\varphi \otimes \psi)(a \otimes b)^* &= (\varphi(a)\psi(b))^* = \psi(b)^*\varphi(a)^* \\ &= [\psi(b)^*, \varphi(a)^*] + (-1)^{\partial\psi(b)^*\partial\varphi(a)^*}\varphi(a)^*\psi(b)^*, \end{aligned}$$

with $[\psi(b)^*, \varphi(a)^*] = 0$ and $\partial\varphi(a)^* = \partial a^* = \partial a$, and hence those coincide. □

Example 3.10. Let H be a graded Hilbert space and $H \otimes H$ be a graded Hilbert space tensor product in the similar sense as $\mathfrak{A} \otimes \mathfrak{B}$ above. It follows from Lemma 3.9 above that there is a graded $*$-homomorphism φ from $\mathbb{B}(H) \otimes \mathbb{B}(H)$ into $\mathbb{B}(H \otimes H)$ which takes homogeneous elementary tensors $s \otimes t$ to the operator defined by sending $\xi \otimes \eta \mapsto s\xi \otimes (-1)^{\partial\xi\partial t} t\eta$.

Proof. (Added). Note that

$$\begin{aligned} \varphi((a_1 \otimes b_1)(a_2 \otimes b_2))(\xi \otimes \eta) &= \varphi((-1)^{\partial b_1 \partial a_2} a_1 a_2 \otimes b_1 b_2)(\xi \otimes \eta) \\ &= (-1)^{\partial b_1 \partial a_2} a_1 a_2 \xi \otimes (-1)^{\partial\xi\partial(b_1 b_2)} b_1 b_2 \eta, \\ \varphi(a_1 \otimes b_1)\varphi(a_2 \otimes b_2)(\xi \otimes \eta) &= \varphi(a_1 \otimes b_1)(a_2\xi \otimes (-1)^{\partial\xi\partial b_2} b_2\eta) \\ &= a_1 a_2 \xi \otimes (-1)^{\partial(a_2\xi)\partial b_1} b_1 (-1)^{\partial\xi\partial b_2} b_2 \eta) \end{aligned}$$

with $\partial(b_1 b_2) = \partial b_1 + \partial b_2$ and $\partial(a_2\xi) = \partial a_2 + \partial\xi$, so that those coincide. Also,

$$\begin{aligned}
\varphi((a\otimes b)^*)(\xi\otimes\eta) &= \varphi((-1)^{\partial a\partial b}a^*\otimes b^*)(\xi\otimes\eta) \\
&= (-1)^{\partial a\partial b}a^*\xi\otimes(-1)^{\partial\xi\partial b^*}b^*\eta, \\
\varphi(a\otimes b)^*(\xi\otimes\eta) &= [(-1)^{\partial\xi\partial b}a\otimes b]^*(\xi\otimes\eta) \\
&= (-1)^{\partial\xi\partial b}(-1)^{\partial a\partial b}a^*\xi\otimes b^*\eta,
\end{aligned}$$

so that those coincide. $\square$

Definition 3.11. Let $\mathfrak{A}$ and $\mathfrak{B}$ be graded C^*-algebras and let $\mathfrak{A}\otimes\mathfrak{B}$ be their algebraic tensor product. The maximal graded tensor product C^*-algebra $\mathfrak{A}\otimes_{\max}\mathfrak{B}$ is defined to be the completion of $\mathfrak{A}\otimes\mathfrak{B}$ by the C^*-norm

$$\|\sum_j a_j\otimes b_j\| \equiv \sup_{\varphi,\psi}\|\sum_j \varphi(a_j)\psi(b_j)\|$$

where the supremum is taken over graded $*$-homomorphisms $\varphi:\mathfrak{A}\to\mathfrak{C}$ and $\psi:\mathfrak{B}\to\mathfrak{C}$ which commute gradedly.

But we often denote the maximal graded C^*-algebra completion of $\mathfrak{A}\otimes\mathfrak{B}$ by the same symbol $\mathfrak{A}\otimes\mathfrak{B}$ for convenience if not specified.

Remark. Taking the maximal tensor product $\otimes_{\max}$ is functorial as follows. If $\varphi:\mathfrak{A}\to\mathfrak{C}$ and $\psi:\mathfrak{B}\to\mathfrak{D}$ are graded $*$-homomorphisms, then there is a unique graded $*$-homomorphism $\varphi\otimes_{\max}\psi:\mathfrak{A}\otimes_{\max}\mathfrak{B}\to\mathfrak{C}\otimes_{\max}\mathfrak{D}$ mapping $a\otimes b$ to $\varphi(a)\otimes\psi(b)$ for all $a\in\mathfrak{A}$ and $b\in\mathfrak{B}$. Diagram it:

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\varphi"] \arrow[d] & \mathfrak{C} \arrow[d] \\
\mathfrak{A}\otimes_{\max}\mathfrak{B} \arrow[r, "\varphi\otimes_{\max}\psi"] & \mathfrak{C}\otimes_{\max}\mathfrak{D} \\
\mathfrak{B} \arrow[r, "\psi"] \arrow[u] & \mathfrak{D}. \arrow[u]
\end{tikzcd}$$

Example 3.12. If one of $\mathfrak{A}$ or $\mathfrak{B}$ is inner-graded, then the ungraded tensor product $\mathfrak{A}\otimes_{\max}\mathfrak{B}$ is isomorphic to the usual maximal tensor product of ungraded $\mathfrak{A}$ and $\mathfrak{B}$. The isomorphism Φ between them is defined by sending $a\otimes b\mapsto a\varepsilon^{\partial b}\otimes b$.

Proof. (Added). Note that if $(\mathfrak{A} \otimes_{\max} \mathfrak{B})_0 = \mathfrak{A} \otimes_{\max} \mathfrak{B}$ and $(\mathfrak{A} \otimes_{\max} \mathfrak{B})_1 = \{0\}$ and $\mathfrak{A} = \mathfrak{A}_0$ and $\mathfrak{A}_1 = 0$ and $\mathfrak{B} = \mathfrak{B}_0$ and $\mathfrak{B}_1 = 0$, then

$$(\mathfrak{A}_0 \oplus \mathfrak{A}_1) \otimes_{\max} (\mathfrak{B}_0 \oplus \mathfrak{B}_1) \cong \mathfrak{A}_0 \otimes_{\max} \mathfrak{B}_0 = (\mathfrak{A} \otimes_{\max} \mathfrak{B})_0 \cong \mathfrak{A} \otimes_{\max} \mathfrak{B}.$$

Also,

$$\begin{aligned}
\Phi((a_1 \otimes b_1)(a_2 \otimes b_2)) &= \Phi((-1)^{\partial b_1 \partial a_2} a_1 a_2 \otimes b_1 b_2) \\
&= (-1)^{\partial b_1 \partial a_2} a_1 a_2 \varepsilon^{\partial(b_1 b_2)} \otimes b_1 b_2, \\
\Phi(a_1 \otimes b_1)\Phi(a_2 \otimes b_2) &= (a_1 \varepsilon^{\partial b_1} \otimes b_1)(a_2 \varepsilon^{\partial b_2} \otimes b_2) \\
&= (-1)^{\partial b_1 \partial(a_2 \varepsilon^{\partial b_2})} a_1 \varepsilon^{\partial b_1} a_2 \varepsilon^{\partial b_2} \otimes b_1 b_2,
\end{aligned}$$

with $\partial(a_2 \varepsilon^{\partial b_2}) = \partial a_2 + \partial(\varepsilon^{\partial b_2})$ and

$$\varepsilon^{\partial b_1} a_2 = \varepsilon^{\partial b_1} a_2 \varepsilon^{\partial b_1} \varepsilon^{\partial b_1} = (-1)^{\partial(\varepsilon^{\partial b_1})} a_2 \varepsilon^{\partial b_1}$$

and hence, those coincide. As well,

$$\begin{aligned}
\Phi((a \otimes b)^*) &= \Phi((-1)^{\partial a \partial b} a^* \otimes b^*) = (-1)^{\partial a \partial b} a^* \varepsilon^{\partial b^*} \otimes b^*, \\
\Phi(a \otimes b)^* &= [a \varepsilon^{\partial b} \otimes b]^* = (-1)^{\partial(a \varepsilon^{\partial b}) \partial b} \varepsilon^{\partial b} a^* \otimes b^*,
\end{aligned}$$

with $\partial b^* = \partial b$, and

$$\varepsilon^{\partial b} a^* = \varepsilon^{\partial b} a^* \varepsilon^{\partial b} \varepsilon^{\partial b} = (-1)^{\partial(\varepsilon^{\partial b})} a^* \varepsilon^{\partial b},$$

so that those coincide. $\square$

If $\mathfrak{A}$ and $\mathfrak{B}$ are inner-graded C^*-algebras by self-adjoint unitaries $\varepsilon_{\mathfrak{A}}$ and $\varepsilon_{\mathfrak{B}}$ of $M(\mathfrak{A})$ and $M(\mathfrak{B})$ respectively, then the graded tensor product $\mathfrak{A} \otimes_{\max} \mathfrak{B}$ is inner-graded by $\varepsilon_{\mathfrak{A}} \otimes \varepsilon_{\mathfrak{B}} \in M(\mathfrak{A} \otimes_{\max} \mathfrak{B})$.

Because, with definitions,

$$\alpha_{\mathfrak{A} \otimes \mathfrak{B}}(a \otimes b) = \alpha_{\mathfrak{A}}(a) \otimes \alpha_{\mathfrak{B}}(b) = \varepsilon_{\mathfrak{A}} a \varepsilon_{\mathfrak{A}} \otimes \varepsilon_{\mathfrak{B}} b \varepsilon_{\mathfrak{B}} = (\varepsilon_{\mathfrak{A}} \otimes \varepsilon_{\mathfrak{B}})(a \otimes b)(\varepsilon_{\mathfrak{A}} \otimes \varepsilon_{\mathfrak{B}}).$$

Definition 3.13. Let $\mathfrak{A}$ and $\mathfrak{B}$ be graded C^*-algebras and let $\mathfrak{A} \otimes \mathfrak{B}$ be their algebraic tensor product. The minimal graded tensor product of $\mathfrak{A}$ and $\mathfrak{B}$ is the completion of $\mathfrak{A} \otimes \mathfrak{B}$ by the representation $\varphi \otimes \psi$ from $\mathbb{B}(H) \otimes_{\max} \mathbb{B}(H)$ to $\mathbb{B}(H \otimes H)$, where $\varphi : \mathfrak{A} \to \mathbb{B}(H)$ and $\psi : \mathfrak{A} \to \mathbb{B}(H)$ are faithful graded $*$-homomorphisms (or representations). We may write it as $\mathfrak{A} \otimes_{\min} \mathfrak{B}$.

Exercise. For any $\mathfrak{A}$ graded or not, we have $\mathfrak{A} \otimes_{\max} \mathbb{K}(H) = \mathfrak{A} \otimes_{\min} \mathbb{K}(H)$ and $S \otimes_{\max} \mathfrak{A} = S \otimes_{\min} \mathfrak{A}$.

Proof. (Added). We have $\mathfrak{A}\otimes_{\max} M_{2n}(\mathbb{C}) = \mathfrak{A}\otimes_{\min} M_{2n}(\mathbb{C})$, where $M_{2n}(\mathbb{C}) \cong \mathbb{B}(\mathbb{C}^{2n})$ with $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$. Indeed, there is a unique graded (algebraic) $*$-isomorphism φ_{2n} from $\mathfrak{A} \otimes M_{2n}(\mathbb{C})$ to $M_{2n}(\mathfrak{A})$. It extends to the graded $*$-isomorphisms from $\mathfrak{A} \otimes_{\max} M_n(\mathbb{C})$ and $\mathfrak{A} \otimes_{\min} M_n(\mathbb{C})$ to $M_n(\mathfrak{A})$.

Similarly, φ_{2n} extends to a unique graded $*$-isomorphism from $\mathfrak{A}\otimes\mathbb{K}(H)$ to $\mathbb{K}(\mathfrak{A})$ the completion of the union of $M_n(\mathfrak{A})$. It extends to the graded $*$-isomorphisms from $\mathfrak{A} \otimes_{\max} \mathbb{K}(H)$ and $\mathfrak{A} \otimes_{\min} \mathbb{K}(H)$ to $\mathbb{K}(\mathfrak{A})$.

Let X be a compact Hausdorff space and $\mathfrak{A}$ be unital. Then we have $C(X)\otimes_{\max}\mathfrak{A} = C(X)\otimes_{\min}\mathfrak{A}$. Indeed, there is a unique graded (algebraic) $*$-isomorphism from $C(X)\otimes\mathfrak{A}$ to $C(X,\mathfrak{A})$ the graded C^*-algebra of continuous $\mathfrak{A}$-valued functions on X. It extends to the graded $*$-isomorphisms from $C(X) \otimes_{\max} \mathfrak{A}$ and $C(X) \otimes_{\min} \mathfrak{A}$ to $C(X,\mathfrak{A})$.

It then follows that $S^+ \otimes_{\max} \mathfrak{A}^+ = S^+ \otimes_{\min} \mathfrak{A}^+$ with S^+ and $\mathfrak{A}^+$ the unitizations by $\mathbb{C}$. Hence $S \otimes_{\max} \mathfrak{A} = S \otimes_{\min} \mathfrak{A}$ is deduced. □

Exercise. The graded tensor product $S \otimes S$ is not commutative.

Proof. Let $f = f_0 \oplus f_1, f' = f_0' \oplus f_1', g = g_0 \oplus g_1, g' = g_0' \oplus g_1' \in S = S_0 \oplus S_1$. Then

$$
\begin{aligned}
(f_i \otimes g_j)(f_k' \otimes g_l') &= (-1)^{\partial g_j \partial f_k'} f_i f_k' \otimes g_j g_l' \\
&= \begin{cases} f_i f_k' \otimes g_j g_l' & \text{if } \partial g_j \partial f_k' = 0, \\ -f_i f_k' \otimes g_j g_l' & \text{if } \partial g_j \partial f_k' = 1 \end{cases}
\end{aligned}
$$

and similarly,

$$
(f_k' \otimes g_l')(f_i \otimes g_j) = \begin{cases} f_i f_k' \otimes g_j g_l' & \text{if } \partial g_l' \partial f_i = 0, \\ -f_i f_k' \otimes g_j g_l' & \text{if } \partial g_l' \partial f_i = 1 \end{cases}.
$$

Therefore, the multiplication is commutative when $\partial g_j \partial f_k' = \partial g_l' \partial f_i$ and is not when $\partial g_j \partial f_k' \neq \partial g_l' \partial f_i$. Hence, the commutative cases are as follows:

$$
\begin{aligned}
&(S_0 \otimes S_0)(S_0 \otimes S_0), \quad (S_1 \otimes S_0)(S_0 \otimes S_0), \quad (S_0 \otimes S_1)(S_0 \otimes S_0), \\
&(S_1 \otimes S_0)(S_1 \otimes S_0), \quad (S_0 \otimes S_1)(S_0 \otimes S_1), \quad (S_1 \otimes S_1)(S_0 \otimes S_0), \\
&(S_1 \otimes S_1)(S_1 \otimes S_1),
\end{aligned}
$$

which are all the same as $S_0 \otimes S_0$, and the noncommutative cases are as:

$$
\begin{aligned}
&(S_1 \otimes S_1)(S_0 \otimes S_1) \neq (S_0 \otimes S_1)(S_1 \otimes S_1) \quad \text{and} \\
&(S_1 \otimes S_1)(S_1 \otimes S_0) \neq (S_1 \otimes S_0)(S_1 \otimes S_1),
\end{aligned}
$$

both of which (not equal) are equal to $S_1 \otimes S_0$ and $S_0 \otimes S_1$, respectively. □

Exercise. We have $\mathbb{K}(H) \otimes_{\max} \mathbb{K}(H') \cong \mathbb{K}(H \otimes H')$.

Proof. We have

$$M_{2n}(\mathbb{C}) \otimes M_{2m}(\mathbb{C}) \cong \mathbb{B}(\mathbb{C}^{2n}) \otimes \mathbb{B}(\mathbb{C}^{2m}) \cong \mathbb{B}(\mathbb{C}^{2n} \otimes \mathbb{C}^{2m}).$$

Thus, the isomorphisms above extend by density and continuity to the isomorphism in the statement. □

Amplification

Let $S = C_0(\mathbb{R})$ as a (graded or not) C^*-algebra. Define a $*$-homomorphism $\eta = \mathrm{ev}_0 : S \to \mathbb{C}$ by $\eta(f) = f(0)$. If S is ungraded, then η is homotopic to the zero $*$-homomorphism. The homotopy between them may be defined as $\eta_t(f) = f(\frac{1}{1-t} - 1)$ for $t \in [0, 1)$ and $\eta_1(f) = 0$. But if S is graded, then η is non-trivial.

Denote by $S_r = S([-r, r])$ the quotient of S by restriction to the closed interval $[-r, r] \subset \mathbb{R}$. Let $\mathrm{id}_r(x) = x$ for $x \in [-r, r]$. For any $f \in S$ and $\mathrm{id}_r \otimes 1 + 1 \otimes \mathrm{id}_r \in S_r \otimes S_r$ self-adjoint, there is $f(\mathrm{id}_r \otimes 1 + 1 \otimes \mathrm{id}_r) \in S_r \otimes S_r$ by functional calculus.

Lemma 3.14. *There is a unique graded $*$-homomorphism $\Delta : S \to S \otimes S$ whose composition with the quotient map $q_r : S \otimes S \to S_r \otimes S_r$ for every $r > 0$ is given by $f(\mathrm{id}_r \otimes 1 + 1 \otimes \mathrm{id}_r)$, so that*

$$\Delta(f) = \lim_{r \to \infty} f(\mathrm{id}_r \otimes 1 + 1 \otimes \mathrm{id}_r) = f(\mathrm{id}_\infty \otimes 1 + 1 \otimes \mathrm{id}_\infty)$$

as a (norm) limit.

Exercise. The intersection of the kernels of the maps $q_r : S \otimes S \to S_r \otimes S_r$ is zero, which implies the uniqueness of the lemma above.

Proof. It is clear that $\ker(q_r) \subset \ker(q_t)$ if $r > t$, and $f \in \ker(q_r)$ if and only if f is zero on $[-r, r]$. If we assume that there is another Δ' as Δ, then $(q_r \circ (\Delta - \Delta'))(f) = 0$ for any $f \in S$ and any $r > 0$. Hence $(\Delta - \Delta')(f) = 0$. □

Remark. (Corrected). Define self-adjoint homogeneous elements u and v of S by $u(x) = e^{-x^2}$ even and $v(x) = xe^{-x^2}$ odd. This definition for u and v is used later. But here we may redefine as $u(x) = e^{-|x|}$ even and $v(x) = xe^{-|x|}$ odd. It then follows that

$$\Delta(u) = u \otimes u \quad \text{and} \quad \Delta(v) = u \otimes v + v \otimes u.$$

Indeed, for convenience we may assume that $x, y \geq 0$. Then compute

$$\begin{aligned}\Delta(u)(x,y) &= e^{-|x\otimes 1+1\otimes y|} = e^{-x-y} \\ &= e^{-x} \otimes e^{-y} = u(x) \otimes u(y)\end{aligned}$$

with $x \otimes 1 = x(1 \otimes 1) = x$. As well, for $x, y \geq 0$,

$$\begin{aligned}\Delta(v)(x,y) &= (x \otimes 1 + 1 \otimes y)e^{-|x\otimes 1+1\otimes y|} \\ &= (x \otimes 1)e^{-x\otimes 1}e^{-1\otimes y} + e^{-x\otimes 1}(1 \otimes y)e^{-1\otimes y} \\ &= (xe^{-x})e^{-y} + e^{-x}(ye^{-y}) = v(x) \otimes u(y) + u(x) \otimes v(y),\end{aligned}$$

as required. Similarly, one can show case by case the other ones, but some may involve changing signs $\pm$. If so, the formula for $\Delta(u)$ and $\Delta(v)$ may involve other signs case by case, to be corrected.

Lemma 3.15. (*Added*). *The C^*-algebra S is generated by u and v (in both of two definitions).*

Proof. Let $S^+ = S \oplus \mathbb{C}1$ be the unitization of S by the unit 1, which is isomorphic to the C^*-algebra $C(\mathbb{T})$ of all continuous, complex-valued functions on the 1-dimensional torus $\mathbb{T}$. Consider the complex algebra generated by 1, u, and v, denoted as $\mathbb{C}[1, u, v]$. For any distinct $x, y \in \mathbb{R}$, we have $u(x) \neq u(y)$ $(x, y \geq 0)$ and $v(x) \neq v(y)$ (any $x \neq y$). Thus, the Stone-Weierstrass approximation theorem implies that the algebra $\mathbb{C}[1, u, v]$ is dense in $S^+ \cong C(\mathbb{T})$. $\square$

There is a sort of coalgebra structure that the diagrams commute:

$$\begin{array}{ccc} S & \xrightarrow{\Delta} & S \otimes S \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ S \otimes S & \xrightarrow{\Delta\otimes\mathrm{id}} & S \otimes S \otimes S \end{array}$$

and

$$\begin{array}{ccc} S & \xrightarrow{\mathrm{id}} & S \\ \| & & \uparrow{\scriptstyle \mathrm{id}\otimes\eta} \\ S & \xrightarrow{\Delta} & S \otimes S \\ \| & & \downarrow{\scriptstyle \eta\otimes\mathrm{id}} \\ S & \xrightarrow{\mathrm{id}} & S. \end{array}$$

Definition 3.16. Let $\mathfrak{A}$ be a graded C^*-algebra. The **amplification** of $\mathfrak{A}$ is defined to be the graded tensor product $S\mathfrak{A} = S \otimes \mathfrak{A}$.

Definition 3.17. The **amplified** category of graded C^*-algebras is the category such that objects are graded C^*-algebras and morphisms from $\mathfrak{A}$ to $\mathfrak{B}$ are graded $*$-homomorphisms from $S\mathfrak{A}$ to $\mathfrak{B}$. The composition $\psi \circ \varphi$ of morphisms $\varphi : S\mathfrak{A} \to \mathfrak{B}$ and $\psi : S\mathfrak{B} \to \mathfrak{C}$ is given by the composition:

$$S\mathfrak{A} \xrightarrow{\Delta\otimes\mathrm{id}} SS\mathfrak{A} \xrightarrow{\mathrm{id}\otimes\varphi = S\varphi} S\mathfrak{B} \xrightarrow{\psi} \mathfrak{C}.$$

Exercise. The composition law in the amplified category is associative.

The identity morphism in it is given by $\eta \otimes \mathrm{id}_{\mathfrak{A}} : S\mathfrak{A} \to \mathbb{C} \otimes \mathfrak{A} \cong \mathfrak{A}$ with $\eta = \mathrm{ev}_0$.

Proof. (Added). For $\varphi : S\mathfrak{A} \to \mathfrak{B}$, $\psi : S\mathfrak{B} \to \mathfrak{C}$, and $\rho : S\mathfrak{C} \to \mathfrak{D}$, we have

$$\begin{aligned}
&\rho \circ (\psi \circ \varphi) = \rho \circ [\psi \circ (S\varphi) \circ (\Delta \otimes \mathrm{id})] \\
&= \rho \circ S[\psi \circ (S\varphi) \circ (\Delta \otimes \mathrm{id})] \circ (\Delta \otimes \mathrm{id}), \\
&(\rho \circ \psi) \circ \varphi = (\rho \circ \psi) \circ (S\varphi) \circ (\Delta \otimes \mathrm{id}) \\
&= [\rho \circ S\psi \circ (\Delta \otimes \mathrm{id})] \circ (S\varphi) \circ (\Delta \otimes \mathrm{id}).
\end{aligned}$$

For instance, if $\Delta(u) = u \otimes u$, then it follows that for $a \in \mathfrak{A}$,

$$\begin{aligned}
&[\rho \circ (\psi \circ \varphi)](u \otimes a) = \rho \circ S[\psi \circ (S\varphi) \circ (\Delta \otimes \mathrm{id})](u \otimes u \otimes a) \\
&= \rho(u \otimes \psi(u \otimes \varphi(u \otimes a))), \\
&[(\rho \circ \psi) \circ \varphi](u \otimes a) = [\rho \circ S\psi \circ (\Delta \otimes \mathrm{id})] \circ (S\varphi)(u \otimes u \otimes a) \\
&= [\rho \circ S\psi \circ (\Delta \otimes \mathrm{id})](u \otimes \varphi(u \otimes a)) \\
&= \rho(u \otimes \psi(u \otimes \varphi(u \otimes a))).
\end{aligned}$$

If $\Delta(v) = u \otimes v + v \otimes u$, then

$$\begin{aligned}
&[\rho \circ (\psi \circ \varphi)](v \otimes a) = \rho \circ S[\psi \circ (S\varphi) \circ (\Delta \otimes \mathrm{id})](u \otimes v \otimes a + v \otimes u \otimes a) \\
&= \{\rho(u \otimes \psi(u \otimes \varphi(v \otimes a))) + \rho(u \otimes \psi(v \otimes \varphi(u \otimes a)))\} \\
&\quad + \rho(v \otimes \psi(u \otimes \varphi(u \otimes a))), \\
&[(\rho \circ \psi) \circ \varphi](v \otimes a) = [\rho \circ S\psi \circ (\Delta \otimes \mathrm{id})] \circ (S\varphi)(u \otimes v \otimes a + v \otimes u \otimes a) \\
&= [\rho \circ S\psi \circ (\Delta \otimes \mathrm{id})](u \otimes \varphi(v \otimes a) + v \otimes \varphi(u \otimes a)) \\
&= \rho(u \otimes \psi(u \otimes \varphi(v \otimes a))) \\
&\quad + \{\rho(u \otimes \psi(v \otimes \varphi(u \otimes a))) + \rho(v \otimes \psi(u \otimes \varphi(u \otimes a)))\}.
\end{aligned}$$

It is then done by considering multiplicative linear combinations and extension by continuity, that $\rho \circ (\psi \circ \varphi) = (\rho \circ \psi) \circ \varphi$.

As well, for $\varphi : S\mathfrak{A} \to \mathfrak{B}$, we have

$$\begin{aligned}
&(\varphi \circ (\eta \otimes \mathrm{id}_{\mathfrak{A}}))(u \otimes a) = (\varphi \circ S(\eta \otimes \mathrm{id}_{\mathfrak{A}}))(u \otimes u \otimes a) \\
&= \varphi(u \otimes u(0)a) = u(0)\varphi(u \otimes a) = \varphi(u \otimes a), \\
&(\varphi \circ (\eta \otimes \mathrm{id}_{\mathfrak{A}}))(v \otimes a) = (\varphi \circ S(\eta \otimes \mathrm{id}_{\mathfrak{A}}))(u \otimes v \otimes a + v \otimes u \otimes a) \\
&= \varphi(u \otimes v(0)a) + \varphi(v \otimes u(0)a) = \varphi(v \otimes a).
\end{aligned}$$

It then follows that $\varphi \circ (\eta \otimes \mathrm{id}_{\mathfrak{A}}) = \varphi$.

Also, for $\psi : S\mathfrak{B} \to \mathfrak{A}$ and $b \in \mathfrak{B}$, we have

$$\begin{aligned}
&((\eta \otimes \mathrm{id}_{\mathfrak{A}}) \circ \psi)(u \otimes b) = ((\eta \otimes \mathrm{id}_{\mathfrak{A}}) \circ S\psi)(u \otimes u \otimes b) \\
&= (\eta \otimes \mathrm{id}_{\mathfrak{A}})(u \otimes \psi(u \otimes b)) = u(0)\psi(u \otimes b) = \psi(u \otimes b), \\
&((\eta \otimes \mathrm{id}_{\mathfrak{A}}) \circ \psi)(v \otimes b) = ((\eta \otimes \mathrm{id}_{\mathfrak{A}}) \circ S\psi)(u \otimes v \otimes b + v \otimes u \otimes b) \\
&= (\eta \otimes \mathrm{id}_{\mathfrak{A}})(u \otimes \psi(v \otimes b)) + (\eta \otimes \mathrm{id}_{\mathfrak{A}})(v \otimes \psi(u \otimes b)) \\
&= u(0)\psi(v \otimes b) + v(0)\psi(u \otimes b) = \psi(v \otimes b).
\end{aligned}$$

Hence we obtain $(\eta \otimes \mathrm{id}_{\mathfrak{A}}) \circ \psi = \psi$. □

Remark. If $\varphi_1 : S\mathfrak{A}_1 \to \mathfrak{B}_1$ and $\varphi_2 : \mathfrak{A}_2 \to \mathfrak{B}_2$ are graded $*$-homomorphisms, then there is a tensor product morphism from $S(\mathfrak{A}_1 \otimes \mathfrak{A}_2)$ to $\mathfrak{B}_1 \otimes \mathfrak{B}_2$ defined by

$$S(\mathfrak{A}_1 \otimes \mathfrak{A}_2) \xrightarrow{\Delta\otimes\mathrm{id}} SS(\mathfrak{A}_1 \otimes \mathfrak{A}_2) \cong S\mathfrak{A}_1 \otimes S\mathfrak{A}_2 \xrightarrow{\varphi_1\otimes\varphi_2} \mathfrak{B}_1 \otimes \mathfrak{B}_2.$$

Stabilization

The stailization of a C^*-algebra $\mathfrak{A}$ is the tensor product $\mathfrak{A} \otimes \mathbb{K}(H)$ with the C^*-algebra of all compact operators on a Hilbert space H.

If $\mathfrak{A}$ is a trivially graded C^*-algebra with unit, then each projection p of $\mathfrak{A} \otimes \mathbb{K}(H)$ corresponds to a projective (right) module $p(\mathfrak{A} \otimes \mathbb{K}(H))$ over $\mathfrak{A}$. In this way, the set of isomorphism classes of finitely generated $\mathfrak{A}$-modules is identified with the set of homotopy classes of projections of $\mathfrak{A} \otimes \mathbb{K}(H)$, as a central idea in K-theory.

There are graded $*$-homomorphisms

$$\mathbb{C} \to \mathbb{K}(H) \quad \text{and} \quad \mathbb{K}(H) \otimes \mathbb{K}(H) \to \mathbb{K}(H)$$

defined by $\mathbb{C} \ni \lambda \mapsto \lambda e$ with e the projection onto a 1-dimensional grading degree zero subspace of H and by identifying $H \otimes H$ with H by grading degree zero unitary. There is no canonical choice of the projection e or the unitary isomorphism.

Lemma 3.18. *Let H and H' be graded Hilbert spaces. Any two grading preserving isometries from H to H' induce the graded $*$-homomorphisms from $\mathbb{K}(H)$ to $\mathbb{K}(H')$ which are homotopic through graded $*$-homomorphisms.*

As a result, up to homotopy, there are canonical maps $\mathbb{C} \to \mathbb{K}(H)$ and $\mathbb{K}(H) \otimes \mathbb{K}(H) \to \mathbb{K}(H)$. Therefore, there is a **stabilized** homotopy category, in which morphisms from $\mathfrak{A}$ to $\mathfrak{B}$ are the homotopy classes of graded $*$-homomorphisms from $\mathfrak{A}$ to $\mathfrak{B} \otimes \mathbb{K}(H)$. Also, there is a **stabilized and amplified** category, in which morphisms from $\mathfrak{A}$ to $\mathfrak{B}$ are the homotopy classes of graded $*$-homomorphisms from $S\mathfrak{A}$ to $\mathfrak{B} \otimes \mathbb{K}(H)$.

4 Looking the K-theory as the E-theory

Definition 4.1. Let $\mathfrak{A}, \mathfrak{B}$ be graded C^*-algebras. We denote by $[\mathfrak{A}, \mathfrak{B}] = [\mathfrak{A} \to \mathfrak{B}]$ the set of homotopy classes of grading preserving $*$-homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$.

Definition 4.2. For a graded C^*-algebra $\mathfrak{A}$, we define (the **E-theory** group)

$$E_0(\mathfrak{A}) \equiv E(\mathfrak{A}) = [S, \mathfrak{A} \otimes \mathbb{K}].$$

Example 4.3. Let $\mathfrak{A} = \mathbb{C}$. Let D be an unbounded self-adjoint operator on a graded Hilbert space $H = H_0 \oplus H_1$ of the form

$$D\xi = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} D_-\xi_1 \\ D_+\xi_0 \end{pmatrix}$$

with $D^{\mathrm{ev}} = 0$ and $D^{\mathrm{od}} = D$ of grading degree one and assume that D has compact resolvent (or possibly assume that D has the countable discrete, point spectrum in $\mathbb{R}$ with finite multiplicity), so that $(xI - D)^{-1} \in \mathbb{K}(H)$ for $x \notin \sigma(D)$ the spectrum of D (contained in $\mathbb{R}$). Note that an unbounded operator D is self-adjoint if and only if the Cayley transform C of D defined to be $C = (D - iI)(D + iI)^{-1}$ is unitary. For example, D may be a Dirac-type operator on a compact manifold. By the functional calculus, a graded $*$-homomorphism $\psi_D : S \to \mathbb{K}(H)$ is defined to be $\psi_D(f) = f(D)$ (because of the spectrum asssumption), and its class $[\psi_D]$ is in $E_0(\mathbb{C})$.

Remark. Recall from [47] that for an (unbounded) operator t on a Hilbert space H, the resolvent set $\rho(t)$ of t is defined to be the set of all $\lambda \in \mathbb{C}$ for which $\lambda 1 - t$ is bijective from the domain $D(t)$ of t onto H and the norm

$$\|(\lambda 1 - t)x\| \geq C\|x\|, \quad x \in D(t)$$

for some $C > 0$, so that there is a bounded operator $b = (\lambda 1 - t)^{-1}$ from H to $D(s) \subset H$ with the operator norm $\|b\| \leq \frac{1}{C}$ such that

$$D(t) = b(\lambda 1 - t)H \subset (\lambda 1 - t)bH = H.$$

The complement of $\rho(t)$ is the spectrum $\mathrm{sp}(t) = \sigma(t)$ of t. Thus the resolvent function $r(\lambda) = (\lambda 1 - t)^{-1}$ is always bounded on H. It is likely to have compact resolvent such as $x \cdot \frac{1}{x} = 1$ by cancelling unboundeds by compacts to be one.

Example 4.4. Suppose that $\mathfrak{A}$ is a unital, trivially graded C^*-algebra. Then $K_0(\mathfrak{A})$ is generated by equivalence classes of projections of $\mathfrak{A} \otimes \mathbb{K}(H)$. Let p_0 and p_1 be such projections acting on the even and odd parts of the graded Hilbert space $H = H_0 \oplus H_1$. Then define

$$\psi_p(f) = \begin{pmatrix} f^{\mathrm{ev}}(0)p_0 & 0 \\ 0 & f^{\mathrm{od}}(0)p_1 \end{pmatrix} \equiv f(0)p$$

(corrected) to have a grading preserving $*$-homomorphism from $S = S_0 \oplus S_1$ to $\mathfrak{A} \otimes \mathbb{K}(H) \cong [\mathfrak{A} \otimes \mathbb{K}(H)_0] \oplus [\mathfrak{A} \otimes \mathbb{K}(H)_1]$.

Indeed, note that

$$p = p_0 \oplus p_1 = \begin{pmatrix} p_{11} & 0 \\ 0 & p_{22} \end{pmatrix} \oplus \begin{pmatrix} 0 & p_{12} \\ p_{21} & 0 \end{pmatrix}$$

and

$$f(x) = f^{\mathrm{ev}}(x) \oplus f^{\mathrm{od}}(x).$$

Thus,

$$\begin{aligned} \psi_p(\alpha(f)) &= \begin{pmatrix} f^{\mathrm{ev}}(0)p_0 & 0 \\ 0 & -f^{\mathrm{od}}(0)p_1 \end{pmatrix} \\ &= \alpha(\psi_p(f)) \end{aligned}$$

(formally). But $f^{\mathrm{ev}}(0) = f_0(0) = f(0)$ and $f^{\mathrm{od}}(0) = f_1(0) = 0$. Also,

$$\begin{aligned} \psi_p(f) &= \begin{pmatrix} (fg)^{\mathrm{ev}}(0)p_0 & 0 \\ 0 & (fg)^{\mathrm{od}}(0)p_1 \end{pmatrix} \\ &= \begin{pmatrix} (f_0g_0 + f_1g_1)(0)p_0 & 0 \\ 0 & (f_0g_1 + f_1g_0)(0)p_1 \end{pmatrix} \\ &= \begin{pmatrix} (f_0g_0)(0)p_0 & 0 \\ 0 & (f_1g_1)(0)p_1 \end{pmatrix} = \psi_p(f)\psi_p(g) \end{aligned}$$

with $(f_1g_1)(0) = 0$. As well, $\psi_p(f^*) = f_0^*(0)p_0 \oplus f_1^*(0)p_1 = \psi_p(f)^*$.

Remark. Those examples above are related as follows. If D is a self-adjoint, grading degree one, operator on H with compact resolvent, then define by

$$\psi_s : f \mapsto f(sD)$$

for $s \in [0,1]$ (corrected) a homotopy from the $*$-homomorphism ψ_D at $s = 1$ as $\psi_D(f) = f(D)$ to the $*$-homomorphism ψ_p at $s = 0$ as $\psi_p(f) = f(0)p$, where $p = p_0 \oplus p_1$ is the projection (onto the kernel of D) corresponding to $H = H_0 \oplus H_1$.

Note that if s is small enough, then the function $f(sx)$ for $x \in \mathbb{R}$ converges in norm to $f(0)\chi_0(x)$ with $\chi_0(x)$ the characteristic function at zero 0. Hence, $f(sD)$ converges in norm to $f(0)\chi_0(D) = f(0)p$ as well.

Exercise. Prove that $E_0(\mathbb{C}) \cong \mathbb{Z}$ in such a way that the Fredholm index of D (or D_+) is associated to the class of the $*$-homomorphism $\psi_D : S \to \mathbb{K}(H)$.

Proof. (Added). Indeed, define an index map:

$$E_0(\mathbb{C}) \xrightarrow{\text{index}} \mathbb{Z}$$

by $\mathrm{index}([\psi_D]) = \mathrm{index}(D)$ (but normalized as below). Also, $DH = D_- H_1 \oplus D_+ H_0$, and

$$\begin{aligned} \mathrm{index}(D) &= \dim\ker(D) - \dim\operatorname{coker}(D) \\ &= \dim\ker(D) - 0 \in \mathbb{Z}, \quad \text{non-negative} \end{aligned}$$

where the dimention $\dim\ker(D)$ is possibly 2 or finite n. If so, we may consider $\frac{1}{2} \cdot \mathrm{index}$ or $\frac{1}{n} \cdot \mathrm{index}$ as a normalization, or may use isomorphisms $2\mathbb{Z} \cong \mathbb{Z} \cong n\mathbb{Z}$ as a group. It is very likely to have 2 because if both $D_\pm$ are the usual differential operator, and if a function has derivative zero, then it is a constant, as well known in Calculus. □

We define the operation of **addition** on $E_0(\mathfrak{A})$ as follows. Let $\psi_1, \psi_2 : S \to \mathfrak{A} \otimes \mathbb{K}(H)$ be $*$-homomorphisms. Define

$$\psi_1 \oplus \psi_2 : S \to \mathfrak{A} \otimes \mathbb{K}(H \oplus H)$$

by

$$(\psi_1 \oplus \psi_2)(f) = \psi_1(f) \oplus \psi_2(f) = \begin{pmatrix} \psi_1(f) & 0 \\ 0 & \psi_2(f) \end{pmatrix} \in \mathbb{K}(H \oplus H),$$

where $H \oplus H$ is identified with H by some degree zero or one unitaries as

$$H \oplus H = (H_0 \oplus H_1) \oplus (H_0 \oplus H_1) \cong_1 (\oplus^2 H_0) \oplus (\oplus^2 H_1) \cong_0 H_0 \oplus H_1 = H,$$

where $\cong_j$ means an isomorphism by a degree j unitary. Homotopy works well with such identifications. The zero element for $E_0(\mathfrak{A})$ is given by the class of the zero homomorphism $\mathbf{0}$ from S to $\mathfrak{A} \otimes \mathbb{K}(H)$. Before going to prove the existence of additive inverses, we make the following preliminary observation which will be important later on.

The additive inverse for the class of $\psi : S \to \mathfrak{A} \otimes \mathbb{K}(H)$ is also represented by

$$\psi^{\mathrm{op}} = \psi \circ \alpha : S \xrightarrow{\alpha} S \xrightarrow{\psi} \mathfrak{A} \otimes \mathbb{K}(H^{\mathrm{op}}),$$

where $H^{\mathrm{op}} = H_1 \oplus H_0$ is the opposite to $H = H_0 \oplus H_1$, with grading reversed.

The set $E_0(\mathfrak{A}) = E(\mathfrak{A})$ so viewed as a group may be called the E-theory group of $\mathfrak{A}$. Moreover, the E-theory (group) is a functor from the category of graded C^*-algebras as $\mathfrak{A}$ and $\mathfrak{B}$ and $*$-homomorphisms to the category of abelian groups as $E(\mathfrak{A})$ and $E(\mathfrak{B})$, where morphisms as compositions are induced as

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\varphi} & \mathfrak{B} \\ {\scriptstyle E}\downarrow & & \downarrow{\scriptstyle E} \\ E(\mathfrak{A}) & \xrightarrow{\varphi_*} & E(\mathfrak{B}) \end{array}$$

where $\varphi_*([\psi]) = [(\varphi \otimes \mathrm{id}) \circ \psi]$ for $*$-homomorphisms $\psi : S \to \mathfrak{A} \otimes \mathbb{K}(H)$.

Lemma 4.5. *Let $\mathfrak{D}$ be any graded C^*-algebra and $\psi : S \to \mathfrak{D}$ be a grading-preserving $*$-homomorphism. Adjoin units to S and $\mathfrak{D}$, and extend ψ, and form the unitary element: $u_\psi = \psi(\frac{x-i}{x+i})$ in the unitization (or unitalization) of $\mathfrak{D}$. Then the correspondence between ψ and u_ψ is a bijection between the set of all $*$-homomorphisms $\psi : S \to \mathfrak{D}$ and the set of all unitary elements u in the unitization (or unitalization) of $\mathfrak{D}$ which are equal to 1 module $\mathfrak{D}$ and are mapped to their adjoints by the grading automorphism as $\alpha(u) = u^*$.*

Proof. (Added). We have the following commutative diagram:

$$\begin{array}{ccc} S^+ \cong C(\mathbb{T}) & \xrightarrow{\psi} & \mathfrak{D}^+ = \mathfrak{D} \oplus \mathbb{C} \\ \uparrow & & \uparrow \\ S = S_0 \oplus S_1 & \xrightarrow{\psi} & \mathfrak{D} = \mathfrak{D}_0 \oplus \mathfrak{D}_1. \end{array}$$

For $x \in \mathbb{R}$, the Cayley transform defined by

$$c(x) = \frac{x-i}{x+i} = \frac{x^2-1}{x^2+1} - \frac{2xi}{x^2+1}$$

takes values in the unit circle S^1 or $\mathbb{T}$ as $c(0) = -1$, $c(1) = -i$, $c(-1) = i$, and $\lim_{x \to \pm\infty} c(x) = 1$, with $|c(x)| = 1$. Note that the unitization $S^+ \cong C(\mathbb{T})$ of

S is generated by the inclusion unitary $z : \mathbb{T} \subset \mathbb{C}$, identified with $c(x)$. Note as well that $c(x)$ is identified with $(c(x) - 1, 1) \in S^+$, so that $c(x) - 1 \in S$. Hence,

$$\psi(c(x)) = (\psi(c(x) - 1), \psi(1)) = (\psi(c(x) - 1), 1) \in \mathfrak{D}^+,$$

and thus which is equal to 1 module $\mathfrak{D}$. Also,

$$\alpha(u_\psi) = \psi(\alpha(c(x)) = \psi(c(-x)) = \psi(\overline{z}) = \psi(z)^* = u_\psi^*.$$

Again, by the Stone-Weierstrass theorem, $C(\mathbb{T})$ is generated by $z = c(x)$ as a C^*-algebra. Thus, such a $*$-homomorphism $\psi : C(\mathbb{T}) \cong S^+ \to \mathfrak{D}^+$ is determined uniquely by the image $\psi(z) = u_\psi$. □

Definition 4.6. If $\mathfrak{D}$ is a graded C^*-algebra, then a **Cayley transform** for $\mathfrak{D}$ is a unitary in the unitization of $\mathfrak{D}$ which is equal to the identity, modulo $\mathfrak{D}$, and is switched to its adjoint by the grading automorphism.

As for **additive inverses** in $E_0(\mathfrak{A})$, let $u = u_\psi$ be the Cayley transform for $\psi : S \to \mathfrak{A} \otimes \mathbb{K}(H)$. Then u^* corresponds to the Cayley transfrom for $\psi^{\mathrm{op}} = \psi_1 \oplus \psi_0 : S \to \mathfrak{A} \otimes \mathbb{K}(H^{\mathrm{op}})$ opposite to $\psi = \psi_0 \oplus \psi_1$, where $H^{\mathrm{op}} = H_1 \oplus H_0$ is the Hilbert spcase opposite to $H = H_0 \oplus H_1$ with grading reversed. The rotation homotopy as

$$\Phi_t = \begin{pmatrix} (\cos t)u & (\sin t)1 \\ (-\sin t)1 & (\cos t)u^* \end{pmatrix} \quad t \in [0, \frac{\pi}{2}]$$

with

$$\Phi_t^*\Phi_t = \begin{pmatrix} (\cos t)u^* & (-\sin t)1 \\ (\sin t)1 & (\cos t)u \end{pmatrix} \begin{pmatrix} (\cos t)u & (\sin t)1 \\ (-\sin t)1 & (\cos t)u^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\Phi_t\Phi_t^* = \begin{pmatrix} (\cos t)u & (\sin t)1 \\ (-\sin t)1 & (\cos t)u^* \end{pmatrix} \begin{pmatrix} (\cos t)u^* & (-\sin t)1 \\ (\sin t)1 & (\cos t)u \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is then a path of Cayley transforms for $*$-homomorphisms $S \to \mathfrak{A} \otimes \mathbb{K}(H \oplus H^{\mathrm{op}})$ as

$$\begin{pmatrix} (\cos t)\psi & (\sin t)\mathbf{0} \\ (-\sin t)\mathbf{0} & (\cos t)\psi^{\mathrm{op}} \end{pmatrix}$$

with $\mathbf{0}$ the zero homomorphism, connecting

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \quad \text{at } t = 0 \text{ to} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{at } t = \frac{\pi}{2}.$$

The last matrix is in turn connected to the identity via

$$\begin{pmatrix} (\cos t)1 & (\sin t)1 \\ (-\sin t)1 & (\cos t)1 \end{pmatrix} \quad t \text{ goes from } 0 \text{ to } -\frac{\pi}{2}.$$

Finally, note that $u = 1$ identified with $1 \oplus 1$ as above corresponds to the zero homomorphism $\psi = \mathbf{0}$ from S to $\mathfrak{A} \otimes \mathbb{K}(H)$ as $u_0 = 1$.

Proposition 4.7. *In the category of trivially graded, unital C^*-algebras $\mathfrak{A}$, the E-theory functor $E_0(\mathfrak{A})$ is naturally isomorphic to the K-theory functor $K_0(\mathfrak{A})$.*

Proof. It is shown that $E_0(\mathfrak{A})$ is the group of path components of the space of Cayley transforms for $\mathfrak{A} \otimes \mathbb{K}(H)$.

Note that

$$\mathfrak{A} \otimes \mathbb{K}(H) = (\mathfrak{A} \oplus 0) \otimes \mathbb{K}(H) = (\mathfrak{A} \otimes \mathbb{K}(H)) \oplus 0 \cong (\mathfrak{A} \otimes \mathbb{K}(H)_0) \oplus (\mathfrak{A} \otimes \mathbb{K}(H)_1).$$

If $\varepsilon = I \oplus (-I)$ the diagonal sum (with $I = 1 \otimes 1_2$) is the grading operator and if u is a Cayley transform in $(\mathfrak{A} \otimes \mathbb{K}(H))^+$, then εu is a self-adjoint unitary whose $+1$ spectral projection $p = \frac{1}{2}(\varepsilon u + 1)$ is equal to the $+1$ spectral projection $p_\varepsilon = I \oplus 0$ of ε, module $\mathfrak{A} \otimes \mathbb{K}(H)$.

Note that

$$\begin{aligned} (\varepsilon u)^* \varepsilon u &= u^* \varepsilon^2 u = u^* u = 1, \\ (\varepsilon u)(\varepsilon u)^* &= \varepsilon u u^* \varepsilon = \varepsilon^2 = 1, \\ (\varepsilon u)^* &= u^* \varepsilon = \varepsilon^2 u^* \varepsilon = \varepsilon \alpha(u^*) = \varepsilon u. \end{aligned}$$

Since εu is self-adjoint and unitary, so that the spectrum $\sigma(\varepsilon u)$ is contained in $\{\pm 1\}$. By the functional calculus, $\frac{1}{2}((\varepsilon u) + 1) = \chi_1(\varepsilon u)$, with $\chi_1(t)$ is the characteristic function at 1. As well, ε is also self-adjoint and unitary. By the functional calculus, $\frac{1}{2}(\varepsilon + 1) = \chi_1(\varepsilon) = I \oplus 0$. Then $p - p_\varepsilon = \frac{1}{2}(\varepsilon(u-1))$ with $u - 1 \in \mathfrak{A} \otimes \mathbb{K}(H)$ as claimed.

Conversely, if p is a projection which is equal to p_ε modulo $\mathfrak{A} \otimes \mathbb{K}(H)$, then the $u = \varepsilon(2p - I)$ defines a Cayley transform for $\mathfrak{A} \otimes \mathbb{K}(H)$.

Indeed, suppose that $p - p_\varepsilon = \frac{1}{2}(\varepsilon(u-1))$. If $p - p_\varepsilon \in \mathfrak{A} \otimes \mathbb{K}(H)$, then $u - 1 \in \mathfrak{A} \otimes \mathbb{K}(H)$. It follows that

$$u = 2\varepsilon(p - p_\varepsilon) + 1 = 2\varepsilon(p - \frac{1}{2}(\varepsilon + 1)) + 1 = \varepsilon(2p - 1)$$

with

$$\begin{aligned} u^* u &= (2p-1)\varepsilon^2(2p-1) = (2p-1)^2 = 1, \\ uu^* &= u(2p-1)^2 u^* = uu^* = 1. \end{aligned}$$

Therefore, we may view $E_0(\mathfrak{A})$ as the group of path components of the projections which are equal to p_ε module $\mathfrak{A} \otimes \mathbb{K}(H)$. Then the map taking $[p]$ to $[p] - [p_\varepsilon]$ gives an isomorphism between $E_0(\mathfrak{A})$ and $K_0(\mathfrak{A})$ by involving the stability of K-theory groups. □

An exercise put here in the original text is put behind Example 6.2 for convenience to the proof, with needed terminology.

Exercise. If $\mathfrak{B}$ is a graded C^*-algebra that is the closure of the union of a directed system of graded C^*-subalgebras $\mathfrak{B}_\alpha$, then the natural map

$$\varinjlim E_0(\mathfrak{B}_\alpha) \to E_0(\mathfrak{B})$$

is an isomorphism.

Proof. As a hint given, every Cayley transform for $\mathfrak{B} \otimes \mathbb{K}(H)$ is a limit of Cayley tansforms for the subalgebras $\mathfrak{B}_\alpha \otimes \mathbb{K}(H)$, by considering restrictions and inductive extensions. It is then done. □

Long exact sequences

Definition 4.8. Let $\mathfrak{A}$ be a graded C^*-algebra. Denote by $H(\mathfrak{A})$ the space of all graded $*$-homomorphisms from S to $\mathfrak{A} \otimes \mathbb{K}(H)$, equipped with the topology of pointwise convergence, so that the net ψ_α converges to ψ if and only if $\psi_\alpha(f)$ converges to $\psi(f)$ in the norm topology for any $f \in S$. Thus,

$$H(\mathfrak{A}) = \mathrm{tHom}(S, \mathfrak{A} \otimes \mathbb{K}(H))$$

the topological Hom space (we may say so).

The space $H(\mathfrak{A})$ has the zero homomorphism as a natural base-point.

There is a direct sum operation as

$$\oplus : H(\mathfrak{A}) \times H(\mathfrak{A}) \to H(\mathfrak{A})$$

as $\oplus(\varphi_1, \varphi_2) = \varphi_1 \oplus \varphi_2 : S \to A \otimes \mathbb{K}(H \oplus H)$, where $H \oplus H$ is identified with H by some degree zero unitary. This operation gives the addition on the group

$$\pi_0(H(\mathfrak{A})) = E_0(\mathfrak{A}) \cong K_0(\mathfrak{A}).$$

Let $\psi \in H(C_0(\mathbb{R}^n) \otimes \mathfrak{A})$. Then ψ is viewed as a map from $\mathbb{R}^n$ to $H(\mathfrak{A})$ which converges to the zero homomorphim at infinity and is also as a zero-pointed (at the north pole) map from the one-point compacttification S^n of $\mathbb{R}^n$ to $H(\mathfrak{A})$, all of which may consist of the space $\Omega^n H(\mathfrak{A})$. It then follows that

$$\pi_n(H(\mathfrak{A})) = \pi_0(\Omega^n H(\mathfrak{A})) \cong E_0(C_0(\mathbb{R}^n) \otimes \mathfrak{A}).$$

Definition 4.9. Let $\mathfrak{A}$ be a graded C^*-algebra. The **higher** E-theory groups of $\mathfrak{A}$ are the homotopy groups of the space $H(\mathfrak{A})$:

$$E_n(\mathfrak{A}) = \pi_n(H(\mathfrak{A})), \quad n \geq 0.$$

The space $H(\mathfrak{A})$ and $E_n(\mathfrak{A})$ are functorial in $\mathfrak{A}$. If $\varphi : \mathfrak{A}_1 \to \mathfrak{A}_2$ is a graded $*$-homomorphism, then

$$\varphi_* : H(\mathfrak{A}_1) \to H(\mathfrak{A}_2)$$

as $\varphi_*(\psi) = (\varphi \otimes \mathrm{id}_{\mathbb{K}(H)}) \circ \psi$, and

$$\varphi_* : E_n(\mathfrak{A}_1) \cong E_0(C_0(\mathbb{R}^n) \otimes \mathfrak{A}_1) \to E_n(\mathfrak{A}_2)$$

as $\varphi_*(\psi) = (\mathrm{id}_{C_0(\mathbb{R}^n)} \otimes \varphi \otimes \mathrm{id}_{\mathbb{K}(H)}) \circ \psi$.

Lemma 4.10. *If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a surjective homomorphism of graded C^*-algebras, then the induced map $\varphi_* : H(\mathfrak{A}) \to H(\mathfrak{B})$ is a fibration.*

Recall that a map $X \to Y$ is a Serre **fibration** if for every map f from a cube of any finite dimension into Y and for every lifting to X of the restriction of f to a face of the cube, there is an extension to a lifting defined on the whole cube.

Proof. Regard $H(\mathfrak{A})$ as the space of Cayley transforms for $\mathfrak{A} \otimes \mathbb{K}(H)$ and thus as the space of unitary elements. Then note that the map φ of unitary groups of $\mathfrak{A}$ and $\mathfrak{B}$ is a fibration. □

The fiber of the map $\varphi_* : H(\mathfrak{A}) \to H(\mathfrak{B})$ is $H(\mathfrak{J})$, where $\mathfrak{J}$ is the kernel of φ. Then there is a long exact sequences:

$$\cdots \to E_{n+1}(\mathfrak{A}) \to E_{n+1}(\mathfrak{A}) \to E_n(\mathfrak{J}) \to E_n(\mathfrak{B}) \to \cdots$$

which ends at $E_0(\mathfrak{B})$. As well, the Mayer-Vietoris sequence holds:

$$\cdots \to E_{n+1}(\mathfrak{A}) \to E_n(\mathfrak{A}) \to E_n(\mathfrak{A}_1) \oplus E_n(\mathfrak{A}_2) \to E_n(\mathfrak{B}) \to \cdots$$

associated to the pull back $\mathfrak{A} = \mathfrak{A}_1 \oplus_{\mathfrak{B}} \mathfrak{A}_2$.

Products

Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C^*-algebras. There is a product operation on K-theory groups:

$$\otimes_* : K_0(\mathfrak{A}) \times K_0(\mathfrak{B}) \to K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B}) \subset K_0(\mathfrak{A} \otimes \mathfrak{B})$$

defined as $[p] \otimes_* [q] = [p \otimes q]$.

There is a map (or product) of spaces

$$\otimes : H(\mathfrak{A}) \times H(\mathfrak{B}) \to H(\mathfrak{A} \otimes \mathfrak{B})$$

denoted as the same as the tensor product and defined by associating to a pair $(\psi_{\mathfrak{A}}, \psi_{\mathfrak{B}}) \in H(\mathfrak{A}) \times H(\mathfrak{B})$ with $\psi_{\mathfrak{A}} : S \to \mathfrak{A} \otimes \mathbb{K}(H)$ and $\psi_{\mathfrak{B}} : S \to \mathfrak{B} \otimes \mathbb{K}(H)$ the composition:

$$S \xrightarrow{\Delta} S \otimes S \xrightarrow{\psi_{\mathfrak{A}} \otimes \psi_{\mathfrak{B}}} (\mathfrak{A} \otimes \mathbb{K}(H)) \otimes (\mathfrak{B} \otimes \mathbb{K}(H)) \cong \mathfrak{A} \otimes \mathfrak{B} \otimes \mathbb{K}(H).$$

Taking homotopy groups we obtain pairings

$$\otimes_* = ((\cdot, \cdot) \circ \Delta)_* : E_0(\mathfrak{A}) \times E_0(\mathfrak{B}) \to E_0(\mathfrak{A} \otimes \mathfrak{B})$$

as well as

$$\otimes_* = ((\cdot, \cdot) \circ \Delta)_* : E_i(\mathfrak{A}) \times E_j(\mathfrak{B}) \to E_{i+j}(\mathfrak{A} \otimes \mathfrak{B})$$

by taking $C_0(\mathbb{R}^i) \otimes \mathfrak{A}$ and $C_0(\mathbb{R}^j) \otimes \mathfrak{B}$ in the first pairing.

Example 4.11. Suppose that $\mathfrak{A} = \mathfrak{B} = \mathbb{C}$. For $j = 1, 2$, define $\psi_j : S \to \mathbb{C} \otimes \mathbb{K}(H) \cong \mathbb{K}(H)$ by $\psi_j(f) = f(d_j)$ by functional calculus for some self-adjoint operators d_j as in Example 4.3. Then the corresponding product of ψ_1 and ψ_2 is given by $f \mapsto f(d_1 \otimes 1 + 1 \otimes d_2)$. This type of formula is the standard construction of an operator, the Fredholm index of which is the product of the indices of d_1 and d_2, in index theory (unchecked).

Proposition 4.12. *(a) The E-theory product $\otimes_*$ is associative.*

(b) The E-theory product $\otimes_*$ is commutative in the sense that if $x \in E_0(\mathfrak{A})$ and $y \in E_0(\mathfrak{B})$, and if $\tau : \mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{B} \otimes \mathfrak{A}$ is the transposition (or flip) isomorphism, then $\tau_*(x \otimes_* y) = y \otimes_* x$ in $E_0(\mathfrak{B} \otimes \mathfrak{A})$.

(c) The E-theory product is functorial in the sense that if $f : \mathfrak{A} \to \mathfrak{A}'$ and $g : \mathfrak{B} \to \mathfrak{B}'$ are graded $*$-homomorphisms, then $(f \otimes g)_*(x \otimes_* y) = f_*(x) \otimes_* g_*(y)$ in $E_0(\mathfrak{A}' \otimes \mathfrak{B}')$.

Remark. In item (b), if we take $x \in E_i(\mathfrak{A})$ and $y \in E_j(\mathfrak{B})$, then $\tau_*(x \otimes_* y) = (-1)^{ij} y \otimes_* x$.

Proof. (Added). (a) Let $\varphi \in H(\mathfrak{A})$, $\psi \in H(\mathfrak{B})$, and $\rho \in H(\mathfrak{C})$. Then the products (not tensors)

$$\begin{aligned}(\varphi \otimes \psi) \otimes \rho &= ((\varphi \otimes \psi) \otimes \rho) \circ \Delta \\ &= (((\varphi \otimes \psi) \circ \Delta) \otimes \rho) \circ \Delta, \\ \varphi \otimes (\psi \otimes \rho) &= (\varphi \otimes (\psi \otimes \rho)) \circ \Delta \\ &= (\varphi \otimes ((\psi \otimes \rho) \circ \Delta)) \circ \Delta\end{aligned}$$

with $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ and those are the same as the tensor $\varphi \otimes \psi \otimes \rho$ on $S \otimes S \otimes S$.

(b) The product $\varphi \otimes \psi = (\varphi \otimes \psi) \circ \Delta$ is mapped by τ to $\tau \circ (\varphi \otimes \psi) \circ \Delta$, which is equal to $(\psi \otimes \varphi) \circ \Delta = \psi \otimes \varphi$.

(c) The product $\varphi \otimes \psi = (\varphi \otimes \psi) \circ \Delta$ is mapped by $f \otimes g$ to $(f \otimes g) \circ (\varphi \otimes \psi) \circ \Delta$, which is equal to $((f \circ \varphi) \otimes (g \circ \psi)) \circ \Delta = f_*(\varphi) \otimes g_*(\psi)$. □

Denote by $1_p \in E_0(\mathbb{C})$ the class of a homomorphism φ_p which maps each element $f \in S$ to $f(0)p \in \mathbb{K}(H)$, where p is the orthogonal projection onto a 1-dimensional, grading degree zero subspace of H.

Proposition 4.13. *If $\mathfrak{B}$ is any graded C^*-algebra and if $x \in E_0(\mathfrak{B})$, then the class $1_p \otimes_* x$ of $E_0(\mathbb{C} \otimes \mathfrak{B})$ corresponds to x, under $\mathbb{C} \otimes \mathfrak{B} \cong \mathfrak{B}$.*

Proof. (Added). Let $\psi \in H(\mathfrak{B})$. Then $\varphi_p \otimes \psi = (\varphi_p \otimes \psi) \circ \Delta$. If $\Delta(f) = f \otimes f$, then $(\varphi_p \otimes \psi)(f \otimes f) = f(0)p \otimes \psi(p)$, and if $\Delta(f) = u \otimes f + f \otimes u$, then

$$(\varphi_p \otimes \psi)(\Delta(f)) = u(0)p \otimes \psi(f) + f(0)p \otimes \psi(u)$$

with $\mathbb{C}p \otimes \mathfrak{B} \cong \mathfrak{B}$ and some more computation. □

5 Asymptotic morphisms for C^*-algebras

Definition 5.1. (Connes- Higson [11]). Let $\mathfrak{A}$ and $\mathfrak{B}$ be graded C^*-algebras. An **asymptotic morphism** from $\mathfrak{A}$ to $\mathfrak{B}$ is a family of functions $\varphi_t : \mathfrak{A} \to \mathfrak{B}$ for $t \in [1, \infty)$ such that the continuity condition holds as that for all $a \in \mathfrak{A}$

$$\varphi_t(a) \in C^b([1,\infty), \mathfrak{B}), \quad [1,\infty) \ni t \mapsto \varphi_t(a) \in \mathfrak{B},$$

and the asymptotic condition holds as that for $a_1, a_2, a \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned}&\lim_{t\to\infty} (\varphi_t(a_1 a_2) - \varphi_t(a_1)\varphi_t(a_2)) = 0 \quad \text{(in norm)} \\ &\lim_{t\to\infty} (\varphi_t(\lambda a_1 + a_2) - \lambda\varphi_t(a_1) - \varphi_t(a_2)) = 0 \\ &\lim_{t\to\infty} (\varphi_t(a^*) - \varphi_t(a)^*) = 0.\end{aligned}$$

If α is the grading automorphism for $\mathfrak{A}$ and $\mathfrak{B}$, we assume that

$$\lim_{t\to\infty}[\alpha(\varphi_t(a)) - \varphi_t(\alpha(a))] = 0 \quad \text{(in norm)}.$$

We denote by $(\varphi_t) : \mathfrak{A} \rightsquigarrow \mathfrak{B}$ an asymptotic morphism defined as above.

An asymptotic morphism $(\varphi_t) : \mathfrak{A} \rightsquigarrow \mathfrak{B}$ is a one-prameter family of maps from $\mathfrak{A}$ to $\mathfrak{B}$ which asymtotically converges to a $*$-homomorphism.

Definition 5.2. Two asymptotic morhisms (φ_t), $(\psi_t) : \mathfrak{A} \rightsquigarrow \mathfrak{B}$ are asymptotically equivalent if for all $a \in \mathfrak{A}$,

$$\lim_{t\to\infty} \|\varphi_t(a) - \psi_t(a)\| = 0.$$

Definition 5.3. Let $\mathfrak{B}$ be a graded C^*-algebra. The asymptotic C^*-algebra of $\mathfrak{B}$ is defined to be the quotient C^*-algebra

$$C^b([1,\infty),\mathfrak{B})/C_0([1,\infty),\mathfrak{B}) \equiv C^b(I_0,\mathfrak{B})/C_0(I_0,\mathfrak{B}),$$

which is denoted as $C_q([1,\infty),\mathfrak{B}) \equiv C_q(I_0,\mathfrak{B})$, with $[1,\infty) \approx I_0 = (0,1] \subset I = [0,1]$ (homeomorphic).

Proposition 5.4. (*Added*). *Up to equivalence, an asymptotic morphism from $\mathfrak{A}$ to $\mathfrak{B}$ is the same thing as a $*$-homomorphism from $\mathfrak{A}$ into $C_q(I_0,\mathfrak{B})$.*

Proof. (Edited). If $\varphi : \mathfrak{A} \to C_q([1,\infty),\mathfrak{B})$ is a $*$-homomorphism and $s : C_q([1,\infty),\mathfrak{B}) \to C^b([1,\infty),\mathfrak{B})$ is a cross section for the quotient map from $C^b([1,\infty),\mathfrak{B})$ to $C_q([1,\infty),\mathfrak{B})$, then we obtain an asymptotic morphism from $\mathfrak{A}$ to $\mathfrak{B}$ by composing as

$$s\circ\varphi : \mathfrak{A} \xrightarrow{\varphi} C_Q([1,\infty),\mathfrak{B}) \xrightarrow{s} C^b([1,\infty),\mathfrak{B}),$$

and its equivalence class is independent of the choice of a cross section.

Conversely, an asymptotic morphism can be viewed as a function from $\mathfrak{A}$ into $C^b([1,\infty),\mathfrak{B})$, and by composing with the quotient map into $C_q([1,\infty),\mathfrak{B})$ we obtain a $*$-homomorphism from $\mathfrak{A}$ to $C_q([1,\infty),\mathfrak{B})$ as

$$q\circ(\varphi_t) : \mathfrak{A} \overset{(\varphi_t)}{\rightsquigarrow} C^b([1,\infty),\mathfrak{B}) \xrightarrow{q} C_q([1,\infty),\mathfrak{B}),$$

which depends only on the asymptotic equivalence class of the asymptotic morphism. $\square$

Now, the composition as

$$\varphi \circ \psi : S \xrightarrow{\psi} \mathfrak{A} \otimes \mathbb{K} \overset{\varphi}{\rightsquigarrow} \mathfrak{B} \otimes \mathbb{K}$$

of a graded $*$-homomorphism ψ with an asymptotic morphism φ given is an asymptotic morphism from S into $\mathfrak{B} \otimes \mathbb{K}$, with $\mathbb{K} = \mathbb{K}(H)$.

Lemma 5.5. *Every asymptotic morphism from S into a graded C^*-algebra $\mathfrak{D}$ is asymptotic to a family of graded $*$-homomorphisms from S to $\mathfrak{D}$.*

Proof. A $*$-homomorphism from S to $\mathfrak{D}$ is identified with a Cayley transform for $\mathfrak{D}$, that is, a unitary in the unitalization of $\mathfrak{D}$, equal to 1 modulo $\mathfrak{D}$, which is switched to its adjoint by the grading homomorphism.

An asymptotic morphism from S to $\mathfrak{D}$ is the same up to equivalence as a norm continuous family of elements x_t of the unitalization $\mathfrak{D}^+$, equal to 1 modulo $\mathfrak{D}$, which are asymptotically unitary and asymptotically switched to their adjoints by the grading automorphism.

Such a family of asymptotic Cayley transforms can be altered to produce a family of Cayley transforms, for large t. First, replace x_t with $y_t = \frac{1}{2}(x_t + \alpha(x_t^*))$ and then unitarize by forming $u_t = y_t(y_t^* y_t)^{-\frac{1}{2}}$. Note that y_t is invertible for t large. Then u_t for t large correspond to a family of $*$-homomorphisms. □

Definition 5.6. Two asymptotic morphisms $\varphi^0, \varphi^1 : \mathfrak{A} \rightsquigarrow \mathfrak{B}$ are **homotopic** if there is an asympptotic morphism $\varphi : \mathfrak{A} \rightsquigarrow C([0,1], \mathfrak{B})$ such that $\varphi(a)(0) = \varphi^0(a)$ and $\varphi(a)(1) = \varphi^1(a)$ for $a \in \mathfrak{A}$. This homotopy is an equivalence relation and we denote by

$$[\mathfrak{A} \rightsquigarrow \mathfrak{B}]$$

the set of all homopopy classes of asymptotic morphisms from $\mathfrak{A}$ to $\mathfrak{B}$.

There is a natural inclusion map from

$$[\mathfrak{A}, \mathfrak{B}] \equiv [\mathfrak{A} \to \mathfrak{B}] \xrightarrow{\quad i \quad} [\mathfrak{A} \rightsquigarrow \mathfrak{B}]$$

since each $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is regarded as a constant asymptotic morphism. It follows from Lemma 5.5 that

Proposition 5.7. *If $\mathfrak{D}$ is a graded C^*-algebra, then*

$$[S, \mathfrak{D}] \cong [S \rightsquigarrow \mathfrak{D}] \quad \textit{an isomorphism.}$$

Asymptotic morphisms × tensor prodcuts

With $\varphi : \mathfrak{A} \otimes \mathbb{K} \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}$, there is the following diagram:

$$\begin{array}{ccc} E_0(\mathfrak{A}) = [S, \mathfrak{A} \otimes \mathbb{K}] & \xrightarrow{\varphi \circ (\cdot)} & [S \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}] \\ i \big\downarrow \cong & & \cong \big\uparrow i \\ [S \rightsquigarrow \mathfrak{A} \otimes \mathbb{K}] & \xrightarrow{\varphi \circ (\cdot)} & [S \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}] \cong [S, \mathfrak{B} \otimes \mathbb{K}] = E_0(\mathfrak{B}). \end{array}$$

As a conclusion, $\varphi : \mathfrak{A} \otimes \mathbb{K} \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}$ induces an E-theory homomorphism $\varphi_* : E_0(\mathfrak{A}) \to E_0(\mathfrak{B})$.

Lemma 5.8. *Let $\mathfrak{D}$ be a C^*-algebra and let $\varphi : \mathfrak{A} \rightsquigarrow \mathfrak{B}$ be an asymptotic morphism of C^*-algebras. Then there is an asymptotic morphism $\varphi \otimes \mathrm{id} : \mathfrak{A} \otimes \mathfrak{D} \rightsquigarrow \mathfrak{B} \otimes \mathfrak{D}$ for any $\mathfrak{D}$ such that*

$$(\varphi \otimes \mathrm{id})_t(a \otimes d) = \varphi_t(a) \otimes d.$$

Moreover, this formula determines $\varphi \otimes \mathrm{id}$ uniquely, up to asymptotic equivalence.

Proof. Assume for simplicity that $\mathfrak{B}$ and $\mathfrak{D}$ are unital. For the non-unital cases, consider adjoining units. There are graded $*$-homomorphisms from $\mathfrak{A}$ and $\mathfrak{D}$ into the asymptotic C^*-algebra $C_q(I_0, \mathfrak{B} \otimes \mathfrak{D})$, determined by sending $a \mapsto \varphi_t(a) \otimes 1$ and $d \mapsto 1 \otimes d$. They graded commute and so determine a homomorphism $\varphi \otimes \mathrm{id} : \mathfrak{A} \otimes \mathfrak{D} \to C_q(I_0, \mathfrak{B} \otimes \mathfrak{D})$. This in turn determines an asymptotic morphism $\varphi \otimes \mathrm{id} : \mathfrak{A} \otimes \mathfrak{D} \rightsquigarrow \mathfrak{B} \otimes \mathfrak{D}$.

Two asymptotic morphisms which are asymptotic on the elementary tensors as $a \otimes d$ determine $*$-homomorphisms into $C_q(I_0, \mathfrak{B} \otimes \mathfrak{D})$ which are equal on elementary tensors, and hence equal everywhere. It follows from this that such two asymptotic morphisms are equivalent. □

Remark. On the argument above, it is crucial to use the maximal tensor product as $\otimes = \otimes_{\max}$.

Lemma 5.9. (*Edited*). (*a*) *An asymptotic morphism $\varphi : \mathfrak{A} \rightsquigarrow \mathfrak{B}$ determines an asymptotic morphism $\varphi \otimes \mathrm{id}$ from $\mathfrak{A} \otimes \mathbb{K}(H)$ to $\mathfrak{B} \otimes \mathbb{K}(H)$, and hence a E-theory group map $\varphi_* : E_0(\mathfrak{A}) \to E_0(\mathfrak{B})$ is deduced.*

(*b*) *An asymptotic morphism $\varphi : \mathfrak{A} \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}(H)$ determines an asymptotic morphism $\varphi \otimes \mathrm{id}$ from $\mathfrak{A} \otimes \mathbb{K}(H)$ to $(\mathfrak{B} \otimes \mathbb{K}(H)) \otimes \mathbb{K}(H)$, and with $\mathbb{K}(H) \otimes \mathbb{K}(H) \cong \mathbb{K}(H)$, hence a E-theory group map $\varphi_* : E_0(\mathfrak{A}) \to E_0(\mathfrak{B})$ is deduced.*

(c) *An asymptotic morphism $\varphi : S\mathfrak{A} \rightsquigarrow \mathfrak{B}$ determines an asymptotic morphism $\varphi \otimes \mathrm{id}$ from $S\mathfrak{A} \otimes \mathbb{K}(H)$ to $\mathfrak{B} \otimes \mathbb{K}(H)$, and hence a E-theory group map $\varphi_* : E_0(\mathfrak{A}) \to E_0(\mathfrak{B})$ is obtained.*

(d) *An asymptotic morphism $\varphi : S\mathfrak{A} \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}(H)$ determines an asymptotic morphism $\varphi \otimes \mathrm{id}$ from $S\mathfrak{A} \otimes \mathbb{K}(H)$ to $(\mathfrak{B} \otimes \mathbb{K}(H)) \otimes \mathbb{K}(H)$, and hence a E-theory group map $\varphi_* : E_0(\mathfrak{A}) \to E_0(\mathfrak{B})$ is obtained.*

Proof. As for (a) and (b),

$$E_0(\mathfrak{A}) = [S, \mathfrak{A} \otimes \mathbb{K}(H)] \xrightarrow{\varphi_*} [S, \mathfrak{B} \otimes \mathbb{K}(H)] = E_0(\mathfrak{B}).$$

As for (c), if $[\psi] \in [S, \mathfrak{A} \otimes \mathbb{K}(H)]$, then the composition:

$$S \xrightarrow{\Delta} S \otimes S \xrightarrow{\mathrm{id} \otimes \psi} S \otimes (\mathfrak{A} \otimes \mathbb{K}(H)) \cong (S\mathfrak{A}) \otimes \mathbb{K}(H) \overset{\varphi \otimes \mathrm{id}}{\rightsquigarrow} \mathfrak{B} \otimes \mathbb{K}(H)$$

represents a E-theory class of $E_0(\mathfrak{B})$.

As for (d), combine (b) and (c). □

Bott periodicity in E-theory

May refer to Atiyah [2].

Definition 5.10. A graded C^*-algebra $\mathfrak{B}$ has the **rotation property** if the automorphism sending $b_1 \otimes b_2 \mapsto (-1)^{\partial b_1 \partial b_2} b_2 \otimes b_1$, which interchanges the two factors of $\mathfrak{B} \otimes \mathfrak{B}$ is homotopic to a tensor product $*$-homomorphism as $\mathrm{id} \otimes \rho$ for some ρ.

Example 5.11. The trivially graded C^*-algebra $\mathfrak{B} = C_0(\mathbb{R}^n)$ has this property with $\rho = \mathrm{id}$. In this case, note that $\mathfrak{B} \otimes \mathfrak{B} \cong C_0(\mathbb{R}^n) \otimes C_0(\mathbb{R}^n)$.

In fact, any trivially graded unital C^*-algebra has this property with $\rho = \mathrm{id}$, and for the non-unital case, we consider the unitization by $\mathbb{C}$. Because any symmetry, i.e., any self-adjoint unitary is homotopic to the identity (see [57, Proposition 4.7]). The usual filp in the trivially graded case is a symmetry.

Theorem 5.12. *Let $\mathfrak{B}$ be a graded C^*-algebra with the rotation property. Suppose that there exists a class $b \in E_0(\mathfrak{B})$ and an asymptotic morphism $\alpha : S \otimes \mathfrak{B} \rightsquigarrow \mathbb{K}(H)$ such that the induced $\alpha_* : E_0(\mathfrak{B}) \to E_0(\mathbb{C})$ maps b to the unit class 1. Then for every C^*-algebra $\mathfrak{A}$, the maps*

$$\mathrm{id}_* \otimes \alpha_* : E_0(\mathfrak{A} \otimes \mathfrak{B}) \to E_0(\mathfrak{A}) \quad \textit{and} \quad \beta_* = \otimes_* b : E_0(\mathfrak{A}) \to E_0(\mathfrak{A} \otimes \mathfrak{B})$$

are inverse to one another.

Proof. For any C^*-algebra $\mathfrak{C}$, the following diagram commutes:

$$\begin{array}{ccc} E_0(\mathfrak{C}) \otimes E_0(\mathfrak{A} \otimes \mathfrak{B}) & \xrightarrow{\otimes_*} & E_0(\mathfrak{C} \otimes \mathfrak{A} \otimes \mathfrak{B}) \\ {\scriptstyle \mathrm{id}_* \otimes (\mathrm{id}_* \otimes \alpha_*)} \downarrow & & \downarrow {\scriptstyle (\mathrm{id}_* \otimes \mathrm{id}_*) \otimes \alpha_*} \\ E_0(\mathfrak{C}) \otimes E_0(\mathfrak{A}) & \xrightarrow{\otimes_*} & E_0(\mathfrak{C} \otimes \mathfrak{A}). \end{array}$$

In this case, the map $\alpha_* : E_0(\mathfrak{A} \otimes \mathfrak{B}) \to E_0(\mathfrak{A})$ is said to be **multiplicative** in this sense. It follows with $\mathfrak{C} = \mathbb{C}$ trivial that α_* is left inverse to the map $\beta_* : E_0(\mathfrak{A}) \to E_0(\mathfrak{A} \otimes \mathfrak{B})$ as

$$\alpha_*(\beta_*(x)) = \alpha_*(x \otimes_* b) = x \otimes_* \alpha_*(b) = x \otimes_* 1 = x.$$

To prove that α_* is also right inverse to β_* we let

$$\sigma : \mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{B} \otimes \mathfrak{A} \quad \text{and} \quad \tau : \mathfrak{B} \otimes \mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{B} \otimes \mathfrak{A} \otimes \mathfrak{B}$$

the isomorphisms which interchange the first and last factors in the tensor products. Note that for $z \in E_0(\mathfrak{B})$, the diagram:

$$\begin{array}{ccc} E_0(\mathfrak{A} \otimes \mathfrak{B}) & \xrightarrow{\sigma_*} & E_0(\mathfrak{B} \otimes \mathfrak{A}) \\ {\scriptstyle z \otimes_*} \downarrow & & \downarrow {\scriptstyle \otimes_* z} \\ E_0(\mathfrak{B} \otimes \mathfrak{A} \otimes \mathfrak{B}) & \xrightarrow{\tau_*} & E_0(\mathfrak{B} \otimes \mathfrak{A} \otimes \mathfrak{B}) \end{array}$$

commutes, so that $\sigma_*(y) \otimes_* z = \tau_*(z \otimes_* y)$ for $y \in E_0(\mathfrak{A} \otimes \mathfrak{B})$. Since $\mathfrak{B}$ has the rotation property, τ is homotopic to the tensor product $\rho \otimes \mathrm{id} \otimes \mathrm{id}$ for some ρ. By setting $z = b$ we obtain

$$\sigma_*(y) \otimes_* b = \tau_*(b \otimes_* y) = \rho_*(b) \otimes_* y$$

in $E_0(\mathfrak{B} \otimes \mathfrak{A} \otimes \mathfrak{B})$. Applying α_* we deduce that

$$\begin{aligned} &\alpha_*(\sigma_*(y) \otimes_* b) = \sigma_*(y) \otimes \alpha_*(b) = \sigma_*(y) \quad \text{equal to} \\ &\alpha_*(\rho_*(b) \otimes_* y) = \rho_*(b) \otimes \alpha_*(y) \end{aligned}$$

in $E_0(\mathfrak{B} \otimes \mathfrak{A})$. Applying σ_* on $E_0(\mathfrak{B} \otimes \mathfrak{A})$ we obtain that

$$\begin{aligned} \sigma_*(\sigma_*(y)) = (\sigma \circ \sigma)_*(y) = y \quad \text{equal to} \\ \sigma_*(\rho_*(b) \otimes \alpha_*(y)) = \alpha_*(y) \otimes_* \rho_*(b) \end{aligned}$$

in $E_0(\mathfrak{A} \otimes \mathfrak{B})$. This shows that α_* is the right inverse to the multiplication $\otimes_*$ by $\rho_*(b)$. Therefore, α_* is both left and right invertible. By uniqueness of the inverse to α_* invertible, we get

$$\beta_* = \otimes_* b = \otimes \rho_*(b)$$

as maps. □

Remark. It follows that $\rho_*(b) = b$.

6 Clifford algebras as C^*-algebras

We may refer to [25] of Higson, Kasparov, and Trout (unchecked in details).

Definition 6.1. Let V be a finite dimensional Euclidean vector space, that is, a real vector space with a positive definite inner product $\langle\cdot,\cdot\rangle$. The complex **Clifford algebra** of V is the graded complex algebra (or C^*-algebra with C^*-norm) generated by the unit 1 of grading degree zero and a linear copy of V, whose elements are self-adjoint and of grading degree one, subject to the relations $v^2 = \|v\|^2 1 = \langle v, v\rangle 1$ for any $v \in V$. We may write the C^*-algebra as

$$Cl^*(V) = \mathbb{C}[V] = Cl^*(1, V) = \mathbb{C}1 \oplus \mathbb{C}[V]$$

but not graded, only as a direct sum.

Remark. The Clifford algebra $Cl^*(V)$ may be constructed from the complex tensor product $T_{\mathbb{C}}(V)$ of V by dividing it the ideal generated by the elements $v \otimes v - \|v\|^2 1$ for $v \in V$.

If the set $\{e_1, \cdots, e_n\}$ is an orthonormal basis for V, then e_j are regarded as elements of $Cl^*(V)$ such that $e_j^2 = 1$ and $e_ie_j + e_je_i = 0$ if $i \neq j$.

Indeed, note that for any $x, y \in V$,

$$\begin{aligned} xy + yx &= (x+y)^2 - x^2 - y^2 \\ &= (\|x+y\|^2 - \|x\|^2 - \|y\|^2)1, \quad \text{so that} \\ e_ie_j + e_je_i &= (\|e_i + e_j\|^2 - \|e_i\|^2 - \|e_j\|^2)1 \\ &= \begin{cases} 0 & \text{if } i \neq j, \\ 2 \cdot 1 & \text{if } i = j. \end{cases} \end{aligned}$$

Thus, the unit 1 and the monomials (or products) $e_{i_1} \cdots e_{i_p}$ with $1 \leq i_1 < \cdots < i_p \leq n$ for $1 \leq p \leq n$ and of grading degree p (mod 2) span $Cl^*(V)$ as a complex linear space and as a basis as

$$Cl^*(V) = \mathbb{C}1 \oplus [\oplus_{1 \leq i_1 < \cdots < i_p \leq n} \mathbb{C}e_{i_1} \cdots e_{i_p}].$$

Example 6.2. The Clifford algebra $Cl^*(\mathbb{R}) = \mathbb{C}1 \oplus \mathbb{C}[\mathbb{R}]$ is isomorphic to

$$\mathbb{C}^2 \cong \mathbb{C}(1,1) \oplus \mathbb{C}(1,-1) = Cl^*(\mathbb{R})_0 \oplus Cl^*(\mathbb{R})_1,$$

where $e_1 = (1, -1)$, with the grading α on $\mathbb{C}^2$ defined as $\mathrm{id}_{\mathbb{C}^2} \oplus (-\mathrm{id}_{\mathbb{C}^2}) = 1_2 \oplus (-1_2)$. The grading automorphism transposes two copies of $\mathbb{C}$ in $\mathbb{C}^2$. Indeed, for $(x, y) \in \mathbb{C}^2$,

$$\begin{aligned} (x,y) &= \frac{x+y}{2}(1,1) + \frac{x-y}{2}(1,-1), \quad \text{so that} \\ \alpha(x,y) &= \frac{x+y}{2}(1,1) + \frac{y-x}{2}(1,-1) = (y,x). \end{aligned}$$

Also, the grading α is not inner because the algebra is commutative.

Exercise. (Edited here). If $\mathfrak{A}$ is trivially graded and unital C^*-algebra, then $E_0(\mathfrak{A} \otimes Cl^*(\mathbb{R})) \cong K_1(\mathfrak{A})$.

Proof. Note that $K_1(\mathfrak{A}) \cong K_0(S\mathfrak{A})$ and

$$K_0((S\mathfrak{A})^+) \cong E_0((S\mathfrak{A})^+) = [S, (S\mathfrak{A})^+ \otimes \mathbb{K}] \cong [S, S\mathfrak{A} \otimes \mathbb{K}] \oplus [S, \mathbb{C} \otimes \mathbb{K}].$$

Hence, $K_1(\mathfrak{A}) \cong E_0(S\mathfrak{A})$. Note as well that $S = C_0(\mathbb{R})$ is generated as a C^*-algebra by an even function and an odd function in S such as u and v, and that every homomorphism φ of S into S determined by setting as $\varphi(u)$ and $\varphi(v)$. This is equivalent to setting $\varphi'(u)$ and $\varphi'(v)$ in $Cl^*(\mathbb{R}) \cong \mathbb{C}^2$ for $\varphi' : S \to Cl^*(\mathbb{R})$. In this sense, we obtain $E_0(S\mathfrak{A}) \cong E_0(\mathfrak{A} \otimes Cl^*(\mathbb{R}))$ (or by other reasons). $\square$

Example 6.3. The Clifford algebra $Cl^*(\mathbb{R}^2)$ is isomorphic to $M_2(\mathbb{C})$ in such a way that

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

so that $e_1^2 = 1_2 = (1, 1) = e_2^2$ and $e_1 e_2 = (-i, i)$ the digonal matrices. Thus

$$Cl^*(\mathbb{R}^2)_0 \oplus Cl^*(\mathbb{R}^2)_1 = [(\mathbb{C}, \mathbb{C})1_2] \oplus [(\mathbb{C}, \mathbb{C})e_1]$$

with the grading $\alpha = 1_2 \oplus (-1_2)$ inner, because $\varepsilon = ie_1e_2 = (1, -1)$, $\varepsilon^* = -ie_2e_1 = ie_1e_2 = \varepsilon$, and $\varepsilon^2 = -(e_1e_2)^2 = 1_2$, and $\varepsilon 1_2 \varepsilon = 1_2$ and $\varepsilon e_1 \varepsilon = -e_1$.

Example 6.4. More generally, the Clifford algebra $Cl^*(\mathbb{R}^{2k})$ of even dimensional $\mathbb{R}^{2k}$ is isomorphic to the matrix algebra $M_{2^k}(\mathbb{C})$, graded by $\varepsilon = i^k e_1 \cdots e_{2k} = I \oplus (-I)$, while the Clifford algebra $Cl^*(\mathbb{R}^{2k+1})$ of odd dimensional $\mathbb{R}^{2k+1}$ is isomorphic to the direct sum $M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C})$, graded by the automorphism which switches the summands. The proof is added as Corollary 6.7 below.

Definition 6.5. Let V be a finite dimensional Euclidean vector space. Denote by $C_0Cl^*(V) = C_0(V, Cl^*(V))$ the graded C^*-algebra of continuous functions from V to $Cl^*(V)$, vanishing at infinity, such that

$$C_0(V, Cl^*(V))_j = C_0(V, Cl^*(V)_j), \quad j = 0, 1.$$

We may call $C_0Cl^*(V)$ the **continuous field** of Clifford algebras over V.

Example 6.6. We have

$$C_0(\mathbb{R}, Cl^*(\mathbb{R})) = C_0(\mathbb{R}, \mathbb{C}) \oplus C_0(\mathbb{R}, \mathbb{C}) = \oplus^2 C_0(\mathbb{R})$$

with the grading automorphism switching the summands. Also,

$$C_0(\mathbb{R}^2, Cl^*(\mathbb{R}^2)) \cong C_0(\mathbb{R}^2, M_2(\mathbb{C})) \cong M_2(C_0(\mathbb{R}^2))$$

graded by $1_2 \oplus (-1_2)$.

Suppose now that U and V are finite dimensional Euclidean vector space. Each of U and V is a subspace of $U \oplus V$. There are corresponding inclusion maps from $Cl^*(U)$ and $Cl^*(V)$ to $Cl^*(U \oplus V)$. Moreover, it follows that there is an isomorphism as

$$Cl^*(U) \otimes Cl^*(V) \cong Cl^*(U \oplus V).$$

Proof. (Added). Define a map Φ from $Cl^*(U) \otimes Cl^*(V)$ into $Cl^*(U \oplus V)$ by sending the canonical basis as

$$\Phi(e_i \otimes 1) = (e_i, 0) + 1, \quad \Phi(1 \otimes f_j) = (0, f_j) + 1,$$
$$\Phi(e_i \otimes f_j) = (e_i, f_j) + 1, \quad \Phi(1 \otimes 1) = (0, 0) + 1$$

in $U \oplus V \oplus \mathbb{C}1 \subset Cl^*(U \oplus V)$ and by extending by linearity and continuity. Note that

$$\begin{aligned} \Phi(e_i \otimes 1)\Phi(1 \otimes f_j) &= [(e_i, 0) + 1][(0, f_j) + 1] = (e_i, f_j) + 1 \\ &= \Phi(e_i \otimes f_j) = \Phi((e_i \otimes 1)(1 \otimes f_j)). \end{aligned}$$

□

Corollary 6.7. (*Added*). *The Clifford algebra* $Cl^*(\mathbb{R}^{2k})$ *is isomorphic to* $M_{2^k}(\mathbb{C})$, *while the Clifford algebra* $Cl^*(\mathbb{R}^{2k+1})$ *is isomorphic to the direct sum* $M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C})$.

Proof. We obtain

$$\begin{aligned} Cl^*(\mathbb{R}^{2k}) &= Cl^*(\oplus^k \mathbb{R}^2) \cong \otimes^k Cl^*(\mathbb{R}^2) \cong \otimes^k M_2(\mathbb{C}) \cong M_{2^k}(\mathbb{C}), \quad \text{and} \\ Cl^*(\mathbb{R}^{2k+1}) &= Cl^*(\mathbb{R} \oplus \mathbb{R}^{2k}) \cong Cl^*(\mathbb{R}) \otimes Cl^*(\mathbb{R}^{2k}) \\ &\cong \mathbb{C}^2 \otimes M_{2^k}(\mathbb{C}) \cong M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C}). \end{aligned}$$

□

Proposition 6.8. *Let V and W be finite dimensional Euclidean spaces. There is an isomorphism of graded C^*-algebras:*

$$C_0(U, Cl^*(U)) \otimes C_0(V, Cl^*(V)) \cong C_0(U \oplus V, Cl^*(U \oplus V))$$

by sending $f_1 \otimes f_2$ to $f_1 \cdot f_2$ defined by $(f_1 \cdot f_2)(u, v) = f_1(u) \cdot f_2(v)$.

Proof. Note that

$$C_0(U) \otimes C_0(V) \cong C_0(U \oplus V)$$

by sending $g_1 \otimes g_2$ to $g_1 \cdot g_2$ defined by $(g_1 \cdot g_2)(u, v) = g_1(u) g_2(v)$. Note as well that $U \oplus V \cong U \times V$ as a space.

Moreover, for the tensor $f_1 \otimes f_2$, the function $f_1 \cdot f_2$ is defined as above. Set $\Phi(f_1 \otimes f_2) = f_1 \cdot f_2$. Then

$$\begin{aligned} &\Phi((f_1 \otimes f_2)(g_1 \otimes g_2)) = \Phi(f_1 g_1 \otimes f_2 g_2) = f_1 g_1 \cdot f_2 g_2, \\ &\Phi(f_1 \otimes f_2)\Phi(g_1 \otimes g_2) = (f_1 \cdot f_2)(g_1 \cdot g_2) \end{aligned}$$

(possibly in this sense). On the other hand, we obtain

$$\begin{aligned} &C_0(U, Cl^*(U)) \otimes C_0(V, Cl^*(V)) \\ &\cong C_0(U) \otimes Cl^*(U) \otimes C_0(V) \otimes Cl^*(V) \\ &\cong C_0(U) \otimes C_0(V) \otimes Cl^*(U) \otimes Cl^*(V) \\ &\cong C_0(U \oplus V) \otimes Cl^*(U \oplus V) \cong C_0(U \oplus V, Cl^*(U \oplus V)). \end{aligned}$$

□

Let $g : V_1 \to V_2$ be an isometric isomorphism of finite dimensional Euclidean vector spaces. There is a corresponding $*$-isomorphism $g_* : Cl^*(V_1) = \mathbb{C}[V_1] \to Cl^*(V_2) = \mathbb{C}[V_2]$ and also a $*$-isomorphism $g_* : C_0 Cl^*(V_1) \to C_0 Cl^*(V_2)$ defined by $g_*(f)(v) = g_*(f(g^{-1}(v)))$ for $f \in C_0 Cl^*(V_1)$ and $v \in V_2$.

Proposition 6.9. (*Improved*). *Let V be a finite dimensional Euclidean space. The C^*-algebra $C_0Cl^*(V) \otimes C_0Cl^*(V)$ has the rotation property in the sense that the flip automorphism on it is homotopic to $(-\mathrm{id}_V)_* \otimes (\mathrm{id}_V)_* = (-\mathrm{id}_V \oplus \mathrm{id}_V)_*$.*

Proof. Since

$$C_0Cl^*(V) \otimes C_0Cl^*(V) \cong C_0Cl^*(V \oplus V),$$

the flip isomorphism on the tensor product corresponds to the $*$-automorphism τ_* of $C_0Cl^*(V \oplus V)$ associated to the flip map τ which exchanges two copies of V in $V \oplus V$, that is, $\tau(x, y) = (y, x)$. But τ is homotopic through isometric isomorphisms of $V \oplus V$ to the map $(-\mathrm{id}_V) \oplus \mathrm{id}_V$ sending $(x, y) \mapsto (-x, y)$, and thus τ_* is homotopic to $(-\mathrm{id}_V \oplus \mathrm{id}_V)_* = (-\mathrm{id}_V)_* \otimes (\mathrm{id}_V)_*$.

Indeed, note that

$$\tau = \begin{pmatrix} 0 & \mathrm{id}_V \\ \mathrm{id}_V & 0 \end{pmatrix}$$

on $V \oplus V$ with $\tau = \tau^*$ and $\tau^2 = \mathrm{id}_{V \oplus V}$. Define

$$u_\theta = \begin{pmatrix} \cos\theta \,\mathrm{id}_V & -\sin\theta \,\mathrm{id}_V \\ \sin\theta \,\mathrm{id}_V & \cos\theta \,\mathrm{id}_V \end{pmatrix}$$

on $V \oplus V$ with $\theta \in [0, \frac{\pi}{2}]$. It then follows that $u_0\tau = \tau$ and $u_{\frac{\pi}{2}}\tau = (-\mathrm{id}_V) \oplus \mathrm{id}_V$. □

Definition 6.10. Denote by $cl = cl_V : V \to Cl^*(V)$ the (Clifford) **inclusion** map defined by $cl(v) = v$, where V is included as a real linear subspace of self-adjoint elements of $Cl^*(V)$.

The (Clifford) function $cl : V \to Cl^*(V)$ is continuous. Because

$$\|cl(u) - cl(v)\|^2 = \|cl(u - v)\|^2 = \|cl(u - v)^2\| = \|u - v\|^2\|1\|.$$

The function cl does not vanish at infinity. Because for any $\varepsilon > 0$, the set

$$\{v \in V \mid \|cl(v)\| = \|v\| \geq \varepsilon\}$$

is not bounded and so not compact in V. Therefore, $cl \notin C_0Cl^*(V)$. However, if $f \in S = C_0(\mathbb{R})$, then the function $f \circ cl$ defined by

$$f \circ cl = f(cl) : V \to Cl^*(V), \quad (f \circ cl)(v) = f(cl(v))$$

by functional calculus does belong to $C_0Cl^*(V)$. Note that $cl(v)^* = cl(v)$, so that the spectrum $\sigma(cl(v))$ is contained in $\mathbb{R}$, and hence $f(cl(v))$ is defined in $C^*(cl(v))$ generated by $cl(v)$, isomorphic to $C_0(\sigma(cl(v)))$ by Gelfand transform. Also, $f(cl(v))$ is approximated by polynomials with respect to $cl(v)$, so that the function $f(cl)$ is continuous on V. As well, the set

$$\{v \in V \mid \|f(cl(v))\| = \sup_{t \in \sigma(cl(v))} |f(t)| \geq \varepsilon\}$$

is compact in V. Indeed, since $cl(v)^2 = \|v\|^2 1$, we have $\sigma(cl(v)^2) = \{\|v\|^2\}$. Thus, $\sigma(cl(v)) \subset \{\pm\|v\|\}$.

Define a $*$-homomorphism $\beta = \beta_V : S = C_0(\mathbb{R}) \to C_0Cl^*(V)$ by $\beta(f) = f(cl_V)$, which we may call the **Bott** $*$-homomorphism. Check that

$$\begin{aligned}\beta(fg)(v) &= (fg)(cl_V)(v) = (fg)(cl(v)) \\ &= f(cl(v))g(cl(v)) = \beta(f)(v)\beta(g)(v).\end{aligned}$$

Definition 6.11. The **Bott** element b of $E_0(C_0Cl^*(V))$ is the E-theory class of the Bott $*$-homomorphism $\beta_V : S \to C_0Cl^*(V)$ tensored wtih a rank one projection of $\mathbb{K}(H)$.

Remark. The Clifford function cl is an example of an unbounded multiplier of the C^*-algebra $C_0Cl^*(V)$. See the last subsection as unbounded multipliers of Section 7 below.

Example 6.12. If $V = \mathbb{R}$, then we have $cl_{\mathbb{R}}(x) = cl(xe_1) = (x, -x) \in \mathbb{C}^2 = Cl^*(\mathbb{R})$. Hence $\beta_{\mathbb{R}}(f)(x) = f(x, -x)$.

If $V = \mathbb{R}^2$, then we have

$$cl_{\mathbb{R}^2}(x, y) = \begin{pmatrix} 0 & x+iy \\ x-iy & 0 \end{pmatrix} = \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} \in M_2(\mathbb{C}) \cong Cl^*(\mathbb{R}^2)$$

with $\mathbb{R}^2 \cong \mathbb{C}$. Hence

$$\beta_{\mathbb{R}^2}(f)(x, y) = f\begin{pmatrix} 0 & x+iy \\ x-iy & 0 \end{pmatrix} = f\begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}.$$

Note that

$$C^*\begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} \cong \mathbb{C}1_2 \oplus \mathbb{C}\begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}$$

and it seems that any element of it does not vanish at infinity, but it is no problem, with f by functional calculus as checked above.

Theorem 6.13. *For every graded C^*-algebra $\mathfrak{A}$ and every finite-dimensional Euclidean space V, the Bott map:*

$$\beta : E_0(\mathfrak{A}) \to E_0(\mathfrak{A} \otimes C_0Cl^*(V))$$

defined by $\beta(x) = x \otimes_* b$ is an isomorphism of abelian groups.

The theorem is proved in the next section by constructing a suitable asymptotic morphism α and proving that $\alpha_*(b) = 1$.

Remark. If n is even $2k$, then the Clifford algebra $Cl^*(\mathbb{R}^n)$ is isomorphic to $M_{2^k}(C_0(\mathbb{R}^n))$. It follows that if $\mathfrak{A}$ is trivially graded, then

$$E_0(\mathfrak{A} \otimes Cl^*(\mathbb{R}^n)) \cong E_0(\mathfrak{A} \otimes C_0(\mathbb{R}^n))$$

with $n = 2k$. Since

$$E_0(\mathfrak{A} \otimes Cl^*(\mathbb{R}^n)) \cong E_0(\mathfrak{A} \otimes C_0Cl^*(\mathbb{R}^n)),$$

the theorem above implies that with n even,

$$E_0(\mathfrak{A} \otimes C_0(\mathbb{R}^n)) \cong E_0(\mathfrak{A}).$$

7 The Dirac operator and the harmonic oscillator

We may refer to [25] of Higson, Kasparov, and Trout (unchecked in details).

The Dirac operator

Theorem 7.1. *There exists an asymptotic morphism $\alpha : S \otimes C_0Cl^*(V) \rightsquigarrow \mathbb{K}(H)$ such that the induced homomorphism $\alpha_* : E_0(C_0Cl^*(V)) \to E_0(\mathbb{C})$ maps the Bott element $b \in E_0(C_0Cl^*(V))$ to $1 \in E_0(\mathbb{C})$.*

The proof is completed as Corollary 7.21 below.

Definition 7.2. Let V be a finite dimensional Euclidean vector space. We assume that the algebra $Cl^*(V)$ has the Hilbert space structure such that the monomials $e_{i_1} \cdots e_{i_p}$ associated to an orthonormal basis $\{e_j\}$ of V are orthonormal. The Hilbert space structure is indepndent of the choice of the basis $\{e_j\}$ of V. Let $H(V)$ denote the infinite dimensional complex Hilbert space of all square integrable $Cl^*(V)$-valued functions on V, and set

$$H(V) \doteq L^2(V, Cl^*(V)) = L^2Cl^*(V).$$

The Hilbert space $H(V)$ is a graded Hilbert space, with grading inherited from $Cl^*(V)$, as $H(V)_j = L^2(V, Cl^*(V)_j)$ for $j = 0, 1$.

Definition 7.3. Let V be a finite dimensional Euclidean vector space and let $e, f \in V$. Define (**multiplicative**) linear operators on the finite dimensional graded Hilbert space $Cl^*(V)$ by (with notation changed)

$$M_e(x) = e \cdot x \quad \text{and} \quad W_f(x) = (-1)^{\partial x} x \cdot f.$$

Proof. (Added). Let $x, y \in Cl^*(V)$ with $x = x_0 \oplus x_1$ and $y = y_0 \oplus y_1$ wtih $\partial x_j = (-1)^j$ and $\partial y_j = (-1)^j$. Then for α, $\beta \in \mathbb{C}$,

$$\begin{aligned} W_f(\alpha x + \beta y) &= W_f(\alpha x_0 + \beta y_0) + W_f(\alpha x_1 + \beta y_1) \\ &= (\alpha x_0 + \beta y_0) f - (\alpha x_1 + \beta y_1) f \\ &= \alpha(x_0 f + (-1) x_1 f) + \beta(y_0 f + (-1) y_1 f) \\ &= \alpha W_f(x) + \beta W_f(y). \end{aligned}$$

□

Lemma 7.4. (*Added*). *The operator $M_e : Cl^*(V) \to Cl^*(V)$ is self adjoint while the operator $W_f : Cl^*(V) \to Cl^*(V)$ is skew adjoint.*

Proof. For $z = \sum_j z_j e_j \in V$,

$$\begin{aligned} \langle M_z^* x, y \rangle &= \langle x, M_z y \rangle = \langle x, zy \rangle \\ &= \sum_j z_j \langle x, e_j y \rangle = \sum_j z_j \langle e_j x, y \rangle = \langle M_z x, y \rangle, \end{aligned}$$

but we need to prove $\langle x, e_j y \rangle = \langle e_j x, y \rangle$, which seems to be a bit nontrivial (omitted). Also,

$$\begin{aligned} \langle W_z^* x, y \rangle &= \langle x, W_z y \rangle = \langle x, y_0 z - y_1 z \rangle \\ &= \sum_j z_j \langle x, y_0 e_j \rangle - \sum_j z_j \langle x, y_1 e_j \rangle \\ &= -\sum_j z_j \langle x_0 e_j, y \rangle + \sum_j z_j \langle x_1 e_j, y \rangle \\ &= \langle -W_z x, y \rangle, \end{aligned}$$

where the proof between the second and third lines is necessary (omitted).

For instance, if $V = \mathbb{R}$, then $Cl^*(V) \cong \mathbb{C}1 \oplus \mathbb{C}e_1$. Then

$$\begin{aligned} M_{e_1}(x1 + ye_1) &= xe_1 + y\langle e_1, e_1 \rangle 1 = y1 + xe_1 \\ &= \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned}$$

and thus $M_{e_1}^* = M_{e_1}$. Also,

$$\begin{aligned} W_{e_1}(x1 + ye_1) &= xe_1 - y\langle e_1, e_1 \rangle 1 = -y1 + xe_1 \\ &= \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned}$$

and hence, $W_{e_1}^* = -W_{e_1}$. □

Exercise. Let $e_1, \cdots, e_n$ be an orthonormal basis for V. If $1 \leq i_1 < \cdots < i_p \leq n$, then the number operator defined by

$$N = \sum_{i=1}^{n} W_{e_i} M_{e_i}$$

maps the monomial $e_{i_1} \cdots e_{i_p}$ in $Cl^*(V)$ to $(2p-n)e_{i_1} \cdots e_{i_p}$.

Proof. (Added). If $V = \mathbb{R}$, then $Cl^*(V) \cong \mathbb{C}1 \oplus \mathbb{C}e_1$. Then

$$N = W_{e_1} M_{e_1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and hence which maps 1 to $-1 = (2 \cdot 0 - 1)e_0$ and e_1 to $e_1 = (2 \cdot 1 - 1)e_1$ respectively.

In general, we have that if $i \neq i_1, \cdots, i \neq i_p$, then

$$\begin{aligned} W_{e_i} M_{e_i} e_{i_1} \cdots e_{i_p} &= (-1)^{p+1} e_i (e_{i_1} \cdots e_{i_p}) e_i \\ &= (-1)^{p+1} (-1)^p (e_{i_1} \cdots e_{i_p}) e_i^2 \\ &= -e_{i_1} \cdots e_{i_p} \end{aligned}$$

and if $i = i_k$ for some $1 \leq k \leq p$, then

$$\begin{aligned} W_{e_i} M_{e_i} e_{i_1} \cdots e_{i_p} &= (-1)^{p+1} e_i (e_{i_1} \cdots e_{i_p}) e_i \\ &= (-1)^{p+1} (-1)^{p-1} (e_{i_1} \cdots e_i^3 = e_{i_k} \cdots e_{i_p}) \\ &= e_{i_1} \cdots e_{i_p}. \end{aligned}$$

Therefore,

$$N e_{i_1} \cdots e_{i_p} = (n-p)(-1) e_{i_1} \cdots e_{i_p} + p(e_{i_1} \cdots e_{i_p}) = (2p-n) e_{i_1} \cdots e_{i_p}.$$

□

Definition 7.5. Let V be a finite dimensional Euclidean vector space with $n = \dim V$. Denote by $\mathfrak{S}(V) = \mathfrak{S}Cl^*(V)$ the dense subspace of $H(V)$, comprised of Schwartz class $Cl^*(V)$-valued functions, that are rapidly decreasing and infinitely differentiable on V and as for partial derivatives,

$$\lim_{\|v\| \to \infty} \|v\|^m \| \frac{\partial^{|p|}}{\partial v_1^{p_1} \cdots \partial v_n^{p_n}} f(v) \| = 0$$

for any non-negative integers m and $p_1, \cdots, p_n$ with $|p| = p_1 + \cdots + p_n$. The **Dirac** operator of V is a unbounded operator D on $H(V)$ with domain $\mathfrak{S}(V)$, defined by

$$(Df)(v) = \sum_{i=1}^{n} W_{e_i} \frac{\partial}{\partial v_i} f(v)$$

where $e_1, \cdots, e_n$ is an orthonormal basis of V and $v_1, \cdots, v_n$ are the corresponding coordinates on V.

Lemma 7.6. *Let V be a finite dimensional Euclidean vector space. The Dirac operator of V is essentially self-adjoint (possibly in the sense that it is self-adjoint on $\mathfrak{S}(V)$).*

If $f \in S = C_0(\mathbb{R})$, $h \in C_0Cl^*(V)$, and M_h is pointwise multiplication operator on the Hilbert space $H(V) = L^2Cl^*(V)$, then the product $f(D)M_h$ is a compact operator on $H(V)$.

Proof. By definition, it follows that

$$D = \sum_{i=1}^{n} W_{e_i} \frac{\partial}{\partial v_i} = \sum_{i=1}^{n} E_i \frac{\partial}{\partial v_i},$$

with each E_i a skew adjoint matrix, as a constant coefficient. Under the Fourier transform $\mathfrak{F}$ as a unitary isomorphism from $H(V)$ to $H(V^\wedge)$ with $V \cong \mathbb{R}^n$ and $V^\wedge \cong \mathbb{R}^n$ with $(\xi_i)_{i=1}^n$ coordinates, the operator D corresponds to the multiplication operator $D^\wedge = \sqrt{-1}\sum_{i=1}^n E_i M_{\xi_i}$. It then follows that $D^\wedge$ and hence $D = \mathfrak{F}^* \circ D^\wedge \circ \mathfrak{F}$ are essentially self-adjoint. Moreover,

$$\begin{aligned}
(D^\wedge)^2\xi &= (\sqrt{-1}\sum_{i=1}^{n} E_i M_{\xi_i})^2 \xi \\
&= -\sum_{i,j=1}^{n} W_{e_i} M_{\xi_i} W_{e_j} M_{\xi_j} \xi = -\sum_{i,j=1}^{n} \xi_i \xi_j \xi (-1) e_j e_i \\
&= \sum_{i=1}^{n} \xi_i^2 \xi = \|\xi\|^2 \xi
\end{aligned}$$

for any $\xi \in V^\wedge \cong \mathbb{R}^n$. It then follows that for $f(x) = e^{-x^2}$ for $x \in \mathbb{R}$,

$$f(D^\wedge) = e^{-(\mathfrak{D}^\wedge)^2} = e^{-\|\xi\|^2}$$

as a pointwise multiplication. Therefore, the inverse Fourier transform $\mathfrak{F}^* f(D^\wedge) = f(D)$ is the convolution by $e^{-\frac{1}{4}\|v\|^2}$ (up to a constant). It follows that if $h \in C_0Cl^*(V)$ is compactly supported on V, then $f(D)M_h$ is a

Hilbert-Schmidt operator, and is therefore compact. Indeed,

$$\mathfrak{F}(f(D)M_h g) = (\mathfrak{F}f(D))\mathfrak{F}(hg) = f(D^\wedge)\mathfrak{F}(hg)$$

so that

$$\|f(D)M_n g\| = \|f(D^\wedge)\mathfrak{F}(hg)\|.$$

It then follows that the Hilbert-Schmidt norm of $f(D)M_h$ is

$$\sqrt{\sum_{g_j \in H(V)} \|f(D)M_h g_j\|^2} = \sqrt{\sum_{g_j \in H(V)} \|f(D^\wedge)\mathfrak{F}(hg_j)\|^2}$$

where $\{g_j\}$ is an orthonormal basis for $H(V)$.

Since the set of all $f \in S$ such that $f(D)M_h$ is compact for all such h is an ideal of S, while the function e^{-x^2} generates S as an ideal, the lemma is proved. □

Definition 7.7. Let V be a finite dimensional Euclidean space. If $h \in C_0Cl^*(V)$ and if $t \in [1, \infty)$, then define a function $h_t(v) = h(\frac{1}{t}v)$ in $C_0Cl^*(V)$.

Lemma 7.8. *Let V be a finite dimensional Euclidean space with D the Dirac operator. For every $f \in S$ and $h \in C_0Cl^*(V)$, we have*

$$\lim_{t\to\infty} \|[f(\frac{1}{t}D), M_{h_t}]\| = 0,$$

where $M_{h_t} \in \mathbb{B}(H(V))$ and $f(\frac{1}{t}D)$ is defined by the functional calculus of unbounded operators.

Proof. By an approximation argument involving the Stone-Weierstrass theorem, it suffices to consider the case where $f(x) = \frac{1}{x\pm i}$ and h is smooth and compactly supported. We compute the (usual additive) commutator, where the graded case can be done similarly,

$$\begin{aligned}
[\frac{1}{x\pm i}, y] &= \frac{1}{x\pm i}y - y\frac{1}{x\pm i} \\
&= \frac{1}{x\pm i}y(x\pm i)\frac{1}{x\pm i} - \frac{1}{x\pm i}(x\pm i)y\frac{1}{x\pm i} \\
&= \frac{1}{x\pm i}yx\frac{1}{x\pm i} - \frac{1}{x\pm i}xy\frac{1}{x\pm i} \\
&= \frac{1}{x\pm i}[y,x]\frac{1}{x\pm i},
\end{aligned}$$

which implies by functional calculus,

$$[(t^{-1}D \pm iI)^{-1}, M_{h_t}] = (t^{-1}D \pm iI)^{-1}[M_{h_t}, t^{-1}D](t^{-1}D \pm iI)^{-1},$$

which has norm bounded by $t^{-1}\|[M_{h_t}, D]\|$. Note that $|x \pm i| \geq 1$ for any $x \in \mathbb{R}$, so that $|\frac{1}{x \pm i}| \leq 1$ for evert $x \in \mathbb{R}$, and thus $\|(t^{-1}D \pm iI)^{-1}\| \leq 1$. Moreover, the usual commutator $[M_{h_t}, D]$ is

$$\begin{aligned}
[M_{h_t}, D] &= M_{h_t}D - DM_{h_t} \\
&= h(t^{-1}v)\sum_{i=1}^{n} W_{e_i}\frac{\partial}{\partial v_i} - \sum_{i=1}^{n} t^{-1}W_{e_i}\frac{\partial h}{\partial v_i}(t^{-1}v),
\end{aligned}$$

where the second term has norm $O(t^{-1})$ and the first term is compactly supported. □

Proposition 7.9. *There is, up to equivalence, a unique asymptotic morphism (as required)*

$$\alpha_t : S \otimes C_0Cl^*(V) \rightsquigarrow \mathbb{K}(H(V))$$

such that $\alpha_t(f \otimes h) = f(t^{-1}D)M_{h_t}$.

Proof. For $t \in [1, \infty)$, define a linear map $\alpha_t : S \otimes C_0Cl^*(V) \to \mathbb{B}(H(V))$ by $\alpha_t(f \otimes h) = f(t^{-1}D)M_{h_t}$. Lemma 7.8 above shows that α_t defines an asymptotic morphism from $S \otimes C_0Cl^*(V)$ to $\mathbb{B}(H(V))$, and equivalently defines a homomorphism from $S \otimes C_0Cl^*(V)$ to $C_q(I_0, \mathbb{B}(H(V)))$, with universality of the tensor product. Although neither of the operators $f(t^{-1}D)$ nor M_{h_t} are compact, it follows from elementary elliptic operator theory that their product is compact. Thus, the image of the homomorphism is contained in the subalgebra $C_q(I_0, \mathbb{K}(H(V)))$ of $C_q(I_0, \mathbb{B}(H(V)))$. □

Exercise. If $\mathfrak{K}$ is a closed ideal of a C^*-algebra $\mathfrak{A}$, then there is a short exact sequence of asymptotic algebras:

$$0 \to C_q(I_0, \mathfrak{K}) \to C_q(I_0, \mathfrak{A}) \to C_q(I_0, \mathfrak{A}/\mathfrak{K}) \to 0.$$

Proof. (Added). There is a short exact sequence of C^*-algebras:

$$0 \to C^b(I_0, \mathfrak{K}) \to C^b(I_0, \mathfrak{A}) \to C^b(I_0, \mathfrak{A}/\mathfrak{K}) \to 0.$$

We need to show that the following diagram commutes and is short exact on the bottom line:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^b(I_0,\mathfrak{K}) & \longrightarrow & C^b(I_0,\mathfrak{A}) & \longrightarrow & C^b(I_0,\mathfrak{A}/\mathfrak{K}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_q(I_0,\mathfrak{K}) & \longrightarrow & C_q(I_0,\mathfrak{A}) & \longrightarrow & C_q(I_0,\mathfrak{A}/\mathfrak{K}) & \longrightarrow & 0
\end{array}$$

but omitted. □

The harmonic oscillator

Definition 7.10. Let V be a finite dimensional Euclidean vector space. The **Clifford operator** is a unbounded operator C on $H(V)$ with domain $\mathfrak{S}(V)$ the Schwartz space, defined by

$$(Cf)(v) = \sum_{i=1}^{n} v_i M_{e_i} f(v),$$

where v_i are the coordinates on V corresponding to the orthonormal basis e_i of V. The definition of C is independent of the choice of the basis of V.

Lemma 7.11. *(Added). The Clifford operator C is esentially self-adjoint on $H(V)$ with domain $\mathfrak{S}(V)$.*

Thus, if $f \in S$, then $f(C) \in \mathbb{B}(H(V))$ by functional calculus.

Proof. With v_i as a $\mathbb{R}$-valued function, We have

$$\begin{aligned}
\langle C^* f, g\rangle = \langle f, Cg\rangle &= \sum_{i=1}^{n} \langle f, v_i M_{e_i} g\rangle \\
&= \sum_{i=1}^{n} \langle M_{e_i} v_i f, g\rangle = \langle \sum_{i=1}^{n} M_{e_i} v_i f, g\rangle
\end{aligned}$$

and hence

$$(C^* f)(v) = \sum_{i=1}^{n} M_{e_i} v_i f(v) = (Cf)(v).$$

□

Lemma 7.12. *Let V be a finite dimensional Euclidean vector space and let $\beta : S \to C_0 Cl^*(V)$ be the Bott $*$-homomorphism defined by $\beta(f) = f \circ cl$ by* *functional calculus. If $C_0Cl^*(V)$ is represented on the Hilbert space $H(V) = L^2Cl^*(V)$ by pointwise multiplication operators as M_{e_j}, then the composition*

$$S \xrightarrow{\beta} C_0Cl^*(V) \xrightarrow{M} \mathbb{B}(H(V))$$

maps $f \in S$ to $f(C) \in \mathbb{B}(H(V))$.

Proof. (Added). We have

$$\begin{aligned}(M \circ \beta)(f)\xi(v) &= M_{\beta(f)}\xi(v) = \beta(f)\xi(v) = (f \circ cl)\xi(v) \\ &= f(cl(v))\xi(v) = f(v)\xi(v), \quad \text{and} \\ f(C)\xi(v) &= f(\sum_{i=1}^{n} x_i M_{e_i})\xi(v) = f(M_v)\xi(v)\end{aligned}$$

and both of which are equal, with $v = M_v$. □

Definition 7.13. Let V be a finite dimensional Euclidean vector space. Define an unbounded operator $B = B_V = C + D$ on $H(V)$, with domain $\mathfrak{S}(V) = \mathfrak{S}Cl^*(V)$, by

$$(Bf)(v) = (Cf)(v) + (Df)(v),$$

where C is the Clifford operator and D is the Diract operator, and B is called the **Bott-Dirac** (or Clifford-Dirac) operator.

Example 7.14. If $V = \mathbb{R}$, then

$$H(V) = L^2Cl^*(V) = L^2(\mathbb{R}, \mathbb{C}1 \oplus \mathbb{C}e_1) \cong L^2(\mathbb{R}) \oplus L^2(\mathbb{R}),$$

and for $v \in \mathbb{R}$

$$C = vM_{e_1} = v\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix},$$

and

$$D = W_{e_1}\frac{\partial}{\partial v} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial v} = \begin{pmatrix} 0 & -\frac{\partial}{\partial v} \\ \frac{\partial}{\partial v} & 0 \end{pmatrix},$$

and thus

$$B = \begin{pmatrix} 0 & v - \frac{\partial}{\partial v} \\ v + \frac{\partial}{\partial v} & 0 \end{pmatrix}.$$

Observe that $B : \mathfrak{S}(V) \to \mathfrak{S}(V)$, so that B^2 is defined on $\mathfrak{S}(V)$.

Proposition 7.15. *Let V be a finite dimensional Euclidean vector space of dimension n and let $B = C + D$ as above. Within $\mathfrak{S}(V)$, there exists an orthonormal basis for $H(V)$ consisting of eigenvectors for B^2 such that*

(a) their eigenvalues are nonnegative integers, and each eigenvalue has finite multiplicity, and

(b) the zero is an eigenvalue with multiplicity one and corresponds to the exponential eigenfunction $\exp(-\frac{1}{2}\|v\|^2)$ *for* $v \in V$.

Proof. First, let us consider the case where $V = \mathbb{R}$. Then

$$B^2 = \begin{pmatrix} v^2 - \frac{d^2}{dv^2} - \mathrm{id} & 0 \\ 0 & v^2 - \frac{d^2}{dv^2} + \mathrm{id} \end{pmatrix},$$

where $(v - \frac{d}{dv})(v + \frac{d}{dv}) = v^2 - \frac{d}{dv}v + v\frac{d}{dv} - \frac{d^2}{dv^2}$ with

$$-\frac{d}{dv}(vf(v)) + v\frac{d}{dv}f(v) = -f(v).$$

It then suffices to prove that within the Schwartz subspace of $L^2(\mathbb{R})$ there is an orthonormal basis of eigenfunctions for the harmonic oscillator operator $h = v^2 - \frac{d^2}{dv^2}$, for which its eigenvalues are positive integers with multiplicities finite and 1 is an eigenvalue with multiplicity one.

That is a well-known computation and is done as follows. Let $k = v + \frac{d}{dv}$ and $l = v - \frac{d}{dv}$ and let $f_1(v) = e^{-\frac{1}{2}v^2}$ for $v \in \mathbb{R}$. Then

$$h = kl - \mathrm{id} = lk + \mathrm{id}$$

as checked above. We have $kf_1 = vf_1 - vf_1 = 0$, so that $hf_1 = (lk + \mathrm{id})f_1 = f_1$.

Compute that

$$\begin{aligned} hlf &= (v^2 - \frac{d^2}{dv^2})(v - \frac{d}{dv})f \\ &= v^3 f - \frac{d^2}{dv^2}(vf) - v^2\frac{d}{dv}f + \frac{d^3}{dv^3}f \\ &= v^3 f - (2\frac{d}{dv}f + v\frac{d^2}{dv^2}f) - v^2\frac{d}{dv}f + \frac{d^3}{dv^3}f, \\ lhf &= (v - \frac{d}{dv})(v^2 - \frac{d^2}{dv^2})f \\ &= v^3 f - \frac{d}{dv}(v^2 f) - v\frac{d^2}{dv^2} + \frac{d^3}{dv^3}f \\ &= v^3 f - (2vf + v^2\frac{d}{dv}f) - v\frac{d^2}{dv^2}f + \frac{d^3}{dv^3}f \end{aligned}$$

and thus $(hl - lh)f = 2(v - \frac{d}{dv})f = 2lf$, so that $[h, l] = hl - lh = 2l$. It then follows that

$$[h, l^2] = hl^2 - l^2h = hl^2 - lhl + lhl - l^2h$$
$$= [h, l]l + l[h, l] = 2 \cdot 2l^2.$$

Suppose that $[h, l^n] = hl^n - l^nh = 2nl^n$. Then compute

$$[h, l^{n+1}] = hl^{n+1} - l^{n+1}h = hl^{n+1} - l^nhl + l^nhl - l^{n+1}h$$
$$= [h, l^n]l + l^n[h, l] = 2(n+1)l^{n+1}.$$

By induction, $[h, l^n] = 2nl^n$.

If we define $f_{n+1} = l^n f_1$, then

$$hf_{n+1} = hl^n f_1 = (l^nh + 2nl^n)f = (l^n(lk + \mathrm{id}) + 2nl^n)f = (2n+1)f_{n+1}.$$

Therefore, the functions f_n are eigenfunctions of the symmetric operator h with distinct eigenvalues, and thus are orthogonal and nonzero, and they span $L^2(\mathbb{R})$ since each f_{n+1} is a polynomial of degree n times f_1 by induction. So after L^2-normalization we obtain a required basis.

The general case follows from the the purely algebraic calculation to have

$$B^2 = C^2 + D^2 + N = \sum_{i=1}^{n} v_i^2 - \sum_{i=1}^{n} \frac{\partial^2}{\partial v_i^2} + (2p - n)\mathrm{id}$$

on $H_p(V)$, where N is the number operator and $H_p(V)$ denotes the subspace of $H(V)$ comprised of $Cl^*(V)$-valued L^2-functions on V whose values are linear combinations of monomials $e_{i_1} \cdots e_{i_p}$ of degree p, and the set of all of which may be denoted by $Cl_p^*(V)$. It follows from this that an eigenbasis for B^2 may be found by separation of variables v_i.

Indeed,

$$B^2 = (C + D)^2 = C^2 + D^2 + CD + DC.$$

Compute that for $f \in H_p(V) = L^2Cl_p^*(V)$ such as $f = f(v)e_{i_1} \cdots e_{i_p}$,

$$\begin{aligned} CDf(v) &= \sum_{i=1}^{n} v_i M_{e_i}(Df)(v) \\ &= \sum_{i=1}^{n} v_i M_{e_i} \sum_{j=1}^{n} W_{e_j} \frac{\partial}{\partial v_j} f(v) \\ &= \sum_{i=1,j=1}^{n} v_i M_{e_i} W_{e_j} \frac{\partial}{\partial v_j} f(v), \end{aligned}$$

with

$$M_{e_i} W_{e_j} \frac{\partial}{\partial v_j} f(v) = e_i \frac{\partial}{\partial v_j} f(v)(-1)^p e_j = \frac{\partial}{\partial v_j} f(v)(-1)^{2p} e_i e_j,$$

so that

$$CDf(v) = \sum_{i=1}^{n} v_i M_{e_i} W_{e_i} \frac{\partial}{\partial v_j} f(v).$$

On the other hand,

$$\begin{aligned}
DCf(v) &= \sum_{j=1}^{n} W_{e_j} \frac{\partial}{\partial v_j} (Cf)(v) \\
&= \sum_{j=1}^{n} W_{e_j} \frac{\partial}{\partial v_j} (\sum_{i=1}^{n} v_i M_{e_i} f)(v) \\
&= \sum_{j=1}^{n} W_{e_j} M_{e_j} f(v) + \sum_{j=1}^{n} W_{e_j} \sum_{i=1}^{n} v_i \frac{\partial}{\partial v_j} (M_{e_i} f)(v),
\end{aligned}$$

with

$$\begin{aligned}
W_{e_j} v_i \frac{\partial}{\partial v_j} (M_{e_i} f)(v) &= W_{e_j} v_i M_{e_i} \frac{\partial}{\partial v_j} f(v) \\
&= v_i e_i \frac{\partial}{\partial v_j} f(v)(-1)^{p+1} e_j = v_i \frac{\partial}{\partial v_j} f(v)(-1)^{2p+1} e_i e_j,
\end{aligned}$$

so that

$$DCf(v) = Nf(v) + \sum_{j=1}^{n} W_j v_j M_{e_j} f(v).$$

Therefore, $(CD + DC)f(v) = Nf(v) = (2p - n)f(v)$.

Compute as well that for $f \in H_p(V)$

$$\begin{aligned}
(C^2 f)(v) &= \sum_{i=1}^{n} v_i M_{e_i} \sum_{j=1}^{n} v_j M_{e_j} f(v) = \sum_{i,j=1}^{n} v_i v_j M_{e_i} M_{e_j} f(v) \\
&= \sum_{i,j=1}^{n} v_i v_j f(v)(-1)^{2p} e_i e_j = \sum_{i=1}^{n} v_i^2 f(v),
\end{aligned}$$

and also

$$(D^2f)(v) = \sum_{i=1}^{n} W_{e_i}\frac{\partial}{\partial v_i}\sum_{j=1}^{n} W_{e_j}\frac{\partial}{\partial v_j}f(v) = \sum_{i=1}^{n} W_{e_i}\frac{\partial}{\partial v_i}\sum_{j=1}^{n}\frac{\partial}{\partial v_j}f(v)(-1)^p e_j$$
$$= \sum_{i,j=1}^{n}\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}f(v)(-1)^p e_j(-1)^{p+1}e_i = -\sum_{i=1}^{n}\frac{\partial^2}{\partial v_i^2}f(v).$$

For (b), we compute that

$$B^2 e^{-\frac{1}{2}\|v\|^2}1 = (\sum_{i=1}^{n} v_i^2 - \sum_{i=1}^{n}\frac{\partial^2}{\partial v_j^2} + N)e^{-\frac{1}{2}\|v\|^2}1$$
$$= \sum_{i=1}^{n} v_i^2 e^{-\frac{1}{2}\|v\|^2}1 - \sum_{i=1}^{n}\frac{\partial}{\partial v_i}(-v_i e^{-\frac{1}{2}\|v\|^2})1 - ne^{-\frac{1}{2}\|v\|^2}1$$
$$= \sum_{i=1}^{n} v_i^2 e^{-\frac{1}{2}\|v\|^2}1 - \sum_{i=1}^{n} v_i^2 e^{-\frac{1}{2}\|v\|^2}1 + ne^{-\frac{1}{2}\|v\|^2}1 - ne^{-\frac{1}{2}\|v\|^2}1 = 0.$$

□

Corollary 7.16. *Let V be a finite dimensional Euclidean vector space. Let $B = B_V$ be the Bott-Dirac unbounded operator of V on $H(V) = L^2Cl^*(V)$ with domain $\mathfrak{S}(V) = \mathfrak{S}Cl^*(V)$. Then*

(a) B^2 is essentially self-adjoint;

(b) B^2 has compact resolvent;

(c) The kernel of B^2 is 1-dimensional and corresponds to the exponential function $\exp(-\|v\|^2)$.

It follows that the same holds for B as well.

Proof. (Added). For (a), it is shown that all the eigenvalues of B^2 are real and the spectrum $\sigma(B^2)$ of B^2 is contained in $\mathbb{R}$, and hence B is self-adjoint on $\mathfrak{S}(V)$.

For (b), if $\lambda \notin \sigma(B^2)$, then it is shown to be that $(\lambda 1 - B^2)^{-1}$ is compact. Indeed, by functional calculus, the operator may be identified with the function $\frac{1}{\lambda - x}$ on $\sigma(B^2)$ with $x \in \mathbb{R}$. Since $\sigma(B^2)$ is discrete, the claim follows from a fact that compact operators are approximated by finite rank operators in norm.

As well, compute the inner product on $\mathfrak{S}(\mathbb{R}) = \mathfrak{S}Cl^*(\mathbb{R})$ in $H(\mathbb{R}) =$

$\oplus^2 L^2(\mathbb{R})$:

$$\begin{aligned}\langle B\xi,\eta\rangle &= \langle B\begin{pmatrix}\xi_0\\ \xi_1\end{pmatrix}, \begin{pmatrix}\eta_0\\ \eta_1\end{pmatrix}\rangle = \langle\begin{pmatrix}(v-\frac{d}{dv})\xi_1\\ (v+\frac{d}{dv})\xi_0\end{pmatrix}, \begin{pmatrix}\eta_0\\ \eta_1\end{pmatrix}\rangle \\ &= \langle\begin{pmatrix}\xi_1\\ \xi_0\end{pmatrix}, \begin{pmatrix}(v+\frac{d}{dv})\eta_0\\ (v-\frac{d}{dv})\eta_1\end{pmatrix}\rangle = \langle\begin{pmatrix}\xi_0\\ \xi_1\end{pmatrix}, \begin{pmatrix}(v-\frac{d}{dv})\eta_1\\ (v+\frac{d}{dv})\eta_0\end{pmatrix}\rangle = \langle\xi, B\eta\rangle\end{aligned}$$

by integration by parts. Hence B is self-adjoint on $\mathfrak{S}(\mathbb{R})$. More details can be deduced similarly as for B^2, but omitted. □

Theorem 7.17. *Let V be a finite dimensional Euclidean vector space. The composition:*

$$S \xrightarrow{\Delta} S\otimes S \xrightarrow{\mathrm{id}\otimes\beta} S\otimes C_0Cl^*(V) \overset{\alpha}{\rightsquigarrow} \mathbb{K}(H(V))$$

is asymptotically equivalent to the asymptotic morphism $\gamma : S \to \mathbb{K}(H(V))$ defined by $\gamma_t(f) = f(t^{-1}B)$ for $t \geq 1$.

Proof. The idea of the proof is to check the equivalence of the asymptotic morphisms $\alpha_t \circ (\mathrm{id}\otimes\beta)\circ\Delta$ and γ on the generators $u(x) = e^{-x^2}$ and $v(x) = xe^{-x^2}$ of $S = C_0(\mathbb{R})$ (but we should use $u(x) = e^{-|x|}$ and $v(x) = xe^{-|x|}$ for using Δ, and then the story in what follows seems to be modified suitably). We then have $\gamma_t(u) = e^{-t^{-2}B^2}$ and

$$\begin{aligned}(\alpha_t\circ(\mathrm{id}\otimes\beta)\circ\Delta)u &= \alpha_t\circ(\mathrm{id}\otimes\beta)(u\otimes u) = \alpha_t(u\otimes\beta(u)) \\ &= \alpha_t(u\otimes e^{-C^2}) = e^{-t^{-2}D^2}e^{-t^{-2}C^2}\end{aligned}$$

(corrected). Then we need to know that $e^{-sB^2} = e^{-sH}$ is asymptotic to $e^{-sD^2}e^{-sC^2}$ for $0 < s = t^{-1} \leq 1$. For this purpose we invoke the next proposition below. The rest of the proof is to be continued below. □

Proposition 7.18. **(Mehler's formula).** *Let V be a finite dimensional Euclidean space and let C and D be the Clifford and Dirac operators for V. Then the operators D^2, C^2, and $C^2 + D^2$ are essentially self-adjoint on $H(V)$, with domain $\mathfrak{S}(V)$, and if $s > 0$, then*

$$\begin{aligned}e^{-s(C^2+D^2)} &= e^{-\frac{1}{2}s_1(s)C^2}e^{-s_2(s)D^2}e^{-\frac{1}{2}s_1(s)C^2} \\ &= e^{-\frac{1}{2}s_1(s)D^2}e^{-s_2(s)C^2}e^{-\frac{1}{2}s_1(s)D^2},\end{aligned}$$

where

$$s_1(s) = \frac{\cosh(2s)-1}{\sinh(2s)} \quad \textit{and} \quad s_2(s) = \frac{\sinh(2s)}{2}.$$

Proof. See for example [12] (the lacking item not checked).

Note that the second identity follows from the fisrt by taking the Fourier transform on $L^2(\mathbb{R})$, which interchanges the operators C^2 and D^2.

(Added as follows). Compute the inner product for $f \in H_p(V)$, $g \in H_q(V)$ in $H(V) = L^2Cl^*(V)$:

$$\langle C^2 f, g\rangle = \langle \sum_{i=1}^{n} v_i^2 f, g\rangle = \langle f, \sum_{i=1}^{n} v_i^2 g\rangle = \langle f, C^2 g\rangle, \quad \text{and}$$

$$\langle D^2 f, g\rangle = \langle -\sum_{i=1}^{n} \frac{\partial^2}{\partial v_i^2} f, g\rangle = \langle f, -\sum_{i=1}^{n} \frac{\partial^2}{\partial v_i^2} g\rangle = \langle f, D^2 g\rangle,$$

by integration by parts, which also follow from C and D being self-adjoint.

If $V = \mathbb{R}$, then $C^2 = v^2$ and $D^2 = -\frac{d^2}{dv^2}$. But the rest is somewhat non trivial. □

Lemma 7.19. *If X is any unbounded self-adjoint operator, then there are asymptotic equivalences: $e^{-\frac{1}{2}\tau_1(t)X^2} \sim e^{-\frac{1}{2}t^{-2}X^2}$, $e^{-\tau_2(t)X^2} \sim e^{-t^{-2}X^2}$, and*

$$t^{-1}Xe^{-\frac{1}{2}\tau_1(t)X^2} \sim t^{-1}Xe^{-\frac{1}{2}t^{-2}X^2}, \quad t^{-1}Xe^{-\tau_2(t)X^2} \sim t^{-1}Xe^{-t^{-2}X^2},$$

where

$$\tau_1(t) = \frac{\cosh(2t^{-2}) - 1}{\sinh(2t^{-2})} = s_1(t^{-2}) \quad \text{and} \quad \tau_2(t) = \frac{\sinh(2t^{-2})}{2} = s_2(t^{-2}),$$

and the asymptotic equivalence $\sim$ means that the differences between the left and right sides in the equivalence relations above converge to zero in norm as $t \to \infty$.

Proof. By the spectral theorem (or functional calculus) it suffices to consider the same problem with the self-adjoint operator X replaced by a real variable x and the operator norm replaced by the supremum norm on $C_0(\mathbb{R})$. Then the lemma is a simple calculus exercise, based on the Taylor series of $\tau_1(t)$ and $\tau_2(t)$ as $t^{-2} + o(t^{-2})$.

(Added). For convenience, set $\rho_j(s) = \tau_j(s^{-1})$ for $s > 0$. Note that

$$\begin{aligned}\rho_2(s) &= \frac{\sinh(2s^2)}{2} = \frac{e^{2s^2} - e^{-2s^2}}{4} \\ &= \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{2^n s^{2n}}{n!} - \frac{(-1)^n 2^n s^{2n}}{n!}\right) \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{2^{2n+1}s^{2(2n+1)}}{(2n+1)!} = s^2 + o(s^2) \quad (s \to +0),\end{aligned}$$

and also

$$\rho_1(s) = \frac{\cosh(2s^2)-1}{\sinh(2s^2)} = \frac{e^{2s^2}+e^{-2s^2}-2}{e^{2s^2}-e^{-2s^2}}$$
$$= \frac{2+8s^4+o(s^4)-2}{4s^2+o(s^2)} = 2s^2+o(s^2) \quad (s \to +0).$$

Hence, possibly, $\tau_1(t)$ may be replaced with $\frac{1}{2}\tau_1(t)$ to have the same Taylor series as $\tau_2(t)$. □

Lemma 7.20. *If $f, g \in S = C_0(\mathbb{R})$, then*

$$\lim_{t\to\infty} \|[f(t^{-1}C), g(t^{-1}D)]\| = 0,$$

where C and D are the Clifford and Dirac operators for V as $\mathbb{R}^n$.

Proof. For any $f \in S$ fixed (and for f^* as well, probably), the set of all $g \in S$ for which the lemma holds becomes a C^*-subalgebra of $C_0(\mathbb{R})$.

Indeed, for such $g_1, g_2 \in S$ and $\alpha \in \mathbb{C}$, we have

$$\|[f(t^{-1}C), g_1(t^{-1}D) + \alpha g_2(t^{-1}D)]\|$$
$$\leq \|[f(t^{-1}C), g_1(t^{-1}D)]\| + |\alpha|\|[f(t^{-1}C), g_2(t^{-1}D)]\|,$$

and

$$\|[f(t^{-1}C), g_1(t^{-1}D)^*]\| = \|[g_1(t^{-1}D), f(t^{-1}C)^*]\|,$$

and if such g_n converges to $g \in S$, then

$$\|[f(t^{-1}C), g(t^{-1}D)]\| = \|[f(t^{-1}C), (g-g_n)(t^{-1}D) + g_n(t^{-1}D)]\|$$
$$\leq \|[f(t^{-1}C), (g-g_n)(t^{-1}D)]\| + \|[f(t^{-1}C), g_n(t^{-1}D)]\|.$$

By the Stone-Weiesrstrass theorem it suffices to prove the lemma when g is one of the resolvent functions $\frac{1}{x\pm i}$. Furthermore, it suffices to consider the case where f is a smooth function with support compact. In this case the norm $\|[f(t^{-1}C), (t^{-1}D\pm i)^{-1}]\|$ is estimated as follows (corrected). Note that

$$\frac{1}{x\pm i} = \frac{1}{\pm i}\left(\frac{1}{1\mp ix}\right) = \frac{1}{\pm i}\sum_{n=0}^{\infty}\frac{(\pm ix)^n}{n!}$$
$$= \frac{1}{\pm i} + x + o(x) \quad (x \to 0).$$

Hence, if t is large enough, then the norm is close to

$$\begin{aligned}\|[f(t^{-1}C), \frac{1}{\pm i}1 + t^{-1}D]\| &= \|[f(t^{-1}C), t^{-1}D]\| \\ &= t^{-1}\|[f(t^{-1}C), D]\|.\end{aligned}$$

Moreover, since f is smooth, its Taylor expansion at zero is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + o(x) \quad (x \to 0).$$

Therefore, if t is large enough, then the norm $\|[f(t^{-1}C), D]\|$ is close to

$$\|[f(0)1 + f'(0)t^{-1}C, D]\| = t^{-1}f'(0)\|[C, D]\|$$

(corrected), which goes to zero as $t \to \infty$. □

Proof. (For Theorem 7.17 above, continued). Since the number operator N commutes with C^2 and D^2, we obtain

$$e^{-t^{-2}B^2} = e^{-t^{-2}(C^2+D^2+N)} = e^{-t^{-2}(C^2+D^2)}e^{-t^{-2}N}.$$

By Mehler's formula as Proposition 7.18 above,

$$\begin{aligned}e^{-t^{-2}B^2} &= e^{-\frac{1}{2}s_1(t^{-2})C^2}e^{-s_2(t^{-2})D^2}e^{-\frac{1}{2}s_1(t^{-2})C^2}e^{-t^{-2}N} \\ &= e^{-\frac{1}{2}\tau_1(t)C^2}e^{-\tau_2(t)D^2}e^{-\frac{1}{2}\tau_1(t)C^2}e^{-t^{-2}N}.\end{aligned}$$

It follows from Lemma 7.19 that

$$e^{-t^{-2}B^2}s \sim e^{-\frac{1}{2}t^{-2}C^2}e^{-t^{-2}D^2}e^{-\frac{1}{2}t^{-2}C^2}e^{-t^{-2}N}$$

as $t \to \infty$. Since the right hand side above is equal to

$$\begin{aligned}&\{e^{-\frac{1}{2}t^{-2}C^2}e^{-t^{-2}D^2} - e^{-t^{-2}D^2}e^{-\frac{1}{2}t^{-2}C^2} + e^{-t^{-2}D^2}e^{-\frac{1}{2}t^{-2}C^2}\}e^{-\frac{1}{2}t^{-2}C^2}e^{-t^{-2}N} \\ &= [e^{-\frac{1}{2}(t^{-1}C)^2}, e^{-(t^{-1}D)^2}]e^{-\frac{1}{2}t^{-2}C^2}e^{-t^{-2}N} + +e^{-t^{-2}D^2}e^{-t^{-2}C^2}e^{-t^{-2}N},\end{aligned}$$

it follows from Lemma 7.20 that

$$e^{-t^{-2}B^2} \sim e^{-t^{-2}D^2}e^{-t^{-2}C^2}$$

as $t \to \infty$ (corrected), and that similarly,

$$e^{-t^{-2}B^2} \sim e^{-t^{-2}C^2}e^{-t^{-2}D^2}$$

as $t \to \infty$, where $e^{-t^{-2}N}$ converge in norm to the identity operator since the operator N is bounded.

We then have shown that $(\alpha_t \circ (\mathrm{id} \otimes \beta) \circ \Delta)u = e^{-t^{-2}D^2}e^{-t^{-2}C^2}$ and $\gamma_t(u) = e^{-t^2B^2}$ are asymptotic on one another.

A similar computation shows that $(\alpha_t \circ (\mathrm{id} \otimes \beta) \circ \Delta)v$ and $\gamma_t(v)$ are asymptotic to one another, for $v(x) = xe^{-x^2}$.

Indeed, $\gamma_t(v) = t^{-1}Be^{-t^{-2}B^2}$ and

$$\begin{aligned}(\alpha_t \circ (\mathrm{id} \otimes \beta) \circ \Delta)v &= \alpha_t \circ (\mathrm{id} \otimes \beta)(v \otimes v) = \alpha_t(v \otimes \beta(v)) \\ &= \alpha_t(v \otimes Ce^{-C^2}) = t^{-1}De^{-t^{-2}D^2}t^{-1}Ce^{-t^{-2}C^2}.\end{aligned}$$

The rest is omitted, but it seems to be non trivial. □

Corollary 7.21. *The homomorphism $\alpha_* : E_0(C_0Cl^*(V)) \to E_0(\mathbb{C})$ maps the element $b \in E_0(C_0Cl^*(V))$ to the element $1 \in E_0(\mathbb{C})$.*

Proof. The class $\alpha_*(b)$ is represented by the composition $\alpha_t(f \otimes \beta(f))$ with $\beta(f) = f(C) = b$ for some $f \in S$. By Theorem 7.17, this composition is asymptotic to the asymptotic morphism $\gamma_t(f) = f(t^{-1}B)$ as $t \to \infty$. But each map γ_t is a $*$-homomorphism, so that the asymptotic morphism $\gamma = (\gamma_t)$ is homotopic to the $*$-homomorphism $\gamma_1 : f \to f(B)$. Now denote by p the projection onto the kernel of B. Define a homotopy

$$f \mapsto \begin{cases} f(sB), & \text{if } s = t^{-1} \in (0,1], \\ \begin{pmatrix} f(0)p & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } s = 0 \end{cases}$$

proving that $\alpha_*(b) = 1$, where $\mathrm{index}(B) = \dim\ker(B) = \dim p = 1$. □

Proof. (For Theorem 6.13, added). It follows from that there is an E-theory homomorphism

$$\mathrm{id}_* \otimes_* \alpha_* : E_0(\mathfrak{A} \otimes C_0Cl^*(V)) \to E_0(\mathfrak{A} \otimes \mathbb{C})$$

which maps the class $x \otimes_* b$ to $x \otimes 1 = x$. Hence, we obtain $(\mathrm{id}_* \otimes_* \alpha_*) \circ \beta(x) = x$ and as well $\beta \circ (\mathrm{id}_* \otimes_* \alpha_*)(x \otimes_* b) = x \otimes_* b$. It then says that the maps β and $(\mathrm{id}_* \otimes_* \alpha_*)$ are isomorphisms in this sense. □

Unbounded multipliers as an appexdix.

Any C^*-algebra $\mathfrak{A}$ may be regarded as a right Hilbert module over $\mathfrak{A}$. May refer to Lance [39] or [57]. An unbounded (essentially self-adjoint) multiplier of $\mathfrak{A}$ is an essentially self-adjoint operator on the Hilbert module $\mathfrak{A}$ in the following sense (compare [39, Chapter 9]):

Definition 7.22. Let $\mathfrak{A}$ be a C^*-algebra and let E be a Hilbert right $\mathfrak{A}$-module. An essentially self-adjoint operator on E is an $\mathfrak{A}$-linere map T from a dense $\mathfrak{A}$-submodule E_T of E into E such that

(a) $\langle Tv, w\rangle = \langle v, Tw\rangle$ for all $v, w \in E_T$.

(b) The operator $I + T^2$ with $I = \mathrm{id}$ is densely defined and has dense range.

If T is essentially self-adjoint, then the closure T^- of T with the graph as the closure of the graph of T is self-adjoint, and regular in the sense that $T^- \pm iI$ are bijections from the domain of T^- to E, and that the inverses $(T^- \pm iI)^{-1}$ are adjoints of one another.

If T is essentially self-adjoint, then there is a functional calculus $*$-homomorphism φ from $S = C_0(\mathbb{R})$ into the bounded, adjointable operators on E such that $\varphi(\frac{1}{x\pm i}) = (T^- \pm iI)^{-1}$.

In the case where $E = \mathfrak{A}$, if the densely defined operators $(T^- \pm iI)^{-1}$ are given by right multiplication with elements of $\mathfrak{A}$, then the functional calculus homomorphism φ maps S into $\mathfrak{A}$ (acting on $\mathfrak{A}$ as right multiplication operators). If $\mathfrak{A}$ is graded, if the domain D_T of T is graded, and if T has odd grading degree as a map from D_T graded into $\mathfrak{A}$ graded, then φ is a graded $*$-homomorphism.

Example 7.23. If $\mathfrak{A} = S$, then the operator $X : f(x) \to xf(x)$ defined on compact supported functions of S is essentially self-adjoint.

Lemma 7.24. *If X_j are essentially self-adjoint multipliers of C^*-algebras $\mathfrak{A}_j$ for $j = 0, 1$, then the operator $(X_1 \otimes 1) + (1 \otimes X_2)$ with domain $D_{X_1} \otimes D_{X_2}$ (not completed) is an essentially self-adjoint multiplier of $\mathfrak{A}_1 \otimes \mathfrak{A}_2$.*

Example 7.25. Using the lemma above we can define $\Delta : S \to S \otimes S$ by $\Delta(f) = f((X \otimes 1) + (1 \otimes X))$ (in another way without using the method mentioned before).

(This subsection is not well checked, but a point to notice is to treat the domain well).

8 The bivariant E or EE-theory with two variables

May refer to Kasparov [29], [30], and [31] as well as Connes-Higson [11] and Guentner-Higson-Trout [16] as a modification of the Kasparov KK-theory.

The bivariant E or EE-theory groups

Definition 8.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable graded C^*-algebras. We denote by $E(\mathfrak{A}, \mathfrak{B}) = EE(\mathfrak{A}, \mathfrak{B})$ the set of all homotopy classes of asymptotic morphisms from $S \otimes \mathfrak{A} \otimes \mathbb{K}$ to $\mathfrak{B} \otimes \mathbb{K}$. Thus,

$$E(\mathfrak{A}, \mathfrak{B}) = [S \otimes \mathfrak{A} \otimes \mathbb{K} \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}].$$

Example 8.2. Each $*$-homomorphism φ from $\mathfrak{A}$ to $\mathfrak{B}$, or more generally, each $*$-homomorphism φ from $S \otimes \mathfrak{A} \otimes \mathbb{K}$ to $\mathfrak{B} \otimes \mathbb{K}$ determines an element of $E(\mathfrak{A}, \mathfrak{B})$. This class depends only on the homotopy class of φ, and is denoted as $[\varphi] \in E(\mathfrak{A}, \mathfrak{B})$.

An operation of addition for the set $E(\mathfrak{A}, \mathfrak{B})$ is given by direct sum of asymptotic morphisms, and the zero asymptotic morphism provides the zero element for this addition.

Lemma 8.3. *The abelian monoid $E(\mathfrak{A}, \mathfrak{B})$ by addition is an abelian group, and may be called the E or EE-theory group of $\mathfrak{A}$ and $\mathfrak{B}$.*

Proof. Let $\varphi; S \otimes \mathfrak{A} \otimes \mathbb{K}(H) \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}(H)$ be an asymptotic morphism. Define an asymptotic morphism

$$\varphi^{\mathrm{op}} : S \otimes \mathfrak{A} \otimes \mathbb{K}(H) \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}(H^{\mathrm{op}})$$

by $\varphi_t^{\mathrm{op}}(x) = \varphi_t(\alpha(x))$, where α is a grading automorphism. Define an asymptotic morphism

$$\Phi^s : S \otimes S \otimes \mathfrak{A} \otimes \mathbb{K}(H) \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}(H \oplus H^{\mathrm{op}}), \quad \text{where}$$
$$\Phi_t^s(f \otimes x) = f\begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} \begin{pmatrix} \varphi_t(x) & 0 \\ 0 & \varphi_t^{\mathrm{op}}(x) \end{pmatrix}$$

for $f \in S$, $x \in S \otimes \mathfrak{A} \otimes \mathbb{K}(H)$, and $s \in [0,1]$, $t \in [1, \infty)$. By composing Φ^s with the comultiplication $\Delta : S \to S \otimes S$ we obtain a homotopy of asymptotic morphisms

$$\Phi^s \circ (\Delta \otimes \mathrm{id}) : S \otimes \mathfrak{A} \otimes \mathbb{K}(H) \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}(H \oplus H^{\mathrm{op}})$$

connecting $\varphi \oplus \varphi^{\mathrm{op}}$ and the zero asymptotic morphism. □

Remark. The above argument provides another proof that the E and K-theory groups with one variable are in fact groups.

If p is a rank-one projection of $\mathbb{K}(H)$, then by composing asymptotic morphisms φ with the $*$-homomorphism $(\cdot) \otimes p$ from $S \otimes \mathfrak{A}$ to $S \otimes \mathfrak{A} \otimes \mathbb{K}(H)$ by sending $f \otimes a$ to $f \otimes a \otimes p$ we obtain a map of sets or of abelian groups:

$$[S \otimes \mathfrak{A} \otimes \mathbb{K}(H) \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}(H)] \to [S \otimes \mathfrak{A} \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}(H)]$$

with $[\varphi] \mapsto [((\cdot) \otimes p) \circ \varphi]$.

Lemma 8.4. *The above map sending $[\varphi]$ to $[((\cdot)\otimes p)\circ\varphi]$ is a bijection.*

Proof. The inverse is given by taking tensor product with the identity map on $\mathbb{K}(H)$. That is,

$$(\otimes_{\mathbb{K}(H)})_* : [S\otimes\mathfrak{A}\rightsquigarrow\mathfrak{B}\otimes\mathbb{K}(H)]\to[S\otimes\mathfrak{A}\otimes\mathbb{K}(H)\rightsquigarrow\mathfrak{B}\otimes\mathbb{K}(H)]$$

with $[\psi]\mapsto[\psi\otimes\mathrm{id}_{\mathbb{K}(H)}]$. It follows that

$$[((\cdot)\otimes p)\circ(\psi\otimes\mathrm{id}_{\mathbb{K}(H)})]=[\psi]\quad\text{and}\quad[(((\cdot)\otimes p)\circ\varphi)\otimes\mathrm{id}_{\mathbb{K}(H)}]=[\varphi].$$

□

The E-theory groups $E(\mathfrak{A},\mathfrak{B})$ defined above are contravariantly functorial in $\mathfrak{A}$ and covariantly functorial in $\mathfrak{B}$ on the category of graded C^*-algebras, in the sense as follows (added):

If $\rho:\mathfrak{A}_1\to\mathfrak{A}_2$ is a graded $*$-homomorphism, then

$$\rho^*: E(\mathfrak{A}_2,\mathfrak{B})\to E(\mathfrak{A}_1,\mathfrak{B})$$

is induced as $\rho^*[\varphi]=[\varphi\circ\rho]$ and precisely,

$$S\otimes\mathfrak{A}_1\otimes\mathbb{K}(H)\xrightarrow{\mathrm{id}_S\otimes\rho\otimes\mathrm{id}_{\mathbb{K}(H)}}S\otimes\mathfrak{A}_2\otimes\mathbb{K}(H)\overset{\varphi}{\rightsquigarrow}\mathfrak{B}\otimes\mathbb{K}(H).$$

If $\rho:\mathfrak{B}_1\to\mathfrak{B}_2$ is a graded $*$-homomorphism, then

$$\rho_*: E(\mathfrak{A},\mathfrak{B}_1)\to E(\mathfrak{A},\mathfrak{B}_2)$$

is induced as $\rho_*[\varphi]=[\rho\circ\varphi]$ as

$$S\otimes\mathfrak{A}\otimes\mathbb{K}(H)\overset{\varphi}{\rightsquigarrow}\mathfrak{B}_1\otimes\mathbb{K}(H)\xrightarrow{\ \rho\ }\mathfrak{B}_2\otimes\mathbb{K}(H).$$

Proposition 8.5. *The functor $E(\mathbb{C},\mathfrak{B})$ on the category of graded C^*-algebras is naturally isomorphic to $E_0(\mathfrak{B})\cong K_0(\mathfrak{B})$.*

Proof. This follows from Proposition 5.7 and Lemma 8.4.

(Added). Indeed, we have

$$\begin{aligned}E(\mathbb{C},\mathfrak{B})&=[S\otimes\mathbb{C}\otimes\mathbb{K}(H)\rightsquigarrow\mathfrak{B}\otimes\mathbb{K}(H)]\\&\cong[S\otimes\mathbb{C}\rightsquigarrow\mathfrak{B}\otimes\mathbb{K}(H)]\\&\cong[S\rightsquigarrow\mathfrak{B}\otimes\mathbb{K}(H)]\\&\cong[S,\mathfrak{B}\otimes\mathbb{K}(H)]=E_0(\mathfrak{B}).\end{aligned}$$

□

Composition of asymptotic morphisms

As the main feature of the baivariant E-theory, there is a bilinear composition law as

$$E(\mathfrak{A},\mathfrak{B})\times E(\mathfrak{B},\mathfrak{C}) \xrightarrow{\otimes_{\mathfrak{B}}} E(\mathfrak{A},\mathfrak{C})$$

as $\otimes_{\mathfrak{B}}(\alpha,\beta)=\alpha\otimes_{\mathfrak{B}}\beta$, which is associative in the sense that the diagram

$$\begin{tikzcd}
E(\mathfrak{A},\mathfrak{B})\times E(\mathfrak{B},\mathfrak{C})\times E(\mathfrak{C},\mathfrak{D}) \arrow[r, "(\otimes_{\mathfrak{B}},\mathrm{id})"] \arrow[d, "(\mathrm{id},\otimes_{\mathfrak{C}})"'] & E(\mathfrak{A},\mathfrak{C})\times E(\mathfrak{C},\mathfrak{D}) \arrow[d, "\otimes_{\mathfrak{C}}"] \\
E(\mathfrak{A},\mathfrak{B})\times E(\mathfrak{B},\mathfrak{D}) \arrow[r, "\otimes_{\mathfrak{B}}"] & E(\mathfrak{A},\mathfrak{D})
\end{tikzcd}$$

is commutative. We may call the operation the E or EE-theory **product** as the Kasparov KK-theory product in KK-theory. It is obtained below in the subsection: The E-theory category.

As an idea to compute (indirectly) the K-theory groups of a C^*-algebra $\mathfrak{A}$, we may find a C^*-algebra $\mathfrak{B}$ with computable K-theory groups such that there are E-theory elements $\alpha\in E(\mathfrak{A},\mathfrak{B})$ and $\beta\in E(\mathfrak{B},\mathfrak{A})$ such that

$$\alpha\otimes_{\mathfrak{B}}\beta=[\mathrm{id}_{\mathfrak{A}}]\in E(\mathfrak{A},\mathfrak{A}) \quad\text{and}\quad \beta\otimes_{\mathfrak{A}}\alpha=[\mathrm{id}_{\mathfrak{B}}]\in E(\mathfrak{B},\mathfrak{B}).$$

In other words, $\mathfrak{A}$ and $\mathfrak{B}$ are E-theory equivalent. Then

$$K_0(\mathfrak{A})\cong E_0(\mathfrak{A})\cong E(\mathbb{C},\mathfrak{A})\cong E(\mathbb{C},\mathfrak{B})\cong E_0(\mathfrak{B})\cong K_0(\mathfrak{B})$$

in the middle by the E-theory product of either $\otimes_{\mathfrak{A}}\alpha$ or $\otimes_{\mathfrak{B}}\beta$.

May refer to [16] for some details in this subsection.

As did previously, recall that an asymptotic morphism $(\varphi_t):\mathfrak{A}\rightsquigarrow\mathfrak{B}$ defines a $*$-homomorphism

$$\varphi:\mathfrak{A}\to(C_qI_0)(\mathfrak{B})=C_q(I_0,\mathfrak{B})=(C^b/C_0)((0,1],\mathfrak{B}),$$

and two asymptotic morphisms from $\mathfrak{A}$ to $\mathfrak{B}$ define the same $*$-homomorphism from $\mathfrak{A}$ to $C_q(I_0,\mathfrak{B})$ if and only if they are asymptotically equivalent.

The asymptotic algebra construction is a functor as $\mathfrak{B}\mapsto C_q(I_0,\mathfrak{B})$, since a $*$-homomorphism ψ from $\mathfrak{B}$ to $\mathfrak{C}$ induces a $*$-homomorphism from $C_q(I_0,\mathfrak{B})$ to $C_q(I_0,\mathfrak{C})$ by composition, with the following commutative:

$$\begin{tikzcd}
\mathfrak{B} \arrow[r, "\psi"] \arrow[d, "1\otimes(\cdot)"'] & C \arrow[d, "1\otimes(\cdot)"] \\
(C^b/C_0)((0,1],\mathfrak{B}) \arrow[r, "\psi\circ(\cdot)"] & (C^b/C_0)((0,1],\mathfrak{C}).
\end{tikzcd}$$

Definition 8.6. Define the asymptotic functors inductively by $(C_q I_0)^0(\mathfrak{B}) = ((C^b/C_0)I_0)^0(\mathfrak{B}) = \mathfrak{B}$ and

$$(C_q I_0)^n(\mathfrak{B}) = ((C^b/C_0)I_0)^n(\mathfrak{B}) = (C^b/C_0)I_0(((C^b/C_0)I_0)^{n-1}(\mathfrak{B})).$$

Two $*$-homomorphisms $\varphi^0, \varphi^1 : \mathfrak{A} \to (C_q I_0)^n(\mathfrak{B})$ are n-homotopic if there exists a $*$-homomorphism $\Phi : \mathfrak{A} \to (C_q I_0)^n(C([0,1], \mathfrak{B}))$ such that φ^0 and φ^1 are recovered respectively as the compositions:

$$\mathfrak{A} \xrightarrow{\Phi} (C_q I_0)^n(C([0,1], \mathfrak{B})) \xrightarrow{\mathrm{ev}_0, \mathrm{ev}_1} (C_q I_0)^n(\mathfrak{B}),$$

where ev_0 and ev_1 are evaluations at 0 and 1 in $[0,1]$.

Lemma 8.7. ([16]). *The relation of being n-homotopic for the set of all $*$-homomorphisms from $\mathfrak{A}$ to $(C_q I_0)^n(\mathfrak{B})$ is an equivalence relation.*

Definition 8.8. For graded C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, denote by $[\mathfrak{A}, (C_q I_0)^n(\mathfrak{B})]_n$ the set of n-homotopy classes of $*$-homomorphisms from $\mathfrak{A}$ to $(C_q I_0)^n(\mathfrak{B})$.

Example 8.9. Observe that

$$[\mathfrak{A}, (C_q I_0)^0(\mathfrak{B})]_0 = [\mathfrak{A}, \mathfrak{B}], \quad \text{and} \quad [\mathfrak{A}, (C_q I_0)(\mathfrak{B})]_1 = [\mathfrak{A} \rightsquigarrow \mathfrak{B}].$$

Remark. The relation of n-homotopy is not the same as homotopy. Homotopic $*$-homomorphisms from $\mathfrak{A}$ into $(C_q I_0)^n(\mathfrak{B})$ are n-homotopic, but not vice-versa, in general. There is a natural transformation T_1 of functors, from $(C_q I_0)^n(\mathfrak{B})$ to $(C_q I_0)^{n+1}(\mathfrak{B})$, defined by including $(C_q I_0)^n(\mathfrak{B})$ into $(C_q I_0)^{n+1}(\mathfrak{B}) = (C_q I_0)(C_q I_0)^n(\mathfrak{B})$ as constant functions on I_0. There is another natural transformation T_2, defined by including $\mathfrak{B}$ into $C_q I_0(\mathfrak{B})$ as constant functions and then applying the functor $(C_q I_0)^n(\cdot)$ to this inclusion. Both natrual transformations are compatible with homotopy in the sense that:

Lemma 8.10. ([16]). *The above natural transformations T_1 and T_2 define the same map from $[\mathfrak{A}, (C_q I_0)^n(\mathfrak{B})]_n$ to $[\mathfrak{A}, (C_q I_0)^{n+1}(\mathfrak{B})]_{n+1}$.*

Proof. (Added). Note that we have the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\varphi{,}\psi"] \arrow[d, equal] & (C_q I_0)^n(\mathfrak{B}) \arrow[d, "T_1"', "T_2"] \\
\mathfrak{A} \arrow[r, "T_1\circ\varphi{,}T_1\circ\psi", "T_2\circ\varphi{,}T_2\circ\psi"'] & (C_q I_0)^{n+1}(\mathfrak{B}).
\end{tikzcd}$$

It then be proved that if $[\varphi] = [\psi]$, then $[T_1 \circ \varphi] = [T_1 \circ \psi]$ and $[T_2 \circ \varphi] = [T_2 \circ \psi]$, and as well, both of which are equal. $\square$

Definition 8.11. Let $\mathfrak{A}$ and $\mathfrak{B}$ be graded C^*-algebras. We denote by $[\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{B})]_\infty$ the direct limit of the directed system of $[\mathfrak{A}, (C_qI_0)^n(\mathfrak{B})]_n$ with the connecting map in Lemma 8.10 above:

$$[\mathfrak{A}, \mathfrak{B}] \to [\mathfrak{A} \rightsquigarrow \mathfrak{B}] = [\mathfrak{A}, (C_qI_0)(\mathfrak{B})]_1 \to [\mathfrak{A}, (C_qI_0)^2(\mathfrak{B})]_2 \to \cdots [\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{B})]_\infty.$$

We may call the elements of the set $[\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{B})]_\infty$ ∞-homotopy classes.

Proposition 8.12. ([16]). *Let $\varphi : \mathfrak{A} \to (C_qI_0)^n(\mathfrak{B})$ and $\psi : \mathfrak{B} \to (C_qI_0)^m(\mathfrak{C})$ be $*$-homomorphisms. Then the class of the composite $*$-homomorphism:*

$$\mathfrak{A} \xrightarrow{\varphi} (C_qI_0)^n(\mathfrak{B}) \xrightarrow{\psi} (C_qI_0)^n((C_qI_0)^m(\mathfrak{C})) = (C_qI_0)^{n+m}(\mathfrak{C})$$

in the set $[\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{C})]_\infty$ depends only on the class of φ in the direct limit set $[\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{B})]_\infty$ and the class of ψ in the set $[\mathfrak{B}, (C_qI_0)^\infty(\mathfrak{C})]_\infty$.

The composition law:

$$[\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{B})]_\infty \times [\mathfrak{B}, (C_qI_0)^\infty(\mathfrak{C})]_\infty \to [\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{C})]_\infty$$

is so defined and is associative as

$$[\rho[_\infty([\psi]_\infty[\varphi]_\infty) = ([\rho]_\infty[\psi]_\infty)[\varphi]_\infty = [\rho \circ \psi \circ \varphi]_\infty$$

in $[\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{D})]_\infty$, with $[\psi]_\infty[\varphi]_\infty = [\psi \circ \varphi]_\infty$ and $[\rho]_\infty[\psi]_\infty = [\rho \circ \psi]_\infty$, and for $\rho \in \mathfrak{C} \to (C_qI_0)^l(\mathfrak{D})$.

Exercise. The class of the identity $*$-homomorphism from $\mathfrak{A}$ to $\mathfrak{A}$ in $[\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{A})]_\infty$ serves as an identity morphism for the above composition law.

Proof. (Added). Consider the compositions:

$$\begin{array}{ccccc} \mathfrak{A} & \xrightarrow{\mathrm{id}} & \mathfrak{A} & \xrightarrow{\varphi} & (C_qI_0)^n(\mathfrak{B}) \\ \mathfrak{B} & \xrightarrow{\psi} & (C_qI_0)^n(\mathfrak{A}) & \xrightarrow{\mathrm{id}} & (C_qI_0)^n(\mathfrak{A}). \end{array}$$

Since $\varphi \circ \mathrm{id} = \varphi$ and $\mathrm{id} \circ \psi = \psi$ in the sense above, we have $[\varphi]_\infty = [\varphi \circ \mathrm{id}]_\infty = [\varphi]_\infty[\mathrm{id}]_\infty$ in $[\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{B})]_\infty$ and $[\psi]_\infty = [\mathrm{id} \circ \psi]_\infty = [\mathrm{id}]_\infty[\psi]_\infty$ in $[\mathfrak{B}, (C_qI_0)^\infty(\mathfrak{A})]_\infty$. $\square$

Definition 8.13. The **asymptotic** category for graded C^*-algebras is defined to be the category where objects are graded C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, morphisms are classes of the sets $[\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{B})]_\infty$, and identity morphisms are the classes of identity $*$-homomorphisms for $\mathfrak{A}$, and the composition law is given as done above.

Observe that there is a functor from the category of graded C^*-algebras and $*$-homomorphisms into the asymptotic category, which is the identity on objects and which assigns to a $*$-homomorphisms $\varphi : \mathfrak{A} \to \mathfrak{B}$ its class $[\varphi]_\infty$ in $[\mathfrak{A}, (C_q I_0)^\infty(\mathfrak{B})]_\infty$.

Exercise. The K-theory viewed as functors from graded C^*-algeras and $*$-homomorphisms to abelian groups factors through the asymptotic category.

Proof. (Added). This is done as that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}(\mathfrak{A}, \mathfrak{B}) \arrow[r] \arrow[d, "(\cdot)_*"'] & {[\mathfrak{A}, \mathfrak{B}] = [\mathfrak{A}, (C_q I_0)^0(\mathfrak{B})]} \arrow[r] \arrow[d, "{[\cdot]_*}"'] & {[\mathfrak{A}, (C_q I_0)^\infty(\mathfrak{B})]_\infty} \arrow[d, "{([\cdot]_\infty)_*}"] \\
\mathrm{H}(K_0(\mathfrak{A}), K_0(\mathfrak{B})) \arrow[r, equal] & \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{B})) \arrow[r, equal] & \mathrm{H}(K_0(\mathfrak{A}), K_0(\mathfrak{B})),
\end{tikzcd}$$

where $\mathrm{H}(\mathfrak{A}, \mathfrak{B}) = \mathrm{Hom}(\mathfrak{A}, \mathfrak{B})$ and $(\varphi)_* = \varphi_*$ is the induced map on K-theory groups by a $*$-homomorphism $\varphi \in \mathrm{Hom}(\mathfrak{A}, \mathfrak{B})$ and as well, $[\varphi]_*$ and $([\varphi]_\infty)_*$ are induced by the homotopy class $[\varphi]$ and the ∞-homotopy class $[\varphi]_\infty$ respectively. $\square$

Operations

Definition 8.14. Let F be a functor from the category of graded C^*-algebras and $*$-homomorphisms to itself. If $\mathfrak{B}$ is a graded C^*-algebra and $F(f) \in F(C([0,1], \mathfrak{B})) = F(\mathfrak{B}[0,1])$, then define a function $(F \circ \mathrm{ev})(f)$ from $[0,1]$ into $F(\mathfrak{B})$ by $(F \circ \mathrm{ev})(f)(t) = F(\mathrm{ev}_t(f)) = F(\mathrm{ev}_t)F(f)$, where $\mathrm{ev}_t : \mathfrak{B}[0,1] \to \mathfrak{B}$ is the evaluation map at $t \in [0,1]$ and $F(\mathrm{ev}_t) : F(\mathfrak{B}[0,1]) \to F(\mathfrak{B})$ is the induced morphism from ev_t by F as

$$\begin{tikzcd}
C([0,1], \mathfrak{B}) \arrow[r, "\mathrm{ev}_t"] \arrow[d, "F"'] & \mathfrak{B} \arrow[d, "F"] \\
F(\mathfrak{B}[0,1]) \arrow[r, "F(\mathrm{ev}_t)"] & F(\mathfrak{B}),
\end{tikzcd}$$

where note that $F(f)$ may not be unique at all as a function $F(\cdot)$ on $\mathfrak{B}[0,1]$. The functor F is **continuous** if $(F \circ \mathrm{ev})(f)$ is continuous on $[0,1]$ for every $\mathfrak{B}$ and every $F(f) \in F(\mathfrak{B}[0,1])$.

Example 8.15. The maximal and minimal tensor products as functors as $\otimes\mathfrak{B} : \mathfrak{A} \mapsto \mathfrak{A} \otimes \mathfrak{B}$ and $\otimes_{\min}\mathfrak{B} : \mathfrak{A} \to \mathfrak{A} \otimes_{\min} \mathfrak{B}$ are continuous.

Proof. (Added). We have the commutative diagram as

$$\begin{array}{ccc} C([0,1],\mathfrak{A}) & \xrightarrow{\mathrm{ev}_t} & \mathfrak{A} \\ \otimes\mathfrak{B}\big\downarrow & & \big\downarrow\otimes\mathfrak{B} \\ C([0,1],\mathfrak{A})\otimes\mathfrak{B} & \xrightarrow{ev_t\otimes\mathfrak{B}} & \mathfrak{A}\otimes\mathfrak{B}, \end{array}$$

with $C([0,1],\mathfrak{A})\otimes\mathfrak{B}\cong(\mathfrak{A}\otimes\mathfrak{B})[0,1]$, as required. Since $C([0,1])$ is nuclear, its minimal and maximal tensor products coincide. □

Definition 8.16. An operation from the category of graded C^*-algebras to itself, viewed as a functor F, is said to be **exact** if for every short exact sequence of graded C^*-algebras:

$$0\to\mathfrak{J}\to\mathfrak{A}\to\mathfrak{A}/\mathfrak{J}\to 0,$$

there is the induced short exact sequence:

$$0\to F(\mathfrak{J})\to F(\mathfrak{A})\to F(\mathfrak{A}/\mathfrak{J})\to 0.$$

Example 8.17. The operation taking the maximal tensor product for any $\mathfrak{B}$ as $\otimes_{\max}\mathfrak{B}:\mathfrak{A}\to\mathfrak{A}\otimes_{\max}\mathfrak{B}$ is exact as a functor.

In contrast, the operation taking the minimal tensor product for some $\mathfrak{B}$ (corrected) as $\otimes_{\min}\mathfrak{B}:\mathfrak{A}\to\mathfrak{A}\otimes_{\min}\mathfrak{B}$ is not exact as a functor. Note that if $\mathfrak{B}$ is nuclear, by definition, $\mathfrak{A}\otimes_{\min}\mathfrak{B}=\mathfrak{A}\otimes_{\max}\mathfrak{B}$ for every $\mathfrak{A}$, so that $\otimes_{\min}\mathfrak{B}$ is exact. May refer to Simon Wassermann (the lacking item not cited but see [35]).

If F is a continuous functor as mentioned above, then there is a natural transformation:

$$T:F(\mathfrak{B}[0,1])=F(\mathfrak{B}\otimes C([0,1]))\to F(\mathfrak{B})[0,1]=F(\mathfrak{B})\otimes C([0,1])$$

with

$$\begin{array}{ccccc} \mathfrak{B} & \xrightarrow{F} & F(\mathfrak{B}) & \xrightarrow{\otimes C([0,1])} & F(\mathfrak{B})\otimes C([0,1]) \\ \otimes C([0,1])\big\downarrow & & & & \big\uparrow T \\ \mathfrak{B}\otimes C([0,1]) & \xrightarrow{F} & F(\mathfrak{B}\otimes C([0,1])) & = & F(\mathfrak{B}\otimes C([0,1])). \end{array}$$

By the same process, there are also natural transformations:

$T^b:F(C^b([1,\infty),\mathfrak{B})=F(C^bI_0(\mathfrak{B}))\to C^b([1,\infty),F(\mathfrak{B}))=C^bI_0(F(\mathfrak{B}))$ and
$T_0:F(C_0([1,\infty),\mathfrak{B})=F(C_0I_0(\mathfrak{B}))\to C_0([1,\infty),F(\mathfrak{B}))=C_0I_0(F(\mathfrak{B}))$.

In addition, if F is an exact continuous functor, then we obtain an induced map T_q from $F(C_qI_0(\mathfrak{B}))$ to $C_qI_0(F(\mathfrak{B}))$ as $T_q(q(x)) = q'(T^b(x))$, where

$$\begin{CD}
0 @>>> F(C_0I_0(\mathfrak{B})) @>>> F(C^bI_0(\mathfrak{B})) @>{q}>> F(C_qI_0(\mathfrak{B})) @>>> 0 \\
@| @V{T_0}VV @V{T^b}VV @VV{T_q}V @| \\
0 @>>> C_0I_0(F(\mathfrak{B})) @>>> C^bI_0(F(\mathfrak{B})) @>{q'}>> C_qI_0(F(\mathfrak{B})) @>>> 0.
\end{CD}$$

Proposition 8.18. ([16]). *Let F be an exact, continuous functor on the category of graded C^*-algebras and $*$-homomorphisms. To each $*$-homomorphism $\varphi : \mathfrak{A} \to (C_qI_0)^n(\mathfrak{B})$, the composition assigned:*

$$F(\mathfrak{A}) \xrightarrow{F(\varphi)} F((C_qI_0)^n(\mathfrak{B})) \xrightarrow{\cong} (C_qI_0)^nF(\mathfrak{B})$$

defines a functor on the asymptotic category.

Proof. (Added). It says that the following diagram is induced:

$$\begin{CD}
[\mathfrak{A}, (C_qI_0)^n(\mathfrak{B})]_n @>>> [\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{B})]_\infty \\
@V{[F(\cdot)]_n}VV @VV{[F(\cdot)]_\infty}V \\
[F(\mathfrak{A}), (C_qI_0)^n(F(\mathfrak{B}))]_n @>>> [F(\mathfrak{A}), (C_qI_0)^\infty(F(\mathfrak{B}))]_\infty.
\end{CD}$$

□

Applying this to the maximal tensor product functor, we obtain

Proposition 8.19. ([16]). *There is the maximal tensor product functor on the asymptotic category.*

Proof. (Added). It says that the following diagram is induced:

$$\begin{CD}
[\mathfrak{A}, (C_qI_0)^n(\mathfrak{B})]_n @>>> [\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{B})]_\infty \\
@V{[\otimes\mathfrak{C}]_n}VV @VV{[\otimes\mathfrak{C}]_\infty}V \\
[\mathfrak{A}\otimes\mathfrak{C}, (C_qI_0)^n(\mathfrak{B}\otimes\mathfrak{C})]_n @>>> [\mathfrak{A}\otimes\mathfrak{C}, (C_qI_0)^\infty(\mathfrak{B}\otimes\mathfrak{C})]_\infty.
\end{CD}$$

□

Definition 8.20. The **amplified asymptotic** category for graded C^*-algebras is the category where objects are graded C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ and morphisms from $\mathfrak{A}$ to $\mathfrak{B}$ are the elements of $[S\otimes\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{B})]_\infty$. The composition of morphisms $[\varphi]_\infty : \mathfrak{A} \to \mathfrak{B}$ and $[\psi]_\infty : \mathfrak{B} \to \mathfrak{C}$ is given by the following composition of morphisms in the asymptotic category:

$$S\otimes\mathfrak{A} \xrightarrow{[\Delta\otimes\mathrm{id}]_\infty} S\otimes S\otimes\mathfrak{A} \xrightarrow{[\mathrm{id}\otimes\varphi]_\infty} S\otimes\mathfrak{B} \xrightarrow{[\psi]_\infty} \mathfrak{C}.$$

The E-theory category

As the main technical theorem in E-theory,

Theorem 8.21. ([16]). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be graded C^*-algebras and assume that $\mathfrak{A}$ is separable. Then the natural map from $[\mathfrak{A}, (C_q I_0)(\mathfrak{B})]_1$ to $[\mathfrak{A}, (C_q I_0)^\infty(\mathfrak{B})]_\infty$ is a bijection.*

Thus, every morphism from $\mathfrak{A}$ to $\mathfrak{B}$ in the asymptotic category is represented by a unique homotopy class of asymptotic morphisms from $\mathfrak{A}$ to $\mathfrak{B}$.

It follows from Definition 8.1 and Theorem 8.21 that the E or EE-theory group may be identified with the set of morphisms in the amplified asymptotic category from $\mathfrak{A} \otimes \mathbb{K}(H)$ to $\mathfrak{B} \otimes \mathbb{K}(H)$, as

$$E(\mathfrak{A}, \mathfrak{B}) \cong [S \otimes \mathfrak{A} \otimes \mathbb{K}(H), (C_q I_0)(\mathfrak{B} \otimes \mathbb{K}(H))]_1.$$

Consequently, we obtain a pairing

$$E(\mathfrak{A}, \mathfrak{B}) \times E(\mathfrak{B}, \mathfrak{C}) \xrightarrow{\otimes} E(\mathfrak{A}, \mathfrak{C})$$

by the composition law in the asymptotic category from $\mathfrak{A} \otimes \mathbb{K}(H)$ to $\mathfrak{B} \otimes \mathbb{K}(H)$ to $\mathfrak{C} \otimes \mathbb{K}(H)$, which may be called the E or EE-theory **product**.

Theorem 8.22. *The E or EE-theory groups $E(\mathfrak{A}, \mathfrak{B})$ become the groups of morphisms in an additive category $\mathfrak{E}$ with objects separable graded C^*-algebras, called the E-theory category.*

There is a functor from the homotopy category of separable graded C^*-algebra and graded $*$-homomorphisms into the category $\mathfrak{E}$, which is the identity on objects.

Remark. If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a $*$-homomorphism and if $\psi : \mathfrak{B} \rightsquigarrow \mathfrak{C}$ is an asymptotic morphism, then the E-theory classes $[\varphi] \in E(\mathfrak{A}, \mathfrak{B})$ and $[\psi] \in E(\mathfrak{B}, \mathfrak{C})$ are defined. In addition, an asymptotic morphism from $\mathfrak{A}$ to $\mathfrak{C}$ is defined as the composition that may be denoted by $\psi \circ \varphi$, and so $[\psi \circ \varphi] \in E(\mathfrak{A}, \mathfrak{C})$ with $[\psi \circ \varphi] = [\varphi] \otimes [\psi]$ as an E-theory product. The same also holds for compositions of φ as an asymptotic morphism and ψ as a $*$-homomorphism and for the same compositions in the amplified category.

The tensor product functor on the asymptotic category for graded C^*-algebras extends to the amplified asymptotic category as:

$$\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{[\varphi]_\infty} & \mathfrak{B} \\
{\scriptstyle \otimes\mathfrak{C}}\downarrow & & \downarrow{\scriptstyle \otimes\mathfrak{C}} \\
\mathfrak{A} \otimes \mathfrak{C} & \xrightarrow{[\varphi \otimes \mathrm{id}]_\infty = [\varphi]_\infty \otimes \mathfrak{C}} & \mathfrak{B} \otimes \mathfrak{C}
\end{array}$$

for any $[\varphi]_\infty \in [S \otimes \mathfrak{A}, (C_q I_0)^\infty(\mathfrak{B})]_\infty$, so that

$$[\varphi \otimes \mathrm{id}]_\infty = [\varphi]_\infty \otimes \mathfrak{C} \in [S \otimes \mathfrak{A} \otimes \mathfrak{C}, (C_q I_0)^\infty(\mathfrak{B} \otimes \mathfrak{C})]_\infty.$$

Theorem 8.23. *There is a functorial maximal tensor product on the E-theory category* $\mathfrak{E}$ *which is compatible with the maximal tensor product on* C^**-algebras via the functor from the category of separable graded* C^**-algebras and graded* $*$*-homomorphisms into the E-theory category.*

Proof. (Added). It says that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{[\varphi]} & \mathfrak{B} \\ \big\downarrow \otimes\mathfrak{C} & & \big\downarrow \otimes\mathfrak{C} \\ \mathfrak{A} \otimes \mathfrak{C} & \xrightarrow{[\varphi\otimes\mathrm{id}]=[\varphi]\otimes\mathfrak{C}} & \mathfrak{B} \otimes \mathfrak{C} \end{array}$$

for $[\varphi] \in E(\mathfrak{A}, \mathfrak{B})$, so that

$$[\varphi \otimes \mathrm{id}] = [\varphi] \otimes \mathfrak{C} \in E(\mathfrak{A} \otimes \mathfrak{C}, \mathfrak{B} \otimes \mathfrak{C}).$$

□

The minimal tensor product does not carry over to the E-theory category.

Definition 8.24. A graded C^*-algebra $\mathfrak{B}$ is **exact** if for every short exact sequence of graded C^*-algebras: $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{J} \to 0$, the sequence of minimal tensor products:

$$0 \to \mathfrak{J} \otimes_{\min} \mathfrak{B} \to \mathfrak{A} \otimes_{\min} \mathfrak{B} \to (\mathfrak{A}/\mathfrak{J}) \otimes_{\min} \mathfrak{B} \to 0$$

is exact.

In other words, $\mathfrak{B}$ is exact if and only if the minimal tensor product functor for graded C^*-algebaras: $\otimes_{\min}\mathfrak{B} : \mathfrak{A} \mapsto \mathfrak{A} \otimes_{\min} \mathfrak{B}$ is exact.

Theorem 8.25. *Let* $\mathfrak{B}$ *be an exact, separable graded* C^**-algebra. Then there is a minimal tensor product functor* $\otimes_{\min}\mathfrak{B}$ *on the E-theory category* $\mathfrak{E}$.

In particular, if $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *are isomorphic in the E-theory category, then* $\mathfrak{A}_1 \otimes_{\min} \mathfrak{B}$ *and* $\mathfrak{A}_2 \otimes_{\min} \mathfrak{B}$ *are isomorphic there too.*

Proof. (Added). It follows that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{A}_1 & \xrightarrow{[\varphi]} & \mathfrak{A}_2 \\ \big\downarrow \otimes\mathfrak{B} & & \big\downarrow \otimes\mathfrak{B} \\ \mathfrak{A}_1 \otimes \mathfrak{B} & \xrightarrow{[\varphi\otimes\mathrm{id}]=[\varphi]\otimes\mathfrak{B}} & \mathfrak{A}_2 \otimes \mathfrak{B} \end{array}$$

for $[\varphi] \in E(\mathfrak{A}_1, \mathfrak{A}_2)$, so that

$$[\varphi \otimes \mathrm{id}] = [\varphi] \otimes \mathfrak{B} \in E(\mathfrak{A}_1 \otimes \mathfrak{B}, \mathfrak{A}_2 \otimes \mathfrak{B}).$$

If the E-theory class $[\varphi]$ is an E or EE-theory equivalence in the same sense as the KK-theory equivalence, then so is $[\varphi \otimes \mathrm{id}] = [\varphi] \otimes \mathfrak{B}$. □

A return to Bott periodicity

Definition 8.26. Let V be a finite dimensional Euclidean vector space. Denote by $[\beta] \in E(\mathbb{C}, C_0 Cl^*(V))$ the E-theory class of the $*$-homomorphism $\beta : S \to C_0 Cl^*(V)$ defined by $\beta(f) = f(cl)$, with $cl : V \to Cl^*(V)$ the inclusion map. Denote by $[\alpha] \in E(C_0 Cl^*(V), \mathbb{C})$ the E-theory class of the asymptotic morphism $\alpha : S \otimes C_0 Cl^*(V) \rightsquigarrow \mathbb{K}(H(V))$ defined as $\alpha_t(f \otimes h) = f(t^{-1}D)M_{h_t}$.

Proposition 8.27. *The following composition in the E-theory category* $\mathfrak{E}$:

$$\mathbb{C} \xrightarrow{[\beta]} C_0 Cl^*(V) \xrightarrow{[\alpha]} \mathbb{C}$$

is the identity morphism from $\mathbb{C}$ *to* $\mathbb{C}$.

Proof. This follows from Remark after Theorem 8.22 and Theorem 7.17, as in the proof of Corollary 7.21.

(Added). Note that $[\beta] \otimes [\alpha] = [\alpha \circ \beta] \in E(\mathbb{C}, \mathbb{C})$ and that $E(\mathbb{C}, \mathbb{C}) \cong E_0(\mathbb{C}) \cong \mathbb{Z}$. It is proved that the composition of β and α induces the trivial E-theory class. □

As the basic result in E-theory,

Theorem 8.28. *The morphisms* $[\alpha] : C_0 Cl^*(V) \to \mathbb{C}$ *and* $[\beta] : \mathbb{C} \to C_0 Cl^*(V)$ *in the E-theory category are mutual inverses.*

Proof. A small variation of the rotation argument mentioned before proves the statement.

It is shown that $[\alpha]$ is right invertible as well, and so is invertible with $[\beta]$ as inverse. □

In other words, $\mathbb{C}$ and $C_0 Cl^*(V)$ are E or EE-theory equivalent.

Excision as the six term exact sequence in E-theory

Definition 8.29. Let $\mathfrak{A}$ be a C^*-algebra. The suspension of $\mathfrak{A}$ is the C^*-algebra $S\mathfrak{A} = C_0(\mathbb{R}) \otimes \mathfrak{A} \cong C_0(\mathbb{R}, \mathfrak{A})$ with isomorphisms:

$$S\mathfrak{A} \cong \{f \in \mathfrak{A}[0,1] \mid f(0) = f(1) = 0\} \cong C_0((0,1)) \otimes \mathfrak{A} = C_0((0,1), \mathfrak{A}) = \mathfrak{A}(0,1).$$

If $\mathfrak{A}$ is graded, then so is $S\mathfrak{A}$, where $C_0(0,1)$ is given the trivial grading.

Theorem 8.30. *The suspension map*

$$(\otimes S, \otimes S)_* : E(\mathfrak{A}, \mathfrak{B}) \to E(S\mathfrak{A}, S\mathfrak{B})$$

defined as sending $[\varphi]$ *to* $[\mathrm{id}_S \otimes \varphi]$ *is an isomorphism. Moreover, there are natural isomorphisms, with* $S^2 = S(S\mathbb{C})$,

$$E(\mathfrak{A}, \mathfrak{B}) \cong E(S^2\mathfrak{A}, \mathfrak{B}) \quad \text{and} \quad E(\mathfrak{A}, \mathfrak{B}) \cong E(\mathfrak{A}, S^2\mathfrak{B}).$$

Proof. It follows from Bott periodicity that S^2 is isomorphic to $\mathbb{C}$ in the E-theory category, and this proves the second part of the theorem.

(Added). Note that $C_0Cl^*(\mathbb{R}^2) \cong C_0(\mathbb{R}^2) \otimes M_2(\mathbb{C}) \cong M_2(S^2)$, and that the stability of EE-theory groups implies that $S^2\mathfrak{A}$ is isomorphic to $\mathfrak{A}$ in the E-theory category with $[\alpha] \in E(S^2\mathfrak{A}, \mathfrak{A})$ and $[\beta] \in E(\mathfrak{A}, S^2\mathfrak{A})$ the E-theory equivalences, so that the fist of the second claim follows by the E-theory products from the left as

$$\otimes[\alpha] : E(\mathfrak{A}, \mathfrak{B}) \to E(S^2\mathfrak{A}, \mathfrak{B}) \quad \text{and} \quad \otimes [\beta] : E(S^2\mathfrak{A}, \mathfrak{B}) \to E(\mathfrak{A}, \mathfrak{B})$$

are shown to be isomorphisms by E-theory products. Similarly, the second of the second claim follows by the E-theory products from the right for $\mathfrak{B}$.

For the first claim, we obtain by applying the suspension map three times

$$E(\mathfrak{A}, \mathfrak{B}) \to E(S\mathfrak{A}, S\mathfrak{B}) \to E(S^2\mathfrak{A}, S^2\mathfrak{B}) \cong E(\mathfrak{A}, \mathfrak{B}) \to E(S\mathfrak{A}, S\mathfrak{B})$$

by the periodicity obtained above. It is done by that the suspension map is injective by definition. $\square$

Theorem 8.31. ([16]). *Let* $\mathfrak{B}$ *be a graded* C^**-algebra and let* $\mathfrak{J}$ *be a closed ideal of a separable* C^**-algebra* $\mathfrak{A}$. *Then there are functorial six-term exact sequences in E-theory:*

$$\begin{tikzcd}
E(\mathfrak{A}/\mathfrak{J}, \mathfrak{B}) \arrow[r] & E(\mathfrak{A}, \mathfrak{B}) \arrow[r] & E(\mathfrak{J}, \mathfrak{B}) \arrow[d] \\
E(\mathfrak{J}, S\mathfrak{B}) \arrow[u] & E(\mathfrak{A}, S\mathfrak{B}) \arrow[l] & E(\mathfrak{A}/\mathfrak{J}, S\mathfrak{B}) \arrow[l]
\end{tikzcd}$$

and

$$\begin{tikzcd}
E(\mathfrak{B}, \mathfrak{J}) \arrow[r] & E(\mathfrak{B}, \mathfrak{A}) \arrow[r] & E(\mathfrak{B}, \mathfrak{A}/\mathfrak{J}) \arrow[d] \\
E(\mathfrak{B}, S(\mathfrak{A}/\mathfrak{J})) \arrow[u] & E(\mathfrak{B}, S\mathfrak{A}) \arrow[l] & E(\mathfrak{B}, S\mathfrak{J}). \arrow[l]
\end{tikzcd}$$

Proof. The second diagram only is proved below. See [16] for more details. The proof has two parts. The first is a construction borrowed from elementary homotopy theory as follows. (To be continued). □

Definition 8.32. Let $\pi : \mathfrak{B} \to \mathfrak{C}$ be a $*$-homomorphism of graded C^*-algebras. The **mapping cone** of π is defined to be the C^*-algebra as

$$C_\pi = \{(b, f) \in \mathfrak{B} \oplus \mathfrak{C}[0,1] \mid \pi(b) = f(0) \text{ and } f(1) = 0\}$$

with the pull back diagram:

$$\begin{array}{ccccc} C_\pi = \mathfrak{B} \oplus_\pi \mathfrak{C}[0,1] & \xrightarrow{p_2} & \mathfrak{C}[0,1] & \xrightarrow{\mathrm{ev}_1} & 0 \in \mathfrak{C} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} & & \\ \mathfrak{B} & \xrightarrow{\pi} & \mathfrak{C}. & & \end{array}$$

Proposition 8.33. *Let $\pi : \mathfrak{B} \to \mathfrak{C}$ be a $*$-homomorphism of graded C^*-algebras. For every C^*-algebra $\mathfrak{A}$, there is a long exact sequence of sets of asymptotic morphisms*:

$$\begin{aligned} \cdots \to [\mathfrak{A} \rightsquigarrow S\mathfrak{C}_\pi] \to [\mathfrak{A} \rightsquigarrow S\mathfrak{B}] \to [\mathfrak{A} \rightsquigarrow S\mathfrak{C}] \\ \to [\mathfrak{A} \rightsquigarrow \mathfrak{C}_\pi] \to [\mathfrak{A} \rightsquigarrow \mathfrak{B}] \to [\mathfrak{A} \rightsquigarrow \mathfrak{C}]. \end{aligned}$$

Proof. As discussed above before, the proof for homotopy classes of asymptotic morphisms may be replaced with the proof for homotopy classes of ordinary $*$-homomorphisms. There is a long sequence of $*$-homomorphisms:

$$\cdots \to SC_\pi \to S\mathfrak{B} \to S\mathfrak{C} = \mathfrak{C}(0,1) \xrightarrow{i} C_\pi \xrightarrow{p_1} \mathfrak{B} \xrightarrow{\pi} \mathfrak{C},$$

where i is the canonical inclusion map, such that the composition of any two successive $*$-homomorphisms in the sequence is null homotopic, so that the composition of any two successive maps of the sequence in the proposition above is trivial. Note that the cone $\mathfrak{C}[0,1)$ is contractible.

Let us prove exactness at the $[\mathfrak{A} \rightsquigarrow \mathfrak{B}]$ term. If the composition

$$\pi \circ \varphi : \mathfrak{A} \overset{\varphi}{\rightsquigarrow} \mathfrak{B} \xrightarrow{\pi} \mathfrak{C}$$

is null homotopic, then the corresponding null homotopy gives an asymptotic morphism $\Phi : \mathfrak{A} \rightsquigarrow \mathfrak{C}[0,1)$. The pair comprised of φ and Φ determines an asymptotic morphism from $\mathfrak{A}$ into C_π, as required. Hence, if the homotoy class $\pi_*[\varphi] = [\varphi \circ \pi]$ is zero, then the class $[\varphi \oplus \Phi]$ in $[\mathfrak{A} \rightsquigarrow C_\pi]$ is mapped to $[\varphi]$. For more details, see [16]. □

Corollary 8.34. *Let $\pi : \mathfrak{B} \to \mathfrak{C}$ be a $*$-homomorphism of graded C^*-algebras. For every C^*-algebra $\mathfrak{A}$, there is a functorial six-term exact sequence:*

$$\begin{array}{ccccc} E(\mathfrak{A}, C_\pi) & \longrightarrow & E(\mathfrak{A}, \mathfrak{B}) & \longrightarrow & E(\mathfrak{A}, \mathfrak{C}) \\ \uparrow & & & & \downarrow \\ E(\mathfrak{A}, S\mathfrak{C}) & \longleftarrow & E(\mathfrak{A}, S\mathfrak{B}) & \longleftarrow & E(\mathfrak{A}, SC_\pi). \end{array}$$

Proof. This follows from Proposition 8.33 above and Theorem 8.30 as the double suspension stability for E or EE-theory groups. □

Proof. (Continued). To prove Theorem 8.31 above it remains to replace C_π with $\mathfrak{J}$ in Corollary 8.34 above, in the case where $\pi : \mathfrak{B} \to \mathfrak{C}$ is a surjective $*$-homomorphism with kernel $\mathfrak{J}$. To this end, observe that there is an inclusion map of $\mathfrak{J}$ into $\mathfrak{B}$ and so into C_π. Using the following construction one can show that this inclusion map induces an isomorphism in the E-theory category. (Further to be continued). □

Theorem 8.35. ([16]). *Let $\mathfrak{J}$ be a closed ideal of a separable graded C^*-algebra $\mathfrak{A}$. Then there is a norm-continuous family $\{u_t\}_{t\in[1,\infty)}$ of degree zero elements of $\mathfrak{J}$ such that (a) $0 \leq u_t \leq 1$ for all t, (b) $\lim_{t\to\infty} \|u_t x - x\| = 0$ for all $x \in \mathfrak{J}$, and (c) $\lim_{t\to\infty} \|u_t a - a u_t\| = 0$ for all $a \in \mathfrak{A}$.*

If $s : \mathfrak{A}/\mathfrak{J} \to \mathfrak{A}$ is any set-theoretic section for the quotient map from $\mathfrak{A}$ to $\mathfrak{A}/\mathfrak{J}$, then an asymptotic morphism $\varphi = (\varphi_t)$ from $S(\mathfrak{A}/\mathfrak{J})$ to $\mathfrak{J}$ is defined by $\varphi_t(f \otimes y) = f(u_t)s(y)$.

Theorem 8.36. ([16]). *Let $\mathfrak{J}$ be a closed ideal of a separable graded C^*-algebra $\mathfrak{A}$. The asymptotic morphism associated as above to the extension :*

$$0 \to S\mathfrak{J} \to \mathfrak{A}[0,1) \xrightarrow{(\mathrm{id}_{\mathfrak{A}}, q\otimes \mathrm{id})} C_\pi \to 0,$$

where $C_\pi = \mathfrak{A} \oplus_q (\mathfrak{A}/\mathfrak{J})[0,1)$ with $q : \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$ the quotient map, determines an element of $E(SC_\pi, S\mathfrak{J})$ which is inverse to the element of $E(S\mathfrak{J}, SC_\pi)$ which is determined by the inclusion map of $\mathfrak{J}$ into C_π by (id, 0).

Proof. (Continued). To complete the proof of Theorem 8.31, it then follows from $SC_\pi \cong S\mathfrak{J}$ in the E-theory category as obained above that $\mathfrak{J} \cong C_\pi$ (corrected) in the E-theory category. □

9 The equivariant EE-theory with group actions

To keep matters as simple as possible we assume that groups in what follows are countable and discrete, but may consider arbitrary second countable, locally compact groups instead.

Definition 9.1. Let G be a countable discrete group and let $\mathfrak{A}$ and $\mathfrak{B}$ be graded G-C^*-algebras, that are graded C^*-algebras with actions of G by grading preserving $*$-automorphisms. A **G-equivariant** asymptotic morphism from $\mathfrak{A}$ to $\mathfrak{B}$ is an asymptotic morphism $\varphi = (\varphi_t) : \mathfrak{A} \rightsquigarrow \mathfrak{B}$ such that

$$\lim_{t\to\infty} \|\varphi_t(g \cdot a) - g \cdot \varphi_t(a)\| = 0$$

for all $a \in \mathfrak{A}$ and $g \in G$, so that the noncommutative diagram:

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "\varphi_t"] \arrow[d, "G"'] & \mathfrak{B} \arrow[d, "G"] \\ \mathfrak{A} \arrow[r, "\varphi_t"] & \mathfrak{B} \end{tikzcd}$$

commutes asymptotically as $t \to \infty$.

Definition 9.2. We denote by $[\mathfrak{A} \rightsquigarrow \mathfrak{B}]^G$ the set of all homotopy classes of G-equivariant asymptotic morphisms from $\mathfrak{A}$ to $\mathfrak{B}$.

If $\mathfrak{B}$ is a G-C^*-algebra, then so is the asymptotic algebra $(C_qI_0)(\mathfrak{B})$.

A G-equivariant asymptotic morphism φ from $\mathfrak{A}$ to $\mathfrak{B}$ is the same thing up to equivalence, as an equivariant $*$-homomorphism ψ from $\mathfrak{A}$ to $(C_qI_0)(\mathfrak{B})$ as the diagram

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "\psi"] \arrow[d, "G"'] & (C_qI_0)(\mathfrak{B}) \arrow[d, "G"] \\ \mathfrak{A} \arrow[r, "\psi"] & (C_qI_0)(\mathfrak{B}) \end{tikzcd}$$

commutes, where if G is not discrete, then the action of G on $(C_qI_0)(\mathfrak{B})$ is not necessarily continuous.

Thanks to this observation it is a straightforward matter to define an equivariant version of the asymptotic category that we constructed previously.

The higher asymptotic algebras $(C_qI_0)^n(\mathfrak{B})$ are also G-C^*-algebras.

Definition 9.3. (Edited). We define $[\mathfrak{A}, (C_qI_0)^n(\mathfrak{B})]_n^G$ to be the set of n-homotopy classes of G-equivariant $*$-homomorphisms from $\mathfrak{A}$ to $(C_qI_0)^n(\mathfrak{B})$.

Define $[\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{B})]_\infty^G$ to be the direct limit of $[\mathfrak{A}, (C_qI_0)^n(\mathfrak{B})]_n^G$.

These are the morphism sets of a category by the composition law as

$$[\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{B})]^G_\infty \times [\mathfrak{B}, (C_qI_0)^\infty(\mathfrak{C})]^G_\infty \to [\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{C})]^G_\infty.$$

This category may be amplified as done before.

Theorem 9.4. ([16]). (Edited). *If $\mathfrak{A}$ is separable and G is discrete, then the canonical map* (1 *to* ∞) *gives an isomorphism*

$$[\mathfrak{A} \rightsquigarrow \mathfrak{B}]^G \cong [\mathfrak{A}, (C_qI_0)(\mathfrak{B})]^G_1 \xrightarrow{\cong} [\mathfrak{A}, (C_qI_0)^\infty(\mathfrak{B})]^G_\infty.$$

Definition 9.5. Let G be a countable discrete group. The standard G-Hilbert space H_G is defined to be the infinite direct sum $\oplus_{n=0}^\infty l^2(G)$, with the regular representation of G on each direct summand and graded so that the summands numbered even are even and those numbered odd are odd.

As the universal property of H_G,

Lemma 9.6. *If H is any separable graded G-Hilbert space with unitary representations of G on its even and odd grading degree summands, then the tensor product Hilbert space $H \otimes H_G$ is unitarily equivalent to H_G via a grading preserving, G-equivariant unitary isomorphism of Hilbert spaces.*

Proof. Denote by H_0 the same Hilbert space H but equipped with the trivial G-action. Define a unitary isomorphism from $H \otimes l^2(G)$ to $H_0 \otimes l^2(G)$ by sending $v \otimes \chi_g$ to $g^{-1}v \otimes \chi_g$ with χ_g the characteristic function at $g \in G$. From this we obtain a unitary isomorphism Φ from $H \otimes H_G$ to $H_0 \otimes H_G$. Since $H_0 \otimes H_G$ is a direct sum of copies of H_G, because $H_0 \cong l^2(\mathbb{N})$ and $l^2(\mathbb{N}) \otimes H_G \cong \oplus_{\mathbb{N}} H_G$, we then have $H_0 \otimes H_G \cong H_G$. Hence we obtain that the diagram

$$\begin{array}{ccc} H \otimes H_G & \xrightarrow{\Phi} & H_0 \otimes H_G \cong H_G \\ G \Big\downarrow \alpha\otimes\alpha & & G \Big\downarrow t\otimes\alpha \\ H \otimes H_G & \xrightarrow{\Phi} & H_0 \otimes H_G \cong H_G \end{array}$$

with α the given G-action and t the trivial G-action, commutes as

$$h\Phi(v \otimes \chi_g) = h(g^{-1}v \otimes \chi_g) = g^{-1}v \otimes \chi_{h^{-1}g},$$
$$\Phi(h(v \otimes \chi_g)) = \Phi(h^{-1}v \otimes \chi_{h^{-1}g}) = g^{-1}hh^{-1}v \otimes \chi_{h^{-1}g} = g^{-1}v \otimes \chi_{h^{-1}g}.$$

□

Definition 9.7. Let G be a countable discrete group and let $\mathfrak{A}$ and $\mathfrak{B}$ be separable, graded G-C^*-algebras. Define the G-**equivariant** E or EE-theory group $E_G(\mathfrak{A}, \mathfrak{B})$ for $\mathfrak{A}$ and $\mathfrak{B}$ to be the set of homotopy classes of G-equivariant asymptotic morphisms from $S \otimes \mathfrak{A} \otimes \mathbb{K}(H_G)$ to $\mathfrak{B} \otimes \mathbb{K}(H_G)$ as

$$E_G(\mathfrak{A}, \mathfrak{B}) = [S \otimes \mathfrak{A} \otimes \mathbb{K}(H_G) \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}(H_G)]^G.$$

Remarks. By virtue of the Hilbert space H_G, if H is any seprable graded G-Hilbert space and if $\varphi : S \otimes \mathfrak{A} \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}(H)$ is a G-equivariant asymptotic morphism, then φ determines an element of $E_G(\mathfrak{A}, \mathfrak{B})$. To see this, simply tensor φ with the identity map on $\mathbb{K}(H_G)$ and apply Lemma 9.6 above as

$$\begin{aligned}(S \otimes \mathfrak{A}) \otimes \mathbb{K}(H_G) \xrightarrow{\varphi \otimes \mathrm{id}} \quad & (\mathfrak{B} \otimes \mathbb{K}(H)) \otimes \mathbb{K}(H_G) \\ & \cong \mathfrak{B} \otimes \mathbb{K}(H \otimes H_G) \cong \mathfrak{B} \otimes \mathbb{K}(H_G).\end{aligned}$$

This construction has a generalization as follows. Suppose that H is a separable graded Hilbert space, equipped with a continuous family of unitary G-actions $\alpha(t)$, parametrized by $t \in [1, \infty)$. The continuity requirement here is pointwise norm continuity, so that if $g \in G$ and $k \in \mathbb{K}(H)$, then $\alpha(t)_g(k)$ is norm continuous for t. Suppose now that $\mathfrak{A}$ and $\mathfrak{B}$ are G-C^*-algebras and that $\varphi = (\varphi_t) : S \otimes \mathfrak{A} \rightsquigarrow \mathfrak{B} \otimes \mathbb{K}(H)$ is an asymptotic morphism which is equivariant with respect to the given family of G-actions, in the sense that the diagram

$$\begin{tikzcd} S \otimes \mathfrak{A} \arrow[r, "\varphi_t"] \arrow[d, "\mathrm{id} \otimes \alpha"'] & \mathfrak{B} \otimes \mathbb{K}(H) \arrow[d, "\mathrm{id} \otimes \alpha(t)"] \\ S \otimes \mathfrak{A} \arrow[r, "\varphi_t"] & \mathfrak{B} \otimes \mathbb{K}(H) \end{tikzcd}$$

with α the given G-action on $\mathfrak{A}$, commutes asymptotically as $t \to \infty$, so that

$$\lim_{t \to \infty} \|\varphi_t((\mathrm{id} \otimes \alpha_g)(x)) - (\mathrm{id} \otimes \alpha(t)_g)(\varphi_t(x))\| = 0$$

for all $g \in G$ and $x \in S \otimes \mathfrak{A}$. Then φ also determines a class of $E_G(\mathfrak{A}, \mathfrak{B})$. Indeed, tensor φ with the identity map on $\mathbb{K}(H_G)$ and apply the procedure in the proof of Lemma 9.6 above, and we then obtain an asymptotic morphism from $S \otimes \mathfrak{A} \otimes \mathbb{K}(H_G)$ to $\mathfrak{B} \otimes \mathbb{K}(H_0 \otimes H_G)$ which is equivariant in the usual sense for the single, fixed representation μ of G on $H_0 \otimes H_G$ as,

$$\begin{tikzcd} S \otimes \mathfrak{A} \otimes \mathbb{K}(H_G) \arrow[r, "\varphi_t \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \alpha \otimes \lambda"'] & \mathfrak{B} \otimes \mathbb{K}(H \otimes H_G) \arrow[r, "\cong"] \arrow[d, "\mathrm{id} \otimes \alpha(t) \otimes \lambda"] & \mathfrak{B} \otimes \mathbb{K}(H_0 \otimes H_G) \arrow[d, "\mathrm{id} \otimes \mu"] \\ S \otimes \mathfrak{A} \otimes \mathbb{K}(H_G) \arrow[r, "\varphi_t \otimes \mathrm{id}"] & \mathfrak{B} \otimes \mathbb{K}(H \otimes H_G) \arrow[r, "\cong"] & \mathfrak{B} \otimes \mathbb{K}(H_0 \otimes H_G) \end{tikzcd}$$

with the action λ which corresponds to the regular representation action of G on H_G.

In the definition above, it is essential to include $\mathbb{K}(H_G)$ as a tensor factor in both arguments above. If we were to leave one out we would obtain a quite different and not very useful object. $\square$

The sets $E_G(\mathfrak{A}, \mathfrak{B})$ become additive abelian groups as the E or EE-theory groups $E(\mathfrak{A}, \mathfrak{B})$ do, called the E_G or EE_G-theory groups of $\mathfrak{A}$ and $\mathfrak{B}$.

Theorem 9.8. *The E_G-theory groups $E_G(\mathfrak{A}, \mathfrak{B})$ become groups of morphisms in an additive category $\mathfrak{E}_G$ whose objects are separable graded G-C^*-algebras, called the equivariant E-theory or E_G-theory, category.*

There is a functor from the homotopy category of graded G-C^*-algebras and graded G-equivariant $*$-homomoprhisms into the equivariant E-theory category $\mathfrak{E}_G$, which is the identity on objects.

There is a functorial maximal tensor product on the equivariant E-theory category.

There are six-term exact sequences of E_G-theory groups associated to short exact sequences of G-C^*-algebras.

These precise statements and proofs are obtained as modifications from the non-equivarinat case as Theorems 8.23 and 8.31 by just adding the symbol G properly to each item, but omitted. May refer to [16].

10 Crossed products of C^*-algebras by discrete groups

May refer to Pedersen [47] or Williams [58] for crossed products of C^*-algebras.

Full crossed products

Definition 10.1. Let G be a discrete group and let $\mathfrak{A}$ be a G-C^*-algebra. A **covariant** representation of $\mathfrak{A}$ and G based on a C^*-algebra $\mathfrak{B}$ is a pair (φ, π) of a $*$-homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ and a group homomorphism π from G into the unitary group $U(M(\mathfrak{B}))$ of the multiplier algebra $M(\mathfrak{B})$ of $\mathfrak{B}$ such that

$$\pi_g \varphi(a) \pi_{g^{-1}} = \varphi(g \cdot a) \quad \text{for all } a \in \mathfrak{A},\ g \in G,$$

as that the diagram commutes:

$$\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{\varphi} & \mathfrak{B} \\
{\scriptstyle G}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Ad}(\pi_G)} \\
\mathfrak{A} & \xrightarrow{\varphi} & \mathfrak{B}.
\end{array}$$

Definition 10.2. Let G be a discrete group and $\mathfrak{A}$ be a G-C^*-algebra. Denote by $C_c(G, \mathfrak{A})$ the involutive algebra of $\mathfrak{A}$-valued, finitely supported, (so) continuous functions on G with the convolution multiplication and involution defined as for $f_1, f_2, f \in C_c(G, \mathfrak{A})$, $g \in G$,

$$(f_1 * f_2)(g) = \sum_{h \in G} f_1(h)(h \cdot (f_2(h^{-1}g))), \quad \text{and}$$
$$f^*(g) = g \cdot (f(g^{-1})^*).$$

Observe that a covariant representation of $\mathfrak{A}$ and G based on a C^*-algebra $\mathfrak{B}$ determines a $*$-homomorphism $\varphi * \pi$ from $C_c(G, \mathfrak{A})$ into $\mathfrak{B}$ defined by

$$(\varphi * \pi) f = \sum_{g \in G} \varphi(f(g))\pi_g \quad \text{for } f \in C_c(G, \mathfrak{A}).$$

Proof. (Added). Check that

$$\begin{aligned}
&(f_1 * (f_2 * f_3))(g) = \sum_{h \in G} f_1(h)(h \cdot ((f_2 * f_3)(h^{-1}g))) \\
&= \sum_{h \in G} f_1(h)(h \cdot [\sum_{k \in G} f_2(k)(k \cdot (f_3(k^{-1}h^{-1}g)))]) \\
&= \sum_{h \in G} \sum_{k \in G, l = hk} f_1(h)(h \cdot f_2(h^{-1}l)(h \cdot h^{-1}l \cdot (f_3(l^{-1}g)))) \\
&= \sum_{l \in G} (f_1 * f_2)(l)(l \cdot (f_3(l^{-1}g)) = ((f_1 * f_2) * f_3)(g).
\end{aligned}$$

Also, compute that

$$
\begin{aligned}
(f_1 * f_2)^*(g) &= g \cdot ((f_1 * f_2)(g^{-1})^*) \\
&= g \cdot [\sum_{h \in G} f_1(h)(h \cdot (f_2(h^{-1}g^{-1})))]^* \\
&= g \cdot [\sum_{g^{-1}h \in G} f_1(g^{-1}h)(g^{-1}h \cdot (f_2(h^{-1})))]^* \\
&= \sum_{h \in G} (h \cdot f_2(h^{-1})^*)(h \cdot h^{-1}g \cdot f_1(g^{-1}h)^*) \\
&= \sum_{h \in G} f_2^*(h)(h \cdot f_1^*(h^{-1}g)) = (f_2^* * f_1^*)(g).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&(\varphi * \pi)(f_1 * f_2) = \sum_{g \in G} \varphi((f_1 * f_2)(g))\pi_g \\
&= \sum_{g \in G} \varphi[\sum_{k \in G} f_1(k)(k \cdot (f_2(k^{-1}g)))]\pi_g \\
&= \sum_{g \in G} \sum_{k \in G} \varphi(f_1(k))\pi_k \pi_{k^{-1}} \varphi(k \cdot (f_2(k^{-1}g)))\pi_k \pi_{k^{-1}g} \\
&= \sum_{k \in G} \varphi(f_1(k))\pi_k \sum_{k^{-1}g \in G} \varphi(f_2(k^{-1}g))\pi_{k^{-1}g} \\
&= (\varphi * \pi) f_1 \cdot (\varphi * \pi) f_2.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&(\varphi * \pi)(f^*) = \sum_{g \in G} \varphi(f^*(g))\pi_g \\
&= \sum_{g \in G} \varphi(g \cdot f(g^{-1})^*)\pi_g = \sum_{g \in G} \pi_g \pi_{g^{-1}} \varphi(g \cdot f(g^{-1})^*)\pi_g \\
&= \sum_{g \in G} \pi_g \varphi(f(g^{-1}))^* = [\sum_{g \in G} \varphi(f(g^{-1}))\pi_{g^{-1}}]^* = [(\varphi * \pi)(f)]^*.
\end{aligned}
$$

□

Definition 10.3. The **full** crossed product C^*-algebra denoted by $\mathfrak{A} \rtimes G = C^*(G, \mathfrak{A})$ or just $\mathfrak{A} * G = G * \mathfrak{A}$ (as yet another notation created here) is defined to be the C^*-completion of the $*$-algebra $C_c(G, \mathfrak{A})$ in the smallest C^*-algebra norm which makes all the $*$-homomorphisms $\varphi * \pi$ continuous.

Indeed, the C^*-algebra $\mathfrak{A} \rtimes G$ has the C^*-norm defined as

$$\|f\| = \sup_{\varphi * \pi} \|(\varphi * \pi) f\| = \|(\oplus_{\varphi * \pi} \varphi * \pi) f\|,$$

so that $\mathfrak{A} \rtimes G$ is isomorphic to the C^*-algebra generated by the image of the direct sum representation of $C_c(G, \mathfrak{A})$ over covariant representations $(\varphi, \pi) = \varphi * \pi$ of $\mathfrak{A}$ and G.

Example 10.4. If $\mathfrak{A} = \mathbb{C}$, then $\mathbb{C} \rtimes G$ is the **full** group C^*-algebra denoted by $C^*(G)$.

If $\mathfrak{A}$ is graded and if G acts on $\mathfrak{A}$ by grading preserving automorphisms, then $C^*(G, \mathfrak{A})$ has a natural grading, where the grading automorphism acts pointwise on functions in $C_c(G, \mathfrak{A})$.

Remark. The C^*-algebra $G * \mathfrak{A} = \mathfrak{A} \rtimes G$ contains $\mathfrak{A}$ as a C^*-subalgebra and the multiplier algebra $M(G * \mathfrak{A})$ contain G within the unitary group $U(M(G*\mathfrak{A}))$. Elements of $C_c(G, \mathfrak{A})$ can be written as finite sums $\sum_{g \in G} a_g \chi_g$, where only finitely many $a_g \in \mathfrak{A}$ are nonzero. In this way, the grading automorphism is given by sending

$$\sum_{g \in G} a_g \chi_g \mapsto \sum_{g \in G} \alpha_g(a_g) \chi_g,$$

where α is the given action of G on $\mathfrak{A}$.

Taking the full crossed product (FCP) by G as an operation is viewed as a functor from G-C^*-algebras to C^*-algebras, which is continuous and exact:

$$\{\mathfrak{A} : G\text{-}C^*\text{-algebras}\} \xrightarrow{\rtimes G} \{\mathfrak{A} \rtimes G : \text{FCP } C^*\text{-algebras}\}.$$

Namely, if $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{J} \to 0$ is a short exact sequence of G-C^*-algebras, then the induced sequence of C^*-algebras:

$$0 \to \mathfrak{J} \rtimes G \to \mathfrak{A} \rtimes G \to (\mathfrak{A}/\mathfrak{J}) \rtimes G \to 0$$

is exact. Also, we have the commutative diagram as

$$\begin{array}{ccc} C([0,1], \mathfrak{A}) & \xrightarrow{\mathrm{ev}_t} & \mathfrak{A} \\ \big\downarrow{\scriptstyle \rtimes G} & & \big\downarrow{\scriptstyle \rtimes G} \\ C([0,1], \mathfrak{A}) \rtimes G & \xrightarrow{\mathrm{ev}_t} & \mathfrak{A} \rtimes G \end{array}$$

with $C([0,1], \mathfrak{A}) \rtimes G \cong (\mathfrak{A} \rtimes G)[0,1]$.

As a result, there is a descent functor from the G-equivariant asymptotic category to the asymptotic category, denoted as

$$[\mathfrak{A}, (C_q I_0)^\infty \mathfrak{B}]^G_\infty \xrightarrow{\rtimes G} [\mathfrak{A} \rtimes G, (C_q I_0)^\infty(\mathfrak{B}) \rtimes G]_\infty.$$

Lemma 10.5. *Let G be a discrete group, let $\mathfrak{B}$ be a G-C^*-algebra, and let H be a G-Hilbert space on which G acts by $u_g : H \to H$ unitary operators for $g \in G$. Then there is an isomorphism of C^*-algebras from $(\mathfrak{B} \otimes \mathbb{K}(H)) \rtimes G$ to $(\mathfrak{B} \rtimes G) \otimes \mathbb{K}(H)$, defined by sending finite sums:*

$$\sum_{g \in G} (b_g \otimes k_g)\chi_g \mapsto \sum_{g \in G} (b_g \chi_g) \otimes k_g u_g.$$

Proof. Sending finite sums so induces an algebraic $*$-homomorphism from $C_c(G, \mathfrak{B} \otimes \mathbb{K}(H))$ to $C_c(G, \mathfrak{B}) \otimes \mathbb{K}(H)$. It follows from examining the definitions of the norms for the maximal tensor product and the full crossed product that the algebraic $*$-isomorphism extends to a $*$-isomorphism of C^*-algebras.

Indeed, note that

$$\begin{aligned}
&((b_{g_1} \otimes k_{g_1})\chi_{g_1}) * ((b_{g_2} \otimes k_{g_2})\chi_{g_2}) \\
&= \sum_{h \in G} ((b_{g_1} \otimes k_{g_1}) \otimes \chi_{g_1})(h)(h \cdot ((b_{g_2} \otimes k_{g_2})\chi_{g_2})(h^{-1} \cdot)) \\
&= (b_{g_1} \otimes k_{g_1})(g_1 \cdot ((b_{g_2} \otimes k_{g_2})\chi_{g_2})(g_1^{-1} \cdot)) \\
&= (b_{g_1} \otimes k_{g_1})(g_1 \cdot (b_{g_2} \otimes k_{g_2}))\chi_{g_1 g_2} \\
&= (b_{g_1}(g_1 \cdot (b_{g_2})) \otimes k_{g_1} k_{g_2}))\chi_{g_1 g_2} \mapsto (b_{g_1}(g_1 \cdot (b_{g_2}))\chi_{g_1 g_2}) \otimes (k_{g_1} k_{g_2} u_{g_1 g_2})
\end{aligned}$$

and that

$$\begin{aligned}
&\{(b_{g_1}\chi_{g_1}) \otimes k_{g_1} u_{g_1}\}\{(b_{g_2}\chi_{g_2}) \otimes k_{g_2} u_{g_2}\} \\
&= \{(b_{g_1}\chi_{g_1}) * (b_{g_2}\chi_{g_2})\} \otimes k_{g_1} u_{g_1} k_{g_2} u_{g_2} \\
&= \{b_{g_1}(g_1 \cdot (b_{g_2}))\chi_{g_1 g_2}\} \otimes k_{g_1}(u_{g_1} k_{g_2} u_{g_1^{-1}}) u_{g_1} u_{g_2},
\end{aligned}$$

where the inner automorphism on $\mathbb{K}(H)$ as $u_{g_1} k_{g_2} u_{g_1^{-1}} = \mathrm{Ad}(u_{g_1})(k_{g_2})$ may be identified with the identity map on $\mathbb{K}(H)$ up to unitary equivalence. Also,

$$\begin{aligned}
[(b_{g_1}\chi_{g_1}) \otimes k_{g_1} u_{g_1}]^* &= (b_{g_1}\chi_{g_1})^* \otimes (k_{g_1} u_{g_1})^* = g_1 \cdot (b_{g_1}\chi_{g_1}((\cdot)^{-1}))^* \otimes u^*_{g_1} k^*_{g_1} \\
&= g_1 \cdot (b^*_{g_1}\chi_{g_1^{-1}}) \otimes u^*_{g_1} k^*_{g_1}, \\
[(b_{g_1} \otimes k_{g_1})\chi_{g_1}]^* &= (g_1 \cdot (b^*_{g_1} \otimes k^*_{g_1}))\chi_{g_1}((\cdot)^{-1}) \\
&\mapsto (g_1 \cdot (b^*_{g_1}))\chi_{g_1^{-1}} \otimes k^*_{g_1} u^*_{g_1} = (g_1 \cdot (b^*_{g_1}))\chi_{g_1^{-1}} \otimes u^*_{g_1} u_{g_1} k^*_{g_1} u^*_{g_1},
\end{aligned}$$

and hence both of up and down sides are equal up to unitary equivalence. □

Combining Lemma 10.5 above with the decent functor between asymptotic categories above we obtain

Theorem 10.6. *There exists a descent functor from the equivariant E or E_G-theory category to the E-theory category which maps G-C^*-algebras to their full crossed products by G, and which maps the E_G-theory classes of G-equivariant $*$-homomorphisms φ to the E-theory classes of the induced $*$-homomorphisms $\varphi \rtimes G$ of full crossed products by G as*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{[\varphi]\in E_G(\mathfrak{A},\mathfrak{B})} & \mathfrak{B} \\ {\scriptstyle\rtimes G}\big\downarrow & & \big\downarrow{\scriptstyle\rtimes G} \\ \mathfrak{A}\rtimes G & \xrightarrow{[\varphi\rtimes G]\in E(\mathfrak{A}\rtimes G,\mathfrak{B}\rtimes G)} & \mathfrak{B}\rtimes G. \end{array}$$

Corollary 10.7. *Let G be a countable discrete group. If $\mathfrak{A}$ and $\mathfrak{B}$ are separable G-C^*-algebras and if they are isomorphic objects in the equivariant E or E_G-theory category, then $E_0(\mathfrak{A} \rtimes G) \cong K_0(\mathfrak{A} \rtimes G)$ is isomorphic to $E_0(\mathfrak{B} \rtimes G) \cong K_0(\mathfrak{B} \rtimes G)$*

Proof. (Added). We have

$$E_0(\mathfrak{A} \rtimes G) \cong E(\mathbb{C}, \mathfrak{A} \rtimes G) \xrightarrow[\cong]{[\varphi\rtimes G]\in E(\mathfrak{A}\rtimes G,\mathfrak{B}\rtimes G)} E(\mathbb{C}, \mathfrak{B} \rtimes G) \cong E_0(\mathfrak{B} \rtimes G),$$

where $[\varphi \rtimes G]$ is the induced K-theory equivalence corresponding to an E_G-theory equivalence $[\varphi] \in E_G(\mathfrak{A}, \mathfrak{B})$. ☐

Reduced crossed products

Definition 10.8. ([39] or [57] for Hilbert modules). Let $\mathfrak{A}$ be a G-C^*-algebra with G a discrete group and denote by $l^2(G, \mathfrak{A})$ the Hilbert $\mathfrak{A}$-module comprised of functions $\xi : G \to \mathfrak{A}$ for which the series $\sum_{g\in G} \xi(g)^*\xi(g)$ converges in norm in $\mathfrak{A}$. The left **regular** representation of $\mathfrak{A}$ on $l^2(G, \mathfrak{A})$ is the covariant representation $\lambda = (\varphi, \pi)$ into the C^*-algebra $\mathbb{B}(l^2(G, \mathfrak{A}))$ of adjointable, bounded operators on $l^2(G, \mathfrak{A})$ given by the formulas

$$(\varphi(a)\xi)(h) = (h^{-1} \cdot a)\xi(h) \quad \text{and} \quad (\pi_g\xi)(h) = \xi(g^{-1}h)$$

for $h, g \in G$ and $\xi \in l^2(G, \mathfrak{A})$.

The regular representation of $\mathfrak{A}$ determines a $*$-homomorphism $\varphi * \pi = \lambda$ from the full crossed product $\mathfrak{A} \rtimes G$ to $\mathbb{B}(l^2(G, \mathfrak{A}))$.

Definition 10.9. The **reduced** crossed product C^*-algebra of a G-C^*-algebra $\mathfrak{A}$ by G is defined to be the image of $\mathfrak{A} \rtimes G$ by the regular representation, and is denoted by $\mathfrak{A} \rtimes_r G = C_r^*(G, \mathfrak{A})$ or $\mathfrak{A} \rtimes_\lambda G = C_\lambda^*(G, \mathfrak{A})$.

Example 10.10. If $\mathfrak{A} = \mathbb{C}$, then $\mathbb{C} \rtimes_r G$ is the **reduced** group C^*-algebra of G, denoted as $C_r^*(G)$ or $C_\lambda^*(G)$.

Taking the reduced crossed product is a functor from (graded) G-C^*-algebras to (graded) C^*-algebras. However, unlike the full crossed product taking, the reduced crossed product taking is not exact for every G.

Definition 10.11. A discrete group G is said to be **exact** if the functor $\rtimes_r G : \mathfrak{A} \mapsto \mathfrak{A} \rtimes_r G$ is exact as that the sequence of reduced crossed products:

$$0 \to \mathfrak{J} \rtimes_r G \to \mathfrak{A} \rtimes_r G \to (\mathfrak{A}/\mathfrak{J}) \rtimes_r G \to 0$$

is exact for any short exact sequence of G-C^*-algebras $\mathfrak{J}$, $\mathfrak{A}$, and $\mathfrak{A}/\mathfrak{J}$.

Proposition 10.12. (Kirchberg-Wassermann [36]). *A discrete group G is exact if and only if the reduced group C^*-algebra $C_r^*(G)$ is exact as that*

$$0 \to \mathfrak{J} \otimes_{\min} C_r^*(G) \to \mathfrak{A} \otimes_{\min} C_r^*(G) \to (\mathfrak{A}/\mathfrak{J}) \otimes_{\min} C_r^*(G) \to 0$$

is exact for any short exact sequence of C^*-algebras $\mathfrak{J}$, $\mathfrak{A}$, and $\mathfrak{A}/\mathfrak{J}$.

Proof. (Sketch). Exactness of $C_r^*(G)$ is implied by exactness of G since the reduced crossed product $\mathfrak{A} \rtimes_r G$ with the trivial G-action is isomorphic to the minimal tensor product $\mathfrak{A} \otimes_{\min} C_r^*(G)$, where $C_r^*(G)$ is trivially graded.

The reverse implication is implied as follows. If $C_r^*(G)$ is exact, then

$$0 \to (\mathfrak{J} \rtimes G) \otimes_{\min} C_r^*(G) \to (\mathfrak{A} \rtimes G) \otimes_{\min} C_r^*(G) \to ((\mathfrak{A}/\mathfrak{J}) \rtimes G) \otimes_{\min} C_r^*(G) \to 0$$

is exact. But for any G-C^*-algebra $\mathfrak{D}$, there is a functorial embedding

$$\mathfrak{D} \rtimes_r G \to (\mathfrak{D} \rtimes G) \otimes_{\min} C_r^*(G)$$

defined by sending $G \ni g = \chi_g \mapsto g \otimes g$ and $\mathfrak{D} \ni d \mapsto d \otimes 1$. Moreover, there is a functorial, continuous and linear left-inverse defined by sending $d \otimes 1 \mapsto d$, $g \otimes g \mapsto g$, and $g \otimes h \mapsto 0$ if $g \neq h \in G$. It follows that the sequence

$$0 \to \mathfrak{J} \rtimes_r G \to \mathfrak{A} \rtimes_r G \to (\mathfrak{A}/\mathfrak{J}) \rtimes_r G \to 0$$

is a direct summand of the above short exact sequence of the minimal tensor products with $C_r^*(G)$, and is therefore exact as well. $\square$

Exercise. If $\mathfrak{C}$ is a C^*-subalgebra of an exact C^*-algebra $\mathfrak{B}$, then $\mathfrak{C}$ is also exact.

Proof. (Added). The following diagram proves the statement:

$$\begin{array}{ccccc} 0 \to \mathfrak{J} \otimes_{\min} \mathfrak{B} & \longrightarrow & \mathfrak{A} \otimes_{\min} \mathfrak{B} & \longrightarrow & (\mathfrak{A}/\mathfrak{J}) \otimes_{\min} \mathfrak{B} \to 0 \\ \uparrow & & \uparrow & & \uparrow \\ 0 \to \mathfrak{J} \otimes_{\min} \mathfrak{C} & \longrightarrow & \mathfrak{A} \otimes_{\min} \mathfrak{C} & \longrightarrow & (\mathfrak{A}/\mathfrak{J}) \otimes_{\min} \mathfrak{C} \to 0 \end{array}$$

where exactness for the first horizontal sequence for any short exact sequence of C^*-algebras $\mathfrak{J}$, $\mathfrak{A}$, and $\mathfrak{A}/\mathfrak{J}$ implies the second, and three up arrows are injective. □

Example 10.13. (Edited). All discrete subgroups of connected Lie groups are exact.

All hyperbolic groups are exact.

Every amenable group is exact. In fact, a group G is amenable if and only if $C^*(G) \cong C_r^*(G)$ and as well if and only if $\mathfrak{A} \rtimes G \cong \mathfrak{A} \rtimes_r G$ for any G-C^*-algebra $\mathfrak{A}$.

Those of interest are not checked. Refer to Wassermann [56] or more.

Theorem 10.14. *Let G be an exact, countable discrete group. There is a descent functor from the G-equivariant E-theory category to the E-theory category which maps a G-C^*-algebra $\mathfrak{A}$ to the reduced crossed product C^*-algebra $\mathfrak{A} \rtimes_r G$, and which maps the class of a G-equivariant $*$-homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ to the class of the induced $*$-homomorphism from $\mathfrak{A} \rtimes_r G$ to $\mathfrak{B} \rtimes_r G$ as*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{[\varphi] \in E_G(\mathfrak{A},\mathfrak{B})} & \mathfrak{B} \\ {\scriptstyle \rtimes_r G}\downarrow & & \downarrow{\scriptstyle \rtimes_r G} \\ \mathfrak{A} \rtimes_r G & \xrightarrow{[\varphi \rtimes_r G] \in E(\mathfrak{A} \rtimes_r G, \mathfrak{B} \rtimes_r G)} & \mathfrak{B} \rtimes_r G. \end{array}$$

Corollary 10.15. *Let G be an exact, countable discrete group. If $\mathfrak{A}$ and $\mathfrak{B}$ are separable G-C^*-algebras and if they are isomorphic objects in the G-equivariant E or E_G-theory category, then $E_0(\mathfrak{A} \rtimes_r G) \cong K_0(\mathfrak{A} \rtimes_r G)$ is isomorphic to $E_0(\mathfrak{B} \rtimes_r G) \cong K_0(\mathfrak{B} \rtimes_r G)$*

Proof. (Added). We have

$$E_0(\mathfrak{A} \rtimes_r G) \cong E(\mathbb{C}, \mathfrak{A} \rtimes_r G) \xrightarrow[\cong]{[\varphi \rtimes_r G] \in E(\mathfrak{A} \rtimes_r G, \mathfrak{B} \rtimes_r G)} E(\mathbb{C}, \mathfrak{B} \rtimes_r G) \cong E_0(\mathfrak{B} \rtimes_r G),$$

where $[\varphi \rtimes_r G]$ is the induced K-theory equivalence corresponding to an E_G-theory equivalence $[\varphi] \in E_G(\mathfrak{A}, \mathfrak{B})$. □

11 The Baum-Connes (BC) conjecture

May refer to [16] as well as Baum-Connes-Higson [4].

Proper G-spaces

Let G be a countable discrete group and let X be a paracompact, Hausdorff topological space with an action of G by homeomorphisms, called a G-**space**.

Definition 11.1. A G-space is said to be **proper** if for every $x \in X$, there is a G-invariant open subset U of X with $x \in U$, a finite subgroup H of G, and a G-equivariant map φ from U to G/H, as that the diagram commutes:

$$\begin{tikzcd} U \arrow[r, "G"] \arrow[d, "\varphi"'] & U \arrow[d, "\varphi"] \\ G/H \arrow[r, "G"] & G/H. \end{tikzcd}$$

Example 11.2. If H is a finite subgroup of G, then the discrete homogeneous space G/H is proper.

Because we may take $U = G/H$ and $\varphi = \mathrm{id}_{G/H}$ the identity map on G/H.

Moreover, if Y is any paracompact, Hausdorff space with an H-action, then the induced space $X = G \times_H Y$, defined to be the quotient $(G \times Y)/H$ space of $G \times Y$ by the diagonal action of H, with H acting on G by right multiplication, is proper.

Lemma 11.3. *A G-space X is proper if and only if for every $x \in X$, there is a G-invariant open subset U of X with $x \in U$, a finite subgroup H of G, an H-space Y, and a G-equivariant homeomorphism from U to $G \times_H Y$.*

Lemma 11.4. *A locally compact G-space X is proper if and only if the map from $G \times X$ to $X \times X$ which takes (g, x) to $(g \cdot x, x)$ is a proper map of locally compact spaces, in the sense that the inverse image of every compact set is compact.*

Example 11.5. If G is a discrete subgroup of a Lie group L, and if K is a compact subgroup of L, then the quotient space L/K is a proper G-space.

Proof. (Added). Consider the map as $G \times (L/K) \to (L/K) \times (L/K)$ defined as above. If an inverse image is non-compact, then the image itself should be non-compact. $\square$

Universal proper G-spaces

Definition 11.6. A proper G-space is said to be **universal** if for every proper G-space Y, there exists a G-equivariant continuous map $Y \to X$, and if moreover this map is unique up to G-equivariant homotopy.

It follows from the definition that any two universal proper G-spaces X_j are G-equivariantly homotopy equivalent, and be also to a universal proper G-space U_G unique up to G-equivariant homotopy, as the diagram commutes:

$$\begin{array}{ccccc} Y & \longrightarrow & X_j & \xrightarrow{\approx} & U_G \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle G} \\ Y & \longrightarrow & X_j & \xrightarrow{\approx} & U_G. \end{array}$$

Proposition 11.7. *Let G a countable discrete group. Then there exists a universal proper G-space.*

The proof below is due to Kasparov and Skandalis [32].

Proof. Let M_1 be the space of countably additive measures on G with total mass ≤ 1, that is a compact space with the topology of pointwise convergence. Let $M_{\frac{1}{2}}$ be the closed subspace of M_1 consisting of measures of total mass $\leq \frac{1}{2}$. The difference set $X = M_1 \setminus M_{\frac{1}{2}}$ is a locally compact proper G-space which is universal. $\square$

One can provide a much more concrete model as examples as $\underline{E}G$ or BG the classifying space for G a torsion free, discrete group. See [4].

Proposition 11.8. *Let M be a complete and simply connected Riemannian manifold of nonpositive sectional curvature. If a discrete group G acts properly and isometrically on M, then M is a universal G-space.*

Remark. The manifold could be infinite dimensional.

G-compact spaces

Definition 11.9. A proper G-space X is said to be **G-compact** if there is a compact subset K of X such that $G \cdot K = \cup_{g \in G} g \cdot K = X$.

If X is a G-compact, proper G-space, then X is locally compact and the quotient space X/G is compact, because for any $x \in X$, $x \in g \cdot K$ for some $g \in G$, and $X/G = K/G$ as a quotient space.

Definition 11.10. Let X be a G-compact proper G-space. A **cut off** function for X is a continuous function f from X to the closed interval $[0,1]$ such that the support $\mathrm{supp}(f)$ of f is compact and $\sum_{g\in G} f^2(g\cdot x) = 1$ for all $x \in X$.

The sum $\sum$ over G is locally finite, namely, a finite sum. Every G-compact proper G-space admits a cut off function. Moreover, any two cut off functions f_0 and f_1 are homotopic in the sense that the functions defined by

$$f_t(x) = \sqrt{tf_1^2(x) + (1-t)f_0^2(x)}, \quad t \in [0,1], x \in X$$

are all cut off functions on X.

Proof. (Added). Note that $\mathrm{supp}(f^2) = \mathrm{supp}(f)$ is compact, so that it is covered by a finite union $\cup_j g_j \cdot K$ for some compact subset K of X. Also, the G-orbit $G\cdot x$ of each $x \in X$ has a finite intersection with each compact $g_j \cdot K$, since G is discrete.

Note that $0 \le f_t(x) \le 1$ since $0 \le f_0(x) \le 1$ and $0 \le f_1(x) \le 1$ for any $x \in X$.

Recall that $\mathrm{supp}(f)$ is the closure of the complement $\ker(f)^c$ of the kernel $\ker(f)$ of f. Since the intersection $\ker(f_0) \cap \ker(f_1)$ is contained in $\ker(f_t)$, then $\ker(f_t)^c$ is contained in the union $\ker(f_0)^c \cup \ker(f_1)^c$. Thus,

$$\mathrm{supp}(f_t) \subset \mathrm{supp}(f_0) \cup \mathrm{supp}(f_1).$$

On the other hand, $\ker(f_0)^c$ and $\ker(f_1)^c$ are contained in $\ker(f_t)^c$, so that $\mathrm{supp}(f_0)$ and $\mathrm{supp}(f_1)$ are contained in $\mathrm{supp}(f_t)$. It follows that

$$\mathrm{supp}(f_t) = \mathrm{supp}(f_0) \cup \mathrm{supp}(f_1).$$

Also, we compute

$$\begin{aligned}\sum_{g\in G} f^2(g\cdot x) &= \sum_{g\in G}(tf_1^2(g\cdot x) + (1-t)f_0^2(g\cdot x)) \\ &= t\sum_{g\in G} f_1^2(g\cdot x) + (1-t)\sum_{g\in G} f_0^2(g\cdot x) = 1.\end{aligned}$$

□

Lemma 11.11. *Let f be a cut off function for a G-compact, proper G-space X. The function defined by*

$$p(g)(x) = f(g^{-1}\cdot x)f(x)$$

defines a projection in $C_c(G, C_c(X))$, and hence in $C_0(X) \rtimes G$.

The K-theory class of this projection is independent of the choice of a cut off function.

Proof. (Edited with details). We compute that for any $g \in G$ and $x \in X$,

$$\begin{aligned}
(p * p)(g)(x) &= \sum_{h \in G} p(h)(x)(h \cdot p(h^{-1}g))(x) \\
&= \sum_{h \in G} p(h)(x)p(h^{-1}g)(h^{-1}x) \\
&= \sum_{h \in G} f(h^{-1}x)f(x)f(g^{-1}hh^{-1}x)f(h^{-1}x) \\
&= f(g^{-1}x)f(x)\sum_{h \in G} f^2(h^{-1}x) = p(g)(x),
\end{aligned}$$

and thus $p^2 = p$. Also,

$$\begin{aligned}
p^*(g)(x) &= g \cdot (p(g^{-1})^*)(x) \\
&= p(g^{-1})^*(g^{-1}x) \\
&= \overline{f(gg^{-1}x)}\,\overline{f(g^{-1}x)} = f(g^{-1}x)f(x) = p(g)(x),
\end{aligned}$$

and thus, $p^* = p$.

If f_0 and f_1 are cut off functions on X, then there is a homotopy (f_t) between f_0 and f_1. Then the projections p_0 and p_1 associated to f_0 and f_1 respectively have a homotopy given by projections p_t defined from f_t for $t \in [0, 1]$. Therefore, $[p_0] = [p_1] = [p_t]$ in the K_0-group of $C_0(X) \rtimes G$. □

Definition 11.12. We denote by $[p_c]$ the K-theory class of $K_0(C_0(X) \rtimes G) \cong E(\mathbb{C}, C_0(X) \rtimes G)$ associated to projections done to cut off functions on X, and call it the cut off class.

Exercise. (N. C. Phillips [48] (the lacking item)). If X is a proper G-space, then $C_0(X) \rtimes G \cong C_0(X) \rtimes_r G$, so that $K_0(C_0(X) \rtimes G) \cong K_0(C_0(X) \rtimes_r G) \ni [p_c]$.

Proof. (Edited). An approach is to show that if $f \in C_c(G, C_c(X))$ and if $1 + f$ is invertible in $C_0(X) \rtimes_r G$ (or $(C_0(X) \rtimes_r G)^+$ the unitization), then the inverse actually lies in $C_c(G, C_c(X))^+$. It follows that $1 + f$ is invertible in $C_0(X) \rtimes G$, and therefore, the map from $C_0(X) \rtimes G$ to $C_0(X) \rtimes_r G$ is spectrum preserving, and hence isometric. □

The assembly map.

Definition 11.13. Let G be a countable discrete group and let $\mathfrak{D}$ be separable G-C^*-algebra. The **assembly** map

$$\mu : E_G(C_0(X), \mathfrak{D}) \to E_0(\mathfrak{D} \rtimes G)$$

is defined to be the composition $([p_c]\otimes) \circ (\rtimes G)$:

$$E_G(C_0(X), \mathfrak{D}) \xrightarrow[\text{descent}]{\rtimes G} E(C_0(X) \rtimes G, \mathfrak{D} \rtimes G) \xrightarrow[\text{product}]{[p_c]\otimes(\cdot)} E(\mathbb{C}, \mathfrak{D} \rtimes G),$$

where $[p_c] \in E(\mathbb{C}, C_0(X) \rtimes G)$.

Definition 11.14. Let G be a countable discrete group and let $\mathfrak{D}$ be a G-C^*-algebra. The (topological) **K-theory** group of G with coefficients in $\mathfrak{D}$ is defined to be the inductive limit:

$$K(G, \mathfrak{D}) = K(G, U_G, \mathfrak{D}) = \varinjlim_X E_G(C_0(X), \mathfrak{D}),$$

where the limit is taken over the collection of G-invariant and G-compact, (proper G-)subspaces X in the universal proper G-space U_G, directed by inclusion.

Remarks. Note that if $X \subset Y \subset U_G$ are G-compact, proper G-spaces, then X is a closed subset of Y and the restriction of functions on Y to X defines a G-equivariant $*$-homomorphism Res from $C_0(Y)$ to $C_0(X)$. This induces a homomorphism from $E_G(C_0(X), \mathfrak{D})$ to $E_G(C_0(Y), \mathfrak{D})$:

$$E_G(C_0(X), \mathfrak{D}) \xrightarrow{X \subset Y} E_G(C_0(Y), \mathfrak{D}) \longrightarrow \cdots \longrightarrow K(G, \mathfrak{D}).$$

If $X \subset Y \subset U_G$ are G-compact, proper G-spaces, then the restriction map from $E(\mathbb{C}, C_0(Y) \rtimes G)$ to $E(\mathbb{C}, C_0(X) \rtimes G)$ maps the unit class for Y to the unit class for X. Consequently the assembly maps for the various G-compact subsets of U_G are compatible and pass to the direct limit, so that as below:

Definition 11.15. The (full) Baum-Connes **assembly** map coefficients in a separable G-C^*-algebra $\mathfrak{D}$ is the map:

$$K(G, \mathfrak{D}) \xrightarrow{\mu} K_0(\mathfrak{D} \rtimes G) \cong E_0(\mathfrak{D} \rtimes G)$$

which is obtained as the limit of the assembly maps for G-compact (proper G-)subspaces X of U_G :

$$\begin{array}{ccc} K(G,\mathfrak{D}) = \varinjlim E_G(C_0(X),\mathfrak{D}) & \xrightarrow{\mu} & K_0(\mathfrak{D} \rtimes G) \\ \Big\uparrow \lim & & \Big\| \\ E_G(C_0(Y),\mathfrak{D}) & \xrightarrow{\mu} & E(\mathbb{C},\mathfrak{D} \rtimes G) \\ \mathrm{Res}^* \Big\uparrow X\subset Y\subset U_G & & \Big\| \\ E_G(C_0(X),\mathfrak{D}) & \xrightarrow{\mu=[p_c]\otimes\circ\rtimes G} & E(\mathbb{C},\mathfrak{D} \rtimes G). \end{array}$$

Definition 11.16. The **reduced** Baum-Connes **assembly** map with coefficients in a separable G-C^*-algebra $\mathfrak{D}$ is the map

$$K(G,\mathfrak{D}) \xrightarrow{\mu_\lambda=\mu_r} K_0(\mathfrak{D} \rtimes_r G)$$

obtained by composing the full Baum-Connes assembly map μ with the map λ_* from $K_0(\mathfrak{D} \rtimes G)$ to $K_0(\mathfrak{D} \rtimes_r G)$ induced from the quotient map λ from $\mathfrak{D} \rtimes G$ to $\mathfrak{D} \rtimes_r G$:

$$\begin{array}{ccc} K(G,\mathfrak{D}) & \xrightarrow{\mu} & K_0(\mathfrak{D} \rtimes G) \\ \Big\| & & \Big\downarrow \lambda_*=r_* \\ K(G,\mathfrak{D}) & \xrightarrow{\mu_\lambda=\lambda_*\circ\mu} & K_0(\mathfrak{D} \rtimes_r G). \end{array}$$

Remark. If G is exact and if X is a G-compact, proper G-space, then there is a reduced assembly map

$$E_G(C_0(X),\mathfrak{D}) \xrightarrow{\mu_\lambda} K_0(\mathfrak{D} \rtimes_r G)$$

defined by means of a composition:

$$E_G(C_0(X),\mathfrak{D}) \xrightarrow[\text{descent}]{\rtimes_r G} E(C_0(X) \rtimes_r G,\mathfrak{D} \rtimes_r G) \xrightarrow[\text{product}]{[p_c]\otimes(\cdot)} E(\mathbb{C},\mathfrak{D} \rtimes_r G),$$

where $[p_c] \in E(\mathbb{C}, C_0(X) \rtimes_r G)$. Then the reduced Baum-Connes assembly map μ_λ in the definition above may be defined equivalently as a direct limi of such maps:

$$\begin{array}{ccc} K(G,\mathfrak{D}) = \varinjlim E_G(C_0(X),\mathfrak{D}) & \xrightarrow{\mu_\lambda} & K_0(\mathfrak{D} \rtimes_r G) \\ \Big\uparrow \lim & & \Big\| \\ E_G(C_0(Y),\mathfrak{D}) & \xrightarrow{\mu_\lambda} & E(\mathbb{C},\mathfrak{D} \rtimes_r G). \\ \mathrm{Res}^* \Big\uparrow X\subset Y\subset U_G & & \Big\| \\ E_G(C_0(X),\mathfrak{D}) & \xrightarrow{\mu_\lambda} & E(\mathbb{C},\mathfrak{D} \rtimes_r G). \end{array}$$

The Baum-Connes (BC) conjecture as statements

The BC conjecture with coefficients. *Let G be a countable discrete group. The reduced Baum and Connes assembly map:*

$$\mu_r : K(G, \mathfrak{D}) \to E_0(\mathfrak{D} \rtimes_r G) \cong K_0(\mathfrak{D} \rtimes_r G)$$

is an isomorphism for every separable G-C^*-algebra $\mathfrak{D}$.

Remark. This conjecture above is false in general, by thanks to Gromov. Counter examples are given later below in Section 17. We may refer to [26] of Higson, Lafforgue, and Skandaklis for more.

The BC conjecture with coefficients trivial. *Let G be a countable discrete group. The reduced Baum and Connes assembly map with $\mathfrak{D} = \mathbb{C}$:*

$$\mu_r : K_*(G) \to K_*(C_r^*(G))$$

is an isomorphism for $* = 0, 1$.

Remark. This conjecture above is proved for all hyperbolic groups by Lafforgue [38] (but not at hand), where from abstract of which, the BC assembly maps are proved to be injective by Kasparov and Skandalis and proved to be surjective by Lafforgue.

Beyond discrete groups, the Baum-Connes conjecture is proved for G all reductive Lie and p-adic groups (by Lafforgue [37] in part).

The major open question seems to be:

Question. The Baum-Connes conjecture is true or not for discrete subgroups of connected Lie groups, as in the case of uniform lattices in semi-simple Lie groups?

The BC conjecture for finite groups

There is a well known result of Green and Julg that the equivariant K-theory for C^*-algebras is equivalent to the K-theory of crossed product C^*-algebras in the case of finite or compact groups (see Green [17] and Julg [28]). This is basically equivalent to that the BC conjecture is true in that case.

Theorem 11.17. **(Green-Julg).** *Let G be a finite group. Then the Baum-Connes assembly map is an isomorphism for every G-algebra $\mathfrak{D}$:*

$$K(G, \mathfrak{D}) \xrightarrow[\cong]{\mu = \mu_r} E_0(\mathfrak{D} \rtimes G) \cong K_0(\mathfrak{D} \rtimes G).$$

Proof. (Edited). (Sketch). If G is finite, $\mathfrak{D} \rtimes G \cong \mathfrak{D} \rtimes_r G$.

If G is finite, then the universal proper G-space U_G can be taken to be the one point space. Thus the theorem becomes an isomorphism

$$E_G(\mathbb{C}, \mathfrak{D}) \xrightarrow[\cong]{\mu} E(\mathbb{C}, \mathfrak{D} \rtimes G).$$

The unit projection of $C^*(G)$ is given by the function $p(g) = \frac{1}{|G|}$, which is central in $C^*(G)$ corresponding to the trivial representation of G. Note that

$$\begin{aligned}(p * p)(g) &= \sum_{h \in G} p(h)(h \cdot p(h^{-1}g)) \\ &= |G| \cdot \frac{1}{|G|^2} = \frac{1}{|G|} = p(g), \\ p^*(g) &= g \cdot (p(g^{-1})^*) = \frac{1}{|G|} = p(g).\end{aligned}$$

The statement is proved by defining an inverse to the assembly map μ. For this purpose, we note that $\mathfrak{D} \rtimes G$ may be identified with the following fixed point algebra:

$$\mathfrak{D} \rtimes G \xrightarrow{\cong} [\mathfrak{D} \otimes \mathrm{End}(l^2(G))]^G,$$

where an equivariant $*$-homomorphism from $\mathfrak{D} \rtimes G$, equipped with the trivial G-action, to $\mathfrak{D} \otimes \mathrm{End}(l^2(G))$ is defined by mapping $d \in \mathfrak{D}$ to $\sum_{g \in G} g \cdot d \otimes \chi_g$ and by mapping $g \in G$ to $1 \otimes \rho_g$, where χ_g is the projection supported at $g \in G$ and ρ is the right regular representation. Then the following homomorphism is induced as

$$\begin{array}{ccc} E(\mathbb{C}, \mathfrak{D} \rtimes G) & \longrightarrow & E_G(\mathbb{C}, \mathfrak{D} \otimes \mathrm{End}(l^2(G))) \\ \cong \downarrow & & \downarrow \cong \\ K_0(\mathfrak{D} \rtimes G) & & K(G, \mathfrak{D}). \end{array}$$

Composing the maps in the diagram we obtain the map from the left bottom to the right bottom as the inverse to μ. See [16] for details.

Since $l^2(G)$ is a finite dimensional vector space, namely, $\mathrm{End}(l^2(G))$ of endomorphisms on $l^2(G)$ is a matrix algebra acting on $l^2(G)$. □

Proper G-C^*-algebras

Definition 11.18. A G-C^*-algebra $\mathfrak{B}$ is said to be **proper** if there exists a locally compact proper G-space X and an equivariant $*$-homomorphism φ

from $C_0(X)$ into the center of the multiplier algebra $M(\mathfrak{B})$ of $\mathfrak{B}$ of degree zero such that $\varphi(C_0(X)) \cdot \mathfrak{B}$ is norm dense in $\mathfrak{B}$.

We may say that $\mathfrak{B}$ is proper over $\varphi(C_0(X))$. It looks like that $\mathfrak{B} = \varphi(C_0(X)) \cdot \mathfrak{B}$. Possibly, φ may be a $*$-isomorphism. Proper algebras are assumed to be separable in what follows.

The notion above is due essentially to Kasparov [30]

Example 11.19. If G is finite, then every G-C^*-algebra $\mathfrak{A}$ is proper over the one point space.

If X is a proper G-space, then $C_0(X)$ is a proper G-C^*-algebra.

If $\mathfrak{B}$ is proper over X, then for every G-C^*-algebra $\mathfrak{D}$, the tensor product $\mathfrak{B} \otimes \mathfrak{D}$ is also proper.

Proof. (Added). We have $C^*(G) \cong C(G^\wedge)$ with the dual group $G^\wedge \cong G$. Let $X = G^\wedge$ and φ the evaluation map at the trivial representation of G in $G^\wedge$. Then

$$\begin{array}{ccc} C(X) & \xrightarrow{\varphi} & \mathbb{C}1 \subset M(\mathfrak{A}) \\ {\scriptstyle g}\downarrow & & \| \\ C(X) & \xrightarrow{\varphi} & \mathbb{C}1, \end{array}$$

with $\mathbb{C}1 \cdot \mathfrak{A} = \mathfrak{A}$. Also, we have

$$\begin{array}{ccc} C_0(X) & \xrightarrow{\varphi=\mathrm{id}} & C_0(X) \subset M(C_0(X)) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ C_0(X) & \xrightarrow{\varphi=\mathrm{id}} & C_0(X) \end{array}$$

with $M(C_0(X)) \cong C^b(X)$ and $C_0(X) \cdot C_0(X) = C_0(X)$. As well,

$$\begin{array}{ccccc} C_0(X) & \xrightarrow{\varphi} & M(\mathfrak{B}) & \xrightarrow{\otimes 1} & M(\mathfrak{B} \otimes \mathfrak{D}) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ C_0(X) & \xrightarrow{\varphi} & M(\mathfrak{B}) & \xrightarrow{\otimes 1} & M(\mathfrak{B} \otimes \mathfrak{D}) \end{array}$$

where $1 \in M(\mathfrak{D})$ is the unit and $C_0(X) \cdot (\mathfrak{B} \otimes \mathfrak{D}) = \mathfrak{B} \otimes \mathfrak{D}$. $\square$

Exercise. If $\mathfrak{B}$ is a proper G-C^*-algebra, then $\mathfrak{B} \rtimes G = \mathfrak{B} \rtimes_r G$.

Proof. (Added). Note that

$$\begin{aligned}\mathfrak{B} \rtimes G &= \varphi(C_0(X))\mathfrak{B} \rtimes G \\ &= \mathfrak{B} \otimes \varphi(C_0(X)) \rtimes G \\ &= \mathfrak{B} \otimes \varphi(C_0(X)) \rtimes_r G = \mathfrak{B} \rtimes_r G\end{aligned}$$

(possibly, in this sense). □

Theorem 11.20. ([16]). *Let G be a countable discrete group and let $\mathfrak{B}$ be a proper G-C^*-algebra. Then the Baum-Connes assembly map is an isomorphism:*

$$K(G, \mathfrak{B}) \xrightarrow[\cong]{\mu} E_0(\mathfrak{B} \rtimes G) \cong K_0(\mathfrak{B} \rtimes G).$$

As well, thanks to the exercise above, the reduced Baum-Connes assembly map is an isomorphism:

$$K(G, \mathfrak{B}) \xrightarrow[\cong]{\mu_r} E_0(\mathfrak{B} \rtimes_r G) \cong K_0(\mathfrak{B} \rtimes_r G).$$

The outline of the proof is given in what follows.

Proposition 11.21. ([16]). *Let H be a finite subgroup of a countable group G and let W be a locally compact space with an action of H by homeomorphisms. If $\mathfrak{D}$ is any G-C^*-algebra, then there is a natural isomorphism:*

$$E_H(C_0(W), \mathfrak{D}) \cong E_G(C_0(G \times_H W), \mathfrak{D}),$$

where $\mathfrak{D}$ on the left hand side is viewed as an H-C^*-algebra by restriction of the given G-action.

Proof. The space W is identified with the open set $\{1_G\} \times W$ of $G \times_H W$. As a result there is an H-equivariant map from $C_0(W)$ to $C_0(G \times_H W)$. Composing with this map defines a restriction homomorphism as:

$$E_G(C_0(G \times_H W), \mathfrak{D}) \xrightarrow{\mathrm{Res}} E_H(C_0(W), \mathfrak{D}).$$

To construct its inverse, observe that every H-equivariant asymptotic morphism from $C_0(W)$ to $\mathfrak{D}$ extends uniquely to a G-equivariant asymptotic morphism from $C_0(G \times_H W)$ to $\mathfrak{D} \otimes \mathbb{K}(l^2(G/H))$. We then obtain an inverse map:

$$E_H(C_0(W), \mathfrak{D}) \to E_G(C_0(G \times_H W), \mathfrak{D}).$$

□

Lemma 11.22. *Let G be a countable group. If the assembly map*

$$\mu : K(G, \mathfrak{B}) \to E_0(\mathfrak{B} \rtimes G) \cong K_0(\mathfrak{B} \rtimes G)$$

is an isomorphism for every G-C^*-algebra $\mathfrak{B}$ which is proper over a G-compact space Z, then it is an isomorphism for every G-C^*-algebra.

Proof. Every proper G-C^*-algebra is a direct limit of G-C^*-algebras which are proper over G-compact spaces. Since K-theory commutes with direct limits, as does the crossed product functor, to prove the lemma it suffices to prove that the same holds for the functor $\mathfrak{D} \mapsto K(G, \mathfrak{D})$. By definition, it then suffices to prove that if Z is a G-compact, proper G-simplicial complex, then the functor $\mathfrak{D} \mapsto E_G(C_0(Z), \mathfrak{D})$ commutes with direct limits. By a Mayer-Vietoris argument, the proof of this reduces to the case where Z is a proper homogeneous space G/H. But we have

$$E_G(C_0(G/H), \mathfrak{D}) \cong E_H(\mathbb{C}, \mathfrak{D}) \cong E_0(\mathfrak{D} \rtimes G).$$

In fact, with $W = \{\text{point}\}$, Proposition 11.21 above implies

$$E_G(C_0(G/H), \mathfrak{D}) \cong E_G(C_0(G \times_H W), \mathfrak{D}) \cong E_H(C_0(W), \mathfrak{D}) \cong E_H(\mathbb{C}, \mathfrak{D}),$$

and Theorem 11.17 implies

$$E_H(\mathbb{C}, \mathfrak{D}) \cong E(\mathbb{C}, \mathfrak{D} \rtimes H) \cong E_0(\mathfrak{D} \rtimes H).$$

The lemma is now proved by continuity of K-theory for direct limits. □

Lemma 11.23. *Let G be a countable group. If the assembly map*

$$\mu : K(G, \mathfrak{B}) \to E_0(\mathfrak{B} \rtimes G) \cong K_0(\mathfrak{B} \rtimes G)$$

is an isomorphism for every G-C^*-algebra $\mathfrak{B}$ which is proper over a proper homogeneous space $Z = G/H$, then it is an isomorphism for every G-C^*-algebra which is proper over a G-compact proper G-space.

Proof. If $\mathfrak{B}$ is proper over a G-compact, proper G-space X, then to each G-invariant open set U of X there corresponds a closed ideal $J_U = C_0(U) \cdot \mathfrak{B}$ of $\mathfrak{B}$. By induction, there are G-invariant open subsets to cover X, each of which admits a G-map to a proper homogeneous space. Therefore, the proof reduces to the case where $\mathfrak{B}$ is proper over some proper homogeneous space G/H.

The proof of Theorem 11.20 is therefore reduced to the case where $\mathfrak{B}$ is proper over some proper homogeneous space G/H.

If $\mathfrak{B}$ is proper over G/H, then $\mathfrak{B}$ is a direct sum of closed ideals corresponding to the points of G/H, and the closed ideal J_{gH} corresponding to $gH \in G/H$ is an H-C^*-algebra. Then there is the following commutative diagram:

$$\begin{array}{ccc} K(G,\mathfrak{B}) & \xrightarrow{\mu} & E_0(\mathfrak{B} \rtimes G) \cong K_0(\mathfrak{B} \rtimes G) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ K(H,J_{gH}) & \xrightarrow{\mu} & E_0(J_{gH} \rtimes H) \cong K_0(J_{gH} \rtimes H). \end{array}$$

See [16] for details. □

The general conjecture with proper G-C^*-algebras

The following provides a strategy for attacking the Baum-Connes conjecture with coefficients in general:

Theorem 11.24. *Let G be a countable discrete group. Suppose that there exists a proper G-C^*-algebra $\mathfrak{B}$ morphisms $\beta \in E_G(\mathbb{C},\mathfrak{B})$ and $\alpha \in E_G(\mathfrak{B},\mathbb{C})$ such that $\alpha \circ \beta = 1 \in E_G(\mathbb{C},\mathbb{C})$. Then the Baum-Connes assembly map is an isomorphism:*

$$K(G,\mathfrak{D}) \xrightarrow{\mu} E_0(\mathfrak{D} \rtimes G) \cong K_0(\mathfrak{D} \rtimes G)$$

for every separable G-C^*-algebra $\mathfrak{D}$.

If in addition G is exact, then the reduced Baum-Connes assembly map is an isomorphism:

$$K(G,\mathfrak{D}) \xrightarrow{\mu_r} E_0(\mathfrak{D} \rtimes_r G) \cong K_0(\mathfrak{D} \rtimes_r G).$$

Proof. Consider the following diagram:

$$\begin{array}{ccc} K(G,\mathbb{C}\otimes\mathfrak{D}) & \xrightarrow{\mu} & E_0(\mathbb{C}\otimes\mathfrak{D} \rtimes G) \\ {\scriptstyle(\beta\otimes\mathrm{id}_{\mathfrak{D}})_*}\downarrow & & \downarrow{\scriptstyle(\beta\otimes\mathrm{id}_{\mathfrak{D}})_*} \\ K(G,\mathfrak{B}\otimes\mathfrak{D}) & \xrightarrow{\mu} & E_0(\mathfrak{B}\otimes\mathfrak{D} \rtimes G) \\ {\scriptstyle(\alpha\otimes\mathrm{id}_{\mathfrak{D}})_*}\downarrow & & \downarrow{\scriptstyle(\alpha\otimes\mathrm{id}_{\mathfrak{D}})_*} \\ K(G,\mathbb{C}\otimes\mathfrak{D}) & \xrightarrow{\mu} & E_0(\mathbb{C}\otimes\mathfrak{D} \rtimes G), \end{array}$$

where the horizontal maps are the assembly maps, while the vertical maps are induced from E-theory products by the classes $\beta\otimes\mathrm{id}_{\mathfrak{D}} \in E_G(\mathbb{C}\otimes\mathfrak{D},\mathfrak{B}\otimes\mathfrak{D})$ and $\alpha\otimes\mathrm{id}_{\mathfrak{D}} \in E_G(\mathfrak{B}\otimes\mathfrak{D},\mathbb{C}\otimes\mathfrak{D})$, and the diagram is commutative.

Since $\mathfrak{B}$ is a proper C^*-algebra, so is the tensor product $\mathfrak{B} \otimes \mathfrak{D}$. Therefore, the horizontal map in the middle is an isomorphism by Theorem 11.20.

The composition of the vertical maps on the left is the identity map by the assumption, and hence so is on the right. It follows that the top or bottom horizontal map is an isomorphism as well.

The same holds for the reduced case similarly. $\square$

Crossed products by the group of integers

We consider the case where $G = \mathbb{Z}^n$ the (torsion) free abelian groups. Let G act by translations on $\mathbb{R}^n$ in the usual way and let G act on the graded C^*-algebra $C_0Cl^*(\mathbb{R}^n)$ by $(g \cdot f)(v) = f(g \cdot v)$ for $g \in G$, $v \in \mathbb{R}^n$, and $f \in C_0Cl^*(\mathbb{R}^n)$.

Exercise. With this action by $\mathbb{Z}^n$, the C^*-algebra $C_0Cl^*(\mathbb{R}^n)$ is proper.

Proof. (Added). Note that the $\mathbb{Z}^n$-space $\mathbb{R}^n$ is proper. Because for any $x \in \mathbb{R}^n$, take $U = \mathbb{R}^n$ as an open set containing x. Then there is a $\mathbb{Z}^n$-equivariant map φ from $\mathbb{R}^n$ to $\mathbb{Z}^n/H$ with H the trivial group as

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\mathbb{Z}^n} & \mathbb{R}^n \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ \mathbb{Z}^n & \xrightarrow{\mathbb{Z}^n} & \mathbb{Z}^n \end{array}$$

where $\varphi(g\cdot(h\cdot x)) = \varphi((g+h)\cdot x) = g+h$ and $g\cdot(\varphi(h\cdot x)) = g\cdot h = g+h$ in $\mathbb{Z}^n$. Hence, $C_0(\mathbb{R}^n)$ is a proper $\mathbb{Z}^n$-C^*-algebra. It then follows that $C_0Cl^*(\mathbb{R}^n)$ is also proper since $C_0Cl^*(\mathbb{R}^n) \cong C_0(\mathbb{R}^n) \otimes Cl^*(\mathbb{R}^n)$. $\square$

We are going to produce a factorization in $\mathbb{Z}^n$-equivariant E-theory as

$$\mathbb{C} \xrightarrow{[\beta]} C_0Cl^*(\mathbb{R}^n) \xrightarrow{[\alpha]} \mathbb{C}$$

such that $[\alpha] \circ [\beta] = [\mathrm{id}]$ the identity map class.

Definition 11.25. Denote by $\beta : S \to C_0Cl^*(\mathbb{R}^n)$ the $*$-homomorphism defined as $\beta(f) = f(cl)$ before, and for $t \geq 1$, denote by $\beta_t : S \to C_0Cl^*(\mathbb{R}^n)$ the $*$-homomorphism defined by $\beta_t(f) = \beta(f_t)$, where $f_t(x) = f(t^{-1}x)$.

Thus, $\beta_t(f) = f(t^{-1}C)$, where C is the Clifford operator.

Proof. (Added). In fact, $f(t^{-1}C)\xi(v) = f(t^{-1}M_v)\xi(v)$ and $\beta_t(f)\xi(v) = f_t(v)\xi(v) = f(t^{-1}v)\xi(v)$ for $\xi \in L^2Cl^*(\mathbb{R}^n)$. $\square$

Lemma 11.26. *The asymptotic morphism $(\beta_t) : S \rightsquigarrow C_0Cl^*(\mathbb{R}^n)$ given by the above family of $*$-homomorphisms is $\mathbb{Z}^n$-equivariant.*

Proof. Consider the diagram

$$\begin{array}{ccc} S = C_0(\mathbb{R}) & \xrightarrow{\mathbb{Z}^n} & S \\ \beta_t \downarrow & & \downarrow \beta_t \\ C_0Cl^*(\mathbb{R}^n) & \xrightarrow{\mathbb{Z}^n} & C_0Cl^*(\mathbb{R}^n) \end{array}$$

which may not be commutative, but show that it commutes approximately as $t \to \infty$, so that

$$\|\beta_t(g \cdot f) - g \cdot (\beta_t(f))\| = \|f(g \cdot t^{-1}C) - g \cdot (f(t^{-1}C))\|$$

(corrected) goes to zero for any $f \in S$ and $g \in \mathbb{Z}^n$ (fixed), where the $\mathbb{Z}^n$-action on S may be trivial. Then the set of all $f \in S$ for which this holds is a C^*-subalgebra of S. Check that for $f_1, f_2 \in S$,

$$\begin{aligned} &\|(f_1 + f_2)(g \cdot t^{-1}C) - g \cdot ((f_1 + f_2)(t^{-1}C))\| \\ &\leq \|f_1(g \cdot t^{-1}C) - g \cdot (f_1(t^{-1}C))\| + \|f_2(g \cdot t^{-1}C) - g \cdot (f_2(t^{-1}C))\|, \end{aligned}$$

and

$$\begin{aligned} &\|(f_1f_2)(g \cdot t^{-1}C) - g \cdot ((f_1f_2)(t^{-1}C))\| \\ &= \|f_1(g \cdot t^{-1}C)f_2(g \cdot t^{-1}C) - f_1(g \cdot t^{-1}C)g \cdot f_2(t^{-1}C) \\ &\quad + f_1(g \cdot t^{-1}C)g \cdot f_2(t^{-1}C) - g \cdot f_1(t^{-1}C))g \cdot f_2(t^{-1}C))\| \\ &\leq \|f_1(g \cdot t^{-1}C)\|\|f_2(g \cdot t^{-1}C) - g \cdot (f_2(t^{-1}C))\| \\ &\quad \|f_1(g \cdot t^{-1}C) - g \cdot (f_1(t^{-1}C))\|\|g \cdot (f_2(t^{-1}C))\|, \end{aligned}$$

so that both of which goes to zero as $t \to \infty$. As well, if $f = \lim f_n$ in norm in S, then

$$\begin{aligned} &\|f(g \cdot t^{-1}C) - g \cdot (f(t^{-1}C))\| \\ &= \|f(g \cdot t^{-1}C) - f_n(g \cdot t^{-1}C) + f_n(g \cdot t^{-1}C) \\ &\quad - g \cdot (f_n(t^{-1}C)) + g \cdot (f_n(t^{-1}C)) - g \cdot (f(t^{-1}C))\| \\ &\leq \|f(g \cdot t^{-1}C) - f_n(g \cdot t^{-1}C)\| + \|f_n(g \cdot t^{-1}C) \\ &\quad - g \cdot (f_n(t^{-1}C))\| + \|g \cdot (f_n(t^{-1}C)) - g \cdot (f(t^{-1}C))\| \end{aligned}$$

so that it goes to zero as $t \to \infty$.

It then suffices to prove the first limit for the generators $f_\pm(x) = \frac{1}{x \pm i}$ of S. For these we have

$$\|f(g \cdot t^{-1}C) - g \cdot (f(t^{-1}C))\| = \|(g \cdot t^{-1}C \pm i1)^{-1} - g \cdot (t^{-1}C \pm i1)^{-1}\|,$$

but which is equal to, by action triviality and functional calculus

$$\begin{aligned}
&\|(t^{-1}C \pm i1)^{-1} - (t^{-1}(g \cdot C) \pm i1)^{-1}\| \\
&= \|(t^{-1}C \pm i1)^{-1}(t^{-1}(g \cdot C) \pm i1)^{-1}(t^{-1}(g \cdot C) - t^{-1}C)\| \\
&\leq \|(t^{-1}C \pm i1)^{-1}\| \|(t^{-1}(g \cdot C) \pm i1)^{-1}\| \|t^{-1}((g \cdot C) - C)\| \leq t^{-1}\|(g \cdot C) - C\|,
\end{aligned}$$

and the operator $(g \cdot C) - C$ is identified with $((g \cdot cl) - cl)(v)$ a function on $\mathbb{R}^n$ with values in $Cl^*(\mathbb{R}^n)$, bounded by the norm of g each element fixed. □

Definition 11.27. Denote by $[(\beta_t)] \in E_{\mathbb{Z}^n}(\mathbb{C}, C_0Cl^*(\mathbb{R}^n))$ the $E_{\mathbb{Z}^n}$-theory class of the asymptotic morphism $(\beta_t) : S \rightsquigarrow C_0Cl^*(\mathbb{R}^n)$.

Definition 11.28. Denote by $g \cdot_s v$ a **continuous** family of the translation actions of $v \in \mathbb{R}^n$ by $sg \in \mathbb{R}^n$ for $g \in \mathbb{Z}^n$ and $s \in [0, 1]$. Denote by $g \cdot_s f$ the corresponding continuous family of actions on $f \in C_0Cl^*(\mathbb{R}^n)$ and also on $f \in H(\mathbb{R}^n) = L^2Cl^*(\mathbb{R}^n)$.

Lemma 11.29. *If $f \otimes h \in S \otimes C_0Cl^*(\mathbb{R}^n)$, $g \in \mathbb{Z}^n$, and $t \in [1, \infty)$, then*

$$\lim_{t \to \infty} \|\alpha_t(f \otimes (g \cdot h)) - g \cdot_{t^{-1}} \alpha_t(f \otimes h)\| = 0$$

as that the following diagram commutes asymptotically as $t \to \infty$:

$$\begin{array}{ccc}
S \otimes C_0Cl^*(\mathbb{R}^n) & \xrightarrow{\mathrm{id} \otimes \mathbb{Z}^n} & S \otimes C_0Cl^*(\mathbb{R}^n) \\
\alpha_t \downarrow & & \downarrow \alpha_t \\
\mathbb{K}(H(\mathbb{R}^n)) & \xrightarrow{\mathbb{Z}^n \cdot_{[0,1]}} & \mathbb{K}(H(\mathbb{R}^n))
\end{array}$$

where (α_t) is the asymptotic morphism defined by $\alpha_t(f \otimes h) = f(t^{-1}D)M_{h_t}$.

Proof. Since the Dirac operator D is translation invariant, we have

$$g \cdot_{t^{-1}} \alpha_t(f \otimes h) = g \cdot_{t^{-1}} f(t^{-1}D)M_{h_t} = f(t^{-1}D)M_{(g \cdot h)_t},$$

with

$$\begin{aligned}
g \cdot_{t^{-1}} h_t(v) &= h_t((t^{-1}g)^{-1} + v) = h(t^{-1}(tg^{-1} + v)) \\
&= h(g^{-1} + t^{-1}v) = (g \cdot h)(t^{-1}v) = (g \cdot h)_t(v),
\end{aligned}$$

and

$$\alpha_t(f \otimes (g \cdot h)) = f(t^{-1}D)M_{(g\cdot h)_t}.$$

It follows that the diagram does commute. □

Definition 11.30. Denote by $[(\alpha_t)] \in E_{\mathbb{Z}^n}(C_0Cl^*(\mathbb{R}^n), \mathbb{C})$ the $E_{\mathbb{Z}^n}$-theory class of the $\mathbb{Z}^n$-equivariant asymptotic morphism $(\alpha_t) : S \otimes C_0Cl^*(\mathbb{R}^n) \rightsquigarrow \mathbb{K}(H(\mathbb{R}^n))$, where $\mathbb{K}(H(\mathbb{R}^n))$ is equipped with a continuous family of actons as $g \cdot_{t^{-1}} k$.

Proposition 11.31. *We have* $[(\alpha)_t] \circ [(\beta_t)] = [\mathrm{id}] \in E_{\mathbb{Z}^n}(\mathbb{C}, \mathbb{C})$.

Proof. Let $s \in [0, 1]$ and denote by $C_0Cl^*(\mathbb{R}^n)_s$ the C^*-algebra $C_0Cl^*(\mathbb{R}^n)$ with the scaled $\mathbb{Z}^n$-action as $g \cdot_s h$. The C^*-algebras $C_0Cl^*(\mathbb{R}^n)_s$ form a continuous field of $\mathbb{Z}^n$-C^*-algebras over the unit interval $[0, 1]$, where the fibers are the same and the $\mathbb{Z}^n$-actions vary continuously. Denote by $C([0, 1], \{C_0Cl^*(\mathbb{R}^n)_s\})$ the $\mathbb{Z}^n$-C^*-algebra of continuous sections of the continuous field, equipped with the $\mathbb{Z}^n$-action as $(g \cdot h)(s) = g \cdot_s h(s) \in C_0Cl^*(\mathbb{R}^n)_s$. In the similar way, form the continuous field of $\mathbb{Z}^n$-C^*-algebras $\mathbb{K}(H(\mathbb{R}^n))_s$ over $[0, 1]$ and denote by $C([0, 1], \{\mathbb{K}(H(\mathbb{R}^n))_s\})$ the $\mathbb{Z}^n$-C^*-algebra of continuous sections of the continuous field.

The asymptotic morphism $(\alpha_t) : S \otimes C_0Cl^*(\mathbb{R}^n) \rightsquigarrow \mathbb{K}(H(\mathbb{R}^n))$ induces an asymptotic morphism

$$(\alpha_t^{\sim}) = (\alpha_{t,s}) : S \otimes C([0, 1], \{C_0Cl^*(\mathbb{R}^n)_s\}) \rightsquigarrow C([0, 1], \{\mathbb{K}(H(\mathbb{R}^n))_s\})$$

defined as $\alpha_{t,s}(f)(s) = \alpha_t(f(s))$. Similarly, the asymptotic morphism $(\beta_t) : S \rightsquigarrow C_0Cl^*(\mathbb{R}^n)$ extends to an asymptotic morphism

$$(\beta_t^{\sim}) = (\beta_{t,s}) : S \rightsquigarrow C([0, 1].\{C_0Cl^*(\mathbb{R}^n)_s\})$$

by forming the tensor product of (β_t) with the identity map id on $C([0, 1])$ and then composing with the inclusion $S \subset C([0, 1], S)$ as constant functions, so that $\beta_{t,s}(f)(s) = 1 \otimes \beta_t(f)$. Then consider the diagram of $\mathbb{Z}^n$-equivariant E-theory morphisms

$$\begin{array}{ccccc}
\mathbb{C} & \xrightarrow{[(\beta_t^{\sim})]} & C([0, 1].\{C_0Cl^*(\mathbb{R}^n)_s\}) & \xrightarrow{[(\alpha_t^{\sim})]} & C([0, 1]) \\
\Vert & & \downarrow [\mathrm{ev}_s] & & \downarrow [\mathrm{ev}_s] \\
\mathbb{C} & \xrightarrow{[(\beta_t)]} & C_0Cl^*(\mathbb{R}^n)_s & \xrightarrow{[(\alpha_t)]} & \mathbb{C}
\end{array}$$

where $[\mathrm{ev}_s]$ denotes the equivariant E-theory class corresponding to the evaluation ev_s at $s \in [0, 1]$.

Observe that $[\mathrm{ev}_s]$ is an isomorphism in equivariant E-theory for every s. Indeed, $[\mathrm{ev}_s]$ is an equivariant homotopy (and so E-theory) equivalence.

In fact, the map ev_s is a $*$-homomoprhism and which splits, so that the composition of the splitting with ev_s is the identity map, and as well note that $C([0,1])$ is contractible as a C^*-algebra.

Set $s = 0$. In this case the composition at the bottom line above is the identity class of $E_{\mathbb{Z}^n}(\mathbb{C},\mathbb{C})$. This is because when $s = 0$ the action $\mathbb{Z}^n$ on $\mathbb{R}^n$ is trivial and the asymptotic morphism $(\beta_t) : S \to C_0Cl^*(\mathbb{R}^n)_0$ is homotopic to the trivially equivariant $*$-homomorphism $\beta : S \to C_0Cl^*(\mathbb{R}^n)$. So the required formula $[(\alpha_t)]\circ[(\beta_t)] = [\mathrm{id}]$ follows from that $[\alpha]\circ[\beta] = [\mathrm{id}]$ obtained before. Since the composition at the bottom line above is the identity, it follows that the composition at the top line is the identity as well. Note that the identity may be deduced from that $C([0,1])$ can be identified with $\mathbb{C}$ by evaluation at any s or by contracting to $\mathbb{C}1$ with 1 the unit.

Now set $s = 1$. Since the composition at the top line is the identity, it follows that the composition at the bottom line is the identity, as desired. □

12 Groups with the Haagerup property

Affine Euclidean spaces

A real vector space equipped with a positive-definite inner product is said to be a (real) **Euclidean** vector space.

Definition 12.1. An **affine** Euclidean space over a Euclidean vector space is only a set E equipped with a simply-transitive action of a Euclidean vector space V viewed as an additive group. An affine subspace of E is an orbit in E under the action of a vector subspace of V. A subset X of E generates E if the smallest affine subspace of E which contains X is E.

Remark. Even if E is infinite dimensional, we are not assuming any completeness for E, and moreover, affine subspaces need not be closed.

Example 12.2. Every Euclidean vector space V is an affine Euclidean space over V by forgetting the structure of a Euclidean vector space.

Definition 12.3. Let E be an affine Euclidean space over a Euclidean vector space V. If $e_1, e_2 \in E$, and if v is the unique vector of V such that $v \cdot e_1 = e_1 + v = e_2$, then we define the distance between e_1 and e_2 to be $d(e_1, e_2) = \|v\|$.

Let Z be a subset of an affine Euclidean space E. Let $d^2 : Z \times Z \to \mathbb{R}$ be the square of the distance function so that $d^2(z, w) = d(z, w)^2$ for all

$z, w \in Z$. The squared distance function has the following three properties: (a) $d^2(z,z) = 0$ for all $z \in Z$; (b) $d^2(z,w) = d^2(w,z)$ for all $(z,w) \in Z \times Z$; (c)

$$\sum_{i,j=1}^{n} a_i d^2(z_i, z_j) a_j \le 0$$

for all n, all $z_1, \cdots, z_n \in Z$, and all $a_1, \cdots, a_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} a_i = 0$.

Proof. (Added). Indeed, we compute

$$\begin{aligned}
\sum_{i,j=1}^{n} a_i d^2(z_i, z_j) a_j &= \sum_{i,j=1}^{n} a_i \|z_i - z_j\|^2 a_j \\
&= \sum_{i,j=1}^{n} a_i \langle z_i - z_j, z_i - z_j \rangle a_j \\
&= \sum_{i,j=1}^{n} a_i (\|z_i\|^2 - 2\langle z_i, z_j\rangle + \|z_j\|^2) a_j \\
&= \sum_{i=1}^{n} a_i \|z_i\|^2 \sum_{j=1}^{n} a_j - 2\langle \sum_{i=1}^{n} a_i z_i, \sum_{j=1}^{n} a_j z_j \rangle + \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j \|z_j\|^2 \\
&= -2 \|\sum_{i=1}^{n} a_i z_i\|^2 \le 0.
\end{aligned}$$

□

Proposition 12.4. *Let Z be a set and let $b : Z \times Z \to \mathbb{R}$ be a function with the above three properties (a), (b), and (c). Then there is a map Φ from Z into an affine Euclidean space E such that the image of Φ generates E and that*

$$b(z_1, z_2) = d^2(\Phi(z_1), \Phi(z_2))$$

for all $z_1, z_2 \in Z$.

If $\Phi' : Z \to E'$ is another such map into another affine Euclidean space, then there is a unique isometry $\Psi : E \to E'$ such that $\Psi(\Phi(z)) = \Phi'(z)$ for every $z \in Z$, as that the diagram commutes

$$\begin{tikzcd}
Z \arrow[r, "\Phi"] \arrow[d, equal] & E \arrow[d, "\Psi"] \\
Z \arrow[r, "\Phi'"] & E'.
\end{tikzcd}$$

Proof. Denote by $f_0(Z,\mathbb{R})$ the real vector space of all finitely supported, real-valued functions on Z with sums equal to zero:

$$f_0(Z,\mathbb{R}) = \{f : Z \to \mathbb{R} \,|\, \mathrm{supp}(f) \text{ finite}, \sum_{z\in Z} f(z) = 0\}.$$

Define the positive semidefinite form on $f_0(Z,\mathbb{R})$ by

$$\langle f_1, f_2\rangle = -\frac{1}{2}\sum_{z_1,z_2\in Z} f(z_1)b(z_1,z_2)f(z_2) \equiv b^{\sim}(f_1,f_2).$$

It is clear that the form $b^{\sim}$ is bilinear and $b^{\sim}(f,f) \geq 0$ by the property (c) of the form b.

The set of all $f \in f_0(Z,\mathbb{R})$ with $\langle f,f\rangle = 0$ is a vector subspace of $f_0(Z,\mathbb{R})$, and is denoted by $f_0(Z,\mathbb{R})_0$.

Indeed, if $f_1, f_2 \in f_0(Z,\mathbb{R})_0$, then $\langle f_1+f_2, f_1+f_2\rangle = 2\langle f_1,f_2\rangle$. We then use the Cauchy-Schwarz inequality in this case. In fact, for any $t\in\mathbb{R}$,

$$0 \leq \langle tf_1+f_2, tf_1+f_2\rangle = t^2\langle f_1,f_1\rangle + 2t\langle f_1,f_2\rangle + \langle f_2,f_2\rangle,$$

so that the discriminant is non-positive: $\langle f_1,f_2\rangle^2 - \langle f_1,f_1\rangle\langle f_2,f_2\rangle \leq 0$.

Let $V = f_0(Z,\mathbb{R})/f_0(Z,\mathbb{R})_0$ be the quotient space, with the structure of a Euclidean vector space.

Let $f_1(Z,\mathbb{R})$ be the set of all finitely supported functions on Z, with sums equal to 1. Two functions of $f_1(Z,\mathbb{R})$ are said to be equivalent if their difference function belongs to $f_0(Z,\mathbb{R})_0$. The set of equivalence classes becomes an affine Euclidean space $E = f_1(Z,\mathbb{R})/f_0(Z,\mathbb{R})_0$ over V.

Indeed, for classes $[g]\in E$ and $[h]\in V$, we may define $[g]+[h] = [g+h]$ in E. Because, if $g\sim g'$ and $h\sim h'$, then $g+h, g'+h' \in f_1(Z,\mathbb{R})$, and $(g+h)-(g'+h') = (g-g')+(h-h') \in f_0(Z,\mathbb{R})_0$.

Define $\Phi : Z \to E$ by $\Phi(z) = [\chi_z]$, with χ_z the characteristic function at $z\in Z$. Then

$$\begin{aligned}
&d^2(\Phi(z_1),\Phi(z_2)) = d^2([\chi_{z_1}],[\chi_{z_2}]) = \langle [\chi_{z_1}]-[\chi_{z_2}], [\chi_{z_1}]-[\chi_{z_2}]\rangle \\
&= \langle [\chi_{z_1}],[\chi_{z_1}]\rangle - 2\langle [\chi_{z_1}],[\chi_{z_2}]\rangle + \langle [\chi_{z_2}],[\chi_{z_2}]\rangle \\
&= \langle \chi_{z_1},\chi_{z_1}\rangle - 2\langle \chi_{z_1},\chi_{z_2}\rangle + \langle \chi_{z_2},\chi_{z_2}\rangle \\
&= -\frac{1}{2}\sum_{k_1,k_2\in\mathbb{Z}} \chi_{z_1}(k_1)b(k_1,k_2)\chi_{z_1}(k_2) + \sum_{k_1,k_2\in\mathbb{Z}} \chi_{z_1}(k_1)b(k_1,k_2)\chi_{z_2}(k_2) \\
&\quad -\frac{1}{2}\sum_{k_1,k_2\in\mathbb{Z}} \chi_{z_2}(k_1)b(k_1,k_2)\chi_{z_2}(k_2) \\
&= \frac{1}{2}b(z_1,z_1) + b(z_1,z_2) - \frac{1}{2}b(z_2,z_2) = b(z_1,z_2)
\end{aligned}$$

as required, where the equality $\langle [\chi_{z_1}], [\chi_{z_2}] \rangle = \langle \chi_{z_1}, \chi_{z_2} \rangle$ is well defined.

If $\Phi' : Z \to E'$ is another such map, then the unique isometry $\Psi : E \to E'$ is given by the formula:

$$\Psi(\Phi(f)) = \sum_{z \in Z} f(z) \Phi'(z).$$

In particular, for any $z \in Z$,

$$\Psi(\Phi(z)) = \Psi(\chi_z) = \sum_{k \in Z} \chi_z(k) \Phi'(k) = \Phi'(z).$$

□

Exercise. In an affine space E over V one can form linear combinations so long as the coefficients sum to 1.

If $\Psi : E \to E'$ is an isometry of affine Euclidean spaces, then the finite sum of coefficients $\sum_j a_j = 1$ implies that $\Psi(\sum_j a_j e_j) = \sum_j a_j \Psi(e_j)$.

Proof. (Added). For instance, for any $e_1, e_2 \in E$, there is $v \in V$ such that $e_2 = e_1 + v$. As well, $e_1 + tv = (1-t)e_1 + te_2 \in E$ for $t \in [0,1]$.

Since Ψ is an isometry, any line segment is mapped isometrically to a line segment without no more or less scaling. Hence, it says and holds that $\Psi((1-t)e_1 + te_2) = (1-t)\Psi(e_1) + t\Psi(e_2)$. □

Definition 12.5. Let Z be a set. A function $b : Z \times Z \to \mathbb{R}$ is said to be a **negative type kernel** if b has the properties (a), (b), and (c) listed above.

According Proposition 12.4 above, mappings from Z into affine Euclidean spaces are classified up to isometry, by negative type kernels.

Isometric group actions

Let E be an affine Euclidean space and suppose that a group G acts on E by isometries. For any point $e \in E$, the function from G into E is defined by $g \mapsto g \cdot e$. There is an **associated** negative type kernel (corrected) $b : G \times G \to \mathbb{R}$ as

$$b(g_1, g_2) = d^2(g_1 \cdot e, g_2 \cdot e) = \langle g_1 \cdot e - g_2 \cdot e, g_1 \cdot e - g_2 \cdot e \rangle.$$

Indeed, (a) $b(g,g) = 0$ for any $g \in G$; (b) $b(g_1, g_2) = b(g_2, g_1)$ for all $g_1, g_2 \in G$; (c) if $\sum_{j=1}^n a_j = 0$ in $\mathbb{R}$, then

$$\begin{aligned}
&\sum_{i,j=1}^n a_i b(g_i, g_j) a_j = \sum_{i,j=1}^n a_i \langle g_i \cdot e - g_j \cdot e, g_i \cdot e - g_j \cdot e \rangle a_j \\
&= \sum_{i,j=1}^n a_i \|g_i \cdot e\|^2 a_j - 2\langle \sum_{i=1}^n a_i (g_i \cdot e), \sum_{j=1}^n a_j (g_j \cdot e) \rangle + \sum_{i,j=1}^n a_i \|g_j \cdot e\|^2 a_j \\
&= -2\| \sum_{j=1}^n a_j (g_j \cdot e)\|^2 \leq 0.
\end{aligned}$$

Since G acts by isometries, the function b is G-invariant in the sense that

$$b(g_1, g_2) = b(gg_1, gg_2) \quad \text{for any } g, g_1, g_2 \in G.$$

Indeed,

$$b(g_1, g_2) = \|g_1 \cdot e - g_2 \cdot e\|^2 = \|g(g_1 \cdot e - g_2 \cdot e)\|^2 = b(gg_1, gg_2).$$

As a result, the function b is determined by the one-variable function $b(e, g) \equiv f(g)$ on G. Because $b(g_1, g_2) = b(g_1^{-1} g_1, g_1^{-1} g_2) = f(g_1^{-1} g_2)$.

Definition 12.6. A real-valued, function f on a group G is said to be a **negative-type function** if (a) $f(e) = 0$ for e the unit of G, (b) $f(g) = f(g^{-1})$ for all $g \in G$, and (c)

$$\sum_{i,j=1}^n a_i f(g_i^{-1} g_j) a_j \leq 0$$

for all n, $g_1, \cdots, g_n \in G$ and all $a_1, \cdots a_n \in \mathbb{R}$ such that $\sum_{i=1}^n a_i = 0$.

Example 12.7. (Added). Check that for $f(g) = b(e, g)$ defined, we have (a) $f(e) = b(e, e) = 0$, (b)

$$f(g) = b(e, g) = b(g^{-1}, g^{-1}g) = b(e, g^{-1}) = f(g^{-1}),$$

and (c)

$$\sum_{i,j=1}^n a_i f(g_i^{-1} g_j) a_j = \sum_{i,j=1}^n a_i b(e, g_i^{-1} g_j) a_j = \sum_{i,j=1}^n a_i b(g_i, g_j) a_j \leq 0.$$

Proposition 12.8. *Let G be a group (corrected) and let f be a negative-type function on G. Then there is an isometric action of G on an affine Euclidean space E and a point $e' \in E$ such that the orbit of e' generates E, and such that $f(g) = d^2(e', g \cdot e')$ for all $g \in G$.*

Proof. There is an affine Euclidean space E associated to the negative type kernel $b(g_1, g_2) = f(g_1^{-1} g_2)$ as constructed before as $E = f_1(G, \mathbb{R})/f_0(G, \mathbb{R})_0$ over $V = f_0(G, \mathbb{R})/f_0(G, \mathbb{R})_0$, and there is a map $\Phi : G \to E$ such that the image $\Phi(G)$ generates E and

$$b(g_1, g_2) = d^2(\Phi(g_1), \Phi(g_2)) = \|\Phi(g_1) - \Phi(g_2)\|^2.$$

Now fix $h \in G$ and consider the map $\Phi_h(g) = \Phi(hg)$. Then

$$\begin{aligned} d^2(\Phi_h(g_1), \Phi_h(g_2)) &= d^2(\Phi(hg_1), \Phi(hg_2)) = b(hg_1, hg_2) \\ &= b(g_1, g_2) = d^2(\Phi(g_1), \Phi(g_2)). \end{aligned}$$

It follows from the uniqueness that there is a unique isometry $\Psi_h : E \to E$ such that $\Psi_h(\Phi(g)) = \Phi_h(g)$. The map $h \mapsto \Psi_h$ is the required action on E, and $\Phi(e) = e'$ is the required point in E, with e the unit of G.

Indeed,

$$\Psi_{g_1 g_2}(\Phi(g)) = \Phi_{g_1 g_2}(g) = \Phi(g_1 g_2 g) = \Phi_{g_1}(\Phi_{g_2}(g)) = \Psi_{g_1}(\Psi_{g_2}(\Phi(g))),$$

and

$$f(g) = b(e, g) = d^2(\Phi(e), \Phi(g \cdot e)) = d^2(e', \Phi_g(e)) = d^2(e', \Psi_g(e')).$$

□

Remarks. There is also a uniqueness assertion as that if E' is another affine Euclidean space equipped with an isometric G-action Ψ', and $e'' \in E'$ is a point such that $f(g) = d(e'', \Psi'_g(e''))^2$ for all $g \in G$, then there is a G-equivariant isometry $\Psi'' : E \to E'$ such that $\Psi'(e') = e''$, as that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\Psi_G} & E \\ {\scriptstyle\Psi''}\downarrow & & \downarrow{\scriptstyle\Psi''} \\ E' & \xrightarrow{\Psi'_G} & E'. \end{array}$$

Note that both of the actions Ψ and Ψ' are isometric and apply to the uniqueness.

Proposition 12.8 above is a reminiscent of the GNS construction in C^*-algebra theory, which associates to each state of a C^*-algebra, both of a Hilbert space representation and a unit vector in the representation space.

Exercise. Let E be an affine Euclidean space over a Euclidean vector space V. Suppose that a group G acts on E by isometries. Then there is a linear representation π of G by orthogonal transformations on V such that

$$g \cdot (e+v) = g \cdot e + \pi_g v, \quad g \in G, e \in E, v \in V.$$

Proof. Define $\pi_g v = g \cdot (e+v) - g \cdot e$. This is well defined because

$$\begin{aligned}
&\|g \cdot (e+v) - g \cdot e - g \cdot (e'+v) + g \cdot e'\|^2 \\
&= \langle g \cdot (e+v) - g \cdot e - g \cdot (e'+v) + g \cdot e', g \cdot (e+v) - g \cdot e - g \cdot (e'+v) + g \cdot e' \rangle \\
&= \langle g \cdot (e+v), g \cdot (e+v) \rangle - \cdots + \langle g \cdot e', g \cdot e' \rangle \\
&= \langle e+v, e+v \rangle - \cdots + \langle e', e' \rangle = \cdots = 0.
\end{aligned}$$

Hence e may be changed and be zero when $0 \in E$ but $0 \notin E$ in general. If $0 \in E$, then the following calculations are trivial. Then

$$\|\pi_g v\|^2 = \|g \cdot (e+v) - g \cdot e\|^2 = d^2(g \cdot (e+v), g \cdot e) = d^2(e+v, e) = \|v\|^2.$$

Also,

$$\begin{aligned}
&\|\pi_g(v_1+v_2) - \pi_g v_1 - \pi_g v_2\|^2 \\
&= \|g \cdot (e+v_1+v_2) - g \cdot e - g \cdot (e+v_1) + g \cdot e - g \cdot (e+v_2) + g \cdot e\|^2 \\
&= \langle g \cdot (e+v_1+v_2), g \cdot (e+v_1+v_2) \rangle - \cdots + \langle g \cdot e, g \cdot e \rangle \\
&= \langle e+v_1+v_2, e+v_1+v_2 \rangle - \cdots + \langle e, e \rangle = \cdots = 0,
\end{aligned}$$

and as well,

$$\begin{aligned}
&\|\pi_{g_1 g_2} v - \pi_{g_1}(\pi_{g_2} v)\|^2 \\
&= \|(g_1 g_2) \cdot (e+v) - (g_1 g_2) \cdot e - \pi_{g_1}(g_2 \cdot (e+v) - g_2 \cdot e)\|^2 \\
&= \|(g_1 g_2) \cdot (e+v) - (g_1 g_2) \cdot e - \pi_{g_1}(g_2 \cdot (e+v)) + \pi_{g_1}(g_2 \cdot e)\|^2 \\
&= \|(g_1 g_2) \cdot (e+v) - (g_1 g_2) \cdot e - g_1 \cdot (e' + g_2 \cdot (e+v)) + g_1 \cdot e' \\
&\quad + g_1 \cdot (e' + g_2 \cdot e) - g_1 \cdot e'\|^2 = \cdots = 0,
\end{aligned}$$

where in the last step $\cdots$ we need to calculate inner product expansion but omitted. $\square$

Exercise. If a Euclidean vector space V is viewed as an affine space over itself, then for every isometric action of G on V there is a linear representation π of G by orthogonal transformations on V such that

$$g \cdot v = g \cdot 0 + \pi(g)v, \quad g \in G, v \in V.$$

For every $s \in [0,1]$, the scaled actions $g \cdot_s v = s(g \cdot 0) + \pi_g v$ are also isometric actions of G on E.

Proof. The first part is done in the last proof above, by choosing 0 as a representative for the action π_g.

Note that

$$\begin{aligned} (g_1 g_2) \cdot_s v &= s((g_1 g_2) \cdot 0) + \pi_{g_1 g_2} v = s(g_1 \cdot (g_2 \cdot 0)) + \pi_{g_1} \pi_{g_2} v, \\ g_1 \cdot_s (g_2 \cdot_s v) &= g_1 \cdot_s (s(g_2 \cdot 0) + \pi_{g_2} v) \\ &= s(g_1 \cdot 0) + \pi_{g_1}(s(g_2 \cdot 0) + \pi_{g_2} v) \\ &= s(g_1 \cdot 0) + \pi_{g_1}(s(g_2 \cdot 0)) + \pi_{g_1} \pi_{g_2} v \end{aligned}$$

with $g_1 \cdot s(g_2 \cdot 0) = s(g_1 \cdot 0) + \pi_{g_1} s(g_2 \cdot 0)$ and $g_1 \cdot s(g_2 \cdot 0) = s(g_1 \cdot (g_2 \cdot 0))$ because of being isometric of the given G-action, and hence the first and second equations above are equal. □

The Haagerup property

Definition 12.9. Let G be a countable discrete group. An isometric action of G on an affine Euclidean space E is **metrically proper** if for some (and hence for every) point x of E, for every $r > 0$, there are only finitely many $g \in G$ such that $d(x, g \cdot x) \leq r$. In this case, we may write it as

$$\lim_{g \in F, |F| \to \infty} d(x, g \cdot x) = \infty.$$

Example 12.10. (Added). For instance, the cyclic group of order 2 denoted as $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ identified with the set $\{\pm 1\}$ acts on $\mathbb{R}$ isometrically and metrically properly as $\mathbb{R} \ni x \mapsto -x \in \mathbb{R}$, with $d(x, -x) = 2|x|$.

Definition 12.11. A countable discrete group has the **Haagerup property** if it admits a metrically proper isometric action on an affine Euclidean space.

Proposition 12.12. *A countable discrete group G has the Haagerup property if and only if there exists a proper, negative-type real-valued function on G, that is, a negative-type function for which the inverse image of each bounded set of $\mathbb{R}$ is a finite subset of G.*

Remark. (Extended). Refer to the metric approximation property of Haagerup [22]. Groups with the Haagerup property are also called **a-T-menable**, by Gromov [19, Section 7]. It is proved by Bekka, Cherix, and Valette [6] that the Haagerup approximation property for a countable group Γ, its admitting a proper function of conditionally negative type, and its being a-T-menable are equivalent, where the first condition means that the C^*-algebra $C_0(\Gamma)$ has an approximate unite consisting of positive definite functions on Γ.

Theorem 12.13. *Every countable amenable group has the Haagerup property.*

Proof. A function $\varphi : G \to \mathbb{C}$ is said to be **positive-definite** if $\varphi(e) = 1$ with $e = 1_G$ the unit of G, $\varphi(g) = \varphi(g^{-1})$, and

$$\sum_{i,j=1}^{n} \overline{\lambda_i}\varphi(g_i^{-1}g_j)\lambda_j \geq 0$$

for all $g_1, \cdots, g_n \in G$ and all $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$. Observe that if φ is positive-definite, then $1 - \mathrm{Re}(\varphi)$ is a negative-type function.

Indeed, with $\mathrm{Re}(\varphi) = \frac{1}{2}(\varphi + \overline{\varphi})$, we have $(1 - \mathrm{Re}(\varphi))(e) = 1 - 1 = 0$, and $(1 - \mathrm{Re}(\varphi))(g) = (1 - \mathrm{Re}(\varphi))(g^{-1})$, and

$$\begin{aligned}
&\sum_{i,j=1}^{n} a_i(1 - \mathrm{Re}(\varphi))(g_i^{-1}g_j)a_j \\
&= \sum_{i,j=1}^{n} a_i a_j - \sum_{i,j=1}^{n} a_i \mathrm{Re}(\varphi))(g_i^{-1}g_j)a_j \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j - \mathrm{Re}(\sum_{i,j=1}^{n} \overline{a_i}\varphi(g_i^{-1}g_j)a_j) \leq 0,
\end{aligned}$$

where $a_1, \cdots, a_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} a_i = 0$.

As one among many characterizations of amenability, G is amenable if and only if there exists a sequence of finitely supported, positive-definite functions φ_n on G which converges pointwise to the constant function 1 on G. Then there is a subsequence (φ_{n_k}) such that the series $\sum_k(1 - \mathrm{Re}(\varphi_{n_k}))$ converges at every point of G. The series limit is a proper, negative-type function.

Note that being zero at $e = 1_G \in G$, having a conjugate symmetry, and being negative type for $1 - \mathrm{Re}(\varphi_{n_k})$ positive are preserved under taking their series limit. $\square$

Theorem 12.14. (Delorme [14]). *A discrete group G with both the Kazhdan property T and the Haagerup property is finite.*

Proof. If G has property T, then every isometric action of G on an affine Hilbert space has a fixed point. This is known as Delorme's theorem. In fact, the converse is true as well. But if an isometric action has a fixed point, it can not be metrically proper, unless G is finite. □

Remark. The reader may refer to de la Harpe and Valette [13] for the theory of groups with Kazhdan property T as an introduction.

Here is a (revised but incomplete) list for convenience, but without proofs, which may be added somewhere.

Table 1: Groups with the Haagerup property

Type	Amenable (countable or not)	Non-amenable
Kazhdan Property T	Finite (or compact)	No
Without Property T (a-T class)	Infinite (or non-compact): Abelian groups, Nilpotent groups, Solvable groups, Their inductive limits, $\cdots$	Free groups, Groups with proper actions on locally finite trees, Coxeter groups, Thompson group, Discrete subgroups of $SO(n,1)$, $SU(n,1)$, $\cdots$

For more information about the Haagerup property, may consult [10] (the lacking item).

Remarks. (Added and edited). See [22] for (finitely generated) free groups.

We may call the class of non-T groups with the Haagerup properpty as the **a-T class**.

It is known that any function of negative type on a Kazhdan T group, associated to a kernel of negative type, is bounded. It is obtained by [8] that any infinite subgroup of a Coxeter group has an unbounded kernel of negativ type. A (abstract) **Coxeter** group may be defined as a group with a countably many generators of order 2 such that each product of two generators has a finite (or infinite) order, as relations ([40]).

The **Thompson** group is generated by Thompson functions with composition as a product, where a Thompson function f is a piecewise linear, increasing function from the closed interval $[0,1]$ to $[0,1]$ with respect to the same dyadic division of each $[0,1]$ such as

$$[0,1] = \cup_{m=0}^{2^n-1}\left[\frac{m}{2^n}, \frac{m+1}{2^n}\right],$$

with $f(0)=0$ and $f(1)=1$. For instance, refer to [41] for more details.

It is known that a connected simple Lie group G has the property T if and only if G is not locally isomorphic to either $SO(n,1)$ or $SU(n,1)$, and if G has a discrete subgroup Γ as a lattice, then G has the property T if and only if Γ does ([40]). See also Robertson [49] and [50].

Example 12.15. (Added). Let G be a finite group. Define a function $f : G \to \mathbb{R}$ by $f(1_G)=0$ and $f(g)=1$ otherwise. Then $f(g)=f(g^{-1})$ for any $g \in G$, and

$$\begin{aligned}
&\sum_{i,j=1}^{n} a_i f(g_i^{-1}g_j)a_j = \sum_{i\neq j} a_i a_j \\
&\leq \sum_{i\neq j} a_i a_j + \sum_{i=1}^{n} a_i^2 = \sum_{i,j=1}^{n} a_i a_j = (\sum_{i=1} a_i)^2 = 0
\end{aligned}$$

where $\sum_{i=1}^{n} a_i = 0$. For any bounded set B of $\mathbb{R}$, $f^{-1}(B) \subset G$ and thus $f^{-1}(B)$ is finite. This shows that any finite group has the Haagerup property.

The BC conjecture for a-T-menable groups

As the main theorem of this lecture (paper),

Theorem 12.16. *Let G be a countable discrete group with the Haagerup property. Then there is a proper G-C^*-algebra $\mathfrak{B}$ and E_G-theory classes $\alpha \in E_G(\mathfrak{B},\mathbb{C})$ and $\beta \in E_G(\mathbb{C},\mathfrak{B})$ such that $\alpha \circ \beta = 1 \in E_G(\mathbb{C},\mathbb{C})$.*

Corollary 12.17. *Let G be a countable discrete group with the Haagerup property and let $\mathfrak{D}$ be a G-C^*-algebra. Then the full (or maximal) Baum-Connes assembly map with coefficients in $\mathfrak{D}$ is an isomorphism.*

Moreover, if G is exact, then the reduced Baum-Connes assembly map with coefficients in $\mathfrak{D}$ is an isomorphism.

Remark. Theorem 12.16 and its Corollary 12.17 above are also true for locally compact groups with the Haagerup property. The final conlusion is known to hold without assuming G of being exact.

Problem. Every countable discrete group with the Haagerup property is C^*-exact?

13 The BC conjecture for a-T-menable groups

There are three Parts I, II, III for the proof of Theorem 12.16.

Part I

We may refer to [25] of Higson, Kasparov, and Trout (unchecked in details).

Notation. Let E be an affine Euclidean space over a Euclidean vector space V. Denote by E_a or E_b finite dimensional affine subspaces of E. Denote by V_a the vector subspace of V corresponding to E_a.

If $E_a \subset E_b$, then we denote by $V_{b,a}$ the orthogonal complement of E_a in E_b, which is the orthogonal complement of V_a in V_b, so that

$$E_b = E_a \oplus V_{b,a} \quad \text{and} \quad V_b = V_a \oplus V_{b,a}$$

as a direct sum decomposition, and hence any point e_b of E_b has a unique decomposition as $e_b = e_a + v_{b,a} \in E_a \oplus V_{b,a}$.

For instance, for a real line $L = E_a$ in a real 2-dimensional plane $P = E_b$, we have $P = L \oplus V_{b,a}$ with $V_{b,a} = \mathbb{R}$ a vector space.

Definition 13.1. Let E_a be a finite-dimensional affine Euclidean subspace of E. Set $C_0Cl^*(E_a) = C_0(E_a, Cl^*(V_a))$.

Lemma 13.2. *Let $E_a \subset E_b$ be an inclusion of finite dimensional subspaces of E. There is an isomorphism of graded C^*-algebras*

$$C_0Cl^*(V_{b,a}) \otimes C_0Cl^*(E_a) \cong C_0Cl^*(E_b)$$

by sending $h_1 \otimes h_2$ to $h_1 \cdot h_2$, where $(h_1 \cdot h_2)(v+e) = h_1(v)h_2(e)$ for $v + e \in V_{b,a} \oplus E_a$.

Proof. (Added). As before,

$$\begin{aligned} C_0Cl^*(E_b) &= C_0Cl^*(V_{b,a} \oplus E_a) \\ &\cong C_0(V_{b,a} \oplus E_a) \otimes Cl^*(V_{a,b} \oplus V_a) \\ &\cong C_0(V_{b,a}) \otimes C_0(E_a) \otimes Cl^*(V_{b,a}) \otimes Cl^*(V_a) \\ &\cong C_0(V_{b,a}) \otimes Cl^*(V_{b,a}) \otimes C_0(E_a) \otimes Cl^*(V_a) \\ &\cong C_0Cl^*(V_{b,a}) \otimes C_0Cl^*(E_a). \end{aligned}$$

□

Definition 13.3. (Corrected and editied). Let V_a (or $V_{b,a}$) be a finite dimensional linear subspace of a Euclidean vector space V and denote by C_a

(or $C_{(b,a)}$) : $V_a \to Cl^*(V_a)$ the Clifford operator corresponding to V_a. Define a $*$-homomorphism $\beta_a : S \to S \otimes C_0Cl^*(V_a)$ (or $\beta_a : S \to S \otimes C_0Cl^*(E_a)$ or $\beta_{(b,a)} : S \to S \otimes C_0Cl^*(V_{b,a})$) by

$$\beta_a(f) = ((\mathrm{id} \otimes \beta_a) \circ \Delta)(f) = f(\mathrm{id}_\infty \otimes 1 + 1 \otimes C_a),$$

where $\beta_a = \beta : S \to C_0Cl^*(V_a)$ is defined by $\beta_a(f) = f(C_a) = f(C_a(\cdot))$.

Definition 13.4. (Corrected). Let $E_a \subset E_b$ be an inclusion of finite dimensional affine subspaces of an affine Euclidean space E over a Euclidean vector space V. Define a $*$-homomorphism

$$\beta_{b,a} : S \otimes C_0Cl^*(E_a) \to S \otimes C_0Cl^*(E_b)$$

by using $S \otimes C_0Cl^*(E_b) \cong S \otimes C_0Cl^*(V_{b,a}) \otimes C_0Cl^*(E_a)$ and by

$$S \otimes C_0Cl^*(E_a) \ni f_1 \otimes f_2 \mapsto \beta_{(b,a)}(f_1) \otimes f_2 \in [S \otimes C_0Cl^*(V_{b,a})] \otimes C_0Cl^*(E_a).$$

Lemma 13.5. *Let $E_a \subset E_b \subset E_c$ be double inclusions of finite dimensional affine subspaces of E over V. Then we have $\beta_{c,b} \circ \beta_{b,a} = \beta_{c,a}$.*

Proof. (Detailed). The generators of S as a C^*-algebra are given by $u(x) = e^{-|x|}$ and $v(x) = xe^{-|x|}$. We compute

$$\begin{aligned}
&(\beta_{c,b} \circ \beta_{b,a})(\beta_a(u)) = (\beta_{c,b} \circ \beta_{b,a})[(\mathrm{id} \otimes \beta_a) \circ \Delta](u) \\
&= (\beta_{c,b} \circ \beta_{b,a})(\mathrm{id} \otimes \beta_a)(u \otimes u) = (\beta_{c,b} \circ \beta_{b,a})(u \otimes u(C_a)) \\
&= \beta_{c,b}(\beta_{(b,a)}(u) \otimes u(C_a)) = \beta_{c,b}(u \otimes u(C_{(b,a)}) \otimes u(C_a)) \\
&= u \otimes u(C_{(c,b)}) \otimes u(C_{(b,a)}) \otimes u(C_a),
\end{aligned}$$

and

$$\beta_{c,a}(\beta_a(u)) = \beta_{c,a}(u \otimes u(C_a)) = u \otimes u(C_{(c,a)}) \otimes u(C_a)$$

with

$$C_{(c,b)}(\cdot) \otimes C_{(b,a)}(\cdot) \otimes C_a(\cdot) = C_{(c,a)}(\cdot) \otimes C_a(\cdot)$$

(identified), so that both of computations for u are the same.

Similarly, we compute

$$\begin{aligned}
&(\beta_{c,b} \circ \beta_{b,a})(\beta_a(v)) = (\beta_{c,b} \circ \beta_{b,a})[(\mathrm{id} \otimes \beta_a) \circ \Delta](v) \\
&= (\beta_{c,b} \circ \beta_{b,a})(\mathrm{id} \otimes \beta_a)(v \otimes u + u \otimes v) = (\beta_{c,b} \circ \beta_{b,a})(v \otimes u(C_a) + u \otimes v(C_a)) \\
&= \beta_{c,b}(\beta_{(b,a)}(v) \otimes u(C_a)) + \beta_{c,b}(\beta_{(b,a)}(u) \otimes v(C_a)) \\
&= \beta_{c,b}(v \otimes v(C_{(b,a)}) \otimes u(C_a)) + \beta_{c,b}(u \otimes u(C_{(b,a)}) \otimes v(C_a)) \\
&= v \otimes v(C_{(c,b)}) \otimes v(C_{(b,a)}) \otimes u(C_a) + u \otimes u(C_{(c,b)}) \otimes u(C_{(b,a)}) \otimes v(C_a),
\end{aligned}$$

and

$$\begin{aligned}\beta_{c,a}(\beta_a(v)) &= \beta_{c,a}(v \otimes u(C_a) + u \otimes v(C_a)) \\ &= v \otimes v(C_{(c,a)}) \otimes u(C_a) + u \otimes u(C_{(c,a)}) \otimes v(C_a)\end{aligned}$$

with

$$C_{(c,b)}(\cdot) \otimes C_{(b,a)}(\cdot) = C_{(c,a)}(\cdot)$$

(identified), so that both of computations for v are the same as well. □

Corollary 13.6. (*Added*). *The graded C^*-algebras $S \otimes C_0 Cl^*(E_a)$ where E_a ranges over finite dimensional affine subspaces of E over V form a directed system as*

$$\cdots S \otimes C_0 Cl^*(E_a) \xrightarrow{\beta_{g,a}} S \otimes C_0 Cl^*(E_b) \xrightarrow{\beta_{c,b}} S \otimes C_0 Cl^*(E_c) \cdots$$

Definition 13.7. Let E be an affine Euclidean space over a Euclidean vector space V. Suppose that $\{E_a\}$ be a family of finite dimensional affine subspaces of E directed by inclusions and $\{V_a\}$ be a corresponding family of finite dimensional subspaces of V. We define the C^*-algebra of E over V to be the direct limit C^*-algebra:

$$\varinjlim_{E_a \subset E} S \otimes C_0 Cl^*(E_a) = \varinjlim_{E_a \subset E; V_a \subset V} S \otimes C_0(E_a, Cl^*(V_a)),$$

which may be denoted as $SC_0(E, Cl^*(V))$ (but not suspended).

An action of G by isometries on E makes the (unsuspended) C^*-algebra of E over V into a G-C^*-algebra.

Proof. (Edited). Define $*$-isomorphisms

$$\Phi_g : C_0 Cl^*(E_a) = C_0(E_a, Cl^*(V_a)) \to C_0 Cl^*(gE_a) = C_0(gE_a, Cl^*(gV_a))$$

by $(\Phi_g f)(e) = \Psi_g(f(g^{-1}e))$, where $\Psi_g : Cl^*(V_a) \to Cl^*(gV_a)$ is induced from the linear isometries of V associated to the action of G on E, as shown in an exercise at Page 105 above. □

Lemma 13.8. *The following diagram commutes:*

$$\begin{array}{ccc} S \otimes C_0 Cl^*(E_a) & \xrightarrow{\beta_{b,a}} & S \otimes C_0 Cl^*(E_b) \\ {\scriptstyle \mathrm{id}\otimes\Phi_g}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\Phi_g} \\ S \otimes C_0 Cl^*(gE_a) & \xrightarrow{\beta_{gb,ga}} & S \otimes C_0 Cl^*(gE_b) \end{array}$$

where the map $\beta_{gb,ga} : SC_0 Cl^*(gE_a) \to SC_0 Cl^*(gE_b)$ may be defined as for the diagram to be commutative.

Proof. (Added). Indeed, define

$$\beta_{gb,ga}((\mathrm{id} \otimes \Phi_g)(f_1 \otimes f_2)) = (\mathrm{id} \otimes \Phi_g)(\beta_{b,a}(f_1 \otimes f_2))$$

with $(\mathrm{id} \otimes \Phi_g)(f_1 \otimes f_2) = f_1 \otimes \Phi_g(f_2) = f_1 \otimes \Psi_g(f_2(g^{-1}\cdot))$ and

$$\begin{aligned}
&(\mathrm{id} \otimes \Phi_g)(\beta_{b,a}(f_1 \otimes f_2)) = (\mathrm{id} \otimes \Phi_g)(\beta_{(b,a)}(f_1) \otimes f_2) \\
&= (\mathrm{id} \otimes \Phi_g)((f_1 \otimes f_1(C_{(b,a)}) \otimes f_2) = (\mathrm{id} \otimes \Phi_g)(f_1 \otimes (f_1(C_{(b,a)}) \otimes f_2)) \\
&= f_1 \otimes \Psi_g[f_1(g^{-1}C_{(b,a)}(\cdot)) \otimes f_2(g^{-1}\cdot)].
\end{aligned}$$

We need to check that this definition extends by linearity and by involutive multiplicativity to a $*$-homomorphism. □

The lemma above asserts that the maps $\mathrm{id} \otimes \Phi_g$ are compatible with the maps $\beta_{b,a}$ for the directed system to define $SC_0(E, Cl^*(V))$. Consequently,

Corollary 13.9. *(Added). The C^*-algebra $SC_0(E, Cl^*(V))$ (unsuspended) is a G-C^*-algebra by the action extended from* $\mathrm{id} \otimes \Phi_G$ *as a direct limit.*

Theorem 13.10. *Let E be an affine Euclidean space over a Euclidean vector space V equipped with a metrically proper action of a countable discrete group G. Then the unsuspended C^*-algebra $SC_0(E, Cl^*(V))$ as a direct limit of suspended C^*-algebras $SC_0(E_a, Cl^*(V_a))$ is a proper G-C^*-algebra.*

Proof. Denote by $Z_0SC_0(E_a, Cl^*(V_a))$ the grading degree zero part of the center of the C^*-algebra $SC_0(E_a, Cl^*(V_a))$. It is isomorphic to the algebra of continuous functions on the locally compact space $[0, \infty) \times E_a$ (a direct product) vanishing at infinity.

Note that the grading degree zero part corresponds to that of S of even functions on $\mathbb{R}$, identified with functions on $[0, \infty)$, and that the Clifford C^*-algebra $Cl^*(V_a)$ is isomorphic to a single matrix C^*-algebra if $\dim V_a$ is even, and to the direct sum of two matrix C^*-algebras if $\dim V_a$ is odd, so that the center $ZCl^*(V_a)$ of $Cl^*(V_a)$ is isomorphic to $\mathbb{C}$ if $\dim V_a$ is even, and to $\mathbb{C}^2$ if $\dim V_a$ is odd, and hence the degree zero part $Z_0Cl^*(V_a)$ of $ZCl^*(V_a)$ is isomorphic to $\mathbb{C}$ in both cases.

The linking map $\beta_{b,a}$ embeds as

$$\beta_{b,a} : Z_0SC_0(E_a, Cl^*(V_a)) \to Z_0SC_0(E_b, Cl^*(V_b))$$

to form a directed system and its direct limit $Z_0SC_0(E, Cl^*(V))$, which is a C^*-subalgebra of $SC_0(E, Cl^*(V))$, and in fact is the degree zero part of the center of $SC_0(E, Cl^*(V))$, so that the notation is compatible in this sense.

It holds that the product set $Z_0SC_0(E, Cl^*(V))\cdot SC_0(E, Cl^*(V))$ is dense in $SC_0(E, Cl^*(V))$.

Note that the product $(f_1\otimes 1_k)\cdot(f_2\otimes v) = f_1f_2\otimes v$ in $Z_0SC_0(E_a, Cl^*(V_a))\cdot SC_0(E_a, Cl^*(V_a))$ with $1_k, v \in Cl^*(V_a)$ and 1_k the identity element of $Cl^*(V_a)$. Moreover, replace f_2 with (f_λ) a (countable) approximate unit for $SC_0(E_a)$ to have the limit $\lim_\lambda f_1f_\lambda = f_1$. Then the set of $\mathbb{Q}$-linear spans of simple tensors $f_1f_\lambda \otimes v$ is dense in there.

The Gelfand spectrum of $Z_0SC_0(E, Cl^*(V))$ is the locally compact space $[0,\infty) \times \overline{E}$ a product space, where $\overline{E}$ is the metric space completion of E, and the product space is given the weakest topology for which the projection to $\overline{E}$ is weakly continuous and the function $t^2 + d^2(e_0, e)$ on $[0,\infty) \times E$ is continuous for some and hence any $e_0 \in E$.

Note that $\overline{E}$ is an affine space over the Hilbert space $\overline{V}$ as a l^2-completion of V, and that by identifying $\overline{E}$ as an orbit of $\overline{V}$ one can transfer the weak topology of the Hilbert space $\overline{V}$ to $\overline{E}$.

If G acts merically and properly on V, then the induced action on the locally compact space $[0,\infty) \times \overline{E}$ is proper. □

Remark. The argument above is due to G. Skandalis (but not cited in the text).

Part II

We may refer to [25] of Higson, Kasparov, and Trout (unchecked in details).

We assume that E is a countably infinite dimensional affine Euclidean space on which a countable discrete group G acts by isometries.

If G has the Haagerup property, then G acts properly and isometrically on some countably infinite dimensional affine space E.

Fix a point $e_0 \in E$. This point is viewed as an affine subspace of E, and thus there is an inclusion $*$-homomorphism as the composition:

$$\beta : S = S \otimes \mathbb{C} \xrightarrow{\beta_{a,0}} SC_0(E_a, Cl^*(V_a)) \to SC_0(E, Cl^*(V)),$$

by $\beta(f) = f(1 \otimes 1 + 1 \otimes C_{a,0})$, where the image of $\beta_{a,0}$ is contained in all C^*-subalgebras $S \otimes C_0(E_a, Cl^*(V_a))$ if $e_0 \in E_a$ for every E_a, and

$$\beta_{a,0}(f) = ((\mathrm{id} \otimes \beta_{(a,0)}) \circ \Delta)(f)$$

where $\beta_{(a,0)}(f) = f(C_{(a,0)})$, with $C_{(a,0)} : E_a \to Cl^*(V_a)$ defined by $C_{(a,0)}(e) = e - e_0 \in V_a$.

Lemma 13.11. *The asymptotic morphism* $(\beta_t) : S \rightsquigarrow SC_0(E, Cl^*(V))$ *defined by* $\beta_t(f) = \beta(f_t)$*, where* $\beta : S \to SC_0(E, Cl^*(V))$ *is defined above and* $f_t(x) = f(t^{-1}x)$*, is* G*-equivariant.*

Proof. Show that if e_0 and e_1 are two points in a finite dimensional affine subspace E_a of E, then for every $f \in S$,

$$\lim_{t\to\infty} \|\Psi_g f(g^{-1}t^{-1}C_{(a,0)}) - f(t^{-1}C_{(ga,1)})\| = 0,$$

where $C_{ga,1}(e) = e - e_1$ for $e \in gE_a$ (corrected as below) with $g \cdot e_0 = e_1$.

Indeed, show that the diagram commutes asymptotically:

$$\begin{array}{ccc} S & \xrightarrow{\beta_t=(\mathrm{id}\otimes\beta_{(a,0)})\otimes\Delta)(\cdot)(t^{-1}\cdot)} & SC_0(E_a, Cl^*(V_a)) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \Phi_g} \\ S & \xrightarrow{\beta_t=(\mathrm{id}\otimes\beta_{(ga,1)})\otimes\Delta)(\cdot)(t^{-1}\cdot)} & SC_0(gE_a, Cl^*(gV_a)) \end{array}$$

with $\beta_{(ga,1)} : gE_a \to Cl^*(gV_a)$ defined as $\beta_{(a,0)}$ similarly.

It suffices to compute the limit for the generating functions $f_\pm(x) = \frac{1}{x\pm i}$ for $S = C_0(\mathbb{R})$. For these one has

$$\begin{aligned} &\|\Psi_g f_\pm(g^{-1}t^{-1}C_{(a,0)}) - f_\pm(t^{-1}C_{(ga,1)})\| \\ &= \|\Psi_g(g^{-1}t^{-1}C_{(a,0)} \pm i1)^{-1} - (t^{-1}C_{(ga,1)} \pm i1)^{-1}\| \\ &= \|(t^{-1}C_{(a,0)} \pm i1)^{-1} - (t^{-1}C_{(ga,1)} \pm i1)^{-1}\| \\ &= \|(t^{-1}C_{(a,0)} \pm i1)^{-1}(t^{-1}C_{(ga,1)} - t^{-1}C_{(a,0)})(t^{-1}C_{(ga,1)} \pm i1)^{-1}\| \\ &\le \|(t^{-1}C_{(a,0)} \pm i1)^{-1}\|\frac{1}{t}\|C_{(ga,1)} - C_{(a,0)}\|\|(t^{-1}C_{(ga,1)} \pm i1)^{-1}\|, \end{aligned}$$

which goes to zero as $t \to \infty$, where note that $f_\pm(x)$ are rational functions of degree one with respect to x, so that the action Ψ_g is cancelled with g^{-1}, and that

$$(C_{(ga,1)} - C_{(a,0)})(e) = (e - e_1) - (e - e_0) = e_0 - e_1$$

and that $\|C_{(ga,1)} - C_{(a,0)}\| = \|e_0 - e_1\|$. □

Definition 13.12. Define the E_G-theory class $[\beta] \in E_G(\mathbb{C}, SC_0(E, Cl^*(V)))$ induced by the G-equivariant asymptotic morphism $(\beta_t) : S \rightsquigarrow SC_0(E, Cl^*(V))$ defined by $\beta_t(f) = \beta(f_t)$.

By broadening Definition 7.2 to the context of affine Euclidean spaces,

Definition 13.13. Let E_a be a finite dimensional affine subspace of an affine Euclidean space E over a Euclidean vector space V, with V_a an associated subspace of V. Define the Hilbert space of E_a as the Hilbert space of all square integrable $Cl^*(V_a)$-valued functions on E_a:

$$H(E_a) = L^2(E_a, Cl^*(V_a)).$$

This is a graded Hilbert space with grading inherited from that of $Cl^*(V_a)$.

Lemma 13.14. *Let $E_a \subset E_b$ be an inclusion of finite dimensional affine subspaces of E over V and let $V_{b,a}$ be the orthogonal complement of E_a in E_b. Then there is an isomorphism of graded Hilbert spaces:*

$$H(V_{b,a}) \otimes H(E_a) \cong H(E_b),$$

by sending $h_1 \otimes h_2$ to $h_1 \cdot h_2$, where $(h_1 \cdot h_2)(v+e) = h_1(v)h_2(e)$ for $v+e \in V_{b,a} \oplus E_a = E_b$.

Proof. (Added). We have

$$\begin{aligned} H(E_b) &= L^2(E_b, Cl^*(E_b)) \\ &\cong L^2(V_{b,a} \oplus E_a) \otimes Cl^*(V_{b,a} \oplus E_a) \\ &\cong L^2(V_{b,a}) \otimes L^2(E_a) \otimes Cl^*(V_{b,a}) \otimes Cl^*(V_a) \\ &\cong L^2(V_{b,a}) \otimes Cl^*(V_{b,a}) \otimes L^2(E_a) \otimes Cl^*(V_a) \\ &\cong H(V_{b,a}) \otimes H(E_a). \end{aligned}$$

□

Definition 13.15. If $W = V_a$ (or $V_{(b,a)}$) is a finite dimensional Euclidean vector space V, then the **basic** vector $f_W = f_a \in H(W)$ (or $f_{(b,a)}$) is defined by

$$f_W(w) = f_a(w) = \pi^{-\frac{1}{4}\dim W} e^{-\frac{1}{2}\|w\|^2} 1 \in Cl^*(W)$$

for $w \in W$.

Remark. The constant $\pi^{-\frac{1}{4}\dim W}$ is chosen so that $\|f_W\| = 1$.

Proof. (Added). Compute that

$$\begin{aligned} \langle f_W, f_W \rangle &= \langle e^{-\frac{1}{2}\|w\|^2}, e^{-\frac{1}{2}\|w\|^2} \rangle \|\pi^{-\frac{1}{4}\dim W} 1\|^2 \\ &= \pi^{-\frac{1}{2}\dim W} \int_W e^{-\sum_{j=1}^{\dim W} w_j^2} dw \\ &= \pi^{-\frac{1}{2}\dim W} \Pi_{j=1}^{\dim W} \int_{\mathbb{R}} e^{-w_j^2} dw_j \\ &= \pi^{-\frac{1}{2}\dim W} \Pi_{j=1}^{\dim W} \sqrt{\pi} = 1. \end{aligned}$$

□

Definition 13.16. If $E_a \subset E_b$ as before, then define an isometry of graded Hilbert spaces $S_{(b,a)} : H(E_a) \to H(E_b)$ by

$$S_{(b,a)}(f) = f_{(b,a)} \otimes f \in H(V_{b,a}) \otimes H(E_a) \cong H(E_b).$$

Proof. (Added). Note that

$$\langle f_{(b,a)} \otimes f, f_{(b,a)} \otimes f \rangle = \langle f_{(b,a)}, f_{(b,a)} \rangle \langle f, f \rangle = \langle f, f \rangle.$$

□

Lemma 13.17. *Let $E_a \subset E_b \subset E_c$ be double inclusions of finite dimensional affine subspaces of E over V. Then $S_{(c,a)} = S_{(c,b)} \circ S_{(b,a)}$.*

Proof. Note that

$$(S_{(c,b)} \circ S_{(b,a)})(f) = f_{(c,b)} \otimes f_{(b,a)} \otimes f \quad \text{and} \quad S_{(c,a)}(f) = f_{(c,a)} \otimes f$$

with

$$\begin{aligned}
&(f_{(c,b)} \otimes f_{(b,a)})(w_{(c,b)}, w_{(b,a)}) = f_{(b,c)}(w_{(c,b)}) f_{(b,a)}(w_{(b,a)}) \\
&= \pi^{-\frac{1}{4}\dim V_{(c,b)}} e^{-\frac{1}{2}\|w_{(c,b)}\|^2} \pi^{-\frac{1}{4}\dim V_{(b,a)}} e^{-\frac{1}{2}\|w_{(b,a)}\|^2} \\
&= \pi^{-\frac{1}{4}(\dim V_{(c,b)} + \dim V_{(b,a)})} e^{-\frac{1}{2}(\|w_{(c,b)}\|^2 + \|w_{(b,a)}\|^2)} \\
&= \pi^{-\frac{1}{4}\dim V_{(c,a)}} e^{-\frac{1}{2}\|w_{(c,a)}\|^2} = f_{(c,a)}(w_{(c,a)})
\end{aligned}$$

with $(w_{(c,b)}, w_{(b,a)}) = w_{(c,a)} \in V_{(c,b)} \oplus V_{(b,a)} = V_{(c,a)}$. □

Corollary 13.18. *(Added). There is a directed system of graded Hilbert spaces as*

$$\cdots H(E_a) \xrightarrow{S_{(b,a)}} H(E_b) \cong H(V_{(b,a)}) \otimes H(E_a) \xrightarrow{S_{(c,b)}} H(E_c) \cdots$$

Definition 13.19. Let E be an affine Euclidean space over a Euclidean vector space V and let $\{E_a\}$ be a family of finite dimensional affine subspaces of E directed by inclusions. Then the graded Hilbert space $H(E)$ is defined to be the direct limit:

$$H(E) = \varinjlim_{E_a \subset E} H(E_a) = \varinjlim_{E_a \subset E; V_a \subset V} L^2(E_a, Cl^*(V_a)) = L^2(E, Cl^*(V))$$

in the category of Hilbert spaces and graded isometric inclusions.

If G acts isometrically on E, then $H(E)$ is equipped with a unitary representation Φ of G as that $SC_0(E_a, Cl^*(V_a))$ is equipped with the G-action $\mathrm{id} \otimes \Phi$.

Suppose now that E is itself finite dimensional. Fix a point $e_0 \in E$ and use it to identity E with its underlying vector space V and use this identification to define scaling maps on E by sending $e \mapsto t^{-1}e \in E$ for $t \geq 1$, with the common fixed point $e_0 \in E$. If $h \in C_0Cl^*(E_a)$ and if $e_a \in E_a \subset E$, then define $h_t \in C_0Cl^*(E_a)$ by $h_t(e) = h(t^{-1}e)$.

Lemma 13.20. *Let E_a be an affine subspace of a finite dimensional affine Euclidean space E. Denote by D_a the Dirac operator for E_a and by $B_a^\perp = C_a^\perp + D_a^\perp$ the Clifford operator plus the Dirac operator for the orthogonal complement $E_a^\perp$ of E_a in E. Then an asymptotic morphism*

$$(\alpha_t^a) : S \otimes C_0(E_a^\perp, Cl^*(V_a^\perp)) \rightsquigarrow \mathbb{K}(H(E))$$

is defined by (to be corrected below)

$$\alpha_t^a(f \otimes h) = f_t(B_a^\perp \otimes 1 + 1 \otimes D_a)(1 \otimes M_{h_t}).$$

Proof. The operator $B_a^\perp$ is essentially seif-adjoint and has compact resolvent (as checked before). So we can define $*$-homomorphisms $\gamma_t : S \to \mathbb{K}(H(E_a^\perp))$ by $\gamma_t(f) = f_t(B_a^\perp)$. As did before, define an asymptotic morphism (α_t) : $S \otimes C_0Cl^*(E_a)) \rightsquigarrow \mathbb{K}(H(E_a))$ by $\alpha_t(f \otimes h) = f_t(D_a)M_{h_t}$. The following composition gives (α_t^a) in the statement:

$$S \otimes C_0Cl^*(E_a) \xrightarrow{\Delta \otimes \mathrm{id}} S \otimes S \otimes C_0Cl^*(E_a) \xrightarrow{\gamma_t \otimes \alpha_t} \mathbb{K}(H(E_a^\perp)) \otimes \mathbb{K}(H(E_a))$$

with $\mathbb{K}(H(E_a^\perp)) \otimes \mathbb{K}(H(E_a)) \cong \mathbb{K}(H(E_a^\perp) \otimes H(E_a)) \cong \mathbb{K}(H(E))$. Indeed, note that if $\Delta(f) = f \otimes f$, then

$$\begin{aligned}(\gamma_t \otimes \alpha_t)(\Delta \otimes \mathrm{id})(f \otimes h) = \gamma_t(f) \otimes \alpha_t(f \otimes h) = \\ f_t(B_a^\perp) \otimes f_t(D_a)M_{h_t} = (f_t(B_a^\perp) \otimes f_t(D_a))(1 \otimes M_{h_t}) \\ = f_t(B_a^\perp \otimes 1 + 1 \otimes D_a)(1 \otimes M_{h_t}),\end{aligned}$$

which is just as the case as for $u(x) = e^{-|x|}$, because $\Delta(u) = u \otimes u$ and $u(x) \otimes u(y) = u(x + y)$. Possibly, the definition in the statement should be slightly changed as for the case as for $v(x) = xe^{-|x|}$ as checked before. □

Lemma 13.21. *Let $E_a \subset E_b$ be an inclusion of affine subspaces of a finite dimensional affine Euclidean space E over V. Then the following diagram* *commutes asymptotically:*

$$\begin{CD}
S \otimes C_0(E_a, Cl^*(V_a)) @>{\alpha_t^a}>> \mathbb{K}(H(E_a^\perp)) \otimes \mathbb{K}(H(E_a)) @>{\cong}>> \mathbb{K}(H(E))\\
@V{\beta_{b,a}}VV @. @|\\
S \otimes C_0(E_b, Cl^*(V_b)) @>{\alpha_t^b}>> \mathbb{K}(H(E_b^\perp)) \otimes \mathbb{K}(H(E_b)) @>{\cong}>> \mathbb{K}(H(E)).
\end{CD}$$

Proof. We do a computation for the generators $u(x) = e^{-|x|}$ and $v(x) = xe^{-|x|}$ of S and (as well for $u(x^2) = e^{-x^2}$ and $\frac{1}{x}v(x^2) = xe^{-x^2}$).

Denote by $E_{b,a}$ the orthogonal complement of E_a in E_b, so that

$$E = E_b^\perp \oplus E_{b,a} \oplus E_a \quad \text{and} \quad H(E) \cong H(E_b^\perp) \otimes H(E_{b,a}) \otimes H(E_a).$$

To do the computation we need to note that under the isomorphism $H(E_b) \cong H(E_{b,a}) \otimes H(E_a)$ of Hilbert spaces, the Dirac operator D_b for E_b corresponds to the operator $D_{b,a} \otimes 1 + 1 \otimes D_a$, and these essentially self-adjoint operators may correspond to their self-adjoint closures. Similarly, the operator $B_a^\perp$ corresponds to the operator $B_b^\perp \otimes 1 + 1 \otimes B_{b,a}$ under the isomorphism $H(E_a^\perp) \cong H(E_b^\perp) \otimes H(E_{b,a})$.

Indeed, note that for $f = f_1 \otimes f_2$,

$$\begin{aligned}
D_b(f)(e) &= \sum_{j=1}^{\dim E_b} W_{e_j} \frac{\partial}{\partial e_j} f(e)\\
&= \sum_{j=1}^{\dim E_{b,a}} \sum_{k=1}^{\dim E_a} W_{e_j \otimes e_k} \frac{\partial}{\partial e_{j,k}} (f_1 \otimes f_2)(e)\\
&= \sum_{j=1}^{\dim E_{b,a}} W_{e_j \otimes 1} (\frac{\partial}{\partial e_j} \otimes \mathrm{id})(f_1 \otimes f_2)(e) + \sum_{k=1}^{\dim E_a} W_{1 \otimes e_k} (\mathrm{id} \otimes \frac{\partial}{\partial e_k})(f_1 \otimes f_2)(e)\\
&= (D_{b,a} \otimes 1 + 1 \otimes D_a)(f_1 \otimes f_2)(e).
\end{aligned}$$

Similarly, the case as for $B_a^\perp = C_a^\perp + D_a^\perp$ is proved.

It then certainly follows that

$$e^{-\frac{1}{t^2} D_b} = e^{-\frac{1}{t^2} D_{b,a}} \otimes e^{-\frac{1}{t^2} D_a} \quad \text{and} \quad e^{-\frac{1}{t^2} B_a^\perp} = e^{-\frac{1}{t^2} B_b^\perp} \otimes e^{-\frac{1}{t^2} B_{b,a}}$$

since $(D_{b,a} \otimes 1)(1 \otimes D_a) = (1 \otimes D_a)(D_{b,a} \otimes 1) = D_{b,a} \otimes D_a$ and the same reason holds for $B_b^\perp$ and $B_{b,a}$. Moreover, it holds that

$$e^{-\frac{1}{t^2} D_b^2} = e^{-\frac{1}{t^2} D_{b,a}^2} \otimes e^{-\frac{1}{t^2} D_a^2} \quad \text{and} \quad e^{-\frac{1}{t^2} (B_a^\perp)^2} = e^{-\frac{1}{t^2} (B_b^\perp)^2} \otimes e^{-\frac{1}{t^2} B_{b,a}^2}$$

because of the multiplication rule for the Clifford algebras involved.

Now applying α_t^a to the element $e^{-x^2} \otimes h \in S \otimes C_0(E_a^\perp, Cl^*(V_a^\perp))$ we obtain

$$\begin{aligned}\alpha_t^a(e^{-x^2} \otimes h) &= e^{-\frac{1}{t^2}(B_a^\perp)^2} \otimes [e^{-\frac{1}{t^2}D_a} M_{h_t}] \\ &= e^{-\frac{1}{t^2}(B_b^\perp)^2} \otimes e^{-\frac{1}{t^2}B_{b,a}^2} \otimes [e^{-\frac{1}{t^2}D_a} M_{h_t}]\end{aligned}$$

in $\mathbb{K}(H(E_b^\perp)) \otimes \mathbb{K}(H(E_{b,a})) \otimes \mathbb{K}(H(E_a))$. Applying $\alpha_t^b \circ \beta_{b,a}$ to $e^{-x^2} \otimes h$ we get

$$\begin{aligned}&(\alpha_t^b \circ \beta_{b,a})(e^{-x^2} \otimes h) = \alpha_t^b((e^{-x^2} \otimes e^{-C_{(b,a)}^2}) \otimes h) \\ &= (e^{-\frac{1}{t^2}(B_b^\perp)^2} \otimes e^{-\frac{1}{t^2}C_{(b,a)}^2}) \otimes [e^{-\frac{1}{t^2}D_b^2} M_{h_t}] \\ &= (e^{-\frac{1}{t^2}(B_b^\perp)^2} \otimes e^{-\frac{1}{t^2}C_{(b,a)}^2}) \otimes [(e^{-\frac{1}{t^2}D_{b,a}^2} \otimes e^{-\frac{1}{t^2}D_a^2}) M_{h_t}].\end{aligned}$$

But we have seen before that the two families of operators $e^{-t^2 B_{b,a}^2}$ and $e^{-\frac{1}{t^2}D_{b,a}^2} e^{-\frac{1}{t^2}C_{b,a}^2}$ as well as $e^{-\frac{1}{t^2}C_{b,a}^2} e^{-\frac{1}{t^2}D_{b,a}^2}$ are asymptotic to one another as $t \to \infty$. It hence follows that $\alpha_t^a(e^{-x^2} \otimes h)$ is asymptotic to $\alpha_t^b(\beta_{b,a}(e^{-x^2} \otimes h)$, as required.

The calculation for $v \otimes h$ is similar. $\square$

From now on we assume that an affine space E over V is infinite dimensional. Let E_a be a finite dimensional affine subspace of E. Denote by $E_a^\perp$ the orthogonal complement of E_a in E, which is an infinite dimensional Euclidean space. We can form the direct limit Hilbert space $H(E_a^\perp)$ as

$$H(E_a^\perp) = \varinjlim_{E_k \subset E_a^\perp} H(E_k)$$

with $\{E_k\}$ a family of finite dimensional affine subspaces of $E_a^\perp$ directed by inclusions.

Lemma 13.22. *There is an isomorphism of graded Hilbert spaces*

$$H(E) \cong H(E_a^\perp) \otimes H(E_a)$$

with $E = E_a^\perp \oplus E_a$.

Proof. (Added). Since we have

$$H(E_k \oplus E_a) \cong H(E_k) \otimes H(E_a)$$

with $\{E_k\}$ a family of finite dimensional affine subspaces of $E_a^{\perp}$ directed by inclusions and $\{E_k \oplus E_a\}_k$ a family of finite dimensional affine subspaces of E, taking the direct limits of both sides implies the isomorphism in the statement. □

Definition 13.23. Let E_a be a finite dimensional subspace of an affine Euclidean space E over V. Define the **Schwarz** space $\mathfrak{S}(E_a)$ to be the dense subspace of $H(E_a)$ of Schwarz class $Cl^*(V_a)$-valued functions on E_a, denoted as

$$\mathfrak{S}(E_a) = \mathfrak{S}(E_a, Cl^*(V_a)) \subset H(E_a) = L^2(E_a, Cl^*(V_a)).$$

Define the **Schwarz** space $\mathfrak{S}(E)$ of E to be the algebraic direct limit of $\mathfrak{S}(E_a)$ as

$$\mathfrak{S}(E) = \mathfrak{S}(E, Cl^*(V)) = \varinjlim_{E_a \subset E} \mathfrak{S}(E_a, Cl^*(V_a))$$

with $\{E_a\}$ a directed family of finite dimensional affine subspaces of E, using the inclusions $S_{(b,a)} : H(E_a) \to H(E_b)$.

Note that the basic vector $f_W \in H(W)$ belongs to $\mathfrak{S}(W)$.

If $V \subset W \subset E_a^{\perp}$ is a double inclusion of finite dimensional subspaces, then the operator $B_V = C_V + D_V$ on $\mathfrak{S}(V)$ also acts on the Schwarz space $\mathfrak{S}(W)$ by the formula:

$$(C_V + D_V)f(w) = \sum_{j=1}^{n} w_j M_{e_j} f(w) + \sum_{k=1}^{n} W_{e_k} \frac{\partial}{\partial w_k} f(w),$$

where $e_1, \cdots, e_n$ is an orthonormal basis for V and $w_1, \cdots, w_n$ are coordinates for V, extended to coordinates on W. The action on $\mathfrak{S}(W)$ by B_V is compatible with the inclusion by $S_{(b,a)}$ as

$$\begin{array}{ccc}
\mathfrak{S}(V) = \mathfrak{S}(E_k) & \xrightarrow{S_{(l,k)}} & \mathfrak{S}(W) = \mathfrak{S}(E_l) = \mathfrak{S}(V_{l,k}) \otimes \mathfrak{S}(E_k) \\
{\scriptstyle B_V}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}\otimes B_V} \\
\mathfrak{S}(V) = \mathfrak{S}(E_k) & \xrightarrow{S_{(l,k)}} & \mathfrak{S}(W) = \mathfrak{S}(E_l) = \mathfrak{S}(V_{l,k}) \otimes \mathfrak{S}(E_k).
\end{array}$$

Indeed,

$$\begin{aligned}
((\mathrm{id} \otimes B_V) \circ S_{(l,k)})(f) &= (\mathrm{id} \otimes B_V)(f_{(l,k)} \otimes f) \\
&= f_{(l,k)} \otimes B_V f = (S_{(l,k)} \circ B_V)(f).
\end{aligned}$$

We then define an unbounded, essentially self-adjoint operator

$$B_a^\perp = \varinjlim_{V=E_k\subset E_a^\perp} B_V$$

on the direct limit Hilbert space $H(E_a^\perp)$ with domain $\mathfrak{S}(E_a^\perp)$.

As a key observation,

Lemma 13.24. *Suppose that $E_a^\perp$ is decomposed as an algebraic direct sum of pairwise orthogonal, finite dimensional subspaces as*

$$E_a^\perp = V_0 \oplus V_1 \oplus \cdots = \oplus_j V_j.$$

If $f = \lim_j f(v_0, v_1, \cdots, v_j) \in \mathfrak{S}(E_a^\perp) \subset H(E_a^\perp)$, then the sum limit

$$\begin{aligned} B_a^\perp f &= \lim(B_0 + B_1 + \cdots B_j) f(v_0, v_1, \cdots, v_j) \\ &= \lim(B_0 f + B_1 f + \cdots + B_j f)(v_0, v_1, \cdots, v_j), \end{aligned}$$

has only finitely many nonzero terms, where each $B_j = C_j + D_j$ is the Clifford-Dirac or Bott-Dirac operator on $H(V_j) = L^2Cl^*(V_j)$ with domain $\mathfrak{S}Cl^*(V_j)$.

The direct limit operator $B_a^\perp$ is essentially self-adjoint on $\mathfrak{S}(E_a^\perp)$ and is independent of the direct sum decomposition of $E_a^\perp$.

Proof. Since $\mathfrak{S}(E_a^\perp) = \varinjlim \mathfrak{S}(V_0\oplus\cdots\oplus V_j)$, if $f \in \mathfrak{S}(E_a^\perp)$, then f belongs to some $\mathfrak{S}(V_0 \oplus \cdots \oplus V_j)$. Its image f_k in $\mathfrak{S}(V_0 \oplus \cdots \oplus V_{n+k})$ under the linking map as $S_{(b,a)}$ in the directed system is a function of the form:

$$f_k(v_0, \cdots, v_{n+k}) = C_k \cdot f(v_0, \cdots, v_n) e^{-\frac{1}{2}\|v_{n+1}\|^2} \cdots e^{-\frac{1}{2}\|v_{n+k}\|^2}$$

for some constant C_k. Since $e^{-\frac{1}{2}\|v_{n+k}\|^2}$ is in the kernel of B_{n+k}, we see that $B_{n+s} f_k = 0$ for all $k \geq s \geq 1$. This proves the first part of the lemma.

Essential self-adjointness follows from the existence of an eigen-basis for $B_a^\perp$, which follows from the existence of eigen-bases in the finite dimensional case (see Proposition 7.15 and Corollary 7.16).

The fact that $B_a^\perp$ is independent of the choice of a direct sum decomposition follows from that

$$B_a^\perp f = B_W f \quad \text{if } f \in \mathfrak{S}(W) \subset \mathfrak{S}(E_a^\perp),$$

which in turn follows from that $B_{W_1\oplus W_2} = B_{W_1} + B_{W_2}$ for both W_1 and W_2 finite dimensional.

Indeed. B_W as well as C_W and D_W are independent of the choice of a basis for W. □

Unfortunately, the operator $B_a^\perp$ does not have compact resolvent. Indeed,

$$\begin{aligned}(B_a^\perp)^2 &= B_0^2 + B_1^2 + \cdots + B_j^2 + \cdots \\ &= (C_0^2 + D_0^2 + N_0) + \cdots + (C_j^2 + D_j^2 + N_j) + \cdots\end{aligned}$$

from which it follows that the eigenvalues for $(B_a^\perp)^2$ are the sums

$$\lambda = \lambda_0 + \lambda_1 + \cdots + \lambda_j + \cdots ,$$

where each λ_j is an eigenvalue for B_j^2 and where all but finitely many λ_j are zero. It then follows from Proposition 7.15 that the eigenvalue 0 has multiplicity 1, but each positive integer is an eigenvalue of $(B_a^\perp)^2$ of infinite multiplicity. For instance,

$$\begin{aligned}n &= n + 0 + 0 + \cdots + 0 + \cdots \\ &= 0 + n + 0 + \cdots + 0 + \cdots = \cdots \\ &= 0 + 0 + 0 + \cdots + n + 0 + \cdots = \cdots .\end{aligned}$$

Because of infinite multiplicity of eigenvalues except zero, $(B_a^\perp)^2$ can not have the resolvent $(\lambda 1 - (B_a^\perp)^2)^{-1}$ compact for $\lambda \notin \sigma((B_a^\perp)^2)$ the spectrum discrete. Note that a compact operator always has eigen-spaces finite dimensional for nonzero eigenvalues. As well, $B_a^\perp$ does not have compact resolvent, since $\sigma(B_a^\perp)^2 = \sigma((B_a^\perp)^2)$.

Notation. Let $E_0 \subset E_1 \subset \cdots \subset E_n \cdots$ be an increasing sequence of finite dimensional affine subspaces of an affine space E whose union $\cup_n E_n$ is E. Set $V_0 = E_0$ and denote by V_n the orthogonal complement of E_{n-1} in E_n, so that $E = \oplus_{n=0}^\infty V_n$ as an algebraic orthogonal direct sum decomposition.

Definition 13.25. Let E_a be a finite dimensional affine subspace of $E = \oplus_{j=0}^\infty V_j$. An algebraic orthogonal direct sum decomposition $E_a^\perp = \oplus_{j=0}^\infty W_j$ is **standard** if $W_0 = V_a$ some finite dimensional vector space and $W_j = V_{n-1+j}$ for $j \geq 1$ and some $n \geq 1$.

Definition 13.26. Let E_a be a finite dimensional affine subspace of an affine space E. An algebraic orthogonal direct sum decomposition $E_a^\perp = \oplus_{j=0}^\infty Z_j$ into finite dimensional **linear** subspaces is **acceptable** if there is a standard decomposition $E_a^\perp = \oplus_{j=0}^\infty W_j$ such that

$$W_0 \oplus \cdots \oplus W_n \subset Z_0 \oplus \cdots \oplus Z_n \subset W_0 \oplus \cdots \oplus W_{n+1}$$

for all sufficiently large n.

Remark. (Added). It may says that the acceptability for the decomposition of affine subspaces of an affine space E over a Euclidean vector space V ensures that its affine structure may be contained or replaced with the vector space structure for the decomposition of linear subspaces of V.

Definition 13.27. Let E_a be a finite dimensional affine subspace of an affine space E and let $E_a^\perp = \oplus_{j=0}^\infty Z_j$ be an acceptable decomposition of $E_a^\perp$ in that sense. For each $t \geq 1$, define a **perturbed** unbounded operator $B_{a,t}^\perp$ on $H(E_a^\perp)$ with domain $\mathfrak{S}(E_a^\perp)$ by

$$B_{a,t}^\perp = \varinjlim(t_0 B_0 + t_1 B_1 + \cdots + t_n B_n)$$

where $t_j = 1+\frac{j}{t}$ and $B_n = C_n + D_n$ the Clifford-Dirac or Bott-Dirac operator on the finite dimensional space Z_n.

The direct limit perturbed operator $B_{a,t}^\perp$ is well defined as $B_a^\perp$, and it does depend on the choice of an acceptable decomposition, as well as on the parameter $t \in [1, \infty)$ (not checked, but a moment of thought may make these clear).

Lemma 13.28. *Let E_a be a finite dimensional affine subspace of an affine space E over a Euclidean vector space V and let $E_a^\perp = \oplus_{j=0}^\infty Z_j$ be an acceptable decomposition of $E_a^\perp$. Then the perturbed operator*

$$B_{a,t}^\perp = \varinjlim(t_0 B_0 + t_1 B_1 + \cdots + t_n B_n)$$

on $H(E_a^\perp)$ with domain $\mathfrak{S}(E_a^\perp)$, is essentially self-adjoint and has **compact** *resolvent, as a key property.*

Proof. The proof of self-adjointness follows from the same argument as for the proof of that for $B_a^\perp$ given above. It is shown that there is an orthonormal eigen-baisis for $B_{a,t}^\perp$ in $\mathfrak{S}(E_a^\perp)$.

As for compactness of the resolvent for $B_{a,t}^\perp$, the formula

$$(B_{a,t}^\perp)^2 = \varinjlim(t_0^2 B_0^2 + t_1^2 B_1^2 + \cdots + t_n^2 B_n^2)$$

implies that the eigenvalues of $(B_{a,t}^\perp)^2$ are the sums

$$\lambda = t_0^2 \lambda_0 + t_1^2 \lambda_1 + \cdots + t_n^2 \lambda_n + \cdots,$$

where each λ_n is an eigenvalue for B_n^2 and all but finitely many λ_n are zero. Since the lowest positive eigenvalue for every B_n is 1, and since $t_j = 1+\frac{j}{t} \to$

∞ as $j \to \infty$ and for t fixed, with $t_j < t_{j+1}$, it follows that for any $R > 0$, there are only finitely many eigenvalues λ for $(B_{a,t}^{\perp})^2$ with $|\lambda| \leq R$. It ensures that each nonzero eigenvalue for $(B_{a,t}^{\perp})^2$ as well as $B_{a,t}^{\perp}$ has multiplicity finite. Hence it follows that $(B_{a,t}^{\perp})^2$ and $B_{a,t}^{\perp}$ have compact resolvent by functional calculus. □

Proposition 13.29. *Let E_a be a finite dimensional affine subspace of E over V as above, for which a point $e_0 \in E_a$ is fixed to define scaling automorphisms of $C_0Cl^*(E_a)$ by sending $h \mapsto h_t$ as before. Let $B_{a,t}^{\perp}$ be the perturbed unbounded operator associated to some acceptable decomposition of $E_a^{\perp}$. Then an* **asymptotic** *morphism*

$$(\alpha_t^a) : S \otimes C_0(E_a, Cl^*(V_a)) \rightsquigarrow \mathbb{K}(H(E_a^{\perp})) \otimes \mathbb{K}(H(E_a)) \cong \mathbb{K}(H(E))$$

is defined by

$$\alpha_t^a(f \otimes h) = f_t(B_{a,t}^{\perp} \otimes 1 + 1 \otimes D_a)(1 \otimes M_{h_t})$$

(to be corrected in some case).

Proof. This is proved in exactly the same way as for $B_a^{\perp}$ involved. But the formula for defining the morphism need to be corrected similarly as in the proof of Lemma 13.20. □

Again, the perturbed operator $B_{a,t}^{\perp}$ does depend on the choice of an acceptable decomposition in some sense. Possibly, it may come from the difference between an affine space and a linear space such as arrangement. But the situation is improved in the limit as $t \to \infty$.

Lemma 13.30. *Let E_a be a finite dimensional affine subspace of E over V as above. Denote by $B_t = B_{a,t}^{\perp}$ and $B_t' = B_{a,t}'^{\perp}$ the operators associated to two acceptable decompositions $\oplus_{j=0}^{\infty} Z_j$ and $\oplus_{j=0}^{\infty} Z_j'$ of $E_a^{\perp}$. Then*

$$\lim_{t \to \infty} \|f(B_t) - f(B_t')\| = 0 \quad \textit{for every } f \in S = C_0(\mathbb{R}).$$

Proof. We prove a special case such that

$$Z_0 \oplus \cdots\cdots \oplus Z_n \subset Z_0' \oplus \cdots \oplus Z_n' \subset Z_0 \oplus \cdots \oplus Z_{n+1}$$

for all n. Then $Z_0 \subset Z_0' \subset Z_0 \oplus Z_1$. Denote by X_0 the orthogonal complement of Z_0 in Z_0' and set $Y_0 = Z_0$. Then $Z_0' = Y_0 \oplus X_0$. Denote by Y_1 the orthogonal complement of Z_0' in Z_1. Then $Z_1 = X_0 \oplus Y_1$ and

$$Y_1 \subset Z_1' \subset Y_1 \oplus Z_2.$$

Denote by X_1 the orthogonal complement of Y_1 in Z_1'. Then $Z_1' = Y_1 \oplus X_1$. Inductively, denote by X_n the orthogonal complement of Y_n (or Z_n) in Z_n' and by Y_n the orthogonal complement of Z_{n-1}' in Z_n. Then $Z_n = X_{n-1} \oplus Y_n$ and $Z_n' = Y_n \oplus X_n$. It then follows that

$$\begin{aligned} E_a^\perp &= \oplus_{j=0}^\infty Z_j = Y_0 \oplus [\oplus_{j=1}^\infty (X_{j-1} \oplus Y_j)] \\ &= \oplus_{j=0}^\infty Z_j' = \oplus_{j=0}^\infty (Y_j \oplus X_j), \end{aligned}$$

with respect to which the operators B_t and B_t' can be written as

$$\begin{aligned} B_t &= t_0 B_{Y_0} + (t_1 B_{X_0} + t_1 B_{Y_1}) + (t_2 B_{X_1} + t_2 B_{Y_2}) + \cdots \\ B_t' &= (t_0 B_{Y_0} + t_0 B_{X_0}) + (t_1 B_{Y_1} + t_1 B_{X_1}) + (t_2 B_{Y_2} + t_2 B_{X_2}) + \cdots . \end{aligned}$$

It then follows that

$$B_t - B_t' = (t_1 - t_0) B_{X_0} + (t_2 - t_1) B_{X_1} + \cdots$$

with each $t_j - t_{j-1} = (1 + t^{-1} j) - (1 + t^{-1}(j-1)) = t^{-1}$. Therefore,

$$(B_t - B_t')^2 = \frac{1}{t^2} B_{X_0}^2 + \frac{1}{t^2} B_{X_1}^2 + \cdots$$

In contrast,

$$B_t^2 = t_0^2 B_{Y_0}^2 + (t_1^2 B_{X_0}^2 + t_1^2 B_{Y_1}^2) + (t_2^2 B_{X_1}^2 + t_2^2 B_{Y_2}^2) + \cdots .$$

and since $t_j^2 \geq 1$, it follows that for every $\xi \in \mathfrak{S}(E_a^\perp)$,

$$\begin{aligned} &\|(B_t - B_t')^2 \xi\|^2 = \|t^{-1} B_{X_0}^2 \xi + t^{-1} B_{X_1}^2 \xi + \cdots \|^2 \\ &= t^{-2} (\|B_{X_0}^2 \xi\|^2 + \|B_{X_1}^2 \xi\|^2 + \cdots) \\ &\leq t^{-2} (\|t_0^2 B_{Y_0}^2 \xi\|^2 + \|t_1^2 B_{X_0} \xi\|^2 + \ \|t_1^2 B_{Y_1} \xi\|^2 + \|B_{X_1} \xi\|^2 + \cdots) \\ &= t^{-2} \|B_t^2 \xi\|^2, \end{aligned}$$

so that $\|(B_t - B_t')^2 \xi\| \leq t^{-1} \|B_t^2 \xi\|$ (corrected).

If $f(x) = \frac{1}{x \pm i}$ in S, then

$$\begin{aligned} &\|(f(B_t) - f(B_t'))\xi\|^2 = \|(B_t \pm i)^{-1} (B_t' - B_t)(B_t' \pm i)^{-1} \xi\|^2 \\ &\leq \|(B_t' - B_t)(B_t' \pm i)^{-1} \xi\|^2 = \langle (B_t' - B_t)^2 (B_t' \pm i)^{-1} \xi, (B_t' \pm i)^{-1} \xi \rangle \\ &\leq \|(B_t' - B_t)^2 (B_t' \pm i)^{-1} \xi\| \|(B_t' \pm i)^{-1} \xi\| \\ &\leq t^{-1} \|B_t^2 (B_t' \pm i)^{-1} \xi\| \|(B_t' \pm i)^{-1} \xi\| \\ &\leq t^{-1} \|B_t^2 (B_t' \pm i)^{-1}\| \|\xi\|^2, \end{aligned}$$

so that $\|f(B_t) - f(B_t')\| \le \frac{1}{\sqrt{t}}\|B_t^2(B_t' \pm i)^{-1}\|^{\frac{1}{2}}$ (corrected), where it is required that the norm of the last compact operator $B_t^2(B_t' \pm i)^{-1}$ is bounded with respect to t (correct or not?).

An approximation argument involving the Stone-Weierstrass theorem may finish the proof. □

By strengthening Lemma 13.30 above,

Lemma 13.31. *With the same hypotheses as in the previous lemma, it holds that for $f \in S$,*

$$\lim_{t\to\infty} \|f(sB_t) - f(sB_t')\| = 0$$

uniformly in $s \in [1, \infty)$.

It follows that the asymptotic morphism (α_t^a) is independent of the choice an acceptable decomposition for $E_a^\perp$, up to asymptotic equivalence (compare the proof of Lemma 13.20). However, the morphism does depend on the choice of an initial direct sum decomposition.

Proposition 13.32. *The following diagram commutes asymptotically:*

$$\begin{tikzcd} S \otimes C_0Cl^*(E_a) \arrow[r, "\alpha_t^a"] \arrow[d, "\beta_{b,a}"'] & \mathbb{K}(H(E)) \arrow[d, equal] \\ S \otimes C_0Cl^*(E_b) \arrow[r, "\alpha_t^b"] & \mathbb{K}(H(E)) \end{tikzcd}$$

under suitable acceptable decompositions for $E_a^\perp$ as well as $E_b^\perp$ chosen.

Proof. As did before, the composition $(\alpha_t^b \circ \beta_{b,a})$ is asymptotic to the asymptotic morphism

$$f \otimes h \mapsto f_t(B_t' \otimes 1 + 1 \otimes D_a)(1 \otimes M_{h_t})$$

where, if (α_t^b) is computed using an acceptable decomposition $E_b^\perp = \oplus_{j=0}^\infty Z_j$, then B_t' is the operator associated to the decomposition

$$E_a^\perp = V_{b,a} \oplus Z_0 \oplus Z_1 \oplus Z_2 \oplus \cdots .$$

But this is an acceptable decomposition for $E_a^\perp$, and thus the statement follows. □

It follows that the asymptotic morphisms (α_t^a) (for any a as a parameter) combine to form an **asymptotic** morphism (α_t) from $SC_0(E, Cl^*(V))$ to $\mathbb{K}(H(E))$, as desired.

Suppose that a countable group G acts isometrially on E. Identify E with its underlying Euclidean space V by using a point $e_0 \in E$. Define a family of actions by G on E by

$$g \cdot_s e = s(g \cdot e_0) + \pi_g v, \quad \text{with } e_0 + v = e \text{ and for } s \in [0,1].$$

Note that if $e = e_0 = e_0 + v$ with $v = 0$, then $g \cdot_0 e_0 = 0(g \cdot e_0) = 0 = e_0$ a global fixed point, and that

$$g \cdot_1 e = (g \cdot e_0) + \pi_g v = g \cdot v = g \cdot (e - e_0).$$

Lemma 13.33. *There exists a direct sum decomposition $E = \oplus_{j=0}^{\infty} V_j$ such that if $E_n = \oplus_{j=0}^{n} V_j$, then for every $g \in G$, there is a natural number $n_0 \in \mathbb{N}$ such that if $n > n_0$, then $g \cdot_s E_n \subset E_{n+1}$ for all $s \in [0,1]$.*

Proposition 13.34. *If the direct sum decomposition $E = \oplus_{j=0}^{\infty} V_j$ is chosen so as in Lemma 13.33 above, then the asymptotic morphism (α_t) : $SC_0(E, Cl^*(V)) \rightsquigarrow \mathbb{K}(H(E))$ deduced above is equivariant in the sense that*

$$\lim_{t\to\infty} \|\alpha_t(g \cdot a) - g \cdot_{t^{-1}} (\alpha_t(x))\| = 0$$

for all $a \in SC_0(E, Cl^*(V))$ and all $g \in G$, as that the diagram:

$$\begin{array}{ccc} SC_0(E, Cl^*(V)) & \xrightarrow{G\cdot} & SC_0(E, Cl^*(V)) \\ {\scriptstyle \alpha_t}\downarrow & & \downarrow{\scriptstyle \alpha_t} \\ \mathbb{K}(H(E)) & \xrightarrow{G\cdot_{t^{-1}}} & \mathbb{K}(H(E)) \end{array}$$

commutes asymptotically as $t \to \infty$.

Proof. The asymptotic morphism on $S \otimes C_0 Cl^*(E_a)$ defined by sending $a \mapsto g^{-1} \cdot_{t^{-1}} (\alpha_t^a(g \cdot a))$ is given by exactly the same formula used to define α_t^a, except for the choice of an acceptable direct sum decomposition of $E_a^{\perp}$. But different choices of acceptable direct sum decompositions of $E_a^{\perp}$ give rise to asymptotically equivalent, asymptotic morphisms.

Note that for $a = f \otimes h$,

$$\begin{aligned} g^{-1} \cdot_{t^{-1}} (\alpha_t(g \cdot (f \otimes h))) &= g^{-1} \cdot_{t^{-1}} (\alpha_t(f \otimes (g \cdot h))) \\ &= g^{-1} \cdot_{t^{-1}} f_t(B_{a,t}^{\perp} \otimes 1 + 1 \otimes D_a)(1 \otimes M_{(g\cdot h)_t}) \\ &= f_t(B_{a,t}^{\perp} \otimes 1 + 1 \otimes D_a)(1 \otimes M_{(g^{-1}\cdot g\cdot h)_t}) \\ &= f_t(B_{a,t}^{\perp} \otimes 1 + 1 \otimes D_a)(1 \otimes M_{h_t}) = \alpha_t^a(f \otimes h). \end{aligned}$$

□

Definition 13.35. Denote by $[\alpha] \in E_G(SC_0(E, Cl^*(V)), \mathbb{C})$ the E_G-class of the asymptotic morphism $(\alpha_t) : SC_0(E, Cl^*(V)) \rightsquigarrow \mathbb{K}(H(E))$.

Part III

We are going to show that a composition in the E_G-theory is the identity class. The proof is almost exactly the same as that of Proposition 11.31.

Lemma 13.36. *Suppose that the isometric action of G on the affine Euclidean space E over V has a fixed point. Then the composition*

$$\mathbb{C} \xrightarrow{[\beta]} SC_0(E, Cl^*(V)) \xrightarrow{[\alpha]} \mathbb{C}$$

in the G-equivariant E-theory is the identity morphism on $\mathbb{C}$.

Proof. First, recall that in the definition of the asymptotic morphism β we fix a point of E. But it is clear from the proof for the construction of β that different choices of an initial point give rise to asymptotically equivalent, asymptotic morphiphisms. Thus we may choose the point that is fixed under the action of G on E. It then implies that each $*$-homomorphism $\beta_t(f) = \beta(f_t)$ is G-equivariant. It follows that the G-equivariant, asymptotic morphism $(\beta_t) : S \rightsquigarrow SC_0(E, Cl^*(V))$ is G-equivariantly homotopy equivalent to the G-equivariant $*$-homomorphism $\beta : S \to SC_0(E, Cl^*(V))$.

Using this fact, it follows that we may compute the composition $[\alpha]\circ[\beta]$ in the G-equivariant E-theory by computing the composition of the asymptotic morphism α with the $*$-homomorphism β. But it is shown that this composition is asymptotic to $\gamma = (\gamma_t) : S \rightsquigarrow \mathbb{K}(H(E))$ defined by $\gamma_t(f) = f_t(B_t)$, where B_t is the operator associated to any acceptable decomposition of E. This in turn is homotopic to the asymptotic morphism sending $f \mapsto f(B_t)$. Finally, this is homotopic to the asymptotic morphism defining the identity class $[\mathrm{id}] \in E_G(\mathbb{C}, \mathbb{C})$ by the homotopy sending

$$S \ni f \mapsto \begin{cases} f(sB_t) & s \in (0,1] \\ f(0)p & s = 0, \end{cases}$$

where p is the projection onto the kernel of B_t. Note that all the operators B_t have the same 1-dimensional, G-fixed kernel. □

Theorem 13.37. *The composition $[\alpha] \circ [\beta] = [\beta] \otimes [\alpha] \in E_G(\mathbb{C}, \mathbb{C})$ is the identity class.*

Proof. Let $s \in [0,1]$ and we denote by $SC_0(E, Cl^*(V))_s$ the C^*-algebra $SC_0(E, Cl^*(V))$, but with the scaled G-action $(g,h) \mapsto g *_s h$. The G-C^*-algebras $SC_0(E, Cl^*(V))_s$ form a continuous field of G-C^*-algebras over the unit interval $[0,1]$ (see Dixmier [15] for continuous fields of C^*-algebras). Denote by $C([0.1], \{SC_0(E, Cl^*(V))_s\})$ the G-C^*-algebra of continuous sections of this continuous field. In a similar way, form a continuous field of G-C^*-algebras $\mathbb{K}(H(E))_s$ as fibers over $[0,1]$ and denote by $C([0,1], \{\mathbb{K}(H(E))_s\})$ the G-C^*-algebra of continuous sections of this continuous field.

The asymptotic morphism $\alpha = (\alpha_t) : SC_0(E, Cl^*(V)) \rightsquigarrow \mathbb{K}(H(E))$ induces an asymptotic morphism

$$\alpha^\sim = (\alpha^\sim_{t,s}) : C([0,1], \{SC_0(E, Cl^*(V))_s\}) \rightsquigarrow C([0,1], \{\mathbb{K}(H(E))_s\})$$

defined as $\alpha^\sim_{t,s}(f)(s) = \alpha_{t,s}(f(s))$ with $(\alpha_{t,s}) : SC_0(E, Cl^*(V))_s \rightsquigarrow \mathbb{K}(H(E))_s$. Similarly, the asymptotic morphism $\beta = (\beta_t) : S \rightsquigarrow SC_0(E, Cl^*(V))$ extends to an asymptotic morphism

$$\beta^\sim = (\beta^\sim_{t,s}) : S \rightsquigarrow C([0,1], \{SC_0(E, Cl^*(V)_s\})$$

defined as $\beta^\sim_{t,s}(f)(s) = \beta_{t,s}(f)$ with $(\beta_{t,s}) : S \rightsquigarrow SC_0(E, Cl^*(V))_s$.

Consider the following diagram of G-equivariant E-theory morphisms:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "{[\beta^\sim]}"] \arrow[d, equal] & C([0,1], \{SC_0(E, Cl^*(V)_s\}) \arrow[r, "{[\alpha^\sim]}"] \arrow[d, "{[\mathrm{ev}_s]}"] & C([0,1]) \arrow[d, "{[\mathrm{ev}_s]}", "\cong"'] \\
\mathbb{C} \arrow[r, "{[\beta]}"] & SC_0(E, Cl^*(V)_s) \arrow[r, "{[\alpha]}"] & \mathbb{C}
\end{tikzcd}$$

where $[\mathrm{ev}_s]$ denotes the G-equivariant E-theory class induced from the evaluation map ev_s at $s \in [0,1]$. If the composition on the bottom line is the identity class for some $s \in [0,1]$, then it is the identity class for all $s \in [0,1]$ by homotopy invariance. But that compostion is the identity class when $s = 0$ since the G-action $(g,e) \mapsto g \cdot_0 e$ has a fixed point. It then follows that the composition is the identity class when $s = 1$, as wanted to prove. □

A generalization to continuous fields on compact Hausdorff spaces

We consider a simple extension of the main Theorem 12.16 to a situation involving continuous fields of affine spaces over a compact parameter space.

Definition 13.38. Let Z be a set. Denote by $\mathfrak{K}_n(Z)$ the set of all **negative type kernels** $b : Z \times Z \to \mathbb{R}$ such that $b(z,z) = 0$ for any $z \in Z$, $b(z_1, z_2) = b(z_2, z_1)$ for any $z_1, z_2 \in Z$, and

$$\sum_{i,j=1}^{n} a_i b(z_i, z_j) a_j \leq 0$$

for all $n \in \mathbb{N}$, $z_1, \cdots, z_n \in Z$, and $a_1, \cdots a_n \in \mathbb{R}$ with $\sum_{j=1}^{n} a_j = 0$. Equip $\mathfrak{K}_n(Z)$ with the topology of pointwise convergence, so that the net (b_μ) converges to b in $\mathfrak{K}_n(Z)$ if and only if $b_\mu(z_1, z_2)$ converges to $b(z_1, z_2)$ for all $z_1, z_2 \in Z$.

Suppose now that X is a compact Hausdorff space and that we are given a continuous map from X into $\mathfrak{K}_n(Z)$ as $x \mapsto b_x$. For each $x \in X$, we can construct a Euclidean vector space V_x and an affine space E_x over V_x. For example, each V_x is a quotient space of the space $f_0(Z, \mathbb{R})$ of all finitely supported functions on Z which have sum 0 (by the subspace $f_0(Z, \mathbb{R})_0$ of zero definite functions with respect to the positive semidefinite form associated to b_x), and each E_x is a quotient space of the space $f_1(Z, \mathbb{R})$ of all finitely supported functions on Z which have sum 1 (by $f_0(Z, \mathbb{R})_0$). We then obtain a sort of **continuous field** of affine Euclidean spaces E_x over X.

Then the (unsuspended, direct limit) C^*-algebras $SC_0(E_x, Cl^*(V_x))$ for $x \in X$ may be put together to form a continuous field of C^*-algebras over X (see [15]). To do this we need to specify which sections of this continuous field are to be continuous.

Definition 13.39. Let U be an open subset of a compact Hausdorff space X and let $s_u : \mathbb{R}^n \to E_u$ be a family of isometries of $\mathbb{R}^n$ into the affine Euclidean spaces E_u for $u \in U$ defined above. The family is said to be **continuous** if there is a finite subset F of the set Z and if there are functions $f_{j,u} : Z \to \mathbb{R}$ where $j = 1, \cdots, n$ and $u \in U$ such that

(a) each function $f_{j,u}$ is supported in F and has sum 1, and thus $f_{j,u}$ determines a point of E_u;

(b) for each $z \in Z$ and each j, the value $f_{j,u}(z)$ is a continuous function of $u \in U$;

(c) the isometry s_u maps the standard basis element e_j of $\mathbb{R}^n$ to the point of E_u determined by $f_{j,u}$.

Note that as a picture,

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{R}^n \ni e_j & \xrightarrow{s_u} & [f_{j,u}] \in E_u = f_1(Z, \mathbb{R})/f_0(Z, \mathbb{R})_0 \\ & & & & \downarrow \\ & & \mathfrak{K}_n(Z) \ni b_u & \longleftarrow & F \subset U \subset X. \end{array}$$

Definition 13.40. Let us say that a function f which assigns to each point $x \in X$ an element f_x of the (unsuspended) C^*-algebra $SC_0(E_x, Cl^*(V_x))$ is a **continuous section** if for every $x \in X$ and every $\varepsilon > 0$ there is an open subset U of X containing x, a continuous family of isometries $s_u : \mathbb{R}^n \to E_u$ as above, and an element $f \in S \otimes C_0(\mathbb{R}^n, Cl^*(\mathbb{R}^n))$ such that the norm $\|s_u^{\sim}(f) - f_u\| < \varepsilon$ for all $u \in U$, where $s_u^{\sim}$ is the following composition as

$$\begin{aligned} s_u^{\sim} : S \otimes C_0(\mathbb{R}^n, Cl^*(\mathbb{R}^n)) &\xrightarrow{\mathrm{id}\otimes \mathrm{Ad}(s_u)} \quad \{S \otimes C_0(E_u, Cl^*(V_u))\}_{u\in U} \\ &\xrightarrow{\quad \subset \quad} \quad C(X, \{SC_0(E_x, Cl^*(V_x))\}_{x\in X}), \end{aligned}$$

where $\mathrm{Ad}(s_u)(h)(e) = (s_u \circ h \circ s_u^{-1})(e)$ on the image $s_u(\mathbb{R}^n) \subset E_u$ and it is equal to zero otherwise (possibly in this sense), the right hand side in the first line means a continuous field of C^*-algebras obtained as Lemma 13.41 below, and the C^*-algebra in the last line is defined in Definition 13.42 below.

Lemma 13.41. *With the above definition of continuous sections, there is a continuous field $\{SC_0(E_x, Cl^*(V_x))\}_{x\in X}$ of C^*-algebras $SC_0(E_x, Cl^*(V_x))$ over a compact Hausdorff space X.*

Definition 13.42. Denote by $C(X, SC_0(E, Cl^*(V)))$ the C^*-algebra of continuous sections of the continuous field $\{SC_0(E_x, Cl^*(V_x))\}_x$ over X.

If a group G acts on a set Z, then G acts on the space $\mathfrak{K}_n(Z)$ by the formula $(g \cdot b)(z_1, z_2) = b(g^{-1}z_1, g^{-1}z_2)$. Indeed,

$$\begin{aligned} (g_1 g_2 \cdot b)(z_1, z_2) &= b((g_1g_2)^{-1}z_1, (g_1g_2)^{-1}z_2) = b(g_2^{-1}g_1^{-1}z_1, g_2^{-1}g_2^{-1}z_2) \\ &= (g_2 \cdot b)(g_1^{-1}z_1, g_2^{-1}z_2) = (g_1 \cdot g_2 \cdot b)(z_1, z_2). \end{aligned}$$

In what follows we are interested in the case where $Z = G$ and the action is the left translation on G.

Definition 13.43. Let X be compact space equipped with an action of a countable discrete group G by homeomorphisms. An G-equivariant map b from X into $\mathfrak{K}_n(G)$ is **proper-valued** if for every $s \geq 0$, there is a finite subset F of G such that $b_x(g_1, g_2) \leq s$ implies $g_1^{-1}g_2 \in F$.

Note that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\ b\ } & \mathfrak{K}_n(G) \\ {\scriptstyle G}\big\downarrow & & \big\downarrow{\scriptstyle G} \\ X & \xrightarrow{\ b\ } & \mathfrak{K}_n(G), \end{array}$$

so that in particular,

$$(g_1 \cdot b_x)(g_1, g_2) = b_x(1_G, g_1^{-1}g_2) = b_{g\cdot x}(g_1, g_2).$$

As a generalization of Theorem 13.10,

Proposition 13.44. *If $b : X \to \mathfrak{K}_n(G)$ is a G-equivariant, proper-valued map, then the associated G-C^*-algebra*

$$C(X, SC_0(E, Cl^*(V))) = C(X, \{SC_0(E_x, Cl^*(V_x))\}_{x\in X})$$

is proper.

By carrying out the previous constructions fiberwise we obtain the following (basically due to Tu [55]):

Theorem 13.45. *Let G be a countable discrete group and let X be a compact metrizable G-space. Assume that there exists a proper-valued, G-equivariant map from X into $\mathfrak{K}_n(G)$. Then $C(X, \{SC_0(E_x, Cl^*(V_x))\}_{x\in X})$ is a proper G-C^*-algebra and there are G-equivriant E-theory classes*

$$[\alpha] \in E_G(C(X, \{SC_0(E_x, Cl^*(V_x))\}_{x\in X}), C(X)) \quad \textit{and}$$
$$[\beta] \in E_G(C(X), C(X, \{SC_0(E_x, Cl^*(V_x))\}_{x\in X}))$$

for which the composition $[\alpha] \circ [\beta] = [\beta] \otimes [\alpha]$ is the E_G-theory identity class in $E_G(C(X), C(X))$.

By trivially adapting the simple argument used to prove Theorem 11.24 we obtain as an important consequence,

Corollary 13.46. *With the same assumption as Theorem 13.45 above, for every G-C^*-algebra $\mathfrak{D}$, the Baum-Connes assembly map*

$$\mu : K(G, C(X, \mathfrak{D})) \to K_0(C(X, \mathfrak{D}) \rtimes G) \cong E_0(C(X, \mathfrak{D}) \rtimes G)$$

is an isomorphism.

If G is exact, then the same is true for the reduced assembly map into $K_0(C(X, \mathfrak{D}) \rtimes_r G) \cong E_0(C(X, \mathfrak{D}) \rtimes_r G)$.

14 Gromov hyperbolic groups with Gromov boundary

Toward the injectivity of the BC assembly map in the next two sections, recall some basics and more about a geometric property of groups which implies the injectivity, known in a number of cases. On the other hand, the surectivity seems to require the understanding of more subtle issues in harmonic analysis.

Geometry of groups

Let G be a finitely generated discrete group with S a set of finite generators of G (with $S = S^{-1}$ for convenience).

The **Cayley graph** Γ for G with S is given by G as the set of vertexes of Γ and by S as the edges of Γ such that two elements $g_1, g_2 \in G$ have one edge if there is $s \in S$ such that $g_1 = g_2 s$ or $g_2^{-1} g_1 = s$. The **word-length** metric of G with Γ is defined by

$$d(g, h) = \min\{\text{the numbers of edges of strings between } g \text{ and } h\},$$

where each string consists of a finite number of directed edges e_j such that the range vertex of e_j is equal to the source vertex of e_{j+1}, and each edge has length 1. Also, the word-metric of G with Γ extends to a metric on Γ in a natural sense that two points of two edges have the distance as the minimum of the usual distances from the points to their vertexes plus the distances of the vertexes.

The **word-length** of an element $g \in G$ is defined by $l(g) = d(g, 1_G)$, where 1_G is the unit of G.

Note that $l(h^{-1}g) = l(g^{-1}h) = d(g, h) = d(h, g)$.

The word-length metric depends on the choice of a generating set S. Nevertheless, the large-scale geometric structure of G with a word-length metric is independent of the choice of S, in the sense as in Definition 14.1 and Example 14.2 below.

Definition 14.1. Let $d(\cdot, \cdot)$ and $\delta(\cdot, \cdot)$ be two distance functions on a set X. They are **coarsely equivalent** if for every $r > 0$, there exits a constant $s > 0$ such that if $d(x_1, x_2) < r$ for $x_1, x_2 \in X$, then $d(x_1, x_2) < s$ and if $\delta(x_1, x_2) < r$, then $d(x_1, x_2) < s$.

Example 14.2. Let d_S and d_T be two word-length metrics on a finitely generated group G, associated to two finite generating sets S and T. Then these are coarsely equivalent. Indeed, there is a constant $c > 0$ such that

$$\frac{1}{c} d_S(g_1, g_2) - c \leq d_T(g_1, g_2) \leq c \cdot d_S(g_1, g_2) + c$$

for any $g_1, g_2 \in G$ (unchecked).

Definition 14.3. A **curve** in a metric space X is a continuous map from a closed interval $[a, b]$ in $\mathbb{R}$ to X.

The **length** of a curve $f : [a, b] \to X$ is defined by

$$l(f) = \sup_{\Delta} \sum_{j=1}^{n} d(f(t_j), f(t_{j-1})),$$

where $\Delta : t_0 = a < t_1 < \cdots < t_n = b$ is a finite partition of $[a, b]$.

A metric space X is said to be a **length** space if

$$d(x_1, x_2) = \inf_{f_{x_1,x_2}} l(f_{x_1,x_2}),$$

where the infimum is taken over functions $f_{x_1,x_2} : [a, b] \to X$, each of which is a curve f in X with $f(a) = x_1$ and $f(b) = x_2$, or $f(a) = x_2$ and $f(b) = x_1$.

Theorem 14.4. *Let G be a finitely generated discrete group acting properly and cocompactly by isometries on a length space (X, d'). Let y be a point of X not fixed by elements of G except 1_G. Then the distance function δ on G defined by*

$$\delta(g_1, g_2) = d'(g_1 \cdot y, g_2 \cdot y)$$

is coarsely equivalent to the word-length metric d on G.

In this case we say that the word-length metric space (G, d) is coarsely equivalent (to (G, δ) and) to the length metric space (X, d'). See Milnor [43] and Roe [51] and [52] (unchecked).

Example 14.5. If G is the fundamental group $\pi_1(M)$ of a closed Riemannian manifold M, then G is coarsely equivalent to the universal covering space $M^{\sim}$.

Any finitely generated discrete group is coarsely equivalent to its Cayley graph.

For example, free groups are coarsely equivalent to trees.

These are unchecked, which may be considered elsewhere.

Gromov hyperbolic groups

We are going to sketch briefly the theory of hyperbolic groups, for the injectivity of the BC assembly map for hyperbolic groups in the next sections. The first injectivity result is due to Connes-Moscovici [5], in which the rational injectivity of the assembly map in the case $\mathfrak{D} = \mathbb{C}$ is proved essentially, with a quite different method from here.

Definition 14.6. Let X be a metric space. A **geodesic segment** in X is a curve $\gamma : [a, b] \to X$ such that

$$d(\gamma(s), \gamma(t)) = |s - t|$$

for any $a \leq s \leq t \leq b$.

If γ is a geodesic segment from x_1 to x_2 in X, then the length $l(\gamma)$ of γ is equal to $d(x_1, x_2)$.

Proof. (Added). Indeed, by definition of $l(\gamma)$,

$$\begin{aligned} l(\gamma) &= \sup_{\Delta} \sum_{j=1}^{n} d(\gamma(t_j), \gamma(t_{j-1})) \\ &= \sup_{\Delta} \sum_{j=1}^{n} |t_j - t_{j-1}| = \sum_{j=1}^{n} (t_j - t_{j-1}) \\ &= t_n - t_0 = d(\gamma(t_n), \gamma(t_0)) = d(x_2, x_1), \end{aligned}$$

where $\Delta : a = t_0 < t_1 < \cdots < t_n = b$ is a finite partition of $[a, b]$ with $\gamma(a) = x_1$ and $\gamma(b) = x_2$. □

Definition 14.7. A **geodesic** metric space is a metric space X in which each two points of X are joined or connected by a geodesic segment.

Definition 14.8. A **geodesic triangle** Δ in a metric space X consists of a triple of points of X and a triple of geodesic segments in X connecting the three points pairwise.

A geodesic triangle is d**-slim** for some $d \geq 0$ if each ponint on each segment is within a distance d (or $\leq d$) for some point on one of the other two segments.

We may write such a geodesic triangle as $\Delta = \cup_{j=0}^{2} \gamma_{j,j+1}$ with $\gamma_{j,j+1}$ a geodesic segment from x_j to x_{j+1} for j mod 3.

Such a triangle Δ is d-slim if for each $x \in \gamma_{j,j+1}$, $d(x, y) < d$ for some $y \in \gamma_{k,k+1}$ with $k \neq j$.

Example 14.9. (Added and edited). Let $G = \mathbb{Z}$ with $S = \{\pm 1\}$. Then $\Gamma = \mathbb{R}$ with $\mathbb{Z}$ as the set of vertexes and $\cup_{n \in \mathbb{Z}}[n, n+1]$ as the set of edges. Also, $d(x, y) = |x - y|$ and $l(x) = |x|$ for $x, y \in \mathbb{Z}$. A geodesic segment in $\mathbb{R}$ is a directed (identically) parametrized closed interval. Thus $\mathbb{R}$ is geodesic as a metric space. Three points of $\mathbb{R}$ gives a geodesic triangle, which is 0-slim.

Similarly, let $G = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$ with $S = \{1 = -1\}$. Then Γ is a circle with length n and is homeomorphic to the 1-dimensional torus $\mathbb{T}$. The similar as above holds.

Example 14.10. (Added). Let $G = \mathbb{Z}^2$ with $S = \{(\pm 1, 0), (0, \pm 1)\}$. Then $\Gamma = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$. Also, $d(x, y)$ and $l(x)$ for $x, y \in \mathbb{Z}^2$ are given by the routes in Γ with the minimum distances. A geodesic segment in Γ is a directed parametrized route. Thus Γ is geodesic as a metric space. The geodesic triangle given by $(0, 0)$, $(1, 0)$, and $(1, 1) \in \mathbb{Z}^2$ such that their geodesic segments make the 1×1 square as a union is 1-slim.

Let $G = \mathbb{Z}_n \times \mathbb{Z}_m$ with $S = \{(1,0),(0,1)\}$. Then Γ is homeomorphic to the 2-torus $\mathbb{T}^2$. The similar as above holds.

Example 14.11. (Detailed). Geodesic triangles in trees are 0-slim.

The two-equilateral triangle of two sides R and of two angles $\frac{\pi}{4}$ in the Euclidean space $\mathbb{R}^2$ is $\frac{R}{2}$. Note that $\frac{R}{2} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}R}{2}$. The equilateral triangle of sides R and of angles $\frac{\pi}{3}$ is $\frac{\sqrt{3}R}{4}$-slim. Note that $R : \frac{R}{2} = \frac{\sqrt{3}R}{2} : x$ implies $x = \frac{\sqrt{3}R}{4}$.

Definition 14.12. A geodesic metric space X is **d-hyperbolic** if every geodesic triangle Δ in X is d-slim.

A geodesic metric space is **hyperbolic** if it is d-hyperbolic for some $d \geq 0$.

For instance, trees are hyperbolic metric spaces, but Euclidean spaces of dimension 2 or more are not.

May refer to Gromov [19] or [18] (not at hand) and more (not cited).

Definition 14.13. A finitely generated discrete group is **word-hyperbolic**, or just hyperbolic if its Cayley graph is a hyperbolic metric space.

Theorem 14.14. *Being hyperbolic of a finitely generated discrete group does not depend on the choice of a finitely generating set.*

Example 14.15. (Added). The groups $\mathbb{Z}$ and $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ are hyperbolic because their $\Gamma = \mathbb{R}$ and $\Gamma = \mathbb{T}$ are 0-hyperbolic.

The group $\mathbb{Z}^2$ is not hyperbolic because its Γ can not be d-hyperbolic for any $d \geq 0$. Indeed, the geodesic triangle Δ given by $(0,0)$, $(1+n, 1+n)$, and $(1+n, -1-n) \in \mathbb{Z}^2$ for $n \in \mathbb{N}$ such that their geodesic segments make the measure of the union of the 1×1 squares in Δ the minimum is n-slim. If we take the geodesic segments to make it the maximum, then Δ is $(n+1)$-slim.

The group $\mathbb{Z}_n \times \mathbb{Z}_m$ is hyperbolic because its $\Gamma \approx \mathbb{T}^2$ is sufficiently $(n+m)$-hyperbolic but more less.

Example 14.16. Every tree is 0-hyperbolic as a geodesic metric space, so that a finitely generated free group is hyperbolic.

The Poincaré disk D_p is a hyperbolic metric space.

If G is a proper cocompact group of isometies of D_p, then G is hyperbolic.

In particular, the fundamental group of a Riemann surface of genus 2 or more is hyperbolic.

(These are unchecked well.)

If G is a finitely generated discrete group and if $k \geq 0$, then the **Rips complex** $\mathrm{Rp}(G, k)$ is a simplicial complex with G as a vertex set such that a tuple $(g_0, \cdots, g_p)$ is a p-simplex if and only if $d(g_i, g_j) \leq k$ for any i, j.

Note that the 0-skeleton of $\mathrm{Rp}(G, k)$ is equal to G.

Theorem 14.17. (Baum-Connes-Higson [4] and Meintrup and Schick [42]). *If G is a δ-hyperbolic group, then the Rips complex $\mathrm{Rp}(G, k)$ for some positive $k \geq 16\delta + 8$ is a universal proper G-space, denoted as U_G.*

Definition 14.18. A **geodesic ray** in a hyperbolic space X is a continuous function f from $[0, \infty)$ to X such that the restriction of f to any closed interval $[0, t]$ for $t > 0$ is a geodesic segment.

Two geodesic rays f_1 and f_2 in X are **equivalent** if

$$\limsup_{t \to \infty} d(f_1(t), f_2(t)) < \infty.$$

The **Gromov** boundary $\partial_g X$ of a hyperbolic metric space X is the set of all equivalence classes of geodesic rays in X.

The **Gromov** boundary ∂G of a hyperbolic group G is the Gromov boundary of its Cayley graph, that is, $\partial G = \partial_g \Gamma$.

The Gromov boundary ∂G does not depend on the choice of a generating set S.

There is an action G on ∂G, such that $g[f] = [gf]$, where $gf(t) = g \cdot f(t)$.

There is a compact metrizable topology on ∂G such that G acts on it by homeomorphisms.

There is a compact metrizable topology on the disjoint union $G \sqcup \partial G = \overline{G}$ so denoted, such that G acts on it by homeomorphisms, and that G is an open dense subset of $\overline{G}$, and that a sequence of points $g_n \in G$ converges to a point $x \in \partial G$ if and only if $g_n \to \infty$ in G in a metric sense and there is a geodesic ray f representating x such that $\sup_n d(g_n, f) < \infty$.

The action of G on $\overline{G}$ is amenable.

(There are unchecked well.)

15 Injectivity of the BC assembly maps with proper C^*-algebras

The first injectivity result is the following (essentially due to Kasparov, but not cited in the text) as an improved version:

Theorem 15.1. *Let G be a countable discrete group. Suppose that there exists a proper G-C^*-algebra $\mathfrak{B}$ and E_G-theory elements $\alpha \in E_G(\mathfrak{B}, \mathbb{C})$ and $\beta \in E_G(\mathbb{C}, \mathfrak{B})$ such that for any finite subgroup H of G, the composite $\gamma = \alpha \circ \beta \in E_G(\mathbb{C}, \mathbb{C})$ restricts to the identity class in $E_H(\mathbb{C}, \mathbb{C})$. Then for every G-C^*-algebra $\mathfrak{B}$, the Baum-Connes assembly map is injective and in fact split injective:*

$$0 \longrightarrow K(G, \mathfrak{D}) \xrightarrow{\mu} E_0(\mathfrak{D} \rtimes G) \cong K_0(\mathfrak{D} \rtimes G).$$

Proof. Consider again the following diagram:

$$\begin{array}{ccc}
K(G, \mathbb{C} \otimes \mathfrak{D}) & \xrightarrow{\mu} & E_0(\mathbb{C} \otimes \mathfrak{D} \rtimes G) \\
{\scriptstyle (\beta \otimes \mathrm{id}_{\mathfrak{D}})_*} \downarrow & & \downarrow {\scriptstyle (\beta \otimes \mathrm{id}_{\mathfrak{D}})_*} \\
K(G, \mathfrak{B} \otimes \mathfrak{D}) & \xrightarrow{\mu} & E_0(\mathfrak{B} \otimes \mathfrak{D} \rtimes G) \\
{\scriptstyle (\alpha \otimes \mathrm{id}_{\mathfrak{D}})_*} \downarrow & & \downarrow {\scriptstyle (\alpha \otimes \mathrm{id}_{\mathfrak{D}})_*} \\
K(G, \mathbb{C} \otimes \mathfrak{D}) & \xrightarrow{\mu} & E_0(\mathbb{C} \otimes \mathfrak{D} \rtimes G),
\end{array}$$

where the assembly map μ in the middle is an isomorphism since $\mathfrak{B} \otimes \mathfrak{D}$ is a proper G-C^*-algebra. We show that the assembly map μ in the top is only injective. For this it suffices to show that the vertical map in the diagram

$$K(G, \mathbb{C} \otimes \mathfrak{D}) \xrightarrow{(\beta \otimes \mathrm{id}_{\mathfrak{D}})_*} K(G, \mathfrak{B} \otimes \mathfrak{D})$$

is injective. (Indeed, if so, $\mu \circ (\beta \otimes \mathrm{id}_{\mathfrak{D}})_*$ is injective, and the commutative diagram as the first square implies that $(\beta \otimes \mathrm{id}_{\mathfrak{D}})_* \circ \mu$ is injective, and hence μ is injective.) For this we show the following composite is an isomorphism:

$$K(G, \mathbb{C} \otimes \mathfrak{D}) \xrightarrow{(\beta \otimes \mathrm{id}_{\mathfrak{D}})_*} K(G, \mathfrak{B} \otimes \mathfrak{D}) \xrightarrow{(\alpha \otimes \mathrm{id}_{\mathfrak{D}})_*} K(G, \mathbb{C} \otimes \mathfrak{D}).$$

In view of the definition, it suffices to show that if X is a G-compact, proper G-space, then the map

$$\gamma_* = \alpha_* \circ \beta_* : E_G(X, \mathfrak{D}) \to E_G(X, \mathfrak{D})$$

is an isomorphism. The proof of this is an induction argument on the number n of G-invariant open subsets U_j to cover X, each of which admits a map to a proper homogeneous space G/H.

If $n = 1$, then X admits such a map, so that $X = G \times_H W$, where W is a compact space with an action of H. Then there is a commutative diagram with restriction isomorphisms Res:

$$\begin{tikzcd}
E_G(G \times_H W, \mathfrak{D}) \arrow[r, "\gamma_*"] \arrow[d, "\mathrm{Res}"', "\cong"] & E_G(G \times_H W, \mathfrak{D}) \arrow[d, "\cong"', "\mathrm{Res}"] \\
E_H(W, \mathfrak{D}) \arrow[r, "\gamma_*=\mathrm{id}", "\cong"'] & E_H(W, \mathfrak{D})
\end{tikzcd}$$

since $\gamma = 1$ in $E_H(\mathbb{C}, \mathbb{C})$. See Proposition 11.21.

If $n \geq 2$, then choose a G-invariant open subset U of X which admits a map to a proper homogeneous space, and the space $X \setminus U$ may be covered by $n-1$ G-invariant open subsets, each of which admits a map to a proper homogeneous space. By induction, we may assume that the map γ_* for $X \setminus U$ is an isomorphism. Applying the Five (or to Three) lemma to the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & E_G(X \setminus U, \mathfrak{D}) \arrow[r] \arrow[d, "\gamma_*"', "\cong"] & E_G(X, \mathfrak{D}) \arrow[r] \arrow[d, "\gamma_*"'] & E_G(U, \mathfrak{D}) \arrow[r] \arrow[d, "\gamma_*"', "\cong"] & \cdots \\
\cdots \arrow[r] & E_G(X \setminus U, \mathfrak{D}) \arrow[r] & E_G(X, \mathfrak{D}) \arrow[r] & E_G(U, \mathfrak{D}) \arrow[r] & \cdots
\end{tikzcd}$$

we conclude that γ_* for X is an isomorphism. □

The second result (possibly taken from [23]) is the following:

Theorem 15.2. *Let X be a compact, metrizable G-space and assume that X is H-equivariantly contractible for any finite subgroup H of G. Let $\mathfrak{D}$ be a separable G-C^*-algebra. If the Baum-Connes assembly map*

$$\mu : K(G, C(X, \mathfrak{D})) \to E_0(C(X, \mathfrak{D}) \rtimes G) \cong K_0(C(X, \mathfrak{D}) \rtimes G)$$

is an isomorphism, then the Baum-Connes assembly map

$$\mu : K(G, \mathfrak{D}) \to E_0(\mathfrak{D} \rtimes G) \cong K_0(\mathfrak{D} \rtimes G)$$

is split injective.

Proof. The inclusion map $i : \mathfrak{D} \to C(X, \mathfrak{D})$ as constant functions gives rise to the following commutative diagram:

$$\begin{tikzcd}
K(G, C(X, \mathfrak{D})) \arrow[r, "\mu"] & K_0(C(X, \mathfrak{D}) \rtimes G) \\
K(G, \mathfrak{D}) \arrow[r, "\mu"] \arrow[u, "i_*"] & K(\mathfrak{D} \rtimes G). \arrow[u, "i_*"']
\end{tikzcd}$$

We show that the left vertical i_* is an isomorphism. For this it suffices to show that if Z is any G-compact, proper G-space, then the map

$$i_* : E_G(C_0(Z), \mathfrak{D}) \to E_G(C_0(Z), C(X, \mathfrak{D}))$$

is an isomorphism. By the Mayer-Vietoris argument as in the last proof above, it suffices to consider the case where Z admits a map to a proper homogeneous space G/H. In this case there is a compact space W with an action of H such that $Z = G \times_H W$. Consider the following commutative diagram with restriction isomorphisms:

$$\begin{array}{ccc} E_G(C_0(G \times_H W), \mathfrak{D}) & \xrightarrow{i_*} & E_G(C_0(G \times_H W), C(X, \mathfrak{D})) \\ \mathrm{Res}\downarrow\cong & & \cong\downarrow\mathrm{Res} \\ E_H(C(W), \mathfrak{D}) & \xrightarrow{i_*} & E_H(C(W), C(X, \mathfrak{D})). \end{array}$$

The bottom horizontal map i_* is an isomorphism since i is a homotopy equivalence of H-C^*-algebras. Therefore, the top horizontal map i_* is also an isomorphism. □

The last injectivity result is an analytic version of the result of Carlsson-Pedersen [9]. As well, may refer to [23].

Definition 15.3. Let G be a discrete group, let X be a G-compact, proper G-space, and let $\overline{X}$ be a metrizable compactification of X with the action extended from that of G on X, by homeomorphisms. The extended action is **small at infinity** if for every compact subset K of X,

$$\lim_{g \to \infty} d(gK) = 0,$$

where the limit is taken as in a metric sense and the diameters $d(gK)$ are computed by a metric on $\overline{X}$.

Theorem 15.4. *Let G be a countable discrete group. Suppose that there is a G-compact space as U_G with a metrizable compactification $\overline{U_G}$ of U_G such that*

(a) *the G-action on U_G extends continuously to $\overline{U_G}$,*
(b) *the action of G on $\overline{U_G}$ is small at infinity, and*
(c) *$\overline{U_G}$ is H-equivariantly contractible, for any finite subgroup H of G.*

Then for every separable G-C^*-algebras $\mathfrak{D}$, the Baum-Connes assembly map

$$\mu : K(G, \mathfrak{D}) \to E_0(\mathfrak{D} \rtimes G) \cong K_0(\mathfrak{D} \rtimes G)$$

is injective.

16 Uniform embeddings of discrete groups into Hilbert spaces

We are going to apply Theorem 15.2 above to prove injectivity of the Baum-Connes assembly map for a quite broad class of groups, specified as in what follows.

Definition 16.1. Let X and Y be metric spaces. A **uniform embedding** from X into Y is a function $f : X \to Y$ with the following two properties:

(*a*) for any $r \geq 0$, there is some $s \geq 0$ such that if $d(x_1, x_2) \leq r$ for $x_1, x_2 \in X$, then $d(f(x_1), f(x_2)) \leq s$, and

(*b*) for any $s \geq 0$, there is $r \geq 0$ such that if $d(x_1, x_2) \geq r$, then $d(f(x_1), f(x_2)) \geq s$.

Remark. (Added). The condition (*a*) seems to say directly that any uniformly bounded band $b_r(X)$ of the diagonal in $X \times X$ is mapped by f to a uniformly bounded band $b_s(Y)$ of the diagonal in $Y \times Y$. The condition (*b*) seems to say that the complement of the interior of $b_s(Y)$ has $b_r(X)$ as an inverse image by f.

Example 16.2. (Detailed). If $f : X \to Y$ is a bi-Lipshitz homeomorphism from X onto the image $f(X)$, then f is a uniform embedding. In particular, if f is an isometry, then it is a uniform embedding.

Indeed, then there are positive constants $c, c' > 0$ such that

$$d(f(x_1), f(x_2)) \leq c\, d(x_1, x_2) \quad \text{and} \quad d(x_1, x_2) \leq c'\, d(f(x_1), f(x_2))$$

for any $x_1, x_2 \in X$. Thus, if $d(x_1, x_2) \leq r$, then $d(f(x_1), f(x_2)) \leq cr$ with $s = cr$, and if $d(x_1, x_2) \geq r = sc'$, then $d(f(x_1), f(x_2)) \geq s$.

If both the metric spaces X and Y (not only X) are bounded, then any function from X to Y is a uniform embedding. In particular, uniform embeddings need not be injective.

Because for the condition (*a*) we may take s to be the diameter of Y:

$$d(Y) = \sup_{y_1, y_2 \in Y} d(y_1, y_2) < \infty.$$

For the condition (*b*), we may take r to be the diameter of X plus 1: $d(X)+1$.

Exercise. Let G be a finitely generated group and H be a finitely generated subgroup of G. If we assume that G and H are metric spaces with word-length metrics, then the inclusion map from H to G is a uniform embedding.

Proof. (Added). Note that in general, for $h_1, h_2 \in H$,

$$d_H(h_1, h_2) = d_H(h_2^{-1}h_1, 1_H) \geq d_G(h_2^{-1}h_1, 1_G) = d_G(h_1, h_2),$$

which implies the condition (a). If we assume that generators of H are contained in those of G, then $d_H(h_1, h_2) = d_G(h_1, h_2)$ for any $h_1, h_2 \in H$, which implies both (a) and (b). This is the case. □

Remark. In the context of groups (with metrics), any function satisfying condition (a) is in fact a Lipschitz function. For (b), there are some examples such that $d(x_1, x_2) \geq e^s$ implies $d(f(x_1), f(x_2)) \geq s$. (Mentioned as being easy to find, but not checked).

If a finitely generated group G acts metrically and properly on an affine Euclidean space E, and if $e \in E$, then the map from G to E defined as $G \ni g \mapsto g \cdot e \in E$ is a uniform embedding.

We are going to prove the following result, which partially extends Corollary 12.17 of the main Theorem 12.16.

Theorem 16.3. (Due to Tu [55] and Yu [59], but not in this form). *Let G be countable discrete group. If G is uniformly embeddable into a Euclidean space, then for any G-C^*-algebra $\mathfrak{D}$, the Baum-Connes assembly map*

$$\mu : K(G, \mathfrak{D}) \to E_0(\mathfrak{D} \rtimes G) \cong K_0(\mathfrak{D} \rtimes G)$$

is split injective.

The proof is given below soon later.

Definition 16.4. Let X be a discrete set. The **Stone-Čech** compactification of X is defined to be the set βX of all nonzero, finitely additive, $\{0, 1\}$-valued probability measures on the (Boolean or σ-) algebra of all subsets of X. With the topology of pointwise convergence, βX becomes a compact Hausdorff space.

A point of βX is a function $\mu : 2^X \to \{0, 1\}$ such that $\mu(\sqcup_{j=1}^n Y_j) = \sum_{j=1}^n \mu(Y_j)$ for finitely many, mutually disjoint subsets $Y_j \in 2^X$ and $\mu(Y) = 1$ for some $Y \in 2^X$, where 2^X is the family of all subsets of X.

A net of functions $\mu_\alpha \in \beta X$ converges to μ if and only if $\mu_\alpha(Y)$ converges to $\mu(Y)$ for any subset Y of X.

Example 16.5. If x is a point of X, then the measure defined as

$$\mu_x(Y) = \begin{cases} 1 & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y \end{cases}$$

for $Y \in 2^X$ belongs to βX. In this way, the set X is embedded into βX as a dense open subset. The measure μ_x is just the point measure δ_x supported at the point x.

Example 16.6. (Added). Similarly, if $x_1, \cdots, x_n$ are distinct points of X, then the measure defined as

$$\mu_{x_1,\cdots,x_n}(Y) = \begin{cases} 1 & \text{if } x_1, \cdots, x_n \in Y, \\ 0 & \text{if otherwise} \end{cases}$$

for $Y \in 2^X$ belongs to βX. In general, for any element $\mu \in \beta X$, there is $Z \in 2^X$ such that

$$\mu(Y) = \mu_Z(Y) = \begin{cases} 1 & \text{if } Z \subset Y, \\ 0 & \text{if otherwise} \end{cases}$$

for $Y \in 2^X$. We may write $\mu_Z = \delta_Z$.

Proof. (Added, as for those properties of X in βX and of βX). Since any $\mu \in \beta X$ is μ_Z for some $Z \in 2^X$, with $Z = \{z_\lambda\}_\lambda$, there is a (one step) net of inclusions from $\sqcup_\lambda z_\lambda$ to Z, so that there is a (one step) net from $\sqcup_\lambda \delta_{z_\lambda}$ converging to μ_Z. Hence, X is dense in βX.

It is also shown that the complement of X is closed in βX. Thus X is open in βX.

It looks like that $\beta X \setminus \mu_X$ is a discrete space (be certainly true), so that it is a locally compact Hausdorff space. Then βX is just the one-point compactification of $\beta X \setminus \mu_X$ by adding μ_X to it. □

Now, let $b \in l^\infty(X)$ be a bounded, complex-valued function on X and μ be a finitely additive probability measure with $\mu(X) = 1$. First, if b is a simple function on X taking only finitely many values λ_j $(1 \leq j \leq n)$, then define the **integral** of b as

$$\int_X b(x) d\mu(x) = \sum_{j=1}^n \lambda_j \mu(\{x \in X \,|\, b(x) = \lambda_j\}).$$

Second, if b is a general bounded function, then b is a uniform limit of simple functions b_k, and thus define the **integral** of b to be the limit of the integrals of the approximants as

$$\int_X b(x) d\mu(x) = \lim_k \int_X b_k(x) d\mu(x).$$

Exercise. If $b \in l^\infty(X)$, then the map $b^\wedge : \mu \mapsto \int_X b(x)d\mu(x)$ defined so is a continuous function from βX to $\mathbb{C}$, so that $b^\wedge \in C(\beta X)$.

Proof. (Added). First, assume that b is a simple function. Then the convergence of a net of measures μ_λ to μ implies the convergence of $b^\wedge(\mu_\lambda)$ to $b^\wedge(\mu)$. Second, if b is bounded, then $b = \lim_k b_k$ a uniform limit of simple functions b_k, and thus the continuity of $b^\wedge$ follows as well. □

Remark. The virtue of using $\{0,1\}$-valued measures is that this integration process makes sense in a great generality, so that it is possible to integrate any function from the set X into any compact space, as well.

Suppose now that G is a finitely generated discrete group. The compact space βG for G as a space has a continuous action of G by $(g\cdot\mu)(E) = \mu(Eg)$ for $g \in G$ and a subset $E \subset G$. Note that for $g, h \in G$,

$$(gh \cdot \mu)(E) = \mu(Egh) = (h \cdot \mu)(Eg) = (g \cdot (h \cdot \mu))(E).$$

Let $f : G \to E$ be a uniform embedding into an affine Euclidean space E and let $b : G \times G \to \mathbb{R}$ be the associated negative type kernel: $b(g_1, g_2) = d(f(g_1), f(g_2))^2$. Then the function defined by $g \mapsto b(gg_1, gg_2)$ is bounded for every $g_1, g_2 \in G$.

(Added). Note that

$$d(gg_1, gg_2) = d((gg_2)^{-1}gg_1, 1_G) = d(g_2^{-1}g_1, 1_G) \equiv r,$$

for which, there is some $s \geq 0$ such that

$$b(gg_1, gg_2) = d^2(f(gg_1), f(gg_2)) \leq s.$$

Define the negative type kernel b_μ for $\mu \in \beta G$ by integration:

$$b_\mu(g_1, g_2) = \int_G b(gg_1, gg_2)d\mu(g).$$

(Detailed). Observe that

$$\begin{aligned}
b_{h\cdot\mu}(g_1, g_2) &= \int_G b(gg_1, gg_2)d(h\cdot\mu)(g) \\
&= \int_G b(ghh^{-1}g_1, ghh^{-1}g_2)d\mu(gh) \\
&= \int_G b(kh^{-1}g_1, kh^{-1}g_2)d\mu(k) = b_\mu(h^{-1}g_1, h^{-1}g_2) \\
&\equiv (h \cdot b_\mu)(g_1, g_2).
\end{aligned}$$

Note that

$$((gh)\cdot b_\mu)(g_1,g_2) = b_\mu((gh)^{-1}g_1,(gh)^{-1}g_2) = (g\cdot(h\cdot b_\mu))(g_1,g_2).$$

Thus there is a G-equivariant map from βG into the space $\mathfrak{K}_n(G)$ of negative type kernels on G as

$$\begin{tikzcd} \beta G \arrow[r, "\int_G b\,d(\cdot)"] \arrow[d, "G\ni h"'] & \mathfrak{K}_n(G) \arrow[d, "h\in G"] \\ \beta G \arrow[r, "\int_G b\,d(\cdot)"] & \mathfrak{K}_n(G). \end{tikzcd}$$

Lemma 16.7. *Let $b_\mu \in \beta G$ as above. For every $s\geq 0$, there is $r\geq 0$ so that if $d(g_1,g_2)\geq r$, then $b_\mu(g_1,g_2)\geq s$.*

Proof. By Definition 16.1 (b) of uniform embeddings, for any $s\geq 0$, there is $r\geq 0$ such that if $d(g_1,g_2)\geq r$, then $\sqrt{b(g_1,g_2)} = d(f(g_1),f(g_2))\geq\sqrt{s}$. Hence,

$$b_\mu(g_1,g_2) = \int_G b(gg_1,gg_2)d\mu(g) \geq \int_G s\cdot d\mu(g) = s,$$

where $d(gg_1,gg_2) = d((gg_2)^{-1}gg_1, 1_G) = d(g_1,g_2)$. □

Since G is finitely generated, for every $r\geq 0$, there is a finite subset F of G, so that if $d(g_1,g_2)<r$, then $g^{-1}g_2\in F$. Therefore, the map $\mu\mapsto b_\mu$ on βG is proper-valued.

It is proved above that

Proposition 16.8. *If a finitely generated discrere group G is uniformly embedded into an affine Euclidean space, then there is a G-equivariant, proper-valued, continuous map from βG to $\mathfrak{K}_n(G)$.*

To prove Theorem 16.3 we apply Theorems 13.45 and 15.2. To do so we need to replace βG by a compact G-space which is smaller and more connected than βG, namely, is second countable and contractible, as done as follows.

Lemma 16.9. *Let G be a countable discrete group, let X be a compact G-space and let $b: X\to\mathfrak{K}_n(G)$ be a continuous, G-equivariant map. Then there is a metrizable compact G-space Y and a G-map f from X to Y such that*

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "f"'] & \mathfrak{K}_n(G) \arrow[d, equal] \\ Y \arrow[r, "c"] & \mathfrak{K}_n(G) \end{tikzcd}$$

for some map c.

Proof. Take Y to be the Gelfand dual of the separable C^*-algebra generated by the functions $b_{g(\cdot)}(g_1, g_2) : x \mapsto b_{gx}(g_1, g_2)$ on X for $g, g_1, g_2 \in G$.

(Added and detailed below.) Then the C^*-algebra is isomorphic to $C_0(Y)$ in general.

Note that

$$\begin{array}{ccccc} X & \xrightarrow{b} & \mathfrak{K}_n(G) & \xrightarrow{(\cdot)(g_1,g_2)} & \mathbb{R} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{b} & \mathfrak{K}_n(G) & \xrightarrow{\cdot(\cdot)(g_1,g_2)} & \mathbb{R}. \end{array}$$

Since G is countable, the set of the functions above is separable.

Is the C^*-algebra unital? This is equivalent to that Y is compact. Possibly, the C^*-algebra may be generated by the unit of $C(X)$ and those functions on X, but when X is compact. Then the C^*-algebra is isomorphic to $C(Y)$, which may not be $C(X)$.

There is an inclusion map from $C(Y)$ to $C(X)$ (or from $C_0(Y)$ to $C_0(X)$) as a C^*-subalgebra.

Each point $x \in X$ corresponds to the functional $\varphi_x : C(X) \to \mathbb{C} \cong C(X)/C_0(X \setminus \{x\})$ the quotient C^*-algebra. The map f may be the restriction map from φ_x to $\varphi_x|_{C_0(Y)} : C_0(Y) \to \mathbb{C}$, which is identified with a point of Y. □

Lemma 16.10. *Let G be a countable discrete group, let Y be a compact metrizable G-space, and let $b : Y \to \mathfrak{K}_n(G)$ be a proper-valued, G-equivariant continuous map. Then there is a metrizable compact G-space Z which is H-equivariantly contractible for any finite subgroup H of G, and a proper-valued, G-equivariant continuous map from Z into $\mathfrak{K}_n(G)$.*

Proof. Let Z be the compact space of countably additive Borel probability measures defined on the Borel σ-algebra of Borel subsets of Y, with the weak∗ topology as a subset of the dual space $C(Y)^*$.

If $\mu \in Z$, then define $b_\mu \in \mathfrak{K}_n(G)$ by integration:

$$b_\mu(g_1, g_2) = \int_Y b_y(g_1, g_2) d\mu(y).$$

The map $\mu \mapsto b_\mu$ on Z has the required properties.

(Added). Note that

$$\begin{array}{ccc} Z & \xrightarrow{b} & \mathfrak{K}_n(G) \\ g\downarrow & & \downarrow g \\ Z & \xrightarrow{b} & \mathfrak{K}_n(G). \end{array}$$

Indeed,

$$\begin{aligned} b_{g\cdot\mu}(g_1,g_2) &= \int_Y b_y(g_1,g_2)d\mu(yg) \\ &= \int_Y b_{y'g^{-1}}(g_1,g_2)d\mu(y') \\ &= \int_Y (g\cdot b_y)(g_1,g_2)d\mu(y') = (g\cdot b_\mu)(g_1,g_2), \end{aligned}$$

where the third and fourth equalities are by definition by this reason. □

Proof. (For Theorem 16.3). Proposition 16.8 and Lemmas 16.9 and 16.10 above show that the hypotheses of Theorem 13.45 and Corollary 13.46 hold. Then Theorem 15.2 implies injectivity of the assembly map, as required. □

Amenable actions

In this subsection we consider a method of constructing uniform embeddings of groups into affine Hilbert spaces.

Definition 16.11. Let G be a discrete group. Denote by $Pb(G)$ the set of all (**probability**) functions $f : G \to [0,1]$ such that $\sum_{g\in G} f(g) = 1$. Equip $Pb(G)$ with the topology of pointwise convergence, so that the net (f_k) converges to f if and only if $f_k(g)$ converges $f(g)$ for every $g \in G$. Equip $Pb(G)$ with an action of G by homeomorphisms by the formula $(g\cdot f)(h) = f(g^{-1}h)$.

(Added). Note that $\sum_{x\in G}(g\cdot f)(x) = \sum_{x\in G} f(g^{-1}x) = \sum_{y\in g^{-1}G} f(y) = 1$; if $(g\cdot f_1) = (g\cdot f_2)$, then $f_1 = f_2$ on $g^{-1}G = G$; any $f = (g\cdot(g^{-1}\cdot f))$; the net $(f_k)\to f$ if and only if $(g\cdot f_k)\to g\cdot f$.

Definition 16.12. Let G be a countable discrete group. An action of G by homeomorphisms on a compact Hausdorff space X is (topologically) **amenable** if there is a sequence of continuous maps $f_n : X \to Pb(G)$ such that for every $g \in G$,

$$\lim_{n\to\infty} \sup_{x\in X} \|f_n(g\cdot x) - g\cdot(f_n(x))\|_1 = 0$$

where we define the norm $\|l\|_1 = \sum_{g\in G} |l(g)|$ for a function l on G.

May refer to Higson and Roe [27], Higson [23], and [1] for more details.

Note that it says that the following diagram commutes asymptotically and uniformly on X in the 1-norm as $n \to \infty$:

$$\begin{array}{ccc} X & \xrightarrow{G} & X \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_n} \\ Pb(G) & \xrightarrow{G} & Pb(G). \end{array}$$

Proposition 16.13. *If a finitely generated group G acts amenably on a compact Hausdorff space X, then G is uniformly embeddable into a Hilbert space.*

The proof is given soon later.

Remark. The method below can be modified to show that if a countable group G which is not necessarily finitely generated acts amenably on some compact Hausdorfff space X, then there is a G-equivariant, proper-valued map from X into the space $\mathfrak{K}_n(G)$ of negative type kernels on G. The methods in the previous section then show that the Baum-Connes assembly map is injective for such G.

Example 16.14. Every hyperbolic group acts amenably on its Gromov boundary.

If G is a discrete subgroup of a connected Lie group H, then G acts amemably on some compact homogeneous space as a quotient space of H.

If G is a discrete group of finite asymptotic dimension, then G acts amenably on the Stone-Čech compactification βG.

See [27] for these facts.

Definition 16.15. Let Z be a set. A function $\varphi : Z \times Z \to \mathbb{C}$ is a **positive definite kernel** on Z if $\varphi(z, z) = 1$ for all $z \in Z$, if $\varphi(z_2, z_1) = \overline{\varphi(z_1, z_2)}$ for all $z_1, z_2 \in Z$, and if

$$\sum_{i,j=1}^{k} \overline{\lambda_i}\varphi(z_i, z_j)\lambda_j \geq 0$$

for all positive integers k, all $\lambda_i, \lambda_j \in \mathbb{C}$ and all $z_i, z_j \in Z$.

Remark. The normalization as $\varphi(z, z) = 1$ is not always assumed, but it is sometimes useful as here. As the case with positive definite functions on groups, the second symmetric condition is implied by the last condition.

Proof. (Added). (Not completed). With $k = 2$ we have

$$|\lambda_1|^2 + \overline{\lambda_1}\varphi(z_1, z_2)\lambda_2 + \overline{\lambda_2}\varphi(z_2, z_1)\lambda_1 + |\lambda_2|^2 \geq 0.$$

If we set $\lambda_1 = 1 = \lambda_2$, then $2 + \varphi(z_1, z_2) + \varphi(z_2, z_1) \geq 0$. It implies that $\mathrm{Im}(\varphi(z_1, z_2)) + \mathrm{Im}(\varphi(z_1, z_2)) = 0$ and $2 + \mathrm{Re}(\varphi(z_1, z_2)) + \mathrm{Re}(\varphi(z_1, z_2)) \geq 0$.

If we set $\lambda_1 = 1$ and $\lambda_2 = -1$, then $2 - \varphi(z_1, z_2) - \varphi(z_2, z_1) \geq 0$. It then follows that

$$-2 \leq \varphi(z_1, z_2) + \varphi(z_2, z_1) = \mathrm{Re}(\varphi(z_1, z_2)) + \mathrm{Re}(\varphi(z_1, z_2)) \leq 2.$$

If we set $\lambda_1 = 1$ and $\lambda_2 = i = \sqrt{-1}$, then $2 + i\varphi(z_1, z_2) - i\varphi(z_2, z_1) \geq 0$. It implies that

$$\mathrm{Im}(i\varphi(z_1, z_2)) - \mathrm{Im}(i\varphi(z_2, z_1)) = \mathrm{Re}(\varphi(z_1, z_2)) - \mathrm{Re}(\varphi(z_2, z_1)) = 0.$$

Therefore, we obtain

$$\begin{aligned}\varphi(z_1, z_2) &= \mathrm{Re}(\varphi(z_1, z_2)) + i\mathrm{Im}(\varphi(z_1, z_2)) \\ &= \mathrm{Re}(\varphi(z_2, z_1)) - i\mathrm{Im}(\varphi(z_2, z_1)) = \overline{\varphi(z_2, z_1)}.\end{aligned}$$

Also, if we set $\lambda_1 = t \in \mathbb{R}$ and $\lambda_2 = 1$, then

$$t^2 + (\varphi(z_1, z_2) + \varphi(z_2, z_1))t + 1 \geq 0.$$

It then follows that

$$(\varphi(z_1, z_2) + \varphi(z_2, z_1))^2 - 4 \leq 0.$$

Consequently, we obtain that $-1 \leq \mathrm{Re}(\varphi(z_1, z_2)) \leq 1$. □

The following is immediate from comparing definitions:

Lemma 16.16. *If φ is a positive definite, kernel on a set Z and if* $\mathrm{Re}(\varphi)$ *denotes its real part, then* $1 - \mathrm{Re}(\varphi)$ *is a negative type kernel on Z.*

Proof. (For Proposition 16.13 above). Suppose that G acts amenably on a compact Hausdorff space X, with $f_n : X \to Pb(G)$ a sequence of continuous functions asymptotically commuting with the G-action uniformly on X in the 1-norm. With suitable approximations to f_n, we may assume that for each n, there is a finite subset F of G such that for every $x \in X$, the function $f_n(x) \in Pb(G)$ is supported in F. Now let $h_n(x, g) = f_n(x)(g)^{\frac{1}{2}}$. Fix a point $x_0 \in X$ and define functions $\varphi_n : G \times G \to \mathbb{C}$ by

$$\varphi_n(g_1, g_2) = \sum_{g \in G} h_n(g_1 x_0, g_1 g) h_n(g_2 x_0, g_2 g).$$

These are positive definite kernels on $G \times G$.

(Added). Check that

$$\begin{aligned}\varphi_n(g_1, g_1) &= \sum_{g \in G} h_n(g_1 x_0, g_1 g) h_n(g_1 x_0, g_1 g) \\ &= \sum_{g \in G} f_n(g_1 x)(g_1 g) = \sum_{g \in G} f_n(g_1 x)(g) = 1.\end{aligned}$$

Also,

$$\begin{aligned}\varphi_n(g_2, g_1) &= \sum_{g \in G} h_n(g_2 x_0, g_2 g) h_n(g_1 x_0, g_1 g) \\ &= \sum_{g \in G} h_n(g_1 x_0, g_1 g) h_n(g_2 x_0, g_2 g) = \varphi_n(g_1, g_2).\end{aligned}$$

As well,

$$\begin{aligned}&\sum_{i,j=1}^{k} \overline{\lambda_i} \varphi_n(g_i, g_j) \lambda_j \\ &= \sum_{i=1}^{k} \overline{\lambda_i} \varphi_n(g_i, g_i) \lambda_i + \sum_{i,j=1, i \neq j}^{k} \overline{\lambda_i} \varphi_n(g_i, g_j) \lambda_j \\ &= \sum_{i=1}^{k} |\lambda_i|^2 \sum_{g \in G} h_n(g_i x, g_i g)^2 + \sum_{i,j=1, i<j}^{k} (\overline{\lambda_i} \lambda_j + \overline{\lambda_j} \lambda_i) \varphi_n(g_i, g_j) \\ &= \sum_{i=1}^{k} |\lambda_i|^2 \sum_{g \in G} f_n(g_i x)(g_i g) + \sum_{i,j=1, i<j}^{k} \mathrm{Re}(\overline{\lambda_i} \lambda_j) \varphi_n(g_i, g_j) \\ &= \sum_{i=1}^{k} |\lambda_i|^2 + \sum_{i,j=1, i<j}^{k} \mathrm{Re}(\overline{\lambda_i} \lambda_j) \varphi_n(g_i, g_j).\end{aligned}$$

But not completed to show that it is positive. (Anyhow, continued below).

(1) For every finite subset F of G and every $\varepsilon > 0$, there is some $n_0 \in \mathbb{N}$ such that if $n > n_0$ and $g_1^{-1} g_2 \in F$, then $|\varphi_n(g_1, g_2) - 1| < \varepsilon$.

(2) In addition, for every $n \in \mathbb{N}$, there exists a finite subset F of G such that if $g_1^{-1} g_2 \notin F$, then $\varphi_n(g_1, g_2) = 0$.

It follows that for a suitable subsequence the series $\sum_j (1 - \mathrm{Re}(\varphi_{n_j}))$ is pointwise convergent everywhere on $G \times G$. But each function $1 - \mathrm{Re}(\varphi_{n_j})$ is a negative type kernel, and therefore so is the series.

(Added). Indeed, by the Cauchy-Shwarz inequality,

$$\begin{aligned}
|\varphi_n(g_1, g_2)| &= |\sum_{g\in G} h_n(g_1x_0, g_1g)h_n(g_2x_0, g_2g)| \\
&= |\sum_{g\in G} f_n(g_1x_0)(g_1g)^{\frac{1}{2}} f_n(g_2x_0)(g_2g)^{\frac{1}{2}}| \\
&\le \sum_{g\in G} f_n(g_1x_0)(g_1g) \sum_{g\in G} f_n(g_2x_0)(g_2g) = 1,
\end{aligned}$$

so that $-1 \le \mathrm{Re}(\varphi_n(g_1, g_2)) \le 1$. Note as well that being zero on the diagonal of $G \times G$, having the conjugate symmetry, and being negative type for $1 - \mathrm{Re}(\varphi_{n_j})$ positive are preserved under taking limits for their series.

The map into an affine Euclidean space which is associated to the series is a uniform embedding, as desired. □

Remark. In fact, it is possible to characterize the amenability of a group action in terms of positive definite kernels (see [1]). The existence of a sequence of positive definite kernels on $G \times G$ with two properties (1) and (2) in the proof above is equivalent to the amenability of the G-action on its Stone-Čech compactification βG.

Remark. The theory of amenable actions is closely related to the theory of exact groups. To see why, suppose that G admits an amenable action on some compact Hausdorff space X. Then using the theory of positive definite kernels it may be shown that $C(X) \rtimes G \cong C(X) \rtimes_r G$, and moreover, the crossed product C^*-algebra is nuclear. This means that for any C^*-algebra $\mathfrak{D}$,

$$[C(X) \rtimes G] \otimes_{\max} \mathfrak{D} \cong [C(X) \rtimes G] \otimes_{\min} \mathfrak{D}.$$

It follows that the crossed product C^*-algebra is exact. But it then follows that $C_r^*(G)$ is also exact since it is a C^*-subalgebra of $C(X) \rtimes G \cong C(X) \rtimes_r G$. Therefore, the group G is exact. To summarize, if G acts amenably on some compact Hausdorff space, then G is exact. In fact, the converse to this is true.

Poincaré duality

We consider a dual formulation of the Baum-Connes conjecture for certain groups.

Theorem 16.17. *Let G be a countable exact group and let $\mathfrak{A}$ be a separable proper G-C^*-algebra. Suppose that there is a class $\alpha \in E_G(\mathfrak{A}, \mathbb{C})$ with the* *property that for every finite subgroup H of G, the resricted class $\alpha|_H \in E_H(\mathfrak{A}, \mathbb{C})$ is invertible. Then the Baum-Connes assembly map*

$$\mu_r : K(G, \mathfrak{D}) \to E_0(\mathfrak{D} \rtimes_r G) \cong K_0(\mathfrak{D} \rtimes_r G)$$

is an isomorphism for a separable G-C^*-algebra $\mathfrak{D}$ if and only if the induced map from α is an isomorphism:

$$(\alpha \otimes [\mathrm{id}_{\mathfrak{D}}])_* : E_0((\mathfrak{A} \otimes \mathfrak{D}) \rtimes_r G) \to E_0(\mathfrak{D} \rtimes_r G).$$

The remark here is omitted (because of no time to do in revising).

Proof. Consider the following diagram:

$$\begin{array}{ccc} K(G, \mathfrak{A} \otimes \mathfrak{D}) & \xrightarrow{\mu_r} & E_0((\mathfrak{A} \otimes \mathfrak{D}) \rtimes_r G) \\ {\scriptstyle (\alpha\otimes[\mathrm{id}_{\mathfrak{D}}])_*}\downarrow & & \downarrow{\scriptstyle (\alpha\otimes[\mathrm{id}_{\mathfrak{D}}])_*} \\ K(G, \mathfrak{D}) & \xrightarrow{\mu_r} & E_0(\mathfrak{D} \rtimes_r G). \end{array}$$

where the horizontal maps are the Baum-Connes assembly maps and the vertical maps are induced by composition with the class α in E_G-theory and by composition with the element from α in the nonequivariant E-theory $E(\mathfrak{A}, \mathbb{C})$. The diagram commutes. The top horizontal map is an isomorphism. The left hand vertical map is also an isomorphism. Therefore, the bottom horizontal map is an isomorphism if and only if the right vertical map is an isomorphism, as required. □

Let M be a complete Riemannian manifold. We denote by $C_0(M, Cl^*(TM))$ the C^*-algebra of sections of the bundle $Cl^*(TM) = \{Cl^*(T_xM)\}_{x\in M}$ of Clifford algebras $Cl^*(T_xM)$ associated to the tangent space $TM = \cup_{x\in M} T_xM$ of M. There is a Dirac operator D on M, that is, an unbounded self-adjoint operator acting on the Hilbert space $L^2(M, Cl^*(TM))$ of L^2-sections of the Clifford algebra bundle on M, and it defines a class

$$[D] = \alpha \in E(C_0(M, Cl^*(TM)), \mathbb{C}).$$

Moreover, if a group G acts isometrically on M, then the Dirac operator defines an equivariant class

$$[D] = \alpha_G \in E_G(C_0(M, Cl^*(TM)), \mathbb{C}).$$

If M is a universal proper G-space, then the hypotheses of the theorem above are satisfied, so that

Proposition 16.18. *Let M be a complete Riemannian manifold and suppose that a countable group G acts on M by isometries. Assume that M is a universal proper G-space. Then the Dirac operator D on M defines an equivariant E-theory class as*

$$[D] \in E_G(C_0(M, Cl^*(TM)), \mathbb{C}).$$

Moreover, its restriction from G to a finite subgroup H of G implies an invertible element as

$$[D|_H] \in E_H(C_0(M, Cl^*(TM)), \mathbb{C}).$$

Remark. For example, the proposition is applied to the case where G is a lattice in a semi-simple group and M is the associated symmetric space.

17 Counter examples for the BC conjecture with coefficients

Property T discrete groups and their C^*-algebras

Definition 17.1. A discrete group G has **property T** if the trivial representation of G is an isolated point in the unitary dual of G.

See [13].

Theorem 17.2. *Let G be a discrete group. The following are equivalent:*
(a) G has property T.
(b) Every isometric action of G on an affine Euclidean space has a fixed point.
(c) There is a central projection $p \in C^*(G)$ such that for any unitary representation of G on a Hilbert space H, the operator p acts as the orthogonal projection onto the G-fixed vectors in H.

The projection $p \in C^*(G)$ is called the **Kazhdan projection** for the property T group G.
Remark. If G is finite, then the Kazhdan projection is given by the sum

$$p = \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{C}[G] = C^*(G)$$

with $ph = hp = p$ for any $h \in G$, where elements of G are regarded as unitaries of $C^*(G)$ via the left regular representation on $l^2(G)$. Note that

$$p^2 = \frac{1}{|G|^2} \sum_{g,h\in G} gh = \frac{|G|}{|G|^2} \sum_{g\in G} g = p.$$

If G is infinite, and if $p = \sum_{g\in G} \alpha_g g$ with $hp = ph = p$, then all the scalars α_g are equal and have sum equal to 1 in the trivial representation.

Lemma 17.3. *If G is an infinite property T group, then the quotient map from $C^*(G)$ onto $C_r^*(G)$ does not induce a K-theory group isomorphism.*

Proof. The central projection p generates a cyclic direct summand of the group $E_0(C^*(G)) \cong K_0(C^*(G))$, which is mapped to zero in $E_0(C_r^*(G)) \cong K_0(C_r^*(G))$. □

Remark. It follows that if G is an infinite, property T group, then the Baum-Connes assembly maps into $E_0(C^*(G)) \cong K_0(C^*(G))$ and $E_0(C_r^*(G)) \cong K_0(C_r^*(G))$ cannot both be isomorphisms. For instance, if G has property T, then each irreducible, finite dimensional unitary representation of G gives a distinct central projection in $C^*(G)$. So if a property T group G has infinitely many irreducible, finite dimensional unitary representations, then $K_0(C^*(G)) \cong E_0(C^*(G))$ contains a free abelian subgroup of infinite rank, whereas $K(G) = K(G, \mathbb{C})$ is very often finitely generated.

Proposition 17.4. *If G is an exact, infinite property T group, then G does not satisfy the hypotheses of Theorem* 11.24 *for the BC conjecture.*

Proof. If G did satisfy the hypotheses, then the canonical quotient map from $E_0(C^*(G)) \cong K_0(C^*(G))$ to $E_0(C_r^*(G)) \cong K_0(C_r^*(G))$ is an isomorphism. □

It says that for proving the BC conjecture, proving an identity class in equivariant, bivariant K or E-theory does not work for such groups G, but the conjecture may hold or not, and still open.

Hyperbolic, property T groups and their C^*-algebras

One can ask whether it is possible to prove the BC conjecture by proving an indentity class in bivariant K or E-theory for crossed product C^*-algebras by actions of G.

Note that if U_G is a complete manifold M, then the Baum-Connes conjecture for G is equivalent to that the map

$$\alpha_* : K_j(C_0(M, Cl^*(TM)) \rtimes_r G) \to K_j(C^*(G))$$

induced from the Dirac operator class $\alpha \in E_G(C_0(M, Cl^*(TM)), \mathbb{C})$ is an isomorphism.

One might hope that in fact the descended class

$$\alpha \in E(C_0(M, Cl^*(TM)), C_r^*(G))$$

is an isomorphism. This is not always the case, as in the following theorem of Skandalis [53]:

Theorem 17.5. *Let G be an infinite, hyperbolic property T group. Then $C_r^*(G)$ is not equivalent in E-theory to any nuclear C^*-algebra.*

The proof is given below soon later.

Recall that a C^*-algebra $\mathfrak{A}$ is **nuclear** if $\mathfrak{A} \otimes_{\min} \mathfrak{D} \cong \mathfrak{A} \otimes_{\max} \mathfrak{D}$ for any C^*-algebra $\mathfrak{D}$. The C^*-algebra $C_0(M, Cl^*(TM)) \rtimes_r G$ is shown to be nuclear. It then follows that

Corollary 17.6. *Let G be an infinite, hyperbolic, property T group and assume that G acts on a complete Riemannian manifold M by isometries. Then the Dirac operator class $\alpha \in E(C_0(M, Cl^*(TM)) \rtimes_r G, C_r^*(G))$ is not invertible.*

Remark. The corollary applies to discrete, cocompact subgroups of the Lie groups $Sp(n, 1)$, with M as a quaternionic hyperbolic space. See [13]. It follows from the work of Lafforgue [37] that in this case the class α as above does induce an isomorphism on the K-theory. This shows that the E-theory as well as the KK-theory as weapons are not enough to attack the BC conjecture.

To prove the theorem above we use the following result:

Theorem 17.7. *Let G be a hyperbolic group and ∂G be its Gromov boundary. Then there is a compact, metrizable topology on the disjoint union $\overline{G} = G \sqcup \partial G$ with the following properties:*

(a) the set G is an open discrete subset of $\overline{G}$:

(b) the left action of G on itself extends continuously to an amenable action of G on $\overline{G}$;

(c) the right action of G on itself extends continuously to an action on $\overline{G}$ which is trivial on ∂G.

Remark. The property (c) is essentially the assertion that the natural action on the Gromov compactification is small at infinity.

Definition 17.8. Let G be a discrete group. Define the left regular representation λ, the right regular representation ρ, and the **adjoint** representation σ of G on $l^2(G)$ by the formulas:

$$(\lambda_g \xi)(h) = \xi(g^{-1}h), \quad (\rho_g \xi)(h) = \xi(hg), \quad (\sigma_g \xi)(h) = \xi(g^{-1}hg)$$

for $g, h \in G$ and $\xi \in l^2(G)$. The **biregular** representation σ of $G \times G$ on $l^2(G)$ is defined by

$$\sigma(g_1, g_2)\xi(h) = \xi(g_1^{-1}hg_2)$$

for all $g_1, g_2, h \in G$ and $\xi \in l^2(G)$.

The left and right regular representations as well as the adjoint representation of G induce the C^*-algebra representations of $C_r^*(G)$ into $\mathbb{B}(l^2(G))$ by the same symbols λ and ρ as well as σ. Since these representations commute with one another, there is a C^*-algebra representation (by the same symbol):

$$\sigma : C_r^*(G) \otimes_{\max} C_r^*(G) \to \mathbb{B}(l^2(G)),$$

which is the same as the biregular representation on $G \times G$ viewed in the maximal C^*-algebra tensor product.

Proof. (Added). Note that

$$\begin{aligned}(\rho_{g_2}\lambda_{g_1}\xi)(h) &= (\lambda_{g_1}\xi)(hg_2) = \xi(g_1^{-1}hg_2) \\ &= (\rho_{g_2}\xi)(g_1^{-1}h) = (\lambda_{g_1}\rho_{g_2}\xi)(h).\end{aligned}$$

Hence $\rho_{g_2}\lambda_{g_1} = \lambda_{g_1}\rho_{g_2}$. Also,

$$\sigma(g_1, g_2) = \lambda_{g_1}\rho_{g_2} = \sigma(g_1, g_2) = \sigma(\chi_{g_1} \otimes \chi_{g_2}) = (\lambda \otimes \rho)(\chi_{g_1} \otimes \chi_{g_2}),$$

where each $(g_1, g_2) \in G \times G$ is identified with $\chi_{g_1} \otimes \chi_{g_2} \in C_r^*(G) \otimes_{\max} C_r^*(G)$. ☐

Definition 17.9. Denote by $\ker(\Phi)$ the kernel of the canonical surjective $*$-homomorphism Φ from $C_r^*(G) \otimes_{\max} C_r^*(G)$ to $C_r^*(G) \otimes_{\min} C_r^*(G)$. Then there is a short exact sequence of C^*-algebras:

$$0 \to \ker(\Phi) \to C_r^*(G) \otimes_{\max} C_r^*(G) \xrightarrow{\Phi} C_r^*(G) \otimes_{\min} C_r^*(G) \to 0.$$

Lemma 17.10. *The C^*-algebra representation σ maps the closed ideal* $\ker(\Phi)$ *into the closed ideal $\mathbb{K}(l^2(G))$ of $\mathbb{B}(l^2(G))$, where we assume that G is a hyperbolic group with the Gromov boundary ∂G.*

Proof. Denote by $\mathbb{B}/\mathbb{K} = \mathbb{B}(l^2(G))/\mathbb{K}(l^2(G))$ the Calkin C^*-algebra. We prove that there is the following commutative diagram:

$$\begin{array}{ccc}
\ker(\Phi) & \xrightarrow{\sigma} & \mathbb{K}(l^2(G)) = \mathbb{K} \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\
C_r^*(G) \otimes_{\max} C_r^*(G) & \xrightarrow{\sigma} & \mathbb{B}(l^2(G)) = \mathbb{B} \\
{\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle q} \\
C_r^*(G) \otimes_{\min} C_r^*(G) & \xrightarrow{\sigma_{\min}} & \mathbb{B}/\mathbb{K}
\end{array}$$

for some $\sigma_{\min}$ to be constructed, for which the map σ on the first line is well defined, with q the quotient map from $\mathbb{B}$ to $\mathbb{B}/\mathbb{K}$ and with each i the inclusion map. Indeed, if $x \in \ker(\Phi)$, then $\Phi(i(x)) = 0$. Thus, $\sigma_{\min}(\Phi(i(x))) = q(\sigma(i(x))) = 0$. Hence, $\sigma(i(x)) \in \mathbb{K}$.

We begin by constructing a $*$-homomorphism from $C(\partial G)$ into $\mathbb{B}/\mathbb{K}$ as follows. For $f \in C(\partial G)$, we extend it to a continuous function on $\overline{G} = G \sqcup \partial G$, and then restrict the extended function on $\overline{G}$ to the open subset G, and then let the restricted function on G act on $l^2(G)$ by pointwise multiplication.

Two different extended functions $f_1^\sim$ and $f_2^\sim$ on $\overline{G}$ from $f \in C(\partial G)$ induce two pointwise multiplication operators on $l^2(G)$ which only differ by a compact operator. Hence there is a $*$-homomorphism $\varphi : C(\partial G) \to \mathbb{B}/\mathbb{K}$, as required.

Note that it says that there is an open subset U of $\overline{G}$ which contains ∂G and whose complement is a finite subset of G such that $f_1^\sim = f_2^\sim$ on U. Possibly, this is the case in the topology.

Now let G act on $C(\partial G)$ by extending nontrivially the left action of G as in the theorem above. Define a $*$-homomorphism from $C(\partial G) \rtimes G$ to $\mathbb{B}/\mathbb{K}$ by

$$\varphi(\sum_{g\in G} f_g \chi_g) = \sum_{g\in G} \varphi(f_g)\lambda_g$$

where the sum may be a finite sum, each $f_g \in C(\partial G)$, and $\lambda = q \circ \lambda : C^*(G) \to \mathbb{B}(l^2(G)) \to \mathbb{B}/\mathbb{K}$ (with q assumed to be faithful on the pointwise image).

Next, the right regular representation commutes with the C^*-algebra $\varphi(C(\partial G)) \subset \mathbb{B}/\mathbb{K}$ by the theorem above. We then obtain a $*$-homomorphism

$$\varphi \otimes \rho : (C(\partial G) \rtimes G) \otimes_{\max} C_r^*(G) \to \mathbb{B}/\mathbb{K}.$$

But since the action of G on ∂G is amenable, the crossed product C^*-algebra $C(\partial G) \rtimes G$ is nuclear, so that the maximal tensor product $\otimes_{\max}$ above is the same as the minimal on $\otimes_{\min}$. Moreover, the amenability also implies that the full crossed product is isomorphic to the reduced crossed product $C(\partial G) \rtimes_r G$. It then follows that the $*$-homomorphism displayed above is the same as:

$$\varphi \otimes \rho : (C(\partial G) \rtimes_r G) \otimes_{\min} C_r^*(G) \to \mathbb{B}/\mathbb{K}.$$

The lemma now follows by restricting this $*$-homomorphism to the C^*-subalgebra $C_r^*(G) \otimes_{\min} C_r^*(G)$. $\square$

Lemma 17.11. *The K-theory group* $K_0(\ker(\Phi))$ *is nonzero, where we assume that* G *is a hyperbolic, property T group.*

Proof. Now let $\Delta : C^*(G) \to C_r^*(G) \otimes_{\max} C_r^*(G)$ be the $*$-homomorphism defined by sending $\chi_g \mapsto \chi_g \otimes \chi_g$ and by universality of $C^*(G)$. Let $p \in C^*(G)$ be the Kazhdan projection and let $q = \Delta(p)$. Then $q \in \ker(\Phi)$. To see this, observe that the following composition corresponds to the tensor product of two copies of the regular representations, as that

$$\begin{array}{ccc} C^*(G) & \xrightarrow{\Delta} & C_r^*(G) \otimes_{\max} C_r^*(G) \\ \| & & \Phi \downarrow \\ C^*(G) & \xrightarrow{\lambda \otimes \lambda} & C_r^*(G) \otimes_{\min} C_r^*(G) \end{array}$$

with $C_r^*(G) \otimes_{\min} C_r^*(G) \subset \mathbb{B}(l^2(G) \otimes l^2(G))$. Also observe that this representation has no nonzero G-fixed vectors in $l^2(G) \otimes l^2(G)$. Hence, the property T of G implies that the image of the Kazhdan projection p is zero in the minimal tensor product.

Note that the representation $\sigma : C_r^*(G) \otimes_{\max} C_r^*(G) \to \mathbb{B}(l^2(G))$ maps q to a nonzero projection. Indeed, the following composition:

$$C^*(G) \xrightarrow{\Delta} C_r^*(G) \otimes_{\max} C_r^*(G) \xrightarrow{\sigma} \mathbb{B}(l^2(G))$$

is the representation σ of $C^*(G)$ associated to the adjoint representation σ of G, which does have nonzero G-fixed vectors, and the map σ of the composite maps q to the orthogonal projection onto these G-fixed vectors.

For instance, $\chi_{g^{-1}1_G g} = \chi_{1_G} \in l^2(G)$ with 1_G the unit of G.

But it follows from the lemma above that the representation σ maps $\ker(\Phi)$ into $\mathbb{K}(l^2(G))$, and every nonzero projection of $\mathbb{K}$ determines a nonzero K-theory class of the K_0-group of $\mathbb{K}$. Hence the map $\sigma_* : K_0(\ker(\Phi)) \to$

$K_0(\mathbb{K})$ induced from σ takes the K-theory class $[q]$ to a nonzero K-theory class. Therefore, the class $[q]$ is itself nonzero. $\square$

Proof. (For the first theorem in this subsection). Let us suppose that there is a separable nuclear C^*-algebra $\mathfrak{A}$ and an invertible E-theory class $[\varphi] \in E(C_r^*(G), \mathfrak{A})$. Since $C_r^*(G)$ is an exact C^*-algebra, there are invertible E-theory classes:

$$[\varphi] \otimes_{\max} [\mathrm{id}] \in E(C_r^*(G) \otimes_{\max} C_r^*(G), \mathfrak{A} \otimes_{\max} C_r^*(G)) \quad \text{and}$$
$$[\varphi] \otimes_{\min} [\mathrm{id}] \in E(C_r^*(G) \otimes_{\min} C_r^*(G), \mathfrak{A} \otimes_{\min} C_r^*(G)).$$

Therefore, we obtain the following commutative diagram in the E-theory category:

$$\begin{array}{ccc}
C_r^*(G) \otimes_{\max} C_r^*(G) & \xrightarrow[\cong]{[\varphi]\otimes_{\max}[\mathrm{id}]} & \mathfrak{A} \otimes_{\max} C_r^*(G) \\
{\scriptstyle [\Phi]}\big\downarrow & & \big\downarrow{\scriptstyle [\Phi^\sim]} \\
C_r^*(G) \otimes_{\min} C_r^*(G) & \xrightarrow[\cong]{[\varphi]\otimes_{\min}[\mathrm{id}]} & \mathfrak{A} \otimes_{\min} C_r^*(G),
\end{array}$$

where the E-theory classes $[\Phi]$ and $[\Phi^\sim]$ corresponds to the surjective homomorphisms Φ and $\Phi^\sim$ that is defined similarly as Φ. But since $\mathfrak{A}$ is nuclear, the vertical map $[\Phi^\sim]$ is an isomorphism, even at the level of C^*-algebras. It then follows that the vertical map $[\Phi]$ is an isomorphism in the E-theory category. As a result, there are isomorphisms of abelian groups:

$$\begin{array}{ccc}
E(\mathbb{C}, C_r^*(G) \otimes_{\max} C_r^*(G)) & \xrightarrow{\cong} & E(\mathbb{C}, C_r^*(G) \otimes_{\min} C_r^*(G)) \\
{\scriptstyle \cong}\big\uparrow & & \big\downarrow{\scriptstyle \cong} \\
E_0(C_r^*(G) \otimes_{\max} C_r^*(G)) & \xrightarrow[\cong]{} & E_0(C_r^*(G) \otimes_{\min} C_r^*(G)) \\
{\scriptstyle \cong}\big\uparrow & & \big\downarrow{\scriptstyle \cong} \\
K_0(C_r^*(G) \otimes_{\max} C_r^*(G)) & \xrightarrow[\cong]{} & K_0(C_r^*(G) \otimes_{\min} C_r^*(G))
\end{array}$$

where the horizontal isomorphisms in the second and third lines are induced from that in the first line.

But the result above contradicts to the six-term exact sequence of K-theory groups associated to the short exact sequence of C^*-algebras:

$$0 \to \ker(\Phi) \to C_r^*(G) \otimes_{\max} C_r^*(G) \xrightarrow{\Phi} C_r^*(G) \otimes_{\min} C_r^*(G) \to 0$$

with $K_0(\ker(\Phi)) \ncong 0$. $\square$

Another weakness of the Bivariant E-theory and the KK-theory

It is shown in the previous subsection that it is not possible to prove the Baum-Connes conjecture for certain infinite, hyperbolic, property T groups such as uniform lattices in $Sp(n,1)$ by working purely within the bivariant E-theory as well as the KK-theory.

Recall that the bivariant E-theory $E(\mathfrak{A},\mathfrak{B})$ has long but periodic exact sequences in both variables $\mathfrak{A}$ and $\mathfrak{B}$, but we could not equip it with the minimal tensor product operation, since the operation does not preserve exact sequences in general. On the other hand, the Kasparov KK-theory has minimal tensor product operation, but the long but periodic exact sequences are only constructed under some hypothesis or other related to the nuclearity of C^*-algebras. One might ask whether or not there is an ideal theory which has both these desirable properties. But the answer is no such as follows.

Theorem 17.12. ([54]). *There is no bivariant ideal theory functor on separable C^*-algebras, which has both the minimal tensor product operation and the long but periodic exact sequence in both variables.*

The proof is given below soon later.

Remark. By the term as bivariant ideal theory functor, we mean a bifunctor such as the EE-theory and the KK-theory, which is equipped with an associative product such as the E-theory product and the Kasparov product allowing us to create an additive (and multiplicative?) category. The homotopy category of separable C^*-algebras and their $*$-homomorpisms should map to this category, and the ordinary K-theory functor $K_j(\mathfrak{A})$ with one variable $\mathfrak{A}$ should factor through it.

Lemma 17.13. ([33]). *Let G be a residually finite, discrete group. Then the biregular representation σ of $G\times G$ on $l^2(G)$ extends to a representation of the minimal tensor product $C^*(G)\otimes_{\min} C^*(G)$.*

Proof. Let $\{H_n\}$ be a decreasing family of normal subgroups of G with index finite, for which the intersection $\cap_n H_n$ is the trivial subgroup $\{1_G\}$ of G.

If $x\in \mathbb{C}[G]\otimes\mathbb{C}[G]\subset C^*(G)\otimes_{\min} C^*(G)$, then we denote by $(\varphi_n\otimes\varphi_n)(x)$ the corresponding element of $\mathbb{C}[G/G_n]\otimes\mathbb{C}[G/G_n]\subset C^*(G/G_n)\otimes_{\min} C^*(G/G_n)$ by the quotient map $\varphi_n: G\to G/G_n$. Denote by σ_n the biregular representation of $(G/G_n)\times(G/G_n)$ on $l^2(G/G_n)$.

By the functoriality of $\otimes_{\min}$, we have

$$\|x\|\geq \sup_n \|(\varphi_n\otimes\varphi_n)(x)\|.$$

Note that there is a $*$-homomorphism as the following composition:

$$\begin{aligned} C^*(G) \otimes_{\max} C^*(G) &\xrightarrow{\varphi_n \otimes \varphi_n} C^*(G/G_n) \otimes_{\max} C^*(G/G_n) \\ &\xrightarrow[\cong]{\Phi} C^*(G/G_n) \otimes_{\min} C^*(G/G_n) \to 0, \end{aligned}$$

and hence, $\|(\varphi_n \otimes \varphi_n)(x)\| \leq \|x\|$ for $x \in \mathbb{C}[G] \otimes \mathbb{C}[G]$, and the $*$-homomorphism on $\mathbb{C}[G] \otimes \mathbb{C}[G]$ extends continuously to $C^*(G) \otimes_{\min} C^*(G)$ by preserving the norm estimate, where note that $C^*(G/G_n)$ is finite dimensional, so that the map Φ is an isomorphism.

In addition, we have

$$\|(\varphi_n \otimes \varphi_n)(x)\| \geq \|\sigma_n((\varphi_n \otimes \varphi_n)(x))\|,$$

with $\sigma_n : C^*(G/G_n) \otimes C^*(G/G_n) \to \mathbb{B}(l^2(G/G_n)) = \mathbb{K}(l^2(G/G_n))$ with $\otimes = \otimes_{\max} = \otimes_{\min}$.

Now, it is checked that

$$\sup_n \|\sigma_n((\varphi_n \otimes \varphi_n)(x))\| \geq \|\sigma(x)\|.$$

Note that there is a direct sum $*$-homomorphism as

$$\mathbb{C}[G] \otimes \mathbb{C}[G] \xrightarrow{\oplus_n[\sigma_n \circ (\varphi_n \otimes \varphi_n)]} \oplus_n \mathbb{B}(l^2(G/G_n)),$$

which is injective by residual finiteness of G. Hence σ on $\mathbb{C}[G] \otimes \mathbb{C}[G]$ may extend to some C^*-algebra generated by $\mathbb{C}[G] \otimes \mathbb{C}[G]$, which is isomorphic to the C^*-algebra generated by the image of $\mathbb{C}[G] \otimes \mathbb{C}[G]$ in the direct sum C^*-algebra $\oplus_n \mathbb{B}(l^2(G/G_n))$.

It is concluded from putting together all the inequalities thta

$$\|x\| \geq \|\sigma(x)\|,$$

as required. □

Lemma 17.14. *Let $\mathfrak{J} = \ker(\pi)$ denote the kernel of the quotient map π from $C^*(G)$ onto $C_r^*(G)$, so that there is a short exact sequence of C^*-algebras:*

$$0 \to \ker(\pi) \to C^*(G) \xrightarrow{\pi} C_r^*(G) \to 0.$$

If there is a bivariant theory functor $F(\mathfrak{A}, \mathfrak{B})$ which has long but periodic exact sequences in both variables, and if C_π is the mapping cone for π, then *the inclusion map* $i : \ker(\pi) \subset C_\pi$ *determines an invertible element* $[i]$ *of* $F(\mathfrak{J}, C_\pi)$*, where*

$$C_\pi = \{(f, a) \in C_0((0,1], C_r^*(G)) \oplus C^*(G) \mid \pi(a) = f(1)\}$$

with the commutative diagram as a pull back C^**-algebra with* π *and* ev_1 *both surjective*:

$$\begin{array}{ccccc}
0 \oplus \mathfrak{J} & \xrightarrow[\cong]{p_2} & \mathfrak{J} = \ker(\pi) & & \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} & & \\
C_\pi = C_0((0,1], C_r^*(G)) \oplus_\pi C^*(G) & \xrightarrow{p_2} & C^*(G) & & \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} & & \\
C_0((0,1], C_r^*(G)) & \xrightarrow{\mathrm{ev}_1} & C_r^*(G) & \longrightarrow & 0,
\end{array}$$

where each p_j *is the* j*-th coordinate projection and* ev_1 *is the evaluation map at* $1 \in (0,1]$.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{J} & \xrightarrow{i} & C^*(G) & \xrightarrow{\pi} & C_r^*(G) & \longrightarrow & 0 \\
\| & & {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & & \| & & \| \\
0 & \longrightarrow & C_\pi & \xrightarrow{i} & Z_\pi & \xrightarrow{\mathrm{ev}_0 \circ p_1} & C_r^*(G) & \longrightarrow & 0
\end{array}$$

where $Z_\pi = C([0,1], C_r^*(G)) \oplus_\pi C^*(G)$ is a pull back C^*-algebra with π and ev_1, which may be called as the mapping interval for π, and the inclusion map $i : C^*(G) \to Z_\pi$ is defined as $i(a) = (1 \otimes \pi(a), a)$ with $1 \otimes \pi(a)$ constant functions on $[0,1]$. Then the inclusion map is a homotopy equivalence.

Indeed, the C^*-algebra Z_π is contractible over $[0,1]$ to a pull back C^*-algebra $C_r^*(G) \oplus_\pi C^*(G)$, which is isomorphic to $C^*(G)$.

Therefore, by applying the F-theory to the diagram above and using the Five Lemma we obtain that the inclusion map $i : \mathfrak{J} \to C_\pi$ induces an invertible class $[i] \in F(\mathfrak{J}, C_\pi)$ with its inverse class $[j] \in F(C_\pi, \mathfrak{J})$ such that $[j] \otimes [i]$ and $[i] \otimes [j]$ are identity classes of $F(C_\pi, C_\pi)$ and $F(\mathfrak{J}, \mathfrak{J})$ respectively. Moreover, induced (corrected) are the isomorphisms as abelian groups:

$$F(\mathfrak{A}, \mathfrak{J}) \xrightarrow[\cong]{(\cdot)\otimes[i]} F(\mathfrak{A}, C_\pi) \quad \text{and} \quad F(C_\pi, \mathfrak{B}) \xrightarrow[\cong]{[i]\otimes(\cdot)} F(\mathfrak{J}, \mathfrak{B}),$$

$$F(\mathfrak{A}, C_\pi) \xrightarrow[\cong]{(\cdot)\otimes[j]} F(\mathfrak{A}, \mathfrak{J}) \quad \text{and} \quad F(\mathfrak{J}, \mathfrak{B}) \xrightarrow[\cong]{[j]\otimes(\cdot)} F(C_\pi, \mathfrak{B})$$

for every C^*-algebra $\mathfrak{A}$ and $\mathfrak{B}$. □

Proof. (For the theorem in this subsection). If the bivariant F-theory has the minimal tensor product operation, then it follows from the lemma above that the inclusion

$$\mathfrak{J} \otimes_{\min} C^*(G) \subset C_\pi \otimes_{\min} C^*(G)$$

determines an invertible class of F-theory, and therefore, induces an isomorphism on K-theory groups. But it is proved that the map on K-theory induced from the above inclusion fails to be surjective as in the following.

Consider the following short exact sequence:

$$0 \to \mathfrak{L} \to C^*(G) \otimes_{\min} C^*(G) \xrightarrow{\pi \otimes \mathrm{id}} C_r^*(G) \otimes_{\min} C^*(G) \to 0,$$

where the closed ideal $\mathfrak{L}$ is by definition the kernel of the quotient homomorphism $\pi \otimes \mathrm{id}$. The mapping cone for $\pi \otimes \mathrm{id}$ is isomorphic to $C_\pi \otimes_{\min} C^*(G)$.

Note that

$$\begin{aligned} C_{\pi\otimes\mathrm{id}} &= C_0((0,1], C_r^*(G) \otimes_{\min} C^*(G)) \oplus_{\pi\otimes\mathrm{id}} [C^*(G) \otimes_{\min} C^*(G)] \\ &\cong [C_0((0,1], C_r^*(G)) \oplus_\pi C^*(G)] \otimes_{\min} C^*(G) \end{aligned}$$

by sending $(f \otimes (\pi(a) \otimes b), a \otimes b) \mapsto (f \otimes \pi(a), a) \otimes b$ and extending to a $*$-homomorphism, with

$$\begin{aligned} \|(f \otimes (\pi(a) \otimes b), a \otimes b)\| &= \max\{\|(f \otimes (\pi(a) \otimes b)\|, \|a \otimes b\|\} \\ &= \|(f \otimes \pi(a), a) \otimes b\|. \end{aligned}$$

It then follows from the same proof for the lemma above that the inclusion

$$\mathfrak{L} \subset C_{\pi\otimes\mathrm{id}} \cong C_\pi \otimes_{\min} C^*(G)$$

induces an invertible class in the F-theory, so that the class induces an isomorphism of K-theory groups.

Observe now that there is a sequence of inclusions:

$$\mathfrak{J} \otimes_{\min} C^*(G) \subset \mathfrak{L} \subset C_\pi \otimes_{\min} C^*(G).$$

It is proved below that the inclusion

$$\mathfrak{J} \otimes_{\min} C^*(G) \subset C_\pi \otimes_{\min} C^*(G).$$

does not induce a surjective K-theory homomorphism because the inclusion $\mathfrak{J} \otimes_{\min} C^*(G) \subset \mathfrak{L}$ does not so.

There is a diagonal homomorphism:

$$\Delta : C^*(G) \to C^*(G) \otimes_{\min} C^*(G) \subset \mathbb{B}(l^2(G) \otimes l^2(G))$$

by universality of $C^*(G)$. Now suppose that G is a property T group and let p be the Kazhdan projection of $C^*(G)$. Set $q = \Delta(p)$. Consider the composition $(\pi \otimes \mathrm{id}) \circ \Delta$. This representation has no nonzero G-fixed vectors, so that q belongs to $\mathfrak{L}$. Since the biregular representation σ of $G \times G$ on $l^2(G)$ extends to $C^*(G) \otimes_{\min} C^*(G)$, it then follows that the following diagram is commutative:

$$\begin{array}{ccc} C^*(G) \otimes_{\min} C^*(G) & \xrightarrow{\sigma} & \mathbb{B}(l^2(G)) \\ {\scriptstyle \pi\otimes\pi}\downarrow & & \downarrow{\scriptstyle q} \\ C^*_r(G) \otimes_{\min} C^*_r(G) & \xrightarrow{\sigma_{\min}} & \mathbb{B}(l^2(G))/\mathbb{K}(l^2(G)), \end{array}$$

where G is assume to be a hyperbolic group, so that the homomorphism $\sigma_{\min}$ is constructed by hyperbolicity of G as before. Since $\mathfrak{L}$ is the kernel of $\pi \otimes \mathrm{id}$, it is contained in the kernel of $\pi \otimes \pi$. Hence the commutativity of the diagram implies that σ maps $\mathfrak{L}$ into $\mathbb{K}(l^2(G))$.

Consider now the sequence of homomorphisms:

$$\mathfrak{J} \otimes_{\min} C^*(G) \xrightarrow[\subset]{i} \mathfrak{L} \xrightarrow{\sigma} \mathbb{K}(l^2(G)),$$

and the composition $\sigma \circ i$ is zero. But the projections p as well as q are mapped to a nonzero class of $\mathbb{K}(l^2(G))$ because the representations $\sigma \circ \Delta$ as well as σ does have nonzero G-fixed vectors. Therefore, the K-theory class $[q] \in K_0(\mathfrak{L})$ is mapped to a nonzero class of $K_0(\mathbb{K}(l^2(G)))$. This shows that the class $[q]$ is not contained in the image of $K_0(\mathfrak{J} \otimes_{\min} C^*(G))$ under the induced map i_*. Namely,

$$K_0(\mathfrak{J} \otimes_{\min} C^*(G)) \xrightarrow[\not\cong]{i_*} K_0(\mathfrak{L}).$$

□

Expander graphs

Definition 17.15. Let Γ be a finite graph, i.e., a finite, 1-dimensional simplicial complex and let $V = V(\Gamma)$ be the set of vertices of Γ. The Laplace operator $\Delta : l^2(V) \to l^2(V)$ is a positive linear operator defined by the positive quadratic form:

$$\langle f, \Delta f \rangle = \sum_{(v,v'),d(v,v')=1} |f(v) - f(v')|^2 \geq 0,$$

where the sum is over (unordered) pairs of adjacent vertives, or in other words over the edges of Γ, where each (not directed) edge $[g, g']$ (a line segment between adjacent g and g') is identified with a pair (g, g'). We may denote by $E = E(V)$ the set of edges of Γ with length one. Denote by $\lambda_1(\Gamma)$ the first nonzero eigenvalue of Δ.

Remark. (Added). Note that $\langle f, \Delta f\rangle = \langle \Delta f, f\rangle$ with $\Delta = \Delta^*$. Since Δ is positive, the spectrum $\sigma(\Delta)$ is contained in $[0, \infty)$.

If f is a constant function on V, then $\langle f, \Delta f\rangle = 0$. Thus $\|\Delta^{\frac{1}{2}} f\|^2 = 0$ for any f. Hence the square root $\Delta^{\frac{1}{2}}$ as well as Δ are zero. □

If the graph Γ is connected, then the kernel of Δ consists of the constant functions on V.

In this case, if $f \in l^2(V)$ and $\sum_{v\in V} f(v) = 0$, then

$$\|f\|^2 \leq \frac{1}{\lambda_1(\Gamma)}\langle \Delta f, f\rangle.$$

This estimate may be called as the expander property.

Proof. (Added but not completed). For some nonzero $f \in l^2(V)$, suppose that $(\lambda 1 - \Delta)f = 0$ with $\lambda \in \mathbb{C}$ and 1 the identity operator on $l^2(V)$, and namely that λ is a point spectrum of Δ, with $0 \leq \lambda < \infty$. It follows that

$$\lambda\langle f, f\rangle - \langle \Delta f, f\rangle = 0.$$

In particular, if $\lambda = \lambda_1(\Gamma) > 0$, then

$$\|f\|^2 = \frac{1}{\lambda_1(\Gamma)}\langle \Delta f, f\rangle.$$

□

Definition 17.16. Let k be a positive integer and let $\varepsilon > 0$. A finite graph Γ is said to be a (k, ε)-expander if it is connected, if no vertex of Γ is incident to more than k edges, and if $\lambda_1(\Gamma) \geq \varepsilon$.

Proposition 17.17. *Let k be a positive integer and, let $\varepsilon > 0$, and let $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of (k, ε)-expander graphs for which* $\lim_{n\to\infty} |V(\Gamma_n)| = \infty$. *Let V_∞ be the disjoint union of the sets $V_n = V(\Gamma_n)$ and suppose that V_∞ is equipped with a distance function which restricts to the path distance function on each V_n. Then the metric space V_∞ may not be embedded into an affine Euclidean space.*

Proof. Suppose that there is a uniform embedding f from V_∞ into an affine Euclidean space E. We may assume that E is complete and separable, and then we may identify E isometrically with $l^2(\mathbb{N})$. By restricting f to each V_n and by adjusting each $f_n = f$ on V_n by a translation in $l^2(\mathbb{N})$, that is, by adding suitable constant vector-valued functions to each f_n, we can arrange that each f_n is orthogonal to every constant function in the Hilbert space of all functions V_n to $l^2(\mathbb{N})$, and we have to arrange that $\sum_{x\in V_n} f(x) = 0$.

Now the Laplace operator can be defined on $l^2(\mathbb{N})$-valued functions on V_∞, just as it is defined on scalar-valued functions on V_n. Then the expander property as above carries over the vector-valued Laplacian.

However,

$$\begin{aligned}\langle \Delta f_n, f_n\rangle &= \sum_{d(v,v')=1} |f_n(v) - f_n(v')|^2 \\ &\leq \sum_{d(v,v')=1} 1 \leq \frac{k}{2}|V_n|,\end{aligned}$$

It follows from the expander property that

$$\sum_{v\in V_n} \|f_n(v)\|^2 = \|f_n\|^2 \leq \frac{1}{\varepsilon}\langle \Delta f_n, f_n\rangle \leq \frac{k}{2\varepsilon}|V_n|.$$

Thus for all n and for at least half of the points $v \in V_n$, we have $\|f_n(v)\|^2 \leq \frac{k}{\varepsilon}$. This contradicts to the definition of a uniform embedding since among this half, there must be points v_n and v'_n with $\lim_{n\to\infty} d(v_n, v'_n) = \infty$. □

Definition 17.18. Let us say that a finitely generated discrete group G is a Gromov group if for some positive integer k and some $\varepsilon > 0$, there is a sequence of (k,ε)-expander graphs Γ_n and a sequence of maps $\varphi_n : V(\Gamma_n) \to G$ such that

(*a*) there is a constant R such that if v and v' are adjacent vertices in some graph Γ_n, then $d(\varphi_n(v), \varphi_n(v')) \leq R$;

$$(b) \quad \lim_{n\to\infty} \max_{g\in G} \frac{|\varphi_n^{-1}(g)|}{|V(\Gamma_n)|} = 0.$$

The second condition implies that

$$\lim_{n\to\infty} |V(\Gamma_n)| = \infty.$$

As a simple extension of the proposition above,

Proposition 17.19. *If G is a Gromov group, then G cannot be uniformly embedded into an affine Euclidean space.*

The Baum-Connes conjecture with coefficients

It is contingent on the existence of a Gromov group that there exists a separable, commutative C^*-algebra $\mathfrak{D}$ and an action of a countable group G on $\mathfrak{D}$, for which the Baum-Connes map

$$\mu_\lambda : K(G, \mathfrak{D}) \to E_0(\mathfrak{D} \rtimes_r G) \cong K_0(\mathfrak{D} \rtimes_r G)$$

fails to be an isomorphism.

Lemma 17.20. *Let G be a countable group and let $\mathfrak{J}$ be a closed ideal of a G-C^*-algebra $\mathfrak{A}$. If the Baum-Connes assembly map μ_λ is an isomorphism for G, with coefficients as all separable C^*-algebras $\mathfrak{J}$ (corrected), then the K-theroy sequence*

$$K_0(\mathfrak{J} \rtimes_r G) \xrightarrow{i_*} K_0(\mathfrak{A} \rtimes_r G) \xrightarrow{q_*} K_0((\mathfrak{A}/\mathfrak{J}) \rtimes_r G)$$

is exact in the middle, where $i : \mathfrak{J} \rtimes_r G \to \mathfrak{A} \rtimes_r G$ is the inclusion map and $q : \mathfrak{A} \rtimes_r G \to (\mathfrak{A}/\mathfrak{J}) \rtimes_r G$ is the quotient map, but their composition $q \circ i$ may not be zero.

Proof. Since exactness of the sequence is preserved by direct limits it suffices to consider the case where $\mathfrak{A}$ itself is separable. The proof follows from the following diagram with the assembly maps μ and the induced quotient maps λ_* :

$$\begin{array}{ccccc}
K(G,\mathfrak{J}) & \xrightarrow{i_*} & K(G,\mathfrak{A}) & \xrightarrow{q_*} & K(G,\mathfrak{A}/\mathfrak{J}) \\
{\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} \\
E_0(\mathfrak{J} \rtimes G) & \xrightarrow{i_*} & E_0(\mathfrak{A} \rtimes G) & \xrightarrow{q_*} & E_0((\mathfrak{A}/\mathfrak{J}) \rtimes G) \\
{\scriptstyle\lambda_*}\downarrow & & \downarrow{\scriptstyle\lambda_*} & & \downarrow{\scriptstyle\lambda_*} \\
E_0(\mathfrak{J} \rtimes_r G) & \xrightarrow{i_*} & E_0(\mathfrak{A} \rtimes_r G) & \xrightarrow{q_*} & E_0((\mathfrak{A}/\mathfrak{J}) \rtimes_r G)
\end{array}$$

and the fact that the horizontal sequence in the middle is exact in the middle as $q_* \circ i_* = 0$ and the assumption that the compositions $\lambda_* \circ \mu = \mu_r$ in the vertical sequences in the middle and the left are isomorphisms.

Indeed, let $[x]$ be a class of $E_0(\mathfrak{J} \rtimes_r G)$. Since the map μ_r is isomorphism at the left column, one can find a class $[y]$ of $E_0(\mathfrak{J} \rtimes G)$, which is in the inverse image of $[x]$ under λ_*. By exactness at the middle on the middle row, $(q_* \circ i_*)[y] = [0]$. Hence, $(\lambda_* \circ q_* \circ i_*)[y] = [0]$. By commutativity of the diagram, we obtain $(q_* \circ i_*)[x] = [0]$. It shows that the image of $E_0(\mathfrak{J} \rtimes_r G)$ under i_* is contained in the kernel of the map q_* on the bottom line.

It remains to show that the converse inclusion holds. For this, we assume that a nonzero class $[p]$ of $E_0(\mathfrak{A} \rtimes_r G)$ is not contained in the image of i_*. But if $q_*[p] = [0]$, then one can find a class $[p']$ at the center of the diagram which is in the image under i_*. If $[p']$ is mapped to $[p]$ under λ_*, then we have a contradicion, as desired (but unchecked).

Note that in the proof above we are not using whether (or not) the map μ_r is an isomorphism at the right column. $\square$

It is proved below that if G is a Gromov group, then for a suitable $\mathfrak{A}$ and $\mathfrak{J}$ the conclusion of the lemma does not hold.

Definition 17.21. Let $\mathfrak{A}$ be the C^*-algebra of bounded $\mathbb{C}$-valued functions on $G \times \mathbb{N}$ such that the restriction to each subset $G \times \{n\}$ for $n \in \mathbb{N}$ belongs to $c_0(G)$ of functions on G vanishing at infinity. Denote by $\mathfrak{J}$ the closed ideal of $\mathfrak{A}$ consisting of functions on $G \times \mathbb{N}$ vanishing at infinity.

Then we have

$$\mathfrak{A} \cong l^\infty(\mathbb{N}, c_0(G)) \cong l^\infty(\mathbb{N}) \otimes c_0(G) \quad \text{and}$$
$$\mathfrak{J} = c_0(G \times \mathbb{N}) \cong c_0(\mathbb{N}, c_0(G)) \cong c_0(\mathbb{N}) \otimes c_0(G).$$

Note that $\mathfrak{A}$ is not separable but $\mathfrak{J}$ is separable and $\mathfrak{A}/\mathfrak{J}$ is not separable as well.

Now let G act on $\mathfrak{A}$ by the right translation action of G on $G \times \mathbb{N}$ as $g \cdot (h, n) = (hg, n)$.

Lemma 17.22. *The right regular representation ρ of G on $l^2(G \times \mathbb{N})$ induces a faithful covariant representation of the reduced crossed product C^*-algebra $\mathfrak{A} \rtimes_r G$ on $l^2(G \times \mathbb{N}) \cong l^2(\mathbb{N}, l^2(G)) \cong l^2(\mathbb{N}) \otimes l^2(G)$ (corrected).*

Proof. (Added). We may assume that $\mathfrak{A} \cong l^\infty(\mathbb{N}) \otimes c_0(G)$ acts on $l^2(G \times \mathbb{N}) \cong l^2(\mathbb{N}) \otimes l^2(G)$ by the multiplication operators M_x as

$$M_{f_1 \otimes f_2}(\xi \otimes \eta) = M_{f_1}\xi \otimes M_{f_2}\eta = f\xi \otimes g\eta \in l^2(\mathbb{N}) \otimes l^2(G).$$

It is well known that this representation M defined above is faithful. It then follows that

$$\begin{aligned}
(\rho_g M_{f_1 \otimes f_2} \rho_{g^{-1}})(\xi \otimes \eta)(n, h) &= (M_{f_1 \otimes f_2} \rho_{g^{-1}})(\xi \otimes \eta)(n, hg) \\
&= (f_1 \xi \otimes f_2 \rho_{g^{-1}} \eta)(n, hg) \\
&= f_1(n)\xi(n) \otimes f_2(hg)\eta(h) \\
&= M_{f_1 \otimes (g \cdot f_2)}(\xi \otimes \eta)(n, h) = M_{g \cdot (f_1 \otimes f_2)}(\xi \otimes \eta)(n, h),
\end{aligned}$$

which shows that the pair (M, ρ) of the representations is by definition a covariant representation of $\mathfrak{A} \rtimes_r G$ to be induced by continuity. $\square$

From here on, we assume that G is a Gromov group with the maps $\varphi_n : V_n \to G$ injective, for simplicity. Let V be the disjoint union of V_n. Let us map V_n under φ_n to the n-th copy $G \times \{n\}$ of G in $G \times \mathbb{N}$, and thereby embed V into $G \times \mathbb{N}$. We then can identify $l^2(V)$ with a closed subspace of $l^2(G \times \mathbb{N})$.

We may identify the vertex set $V_n = V(\Gamma_n)$ with its image in G under φ_n injective. We denote by $E_n = E(\Gamma_n)$ the set of (not directed) edges in Γ_n. Let $[g, g'] \in E_n$ be an edge $[g, g']$ which corresponds elements $g, g' \in G$ which are adjacent in the graph Γ_n.

If $g \in G$ correspond to a vertex of Γ_n, we write $e_n(g)$ for its valence, which means the number of edges in Γ_n joined at g, and set $k_n(g) = e_n(g) - 1$.

Definition 17.23. Denote by $\Delta_\infty : l^2(G \times \mathbb{N}) \to l^2(G \times \mathbb{N})$ the direct sum of the Laplace operators Δ_n on each $l^2(V_n) \subset l^2(V)$ with the identity operator on the orthogonal complement of $l^2(V)$ in $l^2(G \times \mathbb{N})$.

Lemma 17.24. *The operator $\Delta_\infty - \mathrm{id}$ on $l^2(G \times \mathbb{N})$ belongs to $\mathfrak{A} \rtimes_r G \subset \mathbb{B}(l^2(G \times \mathbb{N}))$, and it is in fact in its algebraic crossed product.*

Proof. Let $\chi_{(g,n)}$ be the characteristic function at $(g, n) \in G \times \mathbb{N}$. The set of all $\chi_{(g,n)}$ for $(g, n) \in G \times \mathbb{N}$ is the canonical basis for the Hilbert space $l^2(G \times \mathbb{N})$. In this basis, we have

$$\begin{cases} (\Delta_\infty - \mathrm{id})\chi_{(g,n)} = k_n(g)\chi_{(g,n)} - \sum_{[g,g'] \in E_n} \chi_{(g',n)} & \text{if } g \in V_n, \\ (\Delta_\infty - \mathrm{id})\chi_{(g,n)} = 0 & \text{if } g \notin V_n. \end{cases}$$

Note that if $g \in \Gamma_n$, then

$$\begin{aligned} &\langle \chi_{(g,n)}, (\Delta_\infty - \mathrm{id})\chi_{(g,n)} \rangle = \langle \chi_{(g,n)}, \Delta_\infty \chi_{(g,n)} \rangle - 1 \\ &= \sum_{(h,h'), d(h,h')=1} |\chi_{(g,n)}(h, n) - \chi_{(g,n)}(h', n)|^2 - 1 \\ &= \sum_{(g,h), d(g,h)=1} |1 - 0|^2 - 1 = e_n(g) - 1 = k_n(g) \\ &= \langle \chi_{(g,n)}, k_n(g)\chi_{(g,n)} - \sum_{[g,g'] \in E_n} \chi_{(g',n)} \rangle. \end{aligned}$$

Therefore, we can write $\Delta_\infty - \mathrm{id}$ as a finite sum

$$\Delta_\infty - \mathrm{id} = k \cdot \mathrm{id} + \sum_{h \neq 1_G} c \cdot \rho_h$$

(corrected) with the coefficient functions k and c defined as

$$k(g,n) = \begin{cases} k_n(g) & \text{if } g \in V_n \\ 0 & \text{if } g \notin V_n \end{cases}$$

and for $h \neq 1_G$,

$$c(gh,n) = \begin{cases} -1 & \text{if } [g,gh] \in E(\Gamma_n), \\ 0 & \text{if } [g,gh] \notin E(\Gamma_n). \end{cases}$$

It follows from the first condition of G being a Gromov group that the sum is finite. □

Since the graphs Γ_n are (k,ε)-expanders, the point zero is isolated in the spectrum of Δ_∞. Therefore, it makes a sense that

Definition 17.25. Let G be a Gromov group and assume that the maps $\varphi_n : V_n \to G$ are injective. We denote by $\chi_0(\Delta_\infty)$ the orthogonal projection onto the kernel of Δ_∞, which belongs to $\mathfrak{A} \rtimes_r G$ by functional calculus.

The operator $\chi_0(\Delta_\infty)$ is the orthogonal projection onto the functions of $l^2(G \times \mathbb{N})$ which are constant on each V_n and zero on the complement of $V = \sqcup_n V_n$.

Proof. (Added). Indeed, if f is such a function, then $\langle f, \Delta_\infty f\rangle = 0$, so that $\Delta_\infty^{\frac{1}{2}} f = 0$ and thus $\Delta_\infty f = 0$. Conversely, if f is not such a function, then $\langle f, \Delta_\infty f\rangle$ can not be zero. Hence, $\Delta_\infty f \neq 0$. □

Lemma 17.26. *The E-theory class $[\chi_0(\Delta_\infty)]$ of $E_0(\mathfrak{A} \rtimes_r G)$ is not in the image of the map $i_* : E_0(\mathfrak{I} \rtimes_r G) \to E_0(\mathfrak{A} \rtimes_r G)$.*

Proof. Let $\mathfrak{B}_n = c_0(G \times \{n\})$. Since $\mathfrak{B}_n$ is a quotient of $\mathfrak{A}$ and is invariant under the action of G, there is a quotient homomorphism as

$$\pi_n : \mathfrak{A} \rtimes_r G \to \mathfrak{B}_n \rtimes_r G.$$

This induces the following:

$$(\pi_n)_* : E_0(\mathfrak{A} \rtimes_r G) \to E_0(\mathfrak{B}_n \rtimes_r G) \cong \mathbb{Z}$$

because

$$\mathfrak{B}_n \rtimes_r G \cong C_0(G) \rtimes G \cong \mathbb{K}.$$

Since $\pi_n(\chi_0(\Delta_\infty))$ is a rank one projection, we get $(\pi_n)_*[\chi_0(\Delta_\infty)] = 1$ for all n. Therefore, the E-theory class $[\chi_0(\Delta_\infty)]$ does not come from $E_0(\mathfrak{J} \rtimes_r G)$.

Note that

$$\mathfrak{J} \rtimes_r G \cong c_0(\mathbb{N}) \otimes (C_0(G) \rtimes G) \cong \oplus_{\mathbb{N}} \mathbb{K}$$

the direct sum. It then follows that

$$E_0(\mathfrak{J} \rtimes_r G) \cong \oplus_{\mathbb{N}} \mathbb{Z}$$

the direct sum. □

Lemma 17.27. *The image $q(\chi_0(\Delta_\infty))$ in $(\mathfrak{A}/\mathfrak{J}) \rtimes_r G$ is zero.*

The proof is given below soon later.

Recall that the reduced crossed product C^*-algebras $\mathfrak{D} \rtimes_r G$ is faithfully represented as operators on the Hilbert $\mathfrak{D}$-module $l^2(G, \mathfrak{D})$.

Exercise. If p_g denotes the orthogonal projection onto the space of constant functions of $l^2(G, \mathfrak{D})$ supported on an element $g \in G$, denoted by $\mathfrak{C}_g$, and if $t \in \mathfrak{D} \rtimes_r G$, then $p_g t p_e$ with $g, e \in G$ is an operator from $\mathfrak{C}_e$ to $\mathfrak{C}_g$. If $p_g t p_e = 0$ for all $g, e \in G$, then $t = 0$.

Proof. (Added). Note that the space $\mathfrak{C}_g$ is generated as a vector space by $\chi_g d \in \mathfrak{D} \rtimes_r G$ as well as $l^2(G, \mathfrak{D})$ for any $d \in \mathfrak{D}$, and is closed. Also, for $g_1, g_2 \in G$ and $d_1, d_2 \in \mathfrak{D}$,

$$(\chi_{g_1} d_1)(\chi_{g_2} d_2) = \chi_{g_1} \chi_{g_2^{-1}} (\chi_{g_2^{-1}} d_1 g_2) d_2 = \chi_{g_1 g_2^{-1}} (g_2^{-1} \cdot d_1) d_2 \in \mathfrak{D} \rtimes_r G.$$

Hence, we obtain $\mathfrak{C}_{g_1} \cdot \mathfrak{C}_{g_2} = \mathfrak{C}_{g_1 g_2^{-1}}$.

The first in the statement is clear by definition. The second follows from that the union of $\mathfrak{C}_g$ for $g \in G$ is dense in $\mathfrak{D} \rtimes_r G$ as well as $l^2(G, \mathfrak{D})$. □

Exercise. The operator $p_g t p_e$ can be identified with an element $t_g \in \mathfrak{D}$ such that

$$(p_g t p_e \xi)(g) = t_g \cdot \xi(e) \quad \xi \in l^2(G, \mathfrak{D}).$$

If t is a finite sum $\sum d_g \cdot \chi_g$ with $d_g \in \mathfrak{D}$ and is in the algebraic crossed product of $\mathfrak{D}$ by G, then $t_g = d_g$.

If $\varphi : \mathfrak{D} \to \mathfrak{D}'$ is a G-equivariant $*$-homomorphism and if Φ is the induced map on their crossed products by φ, then $\Phi(t)_g = \varphi(t_g)$.

Proof. (Added). If ξ is a finite sum $\sum_h \chi_h c_h$ with $c_h \in \mathfrak{D}$ and if $t = \sum_{g'} \chi_{g'} d_{g'}$, then

$$p_e \xi = \begin{cases} \chi_e c_e & \text{if } h = e \text{ for some } h, \\ 0 & \text{otherwise,} \end{cases}$$

which may be identified with $c_e = \xi(e)$, and

$$\begin{aligned} p_g t c_e &= p_g \sum_{g'} \chi_{g'} d_{g'} c_e \\ &= \begin{cases} \chi_g d_g c_e = \chi_g d_g(\xi(e)) & \text{if } g' = g \text{ for some } g', \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly, compute with omitting the zero case as

$$p_g \Phi(t) p_e \xi = p_g (\sum_{g'} \chi_{g'} \varphi(d_{g'})) c_e = \varphi(d_g)(\xi(e));$$

so that $\Phi(t)_g = \varphi(t_g)$, where we may assume as that $\Phi(\chi_{g'} d_{g'}) = \chi_{g'} \varphi(d_{g'})$. Note that for $g \in G$ and $d \in \mathfrak{D}$,

$$\chi_g \varphi(d) \chi_{g^{-1}} = g \cdot (\varphi(d)) = \varphi(g \cdot d) = \varphi(\chi_g d \chi_{g^{-1}}).$$

□

If an operator $t \in \mathfrak{A} \rtimes_r G$ has matrix coefficients $t_{(g,n),(g',n')} \in \mathbb{C}$ for the canonical basis $\{\chi_{(g,n)}\}_{(g,n) \in G \times \mathbb{N}}$ of $l^2(G, \mathbb{N})$ as

$$t \chi_{(g,n)} = \sum_{(g',n') \in G \times \mathbb{N}} t_{(g,n),(g',n')} \chi_{(g',n')},$$

then the functions $t_g \in \mathfrak{A} \cong l^\infty(\mathbb{N}, c_0(G))$ associated to t are defined by

$$t_g(h, n) = t_{(hg,n),(h,n)}.$$

Proof. If t is a finite sum $\sum d_g \chi_g$, then $t_g = d_g$. If $t = d_s \chi_s$, then $t = t_s \chi_s$ with $t_s = d_s \in \mathfrak{A} \cong l^\infty(\mathbb{N}, c_0(G))$. Then

$$t \chi_{(g,n)} = t_s \chi_s \chi_{(g,n)} = t_s \chi_{(gs,n)}.$$

Therefore, we obtain

$$\begin{aligned} (t \chi_{(g,n)})(h, u) &= t_{(g,n),(h,u)} \\ &= t_s(h, u) \chi_{(gs,n)}(h, u). \end{aligned}$$

It then follows that

$$t_s(gs,n) = t_{(g,n),(gs,n)}.$$

If we replace $s = g'$ and $gs = h$, then

$$t_{g'}(h,n) = t_{(h(g')^{-1},n),(h,n)}$$

(corrected). □

Proof. (For the lemma above). The projection $\chi_0(\Delta_\infty)$ on $l^2(G \times \mathbb{N})$ is comprised of the sequence of projections $\chi_0(\Delta_n)$ onto the constant functions in $l^2(V_n)$. The matrix coefficients of $\chi_0(\Delta_\infty)$ (as well as $\chi_0(\Delta_n)$) are given by the formula

$$\begin{cases} \chi_0(\Delta_\infty)(\chi_{(g,n)}) = \sum_{g' \in V_n} \frac{1}{|V_n|}\chi_{(g',n)} & \text{if } g \in V_n, \\ \chi_0(\Delta_\infty)(\chi_{(g,n)}) = 0 & \text{if } g \notin V_n \end{cases}$$

for $n \in \mathbb{N}$, with matrix coefficients $\frac{1}{|V_n|}$ as locally constants (unchecked). As a result, the functions $\chi_0(\Delta_\infty)_g$ (as well as $\chi_0(\Delta_n)_g$) in $\mathfrak{A}$ associated to the projection $\chi_0(\Delta_\infty)$ (and $\chi_0(\Delta_n)$) are given by the formula

$$\chi_0(\Delta_\infty)_g(h,n) = \begin{cases} \frac{1}{|V_n|} & \text{if } hg, h \in V_n, \\ 0 & \text{if } hg \notin V_n \text{ or } h \notin V_n \end{cases}$$

for $n \in \mathbb{N}$. This shows that $\chi_0(\Delta_\infty)_g \in \mathfrak{J}$ for all $g \in G$. It follows that the elements $q(\chi_0(\Delta_\infty)_g) = q(\chi_0(\Delta_\infty))_g \in \mathfrak{A}/\mathfrak{J}$ associated to the image of $\chi_0(\Delta_\infty)$ in $(\mathfrak{A}/\mathfrak{J}) \rtimes_r G$ are zero, so that the projection $q(\chi_0(\Delta_\infty))$ is itself zero in $(\mathfrak{A}/\mathfrak{J}) \rtimes_r G$. □

Corollary 17.28. (*Added*). *The following sequence of the E-theory groups:*

$$E_0(\mathfrak{J} \rtimes_r G) \xrightarrow{i_*} E_0(\mathfrak{A} \rtimes_r G) \xrightarrow{q_*} E_0((\mathfrak{A}/\mathfrak{J}) \rtimes_r G)$$

is not exact in the middle.

Theorem 17.29. *Let G be a Gromov group. Then there is a separable, commutative G-C^*-algebra $\mathfrak{D}$ as $\mathfrak{J}$ above, for which the Baum-Connes assembly map*

$$\mu_\lambda; K(G, \mathfrak{D}) \to E_0(\mathfrak{D} \rtimes_r G)$$

is not an isomorphism.

Non exact groups

Theorem 17.30. (Guentner and Kaminker [20], [21] and Ozawa [46]). *If a finitely generated discrete group G is exact, then G is embedded uniformly into a Hilbert space.*

The proof is given below soon later.

Definition 17.31. (Kirchberg [34]). Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C^*-algebras. A unital linear map $\Phi : \mathfrak{A} \to \mathfrak{B}$ of C^*-algebras is completely positive if for all $k \in \mathbb{N}$, the linear map $\Phi_k : M_k(\mathfrak{A}) \to M_k(\mathfrak{B})$ defined by extending Φ entrywise to a matrix over $\mathfrak{A}$ is positive in the sense that positive matrices over $\mathfrak{A}$ is mapped to positive matrices over $\mathfrak{B}$.

Theorem 17.32. (Kirchberg [34]). *A separable C^*-algebra $\mathfrak{A}$ is exact if and only if every injective $*$-homomorphism $\Phi : \mathfrak{A} \to \mathbb{B}(H)$ can be approximated in the pointwise norm topology by a sequence of unital completely positive maps Φ_n, each of which factors through a matrix algebra $M_{k(n)}(\mathbb{C})$ via unital completely positive maps φ_n and ψ_n, as that*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\Phi} & \mathbb{B}(H) \\ \| & & \uparrow \psi_n \\ \mathfrak{A} & \xrightarrow{\varphi_n} & M_{k(n)}(\mathbb{C}) \end{array}$$

with $\Phi_n = \psi_n \circ \varphi_n$.

Corollary 17.33. *If G is a countable exact group, then there exists a sequence of completely positive maps $\Phi_n : C_r^*(G) \to \mathbb{B}(l^2(G))$ which converges pointwise in norm to the natural inclusion map from $C_r^*(G)$ into $\mathbb{B}(l^2(G))$ and which have the property that for every $n \in \mathbb{N}$, the operator valued function sending $g \mapsto \Phi_n(g)$ is supported on a finite subset of G.*

Proof. By the theorem above, there exists a sequence of unital completely positive maps Φ_n which converges pointwise in norm to the natural inclusion map from $C_r^*(G)$ into $\mathbb{B}(l^2(G))$, and each Φ_n factors through a matrix algebra $M_{k(n)}(\mathbb{C})$, as that

$$\Phi_n : C_r^*(G) \xrightarrow{\theta_n} M_{k(n)}(\mathbb{C}) \xrightarrow{\psi_n} \mathbb{B}(l^2(G)).$$

Now, a linear map $\theta : C_r^*(G) \to M_k(\mathbb{C})$ is completely positive if and only if the linear map $\theta' : M_k(C_r^*(G)) \to \mathbb{C}$ defined by

$$\theta'([f_{ij}]) = \frac{1}{k} \sum_{i,j=1}^{k} \theta(f_{ij})_{ij}$$

is a state.

Check that

$$\theta'([f_{ij}]+[g_{ij}]) = \frac{1}{k}\sum_{i,j=1}^{k}\theta(f_{ij}+g_{ij})_{ij}$$
$$= \frac{1}{k}\sum_{i,j=1}^{k}(\theta(f_{ij})_{ij}+\theta(g_{ij})_{ij}) = \theta'([f_{ij}])+\theta'([g_{ij}]).$$

Also

$$\theta'([f_{ij}]^*[f_{ij}]) = \frac{1}{k}\sum_{i,j=1}^{k}\theta(\sum_{l=1}^{k}f_{li}^*f_{lj})_{ij} = \frac{1}{k}\sum_{i,j=1}^{k}\sum_{l=1}^{k}\theta(f_{li}^*f_{lj})_{ij} \geq 0.$$

As well, if 1_k is the $k\times k$ unit matrix of $M_k(C_r^*(G))$, then

$$\theta'(1_k) = \frac{1}{k}\sum_{i=j=1}^{k}\theta(1)_{ii} = 1 = \|\theta'\|.$$

Moreover, the correspondence between completely positive maps θ and states θ' is a bijection.

In addition, if $h_1,\cdots,h_k$ are finitely supported functions on G which determines a unit vector in the k-fold direct sum $\oplus^k l^2(G)$, then the vector spate on $M_k(C_r^*(G))$ as

$$\theta'([f_{ij}]) = \sum_{i,j=1}^{k}\langle h_i, \lambda(f_{ij})h_j\rangle$$

corresponds to a completely positive map θ which is finitely supported as a function on G as in the statement.

But the convex hull of the vector states associated to a faithful representation of a C^*-algebra is always weak* dense in the set of all states, as a version of the Hahn Banach theorem.

It follows that the set of those completely positive maps θ from $C_r^*(G)$ into $M_k\mathbb{C})$ which are finitely supported as functions on G is dense in the set of all completely positive maps from $C_r^*(G)$ into $M_k\mathbb{C})$.

By approximating the maps θ_n in the compositions Φ_n we obtain completely positive maps from $C_r^*(G)$ into $\mathbb{B}(l^2(G))$ with the required properties. $\square$

Proof. (For the first theorem in this subsection). It follows from the corollary above that there exists a sequence of unital completely positive maps $\Phi_n : C_r^*(G) \to \mathbb{B}(l^2(G))$ which converges pointwise in norm to the inclusion map from $C_r^*(G)$ into $\mathbb{B}(l^2(G))$ and each Φ_n is finitely supported as a function on G. Define a sequence of functions $\varphi_n : G \times G \to \mathbb{C}$ by

$$\varphi_n(g_1, g_2) = \langle \chi_{g_1^{-1}}, \Phi_n(g_1^{-1}g_2)\chi_{g_2^{-1}} \rangle$$

for $g_1, g_2 \in G$ and $\chi_g \in l^2(G)$. The functions φ_n are positive definite kernels on G. To prove the inequality $\sum_{i,j=1}^k \overline{\lambda_i}\varphi_n(g_i, g_j)\lambda_j \geq 0$, write the sum as a matrix product

$$(g_1, \cdots, g_k) \begin{pmatrix} \overline{\lambda_1}\Phi_n(g_1^{-1}g_1)\lambda_1 & \cdots & \overline{\lambda_1}\Phi_n(g_1^{-1}g_k)\lambda_k \\ \vdots & \ddots & \vdots \\ \overline{\lambda_k}\Phi_n(g_k^{-1}g_1)\lambda_1 & \cdots & \overline{\lambda_k}\Phi_n(g_k^{-1}g_k)\lambda_k \end{pmatrix} \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix}$$

and apply the definition of complete positiveness.

Indeed,

$$\varphi_n(g, g) = \langle \chi_{g^{-1}}, \Phi_n(1)\chi_{g^{-1}} \rangle = \langle \chi_{g^{-1}}, \chi_{g^{-1}} \rangle = 1.$$

and

$$\overline{\varphi_n(g_1, g_2)} = \langle \Phi_n(g_1^{-1}g_2)\chi_{g_2^{-1}}, \chi_{g_1^{-1}} \rangle = \langle \chi_{g_2^{-1}}, \Phi_n(g_1^{-1}g_2)^*\chi_{g_1^{-1}} \rangle,$$

which is equal to $\varphi_n(g_2, g_1)$, with $\Phi_n(g_1^{-1}g_2)^* = \Phi_n(g_2^{-1}g_1)$ (if so). Also, with conjugate linearity in the second variable,

$$\sum_{i,j=1}^k \lambda_i\varphi_n(g_i, g_j)\overline{\lambda_j} = \sum_{i,j=1}^k \lambda_i\langle \chi_{g_i^{-1}}, \Phi_n(g_i^{-1}g_j)\chi_{g_j^{-1}} \rangle\overline{\lambda_j}$$

$$= \sum_{i,j=1}^k \langle \chi_{g_i^{-1}}, \overline{\lambda_i}\Phi_n(g_i^{-1}g_j)\lambda_j\chi_{g_j^{-1}} \rangle$$

$$= (g_1^{-1}, \cdots, g_k^{-1}) \star \begin{pmatrix} \overline{\lambda_1}\Phi_n(g_1^{-1}g_1)\lambda_1 & \cdots & \overline{\lambda_1}\Phi_n(g_1^{-1}g_k)\lambda_k \\ \vdots & \ddots & \vdots \\ \overline{\lambda_k}\Phi_n(g_k^{-1}g_1)\lambda_1 & \cdots & \overline{\lambda_k}\Phi_n(g_k^{-1}g_k)\lambda_k \end{pmatrix} \begin{pmatrix} g_1^{-1} \\ \vdots \\ g_k^{-1} \end{pmatrix}$$

where the operation $\star$ means the vector product via the inner product (corrected). Note as well that

$$\begin{pmatrix} \lambda_1 g_1 & \cdots & \lambda_k g_k \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}^* \begin{pmatrix} \lambda_1 g_1 & \cdots & \lambda_k g_k \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} \overline{\lambda_1}\lambda_1 g_1^{-1}g_1 & \cdots & \overline{\lambda_1}\lambda_k g_1^{-1}g_k \\ \vdots & \ddots & \vdots \\ \overline{\lambda_k}\lambda_1 g_k^{-1}g_1 & \cdots & \overline{\lambda_k}\lambda_k g_k^{-1}g_k \end{pmatrix}$$

is a positive $k \times k$ matrix, so that its image under the positive linear map $(\Phi_n)_k : M_k(C_r^*(G)) \to M_k(\mathbb{B}(l^2(G))$, which is equal to the $k \times k$ matrix in the (two sided) products with vectors, is positive.

The functions φ_n converge pointwise to 1. Indeed,

$$\lim_{n\to\infty} \varphi_n(g_1, g_2) = \langle \chi_{g_1^{-1}}, \lim_{n\to\infty} \Phi_n(g_1^{-1}g_2)\chi_{g_2^{-1}} \rangle = \langle g_1^{-1}, g_1^{-1}g_2g_2^{-1} \rangle = 1.$$

Moreover, for every finite subset F of G and every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that if $n > N$ and $g_1^{-1}g_2 \in F$, then $|\varphi_n(g_1, g_2) - 1| < \varepsilon$. In addition, for every $n \in \mathbb{N}$, there exists a finite subset F of G such that if $g_1^{-1}g_2 \notin F$, then $\varphi_n(g_1, g_2) = 0$.

It then follows that for a suitable subsequence, the series $\sum_j (1 - \varphi_{n_j})$ is pointwise convergent everywhere on $G \times G$. (Possibly, φ_{n_j} should be replaced with its real part). But each funciton $1 - \varphi_{n_j}$ is a negative type kernel, and therefore so is the series.

Indeed, consider the following composition:

$$C_r^*(G) \xrightarrow{\Phi_n} \mathbb{B}(l^2(G)) \xrightarrow{\langle g_1^{-1}, (*)g_2^{-1} \rangle} \mathbb{C},$$

which is a positive linear functional on $C_r^*(G)$, so that the composition is positive, and hence

$$\|\langle g_1^{-1}, \Phi_n(\cdot)g_2^{-1} \rangle\| = |\langle g_1^{-1}, \Phi_n(1)g_2^{-1} \rangle| \le \|\chi_{g_1^{-1}}\| \|\chi_{g_2^{-1}}\| = 1.$$

Therefore, with $\|\chi_{g_1^{-1}g_2}\| = 1$ we obtain

$$|\varphi_n(g_1, g_2)| = |\langle \chi_{g_1^{-1}}, \Phi_n(g_1^{-1}g_2)\chi_{g_2^{-1}} \rangle| \le \|\langle g_1^{-1}, \Phi_n(\cdot)g_2^{-1} \rangle\| \le 1.$$

Note as well that being zero on the diagonal of $G \times G$, having complex conjugate symmetry, and being negative type for $1 - \varphi_{n_j}$ positive are preserved under taking limits for their series.

The map into an affine Euclidean space, which is associated to the series, is a uniform embedding. □

Remark. The above argument shows that if a countable group G is exact, then G acts amenably on its Stone-Čech compactification βG. As a result, if a countable group G is exact, then the Baum-Connes assembly map

$$\mu_\lambda : K(G, \mathfrak{D}) \to E_0(\mathfrak{D} \rtimes_r G)$$

is injective, for every G-C^*-algebra $\mathfrak{D}$.

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