A locally commentative and transcriptive exercising in the quantum algebraic, analytic, and geometric garden stadium

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A locally commentative and transcriptive exercising in the quantum algebraic, analytic, and geometric garden stadium

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Abstract

As a back to the past for a return to the future, we review and study by running round, round, and through it, namely the non-commutative geometry (NCG) garden, explored and maintained by Connes-Marcolli.

Dedicated to Professor Muneo Cho on his 68th birthday with gratitude and respect (in advance) ♣

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Keywords: Noncommutative geometry, C*-algebra, von Neumann algebra, K-theory, quantum statistical mechanics, quantum field theory, standard model, dynamical system, algebraic geometry, algebraic topology, differential geometry, index theory, number theory.

1 Introduction as original preface

This is nothing but a running commentary (or a locally commentative and transcriptive exercising (LCTE)) based on Connes-Marcolli [83] primarily, as well as Connes [66] and more items as in the references secondly (but mostly in part). Almost along the story of [83], with some considerable effort within time limited for publication, we do make some additional, proofs, tables, examples, corrections, and (extended) remarks or comments, as inserted as: (Added). The texts ◄ Some notations as well as some texts are slightly changed from the original ones, by our taste. Several (score) tables are worked out, 19 in total. With somewhat time and patience for checking out (or not yet) all the 213 items in the references of [83], even with several items not found in the texts assigned, some items are either collected, slightly corrected, or updated, and some of

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which are neither at hand nor accessible (without fee, at this moment), but included in the references at the end, even with several exceptional items, and added with some related items. Fed in by hands 242 items in total. However, almost the details are only touched or not yet checked well. As a reference at an (early) time of this reviewing, may read Village-Mountain (translated from Kan-ji as Chinese characters) [191] in Japanese (and more items such as [114], [134], [135], [230], [231], [241]) for the exciting story of the theory of elementary particles and its experimental observations. Also refer to [183]. This paper is organized of the following (the same number of rounds) 26 sections with titles a bit changed from original ones. The quantum (deformed) stadium is now open!

Table 1: Contents

There are several books on NCG such as [64] (missing, but its translation in Japanese by Circle-Mountain [181]), [66], [120], [148], [159], [169] (missing), and [176]. May refer to these items.

(Added). In addition, [84], and [95] (missing). ◀

2 Handling quantum algebras as noncommutative (NC) spaces

A quantum (smooth (C^∞) or differentiable (or C^r class for $r \in \mathbb{N} = \{1, 2, \cdots\}$) algebra $\mathcal{A}$ or (continuous) C^*-algebra $\mathfrak{A}$ is defined to be an algebra of (smooth, differentiable, or continuous) function elements as coordinates of a noncommutative (or badly or slightly deformed) space X (such as non-Hausdorff spaces obtained from locally compact Hausdorff spaces). We would like to understand such a noncommutative space X by quantum algebraic datam. There are 4 steps:

[S1]: Resolve quantum algebras and compute cyclic cohomology.

[S2]: Find geometric models of noncommutative spaces up to homotopy.

[S3]: Construct spectral geometry for quantum algebras.

[S4]: Compute time evolution and analyze thermo-dynamics.

Step 1. It means finding a resolution of $\mathcal{A}$ as a $\mathcal{A}$-bimodule by projective $\mathcal{A}$-bimodules making it possible to compute effectively the Hochschild homology of $\mathcal{A}$. In general, such resolutions are of type Koszul. A typical example is the resolution (of the diagonal) for the algebra $C^\infty(X)$ of smooth functions on a compact manifold X as in the C^∞-version [61] of the Hochschild, Kostant, Rosenberg theorem [136]. It then follows to know what is the analogue of differential forms and of de Rham currents on the space X, to compute the cyclic homology and cyclic cohomology for $\mathcal{A}$, which are viewed as natural replacements of the de Rham theory for X. For foliation algebras, this is done some time ago (cf. [62], [37], [102]), and is tied in with the natural double complex of transverse currents.

But it is not always easy to perform this step of finding such a resolution and computing Hochschild and cyclic homology or cohomology. For instance, in the case of algebras given by generators and relations this step uses the whole theory of Koszul duality, which is extented to homogeneous algebras (cf. [109] missing, [110], [111] missing, [112], [113], [23]).

As a specific example, it is interesting to resolve the modular Hecke algebras. In essence, finding a resolution in the algebra of modular forms of arbitrary level, equivariant with respect to the action of the group of finite adèles, would yield formulas for the compatibility of Hecke operators with the algebraic structure. This is a basic and hard problem in the theory of modular forms.

Cyclic homology and cohomology theory is well developed, and is first designed to handle the leaf spaces of foliations as well as group rings of discrete groups (cf. [148]). The theory plays a central role as a purely algebraic version of (analytic) noncommutative geometry, and is also crucial in the analytic set-up to construct cyclic cocycles with compatibility with the topology of (quantum) algebras. For instance, if the definition domain for cocycles is a dense subalgebra of a underlying C^*-algebra, stable under holomorphic functional calculus, then it automatically gives an invariant as the K-theory for C^*-algebras (cf. [62]).

Step 2. A lot of noncommutative spaces defined as quotients with bad (or less) topology can be de-singularized, provided that one is ready to work up to homotopy. For instance, if the space X is defined as a quotient $X = Y/\sim$ of an ordinary space Y by an equivalence relation $\sim$, then one can often view X as another quotient $X = Z/\sim$, where the equivalence classes are contractible spaces. The homotopy type of Z is then uniquely determined and serves as a substitute for that of X. See Baum-Connes [16].

For instance, if the equivalence relation on Y comes from a free action of a torsion free, discrete group Γ, then the space Z is given by a product over Γ of the from

$$Z = Y \times_\Gamma E\Gamma = (Y \times E\Gamma)/\Gamma,$$

where $E\Gamma$ is a contractible space (or the universal principal Γ-space) on which Γ acts freely and properly and the equivalence relation by Γ is given as $(g \cdot y, u) = (y, g \cdot u)$ (cf. [17]).

The main point of this second step is that it gives a starting point for computing the K-theory of the space X as well as the K-theory of the C^*-algebra $\mathfrak{A} = \overline{\mathcal{A}}$ as the norm closure of a dense algebra $\mathcal{A}$, playing the role of some algebra of continuous functions on X.

Indeed, for each element of the K-homology of the classifying space Z, there is a general construction of an index problem for families parameterized by X that yields the assembly map ([16])

$$\mu : K_*(Z) \to K_*(\mathfrak{A}).$$

This Baum-Connes assembly map is an isomorphism in a somewhat large number of cases, with a suitable care without torsion. They include all connected locally compact groups, all amenable groupoids, and all hyperbolic discrete groups ([17]). It shows that the K-theory for X is computable in the sense that the topological K-theory for a geometric space X can be computed by the topological K-theory for an analytic C^*-algebra $\mathfrak{A}$, which is noncommutative in general.

Step 3. The third step is not only to compute $K_*(\mathfrak{A})$ but also to get a good model for the vector bundles over X, i.e., the finite projective modules over $\mathfrak{A}$. This step should be combined with the above first step to compute the Chern character using connections, curvature, and eventually computing moduli spaces of Yang-Mills connections, as done for the noncommutative (NC) 2-torus by Connes-Rieffel [96].

We then pass from the soft part of differential geometry to the harder Riemannian metric aspect. The sought for spectral geometry as a triple $(\mathcal{A}, H, D)$

(of an algebra $\mathcal{A}$ and an operator D both acting on a Hilbert space H with some suitable conditions) has three essential features:

(F1): The K-homology class of $(\mathcal{A}, H, D)$.

(F2): The smooth structure. (F3): The metric.

One should always look for a spectral triple such that its K-homology class is as non-trivial as possible. Ideally, it should extend to a classs for the doubled algebra $\mathcal{A} \otimes \mathcal{A}^{op}$ with $\mathcal{A}^{op}$ the opposite algebra of $\mathcal{A}$ with products exchanged, and then be a generator for Poincaré duality. In general, this is too much to ask for, since spaces do not always fulfill Poincaré duality.

The main tool for determining the stable homotopy class of a spectral triple is the bivariant KK-theory for C^*-algebras by Kasparov. Thus, it is important to already have taken the step 2 and to look for classes such that their pairing with K-theory is as non-trivial as can be.

For the smooth structure, there is often a natural candidate for a dense smooth subalgebra $\mathcal{A}^\infty$ of a C^*-algebra $\mathfrak{A} = \overline{\mathcal{A}}$ for some dense subalgebra $\mathcal{A}$, that plays the role of the algebra of smooth functions on a space. It should in general contain the original algebra $\mathcal{A}$ and should have the further property that it is stable under the holomorphic functional calculus. This ensures that inclusions $\mathcal{A}^\infty \subset \mathcal{A} \subset \mathfrak{A}$ extends to isomorphisms in the respective K-theory groups and makes it possible to complete a classification of smooth vector bundles over a space.

The role of a (unbounded, differential) operator D for the smooth structure is that it defines a geodesic flow by the formula

$$f_t(a) = e^{it|D|}ae^{-it|D|} \equiv \mathrm{Ad}(\exp(it|D|))a, \quad a \in \mathcal{A}^\infty, t \in \mathbb{R}, i = \sqrt{-1},$$

and it is expected that smoothness of the spectral triple is governed by the smoothness of the operator-valued function $f_t(a)$ on $\mathbb{R}$. The main result in the general theory on it is the local index formula of Connes and Moscovici [89], which provides the analogue of the Pontrjagin classes of smooth manifolds in the noncommutative framework.

The problem of determining the operator D from the knowledge of the K-homology class corresponding to a spectral triple is similar to the choice of a connection on a bundle. There are general results that assert the existence of an unbounded self-adjoint operator D on H with bounded commutators with $\mathcal{A}$ from estimates on the commutators with the phase operator F. The strongest is obtained by Connes as in [66, p. 391] just assuming that the (additive) commutators $[F, a] = Fa - aF$ are in an ideal denoted as $\mathrm{Li}(H)$, and it ensures the existence of a θ-summable spectral triple which is what one needs to get started.

It is not always possible to find a finitely summable, spectral triple, first because of growth conditions on an algebra (as in Connes [63]), but also since the finitely summable condition is analogous to being type II in the theory of factors. In a general case, like the noncommutative spaces coming from foliations, one can go from being type III to type II by passing to the total space of the space of transverse metrics and then using the theory of hypo-elliptic operators (CnMs [90]).

Another way to attack the problem of determining the operator D is to consider the algebra generated by $\mathcal{A}$ and D with some relations between $\mathcal{A}$ and D and then look for irreducible representations that fall in the stable homotopy class. Ideally, one should minimize the spectral action functional in this homotopy class, thus coming close to gravity (Chamseddine-Connes [48]). In practice, one should use anything available, but the noncommutative space given by the quantum group $SU_q(2)$ shows that things can be quite subtle ([105]).

Given or once determined a spectral triple $(\mathcal{A}, H, D)$, as the basic steps, one should compute the following:

(C1): The dimension spectrum in $\mathbb{C}$. (C2): The local index formula.

(C3): The inner fluctuations, scalar curvature, and spectral action.

Step 4. It is often that a noncommutative space comes with a measure class, which in turn determines a time evolution $\{\sigma_t\} \subset \mathrm{Aut}(\mathfrak{A})$, that is, a one-parameter family of automorphisms of a C^*-algebra $\mathfrak{A} = \overline{\mathcal{A}}$. In the type II case, one can apply the step 3 above, as well as in the finite dimensional case, use the operator D to represent functionals in the measure class of the form

$$\varphi(a) = \oint a|D|^{-p}, \quad a \in \mathcal{A},$$

where $\oint$ denotes the Dixmier trace as the noncommutative integral and p is the dimension. In the type III (always or often in Physics) or general case, such a time evolution is highly non-trivial.

Given such a data or a C^*-dynamical system $(\mathfrak{A}, \sigma_t, \mathbb{R})$, it is natural to regard it as a quantum statistical mechanical system, with a C^*-algebra $\mathfrak{A}$ as an algebra of observables and the $\mathbb{R}$-action σ as time evolution. One can then look for equilibrium states for the system and for given values β of the thermodynamic parameter as inverse temperature.

The algebra $\overline{\mathcal{A}}$ may not be a C^*-algebra. If the algebra $\overline{\mathcal{A}}$ is concretely realized as a C^*-algebra of bounded operators on a Hilbert space H, then one can consider the Hamiltonian H, that is, a (unbounded) operator on H that is the infinitesimal generator of the time evolution. If the operator $\exp(-\beta H)$ is of trace class, then one has equilibrium states for the system $(\overline{\mathcal{A}}, \sigma_t)$, written in the usual Gibbs form

$$\varphi_\beta(a) = \frac{\mathrm{tr}(a \exp(-\beta H))}{\mathrm{tr}(\exp(-\beta H))}, \quad \text{where } z(\beta) = \mathrm{tr}(\exp(-\beta H))$$

is the partition function of the system. The notion of equilibrium states does make sense even if $\exp(-\beta H)$ is not necessarily of trace class, and is given by the more subtle notion of KMS(Kubo-Martin-Schwinger) states.

The KMS_β states on $\overline{\mathcal{A}}$ are positive continuous functionals $\varphi : \overline{\mathcal{A}} \to \mathbb{C}$, so that $\varphi(a^*a) \geq 0$, with $\varphi(1) = 1$, namely states, satisfying the KMS_β condition such that for any $a, b \in \overline{\mathcal{A}}$, there exists a function $f_{a,b}(z)$ which is holomorphic on the open strip $0 < \mathrm{Im}(z) < \beta$, continuous and bounded on the closed strip $0 \leq \mathrm{Im}(z) \leq \beta$, namely $f_{a,b} \in H(\mathbb{R} \times (0,\beta))$, and

$$f_{a,b}(t) = \varphi(a\sigma_t(b)) \quad \text{and} \quad f_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a), \quad t \in \mathbb{R}.$$

The KMS states at zero temperature may be defined as the weak limits as $\beta \to \infty$ of KMS$_\beta$ states. Using KMS states one can construct refined invariants of noncommutative spaces. For a fixed β, the KMS$_\beta$ states form a simplex, and hence one can consider only the set $\mathcal{E}_\beta$ of extremal KMS$_\beta$ states, from which one can recover all the others by convex combinations.

Proof. (Added). Suppose that $\varphi_1, \cdots, \varphi_n \in \mathcal{E}_\beta$ and $k_1, \cdots k_n \geq 0$ with $\sum_{j=1}^n k_j = 1$. Then $\sum_{j=1}^n k_j \varphi_j$ is a state, and there are $f_{a,b,j} \in H(\mathbb{R} \times (0,\beta))$, so that $\sum_{j=1}^n k_j f_{a,b,j} \in H(\mathbb{R} \times (0,\beta))$ and such that

$$\sum_{j=1}^n k_j f_{a,b,j}(t) = \sum_{j=1}^n k_j \varphi_j(a\sigma_t(b)) \quad \text{and}$$
$$\sum_{j=1}^n k_j f_{a,b,j}(t + i\beta) = \sum_{j=1}^n k_j \varphi_j(\sigma_t(b)a), \quad t \in \mathbb{R}.$$

□

An extremal KMS$_\beta$ state is always factorial, and the type of the factor is an invariant of the state. The simplest situation is of type I. It can be shown under the minimal hypotheses ([73]) that extremal KMS$_\beta$ states continue to survive when the temperature becomes lower, i.e., β increases. Thus, in essence, when cooling down the system, it tends to become more classical, and he 0-temperature limit of $\mathcal{E}_\beta$ gives a good replacement of the notion of classical points for a noncommutative space. In examples related to arithmetic, we see how the classical points described by the zero temperature limit of KMS states of certain quantum statistical mechanical systems recover classical arithmetic varieties. The extremal KMS states at zero temperature, evaluated on suitable arithmetic elements in the noncommutative algebra, in significant cases, can be shown to have an interesting Galois action, related to interesting questions in number theory ([29], [86], and [81] missing).

It is shown in [73] how to define an analogue in characteristic zero, of the action of the Frobenius on the étale cohomology by a process involving the above thermo-dynamics. One key feature is that the analogue of the Frobenius is given by the dual of the above time evolution σ_t. The process involves cyclic homology, and its three basic steps are

(1) Cooling. (2) Distillation.

(3) Dual action of $\mathbb{R}_+^*$ on the cyclic homology of the distilled space.

When applied to the simplest system as the Bost-Connes system of [29], this yields a cohomological interpretation of the spectral realization of the zeros of the Riemann zeta function ([70], [73]).

3 Phase spaces in microscopic systems

What can be historically regarded as the first example of a noncommutative space is the Heisenberg formulation of the observational Ritz-Rydberg law of spectroscopy. In fact, it is shown by quantum mechanics that indeed the parameter space as the phase space of the mechanical system given by a single atom fails to be a manifold. It is important to convince oneself of this fact and to understand that this conclusion is indeed dictated by the **experimental** findings of spectroscopy.

At the beginning of the twentieth century, a wealth of experimental data is collected, on the spectra of various chemical elements. These spectra obey experimentally discovered laws, like the most notable as the Ritz-Rydberg combination principle. The RR principle can be stated as follows.

Spectral lines are indexed by pairs of labels. Then certain pairs of spectral lines, when expressed in terms of frequencies, do add up to give another line in the spectrum. Moreover, this happens precisely when the two labels are of the form i, j and of j, k.

In the seminal paper [131], W. Heisenberg considers the classical prediction for the radiation emitted by a moving electron in a field, where the observable dipole moment can be computed, with the motion of the electron given in Fourier expansion. The classical model predicts frequencies distributed according to the law:

$$\nu(n,\alpha) = \alpha\nu(n) = \alpha\frac{1}{h}\frac{dW}{dn}.$$

When comparing the frequencies obtained in this classical model with the data, it is noticed that the classical law does not match the phenomenon observed.

The spectral rays provide a picture of an atom as follows. If atoms are in classical systems, then the picture formed by the spectral lines becomes a group in our modern mathematical language, which is the law above predicts. That is what the classical model predicts that the observed frequencies should simply add, obeying a group law, or in Heisenberg notation that

$$\nu(n,\alpha) + \nu(n,\beta) = \nu(n,\alpha+\beta).$$

Correspondingly, the observables form the convolution algebra of a group.

What the spectral lines are instead providing is the picture of a groupoid. It is realized by Heisenberg [131] that the classical laws above would have to be replaced with the quantum mechanical laws:

$$\begin{aligned} &\nu(n, n-\alpha) = \frac{1}{h}(W(n) - W(n-\alpha)) \quad \text{and} \\ &\nu(n, n-\alpha) + \nu(n-\alpha, n-\alpha-\beta) = \nu(n, n-\alpha-\beta). \end{aligned}$$

These replace the group law with the groupoid law. Similarly, the classical Fourier modes $f_\alpha(n)e^{iw(n)\alpha t}$ are replaced with $f(n, n-\alpha)e^{iw(n,n-\alpha)t}$.

The analysis of the emission spectrum given by Heisenberg is in a good agreement with the Ritz-Rydberg law, or combination principle, for spectral lines in emission or absorption spectra.

Heisenberg [131] also extends the re-definition of the multiplication law for the Fourier coefficients to coordinates and momenta, by introducing transition amplitudes that satisfy similar product rules. This is the most audacious step in Born words, and that brings noncommutative geometry on the scene.

It is Born who realizes that what Heisenberg described in his paper correspond to replacing classical coordinates with another quantum coordinates which no longer commute, but which obey the laws as the matrix multiplication.

The words reported by B. L van der Waerden ([234] missing) are omitted except citing as:

Heisenberg's symbolic multiplication is nothing but the matrix calcucus.

Thus, spectral lines are parameterized by two indices $l_{\alpha\beta}$ satisfying a co-cycle relation:

$$l_{\alpha\beta} + l_{\beta\gamma} = l_{\alpha\gamma},$$

and a co-boundary relation expresses each line as a difference:

$$l_{\alpha\beta} = \nu_\alpha - \nu_\beta.$$

In other words, the RR law gives the groupoid law above, or equivalently, $(i,j)\cdot(j,k) = (i,k)$, and the convolution algebra of a group is replaced by observables satisfying the matrix product:

$$(ab)_{ik} = \sum_j a_{ij} b_{jk},$$

In general, commutativity is lost as $ab \neq ba$.

The Hamiltonian H is a matrix with the frequencies on the diagonal, and observables obey the evolution equation:

$$\frac{d}{dt}a = i[H, a].$$

Out of the Heisenberg paper and of the Born interpretation in terms of matrix calculus, is emerged the statement of the Heisenberg uncertainty principle in the form of a commutation relation of matrices:

$$[p, q] = \frac{h}{2\pi i}1.$$

The matrix calculus and the uncertainty principle are formulated by Born and Jordan in a subsequent paper, published in [27]. This viewpoint on quantum mechanics is later somewhat obscured by the advent of the Schrödinger equation. The Schrödinger approach shifts the emphasis back to the more traditional technique of solving partial differential equations, while the more modern viewpoint of Heisenberg implies a much more serious change of paradigm, affecting our most basic understanding of the notion of spaces. The Heisenberg appproach can be regarded as the historic origin of noncommutative stadium.

Remark. (Added). If necessary, one may recall the following basic facts in quantum mechanics from a Japanese exercising book [119].

Let ν denotes a frequency of a photon. Its energy is given as $h\nu$ and its momentum is $\frac{h\nu}{c} = \frac{h}{\lambda}$, where h denotes the Planck constant and c is the light speed, and λ is its wavelength. Hence $\frac{1}{\lambda} = \frac{\nu}{c}$ may be another frequency.

Let $\lambda[m]$ denotes a wavelength of a light or photon of an atom such as hydrogens. The spectral sequences are given as

$$\frac{1}{\lambda} = R_H \left(\frac{1}{n_i^2} - \frac{1}{n_f^2} \right), \quad n_f = n_i + k, k \in \mathbb{N},$$

where R_H is the Rydberg constant. It is called that the sequence is the Lyman sequence for $n_i = 1$, the Balmer for $n_i = 2$, the Paschen for $n_i = 3$, the Bracket for for $n_i = 4$, and the Pfund for $n_i = 5$.

For general atoms, the spectral rays are approximately obtained as

$$\frac{1}{\lambda} = R\left(\frac{1}{(n_i+\delta_i)^2} - \frac{1}{(n_f+\delta_f)^2} \right) = \frac{R}{(n_i+\delta_i)^2} - \frac{R}{(n_f+\delta_f)^2} \equiv T_i - T_f,$$

where each term T_j is said to be a spectral term. That is, to say that each spectral ray is represented as the difference or combination of two spectral terms or numbers, used by Ritz (1908), known as the Ritz combination law.

In the classical model, frequencies can take integer multiples of the basic frequency. While in the quantum model, frequencies can take combination values under the law above.

Now set $\nu_{nm} = c(T_n - T_m)$. Then it holds that the relation:

$$\nu_{nl} = \nu_{nm} + \nu_{ml}.$$

In the classical model, any addition between two frequencies is taken as another frequency, but in the quantum model, such a special addition only is allowed.

On the other hand, if we set $W_j = T_j ch$, then

$$\nu = \frac{1}{h}(W_i - W_f) \Leftrightarrow h\nu = W_i - W_f.$$

Hence, each W_j is an energy as a a stationary state.

An electron of a hydrogen atom has as angular momentum L among the set $\{n\frac{h}{2\pi} \mid n \in \mathbb{N}\}$. Each integer n is said to be a principal quantum number, and it corresponds to such an energy W_n, called as energy level. ◀

4 Noncommutative spaces as quotients

A large source of examples of noncommutative spaces is given by quotients of equivalence relations. Let X be an ordinary space such as a smooth manifold or a locally compact Hausdorff space. If X is compact, then this space can be described via $C(X)$ the algebra of continuous, complex-valued functions on X, that is a unital abelian C^*-algebra. If X is non-compact, this space can be done via $C_0(X)$ the algebra of continuous, complex-valued functions on X vanishing at infinity, that is a non-unital abelian C^*-algebra.

Suppose then that we are interested in taking a quotient space $Y = X/\sim$ of X with respect to an equivalence relation. In general, it is expected that the quotient may have a less topology with respect to separating of points. Even when X is a smooth compact manifold, the quotient Y may not even be a Hausdorff space. In such a case, one may consider to characterize the space Y through its ring of functions, defined as follows. That consists of functions in $C(X)$ invariant under the equivalence relation, so that the functions are constant on each equivalence class or an orbit of a point. If each of orbits are dense in X, in other words, an equivalence relation is (topologically) minimal, then the algebra of such orbit-wise constant, continuous functions on X becomes $\mathbb{C}$ of constant functions, the trivial C^*-algebra.

There is another better way to associate to the quotient space Y a ring of functions which is nontrivial for any equivalence relation. This requires dropping commutativity of the algebra. Consider functions f_{ab} of two variables a and b defined on the graph of an equivalence relation, with a product which is no longer the commutative point-wise product, but the noncommutative convolution product, dictated by the groupoid of the equivalence relation. In general, the elements of the algebra of functions (f_{ab}) with $a \sim b$ act as bounded operators on the Hilbert space, i.e. the L^2-space on the equivalence classes. It also guarantees the convergence in the operator norm of the convolution product as

$$((f_{ab}) * (g_{ab}))_{ac} = \sum_{b: a \sim b \sim c} f_{ab} g_{bc}.$$

Given below are a few examples to illustrate the difference between the classical construction and that in noncommutative geometry.

Example 4.1. Let $X = \{x_0; x_1\}$ be a set of two points. Let Y be the quotient space of X with the equivalence relations $x_0 \sim x_1$ and $x_j \sim x_j$ $(j = 0, 1)$. The algebra of continuous functions on X invariant under the relation becomes $\mathbb{C}$. While the graph of the relation is just the product space $X \times X$. Hence functions on $X \times X$ may be regarded as 2×2 matrices as:

$$f = \begin{pmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{pmatrix},$$

and their products are given by the matrix multiplication as

$$f * g = \begin{pmatrix} f_{00}g_{00} + f_{01}g_{10} & f_{00}g_{01} + f_{01}g_{11} \\ f_{10}g_{00} + f_{11}g_{10} & f_{10}g_{01} + f_{11}g_{11} \end{pmatrix}$$

Therefore, the algebra of these functions is the 2×2 matrix algebra over $\mathbb{C}$, $M_2(\mathbb{C})$, the simple unital non-trivial C^*-algebra acting on the complex Euclidean space $\mathbb{C}^2$. Note that $\mathbb{C}$ and $M_2(\mathbb{C})$ are not isomorphic as an algebra or a C^*-algebra, but they are Morita equivalent as an algebra. Note as well that the spectrums of $\mathbb{C}$ and $M_2(\mathbb{C})$ are composed of only one point. This example represents the typical situation where the quotient is nice in the sense that two constructions give Morita equivalent algebras. In this sense, Morita equivalent algebras are regarded as the same (commutative or noncommutative) spaces in noncommutative geometry. ◀

Example 4.2. Let $X = [0,1] \times \{0,1\}$ and let $Y = X/\sim$ with the equivalence relation $(x,0) \sim (x,1)$ for $x \in (0,1)$. Then the algebra of continuous, $\mathbb{C}$-valued functions on X invariant under the relation becomes $C([0,1])$ (not $\mathbb{C}$, corrected) the C^*-algebra of continuous, complex-valued functions on $[0,1]$. On the other hand, the graph of the relation becomes $(X \times X)/\sim$, which consists of both the dense subset

$$[((0,1) \times \{0,1\}) \times ((0,1) \times \{0,1\})]/\sim \cong (0,1) \times \{(0,0),(0,1),(1,0),(1,1)\}$$

and the 4 points

$$\{(0,0)\} \times \{(0,0)\}, \{(0,1)\} \times \{(0,1)\}, \{(1,0)\} \times \{(1,0)\} \quad \text{and} \quad \{(1,1)\} \times \{(1,1)\}.$$

Then the corresponding algebra of continuous, 2×2 matrix valued functions on $[0,1]$, implied by continuity for that of the restrictions on $(0,1)$, becomes the C^*-subalgebra of the C^*-algebra $C([0,1], M_2(\mathbb{C})) \cong C([0,1]) \otimes M_2(\mathbb{C})$ of continuous, $M_2(\mathbb{C})$-valued functions f on $[0,1]$, with $f(0)$ and $f(1)$ diagonal. Such an algebra is so called as a dimension drop algebra. We may say it as a C^*-algebra of matrix-valuned, continuous functions on the interval, partially vanishing and non-vanishing to the diagonals at infinity.

In this case, these C^*-algebras are not Morita equivalent. This can be seen by computing their K-theory groups (appeared). It means that the approach of noncommutative spaces can produce something genuinely new when the quotient space ceases to be nice at the boundary.

In general, the first kind of the construction of functions on the quotient space is cohomological in nature that if one seeks for functions satisfying certain equations, but then there are very few solutions. Instead, the second approach in noncommutative geometry can typically produce a large class of functions on noncommutative spaces. ◀

5 Spaces of leaves of foliations

There is a rich collection of examples of noncommutative spaces given by the leaf spaces of foliations. The conection between noncommutative geometry and the geometric theory of foliations is far-reaching obtained, for instance, through the role of Gelfand-Fuchs cohomology, of the Godbillon-Vey invariant, and of the passage from type III to type II using the transverse frame bundle. It is the class of such examples that triggered the initial development of cyclic cohomology (cf. [148]), of the local index formula in noncommutative geometry, as well as the theory of characteristic classes for Hopf algebra actions.

The construction of the algebra associated to a foliation is a special case of the construction in the previous section, but with the presence of holonomy and the special care for the case where the graph of a foliation is non-Hausdorff. Recall the basic steps below.

Remark. (Added). Before doing so, recall some notions in manifolds from [183].

For f a smooth function on a smooth manifold M, the **differential** of f at $p \in M$ is the linear map $df_p : T_p(M) \to \mathbb{R}$ defined by $df_p(X) = X(f)$, which is identified with an element of $T_p(M)^*$.

Let $\varphi : M \to M'$ be a continuous map between smooth manifolds. If $f \circ \varphi$ for any $f \in C^\infty(M', \mathbb{R})$ is smooth, then φ is said to be smooth.

The **differential** of such a smooth map φ at $p \in M$ is the linear map $d\varphi_p : T_p(M) \to T_{\varphi(p)}(M')$ defined by $d\varphi_p(X) = X'$ with $X'(g) = X(g \circ \varphi)$ for $g \in C^\infty(M', \mathbb{R})$.

If $d\varphi_p$ is surjective, the point p is said to be a regular point and $\varphi(p)$ is to be a regular value. If not, p is to be a critical point and $\varphi(p)$ to be a critical value.

If $d\varphi_p$ at every $p \in M$ is injective, $\varphi : M \to M'$ is said to be an **immersion.** If so, for any $p \in M$, there is a neighbourhood U_p of p such that the restriction of φ to U_p is a homeomorphism. An immersion is said to be an **embedding** if φ is also injective.

A **submanifold** M of a smooth manifold M' is a smooth manifold $M \subset M'$ such that the inclusion map is an immersion. Such a submanifold is regular if the inclusion map is an embedding.

A smooth map between smooth manifolds without critical points is said to be a **submersion.** If $\varphi : M \to M'$ is a submersion, then for any $q \in M'$, $\varphi^{-1}(q)$ is a regular submanifold of M, and M is covered by a family of mutually disjoint submanifolds, as $M = \sqcup_{q \in M'} \varphi^{-1}(q)$.

Let $\varphi : M \to M'$ be a smooth map and N' be a submanifold of M'. Then $T_q(N')$ is a subspace of $T_q(M')$ for $q \in N'$ and thus there is a quotient map $\pi_q : T_q(M') \to T_q(M')/T_q(N')$. The map φ is **transverse** to N' if $\pi_{\varphi(p)} \circ d\varphi_p : T_p(M) \to T_q(M')/T_q(N')$ is surjective. If so, $\varphi^{-1}(N')$ is a submanifold of M.

The **transversality** theorem states that for any smooth map $\varphi : M \to M'$ and any submanifold N' of M', there is a smooth map $\psi : M \to M'$ transverse to N', which is arbitrary approximately near to φ.

If M, N are submanifolds of M', then M and N **intersect transversely** if the inclusion map $M \subset M'$ is transverse to N.

A vector field X on M (or its subset) is defined to be a function sending each point $p \in M$ to a tangent vector $X_p \in T_pM$. It is also defined to be a section of the tangent bundle TM over M. For any $f \in C^\infty(M, \mathbb{R})$ and $X : M \to TM$ as a section, the function Xf on M is defined by $(Xf)(p) = X_pf$ for $p \in M$. If each function Xf is smooth (or C^r class), then X is said to be a smooth (or C^r class) vector field on M. ◀

Let M be a smooth manifold and TM its tangent bundle over M, so that for each $x \in M$, T_xM is the tangent space of M at x. A smooth subbundle F of TM is said to be **integrable** it one of the following equivalent conditions is satisfied (cf. [66, I 4.β]):

(a) Any $x \in M$ is contained in a submanifold $W(x)$ (a leaf) of M such that $T_y(W(x)) = F_y$ for $y \in W(x)$.

(b) Any $x \in M$ is in the domain $U \subset M$ of a submersion $p : U \longrightarrow \mathbb{R}^q$ with $q = \mathrm{codim}(F)$ and with $F_y = \ker(p_*)_y$ for $y \in U$.

(c) $C^\infty(F) = \{f \in C^\infty(TM) \mid f_x \in F_x, x \in M\}$ is a Lie algebra. (In other words, if $f, g \in C^\infty(F)$, then $[f, g] \in C^\infty(F)$: **(Frobenius)**.)

(d) The ideal $J(F)$ of smooth exterior differential forms which vanish on F is stable under exterior differentiation.

Any 1-dimensional subbundle F of TM is integrable. But if $\dim F \geq 2$, then the condition is non-trivial. For instance, if $p : P \longrightarrow B$ is a principal H-bundle over B with H a compact structure group, then the bundle of horizontal vectors for a given connection is integrable if and only if the connection is flat.

A foliation of M is given by an integrable subbundle F of TM. The leaves of the foliation (M, F) are the maximal connected submanifolds L of M with $T_x(L) = F_x$ for $x \in L$. The partition of M into leaves as $M = \cup_{\alpha \in X} L_\alpha$ is characterized geometrically by its local triviality in the sense that every point $x \in M$ has a neighbourhood U and a system of local coordinates $(x^j)_{j=1,\cdots,\dim M}$ called foliation charts, so that the partition of U in connected components of leaves corresponds to the partition of

$$\mathbb{R}^{\dim M} = \mathbb{R}^{\dim F} \times \mathbb{R}^{\mathrm{codim(F)}}$$

into the parallel affine subspaces of the form $\mathbb{R}^{\dim F} \times \{\text{point}\}$. These are the leaves of the restriction of F, called plaques.

The set $\mathcal{F} = M/F$ of leaves of a foliation (M, F) is in most cases a noncommutative space (to be corresponded). In other words, even though as a set it has the cardinality of the continuum, it is in general not so at the effective level and it is in general impossible to construct a countable set of measurable functions on M that form a complete set of invariants for the equivalence relation coming from the partition of M into leaves as $M = \cup_{L_\alpha \in \mathcal{F}} L_\alpha$. Even in the simple cases in which the set $\mathcal{F} = M/F$ of leaves is classical, it helps to introduce associated algebraic tools in order to get a feeling for their role in the singular case.

(Added). In the literature as [183], a **foliated** manifold is defined to be M of $(M, \mathcal{F})$ with $\mathcal{F}$ as such a foliation. In this case, the set of all tangent vectors of M tangent to leaves of $\mathcal{F}$ defines such a subbundle F of TM, called the tangent bundle of $\mathcal{F}$ denoted as $T(\mathcal{F})$. Also, the set of all tangent vectors of M orthogonal to leaves of $\mathcal{F}$ defines the normal bundle of $\mathcal{F}$ denoted as $N(\mathcal{F})$, which is isomorphic to $TM/T(\mathcal{F})$. ◀

To each foliation (M, F) associated is canonically a foliation C^*-algebra $C^*(M, F)$ which encodes the topology of the space of leaves. The construction is basically the same as the general one for quotient spaces mentioned above. But there are interesting nuances coming from the presence of holonomy in the foliation context. To take it into account, first construct a manifold N with $\dim N = \dim M + \dim F$, called the graph or holonomy groupoid of the foliation, which refines the equivalence relation coming from the partition of M into leaves as $M = \cup_{L_\alpha \in \mathcal{F}} L_\alpha$. This construction is due to Thom, Pradines, and Winkelnkemper (cf. [238] missing).

An element γ of N is given by two points $x = s(\gamma)$ and $y = r(\gamma)$ of M together with an equivalence class of smooth paths: $\gamma(t) \in M$ for $t \in [0, 1]$ such that $\gamma(0) = x$ and $\gamma(1) = y$, tangent to the bundle F, so that $\frac{d}{dt}\gamma(t) \in F_{\gamma(t)}$ for $t \in [0, 1]$, up to the following equivalence in the sense that γ_1 and γ_2 are equivalent if and only if the holonomy of the path $\gamma_2 \circ \gamma_1^{-1}$ at the point x is the identity. (Namely, those elements may be identified with classes of $\pi_1(L)$ for a left L in M and some base point in L, and which may be identified with the holonomly group for L, of germs of diffeomorphisms).

The graph N has an obvious composition law that the composition $\gamma \circ \gamma$ for $\gamma, \gamma' \in N$ makes sense if $s(\gamma) = r(\gamma')$. If the leaf L which contains both x and y has no holonomy, then the class of the path $\gamma(t)$ in N only depends on the pair of x and y. The condition of trivial holonomy is generic in the topological sense of dense G_δ. In general, for $x = s(\gamma)$ fixed, the map from $N_x = \{\gamma \in N \mid s(\gamma) = x\}$ to the leaf L through x, given by sending $\gamma \in N_x$ to $y = r(\gamma)$, is the holonomy covering of L.

Both the range and source maps r and s from the manifold N to M are smooth submersions and the paired map $(r, s) : N^2 \to M^2$ is an immersion whose image in M^2 is the often singular, subset

$$\{(y, x) \in M \times M \mid y \text{ and } x \text{ are on the same leaf}\}.$$

In the first approximation, elements of $C^*(M, F)$ are viewed as continuous matrices or sections $k(x, y)$, where (x, y) varies in the subset above. Now describe the foliation C^*-algebra in more details. For the notational convenience, assume that the manifold N is Hausdorff. Since it fails in the case of interesting examples, also explain briefly how to remove this hypothesis.

The basic elements of $C^*(M, F)$ are smooth half-densities $f \in C_c^\infty(N, \Omega_N^{\frac{1}{2}})$ with compact support on N. The bundle $\Omega_N^{\frac{1}{2}}$ of half-densities over N is defined as follows. First define a line bundle $\Omega_M^{\frac{1}{2}}$ over M, such that for $x \in, M$, one lets $\Omega_x^{\frac{1}{2}}$ be the 1-dimensional complex vector space of maps from the exterior power $\wedge^k F_x$ with $k = \dim F$ to $\mathbb{C}$ satisfying

$$\rho(\lambda v) = |\lambda|^{\frac{1}{2}} \rho(v), \quad v \in \wedge^k F_x, \lambda \in \mathbb{R}.$$

Then, for $\gamma \in N$, identify $\Omega_\gamma^{\frac{1}{2}}$ with 1-dimensional complex vector space $\Omega_y^{\frac{1}{2}} \otimes \Omega_x^{\frac{1}{2}}$, where γ is such a path between x and y. In other words, define

$$\Omega_N^{\frac{1}{2}} = r^*(\Omega_M^{\frac{1}{2}}) \otimes s^*(\Omega_M^{\frac{1}{2}}).$$

The bundle $\Omega_M^{\frac{1}{2}}$ is trivial on M. Thus we could choose once and for all a tirvialization ν turning elements of $C_c^\infty(N, \Omega_N^{\frac{1}{2}})$ into functions. The use of half densities makes all the constructions canonical.

For $f, g \in C_c^\infty(N, \Omega_N^{\frac{1}{2}})$, the convolution product $f*g$ is defined as the equality

$$(f * g)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2).$$

This makes sense because, for fixed γ such a path between x and y and fixing $v_x \in \wedge^k F_x$ and $v_y \in \wedge^k F_y$, the product $f(\gamma_1)g(\gamma_1^{-1}\gamma)$ defines a one-density on $N^y = \{\gamma_1 \in N \mid r(\gamma_1) = y\}$, which is smooth with compact support and vanishes if γ_1 is not contained in the support of f, and hence can be integrated over N^y to give a scalar, namely the right hand side $(f * g)(\gamma)$ evaluated on v_x and v_y.

The involutive operation is defined as $f^*(\gamma) = \overline{f(\gamma^{-1})}$. Namely, for fixed γ such a path between x and y and fixing $v_x \in \wedge^k F_x$ and $v_y \in \wedge^k F_y$, then $f^*(\gamma)$ evaluated on v_x and v_y is equal to $\overline{f(\gamma^{-1})}$ evaluated on v_y and v_x.

It then follows that $C_c^\infty(N, \Omega_N^{\frac{1}{2}})$ becomes an involutive or $*$-algebra.

For each leaf L of (M, F), one can define a natural representaion of this $*$-algebra on the L^2-space of the holonomy covering $L^\sim$ of L, as follows. Fix a base point $x \in L$, identify $L^\sim$ with $N_x = \{\gamma \in N \mid s(\gamma) = x\}$ and define

$$(\pi_{x\in L}(f)\xi)(\gamma) = \int_{\gamma_1 \circ \gamma_2} f(\gamma_1)\xi(\gamma_2), \quad \xi \in L^2(N_x),$$

which is the L^2-space of all square integrable, half-densities on N_x. Given such a path γ between x and y, there is a natural isometry between $L^2(N_x)$ and $L^2(N_y)$, which transforms the representation π_x to π_y.

By definition, $C^*(M, F)$ is the C^*-algebra completion of $C_c^\infty(N, \Omega_N^{\frac{1}{2}})$ by the universal norm with respect to leaves L in M

$$\|f\| = \sup_{x \in L, L \subset M} \|\pi_{x\in L}(f)\|.$$

Note that the foliation C^*-algebra $C^*(M, F)$ is always separable with respect to the norm and admits a natural smooth dense subalgebra as $C_c^\infty(M, F) \equiv C_c^\infty(N, \Omega_N^{\frac{1}{2}})$ of smooth compactly supported half-densities on N.

If such a leaf L has trivial holonomy, the corresponding representation $\pi_{x\in L}$ is irreducible. In general, its commutant (algebra) may not be only scalars and is generated by the action of the (discrete) holonomy group N_x^x on $L^2(N_x)$.

Example 5.1. (Edited). If the foliation comes from a submersion $p : M \to B$, then its graph N is

$$N = \{(x, y) \in M \times M \mid p(x) = p(y)\},$$

which is a submanifold of $M \times M$, and then $C^*(M, F)$ is identical to the algebra of the continuous field of Hilbert spaces $L^2(p^{-1}(\{x\}))$ for $x \in B$. Thus, unless $\dim F = 0$, it is isomorphic to the tensor product of $C_0(B)$ with $\mathbb{K}(H)$ the elementary C^*-algebra of compact operators on a Hilbert space H.

If the foliation comes from an action of a Lie group G in such a way that the graph is identical to $M \times G$ (this is not always true, even for flows with

$G = \mathbb{R}$), then $C^*(M,F)$ is identical to the reduced crossed product $C_0(M) \rtimes_r G$ of $C_0(M)$ by G. ◀

Moreover, the construction of $C^*(M,F)$ is local in the following sense. If $U \subset M$ is an open subset and F' is the restriction of F to U, then the graph N' of (U,F') is an open subset in the graph N of (M,F), and the inclusion $C_c^\infty(N',\Omega_{N'}^{\frac{1}{2}}) \subset C_c^\infty(N,\Omega_N^{\frac{1}{2}})$ extends to an isometric $*$-homomorphism from $C^*(U,F')$ to $C^*(M,F)$. The proof is straightforward and also applies to the case of non-Hausdorff graph.

Let us now briefly explain how the construction of the foliation C^*-algebra $C^*(M,F)$ is done in the case where the graph of the foliation is not Hausdorff. This case is rather rare, since it never occurs if the foliation is real analytic. However, it does occur in the cases with topologically interesting foliations, such as the Reeb foliation of the 3-sphere, which are constructed by patching together foliated manifolds (M_j,F_j) with boundaries, where the boundary ∂M_j is a leaf of F_j. In fact, most of the constructions done in geometry to produce smooth foliations of given co-dimension on a given manifold give non-Hausdorff graphs. The C^*-algebra $C^*(M,F)$ turns out in this case to be obtained as a fibered product of the C^*-algebras $C^*(M_j,F_j)$.

Example 5.2. (Added). Recall from Foliation in [183] the **Reeb** foliation. Let $f(x)$ be a smooth, even function on the open interval $(-1,1)$ such that

$$\lim_{|x|\to 1} \frac{d^k}{dx^k}\frac{1}{f'(x)} = 0, \quad k = 0, k \in \mathbb{N}.$$

For instance, may let $f(x) = |\tan(\frac{\pi x}{2})|$. Then $\frac{1}{f'(x)} = \frac{2}{\pi}\cos^2(\frac{\pi x}{2})$ $(x > 0)$, which goes to zero as $x \to 1$ with $x < 1$.

Consider the family of the ordinary graphs $L_c \subset \mathbb{R}^2$ of functions $y = f(x)+c$ for $c \in \mathbb{R}$ and $x \in (-1,1)$ and two lines $L'_{\pm 1}$: $x = \pm 1$. These define the smooth (Reeb) foliation for the product space $M_2 = [-1,1] \times \mathbb{R} \subset \mathbb{R}^2 \subset \mathbb{R}^3$, so that

$$M_2 = (\sqcup_{c\in\mathbb{R}} L_c) \sqcup (L'_1 \sqcup L'_{-1}).$$

Rotating M_2 around the y-axis of $\mathbb{R}^2$ in $\mathbb{R}^3$ defines the smooth (Reeb) foliation for $M_3 = D^2 \times \mathbb{R} \subset \mathbb{R}^3$ with co-dimension 1, as

$$M_3 = D^2 \times \mathbb{R} = (\sqcup_{c\in\mathbb{R}} L_c^\sim) \sqcup (S^1 \times \mathbb{R}),$$

where D^2 is the closed unit disk in $\mathbb{R}^2$ with the boundary $\partial D^2 = S^1$, and the leaf $L_c^\sim$ as the rotation of L_c has a bundle structure over the interval $[c,\infty)$ with fibers given as a point at c and as a circle at other points.

That foliation for M_3 is invariant under translation along $\mathbb{R}$ the y-axis. Hence it induces the similar (Reeb) foliation for $D^2 \times S^1$ as the Reeb component, as

$$D^2 \times S^1 = (\sqcup_{c\in\mathbb{R}} L_c^\sim) \sqcup \mathbb{T}^2.$$

Note as well that the induced leaf $L_c^\sim$ in $D^2 \times S^1$ has the closure with the boundary equal to the real 2-dimensional torus $\mathbb{T}^2$.

Since the real 3-dimensional sphere S^3 is obtained by attaching the boundary as the 2-torus $\mathbb{T}^2$ of two copies of $D^2 \times S^1$, the **Reeb** foliation for S^3 is given by taking the Reeb foliation for $D^2 \times S^1$ component-wise. ◄

In the general, non-Hausdorff case, the graph N of (M, F) being non-Hausdorff may have only few continuous functions with compact support. However, by being a manifold, we can give a local chart as $\chi : U \to \mathbb{R}^{\dim N}$. Then take a smooth function $f \in C_c^\infty(\mathbb{R}^{\dim N})$ with the support of f contained in $\chi(U)$, and consider the function on N equal to $f \circ \chi$ on U and to 0 outside of U. If N were Hausdorff, then this would generate all of $C_c^\infty(N)$ by taking linear combinations. In general, we take this linear span as the definition for $C_c^\infty(N)$. Note that we do not get continuous functions, since there may well be a sequence $u_n \in U$ with two limits, one in the support of $f \circ \chi$ and the other in the complement of U.

The above definition of $C_c^\infty(N)$ extends to define $C_c^\infty(N, \Omega_N^{\frac{1}{2}})$ the space of smooth half-densities on N with compact support. It then follows that the convolution product and the involution are defined for elements of $C_c^\infty(N, \Omega_N^{\frac{1}{2}})$.

Moreover, proceed exactly as in the Hausdorff case and construct the representation $\pi_{x \in L}$ of the $*$-algebra $C_c^\infty(N, \Omega_N^{\frac{1}{2}})$ on the Hilbert space $L^2(N_x)$. Note that though N is not Hausdorff, each N_x is Hausdorff, being the holonomy covering of the leaf L through x.

For each $f \in C_c^\infty(N, \Omega_N^{\frac{1}{2}})$ and $x \in L \subset M$, the operator $\pi_{x \in L}(f)$ is smooth and bounded on $L^2(N_x)$.

Exactly as in the Hausdorff case, the foliation C^*-algebra $C^*(M, F)$is defined to be the C^*-completion of $C_c^\infty(N, \Omega_N^{\frac{1}{2}})$ with the norm

$$\|f\|_{M,F} = \sup_{x \in L, L \subset M} \|\pi_{x \in L}(f)\|.$$

There is the so obtained functor from foliations to foliation C^*-algebras, which makes it possible, first of all, to translate from basic geometric properties to corresponding algebraic ones. The simplest examples of foliations already exhibit remarkable C^*-algebras.

Example 5.3. (Edited). For instance, the horocycle foliation of the unit sphere bundle of a Riemann surface with genus ≥ 2 gives a simple C^*-algebra without idempotents.

The Kronecker foliation with angle θ (as slope, ratio, or $2\pi\theta$ radian) nonzero gives rise to the noncommutative 2-torus, which is simple if and only if θ is irrational (cf. §6 below). ◄

In the type II situation with the presence of a holonomy invariant transverse measure λ, the basic result of the theory is the longitudinal index theorem which computes the L^2-index of differential operators D on a foliated manifold (M, F), which are elliptic in the longitudinal direction, i.e., the restrictions D_L of D to the leaves L are elliptic operators.

Let us start with a pair of smooth vector bundles E_1, E_2 on M together with a differential operator D on M from sections of E_1 to those of E_2 such that:

(1) D is restricted to leaves in the sense that $(D\xi)_x$ only depends on the restriction of a section ξ of E_1 to a neighbourhood of x in the leaf of x, i.e., D only uses partial differentiation in the leaf direction.

(2) D is elliptic when restricted to any leaf.

Theorem 5.4. ([58]). *(a) There exist Borel transversals B and B' respectively such that the bundles $(l^2(L \cap B))_{L \in M/F}$ and $(l^2(L \cap B'))_{L \in M/F}$ are measurably isomorphic to the bundles $(\ker(D_L))_{L \in M/F}$ and to $(\ker(D_L^*))_{L \in M/F}$.*

(b) The scalar $\lambda(B) < \infty$ is independent of the choice of B and is denoted as $\dim_\lambda(\ker(D))$.

$$(c) \quad \dim_\lambda(\ker(D)) - \dim_\lambda(\ker(D^*)) = (-1)\frac{k(k+1)}{2}\langle \mathrm{ch}(\sigma_D)\mathrm{Td}(F_{\mathbb{C}}), [C]\rangle,$$

where $k = \dim F$, $\mathrm{ch}(\sigma_D)$ is the Chern character (cohomology) class (or form) for the symbol σ_D of D, $\mathrm{Td}(F_{\mathbb{C}})$ is the Todd genus class (or form), and $[C] \in H_k(M, \mathbb{C})$ is the homology class of the Ruelle-Sullivan current C, which is a closed de rham current of dimension k and encodes the transverse measure λ by integtration of a k-dimensional differential form ω on M along the plaques of foliation charts (and as well, $\langle \cdot, \cdot \rangle$ means the pairing between cohomology and homology theories)

In particular, the Betti numbers β_j of a measured foliation are defined by [58], and given is the L^2-dimension of the space of L^2-harmonic forms along the leaves. More precisely,

Theorem 5.5. ([58]). (Edited). *(a) For each integer j with $0 \leq j \leq \dim F$, there exists a Borel transversal B_j such that the bundle $(H^j(L.\mathbb{C}))_{L \in M/F}$ of j-th square integrable harmonic forms on L is measurably isomorphic to $(l^2(L \cap B))_{L \in M/F}$.*

(b) The scalar $\beta_j = \lambda(B_j)$ is finite, independent of the choice of B_j and of the choice of the Euclidean structure on F.

(c) One has $\sum_j (-1)^j \beta_j = \chi(F, \lambda)$, which is the Euler characteristic, given by the pairing of the Ruelle-Sullivan current C with the Euler class $e(F)$ of the oriented bundle F over M.

Extending ideas of Cheeger and Gromov [51] in the case of discrete groups, It is shown by D. Gaboriau [115] (and [116] missing) as a recent remarkable result that the Betti numbers $\beta_j(F, \lambda)$ of a foliation with contractible leaves are invariants of the measured equivalence relation $\mathcal{R} = \{(x, y) \,|\, y \in L, x \in L\}$.

In the general case, it can not be expected to have a holonomy invariant transverse measure. In fact, the simplest foliations are of type III from the measure theoretic point of view. Obtaining an analogue in general as the theorems above is the basic motivation for the construction of the assembly map.

Let us briefly state the **longitudinal** index theorem as follows. Let D be as above an elliptic differential operator along the leaves of the foliation (M, F).

Since D is elliptic, it has an inverse modulo $C^*(M, F)$, and hence it gives an element $\mathrm{ind}_a(D)$ of $K_0(C^*(M, F))$, which is the **analytic** index of D.

The **topologial** index is obtained as follows. Let i be an auxiliary embedding of a manifold M into $\mathbb{R}^{2n}$. Let now N be the total space of the normal bundle to the leaves as $N_x = i_*(F_x)^\perp \subset \mathbb{R}^{2n}$. Foliate $M^\sim = M \times \mathbb{R}^{2n}$ by $F^\sim$ with $F^\sim_{(x,t)} = F_x \times \{0\}$, so that the leaves of $(M^\sim, F^\sim)$ are just $L^\sim = L \times \{t\}$, where L is a leaf of (M, F) and $t \in \mathbb{R}^{2n}$. The map sending (x, ξ) to $(x, i(x) + \xi)$ sends an open neighbourhood of the 0-section in N into an open transversal $T$of the foliation $(M^\sim, F^\sim)$. For a suitable open neighbourhood Ω of T in $M^\sim$, the foliation C^*-algebra $C^*(\Omega, F^\sim_\Omega)$ of the restriction $F^\sim_\Omega$ of $F^\sim$ to Ω is Morita equivalent to $C_0(T)$. Hence the inclusion $C^*(\Omega, F^\sim_\Omega) \subset C^*(M^\sim, F^\sim)$ yields a K-theory map: $K^0(N) \to K_0(C^*(M^\sim, F^\sim))$. Since $C^*(M^\sim, F^\sim) \cong C^*(M, F) \otimes C_0(\mathbb{R}^{2n})$, then the Bott periodicty implies the equality $K_0(C^*(M^\sim, F^\sim)) \cong K_0(C^*(M, F))$.

Using the Thom isomorphism, $K^0(F^*)$ is identified with $K^0(N)$, so that we get by the above construction the topological index:

$$\mathrm{ind}_t : K^0(F^*) \to K_0(C^*(M, F)).$$

Theorem 5.6. (Edited). *The longitudianl index theorem of Connes and Skandalis* ([97]) *is the equality:*

$$\mathrm{ind}_a(D) = \mathrm{ind}_t([\sigma_D]) \quad \text{in } K_0(C^*(M, F)),$$

where σ_D is the longitudinal symbol of D and its class $[\sigma_D] \in K^0(F^*)$.

Since the K-theory group $K_0(C^*(M, F))$ is hard to compute, one needs more computable invariants of its elements, and this is where cyclic cohomology enters the scene. In fact, its early development is already completed in 1981 for that precise goal (cf. [148]). The role of the trace on $C^*(M, F)$ associated to the transverse measure λ is now played by cyclic cocycles on a dense subalgebra of $C^*(M, F)$. A hard analytic problem is to show that these cocycles have enough semi-continuity properties to define invariants of $K_0(C^*(M, F))$. This is achived as in [62] and makes it possible to formulate corollaries whose statements are independent of the general theory, like the following:

Theorem 5.7. ([62]). *Let M be a compact, oriented manifold and assume that the $A^\wedge$-genus $A^\wedge(M)$ is non-zero, where M is not assumed to be a spin manifold, so that $A^\wedge(M)$ need not be an integer. Let then F be an integrable spin sub-bundle of TM. Then there exists no metic on F, for which the scalar curvature of the leaves is strictly positive on M.*

There is a rich interplay between the theory of foliations and their characteristic classes and operator algebras even at the measure theoretic level, as the classification of von Neumann factors.

In a remarkable series of paper, J. Heitsch and S. Hurder [132] (cf. [138] missing and [139]) have analyzed the interplay between the vanishing of the Godbillon-Vey (GV) invariant of a compact foliated manifold (M, F) and the type of the von Neumann algebra of the foliation. Their work culminates in the following result of S. Hurder ([138] missing):

Theorem 5.8. *If the von Neumann algebra of a foliated compact manifold (M, F) is semi-finite, then the Godbillon-Vey invariant vanishes.*

In fact, it is shown that cyclic cohomology yields a stronger result, by proving that if $GV \neq 0$, then the central decomposition of the von Neumann algebra necessarily contains facctors whose virtual modular spectrum is of finite co-volume in $\mathbb{R}_+^*$. Indeed,

Theorem 5.9. ([62]). *Let (M, F) be an oriented, transeversally oriented, compact, foliated manifold with* $\mathrm{codim}(F) = 1$. *Let $\mathfrak{M}$ be the assocaiated von Neumann algebra, and* $\mathrm{mod}(\mathfrak{M})$ *be its flow of weights. Then, if the Godbillon-Vey class of (M, F) is non-zero, then there exists an invariant probability measure for* $\mathrm{mod}(\mathfrak{M})$.

Actually constructed is an invariant measure for the flow $\mathrm{mod}(\mathcal{M})$, exploiting the following remarkable property of the natural cyclic 1-cocycle τ on the algebra $\mathcal{A}$ of the transverse 1-jet bundle for the foliation:

When viewed as a linear map δ from $\mathcal{A}$ to its dual, δ is an unbounded derivation, which is closable, and whose domain extends to the center $\mathfrak{Z}$ of the von Neumann algebra generated by $\mathcal{A}$. Moreover, δ vanishes on this ceter, and elements $h \in \mathfrak{Z}$ can then be used to obtain new cyclic cocycles τ_h on $\mathcal{A}$. The pairing defined as $l(h) = \langle \tau_h, \mu(x) \rangle$ with the K-theory classes $\mu(x)$ obtained under the assembly map μ, which is constructed by [16], does give a measure on $\mathfrak{Z}$, whose invariance under the flow of weights follows from discreteness of the K-theory group. To show that it is non-zero, use an index formula that evaluates the cyclic cocycles, assocaiated as above to the Gelfand-Fuchs classes, on the range of the assembly map μ.

The central question in the analysis of the noncommutative leaf space of a foliation is the step 3, namely the metric aspect which entails in particular constructing a spectral triple describing the transverse geometry. The reason why such a problem is so difficult is that it essentially amounts to doing metric geometry on manifolds in a way, which is background independent, by using the terminology of physicists, i.e., which is invariant under diffeomorphisms rather than covariant as in traditional Riemannian geometry.

Indeed, the transverse space of a foliation is a manifold endowed with the action of a large pseudo-group of partial diffeomorphisms implementing the hoonomy. Thus, in particular, no invariant metric exists in the general case, and the situation is similar to trying to develop gravity without making use of any particular background metric, which automatically destroys the invariance under the action of diffeomorphisms (cf. [94] and [190] both missing).

Using both the theory of hypo-elliptic differential operators and the basic technique of reduction from type III to type II, a general construction of a spectral triple is done by Connes-Moscocivi [89]. The remaining problem of the computation of the local index formula in cyclic cohomology is solved by [90] and leada in particular to the discovery of new symmetries given by an action of a Hopf algebra which only depends upon the transverse dimension of the foliation. This also leads to the development of the noncommutative analogue of the Chern-Weil theory of characteristic classes [91] in the general context of Hopf algebra actions on noncomutative spaces and cyclic cohomology, a subject which is under-going in rapid progress, in particular thanks to the recent works by M. Khalkhali [147] (missing), [148] and collaborators ([128], [149], [150], [151]).

6 The noncommutative tori

The noncommutative torus is considered as the prototype example of a noncommutative space, since it illustrates the properties and structures of noncommutative geometries. Noncommutative tori play a key role in the early development of the theory in the 1980's ([59]), giving rise to noncommutative analogues of vector bundles, connections, curvature, etc.

Noncommutative tori can be regarded as a special case of noncommutative spaces arising from foliations (cf. [66, I 4.β]). In this case, consider certain vector fields on the ordinary real 2-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. In fact, consider the **Kronecker** foliation on $\mathbb{T}^2$ as $dx = \theta dy$, where θ is a given real number. The case where θ is irrational is especially interesting. Consider the space of solutions of the differential equation: $dx = \theta dy$ for $x, y \in \mathbb{R}/\mathbb{Z}$. In other words, consider the space of leaves of the **Kronecker** foliation on $\mathbb{T}^2$.

(Added). Namely,

$$\mathbb{T}^2 = \sqcup_{c \in \mathbb{R}/\mathbb{Z}} L_c, \quad L_c = \{(x,y) \in (\mathbb{R}/\mathbb{Z})^2 \,|\, x = \theta y + c\} \approx \mathbb{R}.$$

There is the following vector bundle diagram:

$$\begin{array}{ccc} F \subset T\mathbb{T}^2 & \longleftarrow & F_p \subset T_p\mathbb{T}^2 \\ \downarrow & & \| \\ \mathbb{T}^2 & & \{\left(\frac{\partial}{\partial x}\right)_p = \frac{1}{\theta}\left(\frac{\partial}{\partial y}\right)_p\} \subset \mathbb{R}\left(\frac{\partial}{\partial x}\right)_p \oplus \mathbb{R}\left(\frac{\partial}{\partial y}\right)_p, \end{array}$$

with $F_p^* = \{(dx)_p = \theta(dy)_p\} \subset (T_p\mathbb{T}^2)^* = \mathbb{R}(dx)_p \oplus \mathbb{R}(dy)_p$ the co-tangent vector space of $\mathbb{T}^2$ at $p \in \mathbb{T}^2$. Note that a section of the co-tangent bundle $(T\mathbb{T}^2)^*$ over $\mathbb{T}^2$ such as the function $dx : p \mapsto (dx)_p$ is called a differential 1-form. Note as well that any $p \in \mathbb{T}^2$ is contained in L_c for some $c \in \mathbb{R}$ such that $T_q(L_c) = F_q$ for any $q \in L_c$. Hence, the subbundle F is integrable, namely a foliation bundle. ◀

Choose a transversal T to the Kronecker foliation, given by $T = (\mathbb{R}/\mathbb{Z}) \times \{0\}$, so that $T \approx S^1$ (homeomorphic or diffeomorphic). Two points of the transversal, which differ by an integer multiple of θ give rise to the same leaf. Describe the quotient space $S^1/\theta\mathbb{Z}$ by the equivalence relation which identifies any two points on the orbits by the irrational rotation or shift on T as $R_\theta(x) = x + \theta \bmod 1$.

May regard the circle $S^1 = T$ and the quotient space $T/\theta\mathbb{Z}$ at various levels of regularity such as being smooth, topological, and measurable. This corresponds to different algebras of functions on the space S^1 as

$$C^\infty(S^1) \subset C(S^1) \subset L^\infty(S^1).$$

When passing to the quotient $S^1/\theta\mathbb{Z}$, if we consider invariant functions under the action, then the algebra of such functions at any levels has only constant functions.

Instead, if we consider the algebra of functions on the graph of the equivalence relation with the convolution product, then we obtain a highly non-trivial noncommutative algebra as a noncommutative space, describing the space of leaves of the Kronecker foliation. This is given in the algebraic category by the irrational rotation algebra as

$$\mathcal{A}_\theta = \{(a_{ij}) \mid i, j \in T, L_i = L_j\}.$$

Namely, elements of the algebra are $\infty \times \infty$ matrices, but with finitely many non-zero (rows with) entries associated to the transversal $T = S^1$ mod the action. The algebraic rules become the same as for ordinary matrices (but with product as multiplication of series). Since the equivalence on T is given by a group action by $\mathbb{Z}$, the construction coincides with the crossed product as an algebra.

(Added as a possible corresponding interpretation). Note that the graph of the equivalence relation on T is the set

$$\{(x, y) \in T \times T \mid x \in T, y = R_{n\theta}x, n \in \mathbb{Z}\},$$

which is not discrete. But since the action of $\mathbb{Z}$ on $T = S^1$ is minimal, i.e., any orbit is dense in T, then we may assume that the parameter space is just a single orbit, i.e., a discrete space, to make sense of the matrix representation above. Namely, assume that $(x, y) = (R_{n\theta}(x_0), R_{m\theta}(x_0))$ for some fixed $x_0 \in T$ and $n, m \in \mathbb{Z}$. Moreover, an infinite row of the $\infty \times \infty$ matrix $(a_{x,y})$ as above may be identified with a function on T belonging to $C(S^1)$. In this sense, we can define a $C(S^1)$-valued function on $\mathbb{Z}$ as $\mathbb{Z} \ni n \mapsto a_{n,z} \in C(S^1)$ (coordinated as n), which is viewed as an infinite column $(a_{n,z})_n$. Then the product is defined as the multiplication of series:

$$(a_{n,z})_n * (b_{m,z})_m = \sum_n a_{n,z} \sum_m b_{m,z}. \quad \blacktriangleleft$$

For instance, in the topological category, $\mathcal{A}_\theta$ is identified with a dense $*$-subalgebra in the crossed product C^*-algebra as its norm closure:

$$\mathcal{A}_\theta \subset \overline{\mathcal{A}_\theta} = \mathfrak{A}_\theta = C(S^1) \rtimes_{R_\theta} \mathbb{Z} \equiv \mathbb{T}^2_\theta.$$

This crossed product has two natural generators as $C(S^1)$-valued (continuous) functions on $\mathbb{Z}$: for $z \in S^1$,

$$u = (u_{n,z})_n = \begin{cases} 1 = 1(z) & n = 1, \\ 0 & 1 \neq n \in \mathbb{Z}, \end{cases} \quad \text{and} \quad v = (v_{n,z})_n = \begin{cases} z & n = 0, \\ 0 & \text{otherwise}, \end{cases}$$

where u is identified with the generator of $\mathbb{Z}$ and v is done with that of $C(S^1)$. In fact, any element b of $\mathfrak{A}_\theta$ can be written as a (norm convergent) power series

$$b = (b_{n,z}) = \sum_{n \in \mathbb{Z}} b_n u^n, \quad b_n = b_n(z) \in C(S^1), z \in S^1,$$

where the multiplication rule is given by

$$ufu^{-1} = f \circ R_\theta^{-1} = \alpha_\theta(f), \quad f \in C(S^1), \alpha_\theta \in \mathrm{Aut}(C(S^1)).$$

(Added). Note as well that (cf. Rich Mountain [232])

$$\begin{aligned} a * b &= \sum_n a_n u^n \sum_m b_m u^m = \sum_{n,m} a_n u^n b_m u^m \\ &= \sum_n \sum_m a_n \alpha_\theta^n(b_m) u^{n+m} = \sum_n \sum_{m'=n+m} a_n \alpha_\theta^n(b_{m'-n}) u^{m'} \\ &= \sum_{m=m'} [\sum_n a_n \alpha_\theta^n(b_{m-n})] u^m \equiv \sum_m (a*b)(m), \end{aligned}$$

where the summands are the convolutions with respect to the action α_θ. ◀

Since $C(S^1)$ as well as $C^\infty(S^1)$, $L^\infty(S^1)$ are generated by the function $v(t) = e^{2\pi i t}$ for $t \in \mathbb{R}$ (mod 1), it then follows that $\mathfrak{A}_\theta$ as well as $\mathcal{A}_\theta$ are generated by two unitaries u and v with presentation given by the commutation relation

$$vu = \lambda uv, \quad \lambda = e^{2\pi i \theta}.$$

If we work in the smooth category, then any element of $\mathcal{A}_\theta^\infty$ as a smooth crossed product contained in $\mathfrak{A}_\theta$ and containing $\mathcal{A}_\theta$, called as the smooth noncommutaive 2-torus, is given by a power series

$$\sum_{(n,m)\in\mathbb{Z}^2} b_{n,m} u^n v^m, \quad b_{n,m} \in \mathcal{S}(\mathbb{Z}^2),$$

where $\mathcal{S}(\mathbb{Z}^2)$ is the Schwartz space of sequences on $\mathbb{Z}^2$ of rapid decay.

In the definition of $\mathcal{A}_\theta$, it is not always necessary to restrict to the condition that points i, j are in the transversal T. Instead, it is possible to also form another algebra as

$$\mathcal{B}_\theta = \{(a_{ij}) \,|\, i, j \in \mathbb{T}^2, L_i = L_j\}.$$

Now the parameter of integration is no longer discrete. But this ought to correspond to the same noncommutative space in NG. In fact, the algebras are related as the Morita equivalence, so that their C^*-algebras are stably isomorphic as

$$\mathfrak{B}_\theta \cong \mathfrak{A}_\theta \otimes \mathbb{K},$$

where $\mathbb{K}$ is the C^*-algebra of all compact operators on a Hilbert space.

The tangent space to the ordinary 2-torus $\mathbb{T}^2$ is spanned by the tangent (direction) derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ at any point of $\mathbb{T}^2$. By choosing coordinates u and v with $u = e^{2\pi i x}$ and $v = e^{2\pi i y}$, the tangent vectors are given by

$$\frac{\partial}{\partial x} = 2\pi i u \frac{\partial}{\partial u} \quad \text{and} \quad \frac{\partial}{\partial y} = 2\pi i v \frac{\partial}{\partial v}.$$

These have analogues in terms of derivations of the algebra as the noncommutative torus. The two commuting vector fields which span the tangent space for $\mathbb{T}^2$ correspond algebraically to two commuting derivations of the algebra $C^\infty(\mathbb{T}^2)$ of smooth functions on $\mathbb{T}^2$.

These derivations are extended to make sense by replacing the generators u and v of $C^\infty(\mathbb{T}^2)$ by those of the smooth crossed product algebra $\mathcal{A}_\theta^\infty$, which no longer commute. The corresponding derivations δ_1 and δ_2 are given by the same formulas as for $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, so that

$$\delta_1(\sum_{n,m} b_{n,m}u^n v^m) = 2\pi i \sum_{n,m} n b_{n,m} u^n v^m,$$

$$\delta_2(\sum_{n,m} b_{n,m}u^n v^m) = 2\pi i \sum_{n,m} m b_{n,m} v u^n v^{m-1} = 2\pi i \sum_{n,m} m\lambda^n b_{n,m} u^n v^m.$$

(Added). Hence it follows that

$$\delta_2 \circ \delta_1(\sum_{n,m} b_{n,m}u^n v^m) = 2\pi i \sum_{n,m} nm\lambda^n b_{n,m} u^n v^m = \delta_1 \circ \delta_2(\sum_{n,m} b_{n,m}u^n v^m).$$

Hence the derivations are commuting on $\mathcal{A}_\theta^\infty$ (as well as the other algebras).
(Added). Moreover,

$$\delta_1(u^n v^m)u^{n'}v^{m'} + u^n v^m \delta_1(u^{n'}v^{m'}) = (n+n')u^n v^m u^{n'} v^{m'}$$

$$\delta_1((u^n v^m)(u^{n'}v^{m'})) = \delta_1(\lambda^{mn'}u^{n+n'}v^{m+m'})) = (n+n')\lambda^{mn'}u^{n+n'}v^{m+m'},$$

$$\delta_2(u^n v^m)u^{n'}v^{m'} + u^n v^m \delta_2(u^{n'}v^{m'}) = (m\lambda^n + m'\lambda^{n'})u^n v^m u^{n'} v^{m'}$$

$$\delta_2((u^n v^m)(u^{n'}v^{m'})) = \delta_2(\lambda^{mn'}u^{n+n'}v^{m+m'})) = (m+m')\lambda^{mn'+n+n'}u^{n+n'}v^{m+m'},$$

It then follows that the Leibniz rule certainly holds for δ_1, but not for δ_2 (?). Instead of δ_2, we may consider $\delta_2' = 2\pi i \frac{\partial}{\partial v} v$ as an action from the right, with v as the right multiplication (corrected). Then

$$\delta_2'(\sum_{n,m} b_{n,m}u^n v^m) = \sum_{n,m} m b_{n,m} u^n v^m,$$

and $\delta_2' \circ \delta_1 = \delta_1 \circ \delta_2'$, and moreover,

$$\delta_2'(u^n v^m)u^{n'}v^{m'} + u^n v^m \delta_2'(u^{n'}v^{m'}) = (m+m')u^n v^m u^{n'} v^{m'}$$

$$\delta_2((u^n v^m)(u^{n'}v^{m'})) = \delta_2(\lambda^{mn'}u^{n+n'}v^{m+m'})) = (m+m')\lambda^{mn'}u^{n+n'}v^{m+m'}.$$

Therefore, the Leibniz rule holds for δ_2'. ◀

Just as in the classical case of a usual manifold, what ensures that the derivations considered above are enough to span the whole tangent space is the condition of ellipticity for the Laplacian $\Delta = \delta_1^2 + \delta_2^2$. In Fourier modes, the Laplacian is of the form $n^2 + m^2$, and hence Δ^{-1} is a compact operator.

The geometry of the Kronecker foliation is closely related to the structure of the algebra $\mathcal{A}_\theta$. In fact, a choice of a closed transversal as T of the foliation corresponds canonically to a finite projective module over the algebra $\mathcal{A}_\theta$ or $\mathfrak{A}_\theta$. In fact, the main result on finite projective modules over the noncommutative 2-torus $\mathbb{T}_\theta^2$ is the following classification result, obtained by combing those of [59], [199], and [208]:

Theorem 6.1. *Finite projective modules over $\mathfrak{A}_\theta$ are classified up to isomorphism by a pair of integers (p, q) such that $p+q\theta \geq 0$. For a choice of such a pair, the corresponding module $P_{p,q}$ is obtained from the transversal $T_{p,q}$ given by the closed geodesic of the 2-torus $\mathbb{T}^2$ specified by (p, q), via the following construction. Elements of the module associated to the transversal $T_{p,q}$ are rectangular matrices $(\xi_{i,j})$ with $(i, j) \in \mathbb{T} \times S^1$ with i and j belonging to the same leaf. The right action of $(a_{i,j}) \in \mathcal{A}_\theta$ is by matrix multiplication.*

For instance, from the transversal in the y-axis one can obtain the following module over $\mathcal{A}_\theta$. The underlying linear space is the usual Schwarz space $\mathcal{S}(\mathbb{R})$ of complex-valued, smooth functions on $\mathbb{R}$ such that all of whose derivatives are of rapid decay. The right module structure is given by the action of the two generators u, v:

$$(\xi u)(s) = \xi(s+\theta) \quad \text{and} \quad (\xi v)(s) = e^{2\pi i s}\xi(s), \quad s \in \mathbb{R}.$$

Then the commutation relation $vu = \lambda uv$ is satisfied, and the space $\mathcal{S}(\mathbb{R})$ as a right module over $\mathcal{A}_\theta$ or $\mathcal{A}_\theta^\infty$ is finitely generated and projective, i,e, it complements to a free module.

Finitely generated, projective modules play an important role in noncommutative geometry, as they replace vector bundles in the commutative setting. In fact, in ordinary commutative geometry, vector bundles are equivalently described through their sections, which in turn form a finitely generated, projective module over the algebra of smooth functions. The notion of finitly generated, projective modules contiues to make sense in the noncommutative setting, and provides in this way a notion of noncommutative vector bundles.

Suppose given a vector bundle E over a smooth manifold X, which is described algebraically through the space $C^\infty(X, E) = \mathcal{A}$ of smooth sections on X. The dimension of E is computed by the trace of the identity endomorphism. In terms of the space of smooth sections on X and hence of finitely generated, projective modules $\mathcal{E} = p\mathcal{A}^m$ for some m and some finite projection $p \in M_m(\mathcal{A})$, it is possible to recover the dimension of E as a limit

$$\dim_{\mathcal{A}} \mathcal{E} = \lim_{n\to\infty} \frac{1}{n}\#\{\text{Generators of } \mathcal{E}^n\},$$

where $\#\{\cdots\}$ means the number of a set.

This method is applied to the noncommutative setting. In the case of the noncommutative tori, it then follows that the Schwarz space $\mathcal{S}(\mathbb{R})$ has dimension $\dim_{\mathcal{A}_\theta} \mathcal{S}(\mathbb{R}) = \theta$. Similarly, one finds values $p + q\theta$ for the more general case as in the theorem above.

The appearance of a real-valued dimension is related to the density of transversals in the leaves, that is, the limit

$$\lim_{r\to\infty} \frac{\#(B_r \cap S)}{|B_r|}$$

where B_r is the ball of radius r in a leaf and $S = \{x = 0\}$. In this sense, the dimension θ of the Schwarz space measures the relative densities of the two transversals $S = \{x = 0\}$ and $T = \{y = 0\}$.

In general, the appearance of non-integral dimension is a basic feature of von Neumann algebras of type II. The dimension of a vector bundle is the only invariant that remains when we use the algebra $L^\infty(S^1)$ of measurable functions from the measure theoretic point of view. The von Neumann algebra which describes the quotient space $S^1/\theta\mathbb{Z}$ from the measure theoretic point of view is the crossed product von Neumann algebra

$$R = L^\infty(S^1) \rtimes_{\alpha_\theta} \mathbb{Z}.$$

This is the well known hyperfinite factor of type II_1. In particular, the classification of finitely generated, projective modules $\mathcal{E}$ over R is given by positive real numbers as the Murray-von Neumann dimension

$$\dim_R \mathcal{E} \in \mathbb{R}_+.$$

The simplest way to describe the phenomenon of Morita equivalence for noncommutative 2-tori is given in terms of the Kronecker foliation, where it corresponds to reparameterizing the leaf space in terms of a different closed transversal. Thus, Morita equivalence of the algebras $\mathfrak{A}_\theta$ and $\mathfrak{A}_{\theta'}$ for θ and θ' in the same orbit by $PGL_2(\mathbb{Z})$ becomes simply a statement that the leaf space of the foliation is independent of the transversal used to parameterize it. For instance, Morita equivalence between $\mathfrak{A}_\theta$ and $\mathfrak{A}_{-\theta^{-1}}$ corresponds to changing the parameterization of the space of leaves from the transversal $T = \{y = 0\}$ to the transversal $S = \{x = 0\}$.

More generally, an explicit construction of bimodules $M_{\theta,\theta'}$ is obtained by Connes [59] (cf. Rieffel [207]). These are given by the Schwarz space $\mathcal{S}(\mathbb{R}\times\mathbb{Z}/c)$, with the right action of $\mathcal{A}_\theta$ given by

$$uf(x,n) = f(x - \frac{c\theta + d}{c}, n-1) \quad \text{and} \quad vf(x,n) = e^{2\pi i(x - \frac{nd}{c})} f(x,n),$$

and with the left action of $\mathcal{A}_{\theta'}$ given by

$$u'f(x,n) = f(x - \frac{1}{c}, n-a) \quad \text{and} \quad v'f(x,n) = e^{2\pi i(\frac{x}{c\theta+d} - \frac{n}{c})} f(x,n).$$

The bimodule $M_{\theta,\theta'}$ realizes the Morita equivalence between $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ for

$$\theta' = \frac{a\theta + b}{c\theta + d} = g\theta, \quad g \in PGL_2(\mathbb{Z}).$$

7 Duals of discrete groups

Noncommutative geometry provides naturally a generalization of Pontrjagin duality for discrete groups. The Pontrjagin dual $\Gamma^\wedge$ of a finitely generated, discrete abelian group is a compact abelian group. The dual of a more general, finitely generated, discrete (non-abelian) group can be a noncommutative space.

To see this, recall that the usual Pontrjagin duality assigns to a finitely generated, discrete abelian group Γ its dual group $\Gamma^\wedge = \mathrm{Hom}(\Gamma, U(1) = \mathbb{T})$ of characters of Γ. The duality between Γ and $\Gamma^\wedge$ is given by Fourier transform

$$\langle \varphi_z, \gamma_n \rangle = \varphi_z(\gamma_n) = z_1^{n_1} \cdots z_k^{n_k} \in \mathbb{T} \quad \text{for } \gamma_n = \gamma_1^{n_1} \cdots \gamma_k^{n_k} \in \Gamma, \, \varphi_z \in \Gamma^\wedge,$$

where $\gamma_1, \cdots, \gamma_k$ are generators of Γ, $n = (n_1, \cdots, n_k) \in \mathbb{Z}^k \cong \Gamma$, and $z = (z_1, \cdots, z_k) \in \mathbb{T}^k \cong \Gamma^\wedge$ (refined).

Moreover, the Fourier transform gives an identification between the C^*-algebra of continuous, complex-valued functions on $\Gamma^\wedge$ and the group C^*-algebra of Γ

$$C(\Gamma^\wedge) \cong C^*(\Gamma),$$

which is the universal C^*-algebra completion of the group algebra $\mathbb{C}[\Gamma] = C_c(\Gamma)$ or the Banach $*$-algebra $l^1(\Gamma)$ of absolutely summable functions on Γ. Since Γ is commutative, $C^*(\Gamma)$ coincides with the reduced group $C_r^*(\Gamma)$, which is the C^*-algebra generated by Γ under the left regular representation on the Hilbert space $l^2(\Gamma)$ of square summable functions on Γ, and $C_r^*(\Gamma)$ is a quotient of $C^*(G)$ in general.

When Γ is non-abelian, the Pontrjagin duality is no longer applied in the classical sense as above. Indeed, $\Gamma^\wedge$ is almost not Hausdorff, so that the left hand side makes no sense. However, the right hand side still makes sense and it behaves like the algebra of functions on a noncommutative space (as well as on the unitary dual $\Gamma^\wedge$, which is the space of equivalence classes of unitary representations of Γ). It can be said that for a non-abelian group Γ, the Pontrjagin dual is certainly generalized to the unitary dual, but this is also rather limited to such as the case where the unitary dual has a composition series with Hausforff sub-quotients. More generally, the reduced group C^*-algebra $C_r^*(\Gamma)$ as well as $C^*(\Gamma)$ are viewed as a noncommutative space as an algebra of coordinates. In fact, the representation theory of Γ is just identified with that of $C^*(\Gamma)$, not of $C_r^*(\Gamma)$. (Detailed).

As an example that illustrates the general philosophy above,

Example 7.1. Recall below that the dihedral group can be written as a semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}_2$, which is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_2$, with $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

First note that any representation of the free product group $\mathbb{Z}_2 * \mathbb{Z}_2 = \Gamma$ is identified with a pair of subspaces E and F in the Hilbert space H. Define the operators $u = 1 - 2p_E$ and $v = 1 - 2p_F$ with p_E and p_F the projections corresponding to E and F respectively. Then $u = u^*$, $v = v^*$, and

$$u^2 = 1 - 4p_E + 4p_E = 1 \quad \text{and} \quad v^2 = 1 - 4p_F + 4p_F = 1.$$

Hence u and v represent reflections.

The group Γ is realized as and generated by words in the generators u and v. Equivalently, the group Γ can be described as the semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}_2$, by setting $x = uv$, with the action $vxv^{-1} = x^{-1}$ (corrected).

The regular representation of Γ is analyzed by using the Mackey machine for semi-direct product groups. First consider representations of the normal subgroup $\mathbb{Z}$ Then consider orbits of the action of $\mathbb{Z}_2$. The irreducible representations of $\mathbb{Z}$ are labeled by $S^1 = \mathbb{T}$ and given by sending x^n to z^n for $n \in \mathbb{Z}$ and $z \in S^1$. The action of $\mathbb{Z}_2$ is the involution given by conjugation sending z to $\bar{z}$. The quotient of S^1 by the $\mathbb{Z}_2$-action is identified with the closed interval $[-1, 1]$ by sending z to the real part $\mathrm{Re}(z)$. For points in the open interval $(-1, 1)$, the corresponding irreducible representations of Γ are two-dimensional. At each of the two end-points ± 1, two inequivalent 1-dimensional representations of Γ correspond. Then it follows that $C^*(\Gamma)$ is isomorphic to the dimension drop C^*-subalgebra of $C([-1, 1], M_2(\mathbb{C}))$, converging to the trivial C^*-algebra at ± 1. ◀

In the general theory for arbitrary discrete groups Γ, the first two basic steps are known as follows:

(1) The resolution of the diagonal and computation of the cyclic cohomology are provided by the geometric model due to Burghelea [39], given by the free loop space of the classifying space $B\Gamma$.

(2) The assembly BC-map ([16]) from the K-homology of the classifying space $B\Gamma$ to the operator K-theory of the reduced C^*-algebra $C_r^*(\Gamma)$ refined to take care of torsion in the group Γ and gives an approximation to the K-theory of $C_r^*(\Gamma)$ (cf. [220]).

In the presence of a natural smooth subalgebra of $C_r^*(\Gamma)$ containing the group ring $\mathbb{C}[\Gamma]$ and stable under holomorphic functional calculus, the combination of the two steps above makes it possible to prove an index theorem which is a higher dimensional form of Atiyah L^2-index theorem for coverings. This gives the first proof of the Novikov conjecture for hyperbolic groups ([88]). Since then, the analysis of dense smooth subalgebras has played a key role, an in particular, in the ground-breaking work of Vincent Lafforgue ([158]). See also [16], [142], [220], and [221].

The next step (3) as the construction of a spectral geometry is directly related to the geometric group theory. In general, it can not be expected to obtain a finite dimensional spectral triple since the growth properties of a group, except for groups with polynomial growth, give a basic obstruction (cf. [63]). A general construction of a θ-summable spectral triple is given in [66, IV. 9]. Basically, the transition from finitely summable spectral triples to the θ-summable ones corresponds to that from the finite dimensional geometry to the infinite dimensional case. In the θ-summable case, the Chern character is no longer a finite dimensional cyclic cocycle and it is needed to extend to the cyclic cohomology using cocycles with infinite support in the (b, B) bicomplex, fulfilling a subtle growth condition. The general theory of the entire cyclic cohomology is developed in [65]. It is in general difficult to compute the Chern character in the θ-summable case and it would take a long time until it would done for the basic example as discrete subgroups of semi-simple Lie groups. In the case of real rank one, it has been achieved by a remarkable paper of M. Puschnigg [201].

The fourth step as the thermo-dynamics might seem irrelevant in the type II context of discrete groups. However, a small variant of the construction of the group rings, namely, as the Hecke algebra associated to an almost normal inclusion of discrete groups in the sense considered in [29], suffices to meet the type III world. One of the open fields is to extend the above steps (1), (2), and (3) in the general context of almost normal inclusions of discrete groups, and to perform the thermo-dynamical analysis in the spirit of [73] in that context.

8 Brillouin zone and the quantum Hall effect

An important application to physics of the theory of noncommutative 2-tori is the development of a rigorous mathematical model for the integer quantum Hall effect (IQHE) obtained by Bellissard and collaborators (cf. [19] and [20] both missing and [66]) (cf. [46] as well).

The classical Hall effect is a physical phenomenon first observed in the 19th century ([129] missing). A thin metal (sheet) sample is now immersed in a constant uniform strong magnetic field, orthogonal to the surface of the sample. By forcing a constant current to flow through the sample in a direction, the flow of charge carriers in the metal is subject to a Lorentz force perpendicular both to the current and and to the magnetic field. Then the equation for the equilibrium of forces in the sample is given of the form

$$Ne\vec{E} + \vec{j} \wedge \vec{B} = 0,$$

where $\vec{E}$ is the electric field, e and N are the charge and number of the charge carriers in the metal respectively, $\vec{B}$ is the magnetic field, and $\vec{j}$ is the current.

The equation above defines a linear relation, so that the ratio of the intensity of the Hall current to the intensity of the electric field defines the Hall conductance as

$$\sigma_H = \frac{Ne\delta}{B},$$

with $B = |\vec{B}|$ (as norm) the intensity of the magnetic field and δ the sample (sheet) width. The dimension-less quantity

$$\nu_H = \frac{N\delta h}{Be} = \sigma_H R_H$$

is called the filling factor, while the quantity $R_H = \frac{h}{e^2}$ is the Hall resistance. The filling factor measures the fraction of Landau level filled by conducting electrons in the sample. Thus, classically, the Hall conductance, measured in units of $\frac{e^2}{h}$, equals the filling factor.

In 1980, about a century after the classical Hall effect was observed, it is shown by the experiment of von Klitzing that lowering the temperature below $1\ K$, the relation of Hall conductance to filling factor shows plateaux at integer values ([153]). The integer values of the Hall conductance are observed with a surprising experimental accuracy of the order of 10^{-8}. This phenomenon of quantization of the Hall conductance is known as the integer quantum Hall effect (iqHe, changed).

It is first suggested by Laughlin [161] that the iqHe should be of a geometric origin. A detailed mathematical model of the iqHe is developed by Bellissrd and collaborators ([19] and [20] both missing). The model accounts for all the important features of the experiment such as quantization, localization, insensitivity to the presence of disorder, vanishing of direct conductance at plateaux levels, improving over the earlier Laughlin model.

The Bellissard approach to the iqHe is based on noncommutative geometry. The quantization of the Hall condunctance at integer values is indeed geometric in nature in the sense that it resembles another well known quantization phenomenon that happens in the more familiar setting of the geometry of compact two-dimensional manifolds, namely the Gauss-Bonnet theorem, where the integral of the curvature is an integer multiple of 2π, a property that is stable under deformations. In the same spirit, the values of the Hall conductance are related to the evaluation of a certain characteristic class, or in other words, to an index theorem for a Fredholm operator.

More precisely, in the physical model, made is the simplifying assumption that the iqHe can be described by non-interacting particles. Then the Hamiltonian describes the motion of a single electron subject to the magnetic field and an additional potential representing the lattice of ions in the conductor.

In a perfect crystal and in the absence of a magnetic field, there is a group of translational symmetries. This corresponds to a group of unitary operators u_a for $a \in G$, where G is the locally compact group of symmetries. Turning on the magnetic field breaks this symmetry, in the sense that the translates $H_a = u_a H u_a^{-1}$ $(a \neq 1_G)$ of the Hamiltonian H no longer commute with H. Since there is no preferred choice of one translate over the others, the algebra of observables must include all translates of the Hamiltonian, or more better, their (translated) resolvents, namely the bounded operators

$$R_a(z) = u_a(z1 - H)^{-1}u_a^{-1}, \quad z \notin \sigma(H) \text{ the spectrum of } H.$$

For a particle of effective mass m and charge e, confined to the plane, subject to a magnetic field of vector potential $\overrightarrow{A}$ and to a bounded potential v, the Hamiltonian is of the form

$$H = \frac{1}{2m} \sum_{j=1,2} (p_j - eA_j)^2 + v \equiv H_0 + v,$$

where the unperturbed part H_0 is invariant under the magnetic translations, namely the unitary representation of the translation group $\mathbb{R}^2$ given by

$$u_a\psi(x) = \exp(\frac{-ieB}{2\hbar} w(x,a))\psi(x-a),$$

with w the standard symplectic form in the plane.

The hull of the translated resolvents above as the strong closure yields a topological space, whose homeomorphism type is independent of the point z in the resolvent (complement) set $\sigma(H)^c$ of H. This provides a noncommutative version of the Brillouin zone.

Recall that the Brillouin zones of crystals are fundamental domains for the reciprocal lattice obtained via the following inductive procedure. The Bragg hyperplanes of a crystal are the hyperplanes, along which a pattern of diffraction of maximal intensity is observed, when a beam of radiation such as X-rays is shone at the crystal. (For instance, for a convex part of a crystal which may not be convex, such a hyperplane is just a line through on a face of the local convex part). The n-th Brillouin zone consists of all the points such that the line from that point to the origin crosses exactly $n-1$ Bragg hyperplanes of the crystal (where the hyperplaces are assumed not to contain the origin. But if they do contain the origin, then we may choose the other point, not contained, instead. In fact, the zones correspond to cutting as well as gradation of the crystal.)

More precisely, in the case above, if e_1 and e_2 are generators of the periodic lattice, then there is a commutation relation

$$u_{e_1}u_{e_2} = e^{2\pi i\theta}u_{e_2}u_{e_1},$$

where θ is the flux (or bundle) of the magnetic field through a fundamental domain for the lattice, in dimension-less units. Hence the noncommutative Brillouin zone is described by a noncommutative 2-torus.

This can also be seen in a discrete model, where the Hamiltonian is given by the operator

$$\begin{aligned}(H_a f)(m,n) &= e^{-ia_1 m}f(m,n+1) + e^{ia_1 m}f(m,n-1)\\ &\quad + e^{-ia_2 n}f(m+1,n) + e^{ia_2 n}f(m-1,n)\end{aligned}$$

for $f \in L^2(\mathbb{Z}^2)$ (modified). This is a discrete version of the magnetic Laplacian. Note that the equation above can be written in the (corresponding) form

$$H_a = u + u^* + v + v^*,$$

where

$$(uf)(m,n) = e^{-ia_1 m}f(m,n+1) \quad \text{and} \quad (vf)(m,n) = e^{-ia_2 n}f(m+1,n).$$

These satisfy the commutation relation of $\mathbb{T}^2_\theta$ with $\theta = \frac{a_2 - a_1}{2\pi}$ (corrected).

Proof. (Added). For $f, g \in L^2(\mathbb{Z}^2)$,

$$\begin{aligned}\langle uf, g\rangle &= \sum_{m,n} e^{-ia_1 m}f(m,n+1)\overline{g(m,n)}\\ &= \sum_{m,k=n+1} f(m,k)\overline{e^{ia_1 m}g(m,k-1)} = \langle f, u^* g\rangle.\end{aligned}$$

It then follows that $u^*u = uu^* = 1$. Similarly, the same holds for v and v^*.

$$\begin{aligned}(vuf)(m,n) &= v(uf)(m,n) = e^{-ia_2 n}(uf)(m+1,n) \\ &= e^{-ia_2 n}e^{-ia_1(m+1)}f(m+1,n+1), \\ (uvf)(m,n) &= u(vf)(m,n) = e^{-ia_1 m}(vf)(m,n+1) \\ &= e^{-ia_1 m}e^{-ia_2(n+1)}f(m+1,n+1).\end{aligned}$$

Therefore, obtained is $vu = e^{i(a_2-a_1)}uv = e^{2\pi i\theta}uv$. □

In the zero-temperature limit, the Hall conductance satisfies the Kubo formula

$$\sigma_H = \frac{1}{2\pi i R_H}\tau(p_\mu[\delta_1 p_\mu, \delta_2 p_\mu]),$$

where p_μ is the spectral projection of the Hamiltonian on energies smaller than or equal to the Fermi level (energy) E_μ, and τ is the trace on $\mathcal{A}_\theta$ given by

$$\tau(\sum_{n,m} a_{n,m}u^n v^m) = a_{0,0},$$

and δ_1, δ_2 are the derivations (where δ_2 should be replaced with δ_2'). Here we assume that the Fermi level μ (or E_μ) is in a gap in the (possibly discrete) spectrum of the Hamiltonian. In this case, the spectral projection p_μ belong to the C^*-algebra of observables. The Kubo formula above can be derived from purely physical considerations, such as transport theory and the quantum adiabatic limit.

The main result is then the fact that the integrality of the conductance observed in the integer quantum Hall effect is expained topologically in terms of the integrality of the cyclic cocycle $\tau(a^0(\delta_1 a^1 \delta_2 a^2 - \delta_2 a^1 \delta_1 a^2))$ ([59]).

The **fractional** quantum Hall effect (fqHe) is discovered by Størmer and Tsui in 1982. The setup is as in the integer quantum Hall effect as follows. In a high quality semi-conductor interface, which is modelled by an infinite (area), 2-dimensional surface, with low carrier concentration and extremely low temperatures similar to $10mK$, in the presence of a strong magnetic field, the experiment shows that the graph of $\frac{h}{e^2}\sigma_H$ against the filling factor ν exhibits plateaux (which looks like bottoms) at certain (easy) fractional (or rational) values.

The independent electron approximation, in the case of the iqHe, that reduces the problem to a single electron wave-function, is no longer viable in the fqHe. So we need to incorporate the Coulomb interaction between the elections in a many-electron theory. Nonetheless, it is possible to use a crude approximation, whereby we need to alter the underlying geometry to account for an average effect of the multi-electron interactions. Can be obtained in this way a model of the fqHe via noncommutative geometry, where we use hyperbolic geometry to simulate the interactions (cf. Marcolli-Mathai [179], and [178] and [180] both missing).

The noncommutative geometry approach to the integer quantum Hall effect described above is extended to hyperbolic geometry in [40]. The analog of the operator H_a is given by the Harper operator on the Cayley graph of a finitely generated, discrete subgroup Γ of $PSL_2(\mathbb{R})$. Given a (normalized) 2-cocycle (or multiplier) $\sigma : \Gamma \times \Gamma \to U(1)$ satisfying $\sigma(\gamma, 1) = \sigma(1, \gamma) = 1$ for $\gamma \in \Gamma$ and

$$\sigma(\gamma_1, \gamma_2)\sigma(\gamma_1\gamma_2, \gamma_3) = \sigma(\gamma_1, \gamma_2\gamma_3)\sigma(\gamma_2, \gamma_3), \quad \gamma_1, \gamma_2, \gamma_3 \in \Gamma,$$

consider the right σ-regular representation on the Hilbert space $l^2(\Gamma)$ of all square summable, $\mathbb{C}$-valued functions on Γ, of the form

$$r^\sigma_\gamma \psi(\gamma') = \psi(\gamma'\gamma)\sigma(\gamma', \gamma), \quad \gamma, \gamma' \in \Gamma$$

satisfying $r^\sigma_\gamma r^\sigma_{\gamma'} = \sigma(\gamma, \gamma')r^\sigma_{\gamma\gamma'}$.

Proof. (Added). Indeed, for $g_1, g_2, g_3 \in \Gamma$,

$$\begin{aligned}(r^\sigma_{g_1} r^\sigma_{g_2}\psi)(g_3) &= (r^\sigma_{g_2}\psi)(g_3g_1)\sigma(g_3, g_1)\\ &= \psi((g_3g_1)g_2)\sigma(g_3g_1, g_2)\sigma(g_3, g_1)\\ &= \psi((g_3(g_1g_2))\sigma(g_3, g_1g_2)\sigma(g_1, g_2) = \sigma(g_1, g_2)(r^\sigma_{g_1g_2}\psi)(g_3). \quad \blacktriangleleft\end{aligned}$$

□

For $\{g_i\}_{i=1}^k$ a symmetric set of generators of Γ (together with inverses), the Harper operator is of the form

$$r_\sigma = \sum_{i=1}^k r^\sigma_{g_i},$$

and the operator $k - r_\sigma$ is the discrete analogue of the magnetic Laplacian (cf. Sunada (or Sand-Field) [228] missing).

The idea is that by the effect of the strong interaction with the other electrons, a single electron sees the surrounding geometry as a hyperbolic world, with lattice sites that appear as a multiple image effect, as the points in a lattice Γ in $PSL_2(\mathbb{R})$. Thus, consider the general form of such a lattice as $\Gamma = \Gamma(g; \nu_1, \cdots, \nu_n)$, with generators a_i, b_i, c_j for $i = 1, \cdots, g$ and $j = 1, \cdots, n$ and with a presentation of the form

$$\Gamma = \langle \{a_i, b_i\}_{i=1}^g, \{c_j\}_{j=1}^n \mid \Pi_{i=1}^g [a_i, b_i]c_1 \cdots c_n = 1, c_j^{\nu_j} = 1 \rangle,$$

with $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ the multiplicative commutator and with $\nu_1, \cdots, \nu_j$ torsions and with $2g$ the number of torsion free generators. The quotient of the upper half-plane $\mathbb{H}$ in $\mathbb{C}$ by the action of Γ as isometries

$$\Sigma(g, \nu_1, \cdots, \nu_n) \equiv \Gamma\backslash\mathbb{H}$$

is a hyperbolic orbifold.

Now let p_E denote the spectral projection associated to the Fremi level E, i.e., $p_E = \chi_{(-\infty, E]}(H)$ by functional calculus. Then, in the zero temperature limit, the Hall conductance is given by

$$\sigma_H = \mathrm{tr}_K(p_E, p_E, p_E),$$

where tr_K denotes the conductance (Kubo) 2-cocycle. It is a cyclin 2-cocycle on the twiseted group algebra $\mathbb{C}[\Gamma, \sigma]$ of the form

$$\mathrm{tr}_K(f_0, f_1, f_2) = \sum_{j=1}^{g} \mathrm{tr}(f_0[\delta_j(f_1)\delta_{j+g}(f_2) - \delta_{j+g}(f_1)\delta_j(f_2)]),$$

where δ_j are derivations associated to the 1-cocycles associated to a symplectic basis $\{a_j, b_j\}_{j=1}^{g}$ of $H^1(\Gamma, \mathbb{R})$ (cf. [180] missing).

Within this model, obtained are the fractional values of the Hall conductance as integer multiples of orbifold Euler characteristics

$$\chi_{ob}(\Sigma(g; \nu_1, \cdots, \nu_n)) = 2 - 2g + \nu - n \in \mathbb{Q}.$$

In fact, it is shown by Marcolli-Mathai ([178] and [180] both missing) that the conductance 2-cocycle is cohomologous to another cocycle, i.e., the area 2-cocycle, for which can be computed the values on K-theory and hence the value of σ_H, by applying a twisted version of the Connes-Moscovici higher index theorem [88].

While in the case of the integer quantum Hall effect, the noncommutative geometry model is satisfactory enough to explain all the physical properties of the system, but in the fractional case, the orbifold model can be considered as a first rough approximation to the quantum field theory that governs the fqHe. For instance, the geometry of 2-dimensional hyperbolic orbifolds is related to the Chern-Simons theory through the moduli spaces of vortex equations. This remains an interesting open question.

9 Tilings in Euclidean spaces

Recall now a tiling in $\mathbb{R}^d$ as follows. Let $\{b_1, \cdots, b_n\}$ be a finite collection of closed bounded subsets of $\mathbb{R}^d$, homeomorphic to the closed unit ball. These b_j are called proto-tiles. Usually assume that the proto-tiles are polytopes (or a connected union of cubes) in $\mathbb{R}^d$ with a single d-dimensional cell (fixed) as the interiors of the proto-tiles. But this assumption can be relaxed. A **tiling** of $\mathbb{R}^d$ is then defined to be a covering $\mathfrak{T}$ of subsets with mutually disjoint interior, each of which is a tile, which is defined to be a translate of one of the proto-tiles.

Given a tiling $\mathfrak{T}$ of $\mathbb{R}^d$, can be formed its orbit closure under translations. The metric on tilings of $\mathbb{R}^d$ is defined as that two tilings are close if they almost agree on a large ball centered at the origin in $\mathbb{R}^d$. Fore more details and equivalent definitions, may refer to [4] (and [21] missing).

Tilings can be either periodic or aperiodic. There are many familiar examples of periodic tilings. The best known examples of aperiodic tilings are the Penrose tilings ([197] missing). Similar types of aperiodic tilings have been widely studied in the physics of quasi-crystals (cf. [14], [21], and also [167] triple missing).

It is understood early on in the development of noncommutative geometry (cf. [63] and [66]) that Penrose tilings provide an interesting class of noncommutative spaces. In fact, let Ω be the set of tilings $\mathfrak{T}_j$ of $\mathbb{R}^d$ with given prototiles $\{b_1, \cdots, b_n\}$. Define the equivalence relation on Γ given by the action of $\mathbb{R}^d$ by translations. Namely, identify tilings that can be obtained from one another by translations (or vector shifts). In the case of aperiodic tilings, this yields the type of quotient construction described before, which leads naturally to noncommutative spaces. An explicit description of the noncommutative space for the case of Penrose tilings can be found at [66, II. 3].

To simplify the picture slightly, we can consider the similar problem dually with arrangements of points of $\mathbb{R}^d$ instead of tilings. This is the formulation used in the theory of aperiodic solids and quasi-crystals (cf. [21] missing). Instead of those, we consider discrete subsets $\mathfrak{L}_j$ of points of $\mathbb{R}^d$. Such an $\mathfrak{L}$ is said to be a Delaunay set if there are radii $r > 0$ and $R > 0$ such that every open ball $U(x, r)$ of radius r meets $\mathfrak{L}$ at at most one point and every closed ball $B(x, R)$ of radius R meets $\mathfrak{L}$ at at least one point.

(Added). Namely, for any $x \in \mathbb{R}^d$, $U(x, r) \cap \mathfrak{L}$ is either a one point set or empty, and $B(x, r) \cap \mathfrak{L}$ always contains a point. ◀

Define the counting measure associated to the set $\mathfrak{L}$ as

$$\mu_{\mathfrak{L}}(f) = \sum_{x \in \mathfrak{L}} f(x), \quad f \in C_c(\mathbb{R}^d)$$

with f any continuous, real-valued function on $\mathbb{R}^d$ with compact support. Define the action of $\mathbb{R}^d$ by translations as

$$\mu_{\mathfrak{L}} \mapsto t_{-a}\mu_{\mathfrak{L}} = \mu_{\mathfrak{L}} \circ t_a, \quad a \in \mathbb{R}^d,$$

where t_a is the translation by a. Then take Ω as the orbit closure of the measures $\mu_{\mathfrak{L}}$ in the space $\mathcal{R}(\mathbb{R}^d)$ of Radon measures, with the weak-$*$ topology. Then obtained is the (topological) dynamical system (Ω, t), where t is the action of $\mathbb{R}^d$ by translations.

That dynamical system does determine a corresponding noncommutative space, which is described as the quotient of Ω by translations. Namely, the crossed product C^*-algebra

$$\mathfrak{A} = C(\Omega) \rtimes_t \mathbb{R}^d$$

do arise. In fact, also consider the groupoid with set of units the transversal

$$X = \{\omega \in \Omega \,|\, \text{The supp}(\omega) \text{ containis } 0\},$$

arrows of the form $(\omega, a) \in \Omega \times \mathbb{R}^d$, with source and range maps

$$s(\omega, a) = t_{-a}\omega, r(\omega, a) = \omega \text{ and } (\omega, a) \circ (t_{-a}\omega, b) = (\omega, a + b)$$

(cf. [21] missing). This defines a locally compact groupoid $G(\mathfrak{L}, X)$.

Then the grouppoid C^*-algebra $C^*(G(\mathfrak{L}, X))$ and $C(\Omega) \rtimes_t \mathbb{R}^d$ are Morita equivalent.

In the case where $\mathfrak{L}$ is a periodic arrangement points with a cocompact symmetry (or commutative lattice) group Γ in $\mathbb{R}^d$, the space Ω is an ordinary commutative space, which is topologically homeomorphic to a torus, so that $\Omega = \mathbb{R}^d/\Gamma$. The C^*-algebra $\mathfrak{A}$ in this case is isomorphic to $C(\Gamma^\wedge) \otimes \mathbb{K}$, where $\mathbb{K}$ is the C^*-algebra of compact operators on a Hilbert space and $\Gamma^\wedge \cong \mathbb{T}^d$ is the Pontrjagin dual of the abelian group $\Gamma \cong \mathbb{Z}^d$, which is obtained by taking the dual of $\mathbb{R}^d$ module the reciprocal lattice

$$\Gamma^\sharp = \{k \in \mathbb{R}^d \cong (\mathbb{R}^d)^* \mid \langle k, \gamma \rangle \in \exp(2\pi \mathbb{Z}) \subset \mathbb{T}, \gamma \in \Gamma\}$$

(corrected). Thus, in physical language, $\gamma^\wedge$ is identified with the Brillouin zone $B = \mathbb{R}^d/\Gamma^\sharp$ of the periodic crystal corresponding to $\mathfrak{L}$. In this periodic case, the transversal $X = \mathfrak{L}/\Gamma$ is a finite set of points. As well, the groupoid C^*-algebra $C^*(G(\mathfrak{L}, X))$ is isomorphic to $C(\Gamma^\wedge) \otimes M_k(\mathbb{C})$, where k is the cardinality of the transversal X. Thus, the periodic case falls back into the realm of commutative spaces, in the noncommutative geometry, while the aperiodic patterns give rive to truly noncommutative spaces, which are highly non-trivial and interesting.

Two of the richest sources of interesting tilings are the zellijs and the muqarnas, widely used in ancient architecture. Also, those patterns, collectively defined as arabesques, not only do exhibit highly nontrivial geometry, but they reflect the intricate interplay between philosophy, mathematics, and aesthetics (cf. [9] and [38] both missing). Some of the best studies on the zellijs and the muqarnas concentrate on 2-dimensional periodic patterns.

For instance, may find a quoted sentence in [9, p. 43] that:

"As Nature is based on rhythm, so the arabesue is thythmic in concept. It reflects movement marked by the regular recurrence of features, elements, phenomena; hence it has periodicity."

It seems from that viewpoint that only the theory of periodic tilings as commutative geometry should be relevant in that context. However, more recent studies (cf. [38], [43], [44], and [193] all missing) suggest that the design of zellijs and muqarnas is not limited to two-dimensional crystallo-graphic groups, but, especially during the Timurid period, it involves also aperiodic patterns with fivefold symmetry, analogous to those (non included) observed in quasicrystals. This is not an accident and is certainly due to the result of a highly developed geometric theory. Indeed, already in the historic textbook of Abu'l-Wafa' al-Buzjani (940-998) on geometric constructions ([235] missing), there is an explicit mention of meetings and discussions, where mathematicians are directly involved alongside artisans in the design of arabesque patterns.

The appearance of aperiodic tilings is documented in the anonymous Persian manuscript ([5] missing) titled as *"On interlocking similar and congruent figures"*, which dates back to the 11th-13th century. Some of these aperiodic aspects of zellijs and muqarnas are studied by Bulatov in a book ([38] missing), which also contains Vil'danova's Russian translation of the ancient Persian text.

For a more recent study of quasi-periodic tilings in Persian architecture, may find it.

Remark. (Added). Recall from [237] the following facts. **Zellige (zellij in Arabic)** is mosaic tile-work made from individually chiseled geometric tiles set into a plaster base. This form of Islamic art is one of the main characteristics of Moroccan architecture. It consists of geometrically patterned mosaics, used to ornament walls, ceilings, fountains, floors, pools, and tables. **Muqarnas** is a form of ornamented vaulting in Islamic architecture, the geometric subdivison of either a squinch, cupola, or corbel into a large number of miniature squinches, producing a sort of cellular structure, sometimes also called a honeycomb vault. It is used for domes, and especially half-domes in entrances, iwans, and apses, mostly in traditional Persian architecture. ◀

10 NC spaces from dynamical systems

Let us look at some examples of noncommutative geometry (NCG) spaces associated to a dynamical system on a discrete set. For instance, such a discrete dynamical system is given by a self-mapping of a Cantor set. Such noncommutative spaces have been (first) extensively studied in (a paper of GPS [118] or [222] (missing) for a survey) a series of papers, where C. Skau and his coworkers have obtained remarkable results on the classification of minimal actions of $\mathbb{Z}$ on Cantor sets using the K-theory of the associated (crossed product) C^*-algebras.

It is found recently (cf. [99], [100] (missing), [174, §8] (missing), and [176, §4]) that the mapping torus of such systems can be used to model the dual graph of the fibers at the archimedean primes of arithmetic surfaces, in Arakelov geometry, as in the particular case in which the dynamical system is given by a subshift of finite type, encoding the action of a Schottky group Γ in $SL_2(\mathbb{C})$ on its limit set Λ_Γ in $\mathbb{P}^1(\mathbb{C})$. In fact, the results of [99] are motivated by earlier results of Manin [170] that provide a geometric model for such dual graphs in terms of hyperbolic geometry and Schottky uniformizations.

Remark. Recall from [99] the following definitions. A **Fuchsian** group is a discrete subgroup of $PSL_2(\mathbb{R})$ the group of orientation preserving isometries of the hyperbolic plane $\mathbb{H}^2$. A **Kleinian** group is a discrete subgroup of $PSL_2(\mathbb{C})$ the group of orientation preserving isometries of the 3-dimensional real hyperbolic space $\mathbb{H}^3 = PSL_2(\mathbb{C})/SU(2)$. For $g \geq 1$, a **Schottky** group of rank g is a discrete subgroup of $PSL_2(\mathbb{C})$, which is purely loxodromic and isomorphic to a free group of rank g. Schottky groups are Kleinian as particular examples.

Let Ω_Γ the domain of discontinuity of Γ, defined as the complement of Λ_Γ in $\mathbb{P}^1(\mathbb{C})$. Let $X_\Gamma = \Gamma\backslash\Omega_\Gamma$, which is a Riemann surface of genus g. A **Schottky uniformization** of X_Γ is the covering $\Omega_\Gamma \to X_\Gamma$. ◀

More generally, given an alphabet with letters $\{l_1, \cdots, l_N\}$, we let S_A^+ the space of a subshift of finite type consist of all right-infinite, admissible sequences

$$S_A^+ \ni w = [a_k]_{k=0}^\infty = a_0 a_1 a_2 \cdots a_n \cdots$$

in the letters of the alphabet. Namely, each a_j is one of the letters $l_1, \cdots, l_N$, subject to an admissibility condition specified by an $N \times N$ matrix $A = (A_{ij})$ with entries in $\{0,1\}$, so that two letters l_i and l_j in the list can appear as consecutive digits as $a_k a_{k+1}$ in a word w if and only if the entry A_{ij} of the admissibility matrix A is equal to 1.

Similarly, define the space S_A as the set of doubly-infinite admissible sequences as

$$S_A \ni w = [a_k]_{k\in\mathbb{Z}} = \cdots a_{-m} \cdots a_{-2}a_{-1}a_0a_1a_2 \cdots a_n \cdots .$$

The sets S_A^+ and S_A have a natural choice of topology. The topology on S_A is generated by the sets as neighbourhoods of $x \in S_A$

$$W^s(x,k_0) = \{y \in S_A \mid x_k = y_k, k \geq k_0\} \quad \text{and}$$
$$W^u(x,k_0) = \{y \in S_A \mid x_k = y_k, k \leq k_0\}$$

for $k_0 \in \mathbb{Z}$. This induces the topology on S_A^+ by realizing it as a subset of S_A, for instance, by extending each sequence of S_A^+ to the left as a constant sequence. Then consider the action T on S_A by the two-sided shift and that on S_A^+ by the one-sided shift, both of which is defined by

$$(Tw)_k = (T[a_k])_k = a_{k+1}, \quad w = [a_k]_k \in S_A, S_A^+.$$

(Note that the action T on S_A^+ erases the first letter a_0 of a word w and then shifts, so that it looks like the adjoint of the unilateral shift on a Hilbert space.) Typically, the spaces S_A^+ and S_A are topologically Cantor sets. The one-sided shift T on S_A^+ is a continuous surjective map, while the T on S_A is a homeomorphism. (Note as well that the only shifting action on S_A^+ viewed in S_A by extending is different from that on elements of S_A^+ by erasing, shifting, and extending.)

Remark. (Added). By definition, if $y \in W^s(x,k_0)$ in S_A^+ with $k_0 \geq 0$, then $T^{k_0}(y) = T^{k_0}(x)$. If $y \in W^s(x,k_0)$ in S_A, then $T^s(y)$ (improperly) converges to $T^s(x)$ as $s \to +\infty$ (on compact subsets). Such a neighourhood $W^s(x,k_0)$ of x may be said to be a **stable** subset. On the other hand, if $y \in W^u(x,k_0)$ in S_A (and in $S_A^+ \subset S_A$), then $T^{-s}(y)$ (improperly) converges to $T^{-s}(x)$ as $s \to +\infty$ (on compact subsets). Such a neighourhood $W^u(x,k_0)$ of x may be said to be an **unstable** subset. ◀

Example 10.1. (Added). For instance, let $l_1 = a$ and $l_2 = b$ with $N = 2$. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in M_2(\{0,1\}).$$

Then

$$S_A^+ = \{w = aa\cdots\} \sqcup \{w = a\cdots abb\cdots\} \sqcup \{w = bb\cdots\}.$$

As well,

$$S_A = \{w = \cdots aa\cdots\} \sqcup \{w = \cdots aabb\cdots\} \sqcup \{w = \cdots bb\cdots\}.$$

But these sets are countable. Since

$$T(abb\cdots) = bb\cdots = T(bb\cdots).$$

Hence T on S_A^+ is not injective.

If let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(\{0,1\}).$$

Then the space S_A^+ consists of the words of the form

$$\begin{aligned} w &= a\cdots ab\cdots ba\cdots ab\cdots, \quad \text{or} \\ w &= b\cdots ba\cdots ab\cdots ba\cdots, \end{aligned}$$

where the last $\cdots$ of each sequence may continue constantly, or alternatively (finitely or infinitely). Therefore, S_A^+ is uncountable and is homeormorphic to the Cantor set as the inifinite product space $\Pi^\infty\{0,1\}$. ◀

For example, let $\Gamma = F_g$ be a free group of g generators $\{\gamma_1, \cdots, \gamma_g\}$. Take the set

$$\{\gamma_1, \cdots \gamma_g, \gamma_1^{-1}, \cdots, \gamma_g^{-1}\} \equiv \{l_1, \cdots, l_g, l_{g+1}, \cdots, l_{2g}\}$$

as an alphabet. Then consider the right-infinite or doubly-infinite words $w = [a_k]_k$ in these letters, without cancellations in the sense of being subject to the admissibility rule that $a_{k+1} \neq a_k^{-1}$. This implies a subshift of finite type, where the admissibility matrix $A = (A_{ij})$ becomes the symmetric $2g \times 2g$ matrix with $A_{ij} = 0$ for $|i-j| = g$ and $A_{ij} = 1$ otherwise.

(Added). For instance, and for convenience, let $g = 2$. Then

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

Any word in S_A^+ has the form

$$w = (\gamma_k^{\pm}\cdots\gamma_k^{\pm})(\gamma_{k'}^{\pm}\cdots\gamma_{k'}^{\pm})(\gamma_k^{\pm}\cdots\gamma_k^{\pm})(\gamma_{k'}^{\pm}\cdots\gamma_{k'}^{\pm})\cdots,$$

where $k \neq k'$ in $\{1,2\}$ and signs ± 1 take the same $(\cdots)$-wise. ◀

Suppose now that Γ is a Schottky group of genus g, i.e., a finitely generated, discrete subgroup of $SL_2(\mathbb{C})$, isomorphic to the free group F_g of g generators, where all non-trivial elements are hyperbolic. Then the points in S_A^+ defined as above parameterize points in the limit set Λ_Γ in $\mathbb{P}^1(\mathbb{C})$, that is the set of accumulation points of orbits of Γ. The points of S_A parametrize geodesics in the 3-dimensional real hyperbolic space $\mathbb{H}^3$ with ends at points on the limit set Λ_Γ.

(Added). Note that the projective line over $\mathbb{C}$ is defined to be

$$P^1(\mathbb{C}) = \{x \in M_2(\mathbb{C}) \,|\, x = x^*, x^2 = x, \mathrm{tr}(x) = 1\} \approx (\mathbb{C}^2 \setminus \{0\})/\sim,$$

where for $z, w \in \mathbb{C}^2$, $z \sim w$ if and only if $z = \lambda w$ for some nonzero $\lambda \in \mathbb{C}$. ◀

The dynamical system (S_A, T) as a typical example on an interesting class of DS is said to be a **Smale** space. It also means that the space S_A can be locally decomposed as the product of expanding and contracting directions for T. More precisely, the following properties are satisfied:

(i) For every point $x \in S_A$, there exist subsets $W^s(x)$ and $W^u(x)$ of S_A such that $W^s(x) \times W^u(x)$ is homeomorphic a neighbourhood of x;

(ii) The map T is contracting on $W^s(x)$ and expanding on $W^u(x)$, and $W^s(T(x))$ and $T(W^s(x))$ agree in some neighbourhood of x, and so do $W^u(T(x))$ and $T(W^u(x))$.

It follows from a construction of Ruelle [213] that different C^*-algebras can be associated to each of Smale spaces. Refer also to [202] and [203]. For Smale spaces like (S_A, T), there are four associated C^*-algebras as follows. The crossed product C^*-algebra of the dynamical system (S_A, T):

$$C(S_A) \rtimes_T \mathbb{Z}, \quad \text{and} \quad C^*(G^j) \rtimes_T \mathbb{Z} \quad j = s, u, a$$

the crossed products by the action of the shift T on the groupoid C^*-algebras $C^*(G^j)$ of the groupoid G^j for $j = s, u, a$ of the stable, unstable, and asymptotic equivalence relations on (S_A, T) respectively.

The first choice as $C(S_A) \rtimes_T \mathbb{Z}$ is closely related to the continuous dynamical system given by the mapping torus of the action T, while another as $C^*(G^u) \rtimes_T \mathbb{Z}$ is related to the quotient of S_A^+ by the action of T. As in the example of the Schottky group Γ, this corresponds to the action of Γ on its limit set.

Indeed, for the first, consider the suspension flow $\mathfrak{S}_T$ of the dynamical system (S_A, T). This is the mapping torus of (S_A, T), which is defined by

$$\mathfrak{S}_T = S_A \times [0,1]/\sim, \quad \text{where } (x, 0) \sim (T(x), 1).$$

(Added). Topologically, this space is said to be a solenoid, that is, a fiber bundle over the circle S^1 with fiber a Cantor set ([176, §4, 2]). ◀

The first cohomology group $H^1(\mathfrak{S}_T, \mathbb{Z})$ of $\mathfrak{S}_T$ is the ordered cohomology of the dynamical system (S_A, T), in the sense of [32] and [196] (the last missing). There is an identification of $H^1(\mathfrak{S}_T, \mathbb{Z})$ with the even (or zero) K-theory group of the crossed product C^*-algebra as

$$H^1(\mathfrak{S}_T, \mathbb{Z}) \cong K_0(C(S_A) \rtimes_T \mathbb{Z}).$$

This can be deduced from the Pimsner-Voiculescu six-term exact sequence for the K-theory groups of a C^*-algebra crossed product by $\mathbb{Z}$ ([199]), so that

$$\begin{array}{ccccc} K_0(C(S_T)) \cong C(S_T, \mathbb{Z}) & \xrightarrow{(1-T)_*} & K_0(C(S_T)) & \xrightarrow{i_*} & K_0(C(S_A) \rtimes_T \mathbb{Z}) \\ \partial \uparrow & & & & \downarrow \partial = 0 \\ K_1(C(S_A) \rtimes_T \mathbb{Z}) & \xleftarrow{i_* = 0} & K_1(C(S_A)) \cong 0 & \xleftarrow{(1-T)_*} & K_1(C(S_A)) \cong 0, \end{array}$$

where $i : C(S_A) \to C(S_A) \rtimes_T \mathbb{Z}$ is the inclusion map and i_* and $(1-T)_*$ are respectively induced by the maps i and $1 - T$. (Note that $(1-T)f = f - f \circ T$).

(Added). As in [176, §4, 2.1], since the space S_A is totally disconnected, then $K_1(C(S_A)) \cong 0$ and $K_0(C(S_A)) \cong C(S_A, \mathbb{Z})$ the group of locally constant, $\mathbb{Z}$-valued functions on S_A. It then follows from the diagram that

$$K_1(C(S_A) \rtimes_T \mathbb{Z}) \cong \mathrm{im}(\partial) = \ker((1-T)_*) \cong \mathbb{Z}, \quad (\mathrm{im}(\cdot) \text{ as image})$$
$$K_0(C(S_A) \rtimes_T \mathbb{Z}) \cong C(S, \mathbb{Z})/\mathrm{im}((1-T)_*) = \mathrm{coker}((1-T)_*). \quad \blacktriangleleft$$

That can be also obtained in terms of the Thom isomorphism (for K-theory groups of a C^*-algebra crossed product by $\mathbb{R}$) ([60], [62]).

(Added). As done in [176, §4, 2.1], the Thom isomorphism (as above for degree changing by $+1$ (mod 2)) and the μ-map (or the assembly map) (via the Chern character for the isomorphism below) imply that for $j = 0, 1$,

$$\mu : K^{j+1}(\mathfrak{S}_T) \cong H^{j+1}(\mathfrak{S}_T, \mathbb{Z}) \to K_j(C(S_A) \rtimes_T \mathbb{Z}).$$

Hence,

$$K_1(C(S_A) \rtimes_T \mathbb{Z}) \cong H^0(\mathfrak{S}_T) \cong \mathbb{Z} \quad \text{and} \quad K_0(C(S_A) \rtimes_T \mathbb{Z}) \cong H^1(\mathfrak{S}_T),$$

the last which can be identified with the Čech cohomology group, given by the homotopy set $[\mathfrak{S}_T, U(1)]$, by mapping $[f]$ to the homotopy class $[\exp(2\pi i t f(x))]$ for any class $[f] \in C(S, \mathbb{Z})/\mathrm{im}((1-T)_*)$. $\blacktriangleleft$

In fact, as discussed above, one of the fundamental construction of noncommutative geometry is given by that of homotopy quotients (cf. [62]). These are commutative spaces which provide, up to homotopy, geometric models for the corresponding noncommutative spaces. The noncommutative spaces in the case shown below appear as quotient spaces of foliations on the homotopy quotients with contractible leaves.

For the quotient space $S_A/\mathbb{Z}$ as the noncommutative space as the crossed product C^*-algebra $C(S_A) \rtimes_T \mathbb{Z}$, with $\mathbb{Z}$ acting as powers of the invertible two-sided shift T, the homotopy quotient is given by the mapping torus $\mathfrak{S}_T = S_A \times_{\mathbb{Z}} \mathbb{R}$. The noncommutative space $S_A/\mathbb{Z}$ can be identified with the quotient space of the natural foliation on $\mathfrak{S}_T$ whose generic leaves are contractible as a copy of $\mathbb{R}$.

Another noncommutative space associated to a subshift of finite type like T, (which, up to Morita equivalence, corresponds to another choice of the crossed product C^*-algebra of a Smale space, as mentioned above), is the Cuntz-Krieger C^*-algebra $\mathcal{O}_A$, where A is the admissibility matrix of the subshift of finite type (cf. [104], [103]).

(Detailed). A partial isometry is a bounded linear operator S on a Hilbert space H such that there is a closed subspace K of H, and S is an isometry on K, and S is zero on the orthogonal complement $K^\perp$ of K. Equivalently, either the adjoint S^* is a partial isometry, S^*S is a projection, SS^* is a projection, $S = SS^*S$, or $S^* = S^*SS^*$. $\blacktriangleleft$

The Cuntz-Krieger algebra $\mathcal{O}_A$ with $A = (a_{ij})$ an $N \times N$ matrix over $\{0, 1\}$, is defined to be the universal C^*-algebra generated by partial isometris $s_1, \cdots, s_N$

satisfying the relations

$$\sum_{j=1}^{N} s_j s_j^* = 1 \quad \text{and} \quad s_j^* s_j = \sum_{j=1}^{N} a_{ij} s_j s_j^*.$$

(It says that the range projections $s_j s_j^*$ sum to the whole space and the source (or domain or initial) projections $s_j^* s_j$ are decomposed by those range projections).

In the case of a Schottky group Γ in $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm 1\}$ of genus g, the Cuntz-Krieger algebra $\mathcal{O}_A$ (with $A = A_\Gamma$ associated to Γ) can be described in terms of the action of the free group Γ on its limit set Λ_Γ in $\mathbb{P}^1(\mathbb{C})$ (cf. [210], [225]). Then $\mathcal{O}_A$ can be regarded as a noncommutative space replacing the classical quotient Λ_Γ/Γ as

$$\mathcal{O}_A \cong C(\Lambda_\Gamma) \rtimes \Gamma.$$

The quotient space $\Lambda \times_\Gamma \mathbb{H}^3$ is precisely the homotopy quotient of Λ_Γ with respect to the action of Γ, with $E\Gamma = \mathbb{H}^3$ and the classifying space $B\Gamma = \mathbb{H}^3/\Gamma$. Moreover, $\mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold of infinite volume, which is topologically a handle-body of genus g. In this case it is also found that the noncommutative space Λ_Γ/Γ is the quotient space of a foliation on the above homotopy quotient with contractible leaves as $\mathbb{H}^3$.

(Added). The real 3-dimensional hyperbolic space $\mathbb{H}^3$ is defined to be

$$H^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_3 > 0\}$$

the upper-half space with the Riemann metric:

$$\begin{aligned} ds^2 &= (dx_1^2 + dx_2^2 + dx_3^2)(x_3^2)^{-1} = g_{ij} dx_i dx_j \\ &= \langle \left(\frac{1}{x_3^2} \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle \right)_{i,j=1}^{3} (dx_1, dx_2, dx_3)^t, (dx_1, dx_2, dx_3)^t \rangle, \end{aligned}$$

which involves the canonical inner products for the tangent and cotangent spaces of the tangent and cotangent bundles $T\mathbb{H}^3$ and $(T\mathbb{H}^3)^*$ respectively. As well, we have the following Lie group isomorphism:

$$PSL_2(\mathbb{C}) \cong \mathrm{Iso}(\mathbb{H}^3)$$

of isometric transformations of $\mathbb{H}^3$ preserving orientation. There is a bijective corresponding between the group of linear fractional transformations t on the Riemann sphere $\mathbb{C} \cup \{\infty\}$:

$$t(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, ad - bc \neq 0$$

with $t(\infty) = \frac{a}{c}$ and $t(-\frac{d}{c}) = \infty$ $(c \neq 0)$, which is isomorphic to $PSL_2(\mathbb{C})$, and that of orientation preserving, isometric transformations on $\mathbb{H}^3$, extended from linear fractional transformations on $\mathbb{C} \cup \{\infty\}$, where $\mathbb{C} \cup \{\infty\}$ is viewed as the boundary of the corresponding compactification of $\mathbb{H}^3$. ◀

11 NC spaces from string theory

Yang-Mills theory on noncommutative 2-tori $\mathfrak{A}_\theta$ or $\mathcal{A}_\theta$ is first formulated by using suitable notions of connections and curvature for noncommutative spaces (cf. [96]).

In fact, the analogues of connections and curvature of vector bundles are straightforward to be obtained as follows ([59]). A connection is just given by the associated covariant differentiation ∇ on the space of smooth sections. Thus, it is given by a pair of linear operators on the Schwartz space of rapidly decaying functions

$$\nabla_j : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}), \quad j = 1, 2$$

such that

$$\nabla_j(\xi b) = (\nabla_j \xi) b + \xi \delta_j(b), \quad \xi \in \mathcal{S}(\mathbb{R}), b \in \mathcal{A}_\theta.$$

As in the usual case, it is checked that the trace of the curvature

$$\Omega = \nabla_1 \nabla_2 - \nabla_2 \nabla_1$$

is independent of the choice of a connection.

Let us make the following choice for a connection:

$$(\nabla_1 \xi)(s) = -\frac{2\pi i}{\theta} s \xi(s) \quad \text{and} \quad (\nabla_2 \xi)(s) = \frac{d}{ds} \xi(s).$$

Note that, up to the correct powers of $2\pi i$, the total curvature of $\mathcal{S}(\mathbb{R})$ becomes an integer. In fact, the curvature Ω is constant as equal to $\frac{1}{\theta}$. so that the irrational number θ disappears in the total curvature, equal to $\theta\theta^{-1} = 1$. This integrality phenomenon, as that the paring of dimension and curvature, both of which are non-integral, yields an integer:

$$\langle \dim, \Omega \rangle \sim \theta \times \theta^{-1} = 1 \in \mathbb{Z},$$

is the basis for the development of a theory of characteristic classes for noncommutative spaces. In the general case, this requires the development of more sophisticated tools, since the analogues of the derivations δ_j used in the case of the noncommutaive 2-tori are not there in general. The general theory is obtained through cyclic homology, as developed in [61].

Consider the projective module $P_{p,q}$ over $\mathfrak{A}_\theta$ described above. Defiine an $\mathfrak{A}_\theta$-valued inner product $\langle \cdot, \cdot \rangle_{\mathfrak{A}_\theta}$ on $P_{p,q}$, as in [208]. which is used to show that $P_{p,q}$ is a projective module. Connections are required to be compatible with the metric, so that

$$\delta_j(\langle \xi, \eta \rangle_{\mathfrak{A}_\theta}) = \langle \nabla_j \xi, \eta \rangle_{\mathfrak{A}_\theta} + \langle \xi, \nabla_j \eta \rangle_{\mathfrak{A}_\theta}.$$

It is proved by [59] that such connections always exist. The curvature Ω has values in $E = \mathrm{End}_{\mathfrak{A}_\theta}(P_{p,q})$. An E-valued inner product on $P_{p,q}$ is given by

$$\langle \xi, \eta \rangle_E \zeta = \xi \langle \eta, \zeta \rangle_{\mathfrak{A}_\theta}.$$

Also, a canonical faithful trace τ_E is defined as

$$\tau_E(\langle \xi, \eta \rangle_E) = \tau(\langle \eta, \xi \rangle_{\mathfrak{A}_\theta}),$$

where τ is the trace on the C^*-algebra $\mathfrak{A}_\theta$, given above.

The Yang-Mills action is then defined as (in [96])

$$\tau(\langle \Omega, \Omega \rangle_E).$$

Sought are the minima of the Yang-Mills action among metric compatible connections ∇_j given above. The main result of [96] is that this recovers the classical moduli spaces of Yang-Mills connections on the ordinary 2-torus:

Theorem 11.1. *For a choice of a pair (p, q) of integers with $p + q\theta \geq 0$, the moduli space of Yang-Mills connections on the $\mathfrak{A}_\theta$-module $P_{p,q}$ is a classical space given by the symmetric product*

$$s^n(\mathbb{T}^2) = (\mathbb{T}^2)^n / \Sigma_n,$$

where Σ_n is the group of permutations in n elements, with $n = \gcd(p, q)$.

The fact that noncommutativity of space coordinates is relevant for gravity goes back to the analysis of S. Doplicher, K. Fredenhagen, and John Roberts [108], which is independent of string theory and produces in a natural manner the Moyal deformations of space-time, a compact Euclidean version of which is given by the noncommutative 2-tori. Since then, tremendous progress has been made in understanding quantum field theory on noncommutative spaces, thanks mainly to the breakthrough by H. Grosse and R. Wulkenhaar [121].

(Added). May recall the following from GW [122]. The renormalized ϕ^4-model corresponds to the classical action

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi * \partial^\mu \phi + \frac{\Omega^2}{2} x_\mu^\sim \phi * x^{\sim,\mu} \phi + \frac{\mu^2}{2} \phi * \phi + \frac{\lambda}{4!} *^4 \phi \right)(x)$$

with $x_\mu^\sim = 2(\theta^{-1})_{\mu\nu} x^\nu$ and $*^4\phi = \phi^4$. The appearance of the harmonic oscillator second term in the action is a result of the renormalization. ◄

The main aspects of string and D-brane theory that ivolve noncommutative geometry are the bound states of configurations of parallel D-branes (by E. Witten [239]), the matrix models for M-theory (by T. Banks, W. Fischler, S. H. Shenker, and L. Susskind [15]), and the strong coupling limit of string theory (by F. Ardalan and collaborators [8] and [6] both missing). It also plays an important role in the M-theory compactifications (by Connes-Douglas-Schwarz [75]). All these aspects are not discussed in details. Only mentioned is a couple of examples of noncommutative spaces arising from string and D-brane theory.

The noncommutative tori and the components of the Yang-Mills connections do appear in the classification of the BPS states in M-theory (by CDS [75]).

In the matrix formulation of M-theory, the basic equations to obtain two periodicity of the basic coordinates x_i turn out to be

$$u_i x_j u_i^{-1} = x_j + a\delta_i^j, \quad i = 1, 2,$$

where the u_i are unitary gauge transformations. The multiplicative commutator $u_1 u_2 u_1^{-1} u_2^{-1} = [u_1, u_2]$ is then central, and in the irreducible case its scalar value $\lambda = e^{2\pi i\theta}$ corresponds to the algebra of coordinates as the noncommutative 2-torus. The x_j are then the components of the Yang-Mills connections. The same picture is emerged from the other information about M-theory concerning its relation with eleven dimensional super-gravity and that string theory dualities can then be interpreted using Morita equivalence, relating the values of θ on an orbit of $SL_2(\mathbb{Z})$.

It is shown by Nekrasov and Schwarz [194] that Yang-Mills gauge theory on the noncommutative $\mathbb{R}^4$ gives a conceptual understanding of the nonzero B-field desingularization of the moduli space of instantons obtained by perturbing the ADHM equations.

(Added). May quote the following from [194]: The gauge theory on the world-volume of N coincident D-branes is a non-abelian gauge theory. In this theory the scalar fields X_i in the adjoint representation are the non-abelian generalizations of the trasverse coordinates of the branes. The compactification of Matrix theory on a torus $\mathbb{T}^d$ implies that certain constraints are imposed on the matrices X_i as that

$$X_i + 2\pi R_i \delta_{ij} = U_j X_i U_j^{-1}.$$

Also, ADHM stands for Atiyah-Drinfeld-Hitchin-Manin. ◀

Exhibited by Seiberg and Witten [216] is an unexpected relation between the standard gauge theory and the noncommutative one, and clarified is the limit in which the entire string dynamics is described by a gauge theory on a noncommutative space.

Techniques from noncommutative differential and Riemannian geometry, in the sense discussed above are applied to string theory (for instance, as done by F. Ardalan and collaborators [8] missing).

The role of noncommutative geometry in the context of T-duality is considered in an interesting recent work of Mathai and collaborators ([30], [31], and [182]).

(Added). May recall from [31] the following. Let $\pi : E \to M$ be a principal circle bundle, i.e., a circle bundle with a free circle action, with H-flux $[H] \in H^3(E, \mathbb{Z})$. Such bundles are classified by their first Chern class $c_1(E) \in H^2(M, \mathbb{Z})$. It is shown that the T-duality interchanges the fiberwise integral of the H-flux with the first Chearn class, so that the pair (E, H) and its T-dual $(E^\wedge, H^\wedge)$ are related as

$$c_1(E) = \int_{\mathbf{T}^\wedge} H^\wedge \quad \text{and} \quad c_1(E^\wedge) = \int_{\mathbf{T}} H,$$

which can be obtained from the Gysin sequence of the bundles E and $E^\wedge$. In addition, the isomorphisms between the twisted cohomologies and twisted K-theories of (E, H) and $(E^\wedge, H^\wedge)$ are constructed. ◀

Recently, in the context of the holographic description of type IIB string theory on the plane-wave background, obtained by M. M. Sheikh-Jabbari [218] an interesting class of noncommutative spaces from the quantization of Nambu d-brackets. The classical Nambu brackets defined as

$$\{f_1, \cdots, f_k\} = \sum \epsilon^{i_1 \cdots i_k} \frac{\partial f_1}{\partial x^{i_1}} \cdots \frac{\partial f_k}{\partial x^{i_k}}$$

for k real-valued functions with variables $x^1, \cdots, x^k$ is quantized in the even case to the expression in $2k$ operators as

$$\frac{1}{i^k}[F_1, \cdots, F_{2k}] = \sum \frac{1}{i^k (2k)!} \epsilon^{i_1 \cdots i_{2k}} F_{i_1} \cdots F_{i_{2k}}.$$

(The summations above are omitted in the text). This generalizes the quantization of the Poisson bracket defined as

$$\{f_1, f_2\} \mapsto \frac{-i}{\hbar}[F_1, F_2] = \frac{-i}{\hbar}(F_1 F_2 - F_2 F_1).$$

The odd case is more subtle and it involves an additional operator γ related to the chirality. Set

$$\frac{1}{i^k}[F_1, \cdots, F_{2k-1}, \gamma] = \sum \frac{1}{i^k (2k)!} \epsilon^{i_1 \cdots i_{2k}} F_{i_1} \cdots F_{i_{2k-1}} \gamma,$$

where γ is the chirality operator in $2k$ dimensions. For example, for $k = 2$,

$$[F_1, F_2, F_3, \gamma] = \frac{1}{24}\left(\{[F_1, F_2], [F_3, \gamma]\} - \{[F_1, F_3], [F_2, \gamma]\} + \{[F_2, F_3], [F_1, \gamma]\}\right),$$

where $\{S, T\} = ST + TS$.

If the ordinary d-dimensional sphere of radius r is described by the equation $\sum_{i=1}^{d+1}(x^i)^2 = r^2$, then the coordinates satisfy the equation

$$\{x^{i_1}, \cdots, x^{i_d}\} = r^{d-1} \epsilon^{i_1 \cdots i_{d+1}} x^{i_{d+1}}.$$

These equations are then replaced by their quantized version, using the quantization of the Nambu bracket and the introduction of a quantization parameter. This defines algebras generated by unitaries, subject to the relations given by the quantization of those equations. Matrix representations of these algebras correspond to certain fuzzy spheres. It would be interesting to study the general structure of these noncommutative spaces from the point of view of the steps introduced above.

(Added). There are five types of 10 dimensional super string theories involving fermions and bosons and their super symmetries, of type I, type IIA, type IIB, of normal mixed strings, and of abnormal mixed strings, without gravitons as gravity. Any (such) super string theory requires that the space-time is 10 dimensional. Any string may have Plank length 10^{-33} [cm].

The string theory of type I contains both open strings as (low or high dimensional) open intervals and closed strings such as circles and closed intervals. The other theories contain only closed strings. The string theory of type IIA has a symmetry as a space symmetry, and that of IIB has a chirality as without such a symmetry.

All the super string theories are unified by E. Witten in the 11 dimensional space-time. It is shown that both the 10 dimensional super string theory without branes and the 11 dimensional super gravity (SG) theory with branes and without strings are obtained as a sort of limits of an 11 dimensional theory, and the five types of super string theories and the 11 dim SG theory are transformed by the following dualities:

$$\begin{tikzcd}
\text{11 dim Super G} \arrow[r, "\text{S1 Duality}", "\cong"'] & \text{Type IIA} \arrow[r, "\text{T Dulality}", "\cong"'] & \text{Type IIB} \arrow[d, "\Omega\text{ Duality}", "\cong"'] \\
\text{Normal Mixed} \arrow[u, "\text{I Duality}", "\cong"'] & \text{Abnormal M} \arrow[l, "\text{T Duality}"', "\cong"] & \text{Type I} \arrow[l, "\text{S Duality}"', "\cong"]
\end{tikzcd}$$

This 11 dimensional super symmetric theory is the M-theory by Witten.

For example, a strength g of force in the typy I theory is transformed by S-dulity to $\frac{1}{g}$ in the abnormal mixed string theory. ◀

12 Groupoids and the index theorem

Since the construction of the C^*-algebras of foliations is based on the holonomy groupoid, groupoids have played a major role in noncommutative geometry. In fact, the original construction of matrix mechanics by Heisenberg mentioned above is exactly that of the convolution algebra of the groupoid of transitions imposed by experimental results. The convolution algebra of groupoids can be also defined in the context of von Neumann algebras and of C^*-algebras (cf. [58] and [205]). It is particularly simple and canonical in the context of smooth groupoids (cf. [66, II.5]). One virtue of the general construction is that it provides a geometric mental picture of complicated analytical constructions.

The prototype example is given by the tangent groupoid of a manifold (cf. [66, II.5]). It is obtained by blowing up the diagonal in the square $M \times M$ of a manifold M and is given as a set by

$$G_M = M \times M \times (0,1] \cup TM,$$

where TM is the total space of the tangent bundle of M, and a tangent vector $X \in T_x(M)$ appears as the limit of nearby triples (x_1, x_2, ε) provided that in any chart the ratios $(x_1 - x_2)\varepsilon^{-1}$ converge to X (possibly as $x_1 \to x$, $x_2 \to x$, and $\varepsilon \to 0$). When $\varepsilon \to 0$, the Heisenberg matrix law of compositon:

$$(x_1, x_2, \varepsilon) \circ (x_2, x_3, \varepsilon) = (x_1, x_3, \varepsilon)$$

converges to the addition of tangent vectors, so that G_M becomes a smooth groupoid. The functoriality of the construction of the convolution groupoid C^*-algebras $C^*(G)$ for smooth groupoids G, as a functor $G \mapsto C^*(G)$, is then enough to define the Atiyah-Singer analytic index of pseudo-differential operators. It is simply given by using the six-term diagram of K-theory groups for the short exact sequence of C^*-algebras, associated to the geometric sequence

$$M \times M \times (0,1] \to G_M \supset TM,$$

where TM is viewed as a closed subgroupoid of G_M. The corresponding short exact sequence of C^*-algebras can be written as

$$0 \to C_0((0,1]) \otimes \mathbb{K} \to C^*(G_M) \to C_0(T^*M) \to 0,$$

which is a geometric form of the extension of pseudo-differential operators. By construction, the C^*-algebra $C_0((0,1])$ is contractible and the same holds for the C^*-tensor product $C_0((0,1]) \otimes \mathbb{K}$ by the C^*-algebra $\mathbb{K}$ of compact operators. (The being contractible in general implies that its K-theory groups are zero, namely trivial). It then follows that the $*$-homomorphism $C^*(G_M) \to C_0(T^*M)$ as a restriction map induces isomorphisms in K-theory:

$$K_j(C^*(G_M)) \cong K_j(C_0(T^*M)), \quad j = 0, 1.$$

The analytic index by the evaluation map $C^*(G_M) \to \mathbb{K}$ (?) is also obtained as

$$K_0(C^*(G_M)) \to K_0(\mathbb{K}) \cong \mathbb{Z},$$

composed with the isomorphism above. (Possibly, may use the evaluation map as $C_0(T^*M) \to \mathbb{C}$ for a point (or a certain fiber), with $K_0(\mathbb{C}) \cong \mathbb{Z}$.)

As well, using the Thom isomorphism yields a geometric proof (cf. [66]) of the Atiyah-Singer index theorem, where the analyses by the functor $G \mapsto C^*(G)$ need to be done carefully.

This paradigm for a geometric setup of the index theorem has been successfully extended to the cases of manifolds with singularities (cf. [188] and [189] the last missing) and of manifolds with boundary ([1]).

13 Noncommutative Riemannian manifolds

A main property of the homotopy type of a compact oriented manifold is that the Poincaré duality holds not just in ordinary homology but also in K-homology. It fact, the Poincaré duality in ordinary homology is not sufficient to describe the homotopy type of manifolds (cf. [187]). It is proved by Sullivan [227] that for simply connected PL manifolds of dimension at least 5, ignoring 2-torsion, the same property in KO-homology does suffice and the Chern character of the KO-homology fundamental class carries all the rational information on the Pontrjagin classes.

For an ordinary manifold, the choice of the fundamental cycle in K-homology is a refinement of the choice of orientation of the manifold. In its simplest form, it is a choice of spin structure. The role of a spin structure is to allow for the construction of the corresponding Dirac operator, which gives a corresponding Fredholm representation of the algebra of smooth functions. The choice of a square root involed in the Dirac operator corresponds to a choice of K-orientation.

K-homology theory admits a simple definition in terms of Hilbert spaces and Fredholm (module) representations of algebras.

Definition 13.1. ([66, Definition 1]). Let $\mathcal{A}$ be an involutive algebra over $\mathbb{C}$. An **odd Fredholm module** (π, H, F) over $\mathcal{A}$ consists of

(1) a $*$-representation π of $\mathcal{A}$ as (bounded) operators on a Hilbert space H and (2) an operator F with $F = F^*$ and $F^2 = 1$ the identity map on H (or mod $\mathbb{K}(H)$) such that $[F, \pi(a)]$ for any $a \in \mathcal{A}$ is a compact operator on H.

An **even Fredholm module** (π, H, F, γ) over $\mathcal{A}$ is defined to be an odd Fredholm module (π, H, F) together with a $\mathbb{Z}_2$-grading γ with $\gamma = \gamma^*$ and $\gamma^2 = 1$ on H (so $\gamma = \gamma^{-1}$) such that

(a) $\gamma\pi(a) = \pi(a)\gamma$ for all $a \in \mathcal{A}$ and (b) $\gamma F = -F\gamma$. Equivalently, $\gamma\pi(a)\gamma^{-1} \equiv \mathrm{Ad}(\gamma)\pi(a) = \pi(a)$ and $\gamma F \gamma^{-1} = \mathrm{Ad}(\gamma)F = -F$. □

This definition is derived from the Atiyah definition ([11] missing) of abstruct elliptic operators, and agrees with the Kasparov definition [144] for the cycles in K-homology as the KK-theory $KK(\mathfrak{A}, \mathbb{C})$ as an extension theory of C^*-algebras by $\mathbb{K}$ (but it is a cohomology theory for C^*-algebras), when $\mathfrak{A}$ is a C^*-algebra (cf. [24]).

Remark. (Added). In (1), if π is assumed to be faithful, we may replace $\pi(a)$ with $a \in \mathcal{A}$. In (2), it says that F and $\pi(a)$ essentially commute or commute mod $\mathbb{K}(H)$.

For $\mathbb{Z}_2$-graded algebras as $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, the condition (a) becomes $\gamma\pi(a) = (-1)^{\deg(a)}\pi(a)\gamma$.

The conditions $F = F^*$ and $F^2 = 1$ may be replaced with $\pi(a)(F - F^*) \in \mathbb{K}(H)$ and $\pi(a)(F^2 - 1) \in \mathbb{K}(H)$. □

Lemma 13.2. (Added). *If (π, H, F, γ) is an even Fredholm module over $\mathcal{A}$, then the Hilbert space H is $\mathbb{Z}_2$-graded as $H = H_0 \oplus H_1$ with $H_0 = \frac{1}{2}(1+\gamma)H$ and $H_1 = \frac{1}{2}(1-\gamma)H$ and γ as the grading operator, and the $*$-algebra $\mathcal{B} = \mathbb{C}[\pi(\mathcal{A}), F]$ generated by $\pi(\mathcal{A})$ and F is $\mathbb{Z}_2$-graded as $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ with $\mathcal{B}_0 = \frac{1}{2}(1 + \mathrm{Ad}(\gamma))\mathcal{B}$ and $\mathcal{B}_1 = \frac{1}{2}(1 - \mathrm{Ad}(\gamma))\mathcal{B}$ and $\mathrm{Ad}(\gamma)$ as the grading operator, so that $\mathcal{B}_0$ contains $\pi(\mathcal{A})$ and 1 and so on, and $\mathcal{B}_1$ contains F, $\pi(\mathcal{A})F$, and so on.* ◀

Example 13.3. ([66, 288-289]). Let M be a smooth compact manifold (such as the n-dimensional torus $\mathbb{T}^n$) and $C(M)$ the C^*-algebra of all continuous complex-valued functions on M with the supremum norm. Let $E^{\pm}$ be smooth Hermitian complex vector bundles over M and $P : C^\infty(M, E^+) \to C^\infty(M, E^-)$ an elliptic pseudo-differential operator of order 0 on the spaces of smooth sections over M. There is an extension of P to a bounded operator as $P : L^2(M, E^+) \to L^2(M, E^-)$ on the Hilbert spaces of L^2-sections over M, because of being of order 0. There also exists a so-called parametrix $Q : L^2(M, E^-) \to L^2(M, E^+)$ for P such that both $PQ - 1$ and $QP - 1$ are compact operators on $L^2(M, E^{\mp})$, by being elliptic. It then follows that there is an even Fredholm module over $C(M)$ defined by

$$\pi(f) = \begin{pmatrix} \pi^+(f) & 0 \\ 0 & \pi^-(f) \end{pmatrix}, \quad F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

on the Hilbert space $H = L^2(M, E^+) \oplus L^2(M, E^-) \equiv H^+ \oplus H^-$, where $\pi^{\pm}(f)\xi = f\xi \equiv M_f \xi$ for any $\xi \in L^2(M, E^{\pm})$, as multiplication operators.

Proof. (Added). For any $f, g \in C(M)$ and $\xi \oplus \eta \in H$, we have

$$\pi(fg)(\xi \oplus \eta) = (fg\xi) \oplus (fg\eta) = \pi(f)\pi(g)(\xi \oplus \eta)$$

and $\pi(f^*)(\xi \oplus \eta) = (f^*\xi) \oplus (f^*\eta) = (\pi^+(f))^*\xi \oplus (\pi^-(f))^*\eta = \pi(f)^*(\xi \oplus \eta)$, where

$$\langle M_{f^*}\xi, \xi' \rangle = \int_M f^*\xi(\xi')^* = \langle \xi, M_f \xi' \rangle = \langle M_f^* \xi, \xi' \rangle.$$

The condition $F = F^*$ is equivalent to $P = Q^*$ and $Q = P^*$. We have

$$F^2 - (1 \oplus 1) = (QP - 1) \oplus (PQ - 1) \in \mathbb{K}(L^2(M, E^+)) \oplus \mathbb{K}(L^2(M, E^-)),$$

a diagonal sum in a 2×2 matrix, which is contained in $\mathbb{K}(H^+ \oplus H^-)$. Also,

$$\begin{aligned} [F, \pi(f)] &= F\pi(f) - \pi(f)F \\ &= \begin{pmatrix} 0 & Q\pi^-(f) - \pi^+(f)Q \\ P\pi^+(f) - \pi^-(f)P & 0 \end{pmatrix} \in \mathbb{K}(H) \end{aligned}$$

because $P\pi^+(f) - \pi^-(f)P \in \mathbb{K}(H^+, H^-)$ and $Q\pi^-(f) - \pi^+(f)Q \in \mathbb{K}(H^-, H^+)$ by M. F. Atiyah ([11] the lacking item). $\square$

Remark. (More details from Atiyah [12]). Let U be an open subset of $\mathbb{R}^n$. Let $p(x, y)$ be a smooth function on $U \times \mathbb{R}^n$ (called a **symbol of order** m) such that for every compact subset K of U and all multi-indices $\alpha = (\alpha_j)$, $\beta = (\beta_j) \in \mathbb{Z}^n$ with non-negative components, we assume that for some constant $C_{\alpha,\beta,K}$,

$$|D_x^\beta D_y^\alpha p(x, y)| \leq C_{\alpha,\beta,K}(1 + \|y\|)^{m-|\alpha|}, \quad x \in K, y \in \mathbb{R}^n,$$

where $|\alpha| = \sum_{j=1}^n \alpha_j$ and D_x^β, D_y^α are the partial derivatives as

$$D_y^\alpha = \left(-i\frac{\partial}{\partial y_1}\right)^{\alpha_1} \cdots \left(-i\frac{\partial}{\partial y_n}\right)^{\alpha_n}.$$

To each such $p(x, y)$, an associated linear operator P from $C_c^\infty(U)$ of smooth functions on U with compact supports to $C^\infty(U)$ of smooth functions on U (called a **pseudo-differential** operator on U (as a local chart)) is defined by

$$(Pf)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,y \rangle} p(x, y) \hat{f}(y) dy,$$

where $\langle x, y\rangle$ is the real inner product of $\mathbb{R}^n$ and $\hat{f}$ denotes the Fourier transform of f defined as

$$\hat{f}(y) = \int_{\mathbb{R}^n} e^{-i\langle y,t\rangle} f(t)dt.$$

Moreover, for any smooth manifold M, one can define extendedly a pseudo-differential operator $P : C_c^\infty(M) \to C^\infty(M)$ defined locally as above. Furthermore, for any smooth vector bundles E and F over M with dimensions k and l, one can define a pseudo-differential operator $P : C_c^\infty(M, E) \to C^\infty(M, F)$ defined locally as a $l \times k$ matrix $P = (P_{ij})$ with entries P_{ij} pseudo-differential operators on smooth functions with compact supports on local charts of M. ◀

Remark. (Added). As one of the fundamental formuae in Fourier analysis in one real variable in $\mathbb{R}$, we have

$$\frac{d^k}{dx^k} f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy}(iy)^k \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)e^{-ity}dt\right) dy$$

by the Fourier inverse formula. Therefore, a pseudo-differential operator is viewed as a natural generalizaion of differential operators written as the extended Fourier transform with finitely many variables in $\mathbb{R}^n$.

If $P = p(x, D) = \sum_{|\alpha|\le k} g_\alpha(x) D_y^\alpha$ as a differential operator of rank (or order) k with coefficients as C^∞-functions g_α, then it corresponds to $p(x, y) = \sum_{|\alpha|\le k} g_\alpha(x) y^\alpha$ with $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ and the homogeneous polynomial $p_k(x, y) = \sum_{|\alpha|=k} g_\alpha(x) y^\alpha$ is called the principal symbol for P, denoted as $\sigma(P)$. Then P has order k, because, for instance, there are constants $C_{0,0,K}$ and $C_{\alpha,0,K}$ such that

$$|p(x, y)| \le C_{0,0,K}(1 + \|y\|)^k \quad \text{and} \quad |D_y^\alpha p(x, y)| \le C_{0,0,K}(1 + \|y\|)^{k-|\alpha|}$$

for $x \in K$ and $y \in \mathbb{R}^n$.

Anyhow, if such a P has order 0, then $P = \sum_{|\alpha|\le k} g_\alpha(x)1 = M_g$ the multiplication operator with $g = \sum_{|\alpha|\le k} g_\alpha(x)$. If M_g is a Fredholm operator, then there is a parametrix for M_g, also called a pseudo-inverse for M_g, as in the case where M_g or g is invertible (see [192, Theorem 1.4.15]). ◀

Example 13.4. ([66, IV. 5]). Let Γ be a free group and T be a tree on which Γ acts freely and transitively. By definition, the tree T is a 1-dimensional simplicial complex which is connected and simply connected. Let T^j be the set of all j-simlices of T for $j = 0, 1$. Let $p \in T^0$ and define a bijection $\varphi : T^0 \setminus \{p\} \to T^1$ by $\varphi(q) = (p, q)$ the 1-simplex connecting p and q as its end points and contained in the line segment $[p, q]$ in T for $q \in T^0 \setminus \{p\}$. The bijection φ is almost invariant in the sense that for any $g \in \Gamma$, one has

$$\varphi(gq) = (p, gq) = g\varphi(q) = g(p, q) \equiv (p, gq)$$

(by definition) except for finitely many q. Let $H^+ = l^2(T^0)$ and $H^- = l^2(T^1) \oplus \mathbb{C}$. The action of Γ on T^0 and T^1 yields a $C_r^*(\Gamma)$-module structure on $l^2(T^j)$ for

$j = 0, 1$, and hence on $H^{\pm}$, where

$$a(\xi, \lambda) = (a\xi, 0), \quad \xi \in l^2(T^1), \lambda \in \mathbb{C}, a \in C_r^*(\Gamma).$$

Define a unitary operator $P : H^+ \to H^-$ by

$$P\delta_p = (0, 1) \quad \text{and} \quad P\delta_q = \delta_{\varphi(q)}, \quad q \in T^0 \setminus \{p\}.$$

Then an even Fredholm module over $\mathfrak{A} = C_r^*(\Gamma)$ as well as $\mathcal{A} = \mathbb{C}\Gamma$ is defined as

$$\pi(a) = \begin{pmatrix} \pi_+(a) & 0 \\ 0 & \pi_-(a) \end{pmatrix}, \quad F = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix}$$

on $H = H^+ \oplus H^-$, with $\pi_{\pm}(a) = M_a$ the multiplication operator.

Proof. (Added). It holds that $F = F^*$ and $F^2 = 1$ and that

$$\begin{aligned}[F, \pi(a)] &= F\pi(a) - \pi(a)F \\ &= \begin{pmatrix} 0 & P^*\pi_-(a) - \pi_+(a)P^* \\ P\pi_+(a) - \pi_-(a)P & 0 \end{pmatrix}.\end{aligned}$$

In particular, if $a = \delta_g = g$ for $g \in \Gamma$, then

$$\begin{aligned}&[P^*\pi_-(a) - \pi_+(a)P^*](\delta_{\varphi(q)}, \lambda) = P^*a(\delta_{\varphi(q)}, \lambda) - a(\delta_q + \lambda\delta_p) \\ &= P^*(\delta_{\varphi(gq)}, 0) - (\delta_{gq} + \lambda\delta_{gp}) = \delta_{gq} - (\delta_{gq} + \lambda\delta_{gp}) \\ &= -\lambda\delta_{gp}\end{aligned}$$

for any $q \in T^0 \setminus \{p\}$, so that the operator $P^*\pi_-(a) - \pi_+(a)P^*$ has rank 1 and hence, is compact. Similarly, if $a = \delta_g = g$ for $g \in \Gamma$, then

$$\begin{aligned}&[P\pi_+(a) - \pi_-(a)P](\lambda\delta_p + \mu\delta_q) = Pa(\lambda\delta_p + \mu\delta_q) - a(\mu\delta_{\varphi(q)}, \lambda) \\ &= P(\lambda\delta_{gp} + \mu\delta_{gq}) - (\mu\delta_{\varphi(gq)}, \lambda) = (\lambda\delta_{\varphi(gp)} + \mu\delta_{\varphi(gq)}) - (\mu\delta_{\varphi(gq)}, \lambda) \\ &= \lambda(\delta_{\varphi(gp)}, -1)\end{aligned}$$

for any $q \in T^0 \setminus \{p\}$, so that the operator $P\pi_+(a) - \pi_-(a)P$ has rank 1 and hence, is compact. ◄ □

Example 13.5. ([66, IV, 3.α]). With $S^1 \approx P^1(\mathbb{R}) \approx (\mathbb{R}^2 \setminus \{0\})/\sim$ as directions, consider the algebra $C(P^1(\mathbb{R}))$ of functions f, acting on the Hilbert space $L^2(\mathbb{R})$ as multiplication operators M_f as $(f\xi)(s) = f(s)\xi(s)$ for $f \in C(P^1(\mathbb{R}))$ and $\xi \in L^2(\mathbb{R})$, where $\mathbb{R}^+ \approx S^1$ defined by the Cayley transform

$$\mathbb{R} \ni s \mapsto c(s) = \frac{s - i}{s + i} \in S^1,$$

where $|s - i| = |s + i|$. If we assume that $c(s) = e^{i\theta}$ with $0 < \theta < 2\pi$, then $s = -\frac{1}{\tan\frac{\theta}{2}}$ ($\theta \neq \pi$). Define the Hilbert transform F as

$$(F\xi)(s) = \frac{1}{\pi i}\int \frac{\xi(t)}{s - t} dt.$$

This multiples by $+1$ the positive Fourier modes and by -1 the negative Fourier modes. For $f \in C(P^1(\mathbb{R}))$, we have that $[F, f]$ is of finite rank if and only if f is a rational function, denoted as $\frac{P(s)}{Q(s)}$ a fraction of polynomials. This is Kronecker's characterization of rational functions.

(Added). More precisely, any measurable bounded function $f \in L^\infty(\mathbb{R})$ defines a bounded operator M_f on $L^2(\mathbb{R})$. For the quantized calculus by a Fredholm module to be translation invariant, the operator F must commute with translations as T_u and hence be given by a convolution operator. It also requires that F does commute with dilations D_λ as $s \mapsto \lambda s$ with $\lambda > 0$. It then follows that the only nontrivial choice of F with $F^2 = 1$ is the Hilbert transform

$$(F\xi)(s) = \frac{1}{\pi i} \lim_{\varepsilon \to 0} \int_{|s-t|>\varepsilon>0} \frac{\xi(t)}{s-t} dt.$$

Note that

$$(F\xi)(s) = \frac{1}{\pi i}(\frac{1}{t} * \xi(t))(s) = \frac{-1}{\pi i}\langle \frac{1}{t}, \xi(s-t)\rangle,$$

where the $*$ means the convolution with respect to Cauchy principal value, and the right hand side means the bracket as a functional, so that it exists for ξ such as Schwarz functions as rapidly decreasing smooth functions on $\mathbb{R}$ with compact support and for their L^2-extensions. Also,

$$\begin{aligned} T_u(F\xi)(s) = (F\xi)(s-u) &= \frac{1}{\pi i} \lim_{\varepsilon \to 0} \int_{|s-u-t|>\varepsilon>0} \frac{\xi(t)}{s-(t+u)} dt \quad (t+u=v) \\ &= \frac{1}{\pi i} \lim_{\varepsilon \to 0} \int_{|s-v|>\varepsilon>0} \frac{\xi(v-u)}{s-v} dv = F(T_u\xi)(s), \end{aligned}$$

and hence $T_uF = FT_u$, Namely, the range of F is translation invariant. Moreover, similarly, it is invariant under the dilations as

$$\begin{aligned} D_\lambda(F\xi)(s) = (F\xi)(\lambda s) &= \frac{1}{\pi i} \lim_{\varepsilon \to 0} \int_{|\lambda s-t|>\varepsilon>0} \frac{\xi(t)}{\lambda s - t} dt \quad (\lambda^{-1}t = u) \\ &= \frac{1}{\pi i \lambda} \lim_{\varepsilon \to 0} \int_{|s-u|>\lambda^{-1}\varepsilon>0} \frac{\xi(\lambda u)}{s-u} \lambda du = F(D_\lambda \xi)(s). \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} \langle F\xi, \eta\rangle_2 = \int_{\mathbb{R}} (F\xi)(s)\overline{\eta(s)} ds &= \int_{\mathbb{R}} \left(\frac{1}{\pi i} \lim_{\varepsilon \to 0} \int_{|s-t|>\varepsilon>0} \frac{\xi(t)}{s-t} dt \right) \overline{\eta(s)} ds \\ &= \int_{\mathbb{R}} \xi(t) \left(\frac{1}{\pi i} \lim_{\varepsilon \to 0} \int_{|s-t|>\varepsilon>0} \frac{1}{s-t} \overline{\eta(s)} ds \right) dt \\ &= \int_{\mathbb{R}} \xi(t) \left(\overline{\frac{1}{\pi(-i)} \lim_{\varepsilon \to 0} \int_{|s-t|>\varepsilon>0} \frac{1}{s-t} \eta(s) ds} \right) dt \\ &= \int_{\mathbb{R}} \xi(t) \overline{(F\eta)(t)} dt = \langle \xi, F\eta\rangle_2, \end{aligned}$$

and thus $F^* = F$. Also,

$$\begin{aligned}
\langle F\xi, F\eta\rangle_2 &= \int_{\mathbb{R}} (F\xi)(s)\overline{(F\eta)(s)}ds \\
&= \int_{\mathbb{R}} \left(\frac{1}{\pi i}\lim_{\varepsilon\to 0}\int_{|s-t|>\varepsilon>0}\frac{\xi(t)}{s-t}dt\right)\overline{(F\eta)(s)}ds \\
&= \int_{\mathbb{R}} \xi(t)\left(\frac{1}{\pi i}\lim_{\varepsilon\to 0}\int_{|s-t|>\varepsilon>0}\frac{1}{s-t}\overline{\left\{\frac{1}{\pi i}\lim_{\varepsilon\to 0}\int_{|s-u|>\varepsilon>0}\frac{\eta(u)}{s-u}du\right\}}ds\right)dt \\
&= \int_{\mathbb{R}} \xi(t)\overline{\left(\frac{1}{\pi^2}\lim_{\varepsilon\to 0}\int_{|s-u|>\varepsilon>0}\eta(u)\left\{\lim_{\varepsilon\to 0}\int_{|s-t|>\varepsilon>0}\frac{1}{(s-t)(s-u)}ds\right\}du\right)}dt \\
&= \int_{\mathbb{R}} \xi(t)\overline{\eta(t)}dt = \langle \xi, \eta\rangle_2,
\end{aligned}$$

(but the last step seems to be nontrivial and is unchecked) and thus $F^2 = 1$ on the Hilbert space $L^2(\mathbb{R})$.

Compute the commutator as the quantum differential df of f as

$$\begin{aligned}
&[F, M_f]\xi(s) = F(f\xi)(s) - f(s)(F\xi)(s) \\
&= \frac{1}{\pi i}\lim_{\varepsilon\to 0}\int_{|s-t|>\varepsilon>0}\frac{f(t)\xi(t)}{s-t}dt - f(s)\frac{1}{\pi i}\lim_{\varepsilon\to 0}\int_{|s-t|>\varepsilon>0}\frac{\xi(t)}{s-t}dt \\
&= \frac{1}{\pi i}\lim_{\varepsilon\to 0}\int_{|s-t|>\varepsilon>0}\frac{(f(t)-f(s))\xi(t)}{s-t}dt.
\end{aligned}$$

By the Cayley transform, one can transport the above Fredholm module (undetermined) to the algebra of (rational or some) functions on S^1 as follows. Let $H = L^2(S^1)$, on which essentially bounded measurable functions of $L^\infty(S^1)$ act as multiplication operators. Let $F = 2P - 1$, where $1 = 1_H$ and P is the orthogonal projection onto the space

$$H^2(S^1) = \{\xi \in L^2(S^1) \,|\, \xi^\wedge(-n) = 0, n \in \mathbb{N}\}$$

where $\xi^\wedge \in l^2(\mathbb{Z}) = L^2(S^1)^\wedge$ is the Fourier transform of ξ.

Note that $F^* = 2P^* - 1 = F$ and

$$F^2 = (2P-1)^2 = 4P^2 - 2P - 2P + 1 = 1.$$

As well,

$$F = 2P - 1 = P - (1-P) = P \oplus -(1-P) = \begin{pmatrix} 1_{P(H)} & 0 \\ 0 & -1_{(1-P)(H)} \end{pmatrix}$$

on the direct sum $H = P(H) \oplus (1-P)(H)$.

For $f \in L^\infty(\mathbb{R})$ or $L^\infty(S^1)$, the quantum differential $df = [F, f]$ is of finite rank if and only if f is equal a.e. to a rational function with no pole on $\mathbb{R}$ or S^1.

For any interval I of S^1, denote by $I(f)$ the mean $\frac{1}{|I|}\int_I f(x)dx$ of a function $f(x)$ on I. For $a > 0$, the mean oscillation of f is defined by

$$M_a(f) = \sup_{|I|\le a} \frac{1}{|I|}\int_I |f(x) - I(f)|dx.$$

A function f is said to have bounded mean oscillation (BMO) if $M_a(f)$ for all $a > 0$ are bounded. This is true for $f \in L^\infty(S^1)$.

Proof. (Added). Because

$$\frac{1}{|I|}\int_I |f(x) - I(f)|dx \le \frac{1}{|I|}(\|f\|_\infty |I| + \|f\|_\infty |I|) = 2\|f\|_\infty.$$

□

A function f is said to have vanishing mean oscillation (VMO) if $M_a(f) \to 0$ as $a \to 0$. This is true for $f \in C(S^1)$.

Proof. (Added). Because, since S^1 is compact, f and $f(x) - I(f)$ are uniformly continuous on S^1. Hence, for any $\varepsilon > 0$, there is $a > 0$ such that if $|I| \le a$, then $|f(x) - I(f)| < \varepsilon$ for every $x \in I$, and thus, $M_a(f) \le \varepsilon$ as well as $M_{a'}(f) \le \varepsilon$ for $0 < a' < a$. □

For $f \in L^\infty(S^1)$, $[F, f]$ is a compact operator if and only if f has vanishing mean oscillation (VMO). ◀

(Returned). Besides the K-homology class, specified by a Fredholm module, also generalized to the noncommutative setting the infinitesimal line element ds of a Riemannian manifold. In ordinary Riemannian geometry, we deal with the ds^2 given by the usual local expression as $g_{\mu\nu}dx^\mu dx^\nu$.

(Added). In the special theory of relativity, for $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$,

$$\begin{aligned} ds^2 &= g_{ab}dx^a dx^b \equiv c^2dt^2 - dx^2 - dy^2 - dz^2 \\ &= \langle (1 \oplus -1 \oplus -1 \oplus -1)\,(cdt, dx, dy, dz)^t, (cdt, dx, dy, dz)^t \rangle \end{aligned}$$

with $a, b = 0, 1, 2, 3$ and t as time and c as light speed, to define the Minkowski space-time. ◀

However, in order to extend the notion of metric space to the noncommutative setting, it is more natural to deal with ds.

(Added). Recall from [66, VI] the following detailed facts about geometric spaces as manifolds with certain matrics.

Define the metric on a manifold M (such as the 2-dimensional torus $\mathbb{T}^2$ in $\mathbb{R}^3$) as

$$\begin{aligned} d(x, y) &= \inf\{\text{Length } l(\gamma) \text{ of paths } \gamma \text{ between } x \text{ and } y \text{ in } M\} \\ &\equiv \inf_{\gamma \subset M: x \leftrightarrow y} l(\gamma), \end{aligned}$$

where the length is computed as the integral of the square root of a quadratic form in the differential of the path γ:

$$\begin{aligned} l(\gamma) &= \int_x^y \|\gamma'(t)\| dt = \int_x^y \sqrt{\langle \gamma'(t), \gamma'(t) \rangle (t)} dt \\ &= \int_x^y \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt = \int_x^y \sqrt{\sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu} \end{aligned}$$

with $(g_{\mu\nu})_{\mu,\nu}$ a positive definite symmetric matrix.

Let $\mathcal{A}$ be an involutive algebra and (H, F) a Fredholm module over $\mathcal{A}$. To define a unit of length, we consider an operator of the form

$$\begin{aligned} B_G(x, x') &= \sum_{\mu,\nu=1}^{q} (dx^\mu)^* g_{\mu\nu} (dx^\nu) = \sum_{\mu,\nu=1}^{q} ([F, x^\mu])^* g_{\mu\nu} [F, x^\nu] \\ &= ([F, x^\mu])^*_\mu G([F, x^\nu])_\nu = [F, x]^* G[F, x'] \in \mathbb{B}(H) \\ &\text{with } [F, x] = ([F, x^\mu])^q_{\mu=1} = (dx^\mu)^q_{\mu=1}, \ [F, x'] = ([F, x^\nu])^q_{\nu=1} = (dx^\nu)^q_{\nu=1}, \end{aligned}$$

where $dx = [F, x]$ for any $x \in \mathcal{A}$, x^μ, x^ν are elements of $\mathcal{A}$, $x = (x^\mu), x' = (x^\nu) \in \mathcal{A}^q$, and $G = (g_{\mu\nu})^q_{\mu,\nu=1}$ is a positive matrix of the $q \times q$ matrix algebra $M_q(\mathcal{A})$ over $\mathcal{A}$. By construction, each value at (x, x) is a positive infinitesimal, that is, a positive compact operator on H, to viewed as ds^2 in Riemannian geometry.

Note as well that $B_G : \mathcal{A}^q \times \mathcal{A}^q \to \mathbb{K}(H)$ is a sesquilinear form with being conjuagate linear in the first variable and linear in the second.

Define the unit of length as the positive square root:

$$ds = \sqrt{B_G}.$$

We denote by ds the infinitesimal line element of a Riemannian manifold M. For ds^2, the usual local expression is $\sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu$. We may use the **Einstein** (reduced) summation as it to be:

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu &= \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu \\ &= \sum_\mu dx^\mu \sum_\nu g_{\mu\nu} dx^\nu = \langle (dx^\mu), (dx_\mu) \rangle \equiv dx^\mu dx_\mu, \\ (\text{or} \quad & x^2 = x \cdot x = \langle x, Gx \rangle = g_{\mu\nu} x_\mu x_\nu = x_\mu x^\mu) \end{aligned}$$

where the metric $G = (g_{\mu\nu})$ is given in the ordinary (and the super-symmetric) Minkowski space $M = \mathbb{R}^4$, (respectively) as the diagonal sum(s):

$$G = 1 \oplus -1 \oplus -1 \oplus -1 \quad (\text{and} \quad G = -1 \oplus 1 \oplus 1 \oplus 1),$$

so that $x \cdot y = x_0 y_0 - \sum_{j=1,2,3} x_j y_j$ (and $x \cdot y = -x_0 y_0 + \sum_{j=1,2,3} x_j y_j$) (cf. [214] and [183]).

The following table is added:

Table 2: Classification of vectors in the Minkowski space

	T: Timelike	L: Lightlike	S: Spacelike
Inner product	$x \cdot x > 0$:	$x \cdot x = 0$:	$x \cdot x < 0$
Time	$x_0 > 0$: future, $x_0 < 0$: past (past observed)	$x_0 > 0$: future, $x_0 = 0$: now, $x_0 < 0$ (past observable)	$x_0 > 0$: future, $x_0 = 0$, $x_0 < 0$: (to be done)

where we have the following respective decomposition into cones:

$$M = T \sqcup L \sqcup S, \quad \text{with } T = T_{+} \sqcup T_{-} \text{ and so on,}$$

and μ is used for time coordinate x_0 and ν is used for space coordinates x_1, x_2, x_3. ◀

The ds equally corresponds to the fermion propagator in physics, and to the inverse D^{-1} on the Dirac operator D.

In other words, a spin or spin^c structure makes it possible to extract the square root of ds^2, using the Dirac operator as a differential square root of a Laplacian.

This prescription recovers the usual geodesic distance on a Riemannian mainfold as follows.

Lemma 13.6. ([63]). *On a Riemannian spin manifold M, the geodesic distance $d(x,y)$ between two points $x, y \in M$ is computed by the formula:*

$$d(x,y) = \sup\{|f(x) - f(y)| \mid f \in \mathcal{A}, \|[D,f]\| \leq 1\},$$

where D is the Dirac operator, defined as $D = d(\cdot)(ds)^{-1} = \frac{d}{ds}$, and $\mathcal{A}$ is the algebra $C^{\infty}(M)$ of smooth functions on M.

Proof. This essentially follows from the fact that the quantity $\|[D,f]\|$ can be identified with the Lipschitz norm of the function f as

$$\|[D,f]\| = \operatorname{ess\,sup}_{x \in M} \|(\nabla f)_x\| = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x,y)}.$$

(where $[D,f]$ could be identified with $Df = M_{Df}$ in this commutative case). □

Note that points $x, y \in M$ (or a noncommutative space X) are replaced with corresponding pure states φ_x and φ_y on the C^*-algebra closure of an algebra $\mathcal{A}$ such that $f(x) = \varphi_x(f)$ and $f(y) = \varphi_y(f)$ for any $f \in \mathcal{A}$. It then follows that

$$d(x,y) = \sup\{|f(x) - f(y)| \mid f \in \mathcal{A}, \|\frac{df}{ds}\| \leq 1\},$$

with $\frac{df}{ds} = df(ds)^{-1} = [F,f](ds)^{-1}$ (or $= (ds)^{-1}df$), where we assume that $dB_G = [F, B_G] = 0$, that is, B_G commutes with F, similar to the Kähler condition. Define a self-adjoint operator $D = FB_G^{-\frac{1}{2}} = F(ds)^{-1} = (ds)^{-1}F$ (and as well $= F(\cdot)(ds)^{-1} = (ds)^{-1}(\cdot)F$), where we assume that B_G is non-singular, i.e., the kernel of B_G is zero. Then

$$\begin{aligned}\frac{df}{ds} &= [F,f](ds)^{-1} = (Ff - fF)(ds)^{-1} = Ff(ds)^{-1} - fD \\ &= (ds)^{-1}[F,f] = Df - (ds)^{-1}fF,\end{aligned}$$

which should be equal to $[D,f]$ (in the sense as above).

Note that if ds has dimension of a length l, then D had dimension $\frac{1}{l}$ and $d(x,y)$ also has dimension of a length.

On a Riemannian spin manifold M, the condition $\|[D,f]\| \leq 1$ is equivalent to the condition that f is a Lipschitz function with Lipschitz constant no more than 1.

Example 13.7. (Added). Consider the case of $M = [0,1]$ the closed interval (or any $[a,b]$ in the real line $\mathbb{R}$). If $f(x) = x \in C(M)$, then $|f(x) - f(y)| = |x-y|$ and $f'(x) = 1$ with $\|f'\| = 1$.

If $f \in C^1(M)$ the algebra of all continuously differentiable functions on M, which is a dense subalgebra of $C(M)$, and if $|f(x) - f(y)| \leq 1 \cdot |x-y|$ for any $x, y \in M$ with Lipschitz constant 1, then $|f'(x)| \leq 1$ for any $x \in M$. Hence $\|f'\| \leq 1$. Conversely, suppose to the contrary that there are $s, t \in M$ with $s < t$ such that $|f(s) - f(t)| > |s-t|$. Then the mean value theorem for differentiable functions with one variable in Calculus tells us that there is some $c \in M$ with $s < c < t$ such that $|f'(c)| > 1$, a contradiction to $\|f'\| \leq 1$.

Hence we obtain that for $x, y \in M = [0,1]$,

$$|x-y| = \sup\{|f(x) - f(y)| \mid f \in C^1(M), \|f'\| \leq 1\}$$

with $Df = f' = \frac{df}{dx}$.

Since $\frac{d}{dx}(f \cdot g) = \frac{df}{dx}g + f\frac{dg}{dx}$ for $f, g \in C^1(M)$ of continuously differentiable functions on M, we have

$$[D,f]g = [\frac{d}{dx}, f]g = [\frac{d}{dx}, M_f]g = \frac{df}{dx}g = M_{Df}g.$$

Hence $[D,f] = M_{Df}$ may be identified with $Df = \frac{df}{dx}$. Note as well that $C^1(M)$ as well as $C^\infty(M)$ are dense in $L^2(M)$ with respect to the 2-norm. ◀

The advantage of the above definition of the line element ds is that it is of a spectral, operator theoretic nature, and hence it extends to the noncommutative setting. The structure of combining the K-homology fundamental cycle with the spectral definition of the line element is the notion of spectral triple, given as

Definition 13.8. (cf. [68] and [89]). A (compact, initial) noncommutative geometry is a **spectral triple** $(\mathcal{A}, H, D)$, where $\mathcal{A}$ is a unital algebra represented as an algebra of bounded operators on a Hilbert space H, and D is an unbounded Dirac operator defined as the inverse of the line element as $D = \frac{1}{ds}$ or $ds = D^{-1}$, and with the following properties required:

(1) The additive commutator $[D,a]$ is bounded for any $a \in \mathfrak{A}^\infty$, where $\mathfrak{A}^\infty$ is a dense subalgebra of the C^*-algebra $\mathfrak{A}$ as the completion from $\mathcal{A}$.

(2) Self-adjointness $D = D^*$, and the compact resolventness that $(D - \lambda 1)^{-1}$ is a compact operator for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

A spectral triple $(\mathcal{A}, H, D)$ is **even** if the Hilbert space H has a $\mathbb{Z}_2$-grading by an operator γ such that $\gamma = \gamma^*$, $\gamma^2 = 1$ with $\gamma^{-1} = \gamma$, and that $\gamma D = -D\gamma$, and $\gamma a = a\gamma$ for any $a \in \mathcal{A}$, equivalently, $\mathrm{Ad}(\gamma)D = -D$ and $\mathrm{Ad}(\gamma)a = a$ as

$$\begin{tikzcd} H = H_0 \oplus H_1 \arrow[r, "D"] \arrow[d, "\gamma"'] & H_1 \oplus H_0 \arrow[d, "\gamma"] \\ H \arrow[r, "-D"] & H_1 \oplus H_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H = H_0 \oplus H_1 \arrow[r, "a"] \arrow[d, "\gamma"'] & H_0 \oplus H_1 \arrow[d, "\gamma"] \\ H \arrow[r, "a"] & H. \end{tikzcd}$$

Hence, on $H = H_0 \oplus H_1$, for some D_{12}, D_{21} and a_{11}, a_{22},

$$D = \begin{pmatrix} 0 & D_{12} \\ D_{21} & 0 \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}.$$

This definition is entirely spectral. The elements of the algebra are operators and the line element is also an operator.

The polar decomposition $D = F|D|$ gives rise the operator F as in an even Fredholm module (H, F) over $\mathcal{A}$, defining the fundamental class in K-homology.

(Added). Note that for a densely-defined, closable operator T on a Hilbert space H has the polar decomposition as $T = W|T|$ with $W \in \mathbb{B}(H)$ a unique partial isometry with $\ker(T) = \ker(W)$. ◀

The above formula for the geodedic distance is extended to the following context:

Definition 13.9. A state on a unital $*$-algebra $\mathcal{A}$ is a positive linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ such that $\varphi(1) = 1$ and $\varphi(a^*a) \geq 0$ for any $a \in \mathcal{A}$.

The **distance** between two states φ_1, φ_2 on $\mathcal{A}$ is given by the formula

$$d(\varphi_1, \varphi_2) = \sup\{|\varphi_1(a) - \varphi_2(a)| \mid a \in \mathcal{A}, \|[D,a]\| \leq 1\}.$$

(Added). For a positive operator $A \in \mathbb{B}(H)$, the trace of A is defined to be

$$\mathrm{tr}(A) = \sum_n \langle Ae_n, e_n \rangle,$$

where $\{e_n\}$ is any complete orthonomal basis for H.

(Added). Let $1 \leq p < \infty$. Let $\mathcal{L}^p(H)$ denote the Schatten-von Neumann ideal of $\mathbb{B}(H)$ of compact operators T with the p-norm $\|T\|_p$ such that the trace $\mathrm{tr}(|T|^p) = \|T\|_p^p < \infty$ or the p-summablility of the decreasing sequence $\{\mu_n(T)\}_{n=0}^\infty$ of countable, non-negative eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ with finite multiplicities (counted repeatedly and respectively) vanishing at infinity, i.e., $\sum_{n=0}^\infty \mu_n(T)^p < \infty$ with $\mu_n(T) \geq \mu_{n+1}(T) \geq 0$ and $\lim_{n\to\infty} \mu_n(T) = 0$, and that

$$\|T\|_p^p = \mathrm{tr}(|T|^p) = \sum_n \langle |T|^p e_n, e_n \rangle = \sum_{n=0}^\infty \mu_n(T)^p$$

for $T \in \mathbb{K}(H)$, because $|T|^p = \sum_k \mu_k(T)^p e'_k \otimes e'_k$ for some CONB $\{e'_k\}$.

Lemma 13.10. (Added, [133, Propositon 4.2.5]). *Let $1 \leq p < p' < \infty$ and $A, B \in \mathbb{B}(H)$. Then*

$$\|A\| \leq \|A\|_{p'} \leq \|A\|_p, \quad \mathcal{L}^p(H) \subset \mathcal{L}^{p'}(H) \subset \mathbb{K}(H),$$
$$\|A^*\|_p = \|A\|, \quad \|AB\|_p \leq \|B\|\|A\|_p, \quad \|BA\|_p \leq \|B\|\|A\|_p.$$

Let $1 \leq p < \infty$. A Fredholm module (π, H, F) over a $*$-algebra $\mathcal{A}$ is said to be **p-summable** if $[F, \pi(a)] \in \mathcal{L}^p(H)$ for any $a \in \mathcal{A}$, or if the $*$-subalgebra of $a \in \mathcal{A}$ with $[F, \pi(a)] \in \mathcal{L}^p(H)$ is dense in $\mathcal{A}$, or if the $*$-subalgebra of $\pi(a) \in \pi(\mathcal{A})$ with $[F, \pi(a)] \in \mathcal{L}^p(H)$ is dense in $\pi(\mathcal{A})$ (under the operator norm).

Proof. (Added). Note that if $[F, \pi(a)], [F, \pi(b)] \in \mathcal{L}^p(H)$ with $a, b \in \mathcal{A}$, then

$$[F, \pi(a+b)] = [F, \pi(a)] + [F, \pi(b)] \in \mathcal{L}^p(H),$$
$$[F, \pi(ab)] = [F, \pi(a)]\pi(b) + \pi(a)[F, \pi(b)] \in \mathcal{L}^p(H),$$
$$[F, \pi(a^*)] = -[F, \pi(a)]^* \in \mathcal{L}^p(H). \quad \blacktriangleleft$$

$\square$

Let p be a positive real number. A spectral triple $(\mathcal{A}, H, D)$ is of metric dimension p, or **p-summable**, if $|D|^{-1}$ is an infinitesimal of order p (of $\mathcal{L}^p$) (corrected), i.e., $|D|^{-p}$ is an infinitesimal of order 1 (of $\mathcal{L}^1$). Namely, $|D|^{-1} \in \mathcal{L}^p(H)$ (and thus $p \geq 1$).

(Added). Equivalently, $|D|^{-p} \in \mathcal{L}^1(H)$, so that

$$\mathrm{tr}(|D|^{-p}) = \mathrm{tr}((|D|^{-1})^p) < \infty.$$

(For this, we need to assume that the spectrum of $|D|$ is discrete and countable because $|D|^{-1}$ and $|D|^{-p}$ are compact operators).

(Added). By the min-max principle,

$$\mu_n(T) = \min\{\|T|_{E^\perp}\| \mid E \subset H, \dim E = n\},$$

where E is an n-dimensional subspace of H and $E^\perp$ is the orthogonal complement of E in H and $T|_{E^\perp}$ is the restriction of T to $E^\perp$. In fact, this minimum is attained by taking E to be the eigen-space corresponding to the first n eigenvalues $\mu_0(T), \cdots, \mu_{n-1}(T)$ of $|T|$.

Example 13.11. (Added). If $T = (t_{ij})_{i,j=1}^\infty$ is a diagonal $\infty \times \infty$ matrix operator on $l^2(\mathbb{N})$ with diagonal entries (t_{ii}) bounded, decreasing and vanishing at infinity, then T is bounded and compact, with $\mu_n(T) = t_{n+1,n+1}$ for $n \geq 0$. Let $(e_n)_{n=1}^\infty$ be the canonical orthonormal basis for $l^2(\mathbb{N})$ and let E_n be the subspace generated by $e_1, \cdots, e_n$. Then $\|T|_{E_n^\perp}\| = |t_{n+1,n+1}| = \mu_n(T)$. The principle says that the left hand side must be the minimum among $\|T|_E\|$ for subspaces $E \subset l^2(\mathbb{N})$ with $\dim E = n$.

Let $R_n(H)$ denote the set of finite rank operators on H with rank $\leq n$. Then $\mu_n(T)$ is equal to the distance:

$$\mu_n(T) = d(T, R_n) = \inf\{\|T - X\| \mid X \in R_n(H)\}.$$

Proof. (Added). Note that if $X = T|_E$ for E a subspace of H with $\dim E \leq n$, then $T - T|_E = T|_{E^\perp}$. □

It then follows that for any $T_1, T_2 \in \mathbb{K}(H)$,

$$|\mu_n(T_1) - \mu_n(T_2)| \leq \|T_1 - T_2\|.$$

Proof. (Added). Because for $X \in R_n$, we have

$$\mu_n(T_i) - \|T_j - X\| \leq \|T_i - X\| - \|T_j - X\| \leq \|T_1 - T_2\|$$

for $(i, j) = (1, 2)$ or $(2, 1)$, which implies the inequality above. □

The inclusion $R_n(H) + R_m(H) \subset R_{n+m}(H)$ implies

$$\mu_{n+m}(T_1 + T_2) \leq \mu_n(T_1) + \mu_m(T_2).$$

Proof. (Added). Because for $X \in R_n(H)$ and $X' \in R_m(X)$, we have

$$\mu_{n+m}(T_1 + T_2) \leq \|T_1 + T_2 - X_1 - X_2\| \leq \|T_1 - X_1\| + \|T_2 - X_2\|,$$

which implies the inequality above. □

Similarly,

$$\mu_{n+m}(T_1 T_2) \leq \mu_n(T_1)\mu_m(T_2).$$

Proof. (Added). Because for $X_1 \in R_n(H)$ and $X_2 \in R_m(H)$, we have

$$\|T_1 + X_1\|\|T_2 + X_2\| \geq \|T_1 T_2 + T_1 X_2 + X_1(T_2 + X_2)\| \geq \mu_{n+m}(T_1 T_2).$$

□

In particular, since $\mu_0(T_j) = \|T_j\|$, then

$$\mu_n(T_1 T_2) \leq \mu_n(T_1)\|T_2\| \quad \text{and} \quad \mu_n(T_1 T_2) \leq \|T_1\|\mu_n(T_2).$$

Definition 13.12. (Added, [159, Definition 6.1.1]). A compact operator $T \in \mathbb{K}(H)$ is said to be an infinitesimal of order $\alpha \in \mathbb{R}^+$ if $\mu_n(T) = O(\frac{1}{n^\alpha})$ $(n \to \infty)$, i.e., $\mu_n(T) \leq C\frac{1}{n^\alpha}$ for some constant C and for any $n \geq n_0$ for some $n_0 \geq 1$ (corrected).

It follows from the estimates above the definition above that

Lemma 13.13. (Edited and added). *If T_1 and T_2 are infinitesimals of order α_1 and α_2, then $T_1 T_2$ is an infinitesimal of order $\alpha_1 + \alpha_2$.*

The set of all infinitesimals of order α becomes a two-sided ideal of $\mathbb{B}(H)$ (but not closed).

Remark. An infinitesimal of order 1 may not be in $\mathcal{L}^1(H)$. Any infinitesimal T of order α higher than 1 is contained in $\mathcal{L}^1(H)$, because $\sum_{n=n_0}^{\infty} \mu_n(T) \leq C\sum_{n=n_0}^{\infty} \frac{1}{n^\alpha} < \infty$ with $\alpha > 1$. ◀

Let $J^{\frac{1}{2}}$ denote the (two-sided) ideal of $\mathbb{K}(H)$ of compact operators T on H such that $\mu_n(T) = O((\log n)^{-\frac{1}{2}})$ $(n \to \infty)$. Namely, if n large enough, then $|\mu_n(T)| \leq \frac{M}{\sqrt{\log n}}$ for some positive M. Equivalently, $n \leq e^{\frac{M^2}{|\mu_n(T)|^2}}$. Note as well that $T \in J^{\frac{1}{2}}$ if and only if the sequence $\{\sqrt{\log n} \cdot \mu_n(T)\}_{n=1}^{\infty}$ is bounded.

Proof. (Added). Note that for $T_1, T_2 \in J^{\frac{1}{2}}$,

$$\begin{aligned}\sqrt{\log(n+m)} \cdot \mu_{n+m}(T_1+T_2) &\leq \sqrt{\log(n+m)}(\mu_n(T_1)+\mu_m(T_2)) \\ &= \sqrt{\log n + \log(1+\frac{m}{n})} \cdot \mu_n(T_1) + \sqrt{\log m + \log(1+\frac{n}{m})} \cdot \mu_m(T_2).\end{aligned}$$

If $n = m \geq 2$, then

$$\sqrt{\log(2n)} \cdot \mu_{2n}(T_1+T_2) \leq \sqrt{2}(\sqrt{\log n} \cdot \mu_n(T_1) + \sqrt{\log n} \cdot \mu_n(T_2)).$$

If $m = n-1 \geq 3$, then

$$\sqrt{\log(2n-1)} \cdot \mu_{2n-1}(T_1+T_2) \leq \sqrt{2}(\sqrt{\log n} \cdot \mu_n(T_1) + \sqrt{\log(n-1)} \cdot \mu_{n-1}(T_2)).$$

□

(Added). A Fredholm module (π, H, F) over a $*$-algebra $\mathcal{A}$ is said to be **θ-summable** if $[F, \pi(a)] \in J^{\frac{1}{2}}$ for any $a \in \mathcal{A}$. It then follows that $\mathcal{A}$ is stable under holomorphic functional calculus, so that $\mathcal{A}$ and its C^*-algebra completion have the same K-theory ([66, 8.α, Lemma 3]). Moreover, there is a self-adjoint unbounded operator D on H such that $\mathrm{Sign}(D) = D|D|^{-1} = F$ (and thus $D = F|D|$), $[D, \pi(a)]$ is bounded for any $a \in \mathcal{A}$, and $\mathrm{Tr}(e^{-D^2}) < \infty$ ([66, 8.α, Theorem 4]).

(Returned). A spectral triple $(\mathcal{A}, H, D)$ is **θ-summable** if $\mathrm{tr}(e^{-tD^2}) < \infty$ for all $t > 0$, or if $\mathrm{tr}(e^{-D^2}) < \infty$.

(Added). Define the partial sums of the sequence $\{\mu_n(T)\}$ as

$$\sigma_n(T) = \sum_{k=0}^{n-1} \mu_k(T) \geq 0.$$

We have

$$\sigma_n(T) = \sup\{\mathrm{tr}(|TE|) = \|TE\|_1 \mid E \subset H, \dim E = n\},$$

where E is an n-dimensional subspace of H. The supremum is attained if we take E to be the eigen-space corresponding to the first n eigenvalues of $|T|$. It then follows that

$$\sigma_n(T_1+T_2) \leq \sigma_n(T_1) + \sigma_n(T_2).$$

Proof. (Added). Note that

$$\|(T_1 + T_2)E\|_1 \le \|T_1 E\|_1 + \|T_2 E\|_1 \le \sigma_n(T_1) + \sigma_n(T_2),$$

which implies the inequality above. □

As well,

$$\sigma_n(T_1T_2) \le \sigma_n(T_1)\|T_2\| \quad \text{and} \quad \sigma_n(T_1T_2) \le \|T_1\|\sigma_n(T_2).$$

Let

$$\mathcal{L}^{1,\infty}(H) = \{T \in \mathbb{K}(H) \mid \sigma_n(T) = O(\log n)(n \to \infty)\}.$$

By definition, this set consists of $T \in \mathbb{K}(H)$ such that the sequence

$$\{(\log n)^{-1}\sigma_n(T)\}_{n=2}^{\infty}$$

is bounded.

The natural norm on $\mathcal{L}^{1,\infty}(H)$ is given by

$$\|T\|_{1,\infty} = \sup_{n\ge 2} \frac{1}{\log n}\sigma_n(T) = \|\{(\log n)^{-1}\sigma_n(T)\}\|_\infty.$$

The normed space $\mathcal{L}^{1,\infty}(H)$ is a two-sided ideal of $\mathbb{B}(H)$.

Proof. (Added). Note that

$$\frac{1}{\log n}\sigma_n(T_1 + T_2) \le \frac{1}{\log n}\sigma_n(T_1) + \frac{1}{\log n}\sigma_n(T_2) \le \|T_1\|_{1,\infty} + \|T_2\|_{1,\infty}$$

and that for $T_1 \in \mathcal{L}^{1,\infty}(H)$ and $T_2 \in \mathbb{B}(H)$,

$$\frac{1}{\log n}\sigma_n(T_1T_2) \le \|T_1\|_{1,\infty}\|T_2\| \quad \text{and} \quad \frac{1}{\log n}\sigma_n(T_2T_1) \le \|T_2\|\|T_1\|_{1,\infty}.$$

□

Lemma 13.14. (Added). *The ideal $\mathcal{L}^1(H)$ of $\mathbb{B}(H)$ is contained in the ideal $\mathcal{L}^{1,\infty}(H)$.*

For $T \in \mathbb{B}(H)$, we have the norm estimate $\|T\|_{1,\infty} \le \frac{1}{\log 2}\|T\|_1$.

Proof. If $T \in \mathcal{L}^1(H)$, then $\|T\|_1 = \sum_{n=0}^{\infty} \mu_n(T)$ converges. Then for $n \ge 2$,

$$\frac{\sigma_n(T)}{\log n} \le \frac{\|T\|_1}{\log n} \le \frac{\|T\|_1}{\log 2}.$$

Thus, $\|T\|_{1,\infty} \le \frac{1}{\log 2}\|T\|_1$. □

Lemma 13.15. (Added). *The ideal of all infinitesimals of order 1 is contained in the ideal $\mathcal{L}^{1,\infty}(H)$.*

Proof. Note that if $\mu_n(T) \leq C\frac{1}{n}$ for some $C > 0$ and any $n \geq n_0$ for some $n_0 \geq 1$, then for any $n - 1 \geq n_0$,

$$\sigma_n(T) = \sum_{k=0}^{n-1} \mu_k(T) \leq \sum_{k=0}^{n_0-1} \mu_k(T) + C \sum_{k=n_0}^{n-1} \frac{1}{k}$$
$$\leq \sum_{k=0}^{n_0-1} \mu_k(T) + C(\frac{1}{n_0} + \int_{n_0}^{n-1} \frac{1}{x} dx)$$
$$\leq \sum_{k=0}^{n_0-1} \mu_k(T) + C(\frac{1}{n_0} - \log n_0 + \log n)$$

and hence $\frac{\sigma_n(T)}{\log n}$ is bounded by $C + \varepsilon$ as $n \to \infty$, for any $\varepsilon > 0$. □

Remark. As in [159, Section 6.2], one may define $\mathcal{L}^{1,\infty}(H)$ to be the ideal of all infinitesimals of order 1. The converse of the statement above does not hold ? Probably, it does not, but we could not find a suitable proof for that it does hold.

(Added). The **Macaev [Matsaev]** ideal $\mathcal{L}^{\infty,1}(H)$ of $\mathbb{B}(H)$ is defined to be

$$\mathcal{L}^{\infty,1}(H) = \{T \in \mathbb{K}(H) \mid \sum_{n\geq 1} \frac{1}{n}\mu_n(T) < \infty\}.$$

The predual of the ideal $\mathcal{L}^{\infty,1}(H)$ is the ideal $\mathcal{L}_0^{1,\infty}(H)$ of $\mathbb{B}(H)$:

$$\mathcal{L}_0^{1,\infty}(H) = \{T \in \mathbb{K}(H) \mid \sigma_n(T) = o(\log n)(n \to \infty)\}$$

under the pairing given as $(A, B) = \mathrm{tr}(AB)$.

For any bounded sequence $\alpha = (\alpha_n)_{n=1}^{\infty}$, the associated bounded function f_α on $\mathbb{R}_+^* = (0, \infty)$ is defined by

$$f_\alpha(x) = \alpha_n \text{ for } x \in (n-1, n] \text{ for } n \in \mathbb{N}.$$

(Added). The **Cesàro mean** for f a bounded function on $\mathbb{R}_+^*$ is defined by

$$m_C(f)(y) = \frac{1}{\log y} \int_1^y f(x) \frac{dx}{x}.$$

Note that the function $m_C(f)$ is bounded and continuous on $\mathbb{R}_+^*$.

The Cesáro mean satisfies the following scale invariance that for any bounded function f,

$$\lim_{\lambda\to\infty} |m_C(\theta_\mu(f))(\lambda) - m_C(f)(\lambda)| = 0,$$

where $\mu > 0$ and

$$\theta_\mu(f)(\lambda) = f(\lambda\mu) \text{ for } \lambda \in \mathbb{R}_+^*.$$

Proof. (Added but not completed). We then have

$$
\begin{aligned}
&|m_C(\theta_\mu(f))(\lambda) - m_C(f)(\lambda)| \\
&= |\frac{1}{\log\lambda}\int_1^\lambda f(\mu x)\frac{dx}{x} - \frac{1}{\log\lambda}\int_1^\lambda f(x)\frac{dx}{x}| \\
&\leq |\frac{1}{\log\lambda}\int_\mu^{\mu\lambda} f(s)\frac{ds}{s}| + \frac{1}{\log\lambda}\int_1^\lambda \|f\|\frac{dx}{x} \quad (\text{with } s=\mu x) \\
&\leq \frac{\|f\|}{\log\lambda}(\log(\mu\lambda) - \log\mu + \log\lambda) = 2\|f\|.
\end{aligned}
$$

(Possibly, the other estimate is needed. Or it may involve the limit with respect to μ.) Indeed, we also have

$$
\begin{aligned}
&|m_C(\theta_\mu(f))(\lambda) - m_C(f)(\lambda)| = |\frac{1}{\log\lambda}\int_1^\lambda (\theta_\mu(f)(x) - f(x)\frac{dx}{x}| \\
&\leq \|\theta_\mu(f) - f\|,
\end{aligned}
$$

which may go to zero as $\mu \to 1$ in a possible sense. □

Then

$$\theta_{\frac{1}{2}}(f_\alpha)(x) = f_\alpha(\frac{x}{2}) = f_{(\alpha_1,\alpha_1,\alpha_2,\alpha_2,\cdots)}(x) = f_{(\alpha_n,\alpha_n)}(x).$$

If φ is any positive linear form on $C^b(\mathbb{R}_+^*)$ such that $\varphi(f) = \lim_{x\to\infty} f(x)$ for any $f \in C^b(\mathbb{R}_+^*)$ convergent at infinity, then the composition $\omega = \varphi \circ m_C$ is a positive linear form on $l^\infty(\mathbb{R}_+^*)$ such that $(\varphi \circ m_C)(f) = \lim_{x\to\infty} f(x)$ for any $f \in l^\infty(\mathbb{R}_+^*)$ convergent at infinity and that $\omega(f_{(\alpha_n)}) = \omega(f_{(\alpha_1,\alpha_1,\alpha_2,\alpha_2,\cdots)}) = \omega(f_{(\alpha_n,\alpha_n)})$. Moreover, we may assume that $\varphi(1) = 1$ and φ is zero on the subspace $C_0(\mathbb{R}_+^*)$ of continuous functions on $\mathbb{R}_+^*$ vanishing at infinity.

Let $T \in \mathcal{L}^{1,\infty}(H)$ and $T \geq 0$. The Dixmier trace of T is defined to be

$$
\begin{aligned}
\mathrm{tr}_\omega(T) &= \lim_\omega \frac{1}{\log n}\sum_{k=0}^{n-1}\mu_k(T) \\
&= \omega(f_{\{\frac{1}{\log n}\sigma_n(T)\}_{n=2}^\infty}),
\end{aligned}
$$

which may as well be written as $\oint T$ for short as before.

Proposition 13.16. ([66, Proposition 3, Page 306]). (Added and edited).

(1) *The Dixmier trace extends by linearity to the ideal $\mathcal{L}^{1,\infty}(H)$ of $\mathbb{K}(H)$ as a positive functional.*

(2) *If $S \in \mathbb{B}(H)$ and $T \in \mathcal{L}^{1,\infty}(H)$, then $\mathrm{tr}_\omega(ST) = \mathrm{tr}_\omega(TS)$.*

(3) *The Dixmier trace is independent of the choice of an inner product on H and it depends only of the Hilbert space H as a topological vector space.*

(4) *The Dixmier trace vanishes on the ideal $\mathcal{L}_0^{1,\infty}(H)$, which is the $\|\cdot\|_{1,\infty}$-norm closure of the ideal of finite rank operators, and so does on the ideal $\mathcal{L}^1(H)$, contained in $\mathcal{L}_0^{1,\infty}(H)$.*

Proof. (Edited). For (1). If $T_1, T_2 \in \mathcal{L}^{1,\infty}(H)$ are positive, then

$$\mathrm{tr}_\omega(T_1 + T_2) = \mathrm{tr}_\omega(T_1) + \mathrm{tr}_\omega(T_2).$$

Because

$$\begin{aligned}&\mathrm{tr}_\omega(T_1 + T_2) = \omega(f_{\{\frac{1}{\log n}\sum_{k=0}^{n-1}\mu_k(T_1+T_2)\}})\\&\le \omega(f_{\{\frac{1}{\log n}\sum_{k=0}^{n-1}\mu_k(T_1)\}}) + \omega(f_{\{\frac{1}{\log n}\sum_{k=0}^{n-1}\mu_k(T_2)\}}) = \mathrm{tr}_\omega(T_1) + \mathrm{tr}_\omega(T_2).\end{aligned}$$

On the other hand, we have

$$\sigma_n(T_1) + \sigma_n(T_2) \le \sigma_{2n}(T_1 + T_2).$$

Hence

$$\frac{1}{\log n}\sigma_n(T_1) + \frac{1}{\log n}\sigma_n(T_2) \le \frac{\log(2n)}{\log(n)}\frac{1}{\log(2n)}\sigma_{2n}(T_1 + T_2)$$

with

$$\lim_{n\to\infty}\frac{\log(2n)}{\log(n)} = \lim_{n\to\infty}\frac{2}{2n}n = 1.$$

Therefore,

$$\begin{aligned}&\mathrm{tr}_\omega(T_1) + \mathrm{tr}_\omega(T_2) = \omega(f_{\{\frac{1}{\log n}\sigma_n(T_1)\}}) + \omega(f_{\{\frac{1}{\log n}\sigma_n(T_2)\}})\\&\le \omega(f_{\{\frac{\log 2n}{\log n}\frac{1}{\log 2n}\sigma_{2n}(T_1+T_2)\}})\\&= \omega(f_{\{(\frac{\log 2n}{\log n}-1)\frac{1}{\log 2n}\sigma_{2n}(T_1+T_2)\}}) + \omega(f_{\{\frac{1}{\log 2n}\sigma_{2n}(T_1+T_2)\}})\\&= 0 + \omega(f_{\{\frac{1}{\log 2n}\sigma_{2n}(T_1+T_2)\}})\\&= \omega(f_{\{\frac{1}{\log 2n}\sigma_{2n}(T_1+T_2), \frac{1}{\log 2n}\sigma_{2n}(T_1+T_2)\}})\\&= \omega(f_{\{\frac{1}{\log 2n}\sigma_{2n}(T_1+T_2) - \frac{1}{\log(2n-1)}\sigma_{2n-1}(T_1+T_2), 0\}}) + \mathrm{tr}_\omega(T_1 + T_2)\\&= 0 + \mathrm{tr}_\omega(T_1 + T_2),\end{aligned}$$

where note that

$$\begin{aligned}&\frac{1}{\log 2n}\sigma_{2n}(T_1 + T_2) - \frac{1}{\log(2n-1)}\sigma_{2n-1}(T_1 + T_2)\\&\le \frac{1}{\log 2n\log(2n-1)}(\log(2n)\sigma_{2n}(T_1 + T_2) - \log(2n)\sigma_{2n-1}(T_1 + T_2))\\&= \frac{1}{\log(2n-1)}\mu_{2n-1}(T_1 + T_2) \to 0 \quad (n \to \infty).\end{aligned}$$

For (2). Note that for $S \in \mathbb{B}(H)$ and $T \in \mathcal{L}^1(H)$ (such as finite rank operators),

$$\begin{aligned}&|\frac{1}{\log n}\sigma_n(ST) - \frac{1}{\log n}\sigma_n(TS)| \le \frac{1}{\log n}|\sigma_n(ST) - \sigma_n(TS)|\\&\le \frac{1}{\log 2}(|\sigma_n(ST) - \mathrm{tr}(ST)| + |\mathrm{tr}(TS) - \sigma_n(ST)|) \to 0 \quad (n \to \infty).\end{aligned}$$

Hence $\mathrm{tr}_\omega(ST - TS) = 0$. For $T \in \mathcal{L}^{1,\infty}(H)$, we may use the density of $\mathcal{L}^1(H)$ in $\mathcal{L}^{1,\infty}(H)$ under the norm $\|\cdot\|_{1,\infty}$ (if so ?).

For (3). If $S \in \mathbb{B}(H)$ is invertible (or unitary), then for any $T \in \mathcal{L}^{1,\infty}(H)$,

$$\mathrm{tr}_\omega(STS^{-1}) = \mathrm{tr}_\omega(S^{-1}ST) = \mathrm{tr}_\omega(T).$$

For (4). Note that if $T \in \mathcal{L}_0^{1,\infty}(H)$, then the sequence $(\frac{1}{\log n}\sigma_n(T))_{n\geq 2}$ belongs to $c_0(\mathbb{N}+2)$.

Also, if $T \in \mathcal{L}^1(H)$, then

$$\frac{1}{\log n}\sigma_n(T) \leq \frac{1}{\log n}\|T\|_1 \to 0$$

as $n \to \infty$. □

Remark. It is known that for $A \in \mathcal{L}^p(H)$ and $B \in \mathcal{L}^q(H)$ where $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. For $A \in \mathcal{L}^1(H)$ and $B \in \mathbb{B}(H)$, the same formula holds. See [133, Proposition 4.2.12]. ◀

(Returned). Spectral triples also provide a more refined notion of dimension besides the metric dimension summability. It is given by the dimension spectrum, which is not a number but a subset of the complex plane, defined below.

Assume that a spectral triple $(\mathcal{A}, H, D)$ satisfies the regularity condition as that for any $a \in \mathcal{A}^\infty \subset \mathcal{A}$, we have $a, [D, a] \in \cap_k \mathrm{dom}(\delta^k)$, where δ is the derivation defined by $\delta(x) = [|D|, x]$ for $x \in \mathrm{dom}(\delta)$ the domain. Let $\mathcal{B}$ denote the algebra generated by $\delta^k(a)$ and $\delta^k([D, a])$ for any $a \in \mathcal{A}^\infty$. Then the **dimension spectrum** of the triple $(\mathcal{A}, H.D)$ is defined to be the subset $\sigma \subset \mathbb{C}$ consisting of all the singularites of the analytic functions $\zeta_b(z)$ obtained by continuation of

$$\zeta_b(z) = \mathrm{tr}(b|D|^{-z}), \quad \mathrm{Re}(z) > p, b \in \mathcal{B}.$$

Example 13.17. Let M be a smooth compact Riemannian spin manifold. The corresponding spectral triple $(\mathcal{A}, H, D)$ is defined by $\mathcal{A} = C^\infty(M)$ the algebra of all smooth functions on M, H the Hilbert space of spinors:

$$\begin{aligned} H &= L^2(M, S) = L^2(M, \mathbb{C}^\pm) \\ &\cong L^2(M, \mathbb{C}^+) \oplus L^2(M, \mathbb{C}^-) \equiv H^+ \oplus H^-, \end{aligned}$$

with $\mathbb{C}^\pm \cong \mathbb{C} \oplus \mathbb{C}$ as spins $\pm$, and D the Dirac operator:

$$D = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \quad \text{on } H = H^+ \oplus H^- \text{ with } P = Q^*$$

as an unbounded differentiable operator. Then the spectral triple has the metric dimension equal to the dimension $\dim M$ and has the dimension spectrum equal to the set $\{0, 1, \cdots, \dim M\}$.

In the case of an ordinary Riemannian manifold M, it is interesting to check the meaning of the points in the dimension spectrum that are smaller than $n = \dim M$. These are dimensions in which the space manifests itself non-trivially with some interesting geometry.

At $n = \dim M$ of the dimension spectrum of the spectral triple, the volume form of the Riemannian metric is recovered by the equality (valid up to a normalization constant) (cf. [66])

$$\oint f|ds|^n = \int_M f\sqrt{g}d^n x,$$

where the integral in the left hand side is given by the Dixmier trace (cf. [107]), generalizing the Wodzicki residue of pseudo-differential operators (cf. [240]).

(Added). Recall from [66, IV 2.β Proposition 5] that the Wodzicki residue $\mathrm{Res}(T)$ for a pseudo-differential operator T of order $-n$ acting on the space of sections of a complex vector bundle E on an n-dimensional compact manifold M is defined to be

$$\mathrm{Res}(T) = \frac{1}{n(2\pi)^n}\int_{S^*M} \mathrm{tr}_H(\sigma(T))ds,$$

where $\sigma(T) = \sigma_{-n}(T)$ means the principal symbol for T, which is a homogeneous function of degree $-n$ on the cotangent bundle T^*M of M, and the integral above is independent of the choice using a metric on M, of the unit sphere bundle S^*M in T^*M with its induced volume element. Then

(1) The operator T on $H = L^2(M, E)$ of sections of E over M belongs to the ideal $\mathcal{L}^{1,\infty}(H)$.

(2) The Dixmier trace $\mathrm{tr}_\omega(T)$ is independent of ω, and that

$$\oint T = \mathrm{tr}_\omega(T) = \mathrm{Res}(T). \quad \blacktriangleleft$$

One can also has the integration $\int ds^k$ in any other dimension in the dimension spectrum (dim-sp), with $ds = D^{-1}$ the line element.

In the case of a Riemannian manifold, found are other important curvature expressions. For instance, if M is a manifold of dimension 4, by considering integration in dimension 2, found is the Einstein-Hilbert action. In fact, a direct computation implies the following (cf. [145], [143]):

Proposition 13.18. *Let M be a manifold of dimension $\dim M = 4$. Let $dv = \sqrt{g}d^4x$ denote the volume form, $ds = D^{-1}$ the length element as the inverse of the Dirac operator D, and r the scalar curvature. Then*

$$\oint ds^2 = \frac{-1}{48\pi^2}\int_M r dv.$$

In general, the scalar curvature of an n-dimensional manifold is obtained from the integral $\oint ds^{n-2}$.

More refined properties of manifolds carry over to the noncommutative case, such as the presense of a real structure, which makes it possible to distinguish between K-homology and KO-homology and the order one condition for the Dirac operator, as follows (cf. [69] (and [67] missing)). May refer to [120] as well.

Definition 13.19. A **real structure** on an n-dimensional spectral triple $(\mathcal{A}, H, D)$ is defined to be an anti-linear isometry $J : H \to H$ such that

$$\begin{aligned}
&J^2 = \varepsilon(n)1, \\
&JD = \varepsilon'(n)DJ \quad (\text{or} \quad [J, \pm D] = \varepsilon'(n)1), \\
&\text{and} \quad J\gamma = \varepsilon''(n)\gamma J \quad (\text{or} \quad [J, \pm\gamma] = \varepsilon''(n)1) \quad (\text{even case}),
\end{aligned}$$

where the signature functions $\varepsilon(n), \varepsilon'(n)$, and $\varepsilon''(n)$ are defined as in the table (corrected) below:

Table 3: Values of the signature functions

n mod 8	0	1	2	3	4	5	6	7
$\varepsilon = \varepsilon(n)$	1	1	-1	-1	-1	-1	1	1
$\varepsilon' = \varepsilon'(n)$	1	-1	1	1	1	-1	1	1
$\varepsilon'' = \varepsilon''(n)$	1	No	-1	No	1	No	-1	No

Moreover, the action π of $\mathcal{A}$ satisfies the commutation rule that $[a, b^\circ] = 0$ for any $a, b \in \mathcal{A}$ with $a = \pi(a)$, where $b^\circ = Jb^* J^{-1} = \pi^\circ(b)$ for any $b \in \mathcal{A}$, and the operator D satisfies that $[[D, a], b^\circ] = 0$ for any $a, b \in \mathcal{A}$.

The condition $[\mathcal{A}, \mathcal{A}^\circ] = \{0\}$ says that the action of $\mathcal{A}$ commutes with that π° of $\mathcal{A}^\circ$.

Note that the element $b^\circ = \pi^\circ(b)$ is viewed as a representation of the opposite algebra $\mathcal{A}^\circ$ of $\mathcal{A}$ with multiplication as its elements exchanged.

Proof. Note that for any $a, b \in \mathcal{A}$ and $\mathcal{A}^\circ$ with $ba \in \mathcal{A}$ equal to $ab \in \mathcal{A}^\circ$,

$$\pi^\circ(ab) = J\pi(ba)^* J^{-1} = J\pi(a)^*\pi(b)^* J^{-1} = \pi^\circ(a)\pi^\circ(b).$$

□

In ordinary Riemannian geometry, the anti-linear isometry as J is given by the charge conjugation operator acting on spinors. In the noncommutative case, this is replaced with the Tomita-Takesaki (TT) anti-linear conjugation operator (cf. [229]).

There are necessary and sufficient conditions known as that a spectral triple has the following items in order to come from an ordinary compact Riemannian spin manifold ([65] and [120]):

(1) An infinitesimal $ds = D^{-1}$ of order n of $\mathcal{L}^n(H)$, or of order $\frac{1}{n}$ in the sense that $D^{-n} \in \mathcal{L}^1(H)$ or $\in \mathcal{L}^{1,\infty}(H)$ but $\notin \mathcal{L}_0^{1,\infty}(H)$. (Classical dimension).

(2) A real structure as J above.

(3) The commutation relation: $[[D,a],b^0] = 0$ for any $a,b \in \mathcal{A}$,
which is $[[D,a],b] = 0$ for any $a,b \in \mathcal{A}$ commutative.

(4) The regularity hypothesis: $a, [D,a] \in \cap_k \mathrm{dom}(\delta^k)$ for all $a \in \mathcal{A}^\infty$.

(5) A Hochschild cycle $c \in Z_n(\mathcal{A}^\infty, \mathcal{A}^\infty \otimes \mathcal{A}^\circ)$ such that
$\pi(c) = \gamma$ in the even case and $\pi(c) = 1$ in the odd case,
where $\pi(a^0 \otimes \cdots \otimes a^n) = a^0[D,a^1]\cdots[D,a^n]$ on H. (Orientation).

(6) The space $H^\infty = \cap_k \mathrm{dom}(D^k)$ of smooth vectors a finitely generated, projective $\mathcal{A}$-module, endowed with an $\mathcal{A}$-valued inner product $\langle \xi, \eta \rangle$,
with $\langle a\xi, \eta\rangle = \oint a\langle \xi, \eta\rangle ds^n$. (Finiteness).

(7) The intersection form $K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to \mathbb{Z}$ obtained from
the Fredholm index of D with coefficients in $K_*(\mathcal{A} \otimes \mathcal{A}^0)$ invertible,
or nondefenerate. (Poincaré dualiry).

A noncommutative **spin geometry** is defined to be such a real spectral triple satisfying the conditions (1) to (7) above.

When $\mathcal{A} = C^\infty(M)$, the above conditions characterize the Dirac operator associated to both of a Riemannian structure and a spin structure on M (cf. [120]).

(Added). That the number n of order is even corresponds to that the spectral triple is even. If the algebra $\mathcal{A}$ and the Hilbert space H are finite dimensional, then the classical dimension of the geometry is zero. ◀

As formulated in [69], in the commutative case, we could drop the hypothesis that $\mathcal{A} = C^\infty(M)$ and use the orientation condition to construct an embedding of the spectrum of the algebra $\mathcal{A}$ as a submanifold of $\mathbb{R}^n$. There is a recent work by Lord, Rennie and Varilly ([206] and cf. [166] with the title changed), which gives promising results in this direction. Moreover, the conditions can be stated without any commutativity assumption on $\mathcal{A}$. For instance, they are satisfied by the isospectral deformations of [79], discussed later. Another significant noncommutative example is given by the standard model of elementary particles (cf. [69]), also discussed later.

Another example of spectral triple associated to a classical space, which is not classically a smooth manifold, is in the case of manifolds with singularities. In particular, consider the case of an isolated conical singularity as follows.

Example 13.20. (Edited). ([164] and also, cf. [163] missing). Let M be a manifold with an isolated conical singularity. The cone point $c \in M$ has the property that there is a neighbourhood U of c in M such that $U\backslash\{c\}$ has the form $(0,1]\times N$ with N a smooth compact manifold and with metric $g|_U = dr^2 + r^2 g_N$, where g_N is the metric on N.

Note that M looks like an attached disjoint union as:

$$M = \{c\} \sqcup_c ([0,1]\times N) \sqcup_N M_0,$$

where the point c is identified with $\{0\}\times N$ and $1\times N$ identified with N is attached to the boundary ∂M_0 of a submanifold M_0 of M.

A class of differentiable operators on manifolds M with isolated conical singularities is given by the elliptic operators of Fuchs type, action on sections of a bundle E over M. The restriction of such operators to $(0,1]\times N$ has the form

$$r^{-\nu}\sum_{k=0}^{d} a_k(r)(-r\partial_r)^k, \quad \text{for } \nu\in\mathbb{R}, a_k \in C^\infty([0,1].\mathrm{Dif}^{d-k}(N, E|_N),$$

which are elliptic with symbol $\sigma_M(D) = \sum_{k=0}^{d} a_k(0)z^k$ that is an elliptic family parameterized by the imaginary part of z. In particular, operators of Dirac type are elliptic of Fuchs type. For such an operator D being of first order and symmetric, it is shown that its self-adjoint extension has discrete spectrum, with $(n+1)$-summable resolvent, where $n = \dim M$ ([52], [36], [164]).

Let $\mathcal{A} = C_c^\infty(M\setminus\{c\})\oplus\mathbb{C}$ the unitization of the algebra of all smooth functions on $M\setminus\{c\}$ with compact support, which is identified with the algebra of all functions that are smooth on $M\setminus\{c\}$ and constant near the singularity.

The Hilbert space H on which D acts is chosen from a family of weighted Sobolev spaces. A weighted Sobolev space is roughly defined as that its smooth part is the standard Sobolev space and its cone part is defined locally by norms

$$\|f\|_{s,\gamma}^2 = \int_{\mathbb{R}_+^*\times\mathbb{R}^{m-1}} (1+(\log t)^2+\xi^2)^s |(r^{-\gamma+\frac{1}{2}}f)^\wedge(t,\xi)|^2 \frac{dt}{t} d\xi,$$

where $f^\wedge$ denotes the Fourier transform on the group $\mathbb{R}_+^*\times\mathbb{R}^{m-1}$. ◀

Remark. (Added). Recall from [183] that the Sobolev Hilbert space $W_2^l(X)$ on a space X for $l\in\mathbb{Z}$ non-negative is defined to be of functions $f(x)$ on X such that differentials $D^\alpha f\in L^2(X)$ in the weak sense for any $|\alpha|\le l$, with

$$(f,g)_2 = \int_X \sum_{0\le|\alpha|\le l} D^\alpha f(x)\overline{D^\alpha g(x)}dx$$

as the inner product. Note that a partial derivative is converted to the corresponding multication by the Fourier transform, preserving the inner product. ◀

Theorem 13.21. ([164]). *The $(\mathcal{A}, H, D)$ chosen above is a spectral triple.*

In particular, the zeta functions $\mathrm{tr}(a|D|^{-z})$ admit analytic continuation to the complement of the dimension spectrum in $\mathbb{C}$, where the dimension spectrum is of the form $\{\dim M - k \mid k\in\mathbb{N}\}$ with multiplicities ≤ 2.

The zeta functions are related to the heat kernel as

$$\mathrm{tr}(|D|^{-z}) = \frac{1}{\Gamma(\frac{z}{2})}\int_0^\infty t^{\frac{z}{2}-1}\mathrm{tr}(e^{-tD^2})dt,$$

which relys on the results of [52] and [164].

The case of $\mathrm{tr}(a|D|^{-z})$ is treated by splitting $a|D|^{-z}$ as a sum of a contribution from the smooth part and the other from the singularity.

The Chern character from the K-(cohomology) theory for the space M to the cohomology theory for M is applied to the spectral triple as

$$\mathrm{Ch} : K^*(M) \cong K_*(\mathcal{A}) \to H_*(M, \mathbb{C}) \cong PHC^*(\mathcal{A})$$

as the transformation from the K-(homology) theory for the algebra $\mathcal{A}$ to the periodict cyclic homology for $\mathcal{A}$. (Note that homology is assigned to homology, and that cohomology to cohomology.)

The cocycles φ_n in the (b, B)-bicomplex for the algebra $\mathcal{A}$ have been computed explicitly as ([164, Théorème 5.1])

$$\varphi_0(a+\lambda 1) = \int_M aA^\wedge(M) \wedge \mathrm{Ch}(E) + \lambda\,\mathrm{ind}(D_+), \quad \lambda \in \mathbb{C}, a \in C_c^\infty(M)$$

$$\varphi_n(a_0, \cdots, a_n) = \nu_n \int_M a_0 da_1 \wedge \cdots \wedge da_n \wedge A^\wedge(M) \wedge \mathrm{Ch}(E), \quad n \geq 1.$$

(Added). May recall the construction of the fundamental (b, B) bicomplex of entire cyclic cohomology for a unital Banach algebra A over $\mathbb{C}$, following [66, IV 7.α]. For any $n \in \mathbb{N}$, let $C^n = C^n(A, A^*)$ denote the space of continuous $(n+1)$-linear forms φ on A. For $n < 0$, set $C^n = \{0\}$. Define two differentials $b : C^n \to C^{n+1}$ and $B = C_p \circ B_0 : C^n \to C^{n-1} \to C^{n-1}$ as

$$\begin{aligned}
&(b\varphi)(a^0, \cdots, a^{n+1}) = \\
&\sum_{j=0}^{n}(-1)^j\varphi(a^0, \cdots, a^j a^{j+1}, \cdots, a^{n+1}) + (-1)^{n+1}\varphi(a^{n+1}a^0, a^1, \cdots, a^n), \\
&(B_0\varphi)(a^0, \cdots, a^{n-1}) = \\
&\varphi(1, a^0, \cdots, a^{n-1}) - (-1)^n\varphi(a^0, \cdots, a^{n-1}, 1), \quad \varphi \in C^n, \\
&(C_p\psi)(a^0, \cdots, a^{n-1}) = \\
&\sum_{j=0}^{n-1}(-1)^{(n-1)j}\psi(a^j, a^{j+1}, \cdots, a^{j-1}), \quad \psi \in C^{n-1}.
\end{aligned}$$

It then follows that $b^2 = B^2 = 0$ and $b \circ B = -B \circ b$, so that obtained is the bicomplex $(C^{n,m}, d_1, d_2)$, where $C^{n,m} = C^{n-m}$ for any $n, m \in \mathbb{Z}$, and

$$d_1 = (n-m+1)b : C^{n,m} \to C^{n+1,m} \quad \text{and} \quad d_2 = \frac{1}{n-m}B : C^{n,m} \to C^{n,m+1}.$$

Namely, the (b, B) bicomplex diagram with degree $n \pm 1$ is given by

$$\begin{array}{ccccc}
C^n & \xrightarrow{b} & C^{n+1} & \xrightarrow{b} & C^{n+2} \\
B\uparrow & & \uparrow B & & \uparrow B \\
C^{n+1} & \xrightarrow{b} & C^{n+2} & \xrightarrow{b} & C^{n+3} \\
B\uparrow & & \uparrow B & & \uparrow B \\
C^{n+2} & \xrightarrow{b} & C^{n+3} & \xrightarrow{b} & C^{n+4}
\end{array}$$

where the diagram above does not commute, but the diagram can be changed into the (d_1, d_2) bicomplex with bidegree $(n, m) + (1, 0), (0, 1)$ as:

$$\begin{array}{ccccc}
C^{n,m} & \xrightarrow[(n-m+1)b]{d_1} & C^{n+1,m} & \xrightarrow[(n-m+2)b]{d_1} & C^{n+2,m} \\
d_2 \uparrow \frac{B}{n-m+1} & & d_2 \uparrow \frac{B}{n-m+2} & & d_2 \uparrow \frac{B}{n-m+3} \\
C^{n,m-1} & \xrightarrow[(n-m+2)b]{d_1} & C^{n+1,m-1} & \xrightarrow[(n-m+3)b]{d_1} & C^{n+2,m-1} \\
d_2 \uparrow \frac{B}{n-m+2} & & d_2 \uparrow \frac{B}{n-m+3} & & d_2 \uparrow \frac{B}{n-m+4} \\
C^{n,m-2} & \xrightarrow[(n-m+3)b]{d_1} & C^{n+1,m-2} & \xrightarrow[(n-m+4)b]{d_1} & C^{n+2,m-2}
\end{array}$$

where the above diagram still does not commute.

Note from [66, III 1.α] that the complex $(C^n(A, A^*), b)$ is the Hochschild complex of A (with coefficients in A^*). There is the linear map from $C^n(A, A^*)$ to $C^n(A, A^*)$ defined by

$$C_p\varphi = \sum_{\gamma \in \Gamma_{n+1}} \varepsilon(\gamma)\varphi \circ \gamma, \quad \varphi \in C^n,$$

where Γ_{n+1} is the group of cyclic permutations of the set $\{0, 1, 2, \cdots, n\}$. Define $C^n_\lambda(A)$ to be the range of the map C_p as the subspace of $C^n(A)$. Then $(C^n_\lambda(A), b)$ is a subcomplex of the Hochschild complex. In particular, $HC^0(\mathcal{A}) = Z^0_\lambda(\mathcal{A})$ is the linear space of traces on $\mathcal{A}$.

The cyclic cohomology groups $HC^n(\mathcal{A})$ of an algebra $\mathcal{A}$ are defined to be the cohomology groups of the complex $(C^n_\lambda(\mathcal{A}), b)$.

As example, if $\mathcal{A} = \mathbb{C}$, then $HC^{2n}(\mathbb{C}) = \mathbb{C}$ and $HC^{2n-1}(\mathbb{C}) = 0$ for $n \geq 1$, while $H^n(\mathbb{C}) = 0$ for any $n \geq 1$.

Moreover, $HC^*(\mathbb{C}) = \oplus_{n\geq 0} HC^n(\mathbb{C})$ is a polynomial ring with one generator of degree two. Any $HC^*(\mathcal{A}) = \oplus_{n\geq 0} HC^n(\mathcal{A})$ is a module over the ring $HC^*(\mathbb{C})$.

Furthermore, the periodic cyclic cohomology for $\mathcal{A}$ is defined to be

$$PHC^*(\mathcal{A}) = HC^*(\mathcal{A}) \otimes_{HC^*(\mathbb{C})} \mathbb{C}.$$

Now define

$$C^{\mathrm{ev}} = C^{\mathrm{ev}}(A) = \Pi_{n\in \mathbb{N}} C^{2n} \quad \text{and} \quad C^{\mathrm{od}} = C^{\mathrm{od}}(A) = \Pi_{n\in \mathbb{N}} C^{2n+1}.$$

The boundary operator $\partial = d_1 + d_2$ maps C^{ev} to C^{od} and C^{od} to C^{ev} respectively.

An even cochain $(\varphi_{2n})_{n\in\mathbb{N}} \in C^{\mathrm{ev}}$ and an odd cochain $(\varphi_{2n+1})_{n\in\mathbb{N}} \in C^{\mathrm{od}}$ are said to be entire if infinity is the raduis of convergence of respectively

$$\sum_{n\in\mathbb{N}}^{\infty} \frac{1}{n!}\|\varphi_{2n}\|z^n \quad \text{and} \quad \sum_{n\in\mathbb{N}}^{\infty} \frac{1}{n!}\|\varphi_{2n+1}\|z^n,$$

where for $\varphi \in C^m$ for any $m \geq 0$, the norm of φ is given by the Banach space norm:

$$\|\varphi\| = \sup\{|\varphi(a^0, \cdots, a^m)| \mid \|a^j\| \leq 1, 0 \leq j \leq m\}.$$

It follows in particular that any entire even (or odd) cochain $(\varphi_{2n}) \in C^{\mathrm{ev}}$ defines an entire function $f_{(\varphi_{2n})}$ on the Banach space A by

$$f_{(\varphi_{2n})}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\varphi_{2n}(x, \cdots, x), \quad x \in A.$$

(Note that $(-1)^n$ may be replaced with 1).

Define

$$C^{\mathrm{ev}}_{\mathrm{en}} = C^{\mathrm{ev}}_{\mathrm{en}}(A) \text{ and } C^{\mathrm{od}}_{\mathrm{en}} = C^{\mathrm{od}}_{\mathrm{en}}(A)$$

to be the sets of entire even and odd cochains in C^{ev} and C^{od}, respectively.

The entire cyclic cohomology of a Banach algebra A is defined to be the cohomology of the short complex of entire evne and odd cochains on A:

$$C^{\mathrm{ev}}_{\mathrm{en}}(A) \xrightarrow[d_1+d_2]{\partial} C^{\mathrm{od}}_{\mathrm{en}}(A) \xrightarrow[d_1+d_2]{\partial} C^{\mathrm{ev}}_{\mathrm{en}}(A) \xrightarrow[d_1+d_2]{\partial} C^{\mathrm{od}}_{\mathrm{en}}(A).$$

Then define the entire cyclic cohomology groups $H^{\mathrm{ev}}_e(A)$ and $H^{\mathrm{od}}_e(A)$ to be the quotients $\ker(\partial)/\mathrm{im}(\partial)$ at $C^{\mathrm{ev}}_{\mathrm{en}}(A)$ and $C^{\mathrm{od}}_{\mathrm{en}}(A)$, respectively.

As example, if $A = \mathbb{C}$, then $H^{\mathrm{ev}}_e(\mathbb{C}) = \mathbb{C}$ and $H^{\mathrm{od}}_e(\mathbb{C}) = 0$ [66, IV 7.α], where an isomorphism is induced by sending

$$C^{\mathrm{ev}}_e(\mathbb{C}) \ni (\varphi_{2n}) \mapsto \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\varphi_{2n}(1, \cdots, 1).$$

There is an obvious map

$$PHC^*(\mathcal{A}) \to H^*_e(\mathcal{A}) = H^{\mathrm{ev}}_e(\mathcal{A}) \oplus H^{\mathrm{od}}_e(\mathcal{A}).$$

As another example, if $\mathcal{A} = \mathbb{C}[z, z^{-1}]$ the algebra of Laurent polynomials, then $H^{\mathrm{ev}}_e(\mathcal{A}) = \mathbb{C} = H^{\mathrm{od}}_e(\mathbb{C})$, where generatos are given by the cyclic cocycles τ_j [66, IV 7.ϵ]:

$$\tau_0(f) = \int f(z)dz \quad \text{and} \quad \tau_1(f^0, f^1) = \int f^0 df^1. \quad \blacktriangleleft$$

14 Spectral triples from Cantor sets

As an important class of C^*-algebras, consider direct limits of a sequence of finite dimensional C^*-algebras and embeddings. These C^*-algebras are said to be approximately finite dimensional (AFD) or simply AF-algebras.

For an AF C^*-algebra, its isomorphism class is determined by a diagram corresponding to its direct limit system, so called the Bratteli diagram [33] (cf. [39]). From the Bratteli diagram, it is possible to obtain the structure of the algebra, for instance, its ideal structure.

Example 14.1. (Edited). Let X be a Cantor set and $C(X)$ be the C^*-algebra of all continuous, complex-valued functions on X compact. Then $C(X)$ is a unital commutative AF-algebra.

(Added). Note that the Cantor set Y in the closed interval $[0,1] = I_0$ is defined to be the intersection $\cap_n I_n$ of decreasing unions I_n of closed intervals, obtained inductively by removing the middle open interval for each closed interval in I_{n-1}, divided into 3 intervals with the same length such as $I_1 = I_0 \setminus (\frac{1}{3}, \frac{2}{3})$. Viewing each closed interval of I_n as $\mathbb{C}$ we obtain the injective Bratteli diagram for $C(Y)$ as

$$I_0 = \mathbb{C} \xrightarrow{\ 2\ } I_1 = \mathbb{C} \oplus \mathbb{C} \xrightarrow{\ 2\ } I_2 = \oplus^{2^2}\mathbb{C} \xrightarrow{\ 2\ } \cdots \longrightarrow C(Y)$$

where each homomorphism is injective with multiplicity two at each direct summand.

Recall that a topological space X is said to be totally disconnected if every connected component of X consists of a single point of X. The Cantor set Y in $\mathbb{R}$ is homeomorphic to the product space $\Pi\{0,1\}$ with the product topology. In particular, for every point of Y, its open or closed neighbourhoods with more than the point are disconnected, so that its connected component is the point.

A Cantor set X is a totally disconnected, compact Hausdorff (metric) space. A Cantor set X is also the intersection of a family of decreasing coverings of disjoint closed sets. This family provides the injective Bratteli diagram for $C(X)$ as well.

Conversely, a unital commutative AF C^*-algebra A is spanned by its projections, since any finite dimensional commutative C^*-algebras are generated by mutually orthogonal projections. It then follows that the spectrum for A unital is a compact Cantor set. ◀

Example 14.2. Let E be a real Hilbert space and $T : E \to \mathbb{B}(H)$ a linear map sending f to T_f such that

$$\{T_f, T_g\} = T_f T_g + T_g T_f = 0 \quad \text{and} \quad \{T_f^*, T_g\} = \langle g, f \rangle 1.$$

Define the algebra $\mathcal{A}$ to be generated by all the operators T_f for $f \in E$. ◀

May refer to [106] as well.

As described in [66, IV 3.ε], one can construct the Hilbert space for a Cantor set X in $\mathbb{R}$ as follows. We may assume that X has no isolated points and is contained in the closed interval $[0,1]$ and $0,1 \in X$. Let $O = X^c$ be the complement of X in $[0,1]$. Then the open set O in $[0,1]$ is the disjoint union of a sequence of bounded intervals I_j (notation changed). Denote by $l_j = |I_j|$ the length of each interval I_j. We may assume that the lengths are ordered as $l_1 \geq l_2 \geq \cdots > 0$. We let $I_j = (x_j^-, x_j^+)$ for every j with $x_j^\pm$ as the boundary points of I_j. Denote by $V = V^+ \sqcup V^-$ the set of the boundary points $x_j^\pm$ as a disjoint union respectively.

Define the Hilbert space for a Cantor set X as

$$H = l^2(V) = l^2(V^+) \oplus l^2(V^-).$$

Since $V \subset X$ as a set, there is an action of $C(X)$ on H given by

$$(f \cdot \xi)(x) = f(x)\xi(x), \quad \text{for } f \in C(X), \xi \in H, x \in V \subset X.$$

Note that V is countable but X is not.

Define the closed subspace of the piecewise constant functions of H with respect to V as

$$K = \{\xi \in H \,|\, \xi(x_j^-) = \xi(x_j^+) \quad \text{for any } x_j^\pm \in V\},$$

and let p be the orthogonal projection from H to K. Define $F = 2p - 1 = p \oplus -(1-p) = 1_K \oplus -1_{K^\wedge}$ on $K \oplus K^\perp = H$ with $K^\perp$ the orthogonal complement of K in H (cf. [66]). Hence the operator F has K the eigenspace with eigenvalue $+1$ and $K^\perp$ the other with -1.

Denote by H_j the subspace of H corresponding to the coordinates $\xi(x_j^\pm)$ for $\xi \in H$. Then the restriction p_j of p to H_j is

$$p_j \begin{pmatrix} \xi(x_j^-) \\ \xi(x_j^+) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\xi(x_j^-) + \xi(x_j^+)) \\ \frac{1}{2}(\xi(x_j^-) + \xi(x_j^+)) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \xi(x_j^-) \\ \xi(x_j^+) \end{pmatrix}.$$

Thus the restriction F_j of F to H_j is

$$F_j = 2p_j - 1_{H_j} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1 \oslash 1.$$

For any $f \in C(X)$, we have

$$\begin{aligned} df &= [F, f] = [F, M_f] = \oplus_j [F_j, (M_f)_{H_j}] \\ &= \oplus_j [1 \oslash 1, f(x_j^-) \oplus f(x_j^+)] = \oplus_j \left(f(x_j^+) - f(x_j^-)\right)[-1 \oslash 1]\right). \end{aligned}$$

Proposition 14.3. (Edited). *The pair (H, F) is a Fredholm module over $C(X)$.*

The operator $df = [F, f]$ has characteristic values $\pm i(f(x_j^+) - f(x_j^-))$ with respect to the intervals $I_j = (x_j^-, x_j^+)$, with multiplicity one (corrected).

(Added). *If $f \in C(X)$ satisfies $f(x) = x = \mathrm{id}(x)$ on X, then the operator $df = [F, f]$ does have characteristic values $\pm i(x_j^+ - x_j^-) = \pm i l_j$ with $l_j = |I_j|$, or certainly does those l_j with multiplicity two, but up to modulus one.*

Proof. Since $F_j = F_j^*$ and $F_j^2 = 1_2 = 1 \oplus 1$, then $F = F^*$ and $F^2 = 1$ on H. If $f \in C(X)$, then f is uniformly continuous on X a compact space, i.e., for any $\varepsilon > 0$, there is $\delta > 0$ such that if $l_j < \delta$, then $|(f(x_j^+) - f(x_j^-)| < \varepsilon$. It then follows that $df = [F, f]$ is a compact operator on H.

Compute that for $\lambda, \mu \in \mathbb{C}$ with μ fixed,

$$\det(\lambda(1 \oplus 1) - \nu(-1 \oslash 1)) = \lambda^2 + \mu^2 = 0,$$

which implies $\lambda = \pm i\mu$. □

We compute $D = Fdf^{-1}$ or $df^{-1}F$ as the Dirac operator as

$$\begin{aligned} D = Fdf^{-1} &= \oplus_j \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (f(x_j^+) - f(x_j^-))^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \oplus_j (f(x_j^+) - f(x_j^-))^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \\ &= df^{-1}F = \oplus_j (f(x_j^+) - f(x_j^-))^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

(Compare with that in [83]).

Proposition 14.4. *Let X be a Cantor set in $\mathbb{R}$. Let $\mathcal{A} \subset C(X)$ be the dense subalgebra of locally constant functions on X. Then we have a spectral triple $(\mathcal{A}, H, D)$, with the multiplication action on $H = l^2(V) = l^2(X)$ and with D defined as above.*

The zeta function satisfies the equation

$$\operatorname{tr}(|D|^{-s}) = 2\zeta_L(s) \equiv 2\sum_k l_k^s,$$

where $\zeta_L(s)$ is the geometric zeta function of $L = \{l_k\}_{k\geq 1}$ for $s \in \mathbb{C}$ with the real part $\mathrm{Re}(s)$ *positive.*

These zeta functions are related to the theory of Dirichlet series and to other arithmetic zeta functions, and also to Ruelle's dynamical zeta functions (cf. M. L. Lapidus and M. van Frankenhuysen ([160] missing).

Example 14.5. (Edited). If X is the classical or original Cantor set such that $l_1 = \frac{1}{3} = \frac{2}{3} - \frac{1}{3}$, $l_2 = l_3 = \frac{1}{3^2}$, $\cdots$, and $l_{2^k} = \cdots = l_{2^{k+1}-1} = \frac{1}{3^{k+1}}$ with multiplicity 2^k, then the formula above says that

$$\operatorname{tr}(|D|^{-s}) = 2\zeta_L(s) = 2\sum_{k=0}^{\infty} 2^k \frac{1}{3^{(k+1)s}} = \frac{2}{3^s}\sum_{k=0}^{\infty}\left(\frac{2}{3^s}\right)^k = \frac{2\cdot 3^{-s}}{1 - 2\cdot 3^{-s}}.$$

Proof. (Added). Indeed, if $f(x) = x$, then $D = Fdf^{-1}$ satisfies

$$|D|^2 = D^*D = \oplus_j l_j^{-2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad |D|^{-s} = \oplus_j l_j^s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and that $|D|^2$ is unbounded and $|D|^{-s}$ is compact for $s \in \mathbb{R}$ positive or for $s \in \mathbb{C}$ with real part $\mathrm{Re}(s)$ positive. Moreover,

$$\mathrm{tr}(|D|^{-s}) = 2\sum_{j=1}^{\infty} l_j^s.$$

□

It then follows that the dimension spectrum of the spectral triple of the Cantor set has points off the real line. In fact, the set of poles of the right hand side is

$$\left\{\frac{\log 2}{\log 3} + i\frac{2\pi n}{\log 3} \,|\, n \in \mathbb{Z}\right\}.$$

Proof. (Added). Solve the equation $1 - 2 \cdot 3^{-s} = 0$ for $s \in \mathbb{R}$ positive by taking logarithm. Then the solution is $s = \frac{\log 2}{\log 3}$.

Solve the equation $1 - 2 \cdot 3^{-s} = 0$ for $s \in \mathbb{C}$ with $\mathrm{Rs}(s)$ positive by taking exponential logarithm. Then the solutions are given by

$$s = \mathrm{Re}(s) + i\mathrm{Im}(s) \text{ with } \mathrm{Re}(s) = \frac{\log 2}{\log 3} \text{ and } \mathrm{Im}(s) = (\log 3)^{-1} 2n\pi \text{ for } n \in \mathbb{Z}.$$

□

In this case the dimension spectrum lies on a vertical line and it intersects the real axis at the point $D = \log 2 \cdot (\log 3)^{-1} \in \mathbb{R}$, which is the Hausdorff dimension of the ternary Cantor set. The same also holds for other Cantor sets, as long as the self-similarity is given by a unique contraction as in the ternary case where the original interval is replaced by two intervals of lengths scaled by $\frac{1}{3}$. ◄

Remark. (Added). For a subset A of $\mathbb{R}^n$, its Hausdorff dimension is defined to be the infimum

$$\inf\{s > 0 \,|\, H^s(A) = 0\}, \quad \text{where } H^s(A) = \lim_{\delta \to 0} H^s_\delta(H)$$

with $\delta > 0$ and

$$H^s_\delta(A) = \inf\{\sum_j d(U_j)^s \,|\, \{U_j\} \text{ is a } \delta\text{-covering of } A\},$$

where a countable family of subsets $\{U_j\}$ of $\mathbb{R}^n$ is said to be a δ-covering of A if $A \subset \cup_j U_j$ and the diameters of U_j are upper bounded as

$$d(U_j) \equiv \sup\{\|x - y\| \,|\, x, y \in U_j\} \leq \delta. \quad ◄$$

If we consider slightly more complicated fractals in $\mathbb{R}$, where the self-similarity requires more than one scaling map, then the dimension spectrum may be correspondingly more complicated. This can be seen in the case of the Fibonacci Cantor set, for instance, as follows (cf. M. L. Lapidus and M. van Frankenhuysen [160] missing).

Example 14.6. (Edited and extended). The Fibonacci sequence (f_n) is defined by $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ with $f_0 = 0$ and $f_1 = 1$. The Fibonacci Cantor set X_f is defined to be the intersection $\cap_{n\geq 0} I_n$ of unions I_n of closed intervals obtained as that $I_0 = [0,4]$, $I_1 = I_0 \setminus (1,2) = [0,1] \sqcup \underline{[2,4]}$, $I_2 = I_1 \setminus (2+\frac{1}{2}, 3) = \underline{[0,1]} \sqcup [2, 2+\frac{1}{2}] \sqcup \underline{[3,4]}$,

$$\begin{aligned} I_3 &= I_2 \setminus \{(\frac{1}{2^2}, \frac{1}{2}) \sqcup (3 + \frac{1}{2^2}, 3 + \frac{1}{2})\} \\ &= [0, \frac{1}{2^2}] \sqcup \underline{[\frac{1}{2}, 1]} \sqcup \underline{[2, 2+\frac{1}{2}]} \sqcup [3, 3+\frac{1}{2^2}] \sqcup \underline{[3+\frac{1}{2}, 4]}, \end{aligned}$$

and inductively, I_n is defined to be the union of closed intervals obtained from I_{n-1} by removing f_n open intervals of lengths $l_n = \frac{1}{2^{n-1}}$ from closed intervals of I_{n-1}, where the closed intervals in I_{n-1} are chosen (and underlined) with respect to their lengths ordered, higher first, and each chosen closed interval is removed by the open interval at the part of the closed interval as ratios from $\frac{1}{4}$ to $\frac{1}{2}$.

One can associate to the Fibonacci Cantor set X_f the unital commutative AF C^*-algebra $A = C(X_f)$ as that viewing each closed interval of I_n as $\mathbb{C}$ implies that

$$I_0 = \mathbb{C} \xrightarrow{2} I_1 = \mathbb{C}^2 \xrightarrow{1,2} I_3 = \mathbb{C}^3 \xrightarrow{2,1,2} I_4 = \mathbb{C}^5 \longrightarrow \cdots \to C(X_f)$$

where each homomorphism is injective with multiplicity one or two at each direct summand.

We let V denote the set of all endpoints of closed intervals of I_j for $j \geq 0$. Define the Hilbert space for the Fibonacci Cantor set X_f as $H = l^2(V)$. One can define an action of $C(X_f)$ on H as multiplication operators.

Define the closed subspace K of H of functions on X_f that are constant at each pair of endpoints of closed intervals of I_j. The the Fredholm module (F, H) over $C(X_f)$ is associated to the orthogonal projection to K.

Define the Dirac operator as

$$D = Fdf^{-1} \text{ or } df^{-1}F \text{ with } df = [F, M_f].$$

Let $\mathcal{A} \subset C(X_f)$ be the dense involutivie subalgebra of locally constant functions on X_f. Then we obtain a spectral triple $(\mathcal{A}, H, D)$.

Moreover, the zeta function formula is obtained as

$$\begin{aligned} \mathrm{tr}(|D|^{-s}) &= 2\zeta_L(s) = 2\zeta_f(s) \\ &\equiv 2\sum_{n=0}^{\infty} f_{n+1} 2^{-ns} = \frac{2^{-s}}{1 - 2^{-s} - 4^{-s}} \end{aligned}$$

(corrected) for $s \in \mathbb{Z}$ with real part positive, where $\zeta_f(s)$ is the geometric zeta function associated with the Fibonacci numbers f_n.

Proof. (Added). The proof for the first equality is essentially the same as that for the corresponding equality in the previous example. For the second,

$$\zeta_L(s) = \sum_{k=1}^{\infty} l_k^s = 1 + 2^{-s} + (2^{-2s} + 2^{-2s}) + \cdots$$
$$= f_1 + f_2 2^{-s} + f_3 2^{-2s} + \cdots = \sum_{n=0}^{\infty} f_{n+1} 2^{-ns}$$

For the third, it is well known in number theory that

$$f_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\}, \quad n \geq 0.$$

It is proved by induction as in [227]. Note as well that the equation $x^2 - x - 1 = 0$ implies the solutions $\alpha = \frac{1-\sqrt{5}}{2}$ and $\beta = \frac{1+\sqrt{5}}{2}$. Then $f_n = f_{n-1} + f_{n-2} = (\alpha + \beta) f_{n-1} - \alpha\beta f_{n-2}$, which is converted to $f_n - \alpha f_{n-1} = \beta(f_{n-1} - \alpha f_{n-2})$ and to $f_n - \beta f_{n-1} = \alpha(f_{n-1} - \beta f_{n-2})$.

It then implies that

$$\sum_{n=0}^{\infty} f_{n+1} 2^{-ns} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right\} 2^{-ns}$$
$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \left\{ \frac{1+\sqrt{5}}{2} \left(\frac{1+\sqrt{5}}{2^{s+1}} \right)^n - \frac{1-\sqrt{5}}{2} \left(\frac{1-\sqrt{5}}{2^{s+1}} \right)^n \right\}$$
$$= \frac{2^{s+1}}{(2^{s+1} - 1 - \sqrt{5})(2^{s+1} - 1 + \sqrt{5})} = \frac{2^{s+1}}{(2^{s+1} - 1)^2 - 5}$$
$$= \frac{2^{s+1}}{4(2^{2s} - 2^s - 1)} = \frac{2^{-s-1}}{1 - 2^{-s} - 2^{-2s}}.$$

□

The dimension spectrum for the spectral triple is given by the set

$$\left\{ \frac{\log \alpha}{\log 2} + i \frac{2\pi n}{\log 2} \,|\, n \in \mathbb{Z} \right\} \cup \left\{ \frac{\log \beta}{\log 2} + i \frac{2\pi n}{\log 2} \,|\, n \in \mathbb{Z} \right\}$$

(corrected) where $\alpha = \frac{1-\sqrt{5}}{2}$ and $\beta = \frac{1+\sqrt{5}}{2}$ called the golden ratio.

Proof. (Added). Solve the equation $1 - 2^{-s} - 4^{-s} = 0$, which is converted to $(2^{-s})^2 + 2^{-s} - 1 = 0$. The equation $x^2 + x - 1 = 0$ has solutions $\frac{-1\pm\sqrt{5}}{2}$.

We then solve the equation $2^{-s} = \frac{-1\pm\sqrt{5}}{2}$. Taking exponential logarithm implies that

$$2^{-\mathrm{Re}(s)} e^{-i\mathrm{Im}(s)\log 2} = \frac{-1 \pm \sqrt{5}}{2}.$$

It then follows that $\mathrm{Im}(s) = \frac{2\pi n}{\log 2}$ for $n \in \mathbb{Z}$ and $\mathrm{Re}(s) \log 2 = \log\left(\frac{1\pm\sqrt{5}}{2}\right)$.
◀ □

Recent results on the noncommutative geometry of fractals and Cantor sets and spectral triple constructions for AF-algebras can be found in CI [53] (cf. [7]) and GI [123], [124].

For the dual group of the Cantor set viewed as the product of countably many copies of the cyclic group $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$, the corresponding spectral triple is constructed by [53] (cf. [7]). In particular, the Dirac operator has the form $\sum_{n=1}^{\infty} \alpha_n q_n$, where (α_n) is a sequence of positive reals and (q_n) is a sequence of pairwise orthogonal projections.

It is shown by a recent work [54] that it is easy to describe a compact metric space, recovering the metric via a spectral triple. The space is a sum of two-dimensional modules, but spectral triples carry much more information than just regarding the metric. Also provided by the construction of [55] is a natural spectral triple for the Sierpiński gasket, of topological dimension one.

15 Dimensional regularization in QFT

In perturbative, quantum field theory (QFT), computed are expectation values of observables via a formal series, where the terms are parameterized by Feynman graphs and reduced to ordinary finite dimensional integrals in momentum space of expressions assigned to the graphs by the Feynman rules. These expressions typically produce divergent integrals.

For example, in the example of the scalar ϕ^3 theory in dimension $D = 4$ or $D = 4 + 2n$ with $n \in \mathbb{N}$, a divergence appears already in the simplest one-loop diagram, with corresponding integral in Euclidean signature as

$$\xrightarrow[p_1]{l_1^e} \overset{v_1}{\cdot} \underset{l_2:m_2}{\overset{l_1:m_1}{\bigcirc}} \overset{v_2}{\cdot} \xrightarrow[p_2]{l_2^e}$$
$$= \int \frac{\delta^4(p_1 - k_1 + k_2) + \delta^4(-p_2 + k_1 - k_2)}{(k_1^2 - m_1^2 + i(+0))(k_2^2 - m_2^2 + i(+0))} d^D k_1 d^D k_2$$

(corrected as in the sense below).

Remark. (Added). Recall from [183] the Feynman graph (or diagram) (Fg) as follows. The graph Fg is a finite connected graph, consisting of finite points (or vertexes) $v_1, \cdots, v_k$ in the 4-dimensional space-time, and of finite, 1-dimensional (curved, closed) intervals (or edges) $l_1, \cdots, l_N$, and of finite, half-open lines $l_1^e, \cdots, l_{k'}^e$, such that the two end points of each l_j and the one end point of each l_j^e are (two or one) vertexes. Refer to the above graph, where the circle is in fact divided into a union $l^1 \cup l^2$ and $l^1 \cap l^2 = \{v_1, v_2\}$. As well, the half line l_j^e contains v_j as the end point. Each half line l_j^e corresponds to a 4-dimensional vector p_j, and each interval l_j does to a non-negative constant m_j. An orientation for a Fg is given as just the arrows. Then the incidence number $[v_i : l_j]$ with respect to v_i and l_j is defined to be $+1$ when l_j goes into v_i and -1 when l_j starts from v_i and 0 otherwise. Similarly, define $[v_i, l_j^e]$.

The Feynman rule is a correspondence between each Feynman graph and its integral, given as above. In rather general, the correspondence is given as that for such Fg defined above,

$$\mathrm{Fg}(p) = \int \frac{\sum_{i=1}^{k} \delta^4(\sum_{r=1}^{k'} [v_i : l_r^e] p_r + \sum_{j=1}^{N} [v_i, l_j] k_j)}{\Pi_{j=1}^{N} (k_j^2 - m_j^2 + i(+0))} \Pi_{j=1}^{N} d^4 k_j,$$

where $k_j^2 = k_{j0}^2 - \sum_{\nu=1}^{3} k_{j\nu}^2$. ◄

Therefore, we need a regularization procedure for these divergent integrals. The regularization most commonly adopted for computation in quantum field theory is dimensional regularization (or renormalization) (DimR) and minimal subtraction (MinS). The method is introduced in the 1970s in [25] and [137]. It has the advantage of preserving basic symmetries.

The regularization procedure of DimR is essentially based on the use of the formula

$$\int e^{-\lambda k^2} d^d k = \pi^{\frac{d}{2}} \lambda^{-\frac{d}{2}},$$

to define the meaning of the integral in $d = D - z$ dimensions, for $z \in \mathbb{C}$ in a neighbourhood of zero. For instance, in that case above, the procedure of dimensional regularization yields the following:

$$\pi^{\frac{D-z}{2}} \Gamma\left(\frac{4 - D + z}{2}\right) \int_0^1 ((x - x^2)p^2 + m^2)^{\frac{D-z-4}{2}} dx.$$

In a recent survey [172], it is Yuri Manin who refers to DimR as dimensions in search of a space, as a reminiscent of, Six characters in search of an author, a play of Pirandello. Indeed, in the usual approach in perturbative quantum field theory, the dimensional regularization procedure is just regarded as a formal rule of analytic continuation of formal (divergent) expressions in integral dimensions D to complex values of the variable z (corrected).

However, using noncommutative geometry, it becomes possible to construct actual spaces as in the sense of noncommutative Riemanian geometry, as X_z whose dimension in the sense of dimension spectrum is a point $z \in \mathbb{C}$ (cf. [85] missing but checked not to be specified).

It is known in the physics literature that there are problems related to using dimensional regularization in chiral theory, which involves giving a consistent prescription on how to extend the γ_5 as the product of the matrices γ^i when $D = 4$, to non-integer dimension $D - z$. It turns out that a prescription known as Breitenlohner-Maison [35] (and [56] missing) admits an interpretation in terms of the cup product of spectral triples, where we take the product of the spectral triple associated to the ordinary geometry in the integer dimension D, by a spectral triple X_z whose dimension spectrum is reduced to the complex number z (cf. [85] missing).

Illustrate here the construction for the case where $z \in \mathbb{R}_+^*$ of positive reals. The more general case where $z \in \mathbb{C}$ is more delicate.

Example 15.1. (Edited). We need to work in a slightly modified setting for spectral triples, which is given by the type II spectral triples (cf. [22], [41], [42]).

In this setting, the usual type I trace of operators in $\mathcal{L}^1(H)$ the trace class is replaced by the trace on a type II_∞ von Neumann algebra.

Consider a self-adjoint operator Y, affiliated to a type II_∞ factor $\mathcal{N}$, with spectral measure given by, via functional calculus,

$$\mathrm{tr}_{\mathcal{N}}(\chi_E(Y)) = \frac{1}{2}\int_E 1dy$$

for any interval E in $\mathbb{R}$, with χ_E the characteristic function on E.

If $Y = F|Y|$ is the polar decomposition of the operator Y, then set

$$D_z = \rho(z)F|Y|^{\frac{1}{z}}$$

by functional calculus, where the normalization function $\rho(z)$ is chosen to be

$$\rho(z) = \pi^{-\frac{1}{2}}\Gamma\left(\frac{z}{2}+1\right)^{\frac{1}{z}},$$

so that it is obtained as

$$\mathrm{tr}_{\mathcal{N}}(e^{-\lambda D_z^2}) = \pi^{\frac{z}{2}}\lambda^{-\frac{z}{2}}, \quad \lambda \in \mathbb{R}_+^*.$$

This gives a geometric meaning to the basic formula of DimR, given above.

The algebra $\mathcal{A}$ of the spectral triple X_z can be defined to contain any operator a such that the additive commutator $[D_z, a]$ is bounded, and both a and $[D_z, a]$ are smooth for the geodesic flow defined as, for T in the domain,

$$T \mapsto e^{it|D_z|}Te^{-it|D_z|}, \quad t \in \mathbb{R}.$$

The dimension spectrum of X_z is reduced to the single point z, since

$$\mathrm{tr}'_{\mathcal{N}}((D_z^2)^{-\frac{s}{2}}) = \rho(z)^{-s}\int_1^\infty u^{-\frac{s}{z}}du = \rho(z)^{-s}\frac{z}{s-z}$$

has a single simple pole at $s = z$, and is absolutely convergent in the half space as $\mathrm{Re}(\frac{s}{z}) > 1$, where $\mathrm{tr}'_{\mathcal{N}}$ denotes the trace with an infrared cutoff, i.e., integrating outside of $|y| < 1$. ◀

16 Local algebras in super-symmetric QFT

It is striking that the general framework of noncommutative geometry is suitable non only for handling finite dimensional spaces, commutative or not, of dimension, interger or not, but is also compatible with infinite dimensional spaces. (As a note it seems be always that it can be extended to the infinite case without the convergence problem). As already seen before, discrete groups of exponential (or polynomial?) growth naturally give rise to noncommutative spaces which are described by a θ-summable spectral triple, but not by a finitely summable one. This is characteristice of an infinite dimensional space. In that case, as for discrete groups, cyclic cohomology needs to be extended to entire cyclic cohomology. A similar kind of noncommutative space also arises from the quantrum field theory (QTF) in the super-symmetric context, as in [66].

Example 16.1. (Added). ([66, IV 9.β]). (The free massive scalar field theory). Let $X = S^1 \times \mathbb{R}$ as a space-time, with the Lorentzian metric.

A scalar field is defined to be a real-valued function $\varphi = \varphi(x,t)$ on X governed by the Lagrangian $L(\varphi)$:

$$L(\varphi)(x,t) = \frac{1}{2}((\partial_0\varphi)^2 - (\partial_1\varphi)^2) - \frac{m^2}{2}\varphi^2, \quad \partial_0 = \frac{\partial}{\partial t} \quad \text{and} \quad \partial_1 = \frac{\partial}{\partial x}$$

the time derivative for $t \in \mathbb{R}$ and the spacial derivative for $x \in S^1$.

The action functional is given by

$$\begin{aligned} I(\varphi) &= \int_X L(\varphi)(x,t)dxdt = \int_{\mathbb{R}} \left(\int_{S^1} L(\varphi)(x,t)dx \right) dt \\ &\equiv \int_{\mathbb{R}} \left(\int_{S^1} L(\varphi)dx \right)(t)dt. \end{aligned}$$

Thus, at the classical level, one is dealing with a mechanical system with finitely many degrees of freedom, whose configuration space $C = L^2(S^1, \mathbb{R})$ is the space of (square integrable) real-valued functions on S^1.

The Hamiltonian of this classical mechanical system is the functional on the cotangent space T^*C, given by

$$H(\varphi,\pi) = \frac{1}{2}\int_{S^1} (\pi(x)^2 + (\partial_1\varphi(x))^2 + m^2\varphi^2(x))dx,$$

where one uses the linear structure of the space C to identify T^*C with $C \times C^*$, and one views the field π as an element of the dual space C^* of C of all $\pi : C \to \mathbb{C}$ bounded and linear, as

$$\pi(\varphi) = \int_{S^1} \varphi(x)\pi(x)dx \in \mathbb{C}.$$

Consider the components $\varphi_k = \int_{S^1} \varphi(x)e^{-ikx}dx$ under the Fourier transform in C and those π_k of π in C^*., which are subject to the reality condition that $\varphi_{-k} = \overline{\varphi_k}$ and $\pi_{-k} = \overline{\pi_k}$ for any $k \in \mathbb{Z}$. Thus, both spaces C and T^*C are infinite products of finite-dimensional spaces and the Hamiltonian H is converted to the infinite sum:

$$H = \sum_{k\in\mathbb{Z}} \frac{1}{2}\left(\pi_k\overline{\pi_k} + (k^2+m^2)\varphi_k\overline{\varphi_k}\right).$$

In a system with configuration space as $\mathbb{R}$ and Hamiltonian H as $\frac{1}{2}(p^2+\omega^2q^2)$, the quantization of a single harmonic oscillator does quantize the values of the energy in replacing H by the operator $\frac{1}{2}((-i\hbar\frac{\partial}{\partial q})^2 + \omega^2q^2)$ on the Hilbert space $L^2(\mathbb{R})$, with the set $\{n\hbar\omega \mid n \in \mathbb{N}\}$ as the spectrum up to a shift. The algebra of observables of the quantum system is then generated by a single operator a^* and its adjoint a with the commutation relation: $[a, a^*] = 1$, and the Hamiltonian is $H = \hbar\omega a^*a$. These two equations completely describe the quantum system.

Its only unitary representation in $L^2(\mathbb{R})$ is defined to be $a = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial q} + q)$ up to unitary equivalence. The unique up to phase, normalized vector Ω such that $a\Omega = 0$ is called the **vacuum** vector.

It follows from the reality condition for φ_k for k positive that the pair $\{-k, k\}$ corresponds to a pair of harmonic oscillators, whose quantization does to a pair of creation operators a_k^* and a_{-k}^*. The observable algebra of the quantized field has the following presentation. The algebra is generated by a_k^* and a_k for $k \in \mathbb{Z}$, indexed by the canonical orthonormal basis (e_j) for $L^2(\mathbb{R})$, with the commutation relations:

$$[a_k, a_k^*] = 1, \quad [a_k, a_l] = 0, \quad \text{and} \quad [a_k, a_l^*] = 0$$

for $k, l \in \mathbb{Z}$ with $k \neq l$.

Proof. (Added). Indeed, the Fock Hilbert space is

$$H = \mathbb{C}\Omega \oplus [\oplus_{n=1}^{\infty}(\otimes^n L^2(\mathbb{R}))],$$

and the creation operator $a_k^* = a^*(e_k)$ and the annihilation operator $a_k = a(e_k)$ are defined by that for any $k \in \mathbb{Z}$ and $\xi_1, \cdots, \xi_n \in L^2(\mathbb{R})$, we let $a_k^*(\Omega) = e_k$, $a_k(\Omega) = 0$, $a_k(\xi_1) = \langle e_k, \xi_1\rangle\Omega$, and

$$a_k^*(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \sqrt{n+1} e_k \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n, \quad n \geq 1,$$
$$a_k(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \sqrt{n}\langle e_k, \xi_1\rangle \xi_2 \otimes \xi_3 \otimes \cdots \otimes \xi_n, \quad n \geq 2.$$

Therefore, we compute

$$\begin{aligned}
&[a_k, a_k^*]\Omega = (a_k a_k^* - a_k^* a_k)\Omega = a_k e_k = \langle e_k, e_k\rangle\Omega = 1\Omega, \\
&[a_k, a_k^*]\xi_1 = (a_k a_k^* - a_k^* a_k)\xi_1 = a_k(\sqrt{2} e_k \otimes \xi_1) - a_k^*(\langle e_k, \xi_1\rangle\Omega) \\
&= 2\langle e_k, e_k\rangle\xi_1 - \langle e_k, \xi_1\rangle e_k = 2\xi_1 - \langle e_k, \xi_1\rangle e_k,
\end{aligned}$$

which is not equal to ξ_1 in general. □

So, it should be corrected as that the canonical commutation relations (**CCR**) on the basis (e_k) hold:

$$[a_{k,+}, a_{k,+}^*] = 1, \quad [a_{k,+}, a_{l,+}] = 0, \quad \text{and} \quad [a_{k,+}, a_{l,+}^*] = 0$$

for $k, l \in \mathbb{Z}$ with $k \neq l$, where $a_{k,+} = p_+ a_k p_+ = a_+(e_k)$ and $a_{k,+}^* = p_+ a_k^* p_+ = a_+^*(e_k)$, where

$$p_+(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \frac{1}{n!}\sum_{\pi} \xi_{\pi_1} \otimes \xi_{\pi_2} \otimes \cdots \otimes \xi_{\pi_n},$$

where the sum is taken over permutations of the set $\{1, \cdots, n\}$, and the symmetric projection p_+ is defined to be a bounded operator with norm one by extending linearly and continuously. The subspace $p_+ H$ of the Fock space H is said to be the (symmetric) Bose-Fock space. See [34]. (Probably, a_k^* and a_k in [66, IV 9.β] are not the creation and annihilation operators and should mean those cut down by the projection p_+.)

Proof. (Added). Compute that

$$[a_{k,+}, a^*_{k,+}]\Omega = (a_{k,+}a^*_k - a^*_{k,+}a_k)\Omega = a_k e_k = \langle e_k, e_k\rangle\Omega = 1\Omega,$$
$$[a_{k,+}, a^*_{k,+}]\xi_1 = (a_{k,+}a^*_k - a^*_{k,+}a_k)\xi_1 = a_{k,+}(\sqrt{2}e_k \otimes \xi_1) - a^*_{k,+}(\langle e_k, \xi_1\rangle\Omega)$$
$$= p_+ a_k \left(\frac{1}{\sqrt{2}}((e_k \otimes \xi_1) + (\xi_1 \otimes e_k))\right) - \langle e_k, \xi_1\rangle e_k$$
$$= \langle e_k, e_k\rangle\xi_1 + \langle e_k, \xi_1\rangle e_k - \langle e_k, \xi_1\rangle e_k = \xi_1.$$

More computations are omitted. □

On the other hand, the (anti-symmetric) Fermi-Fock space is defined similarly to be the subspace $p_- H$ of the Fock space H, where the anti-symmetric projection p_- is defined by

$$p_-(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \frac{1}{n!}\sum_\pi \varepsilon_\pi \xi_{\pi_1} \otimes \xi_{\pi_2} \otimes \cdots \otimes \xi_{\pi_n},$$

where $\varepsilon_\pi = 1$ if the permutation π is even and $= -1$ if odd. Then the canonical anti-commutation relations (**CAR**) on the basis (e_k) hold:

$$\{a_{k,-}, a^*_{k,-}\} = 1, \quad \{a_{k,-}, a_{l,-}\} = 0, \quad \text{and} \quad \{a_{k,-}, a^*_{l,-}\} = 0$$

for $k, l \in \mathbb{Z}$ with $k \neq l$, where $a_{k,-} = p_- a_k p_- = a_-(e_k)$ and $a^*_{k,-} = p_- a^*_k p_- = a^*_-(e_k)$, and $\{x, y\} = xy + yx$.

The (bosonic) Hamiltonian is then the derivation corresponding to the formal sum

$$H_b = \sum_{k\in\mathbb{Z}} \hbar\omega_k a^*_{k,+}a_{k,+}, \quad \omega_k = \sqrt{k^2 + m^2}.$$

The vacuum representation is given by the tensor product Hilbert space $H_v = \otimes_{k\in\mathbb{Z}}(H_k, \Omega_k)$ and the tensor product algebra representation, corresponding to the tensor product of the vacuum states $\langle(\cdot)\Omega_k, \Omega_k\rangle$ with respect to the vacuum vectors Ω_k. ◀

Example 16.2. (Edited with [66, IV 9.β]). (The free Wess-Zumino model in two dimension). This is a super-symmetric free field theory in a two-dimensional space-time where space is compact. Assume that space is the circle S^1 and space-time is the cylinder set $S^1 \times \mathbb{R}$ endowed with the Lorentzian metric. The fields are given by a complex scalar bosonic field φ of mass m and a spinor field ψ of the same mass. The Lagrangian of the theory is of the form $L = L_b + L_f$, where

$$\begin{cases} L_b = \frac{1}{2}(|\partial_0\varphi|^2 - |\partial_1\varphi|^2 - m^2|\varphi|^2), \\ L_f = i\overline{\psi}\sum_{\mu=0}^1 \gamma_\mu\partial_\mu\psi - m\overline{\psi}\psi, \end{cases}$$

where the spinor field is given by a column matrix

$$\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}, \quad \psi^* = (\psi_1^*, \psi_2^*)$$

with $\overline{\psi} = \psi^*\gamma_0$, and the γ_μ are 2×2 Pauli or Dirac matrices, anticommuting, self-adjoint and of square 1. In fact,

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $\gamma_0^* = \gamma_0$ but $\gamma_1^* = -\gamma_1$, and $\gamma_0^2 = 1$ but $\gamma_1^2 = -1$, and

$$[\gamma_0, \gamma_1] = \gamma_0\gamma_1 - \gamma_1\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = 0,$$

and hence they do commute.

The bosonic part is quantized exactly as in the real example above, except that the field φ is complex, so that we need twice as many creation operators $a^*_{ij,+}(k)$ (cut down by the symmetric projections $p_{ij,+}$) for $k \in \mathbb{Z}$, $i = \sqrt{-1}$, and $j = 0, 1$, for which the commutation relations hold;

$$[a_{ij,+}(k), a_{ij,+}(l)] = [a_{ij,+}(k), a^*_{ij,+}(l)] = 0 \quad \text{and} \quad [a_{ij,+}(k), a^*_{ij,+}(l)] = \delta_{kl}$$

(corrected). We use this notation in what follows.

The time-zero field $\varphi(x)$ is given by

$$\varphi(x) = \frac{1}{\sqrt{4\pi}} \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{\omega(k)}} (a^*_{1,+}(k) + a_{i,+}(-k))e^{-ikx}, \quad \omega(k) = \sqrt{k^2 + m^2}.$$

The conjugate momentum is given as

$$\pi(x) = \frac{i}{\sqrt{4\pi}} \sum_{k \in \mathbb{Z}} \sqrt{\omega(k)} (a^*_{i,+}(k) - a_{1,+}(-k))e^{-ikx}.$$

Then the canonical commutation relations hold:

$$[\varphi(x), \varphi(y)] = [\pi(x), \pi(y)] = [\pi^*(x), \varphi(y)] = 0, \quad [\pi(x), \pi(y)] = -i\delta(x - y).$$

The quantum field $\varphi(x)$ and its conjugate momentum $\pi(x)$ are operator-valued distributions in H_b and the bosonic Hamiltonian has the form:

$$H_b = \int_{S^1} \left(|\pi(x)|^2 + |\partial_1\varphi(x)|^2 + m^2|\varphi(x)|^2 \right) dx,$$

where the Wick ordering (omitted) takes care of an irrelevant additive constant.

The C^*-algebra generated by the fermionic field $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is generated by its Fourier components $\psi_j(k)$ for $i, j = 1, 2$ and $k, l \in \mathbb{Z}$ such that the anti-commutation relations (on the Fermi-Fock space) hold:

$$\{\psi_i(k), \psi_j(l)\} = 0 \quad \text{and} \quad \{\psi_i(k), \psi_j(l)^*\} = \delta_{ij}\delta_{kl},$$

where $\{x, y\} = xy + yx$. In other words, it corresponds to the C^*-algebra A associated to the canonical anti-commutation relations in the Hilbert space H_f of L^2-spinors on S^1.

The quantum fields ψ_1 and ψ_2 are given by

$$\psi_j(x) = \sum_{k\in\mathbb{Z}} \psi_j(k)e^{-ikx} \quad x \in S^1,$$

which are A-valued distributions on S^1. This specifies $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ at time zero. Its time evolution is specified by the Hamiltonian, given by the derivation of A as $\delta(a) = [H_f, a]$, where, with $\overline{\psi} = \psi^*\gamma_0$,

$$\begin{aligned} H_f &= \int_{S^1} (\overline{\psi}\gamma_1 i\partial\psi - m\overline{\psi}\psi)\, dx \\ &= \sum_{k\in\mathbb{Z}} k(\psi_1(k)^*\psi_1(k) - \psi_2(k)^*\psi_2(k)) - m(\psi_1^*(k)\psi_2(k) + \psi_2(k)^*\psi_1(k)). \end{aligned}$$

The derivation above defines a one-parameter group of automorphisms σ_t of A the CAR algebra with respect to $L^2(S^1, S)$ and U_t, where U_t is the one-parameter group generated by the operator

$$H_1 = \begin{pmatrix} i\partial & -m \\ -m & -i\partial \end{pmatrix}.$$

The representation of A associated to the ground state is said to be the Dirac sea representation, described as follows. The Hilbert space H_f is the anti-symmetric Fock space over

$$L^2(S^1, S) = L^2(S^1) \oplus L^2(S^1),$$

i.e., in other words, a suitable spin representation of the infinite dimensional Clifford algebra containing the fermionic quantum fields $\psi_j(x)$. Denote by $b^*_{i_j,-}(k)$ the corresponding creation operators for $k \in \mathbb{Z}$ (cut down by the anti-symmetric projections $p_{i_j,-}$), indexed by the natural orthonormal basis of $L^2(S^1, S)$. The operators satisfy CAR and are related to $\psi_j(k)$ as that

$$\begin{aligned} \psi_1(k) &= \frac{1}{\sqrt{4\pi w(k)}}(\nu(-k)b^*_{-1,-}(k) + \nu(k)b_{1,-}(-k)), \\ \psi_2(k) &= \frac{1}{\sqrt{4\pi w(k)}}(\nu(k)b^*_{-1,-}(k) - \nu(-k)b_{1,-}(-k)), \end{aligned}$$

which define the representation of A on H_f, where

$$w(k) = \sqrt{k^2 + m^2} \quad \text{and} \quad \nu(k) = \sqrt{w(k) + k}.$$

The fermionic Hamiltonian is the positive operator on H_f given by

$$H_f = \int_{S^1} (\overline{\psi}\gamma i\partial\psi - m\overline{\psi}\psi)\, dx,$$

which is the generator for the one-parameter group σ_t of automorphisms of A in that representation.

The Hilbert space of the quantum theory is the tensor product $H = H_b \otimes H_f$ of the bosonic Hilbert space H_b and the fermionic one H_f. The full Hamiltonian of the non-interacting theory acts on the Hilbert space $H = H_b \otimes H_f$ by the positive operator:

$$H = H_b \otimes 1 + 1 \otimes H_f.$$

The self-adjoint square root of H is then defined as in the same way as that the Dirac operator is defined to be the square root of the Laplacian in the case of finite dimensional manifolds. The square root Q of H is called the supercharge operator in supersymmetry, given by

$$\begin{aligned}\sqrt{2}Q = \int_{S^1} \{&\psi_1(x)(\pi(x) - \partial\psi^*(x) - im\varphi(x)) + \\ &\psi_2(x)(\pi^*(x) - \partial\varphi(x) - im\varphi^*(x)) + h.c.\}dx,\end{aligned}$$

where the symbol $(\cdots) + h.c.$ means the adding the Hermitian conjugate of $(\cdots)$. ◄

The basic relation to spectral triples is then given by the following ([66, §IV]):

Theorem 16.3. *For any local region $O \subset C$, let $\mathcal{A}(O)$ be the algebra of functions of quantum fields with support in O acting on the Hilbert space H. Then the triple $(\mathcal{A}(O), H, Q)$ is an even θ-summable spectral triple, with $\mathbb{Z}_2$-grading given by the operator $\gamma = (-1)^{N_f}$ counting the parity of the fermion number operator N_f.*

The algebra $\mathcal{A}(O)$ is generated by the imaginary exponentials $e^{i(\varphi(f)+\varphi(f)^*)}$ and $e^{i(\pi(f)+\pi(f)^*)}$ for $f \in C_c^\infty(O)$.

As shown in [66], exactly as in the case of discrete groups with exponential growth, needed is the entire cyclic cohomology rather than its finite dimensional version, in order to obtain the Chern character of θ-summable spectral triples. Indeed, the index map is non-polynomial in the above example of the Wess-Zumino model in two dimension, and the K-theory of the above local algebras is highly non-trivial. In fact, it is in the framework that the JLO-cocycle is discovered by Jaffe, Lesniewski, and Osterwalder [141].

It is an open problem to extend the above result to interacting theories in higher dimension and to give a full computation of the K-theory of the local algebras as well as of the Chern character in entire cyclic cohomology. The results of Jaffe and collaborators on constructive quantum field theory yield many interacting non-trivial examples of super-symmetric two dimensional models. Moreover, the recent breakthrough by Puschnigg in the case of lattices of semi-simple Lie groups of rank one opens the way to the computation of the Chern character in entire cyclic cohomology.

17 The standard model of elementary particles as noncommutative geometry

The (Glashow-Weinberg-Salam (GWS or WS)) standard model of elementary particle physics provides a surprising example of a spectral triple in the noncommutative setting, which. in addition to the condition of the definition, also has the real structure satisfying all the additional conditions of the definition.

The noncommutative geometry of the standard model, developed by [69] (cf. [47], [48], [80] (missing), [146] (missing)) gives a concise conceptual way to describe the full complexity of the input from physics, through a simple mathematical structure.

The physics of the standard model can be described by a Lagrangian. We consider the standard model minimally coupled to gravity, so that the Lagrangian L is the sum $L_{EH} + L_{SM}$ of the Einstein-Hilbert Lagrangian L_{EH} and the standard model Lagrangian L_{SM}.

The standard model Lagrangian L_{SM} has a somewhat complicated expression, which might take a full page if written in full (cf. M. Veltman [233] missing). It may comprise of terms of five types as

$$L_{SM} = L_G + L_{GH} + L_H + L_{Gf} + L_{Hf},$$

where the terms involve elementary particles in the table below (revised and refined):

Table 4: Elementary particles with spin and Pauli principle

Spin	Pile	Elementary particles
0	OK	Higgs bosons H (almost confirmed) as scalar fields for giving mass
1	OK	Gauge bosons: 8 gluons g_j, photons γ, weak bosons $W^{\pm}$ and Z^0 as vector fields for interactions
2	OK	Gauge bosons as gravitons (not yet confirmed)
$\frac{1}{2}$	No	Fermions f as quarks and leptons for matter
$\frac{2n+1}{2}$	No	Baryons with three quarks
$n \in \mathbb{N}$	No	Mesons with two quarks

and the term L_G is the pure gauge boson part, L_{GH} for the minimal coupling with the Higgs fields, and L_H gives the quartic Higgs self interaction, and the fermion kinetic term L_{Gf} contains the hypercharges Y_L, Y_R. These numbers, which are constant over generations, are assigned phenomenologically, so as to obtain the correct values of the electro-magnetic charges. The term L_{Hf}

contains the Yukawa coupling of the Higgs scalar fields with fermions. A more detailed and explicit description of the various terms above is given by [66, VI.5.β]. See also [233] of Veltman.

The following table is revised and refined:

Table 5: Fermions in the three generations (with charges)

Fermions	I	II	III
q: Quarks as Flavour	u: Up ($\frac{2}{3}e$) d: Down ($-\frac{1}{3}e$)	c: Charm ($\frac{2}{3}e$) s: Strange ($-\frac{1}{3}e$)	t: Top ($\frac{2}{3}e$) b: Botom ($-\frac{1}{3}e$)
l: Leptons (Light)	ν_e: e-Neutrinos (0) e^-: **Electrons** ($-1e$)	ν_μ: μ-Neutrinos (0) μ^-: Muons ($-1e$)	ν_τ: τ-Neutrinos (0) τ^-: Tauons ($-1e$)
$\bar{q}$: Anti-quarks	$\bar{u}$: Up-bar ($-\frac{2}{3}e$) $\bar{d}$: Down-bar ($\frac{1}{3}e$)	$\bar{c}$: Charm-bar ($-\frac{2}{3}e$) $\bar{s}$: Strange-bar ($\frac{1}{3}e$)	$\bar{t}$: Top-bar ($-\frac{2}{3}e$) $\bar{b}$: Botom-bar ($\frac{1}{3}e$)
$\bar{l}$: Anti-leptons	$\bar{\nu}_e$: Anti-ν_e (0) e^+: Positorons ($1e$)	$\bar{\nu}_\mu$: Anti-ν_μ (0) μ^+: Anti-μ ($1e$)	$\bar{\nu}_\tau$: Anti-ν_τ (0) τ^+: Anti-τ ($1e$)

where $u < c < t$, $d < s < b$, and $\nu < e < \mu < \tau$ as mass (after the 4th transition at 10^{-4} second), and electrons have size zero. It is predicted by Dirac that there exist anti-particles such as positorons to electrons, and it is confirmed by C. Anderson that there are e^- and e^+ from the gamma line in the mist box.

Added is the table below:

Table 6: Gauge bosons, forces (as interactions), mass, charges, and objects (as examples), and Higgs bosons

Gauge particles	Forces	m & c	Objects
$g_1, \cdots, g_8$: Gluons	Strong	$m: 0, c: 0$	**Nuclears**, Sun energy
γ: Photons	Electro-magnetic	$m: 0, c: 0$	Atoms, Molecules
$Z^0, W^\pm$: Weak bosons	Weak	$c: 0, \pm 1e$	Nuclear β collapsing
Gravitons	Gravity	$m: 0, c: 0$	Stars, Galaxy
H: Higgs particles	Give mass	$c: 0$	Other particles

where the weak bosons Z^0 and $W^\pm$ do have mass 91 GeV and 80 GeV respectively after the third transition at 10^{-11} second.

As a note, apply quantum mechanics to electro-magnetic fields and interactions by electrons, the quantum electro-dynamics is completed, and the general quantum field theory is formulated. But in application as perturbed, it involves difficulty of divergence such as ultra-violet divergence coming from integral in momentum and infrared divergence from that photos have no mass, so that integral in momentum diverges in a neighbourhood of 0.

The quantum field theories are divided into either the renormalizable thoeies such that infinitely many untra-violet divergences are absorbable to a finite number of constants in renormalization, or the unrernomalizable ones otherwise.

Also added below is the table for interactions between elementary particles in the standard model:

Table 7: Do or not interact or self-interact

	q	l	g_j	γ	Z	$W^{\pm}$	H
q	No	No	Yes	Yes	Yes	Yes	Yes
l	No	No	No	Yes	Yes	Yes	Yes
g_j	Yes	No	Yes	No	No	No	No
γ	Yes	Yes	No	No	No	Yes	No
Z	Yes	Yes	No	No	No	Yes	Yes
$W^{\pm}$	Yes	Yes	No	Yes	Yes	Yes	Yes
H	Yes	Yes	No	No	Yes	Yes	Yes

Yet, added below is the table for elementary particles (at an early stage):

Table 8: Hadrons

Hadrons	Quarks (or anit-), colors	Examples, combinations, (charges)
Baryons (Heavy) (8-fold way)	3 to be fermions in vacuum after the 4th transition at 10^{-4} s, Permutations of (R, G, B) or $(\overline{R}, \overline{G}, \overline{B})$ to white	p: **Protons** u-u-d $(\frac{2}{3}+\frac{2}{3}-\frac{1}{3}=1)$, n: **Neutrons** u-d-d $(\frac{2}{3}-\frac{1}{3}-\frac{1}{3}=0)$, Hyperons λ/σ^0: u-d-s $(\frac{2}{3}-\frac{1}{3}-\frac{1}{3}=0)$, σ^+: u-u-s $(+1)$, σ^-: d-d-s (-1), ξ^0: u-s-s (0), ξ^-: d-s-s (-1), $\cdots$
Mesons (Middle) m	2 to be bosons, $(R, \overline{R})$, $(G, \overline{G})$, or $(B, \overline{B})$	**Yukawa**: π^+/ρ: u-$\overline{d}$ $(\frac{2}{3}+\frac{1}{3}=1)$, π^-: $\overline{u}$-d (-1), π^0: u-$\overline{u}$ (0), Kons with s: K^+: u-$\overline{s}$ $(\frac{2}{3}+\frac{1}{3}=1)$, K^-: $\overline{u}$-s (-1), K^0: d-$\overline{s}$ (0), $\overline{K}^0$: $\overline{d}$-s (0), Cons with c: J/ψ: c-$\overline{c}$ $(\frac{2}{3}-\frac{2}{3}=0)$, $\cdots$ Bones with b: Υ: b-$\overline{b}$ $(-\frac{1}{3}+\frac{1}{3}=0)$, $\cdots$, Tons with t: t-$\overline{t}$ $(\frac{2}{3}-\frac{2}{3}=0)$, $\cdots$

where for instance,

$$\pi^{\pm} \rightarrow \mu^{\pm} + \nu \quad \text{and} \quad \mu^{\pm} \rightarrow e^{\pm} + \nu + \overline{\nu},$$
$$n \rightarrow p + W^- \quad (d \rightarrow u) \quad \text{and} \quad W^- \rightarrow e^- + \overline{\nu}_e$$

in β collapsing for π mesons high up in the atmosphere and for unstable n neutorons, and where $\overline{\nu}$ is the anti-neutrino with charge $0 = -1 \cdot 0$, and moreover,

$\overline{x}$ means the anti-quark to a quark x, with charge -1 times that of x. Protons and neutrons have size 1.6×10^{-15} m. There are more than 300 kinds of hadrons found out, all of which are composed of 3 in 6 kinds of quarks. Note that there are just $6^3 = 216$ combinations of only quarks and $216 \times 2^3 = 1728$ of quarks and anti-quarks, but with ignoring charges (it may happen? or to be nonsense).

A β collapsing in quarks is given as

$$n \rightarrow p + W \rightarrow p + \nu + e.$$

A proton collapsing with taking 10^{37} years is the interaction as

$$p \rightarrow X \text{ (boson) } + \pi^0 \rightarrow e^+ + \pi^0 \text{ (or } \mu + \pi^0).$$

Higgs particles are thought to be created as

$$e^- + e^+ \rightarrow H + Z^0 \quad \text{and} \quad e^- + e^+ \rightarrow H + \overline{\nu}_e + \nu_e.$$

In the interaction by strong force, gluons and colors are exchanged between quarks in baryons such as protons or neutrons, or quarks in mesons, preserving total color white. In the interaction by electro-magnetic force, photons are exchanged between protons and electrons. In the interaction by weak force, the β collapsing of elementary particles is induced such as that unstable neutrons are changed into protons. In the interaction by gravity, gravitons are **assumed** to be exchanged between usual matters. Higgs particles are **assumed** to live as sea in vaccuum to give mass to other elementary particles, especially to weak Bosons.

Added below is the table for four forces:

Table 9: Four forces and their properties

Forces	Strength	Range	Force charge	Objects
Strong	10^3	$\leq 10^{-15}$ m	Color, π	Quarks and hadrons
Electro-magnetic (EM)	1 (ratio)	∞	Electorical $+, -, \gamma$	Quarks and leptons except neutrinos
Weak	10^{-8}	$\leq 10^{-17}$ m	Flavour, W, Z	Quarks and leptons
Gravity	10^{-40}	∞	Mass, g_j	Quarks and leptons

When energy is high (GeV), electro-magnetic force (EM), weak force (W), and strong force (S) are unified to be the grand unified force (GUF). In the standard model, those forces are not exactly identified at any envergy level, but in the super-symmetry theory, they are done at 10^{16} [GeV]. When temperature (K) is further higher, gravity is also unified with GUF to be the universal force (UF).

The gauge theory for quarks and gluons is said to be quantum chromodynamics (QCD).

Added below is the history of our universe at early time:

Table 10: The early history of the universe

Time (second) $\nearrow$	Temperature $\searrow$	Forces events (as 4 transitions)
0 second [s] $\sim$	More than	**Universal** force, Plank era
10^{-44} [s] $\sim$ (Plank Time $\sim$)	10^{32} K $\sim$ (Plank K $\sim$)	**Gravity** separated (first) Grand unification (GU) era
10^{-36} second $\sim$	10^{28} K $\sim$	**Strong** force separated (second) Electric-Weak era ($\sim 10^{-12}$ [s]) Inflation ($\sim 10^{-32}$ [s])
10^{-32} second $\sim$	10^{x} K $\sim$ x?	Gig Bang (10^{-32} [s]), Photons (curved), weak bosons, and Higgs bosons born, (Super-symmetry breaking)
10^{-12} second $\sim$	10^{15} K $\sim$	**Weak** force separated from **EM** force (third), Quarks era
$10^{-6(\text{or } 5)}$ second $\sim$ (around)	$10^{14(\text{or } 13)}$ K $\sim$ (around)	Protons, neutrons born (fourth, QCD) Quarks closed into hadrons, Hadrons era
$1 \sim 10$ second	10^{12} K $\sim$ (around)	Electrons, positorons born or caught Leptons era
$10[s] \sim 38 \times 10^4$ [y] 10^5 [y] around	~ 3000 K	Photons era The universe clear up after Plazma
$\sim 137 \times 10^8$ years	~ 3 K	The present

Note that the Plank time is defined to be $\sqrt{Ghc^{-5}} = 10^{-44}$ second. Also, the Plank length is defined to be $\sqrt{Ghc^{-3}} = 10^{-33}$ cm. Note that x Kelvin (K) $= x - 273.15$ Celsius (°C).

May recall that in the general theory of relativity, the Einstein equation for the universe in 1917 with the universe (third) term is defined as

$$R_{ik} - \frac{1}{2} g_{ik} R + \Lambda g_{ik} = \frac{8\pi G}{c^4} T_{ik},$$

where the summation $\sum_{0 \le i,k \le 3}$ or the 4×4 matrix representation as tensors are omitted, $g_{i,k}$, $R_{i,k}$, and R are respectively, the metric tensor, Riemann curvature tensor, and the scalar curvature for the space-time, and $T_{i,k}$ is the energy momentum tensor of matter and so on, and Λ is the universe constant, to obtain to make the universe stopping. The formula above says that the left hand side in terms of curving on the space-time is just equal to the right hand side in terms of energy or mass in the sense of

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \text{ with } v \text{ vilocity.}$$

A Riemannian metric g on a smooth manifold M is defined to be a smooth, positive definite, inner product-valued function on M, denoted as $g_p(X, Y)$ for

$p \in M$ and $X, Y \in T_p(M)$. The metric can be written locally as

$$g = \sum_{i,j} g_{i,j} dx^i dx^j$$

with $(g_{i,j})$ a $\dim M \times \dim M$ positive definite, symmetric matrix over $\mathbb{R}$. If M is a paracompact space, then there is such a metric tensor.

Define the norm

$$\|L\| = \sqrt{g_p(L, L)} \text{ for } L \in T_p(M).$$

The curvature tensor R on a Riemannian manifold (M, g) is the curvature, of its Levi-Civita (or Riemannian) (linear, unique) connection (with respect to the orthonormal frame bundle on M) corresponding to its covariant derivative ∇, defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z).$$

It can be written locally as

$$R = \sum R^i_{jkl} dx^j \otimes dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^i},$$

where

$$R^i_{jkl} = \left(\frac{\partial \Gamma^i_{lj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^l} \right) + \sum_m (\Gamma^m_{lj} \Gamma^i_{km} - \Gamma^m_{kj} \Gamma^i_{lm}),$$

where the Christoffel symbol means that

$$\Gamma^k_{ij} = \frac{1}{2} \sum_l g^{kl} \left\{ \frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right\},$$

so that

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

The sectional (or Riemanian) curvature is defined to be

$$K_p(P) = g_p(R(X, Y)Y, X),$$

where P is a 2-dimensional subspace of $T_p(M)$ with $\{X, Y\}$ as an orthonormal basis for P, and K_p is independent of the choice of such a basis.

The Ricci tensor is defined to be

$$R_{ij} = -\sum_k R^k_{ijk}.$$

The Ricci curvature at $p \in M$ along $L \in T_p(M)$ with norm 1 is defined to be $\mathrm{Ric}(L, L)$, corresponding to the 2-form associated to (R_{ij}). This is the mean of $K_p(P)$ for a 2-dimensional subspace P containing L in $T_p(M)$.

The scalar curvature R at $p \in M$ is defined to be the mean of $\mathrm{Ric}(L, L)$ with respect to L.

If $\{X_j\}$ is an orthonormal basis for $T_p(M)$, then

$$\mathrm{Ric}(L, L) = \sum_j g_p(R(X_j, L)L, X_j) \quad \text{and} \quad R = \sum_j \mathrm{Ric}(X_j).$$

Further recall that for M a manifold with $\dim M = n$, the frame bundle P over M with $\pi : P \to M$ is defined to be the set of all $p = (x, (e_1, \cdots, e_n))$ for $x \in M$ and $(e_1, \cdots, e_n)$ a basis of $T_p(M)$. There is a right action on P by $GL_n(\mathbb{R})$ such that $pg = (x, (e_1, \cdots, e_n))g = (x, (e_1, \cdots, e_n)g)$, so that $P/GL_n(\mathbb{R}) = M$.

A linear (or affine) connection of M is defined to be a connection of P of M.

The canonical 1-form θ of P of M is defined by $\theta_p(X) = \mathrm{id}_p^{-1}(\pi_p(X))$ for $X \in T_p(P)$, with $\mathrm{id}_p : F \to T_x(M)$ a linear isomorphism and $\pi_p(X) \in T_x(M)$.

The torsion form Θ for such a linear connection P of M is defined by $\Theta = D\theta$ (covariant derivative or curvature form), for which the structure equation holds:

$$d\theta + \theta \wedge \theta = \Theta,$$

where

$$(\omega \wedge \theta)(X, Y) = \omega(X)\theta(Y) - \omega(Y)\theta(X) = [\omega, \theta](X, Y).$$

A linear connection of M implies the parallel transformation φ in TM.

For two vector fields X and Y over M, given a connection, the covariant derivative of Y along X is defined to be

$$(\nabla_X Y)_{p_0} = \lim_{t \to 0} \frac{1}{t}(\varphi_t^{-1}(Y_{p_t}) - Y_{p_0}),$$

where φ_t is the parallel transformation along the integration curve of $\{p_t\}$ of X passing through $p_0 \in M$.

A linear connection ∇ of a real vector bundle E over M is said to be a metric connection if for any (tangent) vector field X over M and sections φ_1, φ_2 of E,

$$X(g(\varphi_1, \varphi_2)) = g(\nabla_X \varphi_1, \varphi_2) + g(\varphi_1, \nabla_X \varphi_2).$$

The Levi-Civita connection is defined to the unique metric connection of TM such that its torsion form is zero.

Consider a differentiable (or smooth) principal fiber bundle P over M:

$$\begin{tikzcd} P \arrow[d, "\pi"'] & G \arrow[l, "r"'] \\ M & \end{tikzcd}$$

so that a Lie group G acts on P from the right as that $r_g(p) = pg$ for $p \in P$ and $g \in G$. The maps r and π induce the linear maps $r_g : T_p(P) \to T_{pg}(P)$ and $\pi : T_p(P) \to T_{\pi(p)}(M)$ by the same symbols respectively. Define $V_p(P) = \ker(\pi)$ in $T_p(P)$, whose elements are said to be vertical.

A connection on P is given by a vector subspace Q_p of $T_p(P)$ for each $p \in P$ such that

$$T_p(P) = V_p(P) \oplus Q_p, \quad r_g(Q_p) = Q_{pg},$$

and the map sending $p \mapsto Q_p$ is differentiable. Each vector in O_p is said to be horizontal.

(Namely, it seems to look like that a connection on P may be a continuous, finite rank projection-valued function on P, and the Levi-Civita one may be a continuous, standard, finite rank projection-valued function.)

There are two parameters to decided the property of the expanding universe, namely, the constant Λ and the other Ω, which represents total of energy and matter of the whole universe. If $\Lambda = 0$, then the universe would be open if $\Omega < 1$, closed if $\Omega > 1$, and flat if $\Omega = 1$. ◀

Table 11: The unified theories (UT) for forces

Force	Theory	Contributors (in part)
E & M	E & M theory	Maxwell
E as M γ	QED: Quantum electro-dynamics, Relativity	Tomonaga-Feynman-Schwinger, Einstein
E, Weak $W^{\pm}$, Z^0	QFD: Quantum flavor dynamics	Glashow-Weinberg-Salam Nanbu, $\cdots$
Strong q, g, m	QCD: Quantum chromo-dynamics, π	Fermi-Yang, Sakata, Gell-Mann, Yukawa, $\cdots$
E, W, S f, b	Standard model (as to QFD + QCD) Super-symmetry (as a) GUT (QFD + QCD)	Dirac, Fermi, GWS, Higgs, Koshiba, Kobayashi-Maskawa, $\cdots$ (Not yet confirmed) (Not yet completed)
Gravity	G theory	Galilei, Kepler, Newton, Einstein
E, W, S, G	Super-GUT Super-string T	(Not yet) Einstein, Weyl, $\cdots$ (Not yet but developed) Nanbu, Green-Schwarz-Witten, $\cdots$

A particle or its field is said to be **bosonic** if its components are commutative as $xy = yx$. A particle or its field is said to be **fermionic** if its components are anti-commutative as $xy = -yx$ (see [214]).

The Maxwell equations in a vacuum are gives as

$$\varepsilon_0 \frac{\partial}{\partial t} E = \mathrm{rot}\, H - J_0, \quad \varepsilon_0 \,\mathrm{div}\, E = \rho_0,$$
$$\mu_0 \frac{\partial}{\partial t} H = -\mathrm{rot}\, E - J_m, \quad \mu_0 \,\mathrm{div}\, H = \rho_m,$$

where ρ_0 and ρ_m are electric and magnetic densities respectively, and J_0 and J_m are electric and magnetic current densities respectively, so that the following equations of continuity are satisfied:

$$\frac{\partial}{\partial t}\rho_0 + \operatorname{div} J_0 = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\rho_m + \operatorname{div} J_m = 0.$$

As well, ε_0 and μ_0 are constants such that

$$\frac{1}{\sqrt{\varepsilon_0\mu_0}} = c = 2.99792 \times 10^8[m/s].$$

In fact, in experience, it may be assumed that $\rho_m = 0$ and $J_m = 0$, which breaks symmetry between electric and magnetic quantities, but there are no contradictions in the classical theory even if those are nonzero. Therefore, the Maxwell equations above may be converted as

$$\mu_0 c^2 \operatorname{rot} H = \frac{\partial}{\partial t}E + \varepsilon_0^{-1}J_0, \quad \operatorname{div} E = \varepsilon_0^{-1}\rho_0,$$
$$\operatorname{rot} E = -\mu_0\frac{\partial}{\partial t}H, \quad \operatorname{div} H = 0.$$

Recall that the divergence and the rotation of a (column) vector-valued function (or a vector field)

$$X = (X_j)_{j=1}^3 = (X_j(t, x_1, x_2, x_3))$$

are written as the inner and exterior products,

$$\operatorname{div} X = \langle \nabla, X\rangle = \sum_{j=1}^3 \frac{\partial}{\partial x_j}X_j,$$
$$\operatorname{rot} X = \nabla \times X = \sum_{j=1}^3 \left(\frac{\partial}{\partial x_{j+1}}X_{j+2} - \frac{\partial}{\partial x_{j+2}}X_{j+1}\right)e_j,$$
$$\nabla = \left(\frac{\partial}{\partial x_j}\right)_{j=1}^3,$$

where $j + k = 3 + j' = j' \pmod 3$ ◀

(Returned). The symmetry group of the Einstein-Hilbert Lagrangian L_{EH} by itself would be, by the equivalence principle, the diffeomorphism group $\mathrm{Dif}(X)$ of the space-time manifold X. In the standard model Lagrangian L_{SM}, on the other hand, the gauge theory has another huge symmetry group which is the group of local gauge transformations. According to our current understanding of elementary particle physics, that is given by

$$G_{SM}(X) = C^\infty(X, U(1) \times SU(2) \times SU(3))$$

at least in the case of a trivial principal bundle, e.g., when the space-time manifold X is contractible.

Then the full symmetry group for the Lagrangian L would be a semi-direct product

$$G(X) = G_{SM}(X) \rtimes \mathrm{Dif}(X).$$

In fact, a diffeomorphism of the manifold X relabels the gauge parameters.

As for a geometrization of the standard model, one would like to have a space X such that

$$G(X) = \mathrm{Dif}(X).$$

It is shown as a result of Thurston-Epstein-Mather (cf. [184]) that the connected component $\mathrm{Dif}_0(X)$ of the identity in the diffeomorphism group $\mathrm{Dif}(X)$ of a connected manifold X is a simple group. A simple group does not have a nontrivial normal subgroup, so that it does not have the semi-direct product structure like $G(X)$.

In the noncommutative setting, the group $\mathrm{Dif}_0(X)$ may be replaced with the group $\mathrm{Aut}^+(\mathcal{A})$ of automorphisms of a noncommutative algebra $\mathcal{A}$, preserving the fundamental class in K-homology, i.e., implementing a unitary compatible with the grading and real structure.

The group $\mathrm{Aut}(\mathcal{A})$ of automorphisms α of an algebra $\mathcal{A}$ has a normal subgroup $\mathrm{Inn}(\mathcal{A})$ of inner automorphisms of $\mathcal{A}$ such that

$$\mathrm{Ad}(u)(x) = uxu^{-1}$$

for $x \in \mathcal{A}$ and $u \in \mathcal{A}^-$ the group of invertible elements of $\mathcal{A}$.

Proof. (Added). We have that

$$\begin{aligned}(\alpha \circ \mathrm{Ad}(u) \circ \alpha^{-1})(x) &= \alpha(u\alpha^{-1}(x)u^{-1}) \\ &= \alpha(u)x\alpha(u)^{-1} \\ &= \mathrm{Ad}(\alpha(u))(x).\end{aligned}$$

□

The group $\mathrm{Aut}^+(\mathcal{A})$ also has $\mathrm{Inn}(\mathcal{A})$ as a normal subgroup.

There is a noncommutative algebra $\mathcal{A}$ whose $\mathrm{Inn}(\mathcal{A})$ corresponds to the group of gauge transformations $G_{SM}(X)$ such that the quotient group

$$\mathrm{Aut}^+(\mathcal{A})/\mathrm{Inn}(\mathcal{A})$$

corresponds to the group of diffeomorphisms (cf. Schücker [215] and [140] as well). The noncommutative space is a product $X \times F$ of an ordinary spacetime manifold X by a finite noncommutative space F. The noncommutative algebra $\mathcal{A}_F$ is a direct sum of $\mathbb{C}$, $\mathbb{H}$ of quaternions, and $M_3(\mathbb{C})$.

The algebra $\mathcal{A}_F$ corresponds to a finite space, where the standard model fermions and the Yukawa parameters as masses of fermions and mixing matrix of Cabibbo-Kobayashi-Masukawa determine the spectral geometry in the following sense. The Hilbert space H_F is finite dimensional and has the set of elementary fermions as a basis. This comprises of the three generations of quarks: ups-downs, charms-stanges, tops-bottoms, with left or right handed and with anti-particles as opposites with overlines and with the additonal color index as red, blue, and green, and of the three generations of leptons: electrons, muons, tauons, and the corresponding neutrions as below. (Edited).

Table 12: Left and right handed, fermions in the three generations

Fermions	I	II	III
Quarks	$u : u_L, u_R$ $d : d_L, d_R$	$c : c_L, c_R$ s: s_L, s_R	$t : t_L, t_R$ $b : b_L, b_R$
Anti-quarks	$\overline{u}$: $\overline{u}_L, \overline{u}_R$ $\overline{d}$: $\overline{d}_L, \overline{d}_R$	$\overline{c}$: $\overline{c}_L, \overline{c}_R$ $\overline{s}$: $\overline{s}_L, \overline{s}_R$	$\overline{t}$: $\overline{t}_L, \overline{t}_R$ $\overline{b}$: $\overline{b}_L, \overline{b}_R$
Leptons	$\nu_e : \nu_L^e$ (only) $e : e_L, e_R$	$\nu_\mu : \nu_L^\mu$ (only) $\mu : \mu_L, \mu_R$	$\nu_\tau : \nu_L^\tau$ (only) $\tau : \tau_L, \tau_R$
Anti-leptons	$\overline{\nu}_e : \overline{\nu}_L^e$ (only) $\overline{e} : \overline{e}_L, \overline{e}_R$	$\overline{\nu}_\mu : \overline{\nu}_L^\mu$ (only) $\overline{\mu} : \overline{\mu}_L, \overline{\mu}_R$	$\overline{\nu}_\tau : \overline{\nu}_L^\tau$ (only) $\overline{\tau} : \overline{\tau}_L, \overline{\tau}_R$

We discuss only the minimal standard model with no right handed neutrinos.

The $\mathbb{Z}_2$ grading γ_F on the Hilbert space H_F has sign $+1$ on left-handed particles and sign -1 on the right-handed particles. The involution J_F giving the real structure is the charge conjugation such that if $H_F = E \oplus \overline{E}$ the direct sum of particles and anti-particles, then J_F acts on the fermion basis as $J_F(f, \overline{h}) = (h, \overline{f})$. Then $J_F^2 = 1$ and $J_F\gamma_F = \gamma_F J_F$.

Proof. (Added). Indeed,

$$\begin{aligned} J_F\gamma_F(f_L, f_R, \overline{h}_L, \overline{h}_R) &= J_F(f_L, -f_R, \overline{h}_L, -\overline{h}_R) = (h_L, -h_R, \overline{f}_L, -\overline{f}_R) \\ &= \gamma_F(h_L, h_R, \overline{f}_L, \overline{f}_R) = \gamma_F J_F(f_L, f_R, \overline{h}_L, \overline{h}_R). \end{aligned}$$

□

The algebra $\mathcal{A}_F$ has a natural representation on H_F as follows. Any element $(z, q, m) \in \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) = \mathcal{A}_F$ acts on H_F as

$$\begin{aligned} &(z, q, m) \cdot \begin{pmatrix} u_L \\ d_L \end{pmatrix} = q \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad (z, q, m) \cdot \begin{pmatrix} u_R \\ d_R \end{pmatrix} = \begin{pmatrix} z u_R \\ \overline{z} d_R \end{pmatrix}, \\ &(z, q, m) \cdot \overline{u}_R = m\overline{u}_R, \quad (z, q, m) \cdot \overline{d}_R = m\overline{d}_R, \\ &(z, q, m) \cdot \begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix} = q \begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix}, \\ &(z, q, m) \cdot e_R = \overline{z} e_R, \quad (z, q, m) \cdot \begin{pmatrix} \overline{e}_L \\ \overline{e}_R \end{pmatrix} = \begin{pmatrix} z\overline{e}_L \\ z\overline{e}_R \end{pmatrix}, \end{aligned}$$

and does similarly for the other generations, where each $q = \alpha + \beta j \in \mathbb{H}$ with $\alpha, \beta \in \mathbb{C}$ acts as the matrix multiplication by

$$q = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix},$$

and each matrix $m \in M_3(\mathbb{C})$ acts on the color indices as (R, G, B) or (r, g, b).

We obtain a spectral triple $(\mathcal{A}_F, H_F, D_F)$, where the Dirac operator as a finite matrix is given by $D_F = Y \oplus \overline{Y}$ the diagonal sum on $H_F = E \oplus \overline{E}$, where Y is the Yukawa coupling matrix, which combines the masses of the fermions with the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix.

The fermionic fields acquire mass through the spontaneous symmetry breaking produced by the Higgs fields. The Yukawa coupling matrix has the form $Y = Y_f \oplus (Y_q \otimes 1)$, where

$$Y_f = \begin{pmatrix} 0 & 0 & M_e \\ 0 & 0 & 0 \\ M_e^* & 0 & 0 \end{pmatrix} \equiv M_e^* \oslash 0 \oslash M_e$$

with respect to the basis as (e_R, ν_L, e_L) and successive generations, while

$$Y_q = \begin{pmatrix} 0 & 0 & M_u & 0 \\ 0 & 0 & 0 & M_d \\ M_u^* & 0 & 0 & 0 \\ 0 & M_d^* & 0 & 0 \end{pmatrix} \equiv (M_u^* \oplus M_d^*) \oslash (M_u \oplus M_d)$$

with respect to the basis as $(u_R, d_R, u_L d_L)$ and successive generations. In the lepton case, up to rotating the fields to mass eigenstates, one obtain a mass term for each fermion, and the off-diagonal terms in M_e of Y_f can be reabsorbed into the definition of the fields. In the quark case, the situation is more complicated and the matrix Y_q can be reduced to the mass eigenvalues and the CKM quark mixing. By rotating the fields, it is possible to eliminate the off-diagonal terms in M_u of Y_q. Then it holds that $VM_dV^* = M_u$, where V is the CKM quark mixing, given by

$$V = \begin{pmatrix} V_{u\text{-}d} & V_{u\text{-}s} & V_{u\text{-}b} \\ V_{c\text{-}d} & V_{c\text{-}s} & V_{c\text{-}b} \\ V_{t\text{-}d} & V_{t\text{-}s} & V_{t\text{-}b} \end{pmatrix}$$

acting on the charge $-\frac{e}{3}$ quarks (of down, strange, and bottom). The entries of this matrix can be expressed in terms of three angles $\theta_{12}, \theta_{23}, \theta_{13}$, and a phase, and can be determined experimentally from weak decays and deep inelestic neutrino scatterings.

The detailed structure of the Yukawa coupling matrix Y_q and in particular the fact that color is not broken allow us to check that the fiinite geometry $(\mathcal{A}_F, H_F, D_F)$ satisfies all the axioms of the definition for a noncommutative spectral manifold.

Any element $a \in \mathcal{A}_F$ and $[D_F, a]$ commute with $J_F \mathcal{A}_F J_F$. These operators preserves the subspace E of H_F. The action of $J_F b^* J_F$ on E for $b = (z, q, m)$ is the multiplication by z or the transpose m^t. It is then not hard to check explicitly the commutation with a or $[D_F, a]$ (cf. [66, §VI 5.δ]). By exchanging the roles of a and b, one sees that a commutes with $J_F b J_F$ and $[D_F, J_F b J_F]$ on $\overline{E}$.

Example 17.1. (Added). (The space X of two points a and b [66, VI:3 Example a]). Let $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ the direct sum of $\mathbb{C}$. An element $f \in \mathcal{A}$ corresponds

$(f(a), f(b)) \in \mathbb{C}^2$. Define a noncommutative geometry $(\mathcal{A}, H, D, \gamma)$ as that $H = H_a \oplus H_b$ the direct sum of $\mathbb{C}^n$ (or a Hilbert space), on which $\mathcal{A}$ is represented as

$$\pi(f) = \begin{pmatrix} f(a)1 & 0 \\ 0 & f(b)1 \end{pmatrix} \in M_{2n}(\mathbb{C}), \quad f \in \mathcal{A}$$

on $H = H_a \oplus H_b$, and

$$D = \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_{2n}(\mathbb{C}),$$

where $M : H_a \to H_b$ and $M^* : H_b \to H_a$ are linear operators. Compute that for $f \in \mathcal{A}$,

$$\begin{aligned}
[D, \pi(f)] &= D\pi(f) - \pi(f)D \\
&= \begin{pmatrix} 0 & M^* f(b) - f(a)M^* \\ Mf(a) - f(b)M & 0 \end{pmatrix} \\
&= (f(b) - f(a)) \begin{pmatrix} 0 & M^* \\ -M & 0 \end{pmatrix}.
\end{aligned}$$

Then the norm of the commutator $[D, f]$ is $|f(b) - f(a)|\lambda$, where λ is the largest eigenvalue $\|M\|$ of $|M| = \sqrt{M^* M}$.

Note that a matrix D' is skew hermitian, i.e., $(D')^* = -D'$ if and only if iD' is hermitian, i.e., $(iD')^* = iD'$, which is diagnalizable by a unitary matrix, and so is D'.

Therefore, the distance between between the points is

$$d(a, b) = \sup\{|f(a) - f(b)| \mid \|[D, f]\| \leq 1\} = \frac{1}{\lambda} = \frac{1}{\|M\|}.$$

Consider the case where $E = \mathcal{A}$, i.e., the trivial bundle over X. Let $\Omega_1(\mathcal{A})$ be the space of universal 1-forms on $\mathcal{A}$, given by the kernel of the multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ defined as $m(f \otimes g) = fg$. Since fg is identified with

$$(f(a)g(a), f(b)g(b)) \in \mathbb{C}^2 = X$$

and $f \otimes g$ is identified with

$$(f(a)g(a), f(a)g(b), f(b)g(a), f(b)g(b)) \in \mathbb{C}^4,$$

the forms are functions on $X \times X$ that vanish on the diagonal. Thus, $\Omega^1(\mathcal{A}) \cong \mathbb{C}^2$ as a space. Let $e \in \mathcal{A}$ denote the idempotent such that $e(a) = 1$ and $e(b) = 0$, i.e., $e = (1, 0) \in \mathbb{C}^2$. This space has a basis as $\omega = \lambda ede + \mu(1-e)d(1-e)$ with

$$ede = e \otimes e \quad \text{and} \quad (1-e) \otimes (1-e) = (1-e)d(1-e)$$

The differential $d : \mathcal{A} \to \Omega^1(\mathcal{A})$ is defined to be the difference

$$df = (\Delta f)ede - (\Delta f)(1-e)d(1-e), \quad \Delta f = f(a) - f(b).$$

It is a derivation. Note that $e(de) = ede$ and $(1-e)(d(1-e)) = (1-e)d(1-e)$ and that $d(1-e) = -de$, so that $ed(1-e) = -ede$ and $(1-e)d(1-e) = -(1-e)de$.

Proof. (Added). Indeed,

$$d1 = (1-1)ede - (1-1)(1-e)d(1-e) = 0.$$

Also, with $x(fdg)y = xfd(gy) - xfgdy$ for $x, y, f, g \in \mathcal{A}$,

$$\begin{aligned}
(df)g + f(dg) &= \{(f(a) - f(b))ede - (f(a) - f(b))(1-e)d(1-e)\}g \\
&\quad + f\{(g(a) - g(b))ede - (g(a) - g(b))(1-e)d(1-e)\} \\
&= \{\Delta f(ed(eg) - e^2dg) - (f(a) - f(b))((1-e)d((1-e)g) - (1-e)^2dg)\} \\
&\quad + \{(g(a) - g(b))(fe)de - (g(a) - g(b))(f(1-e))d(1-e)\} \\
&= \{(f(a) - f(b))(e\{g(a)ede - g(a)(1-e)d(1-e)\} \\
&\quad - e\{(g(a) - g(b))ede - (g(a) - g(b))(1-e)d(1-e)\}) \\
&\quad - (f(a) - f(b))((1-e)\{(-g(b))ede - (-g(b))(1-e)d(1-e)\} \\
&\quad - (1-e)\{(g(a) - g(b))ede - (g(a) - g(b))(1-e)d(1-e)\})\} \\
&\quad + \{f(a)(g(a) - g(b))ede - f(b)(g(a) - g(b))(1-e)d(1-e)\} \\
&= \{(f(a) - f(b))(g(a)ede - (g(a) - g(b))ede) \\
&\quad - (f(a) - f(b))(g(b)(1-e)d(1-e) + (g(a) - g(b))(1-e)d(1-e))\} \\
&\quad + \{f(a)(g(a) - g(b))ede - f(b)(g(a) - g(b))(1-e)d(1-e)\} \\
&= \{(f(a) - f(b))g(b)ede - (f(a) - f(b))g(a)(1-e)d(1-e)\} \\
&\quad + \{f(a)(g(a) - g(b))ede - f(b)(g(a) - g(b))(1-e)d(1-e)\} \\
&= (f(a)g(a) - f(b)g(b))ede - (f(a)g(a) - f(b)g(b))(1-e)d(1-e) = d(fg).
\end{aligned}$$

□

If $f \in \mathcal{A}$ and $\omega \in \Omega^1(\mathcal{A})$, then $f\omega \neq \omega f$ in general.

Proof. (Added). Note that for $\omega = \lambda ede + \mu(1-e)d(1-e)$.

$$\begin{aligned}
f\omega &= \lambda fede + \mu f(1-e)d(1-e) = \lambda f(a)ede + \mu f(b)(1-e)d(1-e), \\
\omega f &= \lambda(ed(ef) - e^2df) + \mu((1-e)d((1-e)f) - (1-e)^2df) \\
&= \lambda\{f(a)ede - e((\Delta f)ede - (\Delta f)(1-e)d(1-e))\} \\
&\quad + \mu\{f(b)(1-e)d(1-e) - (1-e)df\} \\
&= \lambda f(b)ede + \mu f(a)(1-e)d(1-e),
\end{aligned}$$

and hence,

$$f\omega - \omega f = \lambda(\Delta f)ede - \mu(\Delta f)(1-e)d(1-e),$$

which becomes zero if and only if $\lambda(\Delta f)$ and $\mu(\Delta f)$ are zero. In particular, if $\Delta f = 0$, then $f\omega = \omega f$ for any $\omega \in \Omega^1(\mathcal{A})$. □

If M is nonzero, then the $\mathcal{A}$-bimodule representation $\pi : \Omega^1(\mathcal{A}) \to \mathbb{B}(H)$ given as

$$\pi(\lambda ede + \mu(1-e)d(1-e)) = \begin{pmatrix} 0 & -\lambda M^* \\ \mu M & 0 \end{pmatrix} \in \mathbb{B}(H)$$

is injective, and is extended to be the representation $\pi : \Omega^*(\mathcal{A}) \to \mathbb{B}(H)$, and set $\Omega^1(\mathcal{A}) = \Omega^1_D(\mathcal{A})$.

Proof. (Added). The additivity as well as the injectivity are clear. As for the left multiplication by any $f \in \mathcal{A}$ to $\omega = \lambda ede + \mu(1-e)d(1-e)) \in \Omega^1(\mathcal{A})$,

$$\begin{aligned}
\pi(f\omega) &= \pi(\lambda f(a)ede + \mu f(b)(1-e)d(1-e)) \\
&= \begin{pmatrix} 0 & -\lambda f(a)M^* \\ \mu f(b)M & 0 \end{pmatrix} = \begin{pmatrix} f(a) & 0 \\ 0 & f(b) \end{pmatrix} \begin{pmatrix} 0 & -\lambda M^* \\ \mu M & 0 \end{pmatrix} \\
&= \pi(f)\pi(\lambda ede + \mu(1-e)d(1-e)) = \pi(f)\pi(\omega).
\end{aligned}$$

As for the right,

$$\begin{aligned}
\pi(\omega f) &= \pi(\lambda f(b)ede + \mu f(a)(1-e)d(1-e)) \\
&= \begin{pmatrix} 0 & -\lambda f(b)M^* \\ \mu f(a)M & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\lambda M^* \\ \mu M & 0 \end{pmatrix} \begin{pmatrix} f(a) & 0 \\ 0 & f(b) \end{pmatrix} \\
&= \pi(\lambda ede + \mu(1-e)d(1-e))\pi(f) = \pi(\omega)\pi(f).
\end{aligned}$$

□

A vector potential is given by a self-adjoint element V of $\Omega^1_D(\mathcal{A})$, i.e., by a complex number Φ, with

$$\pi(V) = \begin{pmatrix} 0 & \overline{\Phi}M^* \\ \Phi M & 0 \end{pmatrix},$$

so that $V = -\overline{\Phi}ede + \Phi(1-e)d(1-e)$. Its curvature is given by the 2-form in $\Omega^2(\mathcal{A})$:

$$\begin{aligned}
\theta = \nabla^2 = dV + V^2 &= -\overline{\Phi}dede - \Phi dede + (\overline{\Phi}ede - \Phi(1-e)de)^2 \\
&= -(\Phi + \overline{\Phi})dede - (\Phi\overline{\Phi})edede
\end{aligned}$$

(the last term corrected), where

$$\begin{aligned}
&e(de)(1-e) = ed(e(1-e)) - e^2 d(1-e) = ede, \\
&e(de)e = ed(e^2) - e^2 de = 0, \\
&(1-e)(de)(1-e) = (1-e)d(e(1-e)) - (1-e)ed(1-e) = 0.
\end{aligned}$$

Also,

$$\pi(de) = \pi(ede + (1-e)de) = \begin{pmatrix} 0 & -M^* \\ M & 0 \end{pmatrix}$$

and

$$\pi(dede) = \pi(de)\pi(de) = \begin{pmatrix} -M^*M & 0 \\ 0 & -MM^* \end{pmatrix}$$

Recall the following in general. Let E be a Hermitian, finitely generated, projective module over $\mathcal{A}$. A connection on E is given by a linear mapping $\nabla : E \to E \otimes_{\mathcal{A}} \Omega_D^1$ such that

$$\nabla(\xi a) = (\nabla \xi) a + \xi \otimes da, \quad \xi \in E, a \in \mathcal{A}.$$

A connection ∇ is compatible with the metric if and only if

$$\langle \xi, \nabla \eta \rangle - \langle \nabla \xi, \eta \rangle = d \langle \xi, \eta \rangle, \quad \xi, \eta \in E.$$

Extend ∇ to a unique linear mapping $\nabla^{\sim}$ from $E^{\sim} = E \otimes_{\mathcal{A}} \Omega_D^*$ to $E^{\sim}$, such that

$$\nabla^{\sim}(\xi \otimes \omega) = (\nabla \xi)\omega + \xi \otimes d\omega, \quad \xi \in E, \omega \in \Omega_D^*.$$

Define $\theta^{\sim} = (\nabla^{\sim})^2$ and $\theta = \theta^{\sim}|_E$ in $\mathrm{Hom}_{\mathcal{A}}(E, E \otimes_{\mathcal{A}} \Omega_D^2)$.

Recall also the following (cf. [159, 7.3]) in general. Let $(\mathcal{A}, H, D)$ be an n-dimensional spectral triple. Then the tracial state on $\mathcal{A}$ is defined by

$$\varphi(a) = \mathrm{tr}_{\omega}(\pi(a)|D|^{-n}), \quad a \in \mathcal{A}.$$

It is extended to $\pi(\Omega^*(\mathcal{A})) = \oplus_p \pi(\Omega^p(\mathcal{A}))$ by replacing $\mathcal{A}$ with $\pi(\Omega^*(\mathcal{A}))$, under the regularity conditions as that $\mathcal{A}^2 = \mathcal{A}$ or that $\mathcal{A}^2$ is subalgebra large enough, where the subalgebra $\mathcal{A}^2$ is generated by elements $a \in \mathcal{A}$ such that both a and $[D, a]$ are in the domain of the derivation δ defined by $\delta(\cdot) = [|D|, \cdot]$, which is the generator for the automorphisms $\mathrm{Ad}(e^{is|D|})$ for $s \in \mathbb{R}$. Then the inner product on $\pi(\Omega^p(\mathcal{A}))$ is defined by

$$\langle \omega_1, \omega_2 \rangle_p = \mathrm{tr}_{\omega}(\omega_1^* \omega_2 |D|^{-n}), \quad \omega_1, \omega_2 \in \pi(\Omega^p(\mathcal{A})),$$

where π may be removed in such a case where π is faithful.

As well, recall the following (cf. [159, 8.1]) in general. A vector potential V is a self-adjoint element of $\Omega_D^1(\mathcal{A})$. The corresponding field strength is the 2-form $\theta = dV + V^2 \in \Omega_D^2(\mathcal{A})$. Then

$$V = \sum_j a_j [D, b_j] = V^*, \quad a_j, b_j \in \mathcal{A},$$

as a sum, written in several ways. Then

$$dV = \sum_j [D, a_j][D, b_j]$$

modulo a junk 2-form. Then $\theta = \theta^*$, since $dV = (dV)^*$ modulo a junk 2-form.

It then follows in that case that the Yang-Mills action is the formula:

$$\begin{aligned} \mathrm{YM}(V) &= \langle \theta = dV + V^2, dV + V^2 \rangle_2 \equiv YM(\nabla) \\ &= \mathrm{tr}_{\omega}(\pi(\theta)^2 |D|^{-d}) = 2(|\Phi + 1|^2 - 1)^2 \mathrm{tr}((M^* M)^2) \end{aligned}$$

with $d = 0$ and $\mathrm{tr}_{\omega} = \mathrm{tr}$.

Proof. (Added). In fact, we compute

$$\begin{aligned}\pi(edede) &= \pi(ede)\pi(de)\\ &= \begin{pmatrix}0 & -M^*\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & -M^*\\ M & 0\end{pmatrix} = \begin{pmatrix}-MM^* & 0\\ 0 & 0\end{pmatrix}\end{aligned}$$

(cf. [159, Section 7.2.2]) and hence,

$$\begin{aligned}\pi(\theta)^2 &= \left(-(\Phi+\overline{\Phi})\begin{pmatrix}-M^*M & 0\\ 0 & -MM^*\end{pmatrix} - \Phi\overline{\Phi}\begin{pmatrix}-M^*M & 0\\ 0 & 0\end{pmatrix}\right)^2\\ &= (\Phi+\overline{\Phi})^2\begin{pmatrix}(M^*M)^2 & 0\\ 0 & (MM^*)^2\end{pmatrix} + (\Phi\overline{\Phi})^2\begin{pmatrix}(M^*M)^2 & 0\\ 0 & 0\end{pmatrix}\\ &\quad + \{(\Phi+\overline{\Phi})\Phi\overline{\Phi} + \Phi\overline{\Phi}(\Phi+\overline{\Phi})\}\begin{pmatrix}(M^*M)^2 & 0\\ 0 & 0\end{pmatrix}\end{aligned}$$

and thus,

$$\begin{aligned}\mathrm{tr}(\pi(\theta)^2) &= \{(\Phi+\overline{\Phi})^2 + (\Phi+\overline{\Phi})\Phi\overline{\Phi} + \Phi\overline{\Phi}(\Phi+\overline{\Phi}) + (\Phi\overline{\Phi})^2\}\mathrm{tr}((M^*M)^2)\\ &\quad + (\Phi+\overline{\Phi})^2\mathrm{tr}((MM^*)^2)\\ &= \{2(\Phi+\overline{\Phi})^2 + 2\Phi\overline{\Phi}(\Phi+\overline{\Phi}) + (\Phi\overline{\Phi})^2\}\mathrm{tr}((M^*M)^2)\\ &= \{(\Phi+\overline{\Phi})^2 + (\Phi\overline{\Phi}+\Phi+\overline{\Phi})^2\}\mathrm{tr}((M^*M)^2)\end{aligned}$$

with $\mathrm{tr}((MM^*)^2) = \mathrm{tr}((M^*M)^2)$. On the other hand, as in [66, Page 565],

$$\begin{aligned}\mathrm{tr}(\pi(\theta^\sim)^2) &\equiv \mathrm{tr}(\pi(-(\Phi+\overline{\Phi})dede - (\Phi\overline{\Phi})dede)^2)\\ &= 2(\Phi\overline{\Phi}+\Phi+\overline{\Phi})^2\mathrm{tr}((M^*M)^2) = 2(|\Phi+1|^2-1)^2\mathrm{tr}((M^*M)^2)\end{aligned}$$

(Therefore, it seems that the last terms are slightly different.) □

The action of the gauge group $\mathcal{U} = U(1)\times U(1)$ on the space of vector potentials is given by $\gamma_u(V) = udu^* + uVu^*$ for $u = u_a e + u_b(1-e)$.

The fermionic action in this case is given by $\langle\psi, (D+\pi(V))\psi\rangle$, where

$$D + \pi(V) = \begin{pmatrix}0 & (1+\overline{\Phi})M^*\\ (1+\Phi)M & 0\end{pmatrix},$$

which is a term of Yukawa type couling the fields $1+\Phi$ and ψ. ◄

Example 17.2. (Added). (The product space of a 4-dimensional Riemannian manifold V and the two-points space X [66, VI.3 Example b]). Let V be a compact Riemannian spin 4-manifold, $\mathcal{A}_1$ the algebra of functions on V, and (H_1, D_1, γ_5) the Dirac K-cycle on $\mathcal{A}_1$, with its canonical $\mathbb{Z}_2$-grading γ_5 given by the orientation. Let $\mathcal{A}_2 = \mathbb{C}^2$, $H_2 = H_{2,a}\oplus H_{2,b}$, and

$$D_2 = \begin{pmatrix}0 & M^*\\ M & 0\end{pmatrix} \equiv M \oslash M^*$$

(the notation that we made here) as in the example above. Let $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, $H = H_1 \otimes H_2$, and $D = D_1 \otimes 1 + \gamma_5 \otimes D_2$. The algebra $\mathcal{A}$ is commutative and is the algebra of complex-valued functions on the space $Y = V \times X = V_a \sqcup V_b$ the disjoint union as a space, with $V = V_a = V_b$.

The metric on Y associated to the K-cycle (H, D) is given by

$$d(p,q) = \sup_{f \in \mathcal{A}} \{|f(p) - f(q)| \mid \|[D, f]\| \leq 1\}.$$

Since $\mathcal{A} = \mathcal{A}_a \oplus \mathcal{A}_b$ with respect to $Y = V_a \sqcup V_b$, then any $f \in \mathcal{A}$ is a pair (f_a, f_b) of functions on V. Since

$$H = H_a \oplus H_b = (H_1 \otimes H_{2,a}) \oplus (H_1 \otimes H_{2,b}$$

with respect to $H_2 = H_{2,a} \oplus H_{2,b}$, then the action of $f \in \mathcal{A}$ is diagonal as

$$\mathcal{A} \ni f \mapsto \begin{pmatrix} f_a & 0 \\ 0 & f_b \end{pmatrix} \equiv f_a \oplus f_b \in \mathbb{B}(H).$$

The operator D in this decomposition becomes

$$D = \partial_V \otimes (1 \oplus 1) + \gamma_5 \otimes (M \oslash M^*) = \begin{pmatrix} \partial_V \otimes 1 & \gamma_5 \otimes M^* \\ \gamma_5 \otimes M & \partial_V \otimes 1 \end{pmatrix},$$

where ∂_V is the Dirac operator on V and γ_5 is the $\mathbb{Z}_2$-grading of its spinor bundle.

The Connes differential of a function $f \in \mathcal{A}$ is given by

$$\begin{aligned} [D, f] &= D(f_a \oplus f_b) - (f_a \oplus f_b)D \\ &= \begin{pmatrix} (\partial_V \otimes 1)f_a - f_a(\partial_V \otimes 1) & (f_b - f_a)(\gamma_5 \otimes M^*) \\ (f_a - f_b)(\gamma_5 \otimes M) & (\partial_V \otimes 1)f_b - f_b(\partial_V \otimes 1) \end{pmatrix} \end{aligned}$$

with

$$(\partial_V \otimes 1)f_x - f_x(\partial_V \otimes 1) = i^{-1}\gamma(df_x) \otimes 1$$

for $x = a$ or b, where df_x is the usual differential of the restriction of f to V_x of V and the difference $f_a - f_b = f(p_a) - f(p_b) = \Delta f$ for $p_a \in V_a$ and $p_b \in V_b$.

The norm of the operator $[D, f]$ is computed as

$$\|[D, f]\| = \sup_{p \in V} \begin{pmatrix} \|df_a(p)\| & -i\|M\|(\Delta f)(p) \\ i\|M\|(\Delta f)(p) & \|df_b(p)\| \end{pmatrix}$$

where each $\|df_x(p)\|$ is the length of the gradient of f_x at $p \in V_x$. ◀

Example 17.3. (Edited). As described in the example above, we next consider the product space $X \times F$, where X is an ordinary 4-dimensional Riemannian spin manifold and F is the finite geometry described above. This product geometry corresponds to a spectral triple $(\mathcal{A}, H, D)$, obtained as the tensor product of the spectral triple $(C^\infty(X), L^2(X, S), D_1)$ with the spectral triple $(\mathcal{A}_F, H_F, D_F)$, where D_1 is the Dirac operator for X acting on square integrable spinors in $L^2(X,S)$. Namely, the algebra $\mathcal{A}$, the Hilbert space H, and the Dirac operator D are given by

$$\mathcal{A} = C^\infty(X, \mathcal{A}_F), \quad H = L^2(X,S) \otimes H_F, \quad D = D_1 \otimes 1 + \gamma \otimes D_F,$$

where γ is the usual $\mathbb{Z}_2$ grading on the spinor bundle S over X. The induced $\mathbb{Z}_2$ grading on H is the tensor product $\gamma \otimes \gamma_F$, as well as the real structure is given by $J = C \otimes J_F$, where C is the charge conjugation operator on spinors.

Note that we have only used the information on the fermions of the standard model (so far). One sees that the bosons, with the correct quantum numbers, are deduced as inner fluctuations of the metric of the spectral triple $(\mathcal{A}, H, D)$.

It is a general fact that for a noncommutative geometry $(\mathcal{A}, H, D)$, one can consider inner fluctuations of the metric of the form:

$$D \mapsto D + a + JaJ^{-1}, \quad a = \sum a_i[D, a_i'], \quad a_i, a_i' \in \mathcal{A}.$$

In the case of the standard model, a direct computation of the inner fluctuations gives the standard model gauge bosons γ, $W^\pm, Z$, the eight gluons, and the Higgs fields φ with accurate quantum numbers (cf. [69]).

In fact, such a field a of the form above can be separated into a discrete part $a^{(0,1)} = \sum a_i[\gamma \otimes D_F, a_i']$ and a continuous part $a^{(1,0)} = \sum a_i[D_1 \otimes 1, a_i']$, with $a_i = (z_i, q_i, m_i)$ and $a_i' = (z_i', q_i', m_i')$ for $q_i = \alpha_i + \beta_i j$ and $q_i' = \alpha_i' + \beta_i' j$. The discrete part gives a quaternion-valued function

$$q(x) = \sum z_i((\alpha_i' - z_i') + z_i \beta_i' j) = \varphi_1 + \varphi_2 j,$$

which provides the Higgs doublet. The continuous part gives three types of fields as a $U(1)$ gauge field, an $SU(2)$ gauge field, and a $U(3)$ gauge field respectively as

$$U = \sum z_i dz_i', \quad Q = \sum q_i dq_i', \quad M = \sum m_i dm_i',$$

where the last field can be reduced to an $SU(3)$ gauge field M' by subtracting the scalar part of the overall gauge field, which eliminates inessential fluctuations that do not change the metric.

The resulting internal fluctuation of the metric $a + JaJ^{-1}$ is then of the form as the diagonal sum (cf. [69])

$$-2U \oplus \begin{pmatrix} Q_{11} - U & Q_{12} \\ Q_{21} & Q_{22} - U \end{pmatrix} = -2U \oplus [Q - (u \oplus u)]$$

on the basis of leptons (e_R, ν_L, e_L) and successive generations, and

$$[(\frac{4}{3}U + M') \oplus (\frac{-2}{3}U + M')] \oplus [Q + (\frac{1}{3}U + M') \oplus (\frac{1}{3} + M')]$$

on the basis of quarks given by (u_R, d_R, u_L, d_L) and successive generations. As a fact, the expressions above recover all the exact values of the hypercharges Y_L, Y_R that appear in the fermion kinetic term of the standard model Lagrangian.

One can also recover the bosonic part of the standard model Lagrangian from the spectral action principle of Chamseddine-Connes (cf. [47], [48], and [49] the last missing) as a very general principle. As a result (cf. [47]), the Hilbert-Einstein action functional for the Riemannian metric, the Yang-Mills action for the vector potentials, and the self-interaction and the minimal coupling for the Higgs fields all appear with the correct signs in the asymptotic expansion of the number $N(\lambda)$ of eigenvalues of D which are bounded by λ and $-\lambda$ for large postitve λ, namely,

$$N(\lambda) = |\{\text{eigenvalues of } D \text{ in } [-\lambda, \lambda]\}|.$$

The spectral action principle, applied to a spectral triple $(\mathcal{A}, H, D)$, can be stated as saying that the physical action depends only of the spectrum $\sigma(D)$ in $\mathbb{R}$. This spectral datum corresponds to the data (H, D) of the triple, and be independent of the action of $\mathcal{A}$. That different algebras correspond to the same spectral data can be thought of as the noncommutative analogue of isospectral Riemannian manifolds.

A natural expression for an action that depends only on the spectrum $\sigma(D)$ and is additive for direct sums of spaces is of the form

$$\operatorname{tr}(\chi(\lambda^{-1}D)) + \langle\psi, D\psi\rangle,$$

where $\chi(\cdot)$ is a positive even function, and λ is a scale.

In the case of the standard model, the formula above is applied to the full metric including the internal fluctuations and gives the full standard model action minimally coupled with gravity. The fermionic part of the action as the second term above gives (cf. [47], [48])

$$\langle\psi, D\psi\rangle = \int_X (L_{Gf} + L_{Hf})\sqrt{|g|}d^4x.$$

The bosonic part of the action as the first term above, evaluated via hert kernel invariants gives the standard model Lagrangian minimally coupled with gravity. Namely, write the function $\chi(\lambda^{-1}D)$ as the superposition of exponentials. Then compute the trace by a semi-classical approximation from local expressions involving the familiar heat equation expansion. This delivers all the correct terms in the action (cf. [48]).

Note that we here treat the spacetime manifold X in Euclidean signature. The formalism of spectral triples can be extended in various ways to Lorentzian signature (cf. [130]). As the most convenient choice, one may drop the self-adjointness condition for D, while still requiring D^2 to be self-adjoint.

It is remarkable to obtain the standard model action from geometric principles, but there are several shortcomings as follows. (1) First, the finite geometry F is put in by hand, with no conceptual understanding of the representation of $\mathcal{A}_F$ in H_F.

(2) Second, there is a fermion doubling problem in the fermionic part of the action (cf. [165]).

(3) Third, it does not incorporate the neutrino mixing and see-saw mechanism for neutrino masses.

These three problems have been solved in [50] (and [72] missing as a preprint as an early version of which), by keeping the distinction between the following two notions of dimension of a noncommutative space:

• the metric dimension and • the KO-dimension.

The metric dimension manifests itself by the growth of the spectrum of the Dirac operator. It appears that the situation of interest is the 4-dimensional one. In particular, the metric dimension of the finite geometry F is zero.

The KO-dimension as in Definition is only well defined modulo 8 and it takes into account both the $\mathbb{Z}_2$-grading γ of H and the real structure J. The only needed change besides the easy addition of right-handed neutrinos is to do change the $\mathbb{Z}_2$-grading of the finite geometry F to its opposite in the antiparticle sector. It is only due to this that the fermion doubling problem pointed out in [165] can be successfully handled. Moreover, it automatically generate the full standard model, i.e., the model with neutrino mixing and the see-saw mechanism. The fermionic part of the action now involves all the structure of the real spectral triple and takes the form

$$\frac{1}{2}\langle J\xi, D\xi\rangle, \quad \xi \in H^+ = \{\xi \in H \mid \gamma\xi = \xi\}$$

treated as Grassmann variables.

Example 17.4. (Added). (The Glashow-Weinberg-Salam model with $U(1) \times SU(2)$ as unification of electromagnetic and weak forces for leptons as (finite) generators [66, VI.3]). The GWS Lagrangian in the Euclidean (imaginary time) framework with the cooresponding fields is given as

$$L_{GWS} = L_G + L_F + L_\Phi + L_Y + L_V,$$

where the pure gauge boson part L_G is

$$L_G = \frac{1}{4}(G_{\mu\nu a}G_a^{\mu\nu}) + \frac{1}{4}(F_{\mu\nu}F^{\mu\nu}),$$

where

$$G_{\mu\nu a} = \partial_\mu W_{\nu a} - \partial_{nu} W_{\nu a} + g\varepsilon_{abc} W_{\nu b} W_{\nu c} \quad \text{and} \quad F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

are the field strength tensors of an $SU(2)$ gauge field $W_{\mu a}$ and a $U(1)$ gauge field B_μ. (Einstein summation over repeated indices is used here and there.)

And the fermion kinetic term L_F has the form

$$-\sum[\overline{f}_L\gamma^\mu\left(\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + ig'\frac{Y_L}{2}B_\mu\right)f_L + \overline{f}_R\gamma^\mu\left(\partial_\mu + ig'\frac{Y_R}{2}B_\mu\right)f_R],$$

where f_L and f_R are the left and right-handed fermion fields respectively, which for leptons and for each generation are given by a pair, i.e., an isodoublet, of left-handed spinors such as $\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$ and of right-handed spinors as a singlet (e_R), and where Y_L and Y_R are hypercharges, which for leptons are given by $Y_L = -1$ and $Y_R = -2$.

And the kinetic terms for the Higgs fields are

$$L_\Phi = -|\left(\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + i\frac{g'}{2}B_\mu\right)\Phi|^2,$$

where $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$ is an $SU(2)$ doublet of complex scalar fields Φ_1 and Φ_2 with hypercharge $Y_\Phi = 1$.

And the Yukawa coupling of Higgs fields with fermions is

$$L_Y = -\sum[H_{ff'}(\overline{f}_L \cdot \Phi)f'_R + H^*_{ff'}\overline{f'}_R(\Phi^t \cdot f_L)],$$

where $H_{ff'}$ is a general coupling matrix in the space of different fermiones.

And the Higgs self-interaction L_V is the potential

$$L_V = \mu^2(\Phi^t\Phi) - \frac{1}{2}\lambda(\Phi^t\Phi)^2,$$

where $\lambda > 0$ and $\mu^2 > 0$ are scalars. ◀

Example 17.5. (Added). (The dictionary between noncommutative geometry and the quantum filed theory of G.W.S. [66, VI 3])

Table 13: Noncommutative geometry and GWS quantum field theory

Noncommutative geometry	Quantum filed theory
Vector $\psi \in E \otimes_{\mathcal{A}} H, \gamma\psi = \psi$	Chiral fermion f
Differential components of connection ω^a, ω^b	Pure gauge bosons W, B
Finite-difference component of connection $(1+\delta^a)$, δ^b	Higgs field Φ
I_2	Pure gauge boson L_G
I_1	Kinetic terms L_Φ
I_0	Higgs potential L_V
J_0	Fermion kinetic term L_F
J_1	Yukawa coupling L_Y

We have the action

$$\mathrm{YM}(\nabla) = I_2 + I_1 + I_0$$

where each I_j is the integral over M of a Lagrangian density given by the following formulas.

$$\text{For } I_2, \quad |d\omega^a + \omega^a \wedge \omega^a|^2 N_a + |d\omega^b|^2 N_b,$$

where $N_a = \dim H_{2,a}$, $N_b = \dim H_{2,b}$ and the norms are the squared norms for the curvatures of the connections ∇^a and ∇^b, respectively.

$$\text{For } I_1, \quad 2\left|\nabla\begin{pmatrix}1+\varphi_1\\ \varphi_2\end{pmatrix}\right|^2 \operatorname{tr}(M^*M),$$

where ∇ is the covariant differentiation of a pair of scalar fields, given as

$$d + \begin{pmatrix}\omega_{11}^a - \omega_{11}^b & \omega_{12}^a \\ \omega_{21}^a & \omega_{22}^a - \omega_{11}^b\end{pmatrix}.$$

$$\text{For } I_0, \quad \{1 + 2(1 - (|1+\varphi_1|^2 + |\varphi_2|^2))^2\}\operatorname{tr}((\lambda^{\perp}(M^*M))^2),$$

where $\lambda^{\perp}$ is the orthogonal projection of the Hilbert-Schmidt space of matrices onto the orthogonal complement $(\mathbb{C}1)^{\perp}$ of the scalar multiplies of the identity.

The fermionic action is

$$J_0 + J_1 = \langle\psi, D_{\nabla}\psi\rangle,$$

where $\psi \in E \otimes_{\mathcal{A}} H$, $\gamma\psi = \psi$ is given by a pair of left-handed sections of $S \otimes H_{2,a}$ denoted by $\begin{pmatrix}\psi_1^a\\ \psi_2^a\end{pmatrix}$, and a right-handed section of $S \otimes H_{2,b}$ denoted by ψ^b. Both of J_0 and J_1 are given by Lagrangian densities respectively as

$$\begin{aligned} &\overline{\psi}^a(\partial + i^{-1}\gamma(\omega^a))\psi^a + \overline{\psi}^b(\partial + i^{-1}\gamma(\omega^b))\psi^b \\ \text{and} \quad &\overline{\psi}_b M(1+\varphi_1, \varphi_2)\psi_a + h.c. \quad \blacktriangleleft \end{aligned}$$

Example 17.6. (Added). (The dictionary between noncommutative geometry and particle physics [66, VI 5.α]). A translation of a noncommutative geometry or a spectral triple $(\mathcal{A}, H, D)$ is given as

$$\begin{cases} H\text{: the Hilbert space of Euclidean fermions,} \\ D\text{: the inverse of the Euclidean propagator of fermions} \\ U(\mathcal{A})\text{: the gauge group of local gauge transformations,} \end{cases}$$

where $U(\mathcal{A})$ is the unitary group of a $*$-algebra $\mathcal{A}$.

A functor from $*$-algebras to groups is defined by sending $\mathcal{A}$ to $U(\mathcal{A})$. Another functor from algebras to groups is defined by sending an algebra $\mathcal{A}$ to the group $GL(\mathcal{A})$ of invertible elements of $\mathcal{A}$, which plays a fundamental role in Algebraic K-theory. $\blacktriangleleft$

Example 17.7. (Added). (The standard model with $U(1) \times SU(2) \times SU(3)$ [66, VI 5.β]). As with the Glashow-Weinberg-Salam model for leptons, the Lagrangian of the standard model contains five different terms as

$$L = L_G + L_F + L_{\Phi} + L_Y + L_V.$$

The pure gauge boson part L_G is

$$L_G = \frac{1}{4}(G_{\mu\nu a}G_a^{\mu\nu}) + \frac{1}{4}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{4}(H_{\mu\nu b}H_b^{\mu\nu}),$$

where $G_{\mu\nu a}$ is the field strength tensor of an $SU(2)$ gauge field $W_{\mu a}$, F_μ is the field strength tensor of a $U(1)$ gauge field B_μ, and $H_{\mu\nu b}$ is the field strength tensor of an $SU(3)$ gauge field $V_{\mu b}$. This last gauge field, called the gluon field, is the carrier of the strong force, the gauge group $SU(3)$ is the color group, and is the essential new ingredient. The respective coupling constants for the fields W, B, and V are denoted as g, g', and g'', consistent with the previous notation.

The fermion kinetic term L_F is obtained that to the leptonic terms as

$$-\sum_f \left[\overline{f}_L\gamma^\mu(\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + ig'\frac{Y_L}{2}B_\mu)f_L + \overline{f}_R\gamma^\mu(\partial_\mu + ig'\frac{Y_R}{2}B_\mu)f_R\right],$$

one adds the following similar terms involving the quarks:

$$-\sum_f [\overline{f}_L\gamma^\mu(\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + ig'\frac{Y_L}{2}B_\mu + \underline{ig''\lambda_b V_{\mu b}})f_L$$
$$+ \overline{f}_R\gamma^\mu(\partial_\mu + ig'\frac{Y_R}{2}B_\mu + \underline{ig''\lambda_b V_{\mu b}})f_R],$$

where the underline is just for emphasis. For each of the three generations of quarks $\begin{pmatrix} u \\ d \end{pmatrix}$, $\begin{pmatrix} c \\ s \end{pmatrix}$, and $\begin{pmatrix} t \\ b \end{pmatrix}$ one has a left-handed isodoublet such as $\begin{pmatrix} u_L \\ d_L \end{pmatrix}$ and two right-handed $SU(2)$ singlets such as $\begin{pmatrix} u_R \\ d_R \end{pmatrix}$. Each quark field (for a baryon) has 3 colors as red, green, and blue (to make white by 3 colors overlapping) such that for instance, u_R is equal to either u_R^r, u_R^g, or u_R^b. All of these quark fields are thus in the fundamental representation of $SU(3)$. The hypercharges Y_L and Y_R are identical for different generations quarks and leptons and given by the following table below. These mumbers are not explained from the theory but are set by hand so as to get the electro-magnetic charges Q_{em} from the formulas:

$$2Q_{em} = Y_L + 2I_3 \quad \text{and} \quad 2Q_{em} = Y_R,$$

where I_3 is the third generator of the weak isospin group $SU(2)$.

(Added and edited).

Table 14: Hypercharges for fermions in three generations

Fermions	I (Y_L, Y_R)	II (Y_L, Y_R)	III (Y_L, Y_R)
Quarks	u: $(\frac{1}{3}, \frac{4}{3})$ d: $(\frac{1}{3}, -\frac{2}{3})$	c: $(\frac{1}{3}, \frac{4}{3})$ s: $(\frac{1}{3}, -\frac{2}{3})$	t: $(\frac{1}{3}, \frac{4}{3})$ b: $(\frac{1}{3}, -\frac{2}{3})$
Leptons	ν_e: $(-1, \cdot)$ e: $(-1, -2)$	ν_μ: $(-1, \cdot)$ μ: $(-1, -2)$	ν_τ: $(-1, \cdot)$ τ: $(-1, -2)$

The kinetic term L_Φ for the Higgs fields is exactly the same term as in the GWS model for leptons.

The Yukawa coupling of Higgs fields with fermions is given by L_Y as in the GWS model. More explicitly, there is no $H_{ff'} \neq 0$ between leptons and quarks, so that L_Y is a sum of a leptonic part and a quark term.

Since there is no right-handed neutorino in this model, the leptonic part has the form

$$L_{Y,lep} = -G_e\langle\overline{L}_e, \Phi\rangle e_R - G_\mu\langle\overline{L}_\mu, \Phi\rangle\mu_R - G_\tau\langle\overline{L}_\tau, \Phi\rangle\tau_R + h.c.,$$

where L_e is the isodublet $\begin{pmatrix}\nu_{e,L}\\ e_L\end{pmatrix}$, and L_μ, L_τ similarly for the other generations. The coupling constants G_e, G_μ, and G_τ provide the lepton masses through the Higgs vacuum contribution.

The quark Yukawa coupling is more complicated owing to new three terms, providing the masses of the up particles, and to the mixing angles. The first has the form $G\overline{L}u_R\Phi^\sim$, where the isodoublet $L = \begin{pmatrix}u_L\\ q_L\end{pmatrix}$ is obtained from a left-handed up quark and a mixing q_L of left-handed down quarks, and $\Phi^\sim$ has the same isospin but opposite hypercharge to the Higgs doublet Φ and is given by

$$\Phi^\sim = J\Phi^* \text{ with } J = -1 \oslash 1.$$

The Higgs self-interaction L_V is exactly the same form as in the GWS model.

There are essentially, three novel features of this complete standard model with respect to the leptonic case as follows.

(i) The new gauge symmetry as color, with gluons responsible for the strong interaction.

(ii) The new values such as

$$\frac{1}{3}, \frac{4}{3}, -\frac{2}{3}$$

of the hyper-charge for quarks.

(iii) Tne new Yukawa coupling terms as $G\overline{L}u_R\Phi^\sim$. ◄

Added below is the table for super-symmetric (SUSY) particles as super fermions:

Table 15: Super-symmetric partners to fermions in the three generations

Scalar-fermions	I	II	III
S-quarks (unknown)	u_S: $u^\sim$ $(\frac{2}{3}e)$ ss-d: $d^\sim$ $(-\frac{1}{3}e)$	c_S: $c^\sim$ $(\frac{2}{3}e)$ ss-s: $s^\sim$ $(-\frac{1}{3}e)$	t_S: $t^\sim$ $(\frac{2}{3}e)$ ss-b: $b^\sim$ $(-\frac{1}{3}e)$
S-leptons (unknown)	ν_e^S: $\nu_e^\sim$ (0) e_S: $e^\sim$ $(-e)$	ν_μ^S: $\nu_\mu^\sim$ (0) μ_S: $\mu^\sim$ $(-e)$	ν_τ^S: $\nu_\tau^\sim$ (0) τ_S: $\tau^\sim$ $(-e)$

Also added below is the table for super-symmetric (SUSY) particles as super bosons:

Table 16: Super-symmetric partners to gauge bosons and Higgs bosons

Gaugeno particles (unknown)	$g_1^{\sim} \cdots, g_8^{\sim}$ $\gamma^{\sim}, (Z^0)^{\sim}, (W^{\pm})^{\sim}$
Higgsino particles (unknown):	$H^{\sim}$

where the unknown particles $\gamma^{\sim}, (Z^0)^{\sim}$ and (part of) $H^{\sim}$ are thought to be the candidates for dark matter, and super-symmetric partners have spin $\frac{1}{2}$ different from corresponding elementary particles in the standard model.

18 Isospectral deformations of Riemannian manifolds

A rich class of examples of noncommutative manifolds is obtained by cosidering isospectral deformations of a classical Reimannian manifold. These examples satisfy all the axioms of ordinary Riemannian geometry (cf. [69]) except commutativity. Those are obtained by the following by Connes-Landi:

Theorem 18.1. ([79]). *Let M be a compact Riemannian spin manifold. If the isometry group of M has rank $r \geq 2$, then M admits a non-trivial one-parameter isospectral deformation to noncommutative geometries M_θ.*

The main idea of the construction is to deform the standard spectral triples describing the Riemannian geometry M along a two-torus embedded into the isometry group of M, to a family of spectral triples describing noncommutative geometries M_θ.

More precisely, assume that there is an inclusion of the two-torus into the isometry group of a compact Riemannian spin manifold M as

$$\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2 \cong T^2 \subset \mathrm{Iso}(M).$$

For any $s = (s_1, s_2) \in \mathbb{T}^2$ with real parameters, let $u(s)$ be the unitary operator corresponding to s in the subgroup T^2, acting on the Hilbert space $H = L^2(M, S)$ of the real standard spectral triple

$$(\mathcal{A} = C^\infty(M), H = L^2(M, S), D, J),$$

where J is the anti-linear isometry on H as the charge conjugation operator. Equivalently, we may write $u(s) = \exp(i(s_1p_1+s_2p_2))$, where p_1, p_2 are operators corresponding to the Lie algebra generators for $\mathbb{R}^2$ such that the spectrums $\sigma(2p_j) \subset \mathbb{Z}$, $[D, p_j] = 0$, and $\{p_j, J\} = p_jJ + Jp_j = 0$, so that $[u(s), D] = [u(s), J] = 0$.

Proof. (Added). Since D and $i(s_1p_1 + s_2p_2)$ commutes, so does D and its multiples. Hence D and $u(s)$ commute.

We also have

$$[J, i(s_1p_1 + s_2p_2)] = -iJ(s_1p_1 + s_2p_2) - i(s_1p_1 + s_2p_2)J = 0.$$

□

Define the action $\alpha_s(t) = u(s)tu(s)^{-1} = \mathrm{Ad}(u(s))(t)$ for $t \in C^\infty(M)$. Note that $u(s+s') = u(s)u(s')$ since p_1 and p_2 commute.

We say that a bounded operator t on H is of bidegree (n_1, n_2) if $\alpha_s(t) = e^{i(s_1n_1+s_2n_2)}t$ for any $s = (s_1.s_2) \in \mathbb{T}^2$.

For a bounded operator t on H, if the map $\mathbb{T}^2 \ni s \mapsto \alpha_s(t)$ is smooth in norm, then the operator t can be uniquely written as a norm convergent series:

$$t = \sum_{n_1,n_2\in\mathbf{Z}} t^\wedge_{n_1,n_2} \quad \text{with} \quad \alpha_s(t^\wedge_{n_1,n_2}) = e^{i(s_1n_1+s_2n_2)}t^\wedge_{n_1,n_2}$$

for $s = (s_1, s_2) \in \mathbb{T}^2$, i.e., each $t^\wedge_{n_1,n_2}$ is of bidegree (n_1, n_2), where the sequence of norms $\|t^\wedge_{n_1,n_2}\|$ has rapid decay.

For reals $\theta \in \mathbb{R}$, define the left and right twists for such operators t by

$$l_\theta(t) = \sum_{n_1,n_2\in\mathbf{Z}} t^\wedge_{n_1,n_2}\exp(2\pi i\theta n_2p_1) \quad \text{and} \quad r_\theta(t) = \sum_{n_1,n_2\in\mathbf{Z}} \exp(2\pi i\theta n_1p_2)t^\wedge_{n_1,n_2}$$

(corrected). Both series converges in norm, since p_i are self-adjoint operators.

Note that $\|t^\wedge_{n_1,n_2}\exp(2\pi i\theta n_2p_1)\| \leq \|t^\wedge_{n_1,n_2}\|$, and so on.

For reals $\theta \in \mathbb{R}$, we then define the left and right deformed products as

$$x *_{l_\theta} y = e^{2\pi i\theta n_1'n_2}xy \quad \text{and} \quad x *_{r_\theta} y = e^{-2\pi i\theta n_1'n_2}xy$$

(corrected) for $x = x^\wedge_{n_1,n_2}$ a homogeneous operator of bi-degree (n_1, n_2) and $y = y^\wedge_{n_1',n_2'}$ a homogeneous operator of bi-degree (n_1', n_2').

These deformed products with θ omitted satisfy $l(x)l(y) = l(x *_l y)$ and $r(x)r(y) = r(x *_r y)$ (corrected).

Proof. (Added). For $x = x^\wedge_{n_1,n_2}$ and $y = y^\wedge_{n_1',n_2'}$, note that

$$\begin{aligned}\alpha_s(xy) &= \alpha_s(x)\alpha_s(y) = e^{i(s_1n_1+s_2n_2)}xe^{i(s_1n_1'+s_2n_2')}y \\ &= e^{is_1(n_1+n_1')+s_2(n_2+n_2')}xy,\end{aligned}$$

and hence, xy is of bidegree $(n_1 + n_1', n_2 + n_2')$.

We then compute

$$\begin{aligned}l(x *_l y) &= l(e^{2\pi i\theta n_1'n_2}xy) \\ &= e^{2\pi i\theta n_1'n_2}xy\exp(2\pi i\theta(n_2 + n_2')p_1), \\ l(x)l(y) &= (x^\wedge_{n_1,n_2}\exp(2\pi i\theta n_2p_1))(y^\wedge_{n_1',n_2'}\exp(2\pi i\theta n_2'p_1)) \\ &= x^\wedge_{n_1,n_2}\exp(2\pi i\theta n_2p_1)y^\wedge_{n_1',n_2'}\exp(-2\pi i\theta n_2p_1)\exp(2\pi i\theta(n_2 + n_2')p_1) \\ &= x^\wedge_{n_1,n_2}e^{2\pi i\theta n_2n_1'}y^\wedge_{n_1',n_2'}\exp(2\pi i\theta(n_2 + n_2')p_1),\end{aligned}$$

so that we obtain $l(x *_l y) = l(x)l(y)$.

Similarly, we compute

$$\begin{aligned} r(x *_r y) &= r(e^{-2\pi i\theta n_1' n_2} xy) \\ &= e^{-2\pi i\theta n_1' n_2} \exp(2\pi i\theta(n_1 + n_1')p_2)xy, \\ r(x)r(y) &= (\exp(2\pi i\theta n_1 p_2)x^{\wedge}_{n_1,n_2})(\exp(2\pi i\theta n_1' p_2)y^{\wedge}_{n_1',n_2'}) \\ &= \exp(2\pi i\theta(n_1 + n_1')p_2)\exp(-2\pi i\theta n_1' p_2)x^{\wedge}_{n_1,n_2}\exp(2\pi i\theta n_1' p_2)y^{\wedge}_{n_1',n_2'} \\ &= \exp(2\pi i\theta(n_1 + n_1')p_2)e^{-2\pi i\theta n_1' n_2} x^{\wedge}_{n_1,n_2} y^{\wedge}_{n_1',n_2'}, \end{aligned}$$

so that we obtain $r(x *_r y) = r(x)r(y)$. □

May refer to [79] for more details.

The deformed spectral triples M_θ in the theorem above are then obtained as

$$M_\theta = (\mathcal{A}_\theta = l_\theta(C^\infty(X)), H = L^2(X,S), D, J_\theta = J\exp(2\pi i\theta(-p_1 + p_2)))$$

(corrected or revised), where $\mathcal{A}_\theta$ is nothing but $C^\infty(X)$ with pointwise product deformed to the left product $*_l = *_{l_\theta}$.

And J_θ defined so above is an anti-linear isometry on H by definition because J is an anti-linear isometry and $\exp(2\pi i\theta(-p_1 + p_2))$ is a unitary, with $J_\theta = \exp(2\pi i\theta(p_1 - p_2))J$ (corrected or revised) and then $J_\theta^2 = J^2$, and is a twisted involution in the sense that $J_\theta l(x) J_\theta^{-1} = r(J_\theta x J_\theta^{-1})$ (?) for x of bidegree (n_1, n_2).

Proof. (Added, not completed). Note that

$$\begin{aligned} J(2\pi i\theta(-p_1 + p_2))^k &= (2\pi i\theta)^k J(-p_1 + p_2)^k = (2\pi i\theta)^k(p_1 - p_2)^k J \\ &= (2\pi i\theta(p_1 - p_2))^k J. \end{aligned}$$

Therefore, $J_\theta = \exp(2\pi i\theta(p_1 - p_2))J$ as well.

Note that since $\alpha_s(x) = e^{i(s_1 n_1 + s_2 n_2)}x$, then

$$e^{i(s_1(-n_1) + s_2(-n_2))} JxJ^{-1} = J\alpha_s(x)J^{-1} = \alpha_s(JxJ^{-1}).$$

Therefore, JxJ^{-1} is of bidegree $(-n_1, -n_2)$.

Compute that for x of bidegree (n_1, n_2),

$$\begin{aligned} J_\theta l(x) &= J\exp(2\pi i\theta(-p_1 + p_2))x\exp(2\pi i\theta n_2 p_1) \\ &= Je^{2\pi i\theta(-n_1 + n_2)}x\exp(2\pi i\theta(-p_1 + p_2))\exp(2\pi i\theta n_2 p_1), \\ r(JxJ^{-1})J_\theta &= \exp(2\pi i\theta(-n_1)p_2)JxJ^{-1}J\exp(2\pi i\theta(-p_1 + p_2)) \\ &= J\exp(2\pi i\theta n_1 p_2)x\exp(2\pi i\theta(-p_1 + p_2)) \\ &= Je^{2\pi i\theta n_1 n_2}x\exp(2\pi i\theta n_1 p_2)\exp(2\pi i\theta(-p_1 + p_2)), \end{aligned}$$

(so that both do not coincide in our revised case). □

19 Algebraic deformations

There is a general context in which noncommutative spaces are constructed via deformations of commutative algebras. Unlike the isospectral deformations discussed in the previous section, we here proceeds mostly at a formal algebraic level, without involving the operator algebra structure and without invoking the presence of a Riemannian structure.

In what follows, may refer to Kontsevich [156] (cf. [155] (missing)).

The idea of algebraic deformation quantization originates from that for a smooth manifold M as a phase space with a symplectic structure, which defines a Poisson bracket $\{\cdot,\cdot\}$, the system as classical mechanics is quantized by deforming the pointwise product in the algebra $\mathcal{A} = C^\infty(M)$ or its suitable subalgebra to a family of associative products $*_\hbar$ but not necessarily commutative, satisfying $f *_\hbar g \to fg$ as $\hbar \to 0$ and

$$(i\hbar)^{-1}(f *_\hbar g - g *_\hbar f) \to \{f, g\} \quad (\hbar \to 0).$$

Namely, the Poisson bracket is deformed as the limit of products $*_\hbar$ above.

The Poisson bracket for the algebra $C^\infty(M)$ is specified by assigning a section λ of $\Lambda^2(TM)$ such that

$$\{f, g\} = \langle \lambda, df \wedge dg \rangle$$

satisfies the Jacobi identity. Typically this produces a formal deformation as a formal power series in $\hbar$. Namely, the deformation products can be written in terms of a sequence of bi-differential operators $b_k : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, which are bilinear differential maps, such that for $f, g \in \mathcal{A}$,

$$f *_\hbar g = fg + b_1(f,g)\hbar + b_2(f,g)\hbar^2 + \cdots .$$

Moreover, for any two elements of $\mathcal{A}[[\hbar]]$ of formal power series over $\mathcal{A}$ with respect to $\hbar$, the $*_\hbar$ product is defined by

$$\left(\sum_{n\geq 0} f_n\hbar^n\right) *_\hbar \left(\sum_{k\geq 0} g_k\hbar^k\right) = \sum_{n,k\geq 0} f_n g_k \hbar^{n+k} + \sum_{n,k\geq 0, m\geq 1} b_m(f_n, g_k)\hbar^{m+n+k}.$$

Under this perspective, it is proved by Kontsevich [156] (and [155] missing) that such formal deformations always exist, by providing an explicit combinatorial formula that generates all $\{b_2, b_3, \cdots\}$ in the expansion from the b_1, hence in terms of the Poisson structure λ. The formal solution to the equation for $f *_\hbar g$ above can then be written as

$$\sum_{n=0}^{\infty} \hbar^n \sum_{\gamma\in G[n]} \omega_\gamma b_{\gamma,\lambda}(f, g),$$

where $G[n]$ is a set of $n^n(n+1)^n$ labeled graphs with $n+2$ vertices and n edges, ω_γ is a coefficient obtained by integrating a differential form depending on the graph γ on the configuration space of n distinct points in the upper half-plane, and $b_{\gamma,\lambda}$ is a bi-differential operator whose coefficients are derivatives of λ of orders specified by the combinatorial information of the graph γ.

Remark. (Added). Recall from [183] the following facts.

A Poisson structure on a manifold M is defined to be a real bilinear differential operator

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

such that the Jacobi identity holds:

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$$

and $\{f,g\} = -\{g,f\}$ and $\{fg,h\} = f\{g,h\}+g\{f,h\}$. A manifold with a Poisson structure is said to be a Poisson manifold.

The Poisson bracket for a symplectic manifold M defines a Poisson structure on M. Conversely, a non-degenerate Poisson structure on a manifold M defines a symplectic structure for M. Also, the dual space of a Lie algebra has a Poisson structure.

A symplectic manifold is defined to be a differentiable manifold M with a symplectic structure form ω, i.e., a non-degenerate, closed 2-form on M

The Poisson bracket for functions f, g on a symplectic manifold M with ω is defined to be $\{f,g\} = \omega(X_g, X_f)$, so that $[X_f, X_g] = X_{\{f,g\}}$, where X_f, X_g are Hamiltonian vector fields over M associated to f, g respectively.

A deformation quantization for a Poisson manifold M is defined to be a product structure $*$ on $C^\infty[[\hbar]]$ with $\hbar$ a formal element, defined as that for $f, g \in C^\infty(M)$,

$$f * g = \sum_{j=0}^{\infty} \hbar^j f *_j g, \quad f *_j g \in C^\infty(M)$$

where $f *_0 g = fg$, $f *_1 g = \{f,g\}$, and $*_j$ is a real bilinear differential operator on $C^\infty(M)$, so that the associativity holds: $(f * g) * h = f * (g * h)$. ◂

Example 19.1. (Added). ([156, 1.4.1]). As the simplest example as a deformation quantization, we recall the Moyal product for the Poisson structure on $\mathbb{R}^d$ with constant coefficients as

$$\alpha = \sum_{i,j} \alpha_{ij} \partial_i \wedge \partial_j \quad \text{for } \alpha_{ij} = -\alpha_{ji} \in \mathbb{R},$$

where each $\alpha_i = \frac{\partial}{\partial x_i}$ is the partial derivative with respect to the variable x_i $(1 \leq i \leq d)$. The Moyal product $\star_\hbar$ for $C^\infty(\mathbb{R}^d)[[\hbar]]$ is given as that for $f, g \in C^\infty(\mathbb{R}^d)$,

$$\begin{aligned} f \star_\hbar g &= fg + \hbar \sum_{i,j} \alpha_{ij} \partial_i(f) \partial_j(g) + \frac{\hbar^2}{2!} \sum_{i,j,k,l} \alpha_{ij}\alpha_{kl} \partial_i \partial_k(f) \partial_j \partial_l(g) + \cdots \\ &= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{i_1,\cdots,i_n,j_1,\cdots,j_n} \Pi_{k=1}^n \alpha_{i_k j_k} \partial_{i_1} \cdots \partial_{i_n}(f) \partial_{j_1} \cdots \partial_{j_n}(g). \end{aligned}$$ ◂

A setting of deformation quantization, which is compatible with C^*-algebras, is developed by Rieffel [209]. Recall briefly the setting as follows. For simplicity, may restrict to the simple case of a compact manifold.

(Added). Let M be a smooth manifold and let $C^\infty(M)$ the associative $*$-algebra of smooth complex-valued functions on M with pointwise operations. A Poisson bracket on M is a Lie algebra structure on $C^\infty(M)$ as a linear space such that for every $f \in C^\infty(M)$, the linear map $C^\infty(M) \to C^\infty(M)$ sending g to $\{f, g\}$ is a derivation, with $\{f^*, g^*\} = \{f, g\}^*$.

Let TM denote the tangent bundle over M. To give a Poisson structure on M is the same as to give a skew 2-vector field λ on M, which is a cross-section of $\wedge^2 TM$, such that $\{f, g\} = \langle \lambda . df \wedge dg \rangle$ with the Jacobi identity.

Definition 19.2. ([209]). Let M be a compact manifold. A strict Rieffel deformation quantization of $\mathcal{A} = C^\infty(M)$ in the direction of $\{\cdot, \cdot\}$ is obtained by assigning an associative product $\star_\hbar$, an involution $*_\hbar$, and a C^*-norm $\|\cdot\|_\hbar$ on $\mathcal{A}$ for each $\hbar$ in a closed interval I containing 0, such that for every $f \in \mathcal{A}$, the function $\|f\|_\hbar$ on I is continuous (corrected), and for any $f, g \in \mathcal{A}$,

$$\|(i\hbar)^{-1}(f *_\hbar g - g *_\hbar f) - \{f, g\}\|_\hbar \to 0$$

as $\hbar \to 0$.

Remark. (Added). If we denote by $\mathfrak{A}_\hbar$ the C^*-algebra obtained by completing $\mathcal{A}$ by the C^*-norm $\|\cdot\|_\hbar$, then $\mathfrak{A}_0$ becomes $C(M)$ the C^*-algebra of all continuous functions on M with the ordinary pointwise operations, and the family $\{\mathfrak{A}_\hbar\}$ together with $\mathcal{A}$ viewed as a $*$-algebra of continuous cross-sections of this family determines a continuous field of C^*-algebras over I with $\mathfrak{A}_\hbar$ as fibers.

If M is not compact, we may take $\mathcal{A}$ as a $*$-algebra, which contains $C_c^\infty(M)$ with support compact and is contained in $C_0^\infty(M)$ vanishing at infinity. Then it follows that $\mathfrak{A}_0 = C_0(M)$ of continuous functions vanishing at infinity. ◀

The functions involving $\hbar$ are assumed to be analytic with respect to $\hbar$, so that formal power series expansions as in $\mathcal{A}[[\hbar]]$ make sense.

As a remark, the notion of strict deformation quantizations should be regarded as that of integrability for formal solutions.

Rieffel also provides a setting for compatible actions by a Lie group of symmetries and proves that noncommutative tori of higher rank are strict deformation quantization of ordinary tori, that are compatible with the action of the ordinary tori as groups of symmetry. Typically, given a Poisson structure, its strict deformation quantizations are not unique, already in the case of tori.

Using a result of Wassermann [236], Rieffel [209] also produces an example, where formal solutions are not integrable.

Example 19.3. (Added). Let S^2 be the 2-sphere. There is a symplectic structure on S^2 and a corresponding Poisson structure λ, which is invariant under $SO(3)$. ◀

But

Theorem 19.4. ([209, Theorem 7.1]). *There are no $SO(3)$-invariant strict deformation quantizations (should be noncommutative) of the ordinary product on $C^\infty(S^2)$ in the direct to the $SO(3)$-invariant Poisson structure on S^2.*

Remark. It also implies that no $SO(3)$-invariant deformation of the ordinary product in $C(S^2)$ can produce a noncommutative C^*-algebra. This rigidity result reflects a strong rigidity result for $SU(2)$ by Wassermann [236], namely that there are only ergodic actions of $SU(2)$ on von Neumann algebra of type I.

As a fact, there are formal deformations of the Poisson structure that are $SO(3)$-invariant (cf. [18] and [126] both missing), but these only exist as a formal power series in the sense of Kontsevich, and not integrable by the results of Wassermann and Rieffel. ◀

(Added). Recall from [209] the following definition. Let G be a Lie group and α an action of G as a group of diffeomorphisms of a compact, or non-compact, smooth Poisson manifold M, which preserve the Poisson structure. Assume that the corresponding action α of G on $C^\infty(M)$ preserves $\mathcal{A}$. A strict deformation quantization of $\mathcal{A}$ defined above is invariant under the action α if the operator $alpha_x$ on $\mathcal{A}$ with $*_\hbar$ for any $x \in G$ and $\hbar \in I$ is an isometric $*$-automorphism, and the map $G \ni x \mapsto \alpha_x(f)$ is a smooth function on G in the norm $\|\cdot\|_\hbar$ for any $f \in \mathcal{A}$ and $\hbar \in I$, and there is an action α of the Lie algebra $\mathfrak{G}$ of G on $\mathcal{A}$ such that for $X \in \mathfrak{G}$ and $f \in \mathcal{A}$,

$$\alpha_X(f) = \frac{d}{dt}|_{t=0}\alpha_{\exp(tX)}(f)$$

with respect to $\|\cdot\|_\hbar$.

Theorem 19.5. ([209, Theorem 1.3]). *Let $\mathbb{T}^d$ be the d-dimensional torus. Let $\Theta = (\theta_{ij})$ be a skew-symmetric matrix which defines a Poisson structure λ on $C^\infty(\mathbb{T}^d)$ as*

$$\lambda = -\pi^{-1}\sum_{j<k}\theta_{jk}X_j \wedge X_k,$$

where each X_j denotes the vector field on $\mathbb{T}^d$ corresponding to the differentiation in the i-th direction. For each real $\hbar$, let $\mathfrak{A}_\hbar$ the twisted group C^*-algebra of the dual group $\mathbb{Z}^d$ of $\mathbb{T}^d$ obtained by both the Fourier transform carrying $C^\infty(\mathbb{T}^d)$ onto $S(\mathbb{Z}^d)$ the Schwarz space of functions rapidly going to zero at infinity and the corresponding bicharacter $\sigma_\hbar$ for Θ defined as

$$\sigma_\hbar(m,n) = e^{2\pi i\hbar\Theta(m,n)}$$

with Θ viewed as a skew bilinear form on $\mathbb{Z}^d$ by the inner product $\langle\Theta m, n\rangle$. The the family $\{\mathfrak{A}_\hbar\}$ provides a strict deformation quantization for $C^\infty(\mathbb{T}^d)$ in the direction of λ, which is invariant under the evident action of $\mathbb{T}^d$. ◀

Note the following type of phenomenon as a summary. On the one hand, there are formal solutions, formal deformation quantizations about which a lot is known, but for which, in general there may not be an integrability result.

More precisely, in trying to pass from formal to actual solutions, there are cases where the existence fails as in the sphere, and the others as the tori, where the uniqueness fails. The picture that emerges is similar to the case of formal and actual solutions of ordinary differential equations. (Namely, there are formal solutions and actual solutions for ordinary differential equations. Likewise, there are formal noncommutative spaces as $*$-algebras and actual noncommutative spaces as C^*-algebras (integrated) as in the situation above.)

Example 19.6. (Edited). To illustrate that concept, we take a look at the analogous story in the theory of ODE. For a modern viewpoint, may refer to [204] (missing). For instance, there is a formal solution to the Euler equation $x^2y' + y = x$ as a power series expansion $\sum_{n=0}^{\infty}(-1)^n n! x^{n+1}$. But convergent series give existence of actual solutions. Moreover, using summation processes such as Borel summation one can transform a formal solution of an analytic ODE into an actual solution on some local region, but such solution is not unique in general. Also, some divergent series can be summed module functions with some exponential orders. This property known as Gevrey summability holds for formal solutions of analytic ODEs. As stated in a more geometric fashion, it is essentially a cohomological condition. It also shows that whereas on small sectors, there are actual solutions but no uniqueness, on large sectors there is the uniqueness, at the cost of possibly losing existence. A complete answer to summability of formal solutions can then be given in terms of a more refined multi-summability, combining Gevrey series and functions of different order, and the Newton polygon of the equation. ◀

The general flavor of that theory is similar to the problem of formal solutions in noncommutative geometry. It is to be expected that an ambiguity theorem would exist, which accounts for the cases of lack of uniqueness, or of lack of existence, of actual solutions illustrated by the results of Rieffel. Already, in dealing with the noncommutative 2-tori as the first truly non-trivial example of noncommutative spaces, we encounter subtleties related to the difference between the quotient and the deformation approach to the construction of noncommutative spaces.

Example 19.7. (Edited). The smooth $*$-subalgebras in the noncommutative tori (as noncommutative two-torus $\mathbb{T}_\theta^2$ generated by two unitaries u, v such that $uv = e^{2\pi i\theta}vu$) are obtained as deformations of the ordinary product of smooth functions f, g on tori, by setting the (Moyal) products

$$\begin{aligned}(f *_\theta g)(x, y) &= \exp(2\pi i\theta \frac{\partial}{\partial x}\frac{\partial}{\partial y'} f(x, y)g(x', y')|_{x=x', y=y'} \\ &= \sum_{n=0}^{\infty} \frac{(2\pi i\theta)^n}{n!} D_x^n f D_{y'}^n g\end{aligned}$$

for $f, g \in C^\infty(\mathbb{T}^2)$. Note that $u\frac{\partial}{\partial u}$, $v\frac{\partial}{\partial v}$ are derivations for the smooth subalgebras, but not the case for $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial v}$.

The same holds for the quantum plane denoted as $\mathbb{R}_q^2$, that is the algebra (over $\mathbb{R}$) generated by two elements x and y as real variables with relation $xy = qyx$ for $q \neq 0$ a non-zero real (cf. Yu. I. Manin [168] missing).

Those generators u, v for $\mathbb{T}_\theta^2$ can be rotated as $u \mapsto \lambda u$ and $v \mapsto \mu v$ for any $\lambda, \mu \in \mathbb{T}$ without changing the presentation, and to give automorphisms of the noncommutative 2-torus, but such translations of the generators x, y are not defined as automorphisms of the quantum plane. In other words, one can view the noncommutative two-torus as a deformation of the ordinary two-torus, which is a quotient of the classical plane $\mathbb{R}^2$ by a lattice of translations, but the action of translations does not extend to the quantum plane. This is an instance of the fact that operations of quotient and deformation in constructing noncommutative spaces do not satisfy compatibility in that sense.

(Added). In fact, the following diagram does commute:

$$\begin{tikzcd}
\mathbb{C}^2_{e^{2\pi i\theta}} \arrow[r, "t_{\lambda,\mu}"] \arrow[d, "\theta\to 0"'] & \mathbb{C}^2_{e^{2\pi i\theta}} \arrow[r, "i"] \arrow[d, "\theta\to 0"'] & \mathbb{T}^2_\theta \arrow[r, "t_{\lambda,\mu}"] \arrow[d, "\theta\to 0"'] & \mathbb{T}^2_\theta \arrow[d, "\theta\to 0"'] \\
\mathbb{C}[z_1, z_1] \arrow[r, "t_{\lambda,\mu}"] & \mathbb{C}[z_1, z_2] \arrow[r, "i"] & C(\mathbb{T}^2) \arrow[r, "t_{\lambda,\mu}"] & C(\mathbb{T}^2)
\end{tikzcd}$$

where we do define $\mathbb{C}_w^2$ the quantum complex plane as the algebra over $\mathbb{C}$ generated by z_1, z_2 as complex variables with relation $z_1 z_2 = w z_2 z_1$ for w a non-zero complex, and $t_{\lambda,\mu}$ is the translation for $\lambda, \mu \in \mathbb{T}$, and the maps i in the first and second lines are the evident restriction maps and are injective, and both $\mathbb{T}_\theta^2$ and $C(\mathbb{T}^2)$ may be replaced with both the smooth $*$-subalgebra of $\mathbb{T}_\theta^2$ and $C^\infty(\mathbb{T}^2)$, respectively, and down arrows are deformations (reversed).

The Morita equivalence between two noncommutative tori for θ and $\frac{1}{\theta}$ are not detectable in a deformation theoretic perturbative expansion like the Moyal products given above, so that such a phenomenon is not perturbative and cannot seen at the perturbative level as the star products. ◀

In that respect, a version of the structure of spectral triples for non-compact spaces is considered by Gayral, Gracia-Bondía, Iochum, Schücker, and Várilly [117] as follows. In this case, one cannot expect the Dirac operator to have compact resolvent but one can expect a local version to hold, such as that $a(D-i)^{-1}$ are compact for $a \in \mathcal{A}$. Other properties for spectral triples as noncommutative geometry are adapted to this local version, with some difficulties. It is shown that the Moyal product deformation of $\mathbb{R}^{2n}$ fits in the framework of spectral triples and it provides an example of non-unital spectral triples. It then appears that the structure of noncommutative, noncompact Riemannian geometry provided by non-unital spectral triples are adapted to some classes of algebraic deformations in this case.

Example 19.8. (Added). ([117, Theorem 3.2]). It says that the (spectral) triple $(S(\mathbb{R}^k), L^2(\mathbb{R}^k) \otimes \mathbb{C}^{2^{\lfloor \frac{k}{2} \rfloor}}, D)$ with γ, J for $\mathbb{R}^k$ as a spin manifold defines a noncompact, commutative geomerty of spectral dimension k, where for $f \in S(\mathbb{R}^k)$, $[D, f] = Df$, and $f(D+\varepsilon)^{-k}$ are compact operators of Dixmier trace class, with Dixmier trace a scalar multiple of $\int f(x) d^k x$.

Example 19.9. (Added). ([117, Theorem 4.22]). It as the main theorem says that for the Moyal planes $(S(\mathbb{R}^{2n}), *_\theta)$ with the products $*_\theta$, for $\mathbb{R}^{2n}$ as spin manifolds, the triples

$$((S(\mathbb{R}^{2n}), *_\theta), L^2(\mathbb{R}^{2n}) \otimes \mathbb{C}^{2^n}, D) \text{ with } \gamma, J$$

define connected real noncompact, spectral triples of spectral dimension $2n$. ◀

It appears at first that spectral triples may not be the right type of structure to deal with noncommutative spaces associated to algebraic deformations, because a spectral triple corresponds to a form of Riemannian geometry, while such noncommutative spaces originate from Kähler geometry. However, the Kähler structure can often be encoded in the setting of spectral triples, for example, by considering a second Dirac operator as in [28], or through the presence of a Lefshetz operator as in [99].

Noncommutative spaces obtained as deformations of commutative algebras fit in the context of a well developed algebraic theory of noncommutative spaces (cf. Kontsevich [154] and KR [157] both missing, Rosenberg [212] and [211] missing, Manin [168], [169], and [171] triple missing, and Soibelman [224]).

This theory touches on a variety of subjects like quantum groups, and is connected to the theory of mirror symmetry. However, it is often not clear how to integrate this deformation approach with the functional analytic theory of noncommutative geometry, briefly summarized before. Only recently have several results confirmed the existence of an interplay between the algebraic and the functional analytic aspects of noncommutative geometry, especially through the work of Connes and Dubois-Violette (cf. [76], [77], [78]) and of Polishchuk [200]. Also, by the the work of Chakraborty and Pal [45] and of Connes [71] and more recently of [105] and [226], shown is it that quantum groups fit nicely within the framework of noncommutative geometry, described by spectral triples, contrary to what is previously believed. Ultimately, successfully importing tools from the theory of operator algebras into the realm of algebraic geometry might well land within the framework of what Manin refers to as a second quantization of algebraic geometry.

20 Quantum groups

For some time, it is believed that quantum groups could not fit into the setting of noncommutative manifolds, defined in terms of spectral geometry. On the contrary, it is shown by Chakraborty and Pal [45] that the quantum group $SU_q(2)$ for $0 < q < 1$ admits a spectral triple with Dirac operator that is equivariant with respect to its own coaction, as follows.

The $*$-algebra $\mathcal{A}_q$ (or the C^*-algebra $\mathfrak{A}_q$) of (continuous) functions on the quantum group $SU_q(2)$ is generated by two elements α and β with relations

$$\alpha^*\alpha + \beta^*\beta = 1, \quad \alpha\alpha^* + q^2\beta\beta^* = 1,$$
$$\alpha\beta = q\beta\alpha, \quad \alpha\beta^* = q\beta^*\alpha, \quad \beta^*\beta = \beta\beta^*.$$

and with the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ as a group structure defined by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\beta^* \otimes \beta \quad \text{and}$$
$$\Delta(\beta) = \beta \otimes \alpha + \alpha^* \otimes \beta.$$

Note that $SU(2) \cong S^3$, where any element of $SU(2)$ has the form

$$\begin{pmatrix} z & -w^* \\ w & z^* \end{pmatrix}, \quad \text{where}$$
$$\begin{aligned} |z|^2 + |w|^2 &= \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 + \mathrm{Re}(w)^2 + \mathrm{Im}(w)^2 \\ &= 1, \quad z, w \in \mathbb{C}. \end{aligned}$$

If we take $q = 0$, then $\mathcal{A}_0 = \mathfrak{A}_0 = \mathbb{C}$ with $\alpha = 1$ and $\beta = 0$. If we take $q = 1$, then $\mathfrak{A}_1 = C(SU(2))$ the C^*-algebra of continuous functions on $SU(2)$ as a space. The quantum group $SU_q(2)$ may be identified with the $*$-algebra $\mathcal{A}_q$ or the C^*-algebra $\mathfrak{A}_q$. We may also write as $\mathfrak{A}_q = C(SU_q(2))$.

By the representation theory of the quantum group $SU_q(2)$ (cf. [152] missing), similar to its classical counterpart, for each $n = \frac{m}{2}$ for m positive integers, there is a unique irreducible unitary representation $t^{(n)}$ of dimension $2n + 1$. Denote by $t_{ij}^{(n)}$ the (i, j)-entry of $t^{(n)}$ as a $(2n + 1) \times (2n + 1)$ matrix. Define a Hilbert space H with an orthonormal basis $\{e_{ij}^{(n)}\}$ as $t_{ij}^{(n)}$ normalized with $i, j \in \{-n, \cdots, n\}$ for $n \in \frac{1}{2}\mathbb{N}$. Then the unitary representation (omitted) of $SU_q(2)$ on H is defined by

$$\alpha(e_{ij}^{(n)}) = a_+(n,i,j)e_{i-\frac{1}{2},j-\frac{1}{2}}^{(n+\frac{1}{2})} + a_-(n,i,j)e_{i-\frac{1}{2},j-\frac{1}{2}}^{(n-\frac{1}{2})},$$
$$\beta(e_{ij}^{(n)}) = b_+(n,i,j)e_{i+\frac{1}{2},j-\frac{1}{2}}^{(n+\frac{1}{2})} + b_-(n,i,j)e_{i+\frac{1}{2},j-\frac{1}{2}}^{(n-\frac{1}{2})},$$
$$\beta^*(e_{ij}^{(n)}) = b_+^*(n,i,j)e_{i-\frac{1}{2},j+\frac{1}{2}}^{(n+\frac{1}{2})} + b_-^*(n,i,j)e_{i-\frac{1}{2},j+\frac{1}{2}}^{(n-\frac{1}{2})},$$

with coefficients defined by

$$a_+(n,i,j) = q^{2n+i+j+1} F\begin{pmatrix} 2n-2j+2 & 2n-2i+2 \\ 4n+2 & 4n+4 \end{pmatrix},$$
$$a_-(n,i,j) = F\begin{pmatrix} 2n-2j & 2n+2i \\ 4n & 4n+2 \end{pmatrix},$$
$$b_+(n,i,j) = -q^{n+j} F\begin{pmatrix} 2n-2j+2 & 2n+2i+2 \\ 4n+2 & 4n+4 \end{pmatrix},$$
$$b_-(n,i,j) = q^{n+i} F\begin{pmatrix} 2n+2j & 2n-2i \\ 4n & 4n+2 \end{pmatrix},$$
$$b_+^*(n,i,j) = q^{n+i} F\begin{pmatrix} 2n+2j+2 & 2n-2i+2 \\ 4n+2 & 4n+4 \end{pmatrix},$$
$$b_-^*(n,i,j) = -q^{n+j} F\begin{pmatrix} 2n-2j & 2n+2i \\ 4n & 4n+2 \end{pmatrix},$$

where we use the notation as

$$F\begin{pmatrix} k & l \\ s & t \end{pmatrix} = \frac{\sqrt{1-q^k}\sqrt{1-q^l}}{\sqrt{1-q^s}\sqrt{1-q^t}}.$$

The $*$-algebra $\mathcal{A}_q^\wedge$ (or the C^*-algebra $\mathfrak{A}_q^\wedge = C(SU_q(2)^\wedge)$ of (continuous) functions on the dual of $SU_q(2)$ are generated by the following two operators $\alpha^\wedge$, $\beta^\wedge$ on H defined by

$$\alpha^\wedge(e_{ij}^{(n)}) = q^j e_{ij}^{(n)},$$
$$\beta^\wedge(e_{ij}^{(n)}) = \begin{cases} 0 & j = n, \\ \sqrt{q^{-2n}+q^{2n+2}-q^{-2j}-q^{2j+2}}\, e_{i,j+1}^{(n)} & j < n. \end{cases}$$

with

$$\sqrt{q^{-2n}+q^{2n+2}-q^{-2j}-q^{2j+2}} = q^{-n}\sqrt{1-q^{2n-2j}}\sqrt{1-q^{2n+2j+2}}$$

(corrected), satisfying the relations

$$\alpha^\wedge\beta^\wedge = q\beta^\wedge\alpha^\wedge, \quad \alpha^\wedge(\beta^\wedge)^* = q^{-1}(\beta^\wedge)^*\alpha^\wedge, \quad [\beta^\wedge, \alpha^\wedge] = \frac{1}{q-q^{-1}}((\alpha^\wedge)^2 - (\alpha^\wedge)^{-2})$$

and with the coproduct defined by

$$\Delta(\alpha^\wedge) = \alpha^\wedge \otimes \alpha^\wedge - q(\beta^\wedge)^* \otimes \beta^\wedge, \quad \Delta(\beta^\wedge) = (\alpha^\wedge)^{-1} \otimes \beta^\wedge + \beta^\wedge \otimes \alpha^\wedge,$$
$$\Delta((\beta^\wedge)^*) = (\alpha^\wedge)^{-1} \otimes (\beta^\wedge)^* + (\beta^\wedge)^* \otimes \alpha^\wedge.$$

An equivariance condtion means that there is an action on H of the enveloping algebra $\mathcal{U}_q = U_q(SU(2))$, which commutes with the Dirac operator D.

Or an operator on H is said to be equivariant if it commutes with $\alpha^\wedge$, $\beta^\wedge$, and $(\beta^\wedge)^*$ defined above.

Any equivariant self-adjoint (Dirac) operator D on H with discrete spectrum has the form

$$De_{ij}^{(n)} = d(n,i)e_{ij}^{(n)}, \quad d(n,i) \in \mathbb{R}.$$

We then compute the commutators as

$$\begin{aligned}
[D,\alpha]e_{ij}^{(n)} &= D[a_+(n,i,j)e_{i-\frac12,j-\frac12}^{n+\frac12} + a_-(n,i,j)e_{i-\frac12,j-\frac12}^{n-\frac12}] - \alpha(d(n,i)e_{ij}^{(n)}) \\
&= [a_+(n,i,j)(d(n+\frac12, i-\frac12) - d(n,i))]e_{i-\frac12,j-\frac12}^{n+\frac12} \\
&\quad + [a_-(n,i,j)(d(n-\frac12, i-\frac12) - d(n,i))]e_{i-\frac12,j-\frac12}^{n-\frac12}, \\
[D,\beta]e_{ij}^{(n)} &= D[b_+(n,i,j)e_{i+\frac12,j-\frac12}^{n+\frac12} + b_-(n,i,j)e_{i+\frac12,j-\frac12}^{n-\frac12}] - \beta(d(n,i)e_{ij}^{(n)}) \\
&= [b_+(n,i,j)(d(n+\frac12, i+\frac12) - d(n,i))]e_{i+\frac12,j-\frac12}^{n+\frac12} \\
&\quad + [b_-(n,i,j)(d(n-\frac12, i+\frac12) - d(n,i))]e_{i+\frac12,j-\frac12}^{n-\frac12}.
\end{aligned}$$

It then follows that $[D,a]$ is bounded for any $a \in \mathcal{A}_q$ if and only if the following two conditions hold ([45, Propsition 3.1]):

$$\begin{cases} d(n+\frac{1}{2}, i+\frac{1}{2}) - d(n,i) = O(1) & \text{and} \\ d(n+\frac{1}{2}, i-\frac{1}{2}) - d(n,i) = O(n+i+1). \end{cases}$$

The Dirac operator D of that form has compact resolvent if and only if $d(n,i)$ as a single sequence does not have any other limit point than $\pm\infty$.

For instance as the origine, we may define the operator D as

$$De_{ij}^{(n)} = \begin{cases} 2n+1 & n \neq i, \\ -2n-1 & n = i, \end{cases}$$

defined by Connes [71].

It then follows that

Theorem 20.1. *An $SU_q(2)$-equivariant odd 3-summable spectral triple is obtained by $(\mathcal{A}_q, H, D)$ defined above.*

The classical group $SU(2)$ has topological and metric dimension three, and the topological dimension of the $*$-algebra $\mathcal{A}_q$ drops to one, but the metric dimension of the spectral triple remains equal to three (cf. [45]).

Moreover, it is shown by Chakraborty and Pal [45] that the Chern character of the spectral triple is nontrivial.

(Added). Indeed, it is shown that the pairing between the (algebraic or C^*-algebraic) K-theory and the K-homology via the Kasparov product as

$$K_1(\mathcal{A}_q) \times K^1(\mathcal{A}_q) = KK^1(\mathbb{C}, \mathcal{A}_q) \times KK^1(\mathcal{A}_q, \mathbb{C}) \xrightarrow{\otimes} K_0(\mathbb{C}) \cong \mathbb{Z}$$

is non-trivial and onto, where $\mathcal{A}_q$ may be replaced with $\mathfrak{A}_q$. Computed is the pairing between $\mathrm{sign}(D) = D|D|^{-1}$ for some (equivariant or not) spectral triples $(\mathcal{A}_q, H, D)$ obtained above and the generator class $[u]$ of $K_1(\mathcal{A}_q)$ corresponding to the elements $u = \chi_1(\beta^*\beta)(\beta - 1) + 1$ or $u_r = (\beta^*\beta)^r(\beta - 1) + 1 \in \mathcal{A}_q$, where $\chi_1(\cdot)$ means the characteristic function at 1 on the spectrum of $\beta^*\beta$, and $\chi_1(\beta^*\beta)$ is defined by functional calculus, and $r \in \mathbb{N}$ with $q^{2r} < \frac{1}{2} < q^{2r-2}$. Namely,

$$\langle [u], [(\mathfrak{A}, H, D)] \rangle = [u] \otimes [\mathrm{sign}(D)] = -\mathrm{index}(p_0 u_r p_0) \neq 0,$$

where p_0 is the projection onto the eigenspace corresponding to the eigenvalue -1 of $\mathrm{sign}(D)$. Note that $\mathrm{sign}(D)^2 = D^2|D|^{-2} = 1$. ◀

On the other hand, by Connes [71], given is an explicit formula for its local index cocycle, where provided is a cochain whose coboundary is the difference between the Chern characater and the local version in terms of remainders in the rational approximation to the logarithmic derivative of the Dedekind eta function.

The local index formula is obtained by constructing a symbol map $\rho : \mathfrak{B} \to C^\infty(S_q^*)$, where the algebra $C^\infty(S_q^*)$ is a noncommutative version of the cosphere bundle, that is, the cosphere bundle S_q^* of $SU_q(2)$, with a restriction map $r : C^\infty(S_q^*) \to C^\infty(D_{q+}^2 \times D_{q-}^2)$ the algebra of two noncommutative disks, and where $\mathfrak{B}$ is the algebra generated by the elements $\delta^k(a)$ for $a \in \mathcal{A}$, with $\delta(a) = [|D|, a]$. On the cosphere bundle, there is a geodesic flow, induced by the group of automorphisms of $\mathcal{A}$ sending $a \mapsto e^{it|D|}ae^{-it|D|}$. Then $\rho(b)^0$ is defined to be the component of degree zero with respect to the grading induced by this flow.

The algebras $C^\infty(D_{q\pm}^2)$ have extensions

$$0 \to \mathbb{K}^\infty \to C^\infty(D_{q\pm}^2) \xrightarrow{\sigma} C^\infty(S^1) \to 0,$$

where the ideal $\mathbb{K}^\infty$ is the algebra of rapidly decaying matrices in the C^*-algebra $\mathbb{K}$ of compact operators.

There are linear functionals τ_1 and τ_0 on $C^\infty(D_{q\pm}^2)$ defined by

$$\tau_1(a) = \frac{1}{2\pi}\int_0^{2\pi} \sigma(a)d\theta \quad \text{and} \quad \tau_0(a) = \lim_{n\to\infty}\left\{\sum_{k=0}^{n}\langle ae_k, e_k\rangle - n\tau_1(a)\right\},$$

where τ_0 is defined in terms of the representation of $C^\infty(D_{q\pm}^2)$ on the Hilbert space $l^2(\mathbb{N})$ with the canonical orthonormal basis $\{e_k\}$. Note that τ_1 is zero on $\mathbb{K}^\infty$ and $\tau_0 = \mathrm{tr}$ the usual trace on the same.

Now, recall from [66, III.1] that a cycle of dimension n is defined to be a triple $(\Omega, d, \int)$ where $(\Omega = \oplus_{j=0}^n \Omega^j, d)$ is a graded differential algebra over $\mathbb{C}$ with d a graded derivation of degree 1 such that $d : \Omega^j \to \Omega^{j+1}$ and $d^2 = d \circ d = 0$, and $\int : \Omega^n \to \mathbb{C}$ is a closed graded trace on Ω. A cycle over an algebra $\mathcal{A}$ over $\mathbb{C}$ is given by a cycle $(\Omega, d, \int)$ together with a homomorphism $\rho : \mathcal{A} \to \Omega^0$.

Example 20.2. (Added). ([66, III.1]). Let M be a smooth compact manifold, and let C be a closed deRham current on M of dimension $q \leq \dim M$. For $0 \leq j \leq q$, define $\Omega^j = C^\infty(M, \wedge^j T^*M)$ the space of smooth differential forms on M of degree j, where $\Omega^0 = C^\infty(M)$. Then $\Omega = \oplus_{j=0}^q \Omega^j$ becomes a differential algebra with the usual pointwise operations and differentiation. Define a closed graded trace on Ω^q as $\int \omega = \langle C, \omega\rangle$ for $\omega \in \Omega^q$. ◀

In the case of the algebra $\mathcal{A}_q = C^\infty(SU_q(2))$ of $SU_q(2)$, a cycle $(\Omega, d, \int)$ over $\mathcal{A}_q$ is given as in [71] by $\Omega = \Omega^0 \oplus \Omega^1$ with $\Omega^0 = \mathcal{A}_q$, where $\Omega^1 = \mathcal{A}_q \oplus \Omega^{(2)}(S^1)$, with $\Omega^{(2)}(S^1)$ the space of weight two differential forms $f(\theta)d\theta^2$ on S^1 and with the $\mathcal{A}_q$-bimodule structure for Ω^1as

$$a(\xi, f) = (a\xi, \sigma(a)f) \quad \text{and} \quad (\xi, f)a = (\xi a, -i\sigma(\xi)\sigma(a)' + f\sigma(a)),$$

and with differential $d : \mathcal{A}_q \to \Omega^1$

$$da = \partial a + \frac{1}{2}\sigma(a)''d\theta^2$$

with derivation $\partial = \partial_\beta - \partial_\alpha$, and with trace $\int : \Omega^1 \to \mathbb{C}$ as

$$\int(\xi, f) = \tau(\xi) + \frac{1}{2\pi i}\int f d\theta, \quad \tau(a) = \tau_0(\tau_-(a^{(0)}))$$

with $a^{(0)}$ the component of degree zero for ∂ and τ_- the restriction to $C^\infty(D^2_{q-})$. This definition for the cycle corrects for the fact that τ itself as well as τ_0 fail to be traces.

(Added). By construction, τ is ∂-invariant.

The Hochshild co-boundary $b\tau_0$ is computed as

$$b\tau_0(a_0, a_1) = \tau_0(a_0 a_1) - \tau_0(a_1 a_0) = \frac{-1}{2\pi i}\int \sigma(a_0)d\sigma(a_1).$$

Also,

$$b\tau(a_0, a_1) = \frac{1}{2\pi}\int b\tau_0(a_0(t), a_1(t))dt = b\tau_0(a_0, a_1),$$

where $a_j(t) = \nu(t)(a_j) = \exp(it\partial)a_j$.

Define the following trace as the residue at zero of the zeta function with respect to x and $|D|$:

$$\oint x = \mathrm{res}_{z=0}\mathrm{tr}(x|D|^{-z}) = \mathrm{res}_{z=0}\zeta_x(z)$$

defined on the algebra generated by $\mathcal{A} = \mathcal{A}_q$, $[D, \mathcal{A}]$, and $|D|^z$, where $z \in \mathbb{C}$ ◀

It then follows the following by Connes [71]:

Theorem 20.3. (1) *The spectral triple $(\mathcal{A}_q, H, D)$ as in the theorem above has dimension spectrum equal to $\{1, 2, 3\}$.*

(2) *The residue formula for psudo-differential operators $a \in \mathfrak{B}$ in terms of their symbols is given by*

$$\oint a|D|^{-1} = (\tau_0 \otimes \tau_0)(r(\rho(a)^0)), \quad \oint a|D|^{-3} = (\tau_1 \otimes \tau_1)(r(\rho(a)^0)),$$

$$\text{and} \quad \oint a|D|^{-2} = (\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1)(r(\rho(a)^0)).$$

(3) *The character $\chi(a_0, a_1) = \int a_0 da_1$ of the cycle $(\Omega, d, \int)$ is equal to the cyclic cocycle*

$$\psi_1(a_0, a_1) = 2\oint a_0\delta(a_1)p|D|^{-1} - \oint a_0\delta^2(a_1)p|D|^{-1},$$

with $p = \frac{1}{2}(1 + F)$.

The local index formula for the local index cocycle φ is given by

$$\varphi_{\mathrm{odd}} = \psi_1 - (b + B)\varphi_{\mathrm{even}}$$

in the sense that

$$\varphi_1 = \psi_1 - (b + B)\varphi_0 \quad \text{and} \quad \varphi_3 = \psi_1 - (b + B)\varphi_2,$$

where the co-chaines φ_0, φ_2 are given by

$$\varphi_0(a) = \mathrm{tr}(a|D|^{-z}_{z=0} \quad \text{and} \quad \varphi_2(a_0, a_1, a_2) = \frac{1}{24}\oint a_0\delta(a_1)\delta^2(a_2)|D|^{-3}.$$

(4) *The character* $\mathrm{tr}(a_0[F, a_1])$ *differs from the local form* ψ_1 *by the coboundary* $b\psi_0$, *with* $\psi_0(a) = 2\mathrm{tr}(ap|D|^{-s})_{s=0}$. *This cochain is determined by the values* $\psi_0((\beta^*\beta)^n)$ *of the form* $2q^{-2n}(q^2R_n(q^2) - G(q^2))$, *where* $G(q^2)$ *is the logarithmic derivative* $q^2\partial_{q^2}\log(\eta(q^2))$ *(up to sign) of the Dedekind eta function*

$$\eta(q^2) = q^{\frac{1}{12}}\Pi_{k=1}^{\infty}(1 - q^{2k}),$$

so that (up to constant)

$$G(q^2) = \sum_{k=1}^{\infty} \frac{kq^{2k}}{1 - q^{2k}},$$

and $R_n(\cdot)$ *are rational functions with poles only at roots of unity.*

Proof. (Only a part added). We have

$$\log(\eta(q^2)) = \frac{1}{24}\log q^2 + \sum_{k=1}^{\infty}\log(1 - q^{2k}), \quad \text{with } q^2 = x, \text{ so,}$$

$$\partial_x \log(\eta(x)) = \frac{1}{24x} + \sum_{k=1}^{\infty}\frac{-kx^{k-1}}{1 - x^k}, \quad \text{with } x = q^2, \text{ so,}$$

$$q^2\partial_{q^2}\log(\eta(q^2)) = \frac{1}{24} - \sum_{k=1}^{\infty}\frac{kq^{2k}}{1 - q^{2k}}.$$

□

More recently, another breakthrough in the relation between quantum groups and the formalism of spectral triples is obtained by [105] and [226]. Constructed by these is a 3+-summable spectral triple $(\mathcal{A}_q, H, D)$, where $\mathcal{A}_q$ is the algebra of coordinates of the quantum group $SU_q(2)$. The geometry in this case is an isospectral deformation of the classical case, in the sense that the Dirac operator D is the same as the Dirac operator the round metric on the ordinary 3-sphere S^3. Moreover, that spacetral triple is nice as much as that it is equivariant with respect to both left and right action of the (Hopf) enveloping algebra $\mathcal{U}_q(SU(2))$.

The classical Dirac operator for the round metric on S^3 has spectrum $\Sigma = \Sigma_+ \cup \Sigma_-$, with

$$\Sigma_+ = \{2j + \frac{3}{2} \,|\, j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\} \quad \text{and} \quad \Sigma_- = \{-(2j + \frac{1}{2}) \,|\, j = \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$$

with multiplicities $(2j + 1)(2j + 2)$ and $2j(2j + 1)$ respectively. The Hilbert space is defined by taking $H_\lambda \otimes \mathbb{C}^2$, where H_λ is the representation Hilbert space for the left regular representation λ of $\mathcal{A}_q$. It is important to take that Hilbert space, instead of $\mathbb{C}^2 \otimes H_\lambda$. Not only does the latter choice violate the equivariance condition, but also it is shown by Goswami that it produces unbounded commutators $[D, a]$, and hence, not obtained is a spectral triple in that way.

The spectral triple constructed as above ([105] and [226]) has a real structure, and the Dirac operator satisfies a weak form of the order one condition. The local index formula of [71] (cf. the theorem above) is extended to the spectral triple of [226], as proved in [105]. As well, the structures of the cotangent space and the geodesic flow are essentially the same.

Even more recently, the construction of finitely summable spectral triples of [105] and [226] is generalized by Sergey Neshveyev and Lars Tuset [195], to a functional construction that works for any quantum group G_q, obtained as q-deformation of a simply connected, simple compact Lie group G.

21 Noncommutative spherical manifolds

The noncommutative 3-dimensional spheres S^3_φ contained in the noncommutative 4-dimensional Euclidean spaces $\mathbb{R}^4_\varphi$ are obtained as solutions of a problem as the vanishing of the first component of the Chern character of a unitary 2×2 matrix over the quantum $*$-algebra of functions on the noncommutative sphere, where the Chern character is taken in the cyclic homology of the (b, B)-bicomplex. The origin of such a problem is to quantize the volume form of a 3-manifold ([76]). The solutions are parameterized by three angles φ_k for $k = 1, 2, 3$ and the corresponding algebras are obtained by imposing the unit sphere relation $\sum_{\mu=0}^4 x_\mu^2 = 1$ on the four generators x_0, x_1, x_2, x_3 of the quadratic algebra $\mathbb{C}(\mathbb{R}^4_\varphi)$ with the relations

$$\sin(\varphi_k)\{x_0, x_k\} = i\cos(\varphi_l - \varphi_m)[x_l, x_m],$$
$$\cos(\varphi_k)[x_0, x_k] = i\sin(\varphi_l - \varphi_m)\{x_l, x_m\},$$

where $[a, b] = ab - ba$ is the commutator and $\{a, b\} = ab + ba$ is the anticommutator and the indices k, l, m take $1, 2, 3$ cyclicly, or we may use the notation $[a, b]_\pm = ab \pm ba$ ([77]).

The analysis of these algebras is a special case of the general theory of central quadratic forms for quadratic algebas. May refer to [77] and [78].

Let $\mathcal{A} = A(V, R) = T(V)/(R)$ be a quadratic algebra, where V is the linear span of the generators or a finite dimensional vector space over $\mathbb{C}$ and (R) is the two-sided ideal of the tensor algebra $T(V)$ over V generated by the relations, or a subspace R of $V \otimes V$.

(Added). Consider the subset of $V^* \times V^*$ consisting of (α, β) with each nonzero such that $\langle w, \alpha \otimes \beta \rangle = 0$ for any $w \in R$. It then defines a subset Γ of $P(V^*) \times P(V^*)$, where $P(V^*)$ is the complex projective space of all 1-dimensional complex subspaces of V^*. Let E_1 and E_2 be the projection of Γ in the first and second component $P(V^*)$. Assume and set $E = E_1 = E_2$. Define the correspondence σ with graph Γ to be an automorphism of E. Define L to be the pull-back on E of the dual of the tautological line bundle of $P(V^*)$. Then the algebraic variety E is referred to as the characteristic variety, and in many cases, E is the union of elliptic curves, each of which has a finite number of points which are invariant by σ. ◀

The geometric data $\{E, \sigma, L\}$ is (again) given by an algebraic variety E, a correspondence σ on E, and a line bundle L over E. These data are defined so as to yield a homomorphism h from $\mathcal{A}$ to a crossed product algebra constructed from sections of powers of the line bundle L on the graphs of the iterations of the correspondence σ. This crossed product only involves the positive powers of the correspondence σ, hence it remains triangular and far removed from the semi-simple set-up of C^*-algebras. That morphism h can be considerebly refined using the notion of positive central quadratic form.

Definition 21.1. ([77, Definition 6]). Let $q \in S^2(V)$ be a symmetric bilinear form on V^* and C a component of $E \times E$. It is said that Q is central on C if for all $(z, z') \in C$ and $w \in R$, one has

$$w(z, z')q(\sigma(z'), \sigma^{-1}(z)) + q(z, z')w(\sigma(z'), \sigma^{-1}(z)) = 0.$$

It then follows that one can construct purely algebraically a crossed product algebra and a homomorphism from $\mathcal{A} = A(V, R)$ to the crossed product. Also, C^*-algebras arise from positive central quadratic forms that make sense on involutive quadratic algebras.

Let $\mathcal{A} = A(V, R)$ be an involutive quadratic algebra, that is a $*$ algebra over $\mathbb{C}$ with involution x^* preserving the subspace V of the generators. The real structure of V is given by the anti-linear involution $j(v)$ for $v \in V$ as the restriction of x^*. As $(xy)^* = y^*x^*$ for $x, y \in \mathcal{A}$, the space R of relations satisfies

$$(j \otimes j)(R) = t(R) \subset V \otimes V,$$

where $t : V \otimes V \to V \otimes V$ is the transposition defined by $t(v \otimes w) = w \otimes v$. This implies that the characteristic variety is stable under the involution j and that $\sigma(j(z)) = j(\sigma^{-1}(z))$.

Now let C be an invariant component of $E \times E$. It is said that $C$is j-real if it is globally invariant under the involution $j^{\sim}(z, z') = (j(z'), j(z))$.

Let q be a central quadratic form on C. It is said that q is positive on C if $q(z, j(z)) > 0$ for any $z \in C$. One can then endow the line bundle L dual to the tautological bundle on $P(V^*)$ with the Hermitian metric given by

$$\langle fL, gL' \rangle_q(z) = f(z)\overline{g(z)} \frac{L(z)\overline{L'(z)}}{Q(z, j(z))}$$

for $L, L' \in V, z \in K$, $f, g \in C(K)$.

One then defines a generalized crossed product C^*-algebra $C(K) \rtimes_{\sigma,L} \mathbb{Z}$ following M. Pimsner [198] as follows. Given a compact space K, a homeomorphism σ of K, and a Hermitian line bundle L over K, we define the C^*-algebra $C(K) \rtimes_{\sigma,L} \mathbb{Z}$ to be the twisted crossed product of $C(K)$ by the Hilbert C^*-bimodule associated to σ and L ([2], [198]).

For each $n \geq 0$, let L^{σ^n} be the hermitian line bundle pullback of L by σ^n and let (cf. [10], [223] both missing)

$$L_n = L \otimes L^{\sigma} \otimes \cdots \otimes L^{\sigma^{n-1}}.$$

Define a $*$-algebra as the linear span of the monomials ξw^n and $(w^*)^n \eta^*$ for $\xi, \eta \in C(K, L_n)$ with product given as in (cf. [10], [223]), so that

$$(\xi_1 w^{n_1})(\xi_2 w^{n_2}) \equiv (\xi_1 \otimes (\xi_2 \circ \sigma^{n_1}))w^{n_1+n_2}.$$

Using the Hermitian structure of L_n we give meaning to the products $\eta^*\xi$ and $\xi\eta^*$ for $\xi, \eta \in C(K, L_n)$. The product then extends uniquely to an associative product of a $*$-algebra fulfilling the additional rules

$$((w^*)^k \eta^*)(\xi w^k) \equiv (\eta^*\xi) \circ \sigma^{-k} \quad \text{and} \quad (\xi w^k)((w^*)^k \eta^*) \equiv \xi\eta^*.$$

The C^*-norm of $C(K) \rtimes_{\sigma,L} \mathbb{Z}$ is defined as for ordinary crossed product C^*-algebras. Due to the amenability of the group $\mathbb{Z}$, there is no distinction between the maximal and reduced C^*-norms. The maximal C^*-norm is defined to be the supremum of the C^*-norms by C^*-algebra representations on Hilbert spaces. There is a natural positive conditional expectation from the twisted crossed product C^*-algebra onto the C^*-subalgebra $C(K)$, it follows from which that the C^*-norm restricted to $C(K)$ coincides with the usual sup-norm on $C(K)$.

Theorem 21.2. *Let K be a compact σ-invariant subset of E and q be central and strictly positive on the set $\{(z, \bar{z}) \mid z \in K\}$. Let L be the restriction to K of the dual of the tautological line bundle on $P(V^*)$ endowed with the Hermitian metric $\langle \cdot, \cdot \rangle_q$. Then,*

(i) The equality $\sqrt{\theta}(y) = yw + w^*\bar{y}^*$ implies a $*$-homomorphism

$$\theta : \mathcal{A} = A(V, R) \to C(K) \rtimes_{\sigma,L} \mathbb{Z}.$$

(ii) For any $y \in V$, the C^*-norm of $\theta(y)$ fufills

$$\sup_K \|y\| \leq \sqrt{2}\|\theta(y)\| \leq 2 \sup_K \|y\|.$$

(iii) If $\sigma^4 \neq 1$, then $\theta(q) = 1$, where q is viewed as an element of $T(V)/(R)$.

In the above case of the quantum sphere S^3_φ, one lets q be the quadratic form as $q(x, x') \equiv \sum x_\mu x'_\mu$. In the general case one has

Proposition 21.3. (1) *The characteristic variety is the union of 4 points with an elliptic curve F_φ.*

(2) *The quadratic form q is central and positive on $F_\varphi \times F_\varphi$.*

In suitable coordinates, the equations defining the elliptic curve F_φ are given by

$$\frac{z_0^2 - z_1^2}{s_1} = \frac{z_0^2 - z_2^2}{s_2} = \frac{z_0^2 - z_3^2}{s_3},$$

where $s_k = 1 + t_l t_m$ with $t_k = \tan \varphi_k$.

The positivity of q is automatic since in the coordinate x, the involution j_φ of the $*$-algebra $\mathbb{C}[\mathbb{R}^4_\varphi]$ is just $j_\varphi(z) = \bar{z}$, so that $q(x, j_\varphi(x)) > 0$ for $x \neq 0$.

Corollary 21.4. *Let K be a compact σ-invariant subset of F_φ. The homomorphism θ obtained in the theorem above is a unital $*$-homomorphism from $\mathbb{C}[S^3_\varphi]$ to the twisted crossed product $C(K) \rtimes_{\sigma,L} \mathbb{Z}$.*

It follows that a non-trivial C^*-algebra $C^*(S^3_\varphi)$ is obtained as the completion of $\mathbb{C}[S^3_\varphi]$ for the semi-norm $\|p\|_u \equiv \sup \|\pi(p)\|$, where π varies over all unitary representations of $\mathbb{C}[S^3_\varphi]$. The semi-norm defines a finite C^*-semi-norm on $\mathbb{C}[S^3_\varphi]$ since the equation $\sum x_\mu^2 = 1$ as the sphere together with the self-adjointness as $x_\mu = x_\mu^*$ implies that $\|\pi(x_\mu)\| \leq 1$ for any μ in any unitary representation π. What the above corollary gives a lower bound for the C^*-norm such as that given by the theorem above on the linear subspace V of generators.

The correspondence σ on F_φ for φ generic is a translation of modulus η of the elliptic curve F_φ and one can distinguish two cases: the even case where it preserves the two real components of the curve $F_\varphi \cap P^3(\mathbb{R})$ and the odd case where it permutes them.

Proposition 21.5. *Let φ be generic and even. Then,*

(i) The crossed product C^*-algebra $C(F_\varphi) \rtimes_{\sigma,L} \mathbb{Z}$ is isomorphic to the mapping torus of the automorphism β of the noncommutative 2-torus $\mathbb{T}^2_\eta = C_\varphi \rtimes_\sigma \mathbb{Z}$ acting on the generators by the matrix $[1 \oplus 1] + [0 \oslash 4]$.

(ii) The crossed product C^*-algebra $C(F_\varphi) \rtimes_{\sigma,L} \mathbb{Z}$ is a noncommutative 3-manifold with an elliptic action of the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$ and with an invariant trace τ.

May refer to the framework developed in [59]. Also refer to [209] and [3] (missing), where those noncommutative manifolds are analyzed in terms of crossed products by Hilbert C^*-bimodules.

An integration on the translation invariant volume form dv of F_φ gives the $\mathfrak{h}_3$-invariant trace τ, defined as that for $f \in C^\infty(F_\varphi)$ and $k \neq 0$,

$$\tau(f) = \int_{F_\varphi} f dv \quad \text{and} \quad \tau(\xi w^k) = \tau((w^*)^k \eta^*) = 0.$$

It follows, in particular that the following gives the fundamental class as a 3-cyclic cocycle

$$\tau_3(a_0, a_1, a_2, a_3) = \sum \varepsilon_{ijk} \tau(a_0 \delta_i(a_1) \delta_j(a_2) \delta_k(a_3)),$$

where δ_j are the generators of the action of $\mathfrak{h}_3$.

The relation between the noncommutative 3-spheres S^3_φ and the noncommutative nil manifolds $C(F_\varphi) \rtimes_{\sigma,L} \mathbb{Z}$ is analyzed in [77] and [78], thanks to the computation of the Jacobian of the homomorphism θ.

22 Noncommutative spaces from $\mathbb{Q}$-lattices

A class of examples of noncommutative spaces as relevance to number theory is given by the moduli spaces of $\mathbb{Q}$-lattices up to commensurability. These fall within the general framework of noncommutative spaces obtained as quotients of equivalence relations.

May refer to [82].

A $\mathbb{Q}$-lattice in $\mathbb{R}^n$ is defined to be a pair (Λ, φ) of a lattice Λ in $\mathbb{R}^n$, that is a co-compact free abelian subgroup of $\mathbb{R}^n$ with rank n together with a system of labels of its torsion points given by a homomorphism of abelian groups as $\varphi : \mathbb{Q}^n/\mathbb{Z}^n \to \mathbb{Q}\Lambda/\Lambda$.

Two $\mathbb{Q}$-lattices are commensurable, denoted as $(\Lambda_1, \varphi_1) \sim (\Lambda_2, \varphi_2)$, if and only if $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\varphi_1 \equiv \varphi_2 \bmod \Lambda_1 + \Lambda_2$.

In general, the map φ is just a group homomorphism. A $\mathbb{Q}$-lattice is said to be invertible if φ is an isomorphism. Two invertible $\mathbb{Q}$-lattices are commensurable if and only if they are equal.

We denote by $\mathcal{L}_n$ the space of commensurable classes of $\mathbb{Q}$-lattices in $\mathbb{R}^n$. The space $\mathcal{L}_n$ has a typical property as noncommutative spaces in the sense as follows. It has cardinality of continuum but one can not construct a countable collection of measurable functions that separate points of the space $\mathcal{L}_n$. Thus, one can use noncommutative geometry to describe $\mathcal{L}_n$ as a quotient space throught a noncommutative C^*-algebra as $C^*(\mathcal{L}_n)$.

We consider especially the case where $n = 1$ or $n = 2$ in what follows.

One is also interested in the C^*-algebras describing $\mathbb{Q}$-lattices up to scaling as $\mathcal{A}_1 = C^*(\mathcal{L}_1/\mathbb{R}_+^*)$ and $\mathcal{A}_2 = C^*(\mathcal{L}_2/\mathbb{C}^*)$.

Example 22.1. (Edited). (The 1-dimensional case). In this case, a $\mathbb{Q}$-lattice (Λ, φ) in $\mathbb{R}$ can be written in the form $(\lambda\mathbb{Z}, \lambda\rho)$ for some $\lambda > 0$ and some

$$\rho \in (\mathbb{Q}/\mathbb{Z})^\wedge = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{T}) \cong \varprojlim \mathbb{Z}/n\mathbb{Z} \cong \overleftarrow{\mathbb{Z}} \neq \mathbb{Z}^\wedge = \mathrm{Hom}(\mathbb{Z}, \mathbb{T}) \cong \mathbb{T},$$

(where there is an inclusion $\mathbb{Q}/\mathbb{Z} \subset \mathbb{T} = \mathbb{R}/\mathbb{Z}$, and the first dual group is just a subspace of the direct product of $(\mathbb{Z}/n\mathbb{Z})^\wedge \cong \mathbb{Z}/n\mathbb{Z}$, and our notation $\overleftarrow{\mathbb{Z}}$ means the profinite completion of $\mathbb{Z}$). By considering lattices up to scaling, we can eliminate the factor $\lambda > 0$, so that 1-dimensional $\mathbb{Q}$-lattices are completely specified up to scaling by the choice of the element $\rho \in \overleftarrow{\mathbb{Z}}$. Thus, the algebra of coordinates of the space as 1-dimensional $\mathbb{Q}$-lattices up to scaling is the commutative C^*-algebra $C(\overleftarrow{\mathbb{Z}})$, which is isomorphic to the group C^*-algebra $C^*(\mathbb{Q}/\mathbb{Z})$, which is isomorphic to $C((\mathbb{Q}/\mathbb{Z})^\wedge)$ by the Gelfand transform, which is isomorphic to the inductive limit of $C(\mathbb{Z}/n\mathbb{Z})$, where we use Pontrjagin duality between locally compact abelian groups and their dual groups.

The equivalence relation of commensurability is implemented by the action of the semi-group $\mathbb{N}^*$ on $\mathbb{Q}$-lattices. The corresponding action on the C^*-algebra $C^*(\mathbb{Q}/\mathbb{Z}) \cong C((\mathbb{Q}/\mathbb{Z})^\wedge)$ is given as that for $n \in \mathbb{N}^*$,

$$\alpha_n(f)(\rho) = \begin{cases} f(n^{-1}\rho) & \rho \in n\overleftarrow{\mathbb{Z}}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the quotient of the space of 1-dimensional $\mathbb{Q}$-lattices up to scaling by the commensurability relation as well as its algebra of coordinates is given by the semi-group crossed product C^*-algebra $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^*$. This is the Bost-Connes C^*-algebra [29].

It has a natural time evolution given by the covolume of a pair of comensurable $\mathbb{Q}$-lattices. It has symmetries compatible with the time evolution given by the group $(\overleftarrow{\mathbb{Z}})^* = GL_1(\mathbb{A}_f)/\mathbb{Q}^*$, and the KMS (Kubo-Martin-Schwinger) equilibrium states of the system have arithmetic properties, where $\mathbb{A}_f = \overleftarrow{\mathbb{Z}} \otimes \mathbb{Q}$ is the ring of finite adèles of $\mathbb{Q}$. Namely, the partition function of the system is the Riemann zeta function. There is a unique KMS state for sufficiently high temperature, while at low temperature the system undergoes a phase transition with spontaneous symmetry breaking. The pure phases as extremal KMS states at low temperature are parameterized by elements in $(\overleftarrow{\mathbb{Z}})^*$. They have an explicit expression in terms of polylogarithms at roots of unity. At zero temperature, the extremal KMS states, evaluated at the elements of a rational subalgebra, take values that are algebraic numbers. The action on these values of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ factors through its abelianization and is obtained, via the isomorphism $(\overleftarrow{\mathbb{Z}})^* \cong \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ in the class field theory, as the action of symmetries on the algebra (cf. [29], [81] (missing), [82] for details). ◀

Example 22.2. (Edited). (The 2-dimensional case as the GL_2-system). In this case, a $\mathbb{Q}$-lattice (Λ, φ) in $\mathbb{R}^2$ can be written in the form $(\lambda(\mathbb{Z}+\mathbb{Z}\tau), \lambda\rho)$ for some $\lambda \in \mathbb{C}^*$, some $\tau \in \mathbb{H}$, and some $\rho \in M_2(\overleftarrow{\mathbb{Z}}) = \mathrm{Hom}(\mathbb{Q}^2/\mathbb{Z}^2, \mathbb{Q}^2/\mathbb{Z}^2)$. Thus, the space of 2-dimensional $\mathbb{Q}$-lattices, up to the scaling factor $\lambda \in \mathbb{C}^*$ and up to isomorphisms, is given by

$$M_2(\overleftarrow{\mathbb{Z}}) \times \mathbb{H} \mod \Gamma = SL_2(\mathbb{Z}).$$

The commensurability relation giving the space $\mathcal{L}_2/\mathbb{C}^*$ is implemented by the partially defined action of $GL_2^+(\mathbb{Q})$.

In this case, consider the quotient of the space

$$\mathcal{U}^\sim = \{(g, \rho, \alpha) \in GL_2^+(\mathbb{Q}) \times M_2(\overleftarrow{\mathbb{Z}}) \times GL_2^+(\mathbb{R}) \,|\, g\rho \in M_2(\overleftarrow{\mathbb{Z}})\}$$

by the action of $\Gamma \times \Gamma$ given by

$$(\gamma_1, \gamma_2)(g, \rho, \alpha) = (\gamma_1 g \gamma_2^{-1}, \gamma_2\rho, \gamma_2\alpha).$$

The groupoid $\mathcal{R}_2$ of the equivalence classes by commensurability on 2-dimensional $\mathbb{Q}$-lattices, not considered up to scaling at this moment, is a locally compact groupoid, which can be parameterized by the quotient of the space $\mathcal{U}^\sim$ by $\Gamma \times \Gamma$ via the map $r : \mathcal{U}^\sim \to \mathcal{R}_2$ defined by

$$r(g, \rho, \alpha) = ((\alpha^{-1}g^{-1}\Lambda_0, \alpha^{-1}\rho), (\alpha^{-1}\Lambda_0, \alpha^{-1}\rho)).$$

We then consider the quotient by scaling.

The quotient $GL_2^+(\mathbb{R})/\mathbb{C}^*$ can be identified with the hyperbolic plane $\mathbb{H}$ in the usual way. If (Λ_k, φ_k) for $k = 1, 2$ are a pair of commensurable 2-dimensional $\mathbb{Q}$-lattices, then for any $\lambda \in \mathbb{C}^*$, the $\mathbb{Q}$-lattices $(\lambda\Lambda_k, \lambda\varphi_k)$ are also commensurable, with

$$r(g, \rho, \alpha\lambda^{-1}) = \lambda r(g, \rho, \alpha).$$

However, the action of $\mathbb{C}^*$ on $\mathbb{Q}$-lattices is not free due to the presence of lattices such as Λ_0 above with non-trivial automorphisms. Thus, the quotient $Z = \mathcal{R}_2/\mathbb{C}^*$ is no longer a groupoid. But one can still define a convolution algebra for Z by restricting the convolution product of $\mathcal{R}_2$ to homogeneous functions of weight zero, where a function f has weight k if it satisfies

$$f(g, \rho, \alpha\lambda) = \lambda^k f(g, \rho, \alpha), \quad \lambda \in \mathbb{C}^*.$$

The space Z is the quotient of the space

$$\mathcal{U} = \{(g, \rho, z) \in GL_2^+(\mathbb{Q}) \times M_2(\overleftarrow{\mathbb{Z}}) \times \mathbb{H} \mid g\rho \in M_2(\overleftarrow{\mathbb{Z}})\}$$

by the action of $\Gamma \times \Gamma$, where the space $M_2(\overleftarrow{\mathbb{Z}}) \times \mathbb{H}$ has a partially defined action of $GL_2^+(\mathbb{Z})$ given by $g(\rho, z) = (g\rho, g(z))$, where $g(z)$ denotes the action as a fractional linear transformation.

Thus, the algebra $\mathcal{A}_2$ of coordinates for the noncommutative space of commensurable classes of 2-dimensional $\mathbb{Q}$-lattices up to scaling is given by the following convolution algebra. Consider the space $C_c(Z)$ of all continuous, compactly supported, functions on Z. These can be viewed as functions on the space $\mathcal{U}$ invariant under the $\Gamma \times \Gamma$ action as $(g, \rho, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 z)$. Endow $C_c(Z)$ with the convolution product and the involution

$$(f_1 * f_2)(g, \rho, z) = \sum_{s \in \Gamma \backslash GL_2^+(\mathbb{Q}), s\rho \in M_2(\overleftarrow{\mathbb{Z}})} f_1(gs^{-1}, s\rho, s(z)) f_2(s, \rho, z),$$

$$f^*(g, \rho, z) = \overline{f(g^{-1}, g\rho, g(z))}.$$

There is a time evolution on that algebra, which is given by the covolume

$$\sigma_t(f)(g, \rho, z) = \det(g)^{it} f(g, \rho, z).$$

The partition function for this GL_2-system is given by

$$\begin{aligned} Z(\beta) &= \sum_{m \in \Gamma \backslash M_2^+(\overleftarrow{\mathbb{Z}})} \det(m)^{-\beta} \\ &= \sum_{k=1}^{\infty} \sigma(k) k^{-\beta} = \zeta(\beta)\zeta(\beta - 1), \end{aligned}$$

where $\sigma(k) = \sum_{d|k} d$. The partition function of this form suggests that two distinct phase transition might happen at $\beta = 1$ and $\beta = 2$, as a possible sense. ◀

Moreover, the structure of KMS states for the system above is analyzed by Connes-Marcolli [81] (missing) as the following:

Theorem 22.3. *The* KMS_β *states of the* GL_2*-system with* $0 < \beta < \infty$ *the inverse temperature have the following properties:*

(1) *In the range $\beta \leq 1$, there are no* KMS_β *states.*

(2) *In the range $\beta > 2$, the set $\mathcal{E}_\beta$ of extremal* KMS_β *states is given by the classical Shimura variety as*

$$\mathcal{E}_\beta \cong GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/\mathbb{C}^*.$$

The symmetries are more complicated than in the Bost-Connes system. In fact, in additon to symmetries given by automorphisms that commute with the time evolution, there are also symmetries by endomorphisms. The resulting symmetry group is the quotient $GL_2(\mathbb{A}_f)/\mathbb{Q}^*$. It is shown by Shimura [217] that this group is in fact the Galois group of the field F of modular functions. The group $GL_2(\mathbb{A}_f)$ decomposes as a product $GL_2^+(\mathbb{Q})GL_2(\overleftarrow{\mathbb{Z}})$, where $GL_2(\overleftarrow{\mathbb{Z}})$ acts by automorphisms related to the deck transformations of the tower of the modular curves, while $GL_2^+(\mathbb{Q})$ acts by endomorphisms that move across levels in the modular tower.

The modular field F is the field of modular functions over $\mathbb{Q}^{ab}$, namely the union of the fields F_N of modular functions of level N rational over the cyclotomic field $\mathbb{Q}(\zeta_n)$, that is. such that the q-expansion in powers of $q^{\frac{1}{N}} = \exp(\frac{2\pi i \tau}{N})$ has all coefficients in $\mathbb{Q}(e^{\frac{2\pi i}{N}})$.

The action of the Galois group $(\overleftarrow{\mathbb{Z}})^* \cong \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ on the coefficients determines a homomorphism $cycl : (\overleftarrow{\mathbb{Z}})^* \to \mathrm{Aut}(F)$.

If $\tau \in \mathbb{H}$ is a generic point, then the evaluation map $f \mapsto f(\tau)$ determines an embedding of F into $\mathbb{C}$. We denote by F_τ the image in $\mathbb{C}$. This implies an identification

$$\theta_\tau : \mathrm{Gal}(F_\tau/\mathbb{Q}) \xrightarrow{\cong} \mathbb{Q}^*\backslash GL_2(\mathbb{A}_f).$$

There is an arithmetic algebra $\mathcal{A}_{2,\mathbb{Q}}$ defined over $\mathbb{Q}$ of unbounded multipliers of the C^*-algebra $\mathcal{A}_2$, obtained by considering continuous functions on Z with finite support in the variable $g \in \Gamma\backslash GL_2^+(\mathbb{Q})$ and with the properties as follows. Let $p_N : M_2(\overleftarrow{\mathbb{Z}}) \to M_2(\mathbb{Z}/N\mathbb{Z})$ be the canonical projection. With the notation $f_{(g,\rho)}(z) = f(g,\rho,z)$, we say that $f_{(g,\rho)} \in C(\mathbb{H})$ is of level N if $f_{(g,\rho)} = f_{(g,p_N(\rho))}$ for any $(g,\rho) \in GL_2^+(\mathbb{Q}) \times M_2(\overline{\mathbb{Z}})$. We require that any element f of $\mathcal{A}_{2,\mathbb{Q}}$ has $f_{(g,\rho)}$ of finite level, with $f_{(g,m)} \in F$ for all (g,m). Also require that the action $cycl$ on the coefficients of the q-expansion of the $f_{(g,m)}$ satisfies

$$f_{(g,\alpha(u)m)} = cycl(u) f_{(g,m)}$$

for all diagonal $g \in GL_2^+(\mathbb{Q})$ and all $u \in (\overleftarrow{\mathbb{Z}})^*$, with $\alpha(u) = u \oplus 1$ the diagonal sum, to avoid some trivial elements that would spoil the Galois action on values of states (cf. [81] (missing), [82]). The action of symmetries extends to $\mathcal{A}_{2,\mathbb{Q}}$.

It then follows that ([81] (missing)):

Theorem 22.4. *Consider a state $\varphi = \varphi_{\infty,L} \in \mathcal{E}_\infty$ for a generic invertible $\mathbb{Q}$-lattice $L = (\rho,\tau)$. Then the values of the state on elements of the arithmetic subalgebra generate the image in $\mathbb{C}$ of the modular field, and $\varphi(\mathcal{A}_{2,\mathbb{Q}}) \subset F_\tau$, and the isomorphism*

$$\theta_\varphi : \mathrm{Gal}(F_\tau/\mathbb{Q}) \xrightarrow{\cong} \mathbb{Q}^*\backslash GL_2(\mathbb{A}_f),$$

given by $\theta_\varphi(\gamma) = \rho^{-1}\theta_\tau(\gamma)\rho$, *for* θ_τ *given above, intertwines the Galois action on the values of the state with the action of symmetries,* $\gamma\varphi(f) = \varphi(\theta_\varphi(\gamma)f)$ *for* $f \in \mathcal{A}_{2,\mathbb{Q}}$ *and* $\gamma \in \mathrm{Gal}(F_\tau/\mathbb{Q})$.

Remark. (Added). Recall from [82] the following about quantum statistical mechanics. The algebra of observables in quantum statistical mechanics is given by a C^*-algebra $\mathfrak{A}$. Expectation values for observables are obtained by values of states on $\mathfrak{A}$. A state on $\mathfrak{A}$ is a positive, unital linear functional $\varphi : \mathfrak{A} \to \mathbb{C}$, which may be viewed as a probability measure on the correspond noncommutative space X. The time evolution of a system (or a C^*-algebra $\mathfrak{A}$) in quantum statistical mechanics is given as a one-parameter family of automorphisms σ_t of $\mathfrak{A}$ for $t \in \mathbb{R}$. Under a representation of $\mathfrak{A}$ on a Hilbert space, the time evolution is implemented by the Hamiltonian operator H such that

$$\sigma_t(a) = \mathrm{Ad}(e^{itH})(a) = e^{itH}ae^{-itH}, \quad a \in \mathfrak{A}.$$

Look for equilibrium states, which depend on a thermo-dynamical parameter, the inverse temperature $\beta = \frac{1}{kT}$ with k the Boltzmann constant (to be equal 1 for simplicity). In the quantum statistical setting, define a state

$$\varphi(a) = \frac{1}{Z(\beta)}\mathrm{tr}(ae^{-\beta H}), \quad \text{with} \quad Z(\beta) = \mathrm{tr}(e^{-\beta H}),$$

as the partition function, where the definition makes sense under the assumption that the operator $\exp(-\beta H)$ is of trace class. This is only (or certain) the case where the temperature T is low and positive and so β is large.

Given a C^*-dynamical system $(\mathfrak{A}, \sigma, \mathbb{R})$, a state φ on $\mathfrak{A}$ is said to satisfies the KMS condition at the inverse temparature β with $0 < \beta < \infty$, or is a KMS_β state if for any $a, b \in \mathfrak{A}$, there is a function $f_{a,b}(z)$ holomorphic on the strip $0 < \mathrm{Im}(z) < \beta$ and continuous and bounded on the closed strip $0 \leq \mathrm{Im}(z) \leq \beta$ such that for any $t \in \mathbb{R}$,

$$f_{a,b}(t) = \varphi(a\sigma_t(b)) \quad \text{and} \quad f_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a).$$

At zero temperature with $\beta = \infty$ one may define a KMS_∞ state similarly and properly. But a better notion as the KMS_∞ is given by considering states obtained as weak limits of KMS_β states as $\beta \to \infty$. Namely, that is

$$\varphi_\infty(a) = \lim_{\beta\to\infty} \varphi_\beta(a), \quad a \in \mathfrak{A}.$$

Denote by $\mathcal{E}_\infty$ the set of extreme points of the set of KMS_∞ states in this sense. ◀

A notion analogous to that of $\mathbb{Q}$-lattices can be made for other number fields $\mathbb{K}$. This notion is used in [86] to construct a quantum statistical mechanical system for $\mathbb{K}$ an imaginary quadratic field. This system also has properties as for the Bost-Connes system of [29] and the GL_2 system as 2-dimensional $\mathbb{Q}$-lattices of [81] (missing).

Example 22.5. (Edited). We assume that $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ the imaginary quadratic field for d a positive integer. Let $\tau = \sqrt{-d} \in \mathbb{H}$ (the upper half plane) be such that $\mathbb{K} = \mathbb{Q}(\tau)$ and $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\tau$ is the ring of integers of $\mathbb{K}$. There is an embedding from $\mathbb{K}$ into $\mathbb{C}$.

A 1-dimensional $\mathbb{K}$-lattice (Λ, φ) is a finitely generated, $\mathcal{O}$-submodule Λ in $\mathbb{C}$ such that $\Lambda \otimes_{\mathcal{O}} \mathbb{K} \cong \mathbb{K}$, together with a morphism of $\mathcal{O}$-modules: $\varphi : \mathbb{K}/\mathcal{O} \to \mathbb{K}\Lambda/\Lambda$. A 1-dimensional $\mathbb{K}$-lattice (Λ, φ) is invertible if φ is an isomorphism of $\mathcal{O}$-modules. In particular, a 1-dimensional $\mathbb{K}$-lattice is a 2-dimensional $\mathbb{Q}$-lattice.

Two 1-dimensional $\mathbb{K}$-lattices (Λ_1, φ_1) and (Λ_2, φ_2) are said to be commensurable if $\mathbb{K}\Lambda_1 = \mathbb{K}\Lambda_2$ and $\varphi_1 = \varphi_2 \mod \Lambda_1 + \Lambda_2$. In particular, two 1-dimensional $\mathbb{K}$-lattices are commensurable if and only if they are commensurable as 2-dimensional $\mathbb{Q}$-lattices.

The algebra of the corresponding noncommutative space is obtained as a restriction of the algebra of the GL_2-system to the sub-groupoid of the equivalence of commensurability restricted to $\mathbb{K}$-lattices. As well, the time evolution is given by a restriction from the GL_2-system.

The partition function for the resulting system is the Dedekind zeta function $\zeta_{\mathbb{K}}(\beta)$ of the number field $\mathbb{K}$. More precisely,

$$\zeta_{\mathbb{K}}(\beta) = \sum_{J \text{ ideal in } \mathcal{O}} n(J)^{-\beta} = \mathrm{tr}(e^{-\beta H}) = Z(\beta),$$

where $\mathcal{O}$ is the ring of algebraic integers of $\mathbb{K}$, and $n(J)^{-\beta} = \varphi_\beta(e_J)$ for a KMS$_\beta$ state, where $e_J = \mu_J \mu_J^*$ and $\mu_J^* \mu_J = 1$ for some $\mu_J \in \mathcal{A}_{\mathbb{K}}$. As well, $\sigma_t(\mu_J) = n(J)^{it} \mu_J$ for any $t \in \mathbb{R}$.

At the critical temparature $T = 1$ there is a unique KMS state, while at lower temparatures the extremal KMS states are parametrized by elements of $\mathbb{A}_{\mathbb{K}}^*/\mathbb{K}^*$, where $\mathbb{A}_{\mathbb{K}} = \mathbb{A}_{\mathbb{K},f} \times \mathbb{C}$ are the adèles of $\mathbb{K}$, with $\mathbb{A}_{\mathbb{K},f} = \mathbb{A}_f \otimes \mathbb{K}$. The KMS states at zero temparature, evaluated on the restriction to $\mathbb{K}$-lattices of the arithmetic algebra of the GL_2-system, have an action of the Galois group $\mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K})$, realized via the class field theory isomorphism through the action of symmetries of the system as automorphisms and endomorphisms. ◀

For more details, refer to [86].

23 Modular Hecke algebras

The modular Hecke algebras $\mathcal{A}(\Gamma)$ of level Γ, a congruence subgroup of $PSL_2(\mathbb{Z})$ are defined by Connes and Moscovici [92]. These extend both the ring of classical Hecke operators and the algebra of modular forms. Indeed, modular Hecke algebras encode a priori unrelated, two structures on modular forms. One is the algebraic structure given by the pointwise product, the other is the action of the Hecke operators.

To any congruence subgroup Γ of $PSL_2(\mathbb{Z})$ corrresponds a crossed product algebra $\mathcal{A}(\Gamma)$, called the modular Hecke algebra of level Γ, which is a direct extension of both the ring of classical Hecke operators and the algebra $\mathcal{M}(\Gamma)$ of Γ-modular forms.

These algebras can be obtained by considering the action of $GL_2^+(\mathbb{Q})$ on the algebra of modular forms on the full (adelic) modular tower, which yields the holomorphic part of the ring of functions on the noncommutative space of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices.

Denote by $\mathcal{M} = \mathcal{M}(\Gamma)$ the algebra of modular forms of arbitrary level Γ. The elements of $\mathcal{A}(\Gamma)$ are maps with finite support

$$f : \Gamma\backslash GL_2^+(\mathbb{Q}) \to \mathcal{M}, \quad \Gamma\alpha \mapsto f_\alpha \in \mathcal{M},$$

satisfying the covariance condition

$$f_{\alpha\gamma} = f_\alpha|_\gamma, \quad \alpha \in GL_2^+(\mathbb{Q}), \gamma \in \Gamma,$$

and their products are given by convolution.

More in details. Let $G = GL_2^+(\mathbb{Q})$ and $\Gamma \subset PSL_2(\mathbb{Z})$ be a subgroup with finite index. The quotient map from $\Gamma\backslash G$ to $\Gamma\backslash G/\Gamma$ takes finite subsets to one points, and Γ acts on $\mathbb{C}[\Gamma\backslash G]$.

Let H_k be the space of holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ with polynomial growth and with the action for $k \in 2\mathbb{Z}$ of $GL_2^+(\mathbb{Q})$ of the form

$$\left(f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z) = \frac{(ad-bc)^{\frac{k}{2}}}{(cz+d)^k} f\left(\frac{az+b}{cz+d}\right).$$

This implies the induced actions of G and Γ on H_k. The space of modular forms is then obtained as $\mathcal{M}_k(\Gamma) = H_k^\Gamma$ as the invariants under the action of Γ.

Then define $\mathcal{A}_k(\Gamma)$ as the fixed point algebra as

$$\mathcal{A}_k(\Gamma) = (\mathbb{C}[\Gamma \setminus G] \otimes_{\mathbb{C}} H_k)^\Gamma,$$

with respect to the right action of Γ by

$$[\sum_j (\Gamma g_j) \otimes f_j]\gamma = \sum_j (\Gamma g_j \gamma) \otimes (f_j|_k \gamma).$$

Define the graded vector space $\mathcal{A}_*(\Gamma) = \oplus_k \mathcal{A}_k(\Gamma)$. The elements of $\mathcal{A}_k(\Gamma)$ can be thought of as finitely supported, Γ-equivariant maps

$$\varphi : \Gamma\backslash G \to H_k, \quad \varphi(\sum_j (\Gamma g_j) \otimes f_j) = f_j.$$

Let $H_* = \oplus_k H_k$. Consider an embedding

$$\mathcal{A}_*(\Gamma) \subset \mathcal{A}_*^\wedge(\Gamma) \equiv \mathrm{Hom}_\Gamma(\mathbb{C}[\Gamma\backslash G], H_*),$$

where we assume $\mathcal{A}_*(\Gamma) = H_*[\Gamma\backslash G]$ as polynomials in $\Gamma\backslash G$ with H_* as coefficients, and view $\mathcal{A}_*^\wedge(\Gamma) = H_*[[\Gamma\backslash G]]$ as formal power series, that is, Γ-equivariant maps $\varphi : \Gamma\backslash G \to H_*$.

There is an associative multiplication on $\mathcal{A}_*(\Gamma)$ (cf. [92]), which makes it into a noncommutative ring, given by a convolution product, as follows. For any $\varphi \in \mathcal{A}_k(\Gamma)$, we have $\varphi_g = \varphi_{\gamma g}$ with $\varphi_g = 0$ off a finite subset of $\Gamma\backslash G$, and $\varphi_g|_\gamma = \varphi_{g\gamma}$, so that these terms are left Γ-invariant and right Γ-equivariant. For $\varphi \in \mathcal{A}_k(\Gamma)$ and $\psi \in \mathcal{A}_l(\Gamma)$, then define the convolution product as

$$(\varphi * \psi)_g = \sum_{(g_1,g_2)\in G\times_\Gamma G, g_1g_2=g} \langle \varphi_{g_1}, g_2\rangle \varphi_{g_2}.$$

The algebra $\mathcal{A}(\Gamma) = \mathcal{A}_*(\Gamma)$ constructed above has two remarkable subalgebras. One is the algebra of Hecke operators, that is, $\mathcal{A}_0(\Gamma) = \mathbb{C}[\Gamma\backslash G/\Gamma]$, and the other $\mathcal{M}_*(\Gamma) = \oplus_k \mathcal{M}_k(\Gamma)$ since $\mathcal{M}_k(\Gamma) \subset \mathcal{A}_k(\Gamma)$. In particular, observe that all the coefficients φ_g are modular forms. In fact, we have $\varphi_g|_\gamma = \varphi_{g\gamma}$, and thus $\varphi_g|_\gamma = \varphi_g$ for $\gamma \in \Gamma$.

However, note that the convolution product on $\mathcal{A}_*(\Gamma)$ does not agree with the Hecke action h Indeed, the diagram with i the inclusion maps is not commutative, nor is the symmetric one:

$$\begin{tikzcd} M_*(\Gamma)\otimes\mathcal{A}_0(\Gamma) \text{ or } \mathcal{A}_0(\Gamma)\otimes M_*(\Gamma) \arrow[r, "h"] \arrow[d, "i"'] & \mathcal{M}_*(\Gamma) \arrow[d, "i"] \\ \mathcal{A}_*(\Gamma)\otimes\mathcal{A}_*(\Gamma) \arrow[r, "*"] & \mathcal{A}_*(\Gamma). \end{tikzcd}$$

On the other hand, it is obtained that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{A}_0(\Gamma)\otimes M_*(\Gamma) \arrow[r, "h"] \arrow[d, "i"'] & \mathcal{M}_*(\Gamma) \\ \mathcal{A}_*(\Gamma)\otimes\mathcal{A}_*(\Gamma) \arrow[r, "*"] & \mathcal{A}_*(\Gamma), \arrow[u, "\epsilon\otimes 1"'] \end{tikzcd}$$

where $\epsilon : \mathbb{C}[\Gamma\backslash G] \to \mathbb{C}$ is the augmentation map, and is extended to the map

$$\epsilon\otimes 1 : \mathbb{C}[\Gamma\backslash G]\otimes H_k \to H_k, \quad \sum [g]\otimes\varphi_g \mapsto \sum \varphi_g.$$

The **Hopf algebra** $\mathcal{H}_1$ associated to the transverse geometry of co-dimension one foliations is introduced by Connes and Moscovici [90]. This is the universal enveloing algebra of a Lie algebra with basis $\{X, Y, \delta_n, n \geq 1\}$ such that for $n.k.l \geq 1$,

$$[Y, X] = X, \quad [Y, \delta_n] = n\delta_n, \quad [X, \delta_n] = \delta_{n+1}, \quad \text{and} \quad [\delta_k, \delta_l] = 0,$$

and with coproduct as an algebra homomorphism $\Delta : H_1 \to H_1 \otimes H_1$ such that

$$\begin{aligned} \Delta X &= X\otimes 1 + 1\otimes X + \delta_1 \otimes Y, \\ \Delta Y &= Y\otimes 1 + 1\otimes Y, \quad \text{and} \quad \Delta\delta_1 = \delta_1\otimes 1 + 1\otimes\delta_1, \end{aligned}$$

and with antipode as the anti-isomorphism S satisfying

$$S(X) = -X + \delta_1 Y, \quad S(Y) = -Y, \quad \text{and} \quad S(\delta_1) = -\delta_1,$$

and with co-unit $\epsilon(h)$ as the constant term of each $h \in \mathcal{H}_1$.

The Hopf algebra $\mathcal{H}_1$ acts as symmetries of modular Hecke algebras. As a general fact, symmetries of ordinary commutative spaces are encoded by group actions, while those of noncommutative spaces are given by Hopf algebras.

By comparing the actions of the Hopf algebra $\mathcal{H}_1$, derived is an analogy (cf. [92]) between the modular Hecke algebras and the crossed product algebra of the action of a disrecte subgroup of $\mathrm{Dif}(S^1)$ on polynomial functions on the frame bundle of S^1.

Indeed, let Γ be a discrete subgroup of $\mathrm{Dif}(S^1)$ and M be a smooth compact 1-dimensional manifold. Consider the crossed product algebra $\mathcal{A}_\Gamma = C_c^\infty(J_+^1(M)) \rtimes \Gamma$ as in [92], where $J_+^1(M)$ is the oriented 1-jet bundle. This algebra has an action of the Hopf algebra $\mathcal{H}_1$ defined by

$$X(fu_\varphi^*) = y_1 \frac{\partial f}{\partial y} u_\varphi^*, \quad Y(fu_\varphi^*) = y_1 \frac{\partial f}{\partial y_1} u_\varphi^*,$$
$$\text{and} \quad \delta_n(fu_\varphi^*) = y_1^n \frac{d^n}{dy^n}\left(\log \frac{d\varphi}{dy}\right) fu_\varphi^*,$$

with coordinates (y, y_1) on $J_+^1(M) \cong M \times \mathbb{R}^+$. Define the trace τ by the volume form as

$$\tau(fu_\varphi^*) = \begin{cases} \int_{J_+^1(M)} f(y, y_1) \frac{dy \wedge dy_1}{y_1^2}, & \varphi = 1, \\ 0, & \varphi \neq 1, \end{cases}$$

which satisfies $\tau(h(a)) = \nu(h)\tau(a)$ for $h \in \mathcal{H}_1$, where $\nu \in H_1^*$ satisfies $\nu(X) = 0$, $\nu(Y) = 1$, and $\nu(\delta_n) = 0$. The twisted antipode $S^\sim = \nu * S$ satisfies $(S^\sim)^2 = 1$, and

$$S^\sim(X) = -X + \delta_1 Y, \quad S^\sim(Y) = -Y + 1, \quad \text{and} \quad S^\sim(\delta_1) = -\delta_1.$$

The Hopf cyclic cohomology of Hopf algebras is a fundamental tool in noncommutative geometry, and is developed by Connes and Moscovici [90]. It is applied to the computation of the local index formula for transversely hypoelliptic operators on foliations.

An action of a Hopf algebra on an algebra induces a characteristic map from the Hopf cyclic cohomology of the Hopf algebra to the cylic cohomology of the algebra, and hence the index computation can be done in terms of the Hopf cyclic cohomology. The periodic Hopf cyclic cohomology of the Hopf algebra of transverse geometry is related to the Gelfand-Fuchs cohomology of the Lie algebra of formal vector fields [91] (missing).

In the case of the Hopf algebra $\mathcal{H}_1$, there are three basic cyclic cocycles or classes, which correspond respectively to the Schwarzian derivative $\delta_2' \equiv \delta_2 - \frac{1}{2}\delta_1^2$, the Godbillon-Vey class δ_1, and the trranverse fundamental class

$$F \equiv X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y,$$

in the original context of transverse geometry, as in [92].

In particular, the Hopf cyclic cocycle (class) associated to the Schwarzian derivative is of the form

$$\delta_2' \equiv \delta_2 - \frac{1}{2}\delta_1^2 \quad \text{with} \quad \delta_2'(fu_\varphi^*) = y_1^2\{\varphi(y); y\} f u_\varphi^*,$$
$$\text{where} \quad \{F; x\} \equiv \frac{d^2}{dx^2}\left(\log \frac{dF}{dx}\right) - \frac{1}{2}\left(\frac{d}{dx}\left(\log \frac{dF}{dx}\right)\right)^2.$$

The action of the Hopf algebra $\mathcal{H}_1$ on the modular Hecke algebra described in [92] involves the natural derivation on the algebra of modular forms, initially introduced by Ramanujan, which corrects the ordinary differentiation by a logarithmic derivative of the Dedekind η-function

$$X \equiv \frac{1}{2\pi i}\frac{d}{dz} - \frac{1}{2\pi i}\frac{d}{dz}(\log \eta^4) Y, \quad Y(f) = \frac{k}{2} f$$

for $f \in \mathcal{M}_k$. The element Y is the grading operator that multiples by $\frac{k}{2}$ forms of weight k, viewed as sections of the $\frac{k}{2}$-th power of the line bundle of 1-forms. The element δ_1 acts as multiplication by a form-valued cocycle on $GL_2^+(\mathbb{Q})$, which measures the lack of invariance of the section $\eta^4 dz$. Moreover,

Theorem 23.1. ([92]). *There is an action of the Hopf algebra $\mathcal{H}_1$ on the modular Hecke algebra $\mathcal{A}(\Gamma)$ of level Γ, induced by an action on $\mathcal{A}_{G^+(\mathbb{Q})} \equiv \mathcal{M} \rtimes G^+(\mathbb{Q})$, for $\mathcal{M} = \varinjlim_{N\to\infty} \mathcal{M}(\Gamma(N))$, of the form*

$$X(fu_\gamma^*) = X(f)u_\gamma^*, \quad Y(fu_\gamma^*) = Y(f)u_\gamma^*,$$
$$\text{and} \quad \delta_n(fu_\gamma^*) = \frac{d^n}{dZ^n}\left(\log \frac{d(Z|_{0\gamma})}{dZ}\right)(dZ)^n f u_\gamma^*,$$

with $X(f)$ and $Y(f)$ defined just above, and $Z(z) = \int_{i\infty}^z \eta^4 dz$.

The cocycle (class) assocaited to the Schwarzian derivative above is represented by an innter derivation of $\mathcal{A}_{G^+(\mathbb{Q})}$, defined as $\delta_2'(a) = [a, w_4]$, where w_4 is the weight four modular form

$$w_4 = -\frac{1}{72}E_4, \quad E_4(q = e^{2\pi i z}) = 1 + 240\sum_{n=1}^{\infty} n^3 \frac{q^n}{1-q^n},$$

which is expressed as a Schwarzian derivative as $w_4 = \frac{1}{(2\pi i)^2}\{Z : z\}$.

This result is used in [92] to investigate perturbations of the Hopf algebra action. The freedom that one has in modifying the action by a 1-cocycle corresponds exactly to the data introduced by Zagier [242], defining canonical Rankin-Cohen algebras, with the derivation and the element Φ, corresponding respectively to the action of the generator X on modular forms and to $w_4 = 2\Phi$.

The cocycle associated to the Godbillon-Vey class is described in terms of a 1-cocycle on $GL_2^+(\mathbb{Q})$ with values in Eisenstein series of weight two, which measures the lack of $GL_2^+(\mathbb{Q})$-invariance of the connection associated to the generator X. Derived from this is an arithmetic presentation of the rational Euler class in $H^2(SL_2(\mathbb{Q}),\mathbb{Q})$ in terms of generalized Dedekind sums.

On the other hand, the cocycle associated to the transverse fundamental class gives rise to a natural extension of the first Rankin-Cohen bracket [242] from modular forms to the modular Hecke algebras.

Rankin-Cohen algebras can be treated in different perspectives, as introduced and studied by Zagier [242] with a direct algebraic approach. There seems to be a connection to vertex operator algebras, as in a form of duality between these two types of algebras.

Let R be a graded ring with a derivation D of degree two, such that $R = R_* = \oplus_{k\geq 0} R_k$ and $D : R_k \to R_{k+2}$. The Rankin-Cohen brackets $[\cdot,\cdot]_*^{(*,*)}$ on R_* are defined to be a family of brackets

$$[\cdot,\cdot]_n^{(k,l)} : R_k \otimes R_l \to R_{k+l+2n}, \quad \text{for } n \geq 0, \text{ and for } f \in R_k,\ g \in R_l,$$
$$[f,g]_n^{(k,l)} = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s}\binom{n+l-1}{r} D^r f D^s g.$$

(Corrected. In fact, $D^r f \in R_{k+2r}$ and $D^s g \in R_{l+2s}$, and hence $D^r f D^s g \in R_{k+l+2n}$.) Inducing the Rankin-Cohen brackets as (R_*, D) converted to $(R_*, [\cdot,\cdot]_*^{(*,*)})$ gives rise to a standard Rankin-Cohen algebra, by Zagier [242]. There is an isomorphism of categories between graded rings with derivations and standard Rankin-Cohen algebras.

In the case of Lie algebras, we define a standard Lie algebra as the Lie algebra $(\mathcal{A}, [\cdot,\cdot])$ associated to an associative algebra $\mathcal{A}$ with $*$ product, by setting the bracket $[x,y] = x*y - y*x$. An abstract Lie algebra is defined to satisfy all the algebraic identities as in a standard Lie algebra, not necessarily induced by an associative algebra. It then follows as a theorem that the anti-symmetry of the bracket and the Jacobi identity are sufficient to determine all the other algebraic identifies, and hence one can assume these as a definition of an abstract Lie algebra.

As in the case of Lie algebras, we can define a (abstract) Rankin-Cohen algebra as a graded ring R_* with a family of degree 2 brackets $[\cdot,\cdot]_n$ satisfying all the algebraic identifies of the standard Rankin-Cohen algebra. However, in this case there is no simple set of axioms that implies all the algebraic identities. What's more.

Example 23.2. (Edited). There is an example of a Rankin-Cohen algebra which is non-standard, provided by modular forms ([242]).

If $f \in \mathcal{M}_k$ is a modular form satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z),$$

then

$$f'\left(\frac{az+b}{cz+d}\right)\frac{ad-bc}{(cz+d)^2} = kc(cz+d)^{k-1} f(z) + (cz+d)^k f'(z),$$

then its derivative is no longer a modular form, due to the presence (of the first term and not) of the second term (corrected) in

$$f'\left(\frac{az+b}{cz+d}\right) = kc(cz+d)^{k+1}f(z) + (cz+d)^{k+2}f'(z).$$

On the other hand, if $f \in \mathcal{M}_k$ and $g \in \mathcal{M}_l$, then the bracket defined as

$$[f,g](z) = lf'(z)g(z) - kf(z)g'(z)$$

is a modular form in $\mathcal{M}_{k+l+2}$.

Proof. (Added). Indeed, we have

$$\begin{aligned}
&[f,g]\left(\frac{az+b}{cz+d}\right) \\
&= lkc(cz+d)^{k+1}f(z)(cz+d)^l g(z) + l(cz+d)^{k+2}f'(z)(cz+d)^l g(z) \\
&\quad - k(cz+d)^k f(z) lc(cz+d)^{l+1}g(z) - k(cz+d)^k f(z)(cz+d)^{l+2}g'(z) \\
&= (cz+d)^{k+l+2}[f,g](z).
\end{aligned}$$

□

Moreover, we can define the n-th bracket

$$[\cdot,\cdot]_n : \mathcal{M}_k \otimes \mathcal{M}_l \to \mathcal{M}_{k+l+2n}$$

as before. In particular, note that

$$\begin{aligned}
&[f,g]_0 = fg, \quad [f,g]_1 = \binom{k}{1}fg' - \binom{l}{1}f'g = kfg' - lf'g, \\
&[f,g]_2 = \binom{k+1}{2}fg'' - \binom{k+1}{1}\binom{l+1}{1}f'g' + \binom{l+1}{2}f''g \\
&\qquad = \frac{1}{2}(k+1)kfg'' - (k+1)(l+1)f'g' + \frac{1}{2}(l+1)lf''g.
\end{aligned}$$

Note that for the graded ring $\mathcal{M}_*(\Gamma)$ of modular forms, we have the inclusion $\mathcal{M}_*(\Gamma) \subset H$, where $H = Hl(\mathbb{H})_{pl}$ is the vector space of holomorphic functions on the upper half-plane $\mathbb{H}$ with polynomial growth. This H is closed under differention as D, and (H, D) induces a standard Rankin-Cohen algebra $(H, [\cdot,\cdot]_*)$. The inclusion $(\mathcal{M}_*, [\cdot,\cdot]_*) \subset (H, [\cdot,\cdot]_*)$ is not closed under differentiation but it is closed under the brackets.

A way of constructing non-standard Rankin-Cohen algebras is provided by the canonical construction by Zagier ([242]). Consider the data (R_*, D, Φ), where R_* is a graded ring with a derivation D and with a choice of an element $\Phi \in R_4$, as the curvature. Then define the brackets by the formula

$$[f,g]_n^{(k,l)} = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s}\binom{n+l-1}{r} f_r g_s,$$

$$\text{where} \quad f_0 = f \quad \text{and} \quad f_{r+1} = Df_r + r(r+1)\Phi f_{r-1}.$$

Then $(R_*, [\cdot,\cdot]_*)$ is a Rankin-Cohen algebra.

There is a gauge action on the curvature Φ. Namely, for any $\varphi \in R_2$ and $f \in \mathcal{M}_k$, the transformation of $D \mapsto D'$ and $\Phi \mapsto \Phi'$ is defined by

$$D'(f) = D(f) + k\varphi f \quad \text{and} \quad \Phi' = \Phi + \varphi^2 - D(\varphi),$$

which give rise to the same Rankin-Cohen algebra. Thus, all the cases where the curvature Φ can be gauged away to zero correspond to the standard case.

The modular form w_4 defined above provides the curvature element $w_4 = 2\Phi$, and the gauge equivariance condition as for Φ' above can be rephrased in terms of Hopf algebras as the freedom to change the $\mathcal{H}_1$ action by a cocycle. In particular (cf. [92]), for the specified action, the resulting Rankin-Cohen structure is canonical but not standard, in the terminology of Zagier. ◀

The 1-form $dZ = \eta^4 dz$ is, up to scalars, the only holomorphic differential on the elliptic curve $E = X_{\Gamma(6)} \cong X_{\Gamma_0(36)}$ of equation $y^2 = x^3 + 1$, so that $dZ = \frac{dx}{y}$ in Weierstrass coordinates.

The Rankin-Cohen brackets on modular forms can be extended to those RC_n on the modular Hecke algebra, defined in terms of the action of the Hopf algebra $\mathcal{H}_1$of tranverse geometry.

In fact, more generally, it is shown in [93] that one can define such Rankin-Cohen brackets on any associative algebra $\mathcal{A}$ endowed with an action of the Hopf algebra $\mathcal{H}_1$, for which there exists an element $\omega \in \mathcal{A}$ such that $\delta_2'(a) = [\omega, a]$ for any $a \in \mathcal{A}$, with δ_2' defined above, and $\delta_n(\omega) = 0$ for any $n \geq 1$. Under these hypotheses, the following holds:

Theorem 23.3. ([93]). *Suppose that an associative algebra $\mathcal{A}$ has an action of the Hopf algebra $\mathcal{H}_1$ satisfying the conditions that for some $\omega \in \mathcal{A}$, $\delta_2'(a) = [\omega, a]$ holds for any $a \in \mathcal{A}$, and $\delta_n(\omega) = 0$ for any $n \geq 1$. Then,*

(1) *There exist Rankin-Cohen brackets RC_n of the form*

$$RC_n(a,b) = \sum_{k=0}^{n} \frac{A_k}{k!}(2Y+k)_{n-k}(a)\frac{B_{n-k}}{(n-k)!}(2Y+n-k)_k(b),$$

with $(x)_r = x(x+1)\cdots(x+r-1)$ and the coefficients $A_{-1} = 0$, $A_0 = 1$, $B_0 = 1$, and $B_1 = X$, and

$$A_{n+1} = S(X)A_n - n\omega^0(X - \frac{n-1}{2})A_{n-1},$$
$$B_{n+1} = XB_n - n\Omega(Y - \frac{n-1}{2})B_{n-1},$$

with ω^0 the right multiplication by ω, where the antipode $S(X)$ is given by $S(X) = -X + \delta_1 Y$.

(2) *When applied to the modular Hecke algebra $\mathcal{A}(\Gamma)$, with $\omega = \omega_4 = 2\Phi$, the above construction yields the brackets as above that are completely determined by their restriction to modular forms, where they agree with the Rankin-Cohen brackets with respect to the curvature Φ above.*

(3) *The RC brackets $RC_n(\cdot,\cdot)$ determine associative deformations defined as*

$$a * b = \sum_n \hbar^n RC_n(a,b).$$

As for the first step, resolving the diagonal in $\mathcal{A}(\Gamma)$ is not yet done for the modular Hecke algebras. It should shed light on the number theoretic problem of the interrelation of the Hecke operators with the algebraic structure given by the pointwise product.

The algebra $\mathcal{A}(\Gamma)$ is certainly related to the algebra of the space of two-dimensional $\mathbb{Q}$-lattices.

(Added). Recall from [93] that for $a, b \in \mathcal{A}$,

$$RC_1(a,b) = S(X)(a)2Y(b) + 2Y(a)X(b),$$

and

$$\begin{aligned} RC_2(a,b) = {} & S(X)^2(a)Y(2Y+1)(b) + S(X)(2Y+1)(a)X(2Y+1)(b) \\ & + Y(2Y+1)(a)X^2(b) - Y(a)\omega Y(2Y+1)(b) - Y(2Y+1)(a)\omega Y(b). \end{aligned}$$

24 Noncommutative moduli spaces from Shimura varieties

An important source of noncommutative spaces is provided by the boundary of classical algebro-geometric moduli spaces, where one takes into account the possible presense of degenerations of classical algebraic varieties that give rise to objects no longer defined within the context of algebraic varieties, but which still make sense as noncommutative spaces.

Example 24.1. (Edited). An example of algebro-geometric moduli spaces, which is sufficiently simple to describe but at the same time exhibits a rich structure, is given by that of the modular curves. The geometry of modular curves has already appeared behind the discussion of the 2-dimensional $\mathbb{Q}$-lattices and of the modular Hecke algebras, through an assocatiated class of functions: the modular functions that appeared in the discussion of the arithmetic algebra for the quantum statistical mechanical system of 2-dimensionl $\mathbb{Q}$-lattices and the modular forms in the modular Hecke algebras.

The modular curves, defined as quotients of the hyperbolic plane $\mathbb{H}$ by the action of a subgroup Γ of $SL_2(\mathbb{Z})$ with finite index, are complex algebraic curves, which admit an arithmetic structure, as they are defined over cyclotomic number fields $\mathbb{Q}(\zeta_N)$. They are also naturally to be moduli spaces.

The object that they parameterize are elliptic curves with some level structure. The modular curves have an algebra-geometric compactification obained by adding finitly many cusp points, given by the points in $P^1(\mathbb{Q})/\Gamma$. These correspond to the algebro-geometric degeneration of the elliptic curve to $\mathbb{C}^*$.

However, in addition to these degenerations, one can consider degenerations to noncommutative tori, obtained by the limit of $q \to e^{2\pi i\theta}$ in the modulus $q = e^{2\pi i\tau}$ of the elliptic curve, where θ is now allowed to be irrational. The resulting boundary $P^1(\mathbb{R})/\Gamma$ is a noncommutative space. It appeared in the string theory compactifications considered by Connes-Douglas-Schwarz [75]. The arithmetic properties of the noncommutative spaces $P^1(\mathbb{R})/\Gamma$ are studied in [173], [175], and [177] (the last missing). ◀

Example 24.2. (Edited and continued). The modular curves corresponding to finite index subgroups Γ in $SL_2(\mathbb{Z})$ varying form a tower of branched coverings. The projective limit of this tower sits as a connected component in the refined adelic version of the modular tower, given by the quotient

$$GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/\mathbb{C}^*,$$

where $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ denotes the adèles of $\mathbb{Q}$, with $\mathbb{A}_f = \overline{\mathbb{Z}} \otimes \mathbb{Q}$ the finite adèles.

The two-sided quotient space above is also a moduli space. In fact, it belongs to an important class of algebro-geometric moduli spaces of great arithmetic significance, that is, the Shi-mura (Will-Village) varieties $\mathrm{sm}(G, X)$, where the data (G, X) are given by a reductive algebraic group G and a Hermitian symmetric domain X. The pro-variety as the quotient above is the Shimura variety $\mathrm{sm}(GL_2, \mathbb{H}^{\pm})$, where $\mathbb{H}^{\pm} = GL_2(\mathbb{R})/\mathbb{C}^*$ is the union of the upper and lower half-planes in $P^1(\mathbb{C})$.

As mentioned above, the spaces $P^1(\mathbb{R})/\Gamma$ describe degenerations of elliptic curves to noncommutative tori. This type of degeneration corresponds to degenerating a lattice $\lambda = \mathbb{Z} + \mathbb{Z}\tau$ to a pseudo-lattice $l = \mathbb{Z} + \mathbb{Z}\theta$. (May find a detailed discussion of this viewpoint, by Manin [171] missing, and its implications in noncommutative geometry and in arithmetic). In terms of the quotient space above, it corresponds to degenerating the archimedean component, namely replacing $GL_2(\mathbb{R})$ by $M_2(\mathbb{R})^{-0} = M_2(\mathbb{R}) \setminus \{0\}$ of nonzero 2×2 matrices over $\mathbb{R}$. However, with the adelic description as the quotient space, one can equally consider the possibility of degenerating a lattice at the non-archimedean components. This brings back directly to the notion of $\mathbb{Q}$-lattices.

In fact, it is shown by Connes-Marcolli-Ramachandran [87] that the notions of 2-dimensional $\mathbb{Q}$-lattices and commensurability can be reformulated in terms of Tate modules of elliptic curves and isogeny. In these terms, the space of $\mathbb{Q}$-lattices corresponds to non-archimedean degenerations of the Tate module, which corresponds to the bad quotient

$$GL_2(\mathbb{Q})\backslash[M_2(\mathbb{A}_f) \times GL_2(\mathbb{R})]/\mathbb{C}^*.$$

The combination of these two types of degenerations yields a noncommutative compactification of the Shimura variety $\mathrm{sm}(GL_2, \mathbb{H}^{\pm})$, which is the algebra of the bad quotient

$$GL_2(\mathbb{Q}) \setminus M_2(\mathbb{A})^{-0}/\mathbb{C}^*,$$

where $M_2(\mathbb{A})^{-0}$ consists of the elements of $M_2(\mathbb{A})$ with non-zero archimedean component. One can recover the Shimura variety $\mathrm{sm}(GL_2, \mathbb{H}^{\pm})$ as the set of classical points as extremal KMS states at zero temperature of the quantum statistical mechanical system associated to the noncommutative space as the first bad quotient above ([87] and [81] the last missing). ◀

Example 24.3. (Edited and continued). More generally, Shimura varieties are given as moduli spaces for certain types of motives or as moduli spaces of Hodge structures (by Milne [186] (the prepreint missing)). A **Hodge** structure is a pair (W, h) of a finite dimensional $\mathbb{Q}$-vector space W and a homomorphism $h : \mathbb{S} \to GL(W_{\mathbb{R}})$ of the real algebraic group $\mathbb{S} = \mathrm{res}_{\mathbb{C}/\mathbb{R}} G_m$, with $W_{\mathbb{R}} = W \otimes \mathbb{R}$. This determines a decomposition $W_{\mathbb{R}} \otimes \mathbb{C} = \oplus_{p,q} W^{p,q}$ with $\overline{W^{p,q}} = W^{q,p}$ and $h(z)$ action on $W^{p,q}$ by $z^{-p}\bar{z}^{-q}$. This gives a Hodge filtration and a weight filtration $W_{\mathbb{R}} = \oplus_k W_k$, where $W_k = \oplus_{p+q=k} W^{p,q}$.

The Hodge structure (W, h) has weight m if $W_{\mathbb{R}} = W_m$. It is rational if the weight filtration is defined over $\mathbb{Q}$. A Hodge structure of weight m is polarized if there is a morphism of Hodge structures $\psi : W \otimes W \to \mathbb{Q}(-m)$, such that $(2\pi i)^m \psi(\cdot, h(i)\psi)$ is symmetric and positive definite, where $\mathbb{Q}(m)$ is the rational Hodge structure of weight $-2m$, with $W = (2\pi i)^m \mathbb{Q}$ with the action $h(z) = (z\bar{z})^m$. For a rational (W, h), the subspace of $W \otimes \mathbb{Q}(m)$ fixed by $h(z)$ for all $z \in \mathbb{C}^*$ is the space of Hodge cycles.

Then one can view Shimura varieties $\mathrm{sm}(G, X)$ as moduli spaces of Hodge structures in the following way. For a Shimura datum (G, X), let $\rho : G \to GL(V)$ a faithful representation on V. Since G is reductive, there is a finite family of tensors τ_i such that

$$G = \{g \in GL(V) \mid g\tau_i = \tau_i\}.$$

A point $x \in X$ is by construction a $G(\mathbb{R})$ conjugacy class of morphisms $h_x : \mathbb{S} \to G$, with suitable properties.

Consider data of the form $((W, h), \{s_i\}, \varphi)$, where (W, h) is a rational Hodge structure, $\{s_i\}$ a finite family of Hodge cycles, and φ a K-level structure, for some $K \subset G(\mathbb{A}_f)$, namely a K-orbit of $\mathbb{A}_f$-module isomorphisms $\varphi : V(\mathbb{A}_f) \to W(\mathbb{A}_f)$, which maps τ_i to s_i. Isomorphisms of such data are isomorphisms $f : W \to W'$ of rational Hodge structures, sending $s_i \mapsto s'_i$, and such that $f \circ \varphi = \varphi' k$ for some $k \in K$.

Assume that there exists an isomorphism of $\mathbb{Q}$-vector spaces $\beta : W \to V$ mapping $s_i \mapsto \tau_i$ and h to h_x for some $x \in X$.

Denote by $\mathrm{hg}(G, X, K)$ the set data $((W, h), \{s_i\}, \varphi)$.

The Shimura variety

$$\mathrm{sm}_K(G, X) = G(\mathbb{Q}) \backslash [X \times G(\mathbb{A}_f)]/K$$

is the moduli space of isomorphism classes of data $((W, h), \{s_i\}, \varphi)$. Namely, there is a map from $\mathrm{hg}(G, X, K)$ to $\mathrm{sm}_K(G, X)$ over $\mathbb{C}$, that descends to a bijection on isomorphism classes of $\mathrm{hg}(G, X, K)/\sim$.

In such cases one can also consider degenerations of these data, both at the archimedean and at the non-archimedean components. One then considers data $((W, h), \{s_i\}, \varphi, \beta^{\sim})$, with a non-trivial homomorphism $\beta^{\sim} : W \to V$, which is a morphism of Hodge structures, such that $\beta^{\sim}(l_{s_i}) \subset l_{\tau_i}$. This yields noncommutative spaces, inside which the classical Shimura variety sits as the set of classical points.

Quantum statistical mechanical systems associated to Shimura varieties have been studied by Ha and Paugam ([127] missing). Given a faithful representation $\rho : G \to GL(V)$ as above, there is an enveloping semigroup M, that is, a normal irreducible semigroup M in $\mathrm{End}(V)$ such that $M^\times = G$. Such a semigroup can be used to encode the degenerations of the Hodge data described above. The data (G, X, V, M) then determine a noncommutative space which describes the bad quotient $\mathrm{sm}_K^{nc}(G, X) = G(\mathbb{Q})\backslash[X \times M(\mathbb{A}_f)]$ and is a moduli space for the possibly non-invertible data $((W, h), \{s_i\}, \varphi)$. Its set of classical points is the Shimura variety $\mathrm{sm}(G, X)$. The construction of the algebras involves some delicate steps, especially to handle the presence of stacky singularities (cf. [127] missing). ◀

Remark. Recall from [183] the following.

The group $SL_2(\mathbb{Z})$ acts on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} \,|\, \mathrm{im}(\tau) > 0\}$ as

$$g\tau \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

Proof. (Added).

$$\begin{aligned}
(gg')\tau &\equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \tau = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix} \tau \\
&= \frac{(aa' + bc')\tau + ab' + bd'}{(ca' + dc')\tau + cb' + dd'}, \quad \text{and} \\
g(g'\tau) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{a'\tau + b'}{c'\tau + d'} = \frac{a(a'\tau + b') + b(c'\tau + d')}{c(a'\tau + b') + d(c'\tau + d')} = (gg')\tau.
\end{aligned}$$

□

There is a bijection between the quotient space $SL_2(\mathbb{Z})\backslash\mathbb{H}$ and the space of isomorphism classes of elliptic curves over $\mathbb{C}$, by the correspondence of each $\tau \in \mathbb{H}$ to an elliptic curve $E_\tau = \mathbb{C}/L_\tau$ a quotient of $\mathbb{C}$ by a lattice, that is, a complex 1-dimensional torus, where $L_\tau = \mathbb{Z} + \mathbb{Z}\tau$. ◀

25 The adèle class space and the spectral realization

As a noncommutative space we consider the adèle class space $X_{\mathbb{K}}$, associated to any global field $\mathbb{K}$, which leads to a spectral realization of the zeros of the Riemann zeta function for $\mathbb{K} = \mathbb{Q}$, and more generally of L-functions assocaited to Hecke characters. It also gives a geometric interpretation of the Riemann-Weil explicit formulas in number theory as a trace formula. That space for $\mathbb{K} = \mathbb{Q}$ is closely related with the space of commensurability classes of $\mathbb{Q}$-lattices described above.

We first consider the problem of finding the geometry of the set of prime numbers in the proper perspective. Namely, we consider the problem in arithmetic to understand the distribution of the set of all prime numbers as a subset of $\mathbb{Z}$ of integers.

Define the counting function π as $\pi(x)$ to be the number of primes $p \leq x$ for $x \in \mathbb{R}$. The problem is to understand the behavior of the function $\pi(x)$ as $x \to \infty$.

Shown by H. Laurent ([162] missing) is the following formula:

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{e^{2\pi i \Gamma(k) k^{-1}} - 1}{e^{-2\pi i k^{-1}} - 1},$$

where $\Gamma(k) = (k-1)!$.

The asymptotic expansion of $\pi(x)$ guessed by Gauss is the following:

$$\pi(x) = \int_0^x \frac{1}{\log u} du + R(x),$$

where the logarithmic integral has the asymptotic expansion

$$\mathrm{Li}(x) = \int_0^x \frac{1}{\log u} du \sim \sum (k-1)! \frac{x}{\log(x)^k}.$$

The size of the reminder $R(x)$ is governed by

$$R(x) = O(\sqrt{x} \log x) \quad (x \to \infty).$$

A couple of graphs of the functions $\pi(\cdot)$ and $\mathrm{Li}(\cdot)$ are not included. $\Box$.

The Riemann hypothesis is a conjecture on the zeros of the **zeta** function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The definition goes back to Euler, who shows the fundamental factorization

$$\zeta(s) = \Pi_{p:\ \text{prime}} \frac{1}{1 - \frac{1}{p^s}}.$$

It extends to a meromorphic function on the whole complex plane $\mathbb{C}$ and fulfills the functional equation

$$\frac{1}{\sqrt{\pi^s}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{\sqrt{s^{1-s}}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s),$$

so that the function

$$\zeta_{\mathbb{Q}}(s) = \frac{1}{\sqrt{\pi^s}} \gamma\left(\frac{s}{2}\right) \zeta(s)$$

admits the symmetry invariance under sending $s \mapsto 1 - s$.

The **Riemann conjecture** *asserts that all zeros of $\zeta_{\mathbb{Q}}$ are on the critical line* $\frac{1}{2}+i\mathbb{R}$.

The reason why the location of the zeros of $\zeta_{\mathbb{Q}}$ controls the size of the remainder $R(x)$ comes from the explicit formulas that relate primes with the zeros. It is proved by Riemann as the first instance of an explicit formula that

$$\pi'(x)=\mathrm{Li}(x)-\sum_{\rho}\mathrm{Li}(x^{\rho})+\int_{x}^{\infty}\frac{1}{u^2-1}\frac{du}{u\log u}+\log\xi(0),$$

where $\xi(0)=-\frac{1}{8}\Gamma(\frac{1}{4})\frac{1}{\pi^{\frac{1}{4}}}\zeta(\frac{1}{2})$, and the second sum is over non-trivial complex zeros of the zeta function, and

$$\pi'(x)=\pi(x)+\frac{1}{2}\pi(x^{\frac{1}{2}})+\frac{1}{3}\pi(x^{\frac{1}{3}})+\cdots+\frac{1}{n}\pi(x^{\frac{1}{n}})+\cdots,$$

from which the Möbius inversion formula is given as

$$\pi(x)=\sum_{m}\mu(m)\frac{1}{m}\pi'(x^{\frac{1}{m}}).$$

The explicit formulas of Riemann are written as the more modern form by A. Weil as

$$h^{\wedge}(0)+h^{\wedge}(1)-\sum_{\rho}h^{\wedge}(\rho)=\sum_{v\in\sum_{\mathbb{K}}}\int_{\mathbb{K}^*}'\frac{h(u^{-1})}{|1-u|}d^*u,$$

where $\mathbb{K}$ is now an arbitrary global field, $v\in\sum_{\mathbb{K}}$ varies among the places of $\mathbb{K}$, and the integral is over the locally compact field $\mathbb{K}_v$ obtained by the completion of $\mathbb{K}$ at the place v. As well, $\int'$ is also the paring with the distribution on $\mathbb{K}_v$ which agrees with $\frac{du}{|1-u|}$ for $u\neq 1$ and whose Fourier transform relative to a self-dual choice of additive characters α_v vanishes at 1.

By definition, a global field is a countable discrete cocompact subfield in a locally compact ring. This locally compact ring depends functrorially on $\mathbb{K}$ and is the ring of adèles of $\mathbb{K}$, denoted by $\mathbb{A}_{\mathbb{K}}$. The quotient $C_{\mathbb{K}}=GL_1(\mathbb{A}_{\mathbb{K}})/GL_1(\mathbb{K})$ is the locally compact group of idèle classes of $\mathbb{K}$, which plays a central role in class field theory. The multiplicative group $GL_1(\mathbb{K}_v)=\mathbb{K}_v^*$ is embedded canonically as a cocompact subgroup of $C_{\mathbb{K}}$.

In the Weil's explicit formula above, the test function h is in the Bruhat-Schwarz space $\mathcal{S}(C_{\mathbb{K}})$. The sum on the left hand side is over the zeros of L-functions associated to Hecke characters. The function $h^{\wedge}$ is the Fourier transform of h.

The **generalized Riemann conjecture** *asserts that all the zeros of these L-functions are on the critical line* $\frac{1}{2}+i\mathbb{R}$.

This is proved by Weil when the global field $\mathbb{K}$ has non-zero characteristice, namely, some $1+\cdots+1=0$, but remains open in the case where $\mathbb{K}$ is of characteristic zero, namely, any $1+\cdots+1\neq 0$, and as well, in this case, $\mathbb{K}$

is a number field, that is, a finite algebraic extension of the field $\mathbb{Q}$ of rational numbers.

A few graphs of the zeta function $\zeta_{\mathbb{Q}}(\cdot)$ are not included. $\boxdot$.

For $e > 0$, let $n(e)$ be the number of zeros of the Riemann zeta function $\zeta_{\mathbb{Q}}$ whose imaginary parts are in the open interval $(0, e)$. It is proved by Riemann that the step function $n(e)$ can be written as the sum

$$n(e) = n_{sm}(e) + n_{os}(e)$$

of a smooth approximation $n_{sm}(e)$ and a purely oscillatory function $n_{os}(e)$, with the explicit form for the smooth approximation

$$n_{sm}(e) = \frac{e}{2\pi}(\log\frac{e}{2\pi} - 1) + \frac{7}{8} + o(1).$$

(Added). As in [70], the oscillatory part of the step function $n(e)$ is given by

$$n_{os}(e) = \frac{1}{\pi}\mathrm{Im}\log\zeta(\frac{1}{2} + ie),$$

where e is not the imaginary part of any zero and the logarithm takes the (principal) branch (band) which is (or contains) 0 at ∞. ◀

There is a striking analogy between the behavior of the step function $n(e)$ and that of the function counting the number of eigenvalues of the Hamiltonian H of the quantum system obtained by the quantization of a chaotic dynamical system, in the theory of quantum chaos. As a comparison of the asymptotic expansions of the oscillatory terms in the two cases,

$$n_{os}(e) \sim \frac{1}{\pi}\sum_{p}\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{2\sinh\frac{m\lambda_p}{2}}\sin(meT_p)$$

for the quantization of a chaotic dynamical system, and

$$n_{os}(e) = \frac{-1}{\pi}\sum_{p}\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{p^{\frac{m}{2}}}\sin(me\log p)$$

for the Riemann zeta function. It gives an indication on the hypothetical Riemann flow that would make it possible to identify the zeros of the zeta function as the spectrum of a Hamiltonian. For instance, the periodic orbits of the flow should be labeled by prime numbers and the corresponding periods T_p in the spectrum should be given by $\log p$. However, a closer look reveals that an overall minus sign forbids any direct comparison.

The spectral realization as an absorption spectrum. The above major sign obstruction is bypassed by Connes [70] by using the following basic distinction between observed spectra in physics:

When the light coming from a hot chemical element is decomposed through a prism, it gives rise to bright **emission** lines on a dark background (band), and the corresponding frequencies are a signature of its chemical composition.

♭ When the light coming from a distant star is decomposed through a prism, it gives rise to dark lines, called **absorption** lines, on a white background (band).

The spectrum of the light emitted by the sun is first observed as an example of an absorption spectrum. In this case, the absorption lines are discovered by Fraunhofer. The chemicals in the outer atmosphere of the star absorb the corresponding frequencies in the white light coming from the core of the star.

Then the simple idea is that, because of the minus sign above, one should look for the spectral realization of the zeros of the zeta function not as a usual emission spectrum, but as an absorption spectrum. This idea does not suffice to get a solution since one needs such a basic dynamical system.

The adèle class space, defined as the quotient $X_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$ introduced by Connes [70], does do the good job as desired. The action of the idèle class group $C_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}^*/\mathbb{K}^*$ on the adèle class space $X_{\mathbb{K}}$ is given by multiplication. In particular, the idèle classs group $C_{\mathbb{K}}$ also acts on the Hilbert space $L^2(X_{\mathbb{K}})$ suitably defined, and the zeros of L-functions give the absorption spectrum, with non-critical zeros appearing as resonances.

The adèle class space $X_{\mathbb{K}}$ involves all the places of $\mathbb{K}$. In order to simplify, if one consider a restriction to a finite set of places, then one can still get a noncommutative space and one can analyze the action of the analogue of the idèle classs group $C_{\mathbb{K}}$ and compute its trace after performing a suitable cutoff, necessary in all the cases to see the missing lines of an absorption spectrum. Then the following trace formula is obtained:

$$\operatorname{tr}(r_\lambda u(h)) = 2h(1)\log'\lambda + \sum_{v\in S}\int'_{\mathbb{K}_v^*}\frac{h(u^{-1})}{|1-u|}d^*u + o(1),$$

where the second term on the right-hand side is exactly the same as in the Weil explicit formula mentioned above. This is very encouraging because at least it gives rise to a geometric meaning to the same complicated term of the formula as a contribution of the periodic orbits to the computation of the trace.

in particular, it gives a perfect interpretation of the smooth function $n_{sm}(e)$ approximating the counting function $n(e)$ of the zeros of the zeta function, as counting the number of states of the 1-dimensional quantum system with Hamiltonian $h(p,q) = 2\pi pq$, which is just the generator of the scaling group.

Indeed, the function term $\frac{e}{2\pi}(\log\frac{e}{2\pi} - 1)$ of the Riemann explicit formula above appears as the number of missing degrees of freedom in the number of quantum states for the system above, because one obtains

$$\operatorname{Area}(B_+) = \frac{e}{2\pi}\cdot 2\log\lambda - \frac{e}{2\pi}\left(\log\frac{e}{2\pi} - 1\right),$$

from a simple computation of the area of the region B_+:

$$B_+ = \{(p,q)\in[0,\lambda]^2 \mid h(p,q) = 2\pi pq \le e\},$$

while the term $\frac{e}{2\pi}2\log\lambda$ corresponds to the number of degrees of freedom of white light.

(Added). Compute that

$$|B_+| = \int_{\frac{e}{2\pi\lambda}}^{\lambda} \frac{e}{2\pi p} dp + \lambda \cdot \frac{e}{2\pi\lambda} = \frac{e}{2\pi}[\log p]_{p=\frac{e}{2\pi\lambda}}^{\lambda} + \frac{e}{2\pi}$$
$$= \frac{e}{2\pi}\left(\log\lambda - \log\frac{e}{2\pi\lambda} + 1\right) = \frac{e}{2\pi}\left(2\log\lambda - \log\frac{e}{2\pi} + 1\right). \quad \blacktriangleleft$$

A more careful computation gives not only the correction term as $\frac{7}{8}$ in the formula but all the remaining $o(1)$ terms.

It is shown by Connes [70] that the generalized Riemann hypothesis is equivalent to the validity of a global trace formula. But this is one of many equivalent reformulations of the Riemann hypothesis.

26 The Weil proof and thermo-dynamics of endomotives

In many ways, a virtue of a problem as RH comes from the developments that it generates. It does not appear as having any relation with geometry, but its geometric nature gradually does emerge in the 20th century, mainly because of the solution of Weil in the case of global fields of positive characteristic. Outline the program of CCM [74] (missing) (cf. [101]), announced in Tehran in 2005, to adapt the Weil proof for that case to the case of number fields.

Given a global field $\mathbb{K}$ of positive characteristic, there exists a finite field $\mathbb{F}_q$ and a smooth projective curve C defined over $\mathbb{F}_q$ such that $\mathbb{K}$ is the field of $\mathbb{F}_q$-valued, rational functions on C.

The analogue of the zeta function by Artin, Hasse, and Schmidt is

$$\zeta_{\mathbb{K}}(s) = \Pi_{v \in \sum_{\mathbb{K}}} \frac{1}{1 - \frac{1}{q^{f(v)s}}},$$

where $\sum_{\mathbb{K}}$ is the set of places of $\mathbb{K}$ and $f(v)$ is the degree of the place $v \in \sum_{\mathbb{K}}$.

The functional equation takes the form

$$q^{(g-1)(1-s)}\zeta_{\mathbb{K}}(1-s) = q^{(g-1)s}\zeta_{\mathbb{K}}(s),$$

where g is the genus of C.

The analogue of the Riemann conjecture for such global fields is proved by Weil (1942) who developes an algebraic geometry in that context. The proof of Weil rests on two steps: (a) Explicit formula and (b) Positivity.

Both of the steps are based on the geometry of the action of the Frobenius on the set $C(\overline{\mathbb{F}_q})$ of points of C over an algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$. This set $C(\overline{\mathbb{F}_q})$ is mapped canonically to the set $\sum_{\mathbb{K}}$ of places of $\mathbb{K}$, and the degree of a place $v \in \sum_{\mathbb{K}}$ is the number of points in the orbit of the Frobenius acting on the fiber of the projection from $C(\overline{\mathbb{F}_q})$ to $\sum_{\mathbb{K}}$.

The analogue of the Lefschetz fixed point formula:

$$\#\{C(\mathbb{F}_{q^j})\} = \sum(-1)^k \mathrm{tr}(\mathrm{Fr}_j^*|_{H^1_{et}(\overline{C},\mathbb{Q}_l)})$$

makes it possible to compute the left-hand side number of points with coordinates in the finite extension $\mathbb{F}_{q^j}$ from the actin of Fr^* in the étale cohomology group $H^1_{et}(\overline{C}, \mathbb{Q}_l)$, which does not depend upon the choice of the l-adic coefficients $\mathbb{Q}_l$.

This shows that the zeta function is a rational fraction

$$\zeta_{\mathbb{K}}(s) = \frac{p(\frac{1}{q^s})}{(1-\frac{1}{q^s})(1-q^{1-s})},$$

where the polynomial $p(\cdot)$ is the characteristic polynomial of the action of Fr^* in H^1.

The analogue of the Riemann conjecture for global fields of characteristic q means that its eigenvalues $\lambda_j \in \mathbb{C}$ of the factorization

$$p(t) = \Pi(1-\lambda_j t)$$

are of modulus $|\lambda_j| = \sqrt{q}$.

The main ingredient in the proof of Weil is the notion of correspondences, given by divisors in $C \times C$. They can be viewed as multivalued maps:

$$z : C \to C, \quad p \mapsto z(p).$$

Two correspondences u and v are equivalent if they differ by a principal divisor, so that

$$u \sim v \Leftrightarrow u - v = (f).$$

The composition of two correspondences is defined as

$$z = z_1 * z_2, \quad (z_1 * z_2)(p) = z_1(z_2(p)),$$

and their adjoint is given by $z' = \sigma(z)$, using the transposition $\sigma(x,y) = (y,x)$. The degree $d(z)$ of a correpondence is defined by

$$d(z) = z \bullet (p \times C),$$

independently of a generic points $p \in C$, where $\bullet$ is the intersection number. The codegree of a correpondence z is defined similarly as

$$d'(z) = z \bullet (C \times p).$$

The trace of a correspondence z is defined by Weil as

$$\mathrm{tr}(z) = d(z) + d'(z) - z \bullet \Delta,$$

where Δ means the identity correspondence.

Theorem 26.1. (Weil). *The following positivity holds that* $\mathrm{tr}(z * z') > 0$ *unless* z *is a trivial class.*

If one imitates the steps of Weil's proof of RH for the case of number fields, then one clearly needs to have an analogue of the points of $C(\overline{F_q})$ and the action of the Frobenius, of the étale cohomology and of the unramified extensions $\mathbb{K} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ of $\mathbb{K}$.

Endomotives and Galois action

The adèle class space $X_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$ of a global field $\mathbb{K}$ admits a natural action of the idèle class group $C_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}^*/\mathbb{K}^*$, as on the adelic side of the class field theory isomorphism. In order to obtain a description of this space which is closer to geometry, one needs to pass to the Galois side of class field theory.

In the case where $\mathbb{K} = \mathbb{Q}$, it is possible to present the adèle class space in a simple manner not involving adèles, thanks to its intimate relation with the space of 1-dimensional $\mathbb{Q}$-lattices. The direct interpretation of the action of the Galois group of $\overline{\mathbb{Q}}/\mathbb{Q}$ on the values of fabulous states for the BC-system suggests that one should be able to construct directly the space $X_{\mathbb{Q}}$ with a canonical action of the Galois group of $\overline{\mathbb{Q}}/\mathbb{Q}$.

This is done by Connes, Consani, and Marcolli [73], thanks to an extension of the notion of Artin motives, called endomotives. Following Grothendieck, one can reformulate the Galois theory over a field $\mathbb{K}$ as the quivalence of the category of reduced commutative finite dimensional algebras over $\mathbb{K}$ with the category of continuous actions of the Galois group G of $\overline{\mathbb{K}}/\mathbb{K}$ on finite sets.

By construction, the algebra of the BC-system is a crossed product of a commutative algebra $\mathcal{A}$ by a semi-group.

When working over $\mathbb{K} = \mathbb{Q}$ which is essential in the definition of fabulous states, the algebra $\mathcal{A}$ is the group ring $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ of the torsion group $\mathbb{Q}/\mathbb{Z}$. Then

$$\mathcal{A} = \varinjlim \mathcal{A}_n, \quad \mathcal{A}_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}],$$

and we deal with a projective limit of Artin motives.

The key point then is to keep track of the corresponding action of the Galois group G of $\overline{\mathbb{K}}/\mathbb{K}$, with $\mathbb{K} = \mathbb{Q}$.

The Galois-Grothendieck correspondence associates to a reduced commutative finite dimensional algebra $\mathcal{B}$ over $\mathbb{K}$ the set of characters of $\mathcal{B}$ with values in $\overline{\mathbb{K}}$, together with the natural action of G. This action is non-trivial for the algebras $\mathcal{A}_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$, where it corresponds to the cyclotomic theory.

Then it is able to recover the Bost-Connes system with its natural Galois symmetry in a conceptual manner which extends to the general context of semi-group actions on projective systems of Artin motives.

These typically arise from self-maps of algebraic varieties. Given a pointed algebraic variety (Y, y_0) over a field $\mathbb{K}$ and a countable unital abelian semi-group S of finite endomorphisms of (Y, y_0), unramified over $y_0 \in Y$, one constructs a projective system of Artin motives X_s over $\mathbb{K}$ from these data as follows.

For $s \in S$, set

$$X_s = \{y \in Y \mid s(y) = y_0\}.$$

For a pair of $s, s' \in S$, with $s' = s \circ r$, the map $\xi_{s',s} : X_{s \circ r} \to X_s$ is given by $\xi_{s',s}(y) = r(y)$.

(Added). Because $(s \circ r)(y) = y_0$, and thus $s(r(y)) = y_0$. ◀

This defines a projective system indexed by the semigroup S with partial order given by divisibility. Let $X = \varprojlim_s X_s$.

Since $s(y_0) = y_0$, the base point y_0 defines a component Z_s of X_s for all $s \in S$. Let $\xi_{s',s}^{-1}(Z_s)$ be the inverse image of Z_s in $X_{s'}$. It is a union of components of $X_{s'}$. This defines a projection e_s onto an open and closed subset X_{e_s} of the projective limit X.

It is then shown by CCM [73] that the semigroup S acts on the projective limit X by partial isomorphisms $\rho_s : X \to X_{e_s}$ defined by

$$\xi_{sou}(\rho_s(x)) = \xi_u(x), \quad \text{for any } u \in S \text{ and } x \in X.$$

The BC-system is obtained from the pointed algebraic variety $(G_m(\mathbb{Q}), 1)$, where the affine group scheme G_m is the multiplicative group. The semi-group S is given by the semi-group of non-zero endomorphisms of G_m. These correspond to the maps of the form $u \mapsto u^n$ for some non-zero integer $n \in \mathbb{Z}$, restricted to $n \in \mathbb{N}^*$.

In this class of examples, one has an equi-distribution property, by which the uniform normalized counting measures μ_s on X_s are compatible with the projective system. Then define a probability measure on the limit X. Namely, one has

$$\xi_{s',s}\mu_s = \mu_{s'}, \quad \text{for any } s, s' \in S.$$

This follows from the fact that the number of pre-images of a point under $s \in S$ is equal to the degree $\deg s$.

This provides the data which makes it possible to perform the thermodynamical analysis of such endomotives. This gives a rather unexplored new territory since even the simplest examples beyond the BC-system remain to be investigated (at that time).

For instance, let Y be an elliptic curve defined over $\mathbb{K}$. Let S be the semigroup of non-zero endomorphisms of Y. This gives rise to an example in the general class described above.

When the elliptic curve has complex multiplication, this gives rise to a system which agrees with the one constructed by Connes, Marcolli, and Ramachandran [86] in the case of a maximal order.

In the case without complex multiplication, this provides an example of a system, where the Galois action does not factor through an abelian quotient.

The Frobenius as a dual of the time evolution The Frobenius is such a universal symmetry in characteristic p, owing to the linearity of the map sending x to x^p, that it is very hard to find an analogue of such a far-reaching concept in characteristic zero. As we now go to explain, the classification of type III von Neumann factors provides the basic ingredient which, when combined with cyclic cohomology theory, makes it possible to analyze the thermo-dynamics of a noncommutative space and to get an analogue of the action of the Frobenius on étale cohomology.

The key ingredient is of that noncommutativity generates a time evolution as the (noncommutative) measure space theory level. While it is well known in Operator Algebra that the theory of von Neumann algebras represents a far-reaching extension of ordinary measure space theory, the main surprise obtained by Connes [57] following Tomi-ta (Rich, Rice field) theory is that such a von Neumann algebra $\mathfrak{M}$ inherits, from its noncommutativity, a God-given time evolution:

$$\delta : \mathbb{R} \to \mathrm{Out}(\mathfrak{M}) = \mathrm{Aut}(\mathfrak{M})/\mathrm{Inn}(\mathfrak{M}),$$

that is the quotient of the group of automorphisms of $\mathfrak{M}$ by the normal subgroup of inner automorphisms of $\mathfrak{M}$. This leads to the reduction from type III to type II von Neumann algebras and their automorphisms, and eventually to the classification of injective factors. They are classified by their modules:

$$\mathrm{Mod}(\mathfrak{M}) \subset \mathbb{R}_+^*,$$

which are virtual closed subgroups of $\mathbb{R}_+^*$, in the sense of G. Mackey, i.e., ergodic actions of $\mathbb{R}_+^*$, called the flow of weights of Connes and Take-saki (Bamboo-Cape) [98]. This invariant is first defined and used by Connes [57], to show in particular the existence of hyperfinite factors which are no isomorphic to the Ara-ki (Wild Tree)-Woods factors.

The noncommutative measure space theory level as the set-up of von Neumann algebras does not suffice to obtain a relevant cohomology theory. And one needs to be given a weakly dense subalgebra $\mathcal{A}$ in $\mathfrak{M}$ playing the role of smooth functions on the noncommutative space. This algebra plays a key role in cyclic cohomology used at a later stage.

At first one takes its norm closure $\mathfrak{A} = \overline{\mathcal{A}}^n$ in $\mathfrak{M}$ and assumes that it is globally invariant under the modular automorphism group σ_t^φ of a faithful normal state φ on $\mathfrak{M}$. One can then proceed with the thermo-dynamics of the C^*-dynamical system $(\mathfrak{A}, \sigma_t, \mathbb{R})$.

By a simple procedure assuming that KMS states at low temperature are of type I, one obtains a cooling morphism π which is a morphism of algebras from the smooth crossed product $\mathcal{A} \rtimes_\sigma \mathbb{R}$ to a type I algebra of compact operator valued, functions on a canonical $\mathbb{R}_+^*$-principal bundle $\Omega_\beta^\sim$ over the space Ω_β of type I extremal KMS_β states fulfilling a suitable regularity condition ([73]).

Any $\rho \in \Omega_\beta$ gives an irreducible representation π_ρ of $\mathcal{A}$, and the choice of its esentially unique extension to $\mathcal{A} \rtimes_\sigma \mathbb{R}$ determines the fiber of the $\mathbb{R}_+^*$-principal bundle $\Omega_\beta^\sim$. The cooling morphism is then given by

$$\pi_{\rho,H}(\int_{\mathbb{R}} x(t)u_t dt) = \int_{\mathbb{R}} \pi_\rho(x(t))e^{itH} dt.$$

This morphism is equivariant for the dual action $\theta_\lambda \in \mathrm{Aut}(\mathcal{A} \rtimes_\sigma \mathbb{R})$ of $\mathbb{R}_+^*$ given as

$$\theta_\lambda(\int_{\mathbb{R}} x(t)u_t dt) = \int_{\mathbb{R}} \lambda^{it} x(t)u_t dt.$$

The key point is that the range of the morphism π is contained in an algebra of functions on $\Omega_{\mathbb{R}}^\sim$ with values of trace class operators. In other words, modulo the (strong) Morita equivalence, one lands in the commutative world, provided one lowers the temperature.

The interesting space is then obtained by distillation and is given by the co-kernel of the cooling morphism π, but this does not make sense in the category of algebras and algebra homomorphisms, since the latter is not even an additive category. This is where cyclic cohomology enters the scene. Because the category of cyclic modules is an abelian category with a natural functor from the category of algebras and algebra homomorphisms.

Cyclic modules are modules of the cyclic category Λ, which is a small category, obtained by enriching with cyclic morphisms the familiar simplicial category Δ of totally ordered finite sets and increasing maps.

Alternatively, the category Λ can be defined by means of its cyclic covering, the category $E\Lambda$. The latter has one object $(\mathbb{Z}, n)$ for each $n \geq 0$ and the morphisms $f : (\mathbb{Z}, n) \to (\mathbb{Z}, m)$ are given by non-decreasing maps $f : \mathbb{Z} \to \mathbb{Z}$, such that $f(x+n) = f(x) + m$ for any $x \in \mathbb{Z}$.

One has $\Lambda = E\Lambda/\mathbb{Z}$, with respect to the obvious action of $\mathbb{Z}$ by translations.

To any algebra $\mathcal{A}$ one associates a module $\mathcal{A}^{\natural}$ over the category Λ by assigning to each n the $(n+1)$-fold tensor product $\otimes^{n+1}\mathcal{A}$.

The cyclic morphisms correspond to the cyclic permutations of the tensor products, while the face and degeneracy maps correspond to the algebra product of consecutive tensor products and the insertion of the unit.

The corresponding functor sending $\mathcal{A}$ to $\mathcal{A}^{\natural}$ gives a linearization of the category of associative algebras, and cyclic cohomology appears as a derived functor.

One can then define the distilled module $D(\mathcal{A}, \varphi) = (\mathcal{A} \rtimes_{\sigma} \mathbb{R})/\ker(\pi)$ with $\sigma = \sigma^{\varphi}$, as the co-kernel of the cooling morphism π. And consider the dual action of $\mathbb{R}_+^*$ obtained from the equivariance above, in the cyclic homology group $HC_0(D(\mathcal{A}, \varphi))$. As shown by Connes, Consani, and Marcolli [73], this in the simplest case of the BC-system gives a cohomological interpretation of the above spectral realization of the zeros of the Riemann zeta functions and of Hecke L-functions.

One striking feature is that the strip in the KMS condition is canonically identified with the critical strip of the zeta function by multiplication by $i = \sqrt{-1}$, where $\beta > 1$.

This cohomological interpretation combined with the above theory of endomotives gives a natural action of the Galois group G of $\overline{\mathbb{Q}}/\mathbb{Q}$ on the above cohomology. This action factorizes through the abelianization G^{ab}, and the corresponding decomposition according to characters of G^{ab} corresponds to the spectral realization of L-functions.

The role of the invariant $S(\mathfrak{M})$ in the classification of factors, or of the more refined flow of weights mentioned above is similar to the role of the module of local or global fields and the Brauer theory of central simple algebras.

In fact, there are striking parallel worlds (see [73]) between the lattice of unramified extensions from a global field $\mathbb{K}$ of characteristic p to $\mathbb{K} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ and the lattice of extensions of a factor $\mathfrak{M}$ to the crossed product algebras $\mathfrak{M} \rtimes_{\sigma_T} \mathbb{Z}$. Taking the algebraic closure of $\mathbb{F}_q$, i.e., the operation $\mathbb{K} \to \mathbb{K} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ corresponds to passing to the dual algebra as $\mathfrak{M} \to \mathfrak{M} \rtimes_{\sigma} \mathbb{R}$, and the dual action corresponds to the Frobenius automorphism when as above the appropriate cohomological operations such as distillation and cyclic homology HC_0 are performed.

Table 17: Parallel world

World	Global fields $\mathbb{K}$	Factors $\mathfrak{M}$
Modules	$\mathrm{Mod}(\mathbb{K}) \subset \mathbb{R}_+^*$	$\mathrm{Mod}(\mathfrak{M}) \subset \mathbb{R}_+^*$
Extensions (discrete)	$\mathbb{K} \to \mathbb{K} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ tensor, unramified	$\mathfrak{M} \to \mathfrak{M} \rtimes_{\sigma_T} \mathbb{Z}$ $\mathbb{Z}$ crossed products
Extensions (continuous)	$\mathbb{K} \to \mathbb{K} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ by algebraic closure	$\mathfrak{M} \to \mathfrak{M} \rtimes_{\sigma} \mathbb{R}$ by modular automorphism σ
Duals	Frobenius automorphism	Dual action θ

A notable difference from the original Hilbert space theoretic spectral realization of Connes [70] is that while, in the latter case, only the critical zeros are appearing directly in the cyclic homology set-up, and the possible non-critical ones are appearing as resonances, it is more natural to use everywhere the rapid decay framework, advocated by Meyer [185], so that all the zeros appear on the same footing.

This eliminates the difficulty coming from the potential non-critical zeros, so that the trace formula is proved and reduced to the Riemann-Weil formula. However, it is not obvious how to obtain a direct geometric proof of this formula from the S-local trace formula of Connes [70].

This is done by Meyer [185], by showing that the noncommutative geometric framework makes it possible to give a geometric interpretation of the Riemann-Weil explicit formula.

Remark. (Added). Now recall from [185] some basic definitions.

An algebraic number field K is a finite algebraic extension of the field $\mathbb{Q}$ of rational numbers. The set $P(K)$ of places of K consists of equivalence classes of dense embeddings of K into local fields. For instance, $P(\mathbb{Q})$ contains the embeddings of $\mathbb{Q}$ into $\mathbb{R}$and into $\mathbb{Q}_p$ of the p-adic integers for all prime numbers. Let A_K be the adele ring of K, let A_K^* be its ideal group, and let $C_K = A_K^*/K^*$ be its ideal class group. For instance, $C_{\mathbb{Q}}$ is isomorphic to the direct product $\Pi_p \mathbb{Z}_p^* \times \mathbb{R}$, where p runs through the prime numbers and $\mathbb{Z}_p^*$ denotes the multiplicative group of p-adic integers of norm 1. ◀

While the spectral side of the trace formula is given by the action on the cyclic homology of the distilled space, the geometric side is given as follows.

Theorem 26.2. ([73]). *Let $h \in \mathcal{S}(C_{\mathbf{K}})$ the Schwartz space. Then it holds that*

$$\mathrm{tr}(\vartheta(h)|_{H^1}) = h^\wedge(0) + h^\wedge(1) - \Delta \bullet \Delta h(1) - \sum_v \int'_{(\mathbb{K}_v^*, e_{\mathbb{K}_v})} \frac{h(u^{-1})}{|1-u|} d^* u.$$

May refer to [73] for the detailed notations, in particular, for the restricted Schwarz space $\mathcal{S}(C_{\mathbf{K}})$, which are essentially those of Connes [70].

The origin of the terms in the geometric side of the trace formula comes from the Lefschetz formula of Atiyah-Bott [13] and its adaptation by Guillemin-

Sternberg (cf. [125]) to the distribution theoretic trace for flows on manifolds, which is a variation on the theme of AB [13].

(Added). May now recall [125, Introduction] into the following:

Table 18: The systems in physics

Physics	Classical	Quantum
States	Points of a phase space as a symplectic manifold M	Functions or vectors of a Hilbert space H
Dynamics	Hamiltonian energy function H	Self-adjoint operator D
Moving in time	Symplectic vector field X_H	Unitary group $\exp(itD)$
Equilibrium states	A point in M, at which the trajectory of X_H periodic	An eigenstate of D with eigenvalue λ
Oscillating	Period of the trajectory	Period $2\pi\lambda^{-1}$

May also recall from [183] the following facts. A symplectic manifold (M,ω) is a differentiable manifold M with a symplectic form ω, i.e., a non-degenerate closed 2-form., so that $d\omega = 0$.

Under a Hamilton function $H(q,p)$ independent of time, the motion equation is given as

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$

Viewing the right hand side as a vector field $X(= dH)$, then (suppose) $dH = \iota_X(\omega_0)$ (a closed 1-form with $d\omega_0 = 0$), and (then) $L_X\omega_0 = 0$ by Cartan relation, so that integrating X implies to obtain a diffeomorphism preserving $\omega_0 = -\sum dp_i \wedge dq_i$. In general, a symplectic vector field on (M,ω) is a vector field X on M such that $L_X\omega = 0$. There is a bijective correspondence between symplectic vector fields and closed 1-forms on M. A Hamiltonian vector field is defined to be a symplectic vector field corresponding to an exact 1-form η, so that $\eta = d\omega'$ for some ω'. If $\iota_X\omega = df$, then $X = X_f$ is the Hamilton vector field of f, and f is said to be the Hamilton function (Hamiltonian) of the X.

The interior product$\iota_X\omega$ between X and ω is defined by

$$(\iota_X\omega)(Y) = \omega(X,Y).$$

The Lie derivative $L_X\omega$ is defined by

$$(L_X\omega)(Y,Z) = X(\omega(Y,Z)) - \omega([X,Y],Z) - \omega(Y,[X,Z]).$$

Moreover, in particular,

$$L_X = \iota_X d + d\iota_X \quad \text{and} \quad L_X(d\omega) = d(L_X\omega),$$

the first of which is one of Cartan relations.

It then follows that

$$L_X\omega_0 = \iota_X d\omega_0 + ddH = 0. \quad \blacktriangleleft$$

For the action of $C_{\mathbb{K}}$ on the adèle class space $X_{\mathbb{K}}$, the relevant periodic orbits $X_{\mathbb{K},v}$, on which the computation concentrates, turn out to be type III noncommutative spaces. Each of them admits classical points to form a subset $\Xi_{\mathbb{K},v} \subset X_{\mathbb{K},v}$. The union of these classical periodic points is given by

$$\Xi_{\mathbb{K}} = \cup_v \Xi_{\mathbb{K},v}, \quad \Xi_{\mathbb{K},v} = C_{\mathbb{K}}[v], \quad v \in \Sigma_{\mathbb{K}},$$

where for each place $v \in \Sigma_{\mathbb{K}}$, one lets $[v]$ be the adèle defined by $[v]_w = 1$ for any $w \neq v$, and $[v]_v = 0$.

In the function field case, one has a non-canonical isomorphism of the following form:

Proposition 26.3. *Let $\mathbb{K}$ be the function field of an algebraic curve C over $\mathbb{F}_q$. Then the action of the Frobenius on $Y = C(\overline{\mathbb{F}_q})$ is isomorphic to the action of $q^{\mathbb{Z}}$ on the quotient $\Xi_{\mathbb{K}}/C_{\mathbb{K},1}$.*

In the case where $\mathbb{K} = \mathbb{Q}$, the space $\Xi_{\mathbb{Q}}/C_{\mathbb{Q},1}$ appears as the union of periodic orbits of period $\log p$ under the action of $C_{\mathbb{Q}}/C_{\mathbb{Q},1} \sim \mathbb{R}$

The figure as the classical points of the adèles class space, corresponding to $\log 2, \log 3, \log 5, \cdots, \log p, \cdots$, is omitted.

This give a first approximation to the sought for the space $Y = C(\overline{\mathbb{F}_q})$ in characteristice zero. One important refinement is obtained from the subtle nuance between the adelic description of $X_{\mathbb{Q}}$ and the finer description in terms of the endomitive obtained from the pointed algebraic variety $(G_m(\mathbb{Q}), 1)$. The second description keeps track of the Galois symmetry, and as in the proposition above the isomorphism of the two descriptions is non-canonical.

At this point we have, in characteristic zero, several of the geometric notions which are the analogues of the ingredients of the Weil proof, and it is natural to try and imitate the steps in the proof.

The step (a) as the explicit formula corresponds to the formula obtained in the theorem above. What remains is to prove a corresponding positivity result.

A well known result of A. Weil (cf. [26] missing) states that RH is equivalent to the positivity of the distribution entering in the explicit formulae.

Thanks to the above H^1 obtained as the cyclic homology of the distilled module, the reformulation from Weil's can be stated as in the following.

We let $\vartheta(g)|_{H^1}$ denote the induced action of $g \in C_{\mathbb{K}}$ on the co-kernel H^1 described above. We also write $\vartheta(f)|_{H^1}$ for the action of $\int_{C_{\mathbb{K}}} f(g)\vartheta(g)d^*g$ with $f \in \mathcal{S}(C_{\mathbb{K}})$, as in [73] and [74] (missing). Then

Theorem 26.4. *The following two conditions are equivalent:*

(1) *All L-functions with Grössencharakter on $\mathbb{K}$ satisfy the Riemann Hypothesis* (RH).

(2) $\mathrm{tr}(\vartheta(f * f^{\natural})|_{H^1}) \geq 0$ *for all* $f \in \mathcal{S}(C_{\mathbb{K}})$ (Pos : Positivity),

where the convolution of two functions, using the multiplicative Haar measure d^*g is defined by

$$(f_1 * f_2)(g) = \int_{C_\mathbb{K}} f_1(k) f_2(k^{-1}g) d^*g,$$

and the adjoint $f^\natural$ of f is defined by

$$f^\natural(g) = \frac{1}{|g|}\overline{f(g^{-1})}.$$

The role of the specific correspondences used in the Weil proof of RH of positive characteristic is played by the test functions $f \in \mathcal{S}(C_\mathbb{K})$. More precisely, the scaling map which replaces $f(x)$ by $f(g^{-1}x)$ has a graph, namely the set of pairs $(x, g^{-1}x) \in X_\mathbf{K} \times X_\mathbb{K}$, which we view as a correspondence z_g. Then, given a test function f on the idèles classes, one assigns to f the linear combination

$$z(f) = \int_{C_\mathbb{K}} f(g) z_g d^*g$$

of the above graphs, viewed as a divisor on $X_\mathbf{K} \times X_\mathbf{K}$.

The analogs of the degrees $d(z)$ and co-degrees $d'(z) = d(z')$ of correspondences in the context of the Weil proof are given respectively by

$$d(z(h)) = h^\wedge(1) = \int h(u)|u| d^*u,$$

so that the degree $d(z_g)$ of the correspondence z_g is equal to $|g|$, and similarly,

$$d'(z(h)) = d(z(\overline{h^\natural})) = \int h(u) d^*u = h^\wedge(0),$$

so that the co-degree $d'(z_g)$ of the correspondence z_g is equal to 1.

One of the major difficulties is to find the replacement for the principal divisors which in the Weil proof play a key role as an ideal of the algebra of correspondences on which the trace vanishes. At least, already, one can see that there is an interesting subspace $\mathcal{V}$ of the linear space of correspondences described above, on which the trace also vanishes. It is given by the subspace $\mathcal{V}$ of $\mathcal{S}(C_\mathbb{K})$ as

$$\mathcal{V} = \{g(x) = \sum \xi(kx) \,|\, \xi \in \mathcal{S}(\mathbb{A}_\mathbb{K})_0\},$$

where the subspace $\mathcal{S}(\mathbb{A}_\mathbf{K})_0$ of $\mathcal{S}(\mathbb{A}_\mathbf{K})$ is defined by the two boundary conditions $\xi(0) = 0$ and $\int \xi(x) dx = 0$.

This shows that the Weil pairing as in the theorem above admits a huge radical given by all the functions which extend to adèles and gives another justification for working with the above cohomology H^1. In particular, one can modify arbitrarily the degree and co-degree of the correspondence $z(h)$ by adding to h an element of the radical $\mathcal{V}$ using a subtle failure of the Fubini theorem.

It is shown by CCM [74] (missing) that several of the steps of the Weil proof can be transposed in the framework described above.

This constitutes another just motivation to develop noncommutative geometry much more further. One can write a tentative form of a dictionary between the language of algebraic geometry in the case of curves and that of noncommutative geometry. The dictionary is summarized in the following table. It should be stressed that the main problem is to find a correct translation in the right column for the well establised notion of principal divisor in the left column. The table below is somewhat crude in that respect, since one does not expect to be able to work in the usual primary theory which involves periodic cyclic homology and index theorems. Instead, one expects that both the unstable cyclic homology and the finer invariants of spectral triples arising from transgression play an important role. Thus, the table below should be taken as a rough approximation in the first stage and could be a motivation for developing the missing finer notions in the right column.

Table 19: A sort of dictionary between AG and NG

Algeraic Geometry	Noncommutative G
Modulo torsion	$KK(\mathfrak{A}, \mathfrak{B} \otimes \mathrm{II}_1)$
Effective correspondences	Epimorphism of C^*-modules
Principal correspondences	Compact morphisms
Composition	Kasparov product in KK
Degree of correspondence	Pointwise index $d(\Gamma)$
$\deg D(P) \geq g$ implies $\sim$ effective	$d(\Gamma) > 0$ implies $\Gamma + K$ onto for some K
Adjusting the degree by trivial correspondences	Fubini step on the test functions
Frobenius correspondence	Correspondence z_g
Lefschetz formula	Bivariant Chern of $z(h)$ (Localization on graph $z(h)$)
Weil trace unchanged by principal divisors	Bivariant Chern unchanged by compact perturbations

Epilogue. Added and collected mostly is the basic knowledge such as only definitions, related elementary or fundamental properties or results, and facts, as cited. But more other terms, details, or explanations could not be included to be self-contained as aimed. However, hope that it may be some useful as a convenient reference, especially for beginners (including Mr. myself against forgetting) to go into the noncommutative world. So may not to be lost there. Still, at this last moment, there may be found some minor mistakes in typing, because of the limited time and effort for proofing the texts, grown large.

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