

EQUIVARIANT PICARD GROUPS OF C^* -ALGEBRAS WITH FINITE DIMENSIONAL C^* -HOPF ALGEBRA COACTIONS

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ABSTRACT. Let A be a C^* -algebra and H a finite dimensional C^* -Hopf algebra with its dual C^* -Hopf algebra H^0 . Let (ρ, u) be a twisted coaction of H^0 on A . We shall define the (ρ, u, H) -equivariant Picard group of A , which is denoted by $\text{Pic}_H^{\rho, u}(A)$, and discuss basic properties of $\text{Pic}_H^{\rho, u}(A)$. Also, we suppose that (ρ, u) is the coaction of H^0 on the unital C^* -algebra A , that is, $u = 1 \otimes 1^0$. We investigate the relation between $\text{Pic}(A^s)$, the ordinary Picard group of A^s and $\text{Pic}_H^{\rho^s}(A^s)$ where A^s is the stable C^* -algebra of A and ρ^s is the coaction of H^0 on A^s induced by ρ . Furthermore, we shall show that $\text{Pic}_{H^0}^{\widehat{\rho}}(A \rtimes_{\rho, u} H)$ is isomorphic to $\text{Pic}_H^{\rho, u}(A)$, where $\widehat{\rho}$ is the dual coaction of H on the twisted crossed product $A \rtimes_{\rho, u} H$ of A by the twisted coaction (ρ, u) of H^0 on A .

1. INTRODUCTION

Let A be a C^* -algebra and H a finite dimensional C^* -Hopf algebra with its dual C^* -Hopf algebra H^0 . Let (ρ, u) be a twisted coaction of H^0 on A . We shall define the (ρ, u, H) -equivariant Picard group of A , which is denoted by $\text{Pic}_H^{\rho, u}(A)$. Also, we shall give a similar result to the ordinary Picard group as follows: Let $\text{Aut}_H^{\rho, u}(A)$ be the group of all automorphisms α of A satisfying that $(\alpha \otimes \text{id}) \circ \rho = \rho \circ \alpha$ and let $\text{Int}_H^{\rho, u}(A)$ be the normal subgroup of $\text{Aut}_H^{\rho, u}(A)$ consisting of all generalized inner automorphisms $\text{Ad}(v)$ of A satisfying that $\rho(v) = v \otimes 1^0$, where v is a unitary element in the multiplier algebra $M(A)$ of A . Then we have the following exact sequence:

$$1 \longrightarrow \text{Int}_H^{\rho, u}(A) \longrightarrow \text{Aut}_H^{\rho, u}(A) \longrightarrow \text{Pic}_H^{\rho, u}(A).$$

Especially, let A^s be a stable C^* -algebra of a unital C^* -algebra A and ρ a coaction of H^0 on A . Also, let ρ^s be the coaction of H^0 on A^s induced by a coaction ρ of H^0 on A . Then under a certain condition, we can obtain the exact sequence

$$1 \longrightarrow \text{Int}_H^{\rho^s}(A^s) \longrightarrow \text{Aut}_H^{\rho^s}(A^s) \longrightarrow \text{Pic}_H^{\rho^s}(A^s) \longrightarrow 1.$$

In order to do this, we shall extend definitions and results in the case of unital C^* -algebras to ones in the case of non-unital C^* -algebras in the section of Preliminaries. Using this result, we shall investigate the relation between $\text{Pic}(A^s)$, the ordinary Picard group of A^s and $\text{Pic}_H^{\rho^s}(A^s)$, the (ρ^s, H) -equivariant Picard group of A^s . Furthermore, we shall show that $\text{Pic}_{H^0}^{\widehat{\rho}}(A \rtimes_{\rho, u} H)$ is isomorphic to $\text{Pic}_H^{\rho, u}(A)$, where $\widehat{\rho}$ is the dual coaction of H on the twisted crossed product $A \rtimes_{\rho, u} H$ of A by the twisted coaction (ρ, u) .

2. PRELIMINARIES

Let H be a finite dimensional C^* -Hopf algebra. We denote its comultiplication, counit and antipode by Δ , ϵ and S , respectively. We shall use Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for any $h \in H$ which suppresses a possible summation when we write multiplications. We denote by N the dimension of H . Let H^0 be the dual C^* -Hopf algebra of H . We denote its comultiplication, counit and antipode by Δ^0 , ϵ^0 and S^0 , respectively. There is the distinguished projection e in H . We note

that e is the Haar trace on H^0 . Also, there is the distinguished projection τ in H^0 which is the Haar trace on H . Since H is finite dimensional, $H \cong \bigoplus_{k=1}^L M_{f_k}(\mathbf{C})$ and $H^0 \cong \bigoplus_{k=1}^K M_{d_k}(\mathbf{C})$ as C^* -algebras. Let $\{v_{ij}^k \mid k = 1, 2, \dots, L, i, j = 1, 2, \dots, f_k\}$ be a system of matrix units of H . Let $\{w_{ij}^k \mid k = 1, 2, \dots, K, i, j = 1, 2, \dots, d_k\}$ be a basis of H satisfying Szymański and Peligrad's [25, Theorem 2.2,2], which is called a system of *comatrix* units of H , that is, the dual basis of a system of matrix units of H^0 . Also let $\{\phi_{ij}^k \mid k = 1, 2, \dots, K, i, j = 1, 2, \dots, d_k\}$ and $\{\omega_{ij}^k \mid k = 1, 2, \dots, L, i, j = 1, 2, \dots, f_k\}$ be systems of matrix units and comatrix units of H^0 , respectively.

Let A be a C^* -algebra and $M(A)$ its multiplier algebra. Let p, q be projections in A . If p and q are Murray-von Neumann equivalent, then we denote it by $p \sim q$ in A . We denote by id_A and 1_A the identity map on A and the unit element in A , respectively. We simply denote them by id and 1 if no confusion arises. Modifying Blattner, Cohen and Montgomery [3, Definition 2.1], we shall define a weak coaction of H^0 on A .

Definition 2.1. By a weak coaction of H^0 on A , we mean a $*$ -homomorphism $\rho : A \rightarrow A \otimes H^0$ satisfying the following conditions:

- (1) $\overline{\rho(A)(A \otimes H^0)} = A \otimes H^0$,
- (2) $(\text{id} \otimes \epsilon^0)(\rho(x)) = x$ for any $x \in A$,

By a coaction of H^0 on A , we mean a weak coaction ρ such that

- (3) $(\rho \otimes \text{id}) \circ \rho = (\text{id} \otimes \Delta^0) \circ \rho$.

By Condition (1) in Definition 2.1, for any approximate unit $\{u_\alpha\}$ of A and $x \in A \otimes H^0$, $\rho(u_\alpha)x \rightarrow x$ ($\alpha \rightarrow \infty$). Hence $\rho(1) = 1 \otimes 1^0$ when A is unital. Since H^0 is finite dimensional, $M(A \otimes H^0) \cong M(A) \otimes H^0$. We identify $M(A \otimes H^0)$ with $M(A) \otimes H^0$. We also identify $M(A \otimes H^0 \otimes H^0)$ with $M(A) \otimes H^0 \otimes H^0$. Let ρ be a weak coaction of H^0 on A . By Jensen and Thomsen [12, Corollary 1.1.15], there is the unique strictly continuous homomorphism $\underline{\rho} : M(A) \rightarrow M(A) \otimes H^0$ extending ρ .

Lemma 2.1. *With the above notations, $\underline{\rho}$ is a weak coaction of H^0 on $M(A)$.*

Proof. Clearly $\underline{\rho}$ is a $*$ -homomorphism of $M(A)$ to $M(A) \otimes H^0$. Let $\{u_\alpha\}$ be an approximate unit of A . Then by Condition (1) in Definition 2.1, $\{\rho(u_\alpha)\}$ is an approximate unit of $A \otimes H^0$. Hence $\underline{\rho}(1) = 1 \otimes 1^0$. Since H^0 is finite dimensional, $\text{id} \otimes \epsilon^0$ is strictly continuous. Hence $\underline{\rho}$ satisfies Condition (2) in Definition 2.1. \square

Let ρ be a weak coaction of H^0 on A and u a unitary element in $M(A) \otimes H^0 \otimes H^0$. Following Masuda and Tomatsu [19, Section 3], we shall define a twisted coaction of H^0 on A .

Definition 2.2. The pair (ρ, u) is a *twisted coaction* of H^0 on A if the following conditions hold:

- (1) $(\rho \otimes \text{id}) \circ \rho = \text{Ad}(u) \circ (\text{id} \otimes \Delta^0) \circ \rho$,
- (2) $(u \otimes 1^0)(\text{id} \otimes \Delta^0 \otimes \text{id})(u) = (\underline{\rho} \otimes \text{id} \otimes \text{id})(u)(\text{id} \otimes \text{id} \otimes \Delta^0)(u)$,
- (3) $(\text{id} \otimes \text{id} \otimes \epsilon^0)(u) = (\text{id} \otimes \epsilon^0 \otimes \text{id})(u) = 1 \otimes 1^0$.

Remark 2.2. Let (ρ, u) be a twisted coaction of H^0 on A . Since H^0 is finite dimensional, $\text{id}_{M(A)} \otimes \Delta^0$ is strictly continuous. Hence by Lemma 2.1, $(\underline{\rho}, u)$ satisfies Definition 2.2. Therefore, $(\underline{\rho}, u)$ is a twisted coaction of H^0 on $M(A)$. Hence if ρ is a coaction of H^0 on A , $\underline{\rho}$ is a coaction of H^0 on $M(A)$.

Let $\text{Hom}(H, M(A))$ be the linear space of all linear maps from H to $M(A)$. Then by Sweedler [24, pp69-70], it becomes a unital convolution $*$ -algebra. Similarly, we

define $\text{Hom}(H \times H, M(A))$. We note that ϵ and $\epsilon \otimes \epsilon$ are the unit elements in $\text{Hom}(H, M(A))$ and $\text{Hom}(H \times H, M(A))$, respectively.

Modifying Blattner, Cohen and Montgomery, [3, Definition 1.1] we shall define a weak action of H on A .

Definition 2.3. By a weak action of H on A , we mean a bilinear map $(h, x) \mapsto h \cdot x$ of $H \times A$ to A satisfying the following conditions:

- (1) $h \cdot (xy) = [h_{(1)} \cdot x][h_{(2)} \cdot y]$ for any $h \in H, x, y \in A$,
- (2) $[h \cdot u_\alpha]x \rightarrow \epsilon(h)x$ for any approximate unit $\{u_\alpha\}$ of A and $x \in A$.
- (3) $1 \cdot x = x$ for any $x \in A$,
- (4) $[h \cdot x]^* = S(h)^* \cdot x^*$ for any $h \in H, x \in A$.

By an action of H on A , we mean a weak action of such that

- (5) $h \cdot [l \cdot x] = (hl) \cdot x$ for any $x \in A$ and $h, l \in H$.

Since H is finite dimensional, as mentioned in [3, pp. 163], there is an isomorphism ι of $M(A) \otimes H^0$ onto $\text{Hom}(H, M(A))$ defined by $\iota(x \otimes \phi)(h) = \phi(h)x$ for any $x \in M(A), h \in H, \phi \in H^0$. Also, we can define an isomorphism j of $M(A) \otimes H^0 \otimes H^0$ onto $\text{Hom}(H \times H, M(A))$ in the similar way to the above. We note that $\iota(A \otimes H^0) = \text{Hom}(H, A)$ and $j(A \otimes H^0 \otimes H^0) = \text{Hom}(H \otimes H, A)$. For any $x \in M(A) \otimes H^0$ and $y \in M(A) \otimes H^0 \otimes H^0$, we denote $\iota(x)$ and $j(y)$ by \hat{x} and \hat{y} , respectively.

Let a bilinear map $(h, x) \mapsto h \cdot x$ from $H \times A$ to A be a weak action. For any $x \in A$, let f_x be the linear map from H to A defined by $f_x(h) = h \cdot x$ for any $h \in H$. Let ρ be the linear map from A to $A \otimes H^0$ defined by $\rho(x) = \iota^{-1}(f_x)$ for any $x \in A$.

Lemma 2.3. *With the above notations, ρ is a weak coaction of H^0 on A .*

Proof. By its definition, ρ is a $*$ -homomorphism of A to $A \otimes H^0$ satisfying Conditions (2) in Definition 2.1. So, we have only to show that ρ satisfies Condition (1) in Definition 2.1. Let $\{u_\alpha\}$ be an approximate unit of A . We can write that $\rho(u_\alpha) = \sum_j u_{\alpha j} \otimes \phi_j$, where $u_{\alpha j} \in A$ and $\{\phi_j\}$ is a basis of H^0 with $\sum_j \phi_j = 1^0$. Let $\{h_j\}$ be the dual basis of H corresponding to $\{\phi_j\}$. Then for any $x \in A$ and j , $[h_j \cdot u_\alpha]x \rightarrow \epsilon(h_j)x$ by Condition (2) in Definition 2.3. Since $[h_j \cdot u_\alpha]x = (\text{id} \otimes h_j)(\rho(u_\alpha))x = u_{\alpha j}x$, $u_{\alpha j}x \rightarrow \epsilon(h_j)x$ for any j . Also, since $\sum_j \phi_j = 1^0$,

$$1 = \phi_j(h_j) = \sum_i \phi_i(h_j) = 1^0(h_j) = \epsilon(h_j)$$

for any j . Hence $u_{\alpha j}x \rightarrow x$ for any j . Therefore, for any $x \in A$ and $\phi \in H^0$,

$$\rho(u_\alpha)(x \otimes \phi) = \sum_j u_{\alpha j}x \otimes \phi_j \phi \rightarrow \sum_j x \otimes \phi_j \phi = x \otimes \phi.$$

Thus $\overline{\rho(A)(A \otimes H^0)} = A \otimes H^0$. □

For any weak coaction ρ of H^0 on A , we can define the bilinear map $(h, x) \mapsto h \cdot_\rho x$ from $H \times A$ to A by

$$h \cdot_\rho x = (\text{id} \otimes h)(\rho(x)) = \rho(\hat{x})(h).$$

We shall prove that the above map is a weak action of H on A .

Lemma 2.4. *With the above notations, the linear map $(h, x) \mapsto h \cdot_\rho x$ from $H \times A$ to A is a weak action of H on A .*

Proof. We have only to show that the above linear map satisfies Condition (2) in Definition 2.3. Let $\{u_\alpha\}$ be an approximate unit of A . Then for any $x \in A \otimes H^0$,

$\rho(u_\alpha)x \rightarrow x$ by the proof of Lemma 2.1. We can write that $\rho(u_\alpha) = \sum_j u_{\alpha j} \otimes \phi_j$, where $u_{\alpha j} \in A$ and $\{\phi_j\}$ is a basis of H^0 . Then for any $a \in A$,

$$\begin{aligned} [h \cdot_\rho u_\alpha]a &= (\text{id} \otimes h)(\rho(u_\alpha))a = \sum_j u_{\alpha j} \phi_j(h)a \\ &= (\text{id} \otimes h)(\rho(u_\alpha)(a \otimes 1^0)) \rightarrow \epsilon(h)a \end{aligned}$$

since $\text{id} \otimes h$ is a bounded operator from $A \otimes H^0$ to A . \square

Remark 2.5. By the proofs of Lemmas 2.3 and 2.4, Condition (2) in Definition 2.3 is equivalent to the following Condition (2)':

(2)' $[h \cdot u_\alpha]x \rightarrow \epsilon(h)x$ for some approximate unit of A and any $x \in A$.

Also, if A is unital, Condition (2) in Definition 2.3 means that $h \cdot 1 = \epsilon(h)$ for any $h \in H$.

Let ρ be a weak coaction of H^0 on A . Then by Lemma 2.4, there is a weak action of H on A . We call it the weak action of H on A induced by ρ . Also, by Lemma 2.1, there is the weak coaction $\underline{\rho}$ of H^0 on $M(A)$, which is an extension of ρ to $M(A)$. Hence we can obtain the action of H on $M(A)$ induced by $\underline{\rho}$. We can see that this action is an extension of the action induced by ρ to $M(A)$.

Definition 2.4. Let $\sigma : H \times H \rightarrow M(A)$ be a bilinear map. σ is a *unitary cocycle* for a weak action of H on A if σ satisfies the following conditions:

- (1) σ is a unitary element in $\text{Hom}(H \times H, M(A))$,
- (2) σ is normal, that is, for any $h \in H$, $\sigma(h, 1) = \sigma(1, h) = \epsilon(h)1$,
- (3) (cocycle condition) For any $h, l, m \in H$,
 $[h_{(1)} \cdot \sigma(l_{(1)}, m_{(1)})]\sigma(h_{(2)}, l_{(2)}m_{(2)}) = \sigma(h_{(1)}, l_{(1)})\sigma(h_{(2)}l_{(2)}, m)$,
- (4) (twisted modular condition) For any $h, l \in H$, $x \in A$,
 $[h_{(1)} \cdot [l_{(1)} \cdot x]]\sigma(h_{(2)}, l_{(2)}) = \sigma(h_{(1)}, l_{(1)})[(h_{(2)}l_{(2)}) \cdot x]$,

where if necessary, we consider the extension of the weak action to $M(A)$.

We call a pair which is consisting of a weak action of H on A and its unitary cocycle, a *twisted action* of H on A .

Let (ρ, u) be a twisted coaction of H^0 on A . Then we can consider the twisted action of H on A and its unitary cocycle \hat{u} defined by

$$h \cdot_{\rho, u} x = \rho(x)\hat{u}(h) = (\text{id} \otimes h)(\rho(x))$$

for any $x \in A$ and $h \in H$. We call it the twisted action induced by (ρ, u) . Also, we can consider the twisted coaction $(\underline{\rho}, u)$ of H^0 on $M(A)$ and the twisted action of H on $M(A)$ induced by $(\underline{\rho}, u)$. Let $M(A) \rtimes_{\underline{\rho}, u} H$ be the twisted crossed product by the twisted action of H on $M(A)$ induced by $(\underline{\rho}, u)$. Let $x \rtimes_{\underline{\rho}, u} h$ be the element in $M(A) \rtimes_{\underline{\rho}, u} H$ induced by elements $x \in M(A)$, $h \in H$. Let $A \rtimes_{\rho, u} H$ be the set of all finite sums of elements in the form $x \rtimes_{\rho, u} h$, where $x \in A$, $h \in H$. By easy computations, we can see that $A \rtimes_{\rho, u} H$ is a closed two-sided ideal of $M(A) \rtimes_{\underline{\rho}, u} H$. We call it the twisted crossed product by (ρ, u) and its element denotes by $x \rtimes_{\rho, u} h$, where $x \in A$ and $h \in H$. Let $E_1^{\rho, u}$ be the canonical conditional expectation from $M(A) \rtimes_{\underline{\rho}, u} H$ onto $M(A)$ defined by

$$E_1^{\rho, u}(x \rtimes_{\rho, u} h) = \tau(h)x$$

for any $x \in M(A)$ and $h \in H$. Let Λ be the set of all triplets (i, j, k) , where $i, j = 1, 2, \dots, d_k$ and $k = 1, 2, \dots, K$ with $\sum_{k=1}^K d_k^2 = N$. Let $W_I = \sqrt{d_k} \rtimes_{\rho, u} w_{ij}^k$ for any $I = (i, j, k) \in \Lambda$. By [15, Proposition 3.18], $\{(W_I^*, W_I)\}_{I \in \Lambda}$ is a quasi-basis for $E_1^{\rho, u}$. We suppose that A acts on a Hilbert space faithfully and nondegenerately.

Lemma 2.6. *With the above notations, $M(A) \rtimes_{\rho, u} H = M(A \rtimes_{\rho, u} H)$.*

Proof. By the definition of the multiplier algebras $M(A)$ and $M(A \rtimes_{\rho,u} H)$, it is clear that $M(A) \rtimes_{\rho,u} H \subset M(A \rtimes_{\rho,u} H)$ since $M(A) \rtimes_{\rho,u} H$ and $M(A \rtimes_{\rho,u} H)$ act on the same Hilbert space. We show that another inclusion. Let $x \in M(A \rtimes_{\rho,u} H)$. Then there is a bounded net $\{x_\alpha\}_{\alpha \in \Gamma} \subset A \rtimes_{\rho,u} H$ such that $\{x_\alpha\}_{\alpha \in \Gamma}$ converges to x strictly. Since $x_\alpha \in A \rtimes_{\rho,u} H$, $x_\alpha = \sum_I E_1^{\rho,u}(x_\alpha W_I^*) W_I$. By the definition of $E_1^{\rho,u}$, $E_1^{\rho,u}(x_\alpha W_I^*) \in A$. Also, for any $a \in A$,

$$\lim_{\alpha \rightarrow \infty} E_1^{\rho,u}(x_\alpha W_I^*) a = \lim_{\alpha \rightarrow \infty} E_1^{\rho,u}(x_\alpha W_I^* a) = E_1^{\rho,u}(x W_I^* a) = E_1^{\rho,u}(x W_I^*) a.$$

Similarly, $\lim_{\alpha \rightarrow \infty} a E_1^{\rho,u}(x_\alpha W_I^*) = a E_1^{\rho,u}(x W_I^*)$. Hence $E_1^{\rho,u}(x W_I^*) \in M(A)$. Also, by the above discussions, we can see that $E_1^{\rho,u}(\cdot W_I^*)$ is strictly continuous for any $I \in \Lambda$. For any $a \in A$ and $h \in H$,

$$\begin{aligned} (a \rtimes_{\rho,u} h) E_1^{\rho,u}(x_\alpha W_I^*) &= a [h_{(1)} \cdot_{\rho,u} E_1^{\rho,u}(x_\alpha W_I^*)] \rtimes_{\rho,u} h_{(2)} \\ &= a((\text{id} \otimes h_{(1)}) \circ (\underline{\rho} \otimes \text{id}))(E_1^{\rho,u}(x_\alpha W_I^*) \rtimes_{\rho,u} h_{(2)}). \end{aligned}$$

Since $\text{id} \otimes h_{(1)}$, $\underline{\rho} \otimes \text{id}$ and $E_1^{\rho,u}(\cdot W_I^*)$ are strictly continuous for any $I \in \Lambda$, we can see that

$$\lim_{\alpha \rightarrow \infty} (a \rtimes_{\rho,u} h) E_1^{\rho,u}(x_\alpha W_I^*) = (a \rtimes_{\rho,u} h) E_1^{\rho,u}(x W_I^*).$$

Similarly, we can see that for any $a \in A$, $h \in H$,

$$\lim_{\alpha \rightarrow \infty} E_1^{\rho,u}(x_\alpha W_I^*) (a \rtimes_{\rho,u} h) = E_1^{\rho,u}(x W_I^*) (a \rtimes_{\rho,u} h).$$

Thus $E_1^{\rho,u}(x_\alpha W_I^*)$ converges to $E_1^{\rho,u}(x W_I^*)$ in $M(A \rtimes_{\rho,u} H)$ strictly. Therefore, $x = \sum_I E_1^{\rho,u}(x W_I^*) W_I$ since $x_\alpha = \sum_I E_1^{\rho,u}(x_\alpha W_I^*) W_I$. It follows that $x \in M(A) \rtimes_{\rho,u} H$. \square

Remark 2.7. Let $(\underline{\rho})^\wedge$ be the dual coaction of $\underline{\rho}$ of H on $M(A) \rtimes_{\rho,u} H$ and $(\underline{\widehat{\rho}})$ the coaction of H on $M(A \rtimes_{\rho,u} H)$ induced by the dual coaction $\widehat{\rho}$ of H on $A \rtimes_{\rho,u} H$. By Lemma 2.6, we can see that $(\underline{\rho})^\wedge = (\underline{\widehat{\rho}})$. Indeed, by Lemma 2.6, it suffices to show that $(\underline{\widehat{\rho}})(x \rtimes_{\rho,u} h) = (\underline{\rho})^\wedge(x \rtimes_{\rho,u} h)$ for any $x \in M(A)$ and $h \in H$. Since $x \in M(A)$, there is a bounded net $\{x_\alpha\} \subset A$ such that x_α converges to x strictly in $M(A)$. Then since $x_\alpha \rtimes_{\rho,u} h$ converges to $x \rtimes_{\rho,u} h$ strictly in $M(A) \rtimes_{\rho,u} H$ and $(\underline{\widehat{\rho}})$ is strictly continuous,

$$\begin{aligned} (\underline{\widehat{\rho}})(x \rtimes_{\rho,u} h) &= \lim_{\alpha \rightarrow \infty} \widehat{\rho}(x_\alpha \rtimes_{\rho,u} h) = \lim_{\alpha \rightarrow \infty} (x_\alpha \rtimes_{\rho,u} h_{(1)}) \otimes h_{(2)} \\ &= (x \rtimes_{\rho,u} h_{(1)}) \otimes h_{(2)} = (\underline{\rho})^\wedge(x \rtimes_{\rho,u} h), \end{aligned}$$

where the limits are taken under the strict topology. We denote it by $\underline{\widehat{\rho}}$.

Next, we extend [16, Theorem 3.3] to a twisted coaction of H^0 on a (non-unital) C^* -algebra A . Before doing it, we define the exterior equivalence for twisted coactions of a finite dimensional C^* -Hopf algebra H^0 on a C^* -algebra A .

Definition 2.5. Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A . We say that (ρ, u) is exterior equivalent to (σ, v) if there is a unitary element $w \in M(A) \otimes H^0$ satisfying the following conditions:

- (1) $\sigma = \text{Ad}(w) \circ \rho$,
- (2) $v = (w \otimes 1^0)(\underline{\rho} \otimes \text{id})(w)u(\text{id} \otimes \Delta^0)(w^*)$.

The above conditions (1), (2) are equivalent to the following, respectively:

- (1)' $h \cdot_{\sigma,v} a = \widehat{w}(h_{(1)})[h_{(2)} \cdot_{\rho,u} a] \widehat{w}^*(h_{(3)})$ for any $a \in A$ and $h \in H$,
- (2)' $\widehat{v}(h, l) = \widehat{w}(h_{(1)})[h_{(2)} \cdot_{\rho,u} \widehat{w}(l_{(1)})] \widehat{u}(h_{(3)}, l_{(2)}) \widehat{w}^*(h_{(4)} l_{(3)})$ for any $h, l \in H^0$.

If ρ and σ are coactions of H^0 on A , Conditions (1), (2) and (1)', (2)' are following:

- (1) $\sigma = \text{Ad}(w) \circ \rho$,

- (2) $(w \otimes 1^0)(\underline{\rho} \otimes \text{id})(w) = (\text{id} \otimes \Delta^0)(w)$,
(1)' $h \cdot_\sigma a = \widehat{w}(h_{(1)})[h_{(2)} \cdot_\rho a] \widehat{w}^*(h_{(3)})$ for any $a \in A$, $h \in H^0$,
(2)' $\widehat{w}(h_{(1)})[h_{(2)} \cdot_\rho \widehat{w}(l)] = \widehat{w}(hl)$ for any $h, l \in H^0$.

Furthermore, let (ρ, u) be a twisted coaction of H^0 on A and let w be any unitary element in $M(A) \otimes H^0$ with $(\text{id} \otimes \epsilon^0)(w) = 1^0$. Let

$$\sigma = \text{Ad}(w) \circ \rho, \quad v = (w \otimes 1^0)(\underline{\rho} \otimes \text{id})(w)u(\text{id} \otimes \Delta^0)(w^*).$$

Then (σ, v) is a twisted coaction of H^0 on A by easy computations.

In the case of twisted coactions on von Neumann algebras, Vaes and Vanierman [26] and in the case of ordinary coactions on C^* -algebras, Baa'j and Skandalis [1] have already obtained the much more generalized results than the following. We give a proof related to inclusions of unital C^* -algebras of Watatani index-finite type.

Proposition 2.8. *Let A be a C^* -algebra and H a finite dimensional C^* -Hopf algebra with its dual C^* -Hopf algebra H^0 . Let (ρ, u) be a twisted coaction of H^0 on A . Then there are an isomorphism Ψ of $M(A) \otimes M_N(\mathbf{C})$ onto $M(A) \rtimes_{\underline{\rho}, u} H \rtimes_{\widehat{\rho}} H^0$ and a unitary element $U \in (M(A) \rtimes_{\underline{\rho}, u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$ such that*

$$\begin{aligned} \text{Ad}(U) \circ \widehat{\rho} &= (\Psi \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id}_{M_N(\mathbf{C})}) \circ \Psi^{-1}, \\ (\Psi \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(u \otimes I_N) &= (U \otimes 1^0)(\widehat{\rho} \otimes \text{id}_{H^0})(U)(\text{id} \otimes \Delta^0)(U^*), \\ \Psi(A \otimes M_N(\mathbf{C})) &= A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0. \end{aligned}$$

That is, the coaction $\widehat{\rho}$ of H^0 on $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$ is exterior equivalent to the twisted coaction

$$((\Psi \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id}_{M_N(\mathbf{C})}) \circ \Psi^{-1}, (\Psi \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(u \otimes I_N)),$$

where we identify $A \otimes H^0 \otimes H^0 \otimes M_N(\mathbf{C})$ with $A \otimes M_N(\mathbf{C}) \otimes H^0 \otimes H^0$.

Proof. By [16, Theorem 3.3], there are an isomorphism Ψ of $M(A) \otimes M_N(\mathbf{C})$ onto $M(A) \rtimes_{\underline{\rho}, u} H \rtimes_{\widehat{\rho}} H^0$ and a unitary element $U \in (M(A) \rtimes_{\underline{\rho}, u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$ satisfying the required conditions except for the equation

$$\Psi(A \otimes M_N(\mathbf{C})) = A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0.$$

So, we show the equation. By [16, Section 3],

$$\Psi([a_{IJ}]) = \sum_{I, J} V_I^*(a_{IJ} \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} 1^0) V_J$$

for any $[a_{IJ}] \in A \otimes M_N(\mathbf{C})$, where $V_I = (1 \rtimes_{\widehat{\rho}} \tau)(W_I \rtimes_{\widehat{\rho}} 1^0)$ for any $I \in \Lambda$. Since $V_I \in M(A) \rtimes_{\underline{\rho}, u} H \rtimes_{\widehat{\rho}} H^0$ for any $I \in \Lambda$, $\Psi(A \otimes M_N(\mathbf{C})) \subset A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$. For any $z \in A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$, we can write that

$$z = \sum_{i=1}^n (x_i \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau)(y_i \rtimes_{\widehat{\rho}} 1^0),$$

where $x_i, y_i \in M(A) \rtimes_{\underline{\rho}, u} H$ for any i . Let $\{u_\alpha\}$ be an approximate unit of A . Then $(u_\alpha \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} 1^0)(x_i \rtimes_{\widehat{\rho}} 1^0)$ and $(y_i \rtimes_{\widehat{\rho}} 1^0)(u_\alpha \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} 1^0)$ are in $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$ for any i and α . Hence $(A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau)(A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} 1^0)$ is dense in $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$. On the other hand, for any $x, y \in A \rtimes_{\rho, u} H$,

$$\Psi([E_1^{\rho, u}(W_I x) E_1^{\rho, u}(y W_J^*)]_{I, J}) = (x \rtimes_{\widehat{\rho}} 1^0)(1 \rtimes_{\widehat{\rho}} \tau)(y \rtimes_{\widehat{\rho}} 1^0)$$

by the proof of [16, Theorem 3.3]. Since $E_1^{\rho, u}(A \rtimes_{\rho, u} H) = A$ and $E_1^{\rho, u}$ is continuous by its definition, $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0 \subset \Psi(A \otimes M_N(\mathbf{C}))$. \square

We extend [15, Theorem 6.4] to coactions of H^0 on a (non-unital) C^* -algebra. First, we recall a saturated coaction. We say that a coaction ρ of H^0 on a unital C^* -algebra A is *saturated* if the induced action from ρ of H on A is saturated in the sense of [25, Definition 4.2].

Let B be a C^* -algebra and σ a coaction of H^0 on B . Let $B^\sigma = \{b \in B \mid \sigma(b) = b \otimes 1^0\}$, the fixed point C^* -subalgebra of B for the coaction σ . We suppose that B acts on a Hilbert space \mathcal{H} non-degenerately and faithfully. Also, we suppose that $\underline{\sigma}$ is saturated. Then the canonical conditional expectation $E^\underline{\sigma}$ from $M(B)$ onto $M(B)^\underline{\sigma}$ defined by $E^\underline{\sigma}(x) = e \cdot_\underline{\sigma} x$ for any $x \in M(B)$ is of Watatani index-finite type by [25, Theorem 4.3]. Thus there is a quasi-basis $\{(u_i, u_i^*)\}_{i=1}^n$ of $E^\underline{\sigma}$. Let $\{v_\alpha\}$ be an approximate unit of B^σ . For any $x \in B$,

$$v_\alpha x = v_\alpha \sum_{i=1}^n E^\underline{\sigma}(x u_i) u_i^* \rightarrow \sum_{i=1}^n E^\underline{\sigma}(x u_i) u_i^* = x \quad (\alpha \rightarrow \infty)$$

since $E^\sigma(x u_i) \in B^\sigma$. Similarly $x v_\alpha \rightarrow x$ ($\alpha \rightarrow \infty$) since $x = \sum_{i=1}^n u_i E^\underline{\sigma}(u_i^* x)$. Thus $\{v_\alpha\}$ is an approximate unit of B . Hence B^σ acts on \mathcal{H} non-degenerately and faithfully.

Lemma 2.9. *With the above notations, we suppose that $\underline{\sigma}$ is saturated. Then $M(B^\sigma) = M(B)^\underline{\sigma}$.*

Proof. By the above discussions, we may suppose that B and B^σ act on a Hilbert space non-degenerately and faithfully. Let $x \in M(B^\sigma)$. Then there is a bounded net $\{a_\alpha\} \subset B^\sigma$ such that $a_\alpha \rightarrow x$ ($\alpha \rightarrow \infty$) strictly in $M(B^\sigma)$. Since any approximate unit of B^σ is an approximate unit of B by the above discussion, for any $y \in B^\sigma$,

$$\underline{\sigma}(x)(y \otimes 1^0) = \sigma(xy) = \sigma(\lim_{\alpha \rightarrow \infty} a_\alpha y) = \lim_{\alpha \rightarrow \infty} \sigma(a_\alpha y) = \lim_{\alpha \rightarrow \infty} a_\alpha y \otimes 1^0 = xy \otimes 1^0.$$

Thus $x \in M(B)^\underline{\sigma}$. Next, let $x \in M(B)^\underline{\sigma}$. Then for any $b \in B^\sigma$, xb and bx are in B . Thus

$$\begin{aligned} \sigma(xb) &= \underline{\sigma}(x)\sigma(b) = (x \otimes 1^0)(b \otimes 1^0) = xb \otimes 1^0, \\ \sigma(bx) &= \sigma(b)\underline{\sigma}(x) = (b \otimes 1^0)(x \otimes 1^0) = bx \otimes 1^0. \end{aligned}$$

Hence $x \in M(B^\sigma)$. □

We suppose that $\widehat{\underline{\sigma}}(1 \rtimes_\underline{\sigma} e) \sim (1 \rtimes_\underline{\sigma} e) \otimes 1$ in $(M(B) \rtimes_\underline{\sigma} H) \otimes H$. As mentioned in [16, Section 2], without the assumption of saturation for an action, all the statements in Sections 4, 5 and 6 in [15] hold. Hence by [15, Sections 4 and 5], $\underline{\sigma}$ is saturated and there is a unitary element $w^\sigma \in M(B) \otimes H$ satisfying that

$$w^{\sigma*}((1 \rtimes_\underline{\sigma} e) \otimes 1)w^\sigma = \widehat{\underline{\sigma}}(1 \rtimes_\underline{\sigma} e),$$

Let $U^\sigma = w^\sigma(z^{\sigma*} \otimes 1)$, where $z^\sigma = (\text{id}_{M(B)} \otimes \epsilon)(w^\sigma) \in M(B)^\underline{\sigma}$. Then $U^\sigma \in M(B) \otimes H$ and satisfies that

$$\widehat{U}^\sigma(1^0) = 1, \quad \widehat{U}^\sigma(\phi_{(1)})a\widehat{U}^{\sigma*}(\phi_{(2)}) \in M(B)^\underline{\sigma}$$

for any $a \in M(B)^\underline{\sigma}$, $\phi \in H^0$. Let \widehat{u}^σ be a bilinear map from $H^0 \times H^0$ to $M(B)$ defined by

$$\widehat{u}^\sigma(\phi, \psi) = \widehat{U}^\sigma(\phi_{(1)})\widehat{U}^\sigma(\psi_{(1)})\widehat{U}^{\sigma*}(\phi_{(2)}\psi_{(2)})$$

for any $\phi, \psi \in H^0$. Then by [15, Lemma 5.4], $\widehat{u}^\sigma(\phi, \psi) \in M(B)^\underline{\sigma}$ for any $\phi, \psi \in H^0$ and by [15, Corollary 5.3], the map

$$H^0 \times M(B)^\underline{\sigma} \longrightarrow M(B)^\underline{\sigma} : (\phi, a) \mapsto \widehat{U}^\sigma(\phi_{(1)})a\widehat{U}^{\sigma*}(\phi_{(2)})$$

is a weak action of H^0 on $M(B)^\underline{\sigma}$. Furthermore, by [15, Proposition 5.6] \widehat{u}^σ is a unitary cocycle for the above weak action. Let u^σ be the unitary element in $M(B)^\underline{\sigma} \otimes H \otimes H$ induced by \widehat{u}^σ and ρ' the weak coaction of H on $M(B)^\underline{\sigma}$ induced

by the above weak action. Thus we obtain a twisted coaction (ρ', u^σ) of H on $M(B)^\underline{\sigma}$. Let π' be the map from $M(B)^\underline{\sigma} \rtimes_{\rho', u^\sigma} H^0$ to $M(B)$ defined by

$$\pi'(a \rtimes_{\rho', u^\sigma} \phi) = a\widehat{U}^\sigma(\phi)$$

for any $a \in M(B)^\underline{\sigma}$, $\phi \in H^0$. Then by [15, Proposition 6.1 and Theorem 6.4], π' is an isomorphism of $M(B)^\underline{\sigma} \rtimes_{\rho', u^\sigma} H^0$ onto $M(B)$ satisfying that

$$\underline{\sigma} \circ \pi' = (\pi' \otimes \text{id}_{H^0}) \circ \widehat{\rho}', \quad E_1^{\rho', u^\sigma} = E^\underline{\sigma} \circ \pi',$$

where E_1^{ρ', u^σ} is the canonical conditional expectation from $M(B)^\widehat{\underline{\sigma}} \rtimes_{\rho', u^\sigma} H^0$ onto $M(B)^\underline{\sigma}$ and $E^\underline{\sigma}$ is the canonical conditional expectation from $M(B)$ onto $M(B)^\underline{\sigma}$. Let $\rho = \rho'|_{B^\sigma}$.

Lemma 2.10. *With the above notations, (ρ, u^σ) is a twisted coaction of H on B^σ and $\underline{\rho} = \rho'$.*

Proof. By the definition of ρ , for any $a \in B^\sigma$,

$$\rho(a) = U^\sigma(a \otimes 1)U^{\sigma*}.$$

Since $a \in B^\sigma \subset M(B^\sigma) = M(B)^\underline{\sigma}$ by Lemma 2.9, $\rho(a) \in M(B)^\underline{\sigma} \otimes H$. On the other hand, since $U^\sigma \in M(B) \otimes H$, $\rho(a) \in B \otimes H$. Thus $\rho(a) \in (M(B)^\underline{\sigma} \otimes H) \cap (B \otimes H) = B^\sigma \otimes H$. Hence ρ is a homomorphism of B^σ to $B^\sigma \otimes H$. Since $(\rho' \otimes \text{id}) \circ \rho' = \text{Ad}(u^\sigma) \circ (\text{id} \otimes \Delta) \circ \rho'$ and $\rho(a) \in B^\sigma \otimes H$ for any $a \in B^\sigma$, we can see that $(\rho \otimes \text{id}) \circ \rho = \text{Ad}(u^\sigma) \circ (\text{id} \otimes \Delta) \circ \rho$. By the definition of ρ' , ρ' is strictly continuous on $M(B)^\underline{\sigma}$. Hence for any approximate unit $\{u_\alpha\}$ of B^σ ,

$$1 \otimes 1 = \rho'(1) = \rho'(\lim_{\alpha \rightarrow \infty} u_\alpha) = \lim_{\alpha \rightarrow \infty} \rho'(u_\alpha) = \lim_{\alpha \rightarrow \infty} \rho(u_\alpha),$$

where the limits are taken under the strict topologies in $M(B^\sigma)$ and $M(B^\sigma) \otimes H$, respectively. This means that

$$\overline{\rho(B^\sigma)(B^\sigma \otimes H)} = B^\sigma \otimes H.$$

It follows that (ρ, u^σ) is a twisted coaction of H on B^σ . Furthermore, since ρ' is strictly continuous, $\rho' = \underline{\rho}$ on $M(B^\sigma)$. \square

Let $\pi = \pi'|_{B^\sigma \rtimes_{\rho, u^\sigma} H^0}$.

Lemma 2.11. *With the above notations, π is an isomorphism of $B^\sigma \rtimes_{\rho, u^\sigma} H^0$ onto B satisfying that*

$$\sigma \circ \pi = (\pi \otimes \text{id}_{H^0}) \circ \widehat{\rho}, \quad E_1^{\rho, u^\sigma} = E^\sigma \circ \pi,$$

where E_1^{ρ, u^σ} is the canonical conditional expectation from $B^\sigma \rtimes_{\rho, u^\sigma} H^0$ onto B^σ and E^σ is the canonical conditional expectation from B onto B^σ . Furthermore, $\pi' = \underline{\pi}$.

Proof. Let $E^\underline{\sigma}$ be the canonical conditional expectation from $M(B)$ onto $M(B)^\underline{\sigma}$. By [15, Proposition 4.3 and Remark 4.9], $\{(\sqrt{f_k}\widehat{U}^\sigma(\omega_{ij}^k)^*, \sqrt{f_k}\widehat{U}^\sigma(\omega_{ij}^k))\}_{i,j,k}$ is a quasi-basis for $E^\underline{\sigma}$. Hence for any $b \in B$,

$$b = \sum_{i,j,k} f_k E^\underline{\sigma}(b\widehat{U}^\sigma(\omega_{ij}^k)^*)\widehat{U}^\sigma(\omega_{ij}^k).$$

Since $\widehat{U}^\sigma(\omega_{ij}^k) \in M(B)$ for any i, j, k ,

$$E^\underline{\sigma}(b\widehat{U}^\sigma(\omega_{ij}^k)^*) \in B^\sigma$$

for any i, j, k and $b \in B$. Let $a = \sum_{i,j,k} f_k E^\underline{\sigma}(b\widehat{U}^\sigma(\omega_{ij}^k)^*) \rtimes_{\rho, u^\sigma} \omega_{ij}^k$. Then $a \in B^\sigma \rtimes_{\rho, u^\sigma} H^0$ and $\pi(a) = b$. Thus π is surjective. Since π' is an isomorphism of

$M(B)^{\underline{\sigma}} \rtimes_{\underline{\rho}, u^\sigma} H^0$ onto $M(B)$, we can see that π is an isomorphism of $B^\sigma \rtimes_{\rho, u^\sigma} H^0$ onto B . Also, since $\underline{\sigma} \circ \pi' = (\pi' \otimes \text{id}) \circ \widehat{\rho}$ and $E_1^{\underline{\rho}, u^\sigma} = E^\sigma \circ \pi'$, we can see that

$$\sigma \circ \pi = (\pi \otimes \text{id}) \circ \widehat{\rho}, \quad E_1^{\rho, u^\sigma} \circ \pi.$$

Furthermore, by the definition of π' , π' is strictly continuous. Thus $\pi' = \underline{\pi}$. \square

Combining Lemmas 2.9, 2.10 and 2.11, we obtain the following proposition:

Proposition 2.12. *Let B be a C^* -algebra and σ a coaction of H^0 on B . We suppose that $\widehat{\underline{\sigma}}(1 \rtimes_{\underline{\sigma}} e) \sim (1 \rtimes_{\underline{\sigma}} e) \otimes 1$ in $(M(B) \rtimes_{\underline{\sigma}} H) \otimes H$. Then there are a twisted coaction (ρ, u^σ) of H on B^σ and an isomorphism π of $B^\sigma \rtimes_{\rho, u^\sigma} H^0$ onto B satisfying that*

$$\sigma \circ \pi = (\pi \otimes \text{id}_H) \circ \widehat{\rho}, \quad E_1^{\rho, u^\sigma} = E^\sigma \circ \pi,$$

where B^σ is the fixed point C^* -subalgebra of B for σ and E_1^{ρ, u^σ} and E^σ are the canonical conditional expectations from B and $B^\sigma \rtimes_{\rho, u^\sigma} H^0$ onto B^σ , respectively.

3. TWISTED COACTIONS ON A HILBERT C^* -BIMODULE AND STRONG MORITA EQUIVALENCE FOR TWISTED COACTIONS

First, we shall define crossed products of Hilbert C^* -bimodules in the sense of Brown, Mingo and Shen [6] and show their duality theorem, which is similar to [17, Theorem 5.7]. We give the definition of a Hilbert C^* -bimodule in the sense of [6].

Let A and B be C^* -algebras. Let X be a left pre-Hilbert A -bimodule and a right pre-Hilbert B -module. Its left A -valued inner product and right B -valued inner product denote by ${}_A\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_B$, respectively.

Definition 3.1. We call X a *pre-Hilbert A – B -bimodule* if X satisfies the condition

$${}_A\langle x, y \rangle z = x \langle y, z \rangle_B$$

for any $x, y, z \in X$. We call X a *Hilbert A – B -bimodule* if X is complete with the norms.

Remark 3.1. We suppose that X is a pre-Hilbert A – B -bimodule. Then by [6, Remark 1.9], we can see the following:

- (1) For any $x \in X$, $\|{}_A\langle x, x \rangle\| = \|\langle x, x \rangle_B\|$,
- (2) For any $a \in A$, $b \in B$ and $x, y \in X$,

$${}_A\langle x, yb \rangle = {}_A\langle xb^*, y \rangle, \quad \langle ax, y \rangle_B = \langle x, a^*y \rangle_B.$$

- (3) If X is complete with the norm and full with the both-sided inner products, then X is an A – B -equivalence bimodule.

In this paper, by the words “pre-Hilbert C^* -bimodules” and “Hilbert C^* -bimodules”, we mean pre-Hilbert C^* -bimodule and Hilbert C^* -bimodules in the sense of [6], respectively.

Let A and B be C^* -algebras. Let X be a Hilbert A – B -bimodule and let $\mathbf{B}_B(X)$ be the C^* -algebra of all right B -linear operators on X for which there is a right adjoint B -linear operator on X . We note that a right B -linear operator on X is bounded. For each $x, y \in X$, let $\theta_{x,y}$ be a rank-one operator on X defined by $\theta_{x,y}(z) = x \langle y, z \rangle_B$ for any $z \in X$. Then $\theta_{x,y}$ is a right B -linear operator on X . Let $\mathbf{K}_B(X)$ be the closure of all linear spans of such $\theta_{x,y}$. Then $\mathbf{K}_B(X)$ is a closed two-sided ideal of $\mathbf{B}_B(X)$. Similarly, we define ${}_A\mathbf{B}(X)$ and ${}_A\mathbf{K}(X)$. If X is an A – B -equivalence bimodule, we identify A and $M(A)$ with $\mathbf{K}_B(X)$ and $\mathbf{B}_B(X)$, respectively and identify B and $M(B)$ with ${}_A\mathbf{B}(X)$ and ${}_A\mathbf{K}(X)$, respectively. For any $a \in M(A)$, we regard $a \in M(A)$ as an element in $\mathbf{B}_B(X)$ as follows: For any $b \in A$, $x \in X$,

$$a(bx) = (ab)x.$$

Since $X = \overline{AX}$ by [6, Proposition 1.7], we can obtain an element in $\mathbf{B}_B(X)$ induced by $a \in M(A)$. Similarly, we can obtain an element in ${}_A\mathbf{B}(X)$ induced by any $b \in M(B)$.

Lemma 3.2. *With the above notations, we suppose that X is a Hilbert $A - B$ -bimodule. For any $a \in M(A)$, there is a bounded net $\{a_\alpha\}_{\alpha \in \Gamma} \subset A$ such that $ax = \lim_{\alpha \rightarrow \infty} a_\alpha x$ for any $x \in X$.*

Proof. Since $a \in M(A)$, there is a bounded net $\{a_\alpha\}_{\alpha \in \Gamma} \subset A$ such that $\{a_\alpha\}_{\alpha \in \Gamma}$ converges to a strictly. We can prove that $ax = \lim_{\alpha \rightarrow \infty} a_\alpha x$ for any $x \in X$ in a routine way since $X = \overline{AX}$ by [6, Proposition 1.7]. \square

Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A and B , respectively.

Definition 3.2. Let λ be a linear map from a Hilbert $A - B$ -bimodule X to $X \otimes H^0$. Then we say that λ is a *twisted coaction* of H^0 on X with respect to $(A, B, \rho, u, \sigma, v)$ if the following conditions hold:

- (1) $\lambda(ax) = \rho(a)\lambda(x)$ for any $a \in A, x \in X$,
- (2) $\lambda(xb) = \lambda(x)\sigma(b)$ for any $b \in B, x \in X$,
- (3) $\rho(A\langle x, y \rangle) = {}_{A \otimes H^0}\langle \lambda(x), \lambda(y) \rangle$ for any $x, y \in X$,
- (4) $\sigma(\langle x, y \rangle_B) = \langle \lambda(x), \lambda(y) \rangle_{B \otimes H^0}$ for any $x, y \in X$,
- (5) $(\text{id}_X \otimes \epsilon^0) \circ \lambda = \text{id}_X$,
- (6) $(\lambda \otimes \text{id})(\lambda(x)) = u(\text{id} \otimes \Delta^0)(\lambda(x))v^*$ for any $x \in X$,

where u and v are regarded as elements in $\mathbf{B}_B(X)$ and ${}_A\mathbf{B}(X)$, respectively.

We note that the twisted coaction λ of H^0 on the Hilbert $A - B$ -bimodule X with respect to $(A, B, \rho, u, \sigma, v)$ is isometric. Indeed, for any $x \in X$

$$\|\lambda(x)\|^2 = \|{}_{A \otimes H^0}\langle \lambda(x), \lambda(x) \rangle\| = \|\rho(A\langle x, y \rangle)\| = \|{}_A\langle x, y \rangle\| = \|x\|^2.$$

Let λ be a twisted coaction of H^0 on a Hilbert $A - B$ -bimodule X with respect to $(A, B, \rho, u, \sigma, v)$. We define the *twisted action* of H on X induced by λ as follows: For any $x \in X, h \in H$,

$$h \cdot_\lambda x = (\text{id} \otimes h)(\lambda(x)) = \lambda(x)\widehat{h},$$

where $\lambda(x)\widehat{h}$ is the element in $\text{Hom}(H, X)$ induced by $\lambda(x)$ in $X \otimes H^0$. Then we obtain the following conditions which are equivalent to Conditions (1)-(6) in Definition 3.2, respectively:

- (1)' $h \cdot_\lambda ax = [h_{(1)} \cdot_{\rho, u} a][h_{(2)} \cdot_\lambda x]$ for any $a \in A, x \in X$,
 - (2)' $h \cdot_\lambda xb = [h_{(1)} \cdot_\lambda x][h_{(2)} \cdot_{\sigma, v} b]$ for any $b \in B, x \in X$,
 - (3)' $h \cdot_\rho A\langle x, y \rangle = {}_A\langle [h_{(1)} \cdot_\lambda x], [S(h_{(2)}^*) \cdot_\lambda y] \rangle$ for any $x, y \in X$,
 - (4)' $h \cdot_\sigma \langle x, y \rangle_B = \langle [S(h_{(1)}^*) \cdot_\lambda x], [h_{(2)} \cdot_\lambda y] \rangle_B$ for any $x, y \in X$,
 - (5)' $1_H \cdot_\lambda x = x$ for any $x \in X$,
 - (6)' $h \cdot_\lambda [l \cdot_\lambda x] = \widehat{u}(h_{(1)}, l_{(1)})[h_{(2)} l_{(2)} \cdot_\lambda x]\widehat{v}^*(h_{(3)}, l_{(3)})$ for any $x \in X, h, l \in H$,
- where \widehat{u} and \widehat{v} are elements in $\text{Hom}(H \times H, M(A))$ and $\text{Hom}(H \times H, M(B))$ induced by $u \in M(A) \otimes H^0 \otimes H^0$ and $v \in M(B) \otimes H^0 \otimes H^0$, respectively.

Remark 3.3. In Definition 3.2, if ρ and σ are coactions of H^0 on A and B , respectively, then Condition (6) in Definition 3.2 and its equivalent Condition (6)' are following, respectively:

- (6) $(\lambda \otimes \text{id}) \circ \lambda = (\text{id} \otimes \Delta^0) \circ \lambda$,
- (6)' $h \cdot_\lambda [l \cdot_\lambda x] = hl \cdot_\lambda x$ for any $x \in X$.

In this case, we call λ a *coaction* of H^0 on X with respect to (A, B, ρ, σ) .

Next, we shall define crossed products of Hilbert C^* -bimodules by twisted coactions in the same way as in [17, Section 4] and give a duality theorem for them.

Let (ρ, u) and (σ, v) be twisted coactions of H^0 on C^* -algebras A and B , respectively. Let λ be a twisted coaction of H^0 on a Hilbert $A - B$ -bimodule X with

respect to $(A, B, \rho, u, \sigma, v)$. We define $X \rtimes_{\lambda} H$, a Hilbert $A \rtimes_{\rho, u} H - B \rtimes_{\sigma, v} H$ -bimodule as follows: Let $(X \rtimes_{\lambda} H)_0$ be just $X \otimes H$ (the algebraic tensor product) as vector spaces. Its left and right actions are given by

$$\begin{aligned}(a \rtimes_{\rho, u} h)(x \rtimes_{\lambda} l) &= a[h_{(1)} \cdot_{\lambda} x] \widehat{v}(h_{(2)}, l_{(1)}) \rtimes_{\lambda} h_{(3)} l_{(2)}, \\ (x \rtimes_{\lambda} l)(b \rtimes_{\sigma, v} m) &= x[l_{(1)} \cdot_{\sigma, v} b] \widehat{v}(l_{(2)}, m_{(1)}) \rtimes_{\lambda} l_{(3)} m_{(2)}\end{aligned}$$

for any $a \in A, b \in B, x \in X$ and $h, l, m \in H$. Also, its left $A \rtimes_{\rho, u} H$ -valued and right $B \rtimes_{\sigma, v} H$ -valued inner products are given by

$$\begin{aligned}{}_{A \rtimes_{\rho, u} H} \langle x \rtimes_{\lambda} h, y \rtimes_{\lambda} l \rangle &= {}_A \langle x, [S(h_{(2)} l_{(3)}^*)^* \cdot_{\lambda} y] \widehat{v}(S(h_{(1)} l_{(2)}^*)^*, l_{(1)}) \rangle \rtimes_{\rho, u} h_{(3)} l_{(4)}^*, \\ \langle x \rtimes_{\lambda} h, y \rtimes_{\lambda} l \rangle_{B \rtimes_{\sigma, v} H} &= \widehat{v}^*(h_{(2)}^*, S(h_{(1)}^*)) [h_{(3)}^* \cdot_{\sigma, v} \langle x, y \rangle_B] \widehat{v}(h_{(4)}^*, l_{(1)}) \rtimes_{\sigma, v} h_{(5)}^* l_{(2)}\end{aligned}$$

for any $x, y \in X$ and $h, l \in H$. In the same way as in [17, Section 4], we can see that $(X \rtimes_{\lambda} H)_0$ is a pre-Hilbert $A \rtimes_{\rho, u} H - B \rtimes_{\sigma, v} H$ -bimodule. Let $X \rtimes_{\lambda} H$ be the completion of $(X \rtimes_{\lambda} H)_0$. It is a Hilbert $A \rtimes_{\rho, u} H - B \rtimes_{\sigma, v} H$ -bimodule. Let $\widehat{\lambda}$ be a linear map from $(X \rtimes_{\lambda} H)_0$ to $(X \rtimes_{\lambda} H)_0 \otimes H$ defined by

$$\widehat{\lambda}(x \rtimes_{\lambda} h) = (x \rtimes_{\lambda} h_{(1)}) \otimes h_{(2)}$$

for any $x \in X, h \in H$. By easy computations, we can see that $\widehat{\lambda}$ is a linear map from H to $(X \rtimes_{\lambda} H)_0 \otimes H$ satisfying Conditions (1)-(6) in Definition 3.2. Thus for any $x \in (X \rtimes_{\lambda} H)_0$,

$$\|\widehat{\lambda}(x)\|^2 = \|({}_{A \rtimes_{\rho, u} H} \widehat{\lambda}(x), \widehat{\lambda}(x))\| = \|\widehat{\rho}({}_A \langle x, x \rangle)\| = \|{}_A \langle x, x \rangle\| = \|x\|^2.$$

Hence $\widehat{\lambda}$ is an isometry. We extend $\widehat{\lambda}$ to $X \rtimes_{\lambda} H$. We can see that the extension of $\widehat{\lambda}$ is a coaction of H on $X \rtimes_{\lambda} H$ with respect to $(A \rtimes_{\rho, u} H, B \rtimes_{\sigma, v} H, \widehat{\rho}, \widehat{\sigma})$. We also denote it by the same symbol $\widehat{\lambda}$ and call it the dual coaction of λ . Similarly we define the second dual coaction of λ , which is a coaction of H^0 on $X \rtimes_{\lambda} H \rtimes_{\widehat{\lambda}} H^0$. Let Λ be as in Section 2. For any $I = (i, j, k) \in \Lambda$, let W_I^{ρ}, V_I^{ρ} be elements in $M(A) \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$ defined by

$$W_I^{\rho} = \sqrt{d_k} \rtimes_{\rho, u} w_{ij}^k, \quad V_I^{\rho} = (1 \rtimes_{\rho, u} 1 \rtimes_{\widehat{\rho}} \tau)(W_I^{\rho} \rtimes_{\widehat{\rho}} 1^0).$$

Similarly for any $I = (i, j, k) \in \Lambda$, we define elements

$$W_I^{\sigma} = \sqrt{d_k} \rtimes_{\sigma, v} w_{ij}^k, \quad V_I^{\sigma} = (1 \rtimes_{\sigma, v} 1 \rtimes_{\widehat{\sigma}} \tau)(W_I^{\sigma} \rtimes_{\widehat{\sigma}} 1^0)$$

in $M(B) \rtimes_{\sigma, v} H \rtimes_{\widehat{\sigma}} H^0$. We regard $M_N(\mathbf{C})$ as an equivalence $M_N(\mathbf{C}) - M_N(\mathbf{C})$ -bimodule in the usual way. Let $X \otimes M_N(\mathbf{C})$ be the exterior tensor product of X and $M_N(\mathbf{C})$, which is a Hilbert $A \otimes M_N(\mathbf{C}) - B \otimes M_N(\mathbf{C})$ -bimodule. Let $\{f_{IJ}\}_{I, J \in \Lambda}$ be a system of matrix units of $M_N(\mathbf{C})$. Let Ψ_X be a linear map from $X \otimes M_N(\mathbf{C})$ to $X \rtimes_{\lambda} H \rtimes_{\widehat{\lambda}} H^0$ defined by

$$\Psi_X \left(\sum_{I, J} x_{IJ} \otimes f_{IJ} \right) = \sum_{I, J} V_I^{\rho*} (x_{IJ} \rtimes_{\lambda} 1 \rtimes_{\widehat{\lambda}} 1^0) V_J^{\sigma}.$$

Let Ψ_A and Ψ_B be the isomorphisms of $A \otimes M_N(\mathbf{C})$ and $B \otimes M_N(\mathbf{C})$ onto $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0$ and $B \rtimes_{\sigma, v} H \rtimes_{\widehat{\sigma}} H^0$ defined in Proposition 2.8, respectively. Then we have the same lemmas as [17, Lemmas 5.1 and 5.5]. Hence Ψ_X is an isometry from $X \otimes M_N(\mathbf{C})$ to $X \rtimes_{\lambda} H \rtimes_{\widehat{\lambda}} H^0$ whose image is $(X \rtimes_{\lambda} H)_0 \rtimes_{\widehat{\lambda}} H^0$, the linear span of the set

$$\{x \rtimes_{\lambda} h \rtimes_{\widehat{\lambda}} \phi \mid x \in X, h \in H, \phi \in H^0\}.$$

Since $X \otimes M_N(\mathbf{C})$ is complete, so is $(X \rtimes_{\lambda} H)_0 \rtimes_{\widehat{\lambda}} H^0$. Furthermore, we claim that $(X \rtimes_{\lambda} H)_0$ is also complete. In order to show it, we need the following lemma: Let E_1^{λ} be a linear map from $(X \rtimes_{\lambda} H)_0$ onto X defined by

$$E_1^{\lambda}(x \rtimes_{\lambda} h) = \tau(h)x$$

for any $x \in X, h \in H$.

Lemma 3.4. *With the above notations, E_1^λ is continuous.*

Proof. In the same way as in the proof of [17, Lemma 5.6], we can see that

$$E_1^\lambda(x \rtimes_\lambda h) = \tau \cdot_{\widehat{\gamma}}(x \rtimes_\lambda h) = \widehat{V}^{\widehat{\rho}}(\tau_{(1)})(x \rtimes_\lambda h \rtimes_{\widehat{\gamma}} 1^0) \widehat{V}^{\widehat{\sigma}^*}(\tau_{(2)}),$$

where we identify $X \rtimes_\lambda H \rtimes_{\widehat{\gamma}} 1^0$ with $X \rtimes_\lambda H$ and

$$\widehat{V}^{\widehat{\rho}}(\phi) = 1 \rtimes_{\underline{\rho}, u} 1 \rtimes_{\underline{\rho}} \phi, \quad \widehat{V}^{\widehat{\sigma}}(\phi) = 1 \rtimes_{\underline{\sigma}, v} 1 \rtimes_{\underline{\sigma}} \phi$$

for any $\phi \in H^0$. Hence E_1^λ is continuous. \square

Let E_2^λ be a linear map from $(X \rtimes_\lambda H \rtimes_{\widehat{\gamma}} H^0)_0$ to $X \rtimes_\lambda H$ defined by

$$E_2^\lambda(x \rtimes_{\widehat{\gamma}} \phi) = \phi(e)x$$

for any $x \in X \rtimes_\lambda H, \phi \in H^0$.

Lemma 3.5. *With the above notations, $(X \rtimes_\lambda H)_0$ is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $(X \rtimes_\lambda H)_0$. Using Lemma 3.4 and the linear map E_2^λ , we can see that $\{x_n\}$ is convergent in $(X \rtimes_\lambda H)_0$. \square

By Lemma 3.5, $X \rtimes_\lambda H = (X \rtimes_\lambda H)_0$. In the same way as in the proof of [17, Theorem 5.7], we obtain the following proposition using Lemma 3.5:

Proposition 3.6. *Let A, B be C^* -algebras and H a finite dimensional C^* -Hopf algebra with its dual C^* -Hopf algebra H^0 . Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A and B , respectively. Let λ be a twisted coaction of H^0 on a Hilbert $A - B$ -bimodule X with respect to $(A, B, \rho, u, \sigma, v)$. Then there is an isomorphism Ψ_X from $X \otimes M_N(\mathbf{C})$ onto $X \rtimes_\lambda H \rtimes_{\widehat{\gamma}} H^0$ satisfying that*

$$(1) \Psi_X\left(\left(\sum_{I,J} a_{IJ} \otimes f_{IJ}\right)\left(\sum_{I,J} x_{IJ} \otimes f_{IJ}\right)\right) = \Psi_A\left(\sum_{I,J} a_{IJ} \otimes f_{IJ}\right) \Psi_X\left(\sum_{I,J} x_{IJ} \otimes f_{IJ}\right),$$

$$(2) \Psi_X\left(\left(\sum_{I,J} x_{IJ} \otimes f_{IJ}\right)\left(\sum_{I,J} b_{IJ} \otimes f_{IJ}\right)\right) = \Psi_X\left(\sum_{I,J} x_{IJ} \otimes f_{IJ}\right) \Psi_B\left(\sum_{I,J} b_{IJ} \otimes f_{IJ}\right),$$

$$(3) {}_{A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0} \langle \Psi_X\left(\sum_{I,J} x_{IJ} \otimes f_{IJ}\right), \Psi_X\left(\sum_{I,J} y_{IJ} \otimes f_{IJ}\right) \rangle \\ = \Psi_A({}_{A \otimes M_N(\mathbf{C})} \langle \sum_{I,J} x_{IJ} \otimes f_{IJ}, \sum_{I,J} y_{IJ} \otimes f_{IJ} \rangle),$$

$$(4) \langle \Psi_X\left(\sum_{I,J} x_{IJ} \otimes f_{IJ}\right), \Psi_X\left(\sum_{I,J} y_{IJ} \otimes f_{IJ}\right) \rangle_{B \rtimes_{\sigma, v} H \rtimes_{\widehat{\sigma}} H^0} \\ = \Psi_B(\langle \sum_{I,J} x_{IJ} \otimes f_{IJ}, \sum_{I,J} y_{IJ} \otimes f_{IJ} \rangle_{B \otimes M_N(\mathbf{C})})$$

for any $a_{IJ} \in A, b_{IJ} \in B, x_{IJ}, y_{IJ} \in X, I, J \in \Lambda$, where $X \rtimes_\lambda H \rtimes_{\widehat{\gamma}} H^0$ is a Hilbert $A \rtimes_{\rho, u} H \rtimes_{\widehat{\rho}} H^0 - B \rtimes_{\sigma, v} H \rtimes_{\widehat{\sigma}} H^0$ -bimodule and $X \otimes M_N(\mathbf{C})$ is an exterior tensor product of X and the Hilbert $M_N(\mathbf{C}) - M_N(\mathbf{C})$ -bimodule $M_N(\mathbf{C})$. Furthermore, there are unitary elements $U \in (M(A) \rtimes_{\underline{\rho}, u} H \rtimes_{\underline{\rho}} H^0) \otimes H^0$ and $V \in (M(B) \rtimes_{\underline{\sigma}, v} H \rtimes_{\underline{\sigma}} H^0) \otimes H^0$ such that

$$U \widehat{\lambda}(x) V = ((\Psi_X \otimes \text{id}) \circ (\lambda \otimes \text{id}_{M_N(\mathbf{C})}) \circ \Psi_X^{-1})(x)$$

for any $x \in X \otimes M_N(\mathbf{C})$.

The above proposition has already obtained in the case of Kac systems by Guo and Zhang [10], which is a generalization of the above result. Also, we have the following lemmas:

Lemma 3.7. *With the above notations, if X is full with the both-sided inner products, then so is $X \rtimes_{\lambda} H$.*

Proof. Modifying the proof of [17, Lemma 4.5], we can prove the lemma. \square

Lemma 3.8. *With the above notations, if $X \rtimes_{\lambda} H$ is full with the both-sided inner products, then so is X .*

Proof. Since $X \rtimes_{\lambda} H$ is full with the both-sided inner products, so is $X \rtimes_{\lambda} H \rtimes_{\widehat{\lambda}} H^0$ by Lemma 3.7. Thus $X \otimes M_N(\mathbf{C})$ is full with the both-sided inner products by Proposition 3.6. Let f be a minimal projection in $M_N(\mathbf{C})$. Then

$$\begin{aligned} A \otimes f &= (1_{M(A)} \otimes f)(A \otimes M_N(\mathbf{C}))(1_{M(A)} \otimes f) \\ &= (1 \otimes f) \overline{_{A \otimes M_N(\mathbf{C})} \langle X \otimes M_N(\mathbf{C}), X \otimes M_N(\mathbf{C}) \rangle} (1 \otimes f) \\ &= \overline{{}_A \langle X, X \rangle \otimes f M_N(\mathbf{C})} f = \overline{{}_A \langle X, X \rangle} \otimes f. \end{aligned}$$

Hence X is full with the left-sided inner product. Similarly, we can see that X is full with the right-sided inner product. Therefore, we obtain the conclusion. \square

Definition 3.3. Let (ρ, u) and (σ, v) be twisted coactions of H^0 on C^* -algebras A and B , respectively. Then (ρ, u) is *strongly Morita equivalent* to (σ, v) if there are an A – B -equivalence bimodule X and a twisted coaction λ of H^0 on X with respect to $(A, B, \rho, u, \sigma, v)$.

In the same way as in [17, Section 3], we can see that the strong Morita equivalence for twisted coactions of H^0 on C^* -algebras is an equivalence relation. Also, we can obtain the following lemma in the similar way to [17, Lemma 3.12] using approximate units in a C^* -algebra. We give it without its proof.

Lemma 3.9. *Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A . Then the following conditions are equivalent:*

- (1) *The twisted coactions (ρ, u) and (σ, v) are exterior equivalent,*
- (2) *The twisted coactions (ρ, u) and (σ, v) are strongly Morita equivalent by a twisted coaction λ of H^0 on ${}_A A_A$, which is a linear map from ${}_A A_A$ to ${}_{A \otimes H^0} A \otimes H^0_{A \otimes H^0}$, where ${}_A A_A$ and ${}_{A \otimes H^0} A \otimes H^0_{A \otimes H^0}$ are regarded as an A – A -equivalence bimodule and an $A \otimes H^0$ – $A \otimes H^0$ -equivalence bimodule in the usual way.*

Remark 3.10. Let A and B be C^* -algebras and σ a coaction of H^0 on B . Let X be an A – B -equivalence bimodule and λ a linear map from X to $X \otimes H^0$ satisfying that

- (1) $\lambda(xb) = \lambda(x)\sigma(b)$ for any $b \in B, x \in X$,
- (2) $\sigma(\langle x, y \rangle_B) = \langle \lambda(x), \lambda(y) \rangle_{B \otimes H^0}$ for any $x, y \in X$,
- (3) $(\text{id}_X \otimes \epsilon^0) \circ \lambda = \text{id}_X$,
- (4) $(\lambda \otimes \text{id}) \circ \lambda = (\text{id} \otimes \Delta^0) \circ \lambda$.

We call $(B, X, \sigma, \lambda, H^0)$ a *right covariant system* (See [17, Definition 3.4]). Then we can construct an action “ \cdot ” of H on $\mathbf{K}_B(X)$ as follows: For any $a \in \mathbf{K}_B(X)$, $h \in H$ and $x \in X$,

$$[h \cdot a]x = h_{(1)} \cdot_{\lambda} a[S(h_{(2)}) \cdot_{\lambda} x].$$

If $a \in \mathbf{K}_B(X)$, we can see that $h \cdot a \in \mathbf{K}_B(X)$. Thus identifying A with $\mathbf{K}_B(X)$, we can obtain an action of H on A .

4. LINKING C^* -ALGEBRAS AND COACTIONS ON C^* -ALGEBRAS

Let (ρ, u) and (σ, v) be twisted coactions of H^0 on C^* -algebras A and B , respectively. We suppose that there are a Hilbert A – B -bimodule X and a twisted

coaction λ of H^0 on X with respect to $(A, B, \rho, u, \sigma, v)$. Let C be the linking C^* -algebra for X defined in Brown, Mingo and Shen [6]. By [6, Proposition 2.3], C is the C^* -algebra which is consisting of all 2×2 -matrices

$$\begin{bmatrix} a & x \\ \widetilde{y} & b \end{bmatrix}, \quad a \in A, \quad b \in B, \quad x, y \in X,$$

where \widetilde{y} denotes y viewed as an element in \widetilde{X} , the dual Hilbert C^* -bimodule of X . Before we define the coaction of H^0 on C induced by the twisted coaction λ of H^0 on X with respect to $(A, B, \rho, u, \sigma, v)$, we give a remark.

Remark 4.1. We identify the H^0 – H^0 -equivalence bimodule $\widetilde{H^0}$ with H^0 as H^0 – H^0 -equivalence bimodule by the map

$$\widetilde{H^0} \longrightarrow H^0 : \widetilde{\phi} \mapsto \phi^*.$$

Also, we identify the Hilbert $B \otimes H^0 - A \otimes H^0$ -bimodule $\widetilde{X \otimes H^0}$ with $\widetilde{X} \otimes H^0$ by the map

$$\widetilde{X \otimes H^0} \longrightarrow \widetilde{X} \otimes H^0 : \widetilde{x \otimes \phi} \mapsto \widetilde{x} \otimes \phi^*.$$

Furthermore, we identify the linking C^* -algebra for $X \otimes H^0$, the Hilbert $A \otimes H^0 - B \otimes H^0$ -bimodule with $C \otimes H^0$ by the isomorphism defined by

$$\begin{aligned} & \Phi\left(\begin{bmatrix} a \otimes \phi_{11} & x \otimes \phi_{12} \\ y \otimes \phi_{21} & b \otimes \phi_{22} \end{bmatrix}\right) \\ &= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \otimes \phi_{11} + \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \otimes \phi_{12} + \begin{bmatrix} 0 & 0 \\ \widetilde{y} & 0 \end{bmatrix} \otimes \phi_{21}^* + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \otimes \phi_{22}, \end{aligned}$$

where $a \in A, b \in B, x, y \in X$ and $\phi_{ij} \in H^0$ ($i, j = 1, 2$).

Let γ be the homomorphism of C to $C \otimes H^0$ defined by for any $a \in A, b \in B, x, y \in X$,

$$\gamma\left(\begin{bmatrix} a & x \\ \widetilde{y} & b \end{bmatrix}\right) = \begin{bmatrix} \rho(a) & \lambda(x) \\ \lambda(\widetilde{y}) & \sigma(b) \end{bmatrix}.$$

Let w be the unitary element in $M(C)$ defined by $w = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$. By routine computations, (γ, w) is a twisted coaction of H^0 on C .

Remark 4.2. (1) We note that the twisted action of H on C induced by (γ, w) as follows: For any $a \in A, b \in B, x, y \in X$ and $h \in H$,

$$h \cdot \gamma \begin{bmatrix} a & x \\ \widetilde{y} & b \end{bmatrix} = \begin{bmatrix} h \cdot_{\rho, u} a & h \cdot_{\lambda} x \\ S(h)^* \cdot_{\lambda} y & h \cdot_{\sigma, v} b \end{bmatrix}.$$

(2) Let $\widetilde{\lambda}$ be a linear map from X to $X \otimes H^0$ defined by for any $x \in X$,

$$\widetilde{\lambda}(\widetilde{x}) = \widetilde{\lambda(x)}.$$

Then $\widetilde{\lambda}$ is the coaction of H^0 on \widetilde{X} induced by λ . Also, the twisted action of H on \widetilde{X} induced by $\widetilde{\lambda}$ is as follows: For any $x \in X, h \in H$,

$$h \cdot_{\widetilde{\lambda}} \widetilde{x} = S(\widetilde{h^*}) \cdot_{\lambda} y.$$

Let C_1 be the linking C^* -algebra for the Hilbert $A \rtimes_{\rho} H - B \rtimes_{\sigma} H$ -bimodule $X \rtimes_{\lambda} H$. Then we obtain the following lemma by Remarks 4.1 and 4.2:

Lemma 4.3. *With the above notations, there is an isomorphism π_1 of $C \rtimes_{\gamma, w} H$ onto C_1 .*

Proof. Let π_1 be the map from $C \rtimes_{\gamma,w} H$ to C_1 defined by

$$\pi_1\left(\begin{bmatrix} a & x \\ \tilde{y} & b \end{bmatrix} \rtimes_{\gamma,w} h\right) = \left[\widehat{u}(S(h_{(2)}), h_{(1)})^* [h_{(3)}^* \cdot \lambda y] \rtimes_{\lambda} h_{(4)}^* \right] \tilde{\ } b \rtimes_{\sigma,v} h$$

for any $a \in A, b \in B, x, y \in X$ and $h \in H$. Let θ_1 be the map from C_1 to $C \rtimes_{\gamma,w} H$ defined by

$$\begin{aligned} & \theta_1\left(\begin{bmatrix} a \rtimes_{\rho,u} h & x \rtimes_{\lambda} l \\ y \rtimes_{\lambda} k & b \rtimes_{\sigma,v} m \end{bmatrix}\right) \\ &= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \rtimes_{\gamma,w} h + \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \rtimes_{\gamma,w} l + \left[\begin{bmatrix} 0 & 0 \\ \{[S(h_{(3)} \cdot \lambda y \widehat{v}(S(k_{(2)}), k_{(1)})) \tilde{\ } 0] \rtimes_{\gamma,w} k_{(4)}^* \end{bmatrix}\right. \\ & \left. + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \rtimes_{\gamma,w} m \right] \end{aligned}$$

for any $a \in A, b \in B, x, y \in X$ and $h, k, l, m \in H$. Then by routine computations, π_1 is a homomorphism of $C \rtimes_{\gamma,w} H$ to C_1 and θ_1 is a homomorphism of C_1 to $C \rtimes_{\gamma,w} H$. Moreover, we can see that θ_1 is the inverse map of π_1 . Therefore, we obtain the conclusion. \square

By the proof of the above lemma, we obtain the following corollary:

Corollary 4.4. *With the above notations, there is a Hilbert $B \rtimes_{\sigma} H - A \rtimes_{\rho} H$ -bimodule isomorphism π of $\widetilde{X \rtimes_{\lambda} H}$ onto $\widetilde{X} \rtimes_{\lambda} H$.*

Remark 4.5. Let γ_1 be a coaction of H on C_1 defined by

$$\gamma_1 = (\pi_1 \otimes \text{id}_H) \circ \widehat{\gamma} \circ \pi_1^{-1}.$$

Then by routine computations, for any $a \in A, b \in B, x, y \in X$ and $h, l, k, m \in H$,

$$\begin{aligned} & \gamma_1\left(\begin{bmatrix} a \rtimes_{\rho,u} h & x \rtimes_{\lambda} l \\ y \rtimes_{\lambda} k & b \rtimes_{\sigma,v} m \end{bmatrix}\right) \\ &= \begin{bmatrix} a \rtimes_{\rho,u} h_{(1)} & 0 \\ 0 & 0 \end{bmatrix} \otimes h_{(2)} + \begin{bmatrix} 0 & x \rtimes_{\lambda} l_{(1)} \\ 0 & 0 \end{bmatrix} \otimes l_{(2)} + \left[\begin{bmatrix} 0 & 0 \\ (y \rtimes_{\lambda} k_{(1)}) \tilde{\ } & 0 \end{bmatrix} \otimes k_{(2)}^*\right. \\ & \left. + \begin{bmatrix} 0 & 0 \\ 0 & b \rtimes_{\sigma,v} m_{(1)} \end{bmatrix} \otimes m_{(2)}\right]. \end{aligned}$$

We give a result similar to [15, Theorem 6.4] for coactions of H^0 on a Hilbert C^* -bimodule applying Proposition 2.12 to a linking C^* -algebra. Let ρ and σ be coactions of H^0 on A and B , respectively and let X be a Hilbert $A - B$ -bimodule. Let λ be a coaction of H^0 on X with respect to (A, B, ρ, σ) . Let C be the linking C^* -algebra and γ the coaction of H^0 on C induced by ρ, σ and λ . As defined in Section 3, let

$$X^\lambda = \{x \in X \mid \lambda(x) = x \otimes 1^0\}.$$

Then by Lemma 3.8, X^λ is an Hilbert $A^\rho - B^\sigma$ -bimodule. Let C_0 be the linking C^* -algebra for X^λ . We can prove the following lemma in the straightforward way. So, we give it without its proof.

Lemma 4.6. *With the above notations and assumptions, $C^\gamma = C_0$, where C^γ is the fixed point C^* -subalgebra of C for γ .*

Lemma 4.7. *With the above notations, if $\widehat{\rho}(1 \rtimes_{\rho} e) \sim (1 \rtimes_{\rho} e) \otimes 1$ in $(M(A) \rtimes_{\rho} H) \otimes H$ and $\widehat{\sigma}(1 \rtimes_{\sigma} e) \sim (1 \rtimes_{\sigma} e) \otimes 1$ in $(M(B) \rtimes_{\sigma} H) \otimes H$, then $\widehat{\gamma}(1_{M(C)} \rtimes_{\gamma} e) \sim (1_{M(C)} \rtimes_{\gamma} e) \otimes 1$ in $(M(C) \rtimes_{\gamma} H) \otimes H$.*

Proof. By Remark 4.3, we identify $C \rtimes_{\gamma} H$ with C_1 , the linking C^* -algebra for the Hilbert $A \rtimes_{\rho} H - A \rtimes_{\rho} H$ -bimodule $X \rtimes_{\lambda} H$. Also, we identify $\widehat{\gamma}$ with γ_1 , the coaction of H on C_1 defined in Remark 4.5. Hence

$$\widehat{\gamma}(1 \rtimes_{\underline{\gamma}} e) = \begin{bmatrix} 1 \rtimes_{\underline{\rho}} e(1) & 0 \\ 0 & 0 \end{bmatrix} \otimes e(2) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \rtimes_{\underline{\sigma}} e(1) \end{bmatrix} \otimes e(2).$$

By the assumptions,

$$\begin{aligned} \begin{bmatrix} 1 \rtimes_{\underline{\rho}} e(1) & 0 \\ 0 & 0 \end{bmatrix} \otimes e(2) &\sim \begin{bmatrix} 1 \rtimes_{\underline{\rho}} e & 0 \\ 0 & 0 \end{bmatrix} \otimes 1 \quad \text{in} \quad \begin{bmatrix} M(A) \rtimes_{\underline{\rho}} H & 0 \\ 0 & 0 \end{bmatrix} \otimes H, \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \rtimes_{\underline{\sigma}} e(1) \end{bmatrix} \otimes e(2) &\sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \rtimes_{\underline{\sigma}} e \end{bmatrix} \otimes 1 \quad \text{in} \quad \begin{bmatrix} 0 & 0 \\ 0 & M(A) \rtimes_{\underline{\sigma}} H \end{bmatrix} \otimes H. \end{aligned}$$

Since $\begin{bmatrix} M(A) \rtimes_{\underline{\rho}} H & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & M(A) \rtimes_{\underline{\sigma}} H \end{bmatrix}$ are C^* -subalgebras of $M(C_1)$ by the proof of Echterhoff and Raeburn [9, Proposition A.1],

$$\begin{bmatrix} 1 \rtimes_{\underline{\rho}} e(1) & 0 \\ 0 & 0 \end{bmatrix} \otimes e(2) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \rtimes_{\underline{\sigma}} e(1) \end{bmatrix} \otimes e(2) \sim \begin{bmatrix} 1 \rtimes_{\underline{\rho}} e & 0 \\ 0 & 1 \rtimes_{\underline{\sigma}} e \end{bmatrix} \otimes 1$$

in $M(C_1) \otimes H$. Therefore, we obtain the conclusion since $M(C_1) \otimes H$ is identified with $(M(C) \rtimes_{\underline{\gamma}} H) \otimes H$. \square

By [15, Section 4], there is a unitary element $w^{\rho} \in M(A) \otimes H$ satisfying that

$$\begin{aligned} w^{\rho*}((1 \rtimes_{\underline{\rho}} e) \otimes 1)w^{\rho} &= \widehat{\rho}(1 \rtimes_{\underline{\rho}} e), \\ U^{\rho} &= w^{\rho}(z^{\rho*} \otimes 1), \quad z^{\rho} = (\text{id}_{M(A)} \otimes \epsilon)(w^{\rho}) \in M(A)^{\underline{\rho}}. \end{aligned}$$

Also, there is a unitary element $w^{\sigma} \in M(B) \otimes H$ satisfying that

$$\begin{aligned} w^{\sigma*}((1 \rtimes_{\underline{\sigma}} e) \otimes 1)w^{\sigma} &= \widehat{\underline{\sigma}}(1 \rtimes_{\underline{\sigma}} e), \\ U^{\sigma} &= w^{\sigma}(z^{\sigma*} \otimes 1), \quad z^{\sigma} = (\text{id}_{M(A)} \otimes \epsilon)(w^{\sigma}) \in M(A)^{\underline{\sigma}}. \end{aligned}$$

Let $w^{\gamma} = \begin{bmatrix} w^{\rho} & 0 \\ 0 & w^{\sigma} \end{bmatrix} \in M(C) \otimes H$. Then w^{γ} is a unitary element satisfying that $w^{\gamma*}((1 \rtimes_{\underline{\gamma}} e) \otimes 1)w^{\gamma} = \widehat{\underline{\gamma}}(1 \rtimes_{\underline{\gamma}} e)$. Let $U^{\gamma} = w^{\gamma}(z^{\gamma*} \otimes 1)$, where $z^{\gamma} = (\text{id}_{M(C)} \otimes \epsilon)(w^{\gamma}) \in M(C)^{\underline{\gamma}}$. Then by Section 2, U^{γ} satisfies that

$$\widehat{U}^{\gamma}(1^0) = 1, \quad \widehat{U}^{\gamma}(\phi_{(1)})c\widehat{U}^{\gamma*}(\phi_{(2)}) \in M(C)^{\underline{\gamma}}$$

for any $c \in M(C)^{\underline{\gamma}}$, $\phi \in H^0$. Let (η, u^{γ}) be a twisted coaction of H on C^{γ} induced by U^{γ} which is defined in Section 2. Then by the proof of Proposition 2.12, there is the isomorphism π_C of $C^{\gamma} \rtimes_{\eta, u^{\gamma}} H^0$ onto C defined by

$$\pi_C(c \rtimes_{\eta, u^{\gamma}} \phi) = c\widehat{U}^{\gamma}(\phi)$$

for any $c \in C^{\gamma}$, $\phi \in H^0$, which satisfies that

$$\gamma \circ \pi_C = (\pi_C \otimes \text{id}_H) \circ \widehat{\eta}, \quad E^{\eta, u^{\gamma}} = E^{\gamma} \circ \pi_C,$$

where $E^{\eta, u^{\gamma}}$ and E^{γ} are the canonical conditional expectations from $C^{\gamma} \rtimes_{\eta, u^{\gamma}} H^0$ and C onto C^{γ} , respectively. Let $p = \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}$, $q = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}$. Then p and q are projection in $M(C^{\gamma})$. We note that $M(C^{\gamma}) = M(C)^{\underline{\gamma}}$ by Lemma 2.9.

Lemma 4.8. *With the above notations and assumptions,*

$$\begin{aligned} \pi_C(p \rtimes_{\underline{\eta}, u^{\gamma}} 1^0) &= p, \quad u^{\gamma}(p \otimes 1 \otimes 1) = (p \otimes 1 \otimes 1)u^{\gamma}, \\ \pi_C(q \rtimes_{\underline{\eta}, u^{\gamma}} 1^0) &= q, \quad u^{\gamma}(q \otimes 1 \otimes 1) = (q \otimes 1 \otimes 1)u^{\gamma}, \end{aligned}$$

Proof. We note that C^γ is identified with the C^* -subalgebra $C^\gamma \rtimes_{\eta, u^\gamma} 1^0$ of $C^\gamma \rtimes_{\eta, u^\gamma} H^0$. Then by [25, Proposition 2.12],

$$\begin{aligned} p &= E_1^{\eta, u^\gamma}(p \rtimes_{\underline{\eta}, u^\gamma} 1^0) = E^\gamma(\pi_C(p \rtimes_{\underline{\eta}, u^\gamma} 1^0)) = e \cdot_\gamma \pi_C(p \rtimes_{\underline{\eta}, u^\gamma} 1^0) \\ &= \pi_C(e \cdot_{\hat{\eta}}(p \rtimes_{\underline{\eta}, u^\gamma} 1^0)) = \pi_C(p) = \pi_C(p \rtimes_{\underline{\eta}, u^\gamma} 1^0) \end{aligned}$$

since $\gamma \circ \pi_C = (\pi_C \otimes \text{id}_H) \circ \hat{\eta}$. Similarly, we can obtain that $\pi_C(q \rtimes_{\underline{\eta}, u^\gamma} 1^0) = q$.

Furthermore, by the definition of U^γ , $U^\gamma = \begin{bmatrix} U^\rho & 0 \\ 0 & U^\sigma \end{bmatrix} \in M(C) \otimes H$. Hence $U^\gamma(p \otimes 1) = (p \otimes 1)U^\gamma$. Since

$$\hat{u}^\gamma(\phi, \psi) = \hat{U}^\gamma(\phi_{(1)})\hat{U}^\gamma(\psi_{(1)})\hat{U}^{\gamma*}(\phi_{(2)}\psi_{(2)})$$

for any $\phi, \psi \in H^0$, we can see that $u^\gamma(p \otimes 1 \otimes 1) = (p \otimes 1 \otimes 1)u^\gamma$. Similarly $u^\gamma(q \otimes 1 \otimes 1) = (q \otimes 1 \otimes 1)u^\gamma$. \square

Let $\alpha = \eta|_{A^\rho}$, $\beta = \eta|_{B^\sigma}$ and $\mu = \eta|_{X^\lambda}$. Let $u^\rho = u^\gamma(p \otimes 1 \otimes 1)$, $u^\sigma = u^\gamma(q \otimes 1 \otimes 1)$. Furthermore, let $\pi_A = \pi_C|_A$, $\pi_B = \pi_C|_B$, $\pi_X = \pi_C|_X$. Then (α, u^ρ) and (β, u^σ) are twisted coactions of H^0 on A^ρ and B^σ , respectively and μ is a twisted coaction of H^0 on X^λ with respect to $(A, B, \alpha, u^\rho, \beta, u^\sigma)$. Also, π_A and π_B are isomorphisms of $A^\rho \rtimes_{\alpha, u^\rho} H^0$ and $B^\sigma \rtimes_{\beta, u^\sigma} H^0$ onto A and B satisfying the results in Proposition 2.12, respectively. Furthermore, we obtain the following:

Theorem 4.9. *Let A and B be C^* -algebras and H a finite dimensional C^* -Hopf algebra with its dual C^* -Hopf algebra H^0 . Let ρ and σ be coactions of H^0 on A and B , respectively. Let λ be a coaction of H^0 on a Hilbert A – B -bimodule X with respect to (A, B, ρ, σ) . We suppose that $\hat{\rho}(1 \rtimes_{\underline{\rho}} e) \sim (1 \rtimes_{\underline{\rho}} e) \otimes 1$ in $M(A) \rtimes_{\underline{\rho}} H$ and that $\hat{\sigma}(1 \rtimes_{\underline{\sigma}} e) \sim (1 \rtimes_{\underline{\sigma}} e) \otimes 1$ in $M(B) \rtimes_{\underline{\sigma}} H$. Then there are a twisted coaction μ of H^0 on X^λ and a bijective linear map π_X from $X^\lambda \rtimes_{\mu} H^0$ onto X satisfying the following conditions:*

- (1) $\pi_X((a \rtimes_{\alpha, u^\rho} \phi)(x \rtimes_{\mu} \psi)) = \pi_A(a \rtimes_{\alpha, u^\rho} \phi)\pi_X(x \rtimes_{\mu} \psi)$,
- (2) $\pi_X((x \rtimes_{\mu} \phi)(b \rtimes_{\beta, u^\sigma} \psi)) = \pi_X(x \rtimes_{\mu} \phi)\pi_B(b \rtimes_{\beta, u^\sigma} \psi)$,
- (3) $\pi_A(\langle a \rtimes_{\alpha, u^\rho} \phi, y \rtimes_{\mu} \psi \rangle) = \langle \pi_X(x \rtimes_{\mu} \phi), \pi_X(y \rtimes_{\mu} \psi) \rangle$,
- (4) $\pi_B(\langle x \rtimes_{\mu} \phi, y \rtimes_{\mu} \psi \rangle_{B^\sigma \rtimes_{\beta, u^\sigma} H^0}) = \langle \pi_X(x \rtimes_{\mu} \phi), \pi_X(y \rtimes_{\mu} \psi) \rangle_B$,
- (5) $h \cdot_\lambda \pi_X(x \rtimes_{\mu} \phi) = \pi_X(h \cdot_{\hat{\mu}}(x \rtimes_{\mu} \phi))$, for any $x, y \in X^\lambda$, $a \in A^\rho$, $b \in B^\sigma$, $h \in H$, $\phi, \psi \in H^0$.

Proof. Using the above discussions, we can prove the theorem in a straightforward way. \square

Let A be a unital C^* -algebra and ρ a coaction of H^0 on A . Let \mathbf{K} be the C^* -algebra of all compact operators on a countably infinite dimensional Hilbert space. Let $A^s = A \otimes \mathbf{K}$ and $\rho^s = \rho \otimes \text{id}$. We identify $H^0 \otimes \mathbf{K}$ with $\mathbf{K} \otimes H^0$. Then ρ^s is a coaction of H^0 on A^s .

Lemma 4.10. *With the above notations, ρ and ρ^s are strongly Morita equivalent.*

Proof. This is immediate by routine computations. \square

Let A and B be unital C^* -algebras. Let ρ and σ be coactions of H^0 on A and B , respectively. We suppose that ρ and σ are strongly Morita equivalent. Also, we suppose that there are an A – B -equivalence bimodule X and a coaction λ of H^0 on X with respect to (A, B, ρ, σ) . Let C be the linking C^* -algebra for X and γ the coaction of H^0 on C induced by ρ, σ and λ , which is defined in the above. Let $A^s = A \otimes \mathbf{K}$, $B^s = B \otimes \mathbf{K}$ and $C^s = C \otimes \mathbf{K}$. Let $X^s = X \otimes \mathbf{K}$, the exterior tensor product of X and \mathbf{K} , which is an A^s – B^s -equivalence bimodule in the usual way.

Let $\rho^s = \rho \otimes \text{id}$, $\sigma^s = \sigma \otimes \text{id}$ and $\gamma^s = \gamma \otimes \text{id}$. Let $\lambda^s = \lambda \otimes \text{id}$, which is a coaction of H^0 on X^s . Let

$$p = \begin{bmatrix} 1_A \otimes 1_{M(\mathbf{K})} & 0 \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \otimes 1_{M(\mathbf{K})} \end{bmatrix}.$$

Then p and q are full projections in $M(C^s)$ and $A^s \cong pC^s p$, $B^s \cong qC^s q$. We identify A^s and B^s with $pC^s p$ and $qC^s q$, respectively. By Brown [4, Lemma 2.5], there is a partial isometry $w \in M(C^s)$ such that $w^*w = p$, $ww^* = q$. Let θ be a map from A^s to C^s defined by

$$\theta(a) = waw^* = w \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} w^*$$

for any $a \in A$. Since $w^*w = p$ and $ww^* = q$, by easy computations, we can see that θ is an isomorphism of A^s onto B^s .

Proposition 4.11. *With the above notations, there is a unitary element $u \in M(B^s) \otimes H^0$ such that*

$$\begin{aligned} (\theta \otimes \text{id}_{H^0}) \circ \rho^s \circ \theta^{-1} &= \text{Ad}(u) \circ \sigma^s, \\ (u \otimes 1^0)(\underline{\sigma}^s \otimes \text{id}_{H^0})(u) &= (\text{id}_{M(B^s)} \otimes \Delta^0)(u), \end{aligned}$$

where $\underline{\sigma}^s$ is the strictly continuous coaction of H^0 on $M(B^s)$ extending the coaction σ^s of H^0 on B^s .

Proof. We note that $\theta = \text{Ad}(w)$. Since $\rho^s = \gamma^s|_{A^s}$ and $\sigma^s = \gamma^s|_{B^s}$, we can obtain that

$$(\theta \otimes \text{id}_{H^0}) \circ \rho^s \circ \theta^{-1} = \text{Ad}((w \otimes 1^0)\underline{\gamma}^s(w^*)) \circ \sigma^s,$$

where $\underline{\gamma}^s$ is the strictly continuous coaction of H^0 on $M(C^s)$ extending the coaction γ^s of H^0 on C^s . Let $u = (w \otimes 1^0)\underline{\gamma}^s(w^*)$. By routine computations, we can show that u is a desired unitary element in $M(B^s) \otimes H^0$. \square

5. EQUIVARIANT PICARD GROUPS

Following Jansen and Waldmann [11], we shall define the equivariant Picard group of a C^* -algebra.

Let A be a C^* -algebra and H a finite dimensional C^* -Hopf algebra with its dual C^* -Hopf algebra H^0 . Let (ρ, u) be a twisted coaction of H^0 on A . We denote by (X, λ) , a pair of an $A - A$ -equivalence bimodule X and a twisted coaction λ of H^0 on X with respect to (A, A, ρ, u, ρ, u) . Let $\text{Equi}_H^{\rho, u}(A)$ be the set of all such pairs (X, λ) as above. We define an equivalence relation \sim in $\text{Equi}_H^{\rho, u}(A)$ as follows: For $(X, \lambda), (Y, \mu) \in \text{Equi}_H^{\rho, u}(A)$, $(X, \lambda) \sim (Y, \mu)$ if and only if there is an $A - A$ -equivalence bimodule isomorphism π of X onto Y such that $\mu \circ \pi = (\pi \otimes \text{id}_{H^0}) \circ \lambda$, that is, for any $x \in X$ and $h \in H$, $\pi(h \cdot_\lambda x) = h \cdot_\mu \pi(x)$. We denote by $[X, \lambda]$ the equivalence class of (X, λ) in $\text{Equi}_H^{\rho, u}(A)$. Let $\text{Pic}_H^{\rho, u}(A) = \text{Equi}_H^{\rho, u}(A) / \sim$. We define the product in $\text{Pic}_H^{\rho, u}(A)$ as follows: For $(X, \lambda), (Y, \mu) \in \text{Equi}_H^{\rho, u}(A)$,

$$[X, \lambda][Y, \mu] = [X \otimes_A Y, \lambda \otimes \mu],$$

where $\lambda \otimes \mu$ is the twisted coaction of H^0 on X induced by the action “ $\cdot_{\lambda \otimes \mu}$ ” of H on X defined in [17, Proposition 3.1]. By easy computations, we can see that the above product is well-defined. We regard A as an $A - A$ -equivalence bimodule in the usual way. We sometimes denote it by ${}_A A_A$. Also, we can regard a twisted coaction ρ of H^0 on C^* -algebra A as a twisted coaction of H^0 on the $A - A$ -equivalence bimodule ${}_A A_A$ with respect to (A, A, ρ, u, ρ, u) . Then $[{}_A A_A, \rho]$ is the unit element in $\text{Pic}_H^{\rho, u}(A)$. Let $\tilde{\lambda}$ be the coaction of H^0 on \tilde{X} defined by $\tilde{\lambda}(\tilde{x}) = \overline{\lambda(x)}$ for any $x \in X$, which is also defined in Remark 4.2 (2). Then we can see that $[\tilde{X}, \tilde{\lambda}]$

is the inverse element of $[X, \lambda]$ in $\text{Pic}_H^{\rho, u}(A)$. By the above product, $\text{Pic}_H^{\rho, u}(A)$ is a group. We call it the (ρ, u, H) -equivariant Picard group of A .

Let $\text{Aut}_H^{\rho, u}(A)$ be the group of all automorphisms α of A satisfying that $(\alpha \otimes \text{id}_{H^0}) \circ \rho = \rho \circ \alpha$ and $\text{Int}_H^{\rho, u}(A)$ the set of all generalized inner automorphisms $\text{Ad}(v)$ of A satisfying that $\underline{\rho}(v) = v \otimes 1^0$, where v is a unitary element in $M(A)$. By easy computations $\text{Int}_H^{\rho, u}(A)$ is a normal subgroup of $\text{Aut}_H^{\rho, u}(A)$. Modifying [5], for each $\alpha \in \text{Aut}_H^{\rho, u}(A)$, we construct the element $(X_\alpha, \lambda_\alpha) \in \text{Equi}_H^{\rho, u}(A)$ as follows: Let $\alpha \in \text{Aut}_H^{\rho, u}(A)$. Let X_α be the vector space A with the obvious left action of A on X_α and the obvious left A -valued inner product, but define the right action of A on X_α by $x \cdot a = x\alpha(a)$ for any $x \in X_\alpha, a \in A$ and the right A -valued inner product by $\langle x, y \rangle_A = \alpha^{-1}(x^*y)$ for any $x, y \in X_\alpha$. Then by [5], X_α is an $A - A$ -equivalence bimodule. Also, ρ can be regarded as a linear map from X_α to an $A \otimes H^0 - A \otimes H^0$ -equivalence bimodule $X_\alpha \otimes H^0$. We denote it by λ_α . By easy computations, λ_α is a twisted coaction of H^0 on X_α with respect to (A, A, ρ, u, ρ, u) . Thus we obtain the map Φ

$$\Phi : \text{Aut}_H^{\rho, u}(A) \rightarrow \text{Pic}_H^{\rho, u}(A) : \alpha \mapsto [X_\alpha, \lambda_\alpha].$$

Modifying [5], we can see that the map Φ is a homomorphism of $\text{Aut}_H^{\rho, u}(A)$ to $\text{Pic}_H^{\rho, u}(A)$. We have the similar result to [5, Proposition 3.1].

Proposition 5.1. *With the above notations, we have the exact sequence*

$$1 \rightarrow \text{Int}_H^{\rho, u}(A) \xrightarrow{\iota} \text{Aut}_H^{\rho, u}(A) \xrightarrow{\Phi} \text{Pic}_H^{\rho, u}(A),$$

where ι is the inclusion map of $\text{Int}_H^{\rho, u}(A)$ to $\text{Aut}_H^{\rho, u}(A)$.

Proof. Modifying the proof of [5, Proposition 3.1], we shall prove this Proposition. Let v be a unitary element in $M(A)$ with $\underline{\rho}(v) = v \otimes 1^0$. We show that $[X_{\text{Ad}(v)}, \lambda_{\text{Ad}(v)}] = [{}_A A_A, \rho]$ in $\text{Pic}_H^{\rho, u}(A)$. Let π be the map from ${}_A A_A$ to $X_{\text{Ad}(v)}$ defined by $\pi(a) = av^*$ for any $a \in {}_A A_A$. Then π is an $A - A$ -equivalence bimodule isomorphism. Also, for any $a \in {}_A A_A$ and $h \in H$,

$$h \cdot_{\lambda_{\text{Ad}(v)}} \pi(a) = h \cdot_{\lambda_{\text{Ad}(v)}} (av^*) = [h_{(1)} \cdot_{\rho} a][h_{(2)} \cdot_{\underline{\rho}} v^*] = [h \cdot_{\rho} a]v^* = \pi(h \cdot_{\rho} a).$$

Thus $[X_{\text{Ad}(v)}, \lambda_{\text{Ad}(v)}] = [{}_A A_A, \rho]$ in $\text{Pic}_H^{\rho, u}(A)$. Conversely, let $\alpha \in \text{Aut}_H^{\rho, u}(A)$ with $[X_\alpha, \lambda_\alpha] = [{}_A A_A, \rho]$ in $\text{Pic}_H^{\rho, u}(A)$. Then there is an $A - A$ -equivalence bimodule isomorphism π of ${}_A A_A$ onto X_α such that

$$\lambda_\alpha \circ \pi = (\pi \otimes \text{id}) \circ \rho.$$

By the proof of [5, Proposition 3.1], $(\pi \circ \alpha^{-1}, \pi)$ is a double centralizer of A . Hence $(\pi \circ \alpha^{-1}, \pi) \in M(A)$. Let $v = (\pi \circ \alpha^{-1}, \pi)$. Then v is a unitary element in $M(A)$ such that $\alpha = \text{Ad}(v^*)$. Furthermore, since $\lambda_\alpha \circ \pi = (\pi \otimes \text{id}) \circ \rho$, for any $a \in A$, $\lambda_\alpha(\pi(a)) = (\pi \otimes \text{id})(\rho(a))$. It follows that $\rho(av^*) = \rho(a)(v \otimes 1^0)^*$ for any $a \in A$. That is, $\underline{\rho}(v) = v \otimes 1^0$. Therefore, we obtain the conclusion. \square

Next, we shall show a similar result to [5, Corollary 3.5]. Let A be a C^* -algebra and X an $A - A$ -equivalence bimodule. Let ρ be a coaction of H^0 on A and λ a coaction of H^0 on X with respect to (A, A, ρ, ρ) . Let C be the linking C^* -algebra for X and γ the coaction of H^0 on C induced by ρ and λ which is defined in Section 4. Furthermore, we suppose that A is unital and that $\widehat{\rho}(1 \rtimes_{\rho} e) \sim (1 \rtimes_{\rho} e) \otimes 1$ in $(A \rtimes_{\rho} H) \otimes H$. Then ρ is saturated by [15, Section 4]. Let $(\widehat{\rho})^s$ be the coaction of H on $(A \rtimes_{\rho} H)^s \otimes H$ induced by the dual coaction $\widehat{\rho}$ of H on $A \rtimes_{\rho} H$. Also, let (ρ^s) be the dual coaction of ρ^s which is a coaction of H on $A^s \rtimes_{\rho^s} H$. By their definitions, we can see that $(\widehat{\rho})^s = (\rho^s)$, where we identify $(A \rtimes_{\rho} H)^s$ with $A^s \rtimes_{\rho^s} H$. We denote them by $\widehat{\rho}^s$.

Lemma 5.2. *With the above notations, if $\widehat{\rho}(1 \rtimes_{\rho} e) \sim (1 \rtimes_{\rho} e) \otimes 1$ in $(A \rtimes_{\rho} H) \otimes H$, then $\widehat{\rho}^s(1 \rtimes_{\underline{\rho}^s} e) \sim (1 \rtimes_{\underline{\rho}^s} e) \otimes 1$ in $(M(A^s) \rtimes_{\underline{\rho}^s} H) \otimes H$.*

Proof. This is immediate by straightforward computations. \square

Let C be the linking C^* -algebra for an $A^s - A^s$ -equivalence bimodule X^s and γ the coaction of H on C induced by ρ^s and λ^s .

Lemma 5.3. *With the above notations, if $\widehat{\rho}(1 \rtimes_{\rho} e) \sim (1 \rtimes_{\rho} e) \otimes 1$ in $(A \rtimes_{\rho} H) \otimes H$, then $\widehat{\gamma}(1_{M(C)} \rtimes_{\underline{\gamma}} e) \sim (1_{M(C)} \rtimes_{\underline{\gamma}} e) \otimes 1$ in $(M(C) \rtimes_{\underline{\gamma}} H) \otimes H$.*

Proof. This is immediate by Lemmas 4.7 and 5.2. \square

Lemma 5.4. *With the above notations, we suppose that $\widehat{\rho}(1 \rtimes_{\rho} e) \sim (1 \rtimes_{\rho} e) \otimes 1$ in $(A \rtimes_{\rho} H) \otimes H$. Let Φ be the homomorphism of $\text{Aut}_H^{\rho^s}(A^s)$ to $\text{Pic}_H^{\rho^s}(A^s)$ defined by $\Phi(\alpha) = [X_{\alpha}, \lambda_{\alpha}]$ for any $\alpha \in \text{Aut}_H^{\rho^s}(A^s)$. Then Φ is surjective.*

Proof. Let $[X, \lambda]$ be any element in $\text{Pic}_H^{\rho^s}(A^s)$. Let

$$X^{\lambda} = \{x \in X \mid \lambda(x) = x \otimes 1^0\}.$$

Since $\widehat{\rho}(1 \rtimes_{\rho} e) \sim (1 \rtimes_{\rho} e) \otimes 1$ in $(A \rtimes_{\rho} H) \otimes H$, by Lemma 5.2, $\widehat{\rho^s}(1 \rtimes_{\rho^s} e) \sim (1 \rtimes_{\rho^s} e) \otimes 1$ in $(M(A^s) \rtimes_{\rho^s} H) \otimes H$. Since X is an $A^s - A^s$ -equivalence bimodule, by Lemma 3.8 and Theorem 4.9, X^{λ} is an $(A^s)^{\rho^s} - (A^s)^{\rho^s}$ -equivalence bimodule, where $(A^s)^{\rho^s}$ is the fixed point C^* -subalgebra of A^s for the coaction ρ^s . Let C be the linking C^* -algebra for X and γ the coaction of H^0 on C induced by ρ^s and λ . Let C^{γ} be the fixed point C^* -algebra of C for γ . Then by Lemma 4.6, C^{γ} is isomorphic to C_0 , the linking C^* -algebra for X^{λ} . We identify C^{γ} with C_0 . Let

$$p = \begin{bmatrix} 1_A \otimes 1_{M(\mathbf{K})} & 0 \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 & 0 \\ 0 & 1_A \otimes 1_{M(\mathbf{K})} \end{bmatrix}$$

Then p and q are projections in $M(C)^{\underline{\gamma}}$. Since $M(C)^{\underline{\gamma}} = M(C^{\gamma})$ by Lemmas 2.9 and 4.7, p and q are full for C^{γ} . By the proof of [5, Theorem 3.4], there is a partial isometry $w \in M(C)^{\underline{\gamma}}$ such that

$$w^*w = p, \quad q = ww^*.$$

Hence $w \in M(C)$. Let α be the map on A^s defined by

$$\alpha(a) = w^*aw = w^* \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} w$$

for any $a \in A^s$. By routine computations, α is an automorphism of A^s . Let π be a linear map from X to X_{α} defined by

$$\pi(x) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} w = p \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} wp$$

for any $x \in X$. In the same way as in the proof of [5, Lemma 3.3], we can see that π is an $A^s - A^s$ -equivalence bimodule isomorphism of X onto X_{α} . For any $a \in A^s$,

$$\begin{aligned} (\rho^s \circ \alpha)(a) &= \rho^s(w^*aw) = \gamma(w^* \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} w) = \underline{\gamma}(w^*) \begin{bmatrix} 0 & 0 \\ 0 & \rho^s(a) \end{bmatrix} \underline{\gamma}(w) \\ &= (\alpha \otimes \text{id}_{H^0})(\rho^s(a)) \end{aligned}$$

since $w \in M(C)^{\underline{\gamma}}$. Hence $\alpha \in \text{Aut}_H^{\rho^s}(A^s)$. Furthermore, for any $x \in X$,

$$\begin{aligned} (\lambda_{\alpha} \circ \pi)(x) &= \lambda_{\alpha} \left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} w \right) = \rho^s \left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} w \right) \\ &= \gamma \left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} w \right) = \begin{bmatrix} 0 & \lambda(x) \\ 0 & 0 \end{bmatrix} (w \otimes 1^0) = (\pi \otimes \text{id}_{H^0})(\lambda(x)), \end{aligned}$$

where we identify $\mathbf{K} \otimes H^0$ with $H^0 \otimes \mathbf{K}$. Thus $\Phi(\alpha) = [X, \lambda]$. Therefore, we obtain the conclusion. \square

Theorem 5.5. *Let A be a unital C^* -algebra and ρ a coaction of H^0 on A . We suppose that $\widehat{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ in $(A \rtimes_\rho H) \otimes H$. Then we have the following exact sequence:*

$$1 \rightarrow \text{Int}_H^{\rho^s}(A^s) \xrightarrow{\iota} \text{Aut}_H^{\rho^s}(A^s) \xrightarrow{\Phi} \text{Pic}_H^{\rho^s}(A^s) \rightarrow 1,$$

where ι is the inclusion map of $\text{Int}_H^{\rho^s}(A^s)$ to $\text{Aut}_H^{\rho^s}(A^s)$.

Proof. This is immediate by Proposition 5.1 and Lemma 5.4. \square

Since the following lemma is obtained in a straightforward way, we omit its proof:

Lemma 5.6. *Let (ρ, u) and (σ, v) be twisted coactions on C^* -algebras A and B , respectively. We suppose that (ρ, u) is strongly Morita equivalent to (σ, v) . Then $\text{Pic}_H^{\rho, u}(A) \cong \text{Pic}_H^{\sigma, v}(B)$.*

6. ORDINARY PICARD GROUPS AND EQUIVARIANT PICARD GROUPS

In this section, we shall investigate the relation between ordinary Picard groups and equivariant Picard groups. Let ρ be a coaction of H^0 on a C^* -algebra A and let f_ρ be the map from $\text{Pic}_H^\rho(A)$ to $\text{Pic}(A)$ defined by

$$f_\rho : \text{Pic}_H^\rho(A) \rightarrow \text{Pic}(A) : [X, \lambda] \mapsto [X],$$

where $\text{Pic}(A)$ is the ordinary Picard group of A . Clearly f_ρ is a homomorphism of $\text{Pic}_H^\rho(A)$ to $\text{Pic}(A)$. Let $\text{Aut}(A)$ be the group of all automorphisms of A and let $\alpha \in \text{Aut}(A)$. Let X_α be the $A - A$ -equivalence bimodule induced by α defined in Section 5. Let λ be a coaction of H^0 on X_α with respect to (A, A, ρ, ρ) . Then for any $a \in A$ and $x, y \in X_\alpha$,

- (1) $\lambda(ax) = \lambda(a \cdot x) = \rho(a) \cdot \lambda(x) = \rho(a)\lambda(x)$,
- (2) $\lambda(x\alpha(a)) = \lambda(x \cdot a) = \lambda(x) \cdot \rho(a) = \lambda(x)(\alpha \otimes \text{id})(\rho(a))$,
- (3) $\rho(xy^*) = \rho_{(A, \langle x, y \rangle)} = {}_{A \otimes H^0} \langle \lambda(x), \lambda(y) \rangle = \lambda(x)\lambda(y)^*$,
- (4) $\rho(\alpha^{-1}(x^*y)) = \rho(\langle x, y \rangle_A) = \langle \lambda(x), \lambda(y) \rangle_{A \otimes H^0} = (\alpha^{-1} \otimes \text{id})(\lambda(x)^*\lambda(y))$,
- (5) $(\text{id} \otimes \epsilon^0)(\lambda(x)) = x$,
- (6) $(\lambda \otimes \text{id})(\lambda(x)) = (\text{id} \otimes \Delta^0)(\lambda(x))$.

Let $\{u_\gamma\}$ be an approximate unit of A . Then $\lambda(u_\gamma) \in X_\alpha \otimes H^0$. Since $X_\alpha = A$ as vector spaces, we regard $\lambda(u_\gamma)$ as an element in $A \otimes H^0$.

Lemma 6.1. *With the above notations, we regard $\lambda(u_\gamma)$ as an element in $A \otimes H^0$. Then $\{\lambda(u_\gamma)\}$ converges to a unitary element in $M(A \otimes H^0)$ strictly and the unitary element does not depend on the choice of an approximate unit of A .*

Proof. Let $a \in A$ and $x \in A \otimes H^0$. Then by Equation (2),

$$\begin{aligned} \|(\lambda(u_\gamma) - \lambda(u_{\gamma'}))(\alpha \otimes \text{id})(\rho(a)x)\| &= \|\lambda((u_\gamma - u_{\gamma'})\alpha(a))(\alpha \otimes \text{id})(x)\| \\ &\leq \|\lambda((u_\gamma - u_{\gamma'})\alpha(a))\| \|x\| = \|(u_\gamma - u_{\gamma'})\alpha(a)\| \|x\| \end{aligned}$$

since λ is isometric. Since $\rho(A)(A \otimes H^0)$ is dense in $A \otimes H^0$, $\{\lambda(u_\gamma)y\}$ is a Cauchy net for any $y \in A \otimes H^0$. Similarly by Equation (1), $\{y\lambda(u_\gamma)\}$ is also a Cauchy net for any $y \in A \otimes H^0$. Thus $\{\lambda(u_\gamma)\}$ converges to some element $u \in M(A \otimes H^0)$ strictly. We note

$$\begin{aligned} \lim_{\gamma \rightarrow \infty} \rho(u_\gamma) &= \lim_{\gamma \rightarrow \infty} \underline{\rho}(u_\gamma) = \underline{\rho}(\lim_{\gamma \rightarrow \infty} u_\gamma) = \underline{\rho}(1) = 1, \\ \lim_{\gamma \rightarrow \infty} \alpha^{-1}(u_\gamma) &= \lim_{\gamma \rightarrow \infty} \underline{\alpha}^{-1}(u_\gamma) = \underline{\alpha}^{-1}(\lim_{\gamma \rightarrow \infty} u_\gamma) = \underline{\alpha}^{-1}(1) = 1, \end{aligned}$$

where the limits are taken under the strict topologies in $M(A \otimes H^0)$ and $M(A)$, respectively and $\underline{\alpha}^{-1}$ is an automorphism of $M(A)$ extending α^{-1} to $M(A)$, which is strictly continuous on $M(A)$. Hence by Equations (3), (4), we can see that u is a unitary element in $M(A \otimes H^0)$. Let $\{v_\beta\}$ be another approximate unit of A and

let v be the limit of $\lambda(v_\beta)$ under the strict topology in $M(A \otimes H^0)$. Then by the above discussion, we have that

$$\|(\lambda(u_\gamma) - \lambda(v_\beta))(\alpha \otimes \text{id})(\rho(a)x)\| \leq \| (u_\gamma - v_\beta)\alpha(a) \| \|x\|$$

for any $a \in A$ and $x \in A \otimes H^0$. Since $\rho(A)(A \otimes H^0)$ is dense in $A \otimes H^0$, $u = v$. \square

Lemma 6.2. *Let u be as in the proof of Lemma 6.1. Then u satisfies that $\lambda(x) = \rho(x)u$ for any $x \in X_\alpha$, $\rho(\alpha(a)) = u(\alpha \otimes \text{id})(a)u^*$ for any $a \in A$ and that $(\underline{\rho} \otimes \text{id})(u)(u \otimes 1^0) = (\text{id} \otimes \Delta^0)(u)$.*

Proof. Let $\{u_\gamma\}$ be an approximate unit of A . By Equation (1), for any $x \in X_\alpha$, $\lambda(xu_\gamma) = \rho(x)\lambda(u_\gamma)$. Thus $\lambda(x) = \rho(x)u$. Also, by Equation (2) for any $a \in A$,

$$\lambda(u_\gamma\alpha(a)) = \lambda(u_\gamma)(\alpha \otimes \text{id})(\rho(a)).$$

Hence $\lambda(\alpha(a)) = u(\alpha \otimes \text{id})(\rho(a))$. Since $\lambda(\alpha(a)) = \rho(\alpha(a))u$ for any $a \in A$ by the above discussion, for any $a \in A$,

$$\rho(\alpha(a))u = u(\alpha \otimes \text{id})(\rho(a)).$$

for any $a \in A$. Since u is a unitary element in $M(A \otimes H^0)$,

$$\rho(\alpha(a)) = u(\alpha \otimes \text{id})(\rho(a))u^*$$

for any $a \in A$. Furthermore, for any $a \in A$

$$\begin{aligned} (\lambda \otimes \text{id})(\lambda(u_\gamma a)) &= (\lambda \otimes \text{id})(\lambda(u_\gamma)(\alpha \otimes \text{id})(\rho(\alpha^{-1}(a)))) \\ &= (\rho \otimes \text{id})(\lambda(u_\gamma))((\lambda \otimes \text{id}) \circ (\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1})(a) \end{aligned}$$

by Equations (1), (2). Thus Equation (2)

$$\begin{aligned} (\lambda \otimes \text{id})(\lambda(a)) &= (\underline{\rho} \otimes \text{id})(u)((\lambda \otimes \text{id}) \circ (\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1})(a) \\ &= \lim_{\gamma \rightarrow \infty} (\underline{\rho} \otimes \text{id})(u)(\lambda \otimes \text{id})((u_\gamma \otimes 1^0)((\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1})(a)) \\ &= \lim_{\gamma \rightarrow \infty} (\underline{\rho} \otimes \text{id})(u)(\lambda(u_\gamma) \otimes 1^0)((\alpha \otimes \text{id} \otimes \text{id}) \circ (\rho \otimes \text{id}) \circ \rho \circ \alpha^{-1})(a) \\ &= (\underline{\rho} \otimes \text{id})(u)(u \otimes 1^0)((\alpha \otimes \text{id} \otimes \text{id}) \circ (\text{id} \otimes \Delta^0) \circ \rho \circ \alpha^{-1})(a) \\ &= (\underline{\rho} \otimes \text{id})(u)(u \otimes 1^0)((\text{id} \otimes \Delta^0) \circ (\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1})(a). \end{aligned}$$

Also, by Equation (2)

$$\begin{aligned} (\text{id} \otimes \Delta^0)(\lambda(u_\gamma a)) &= (\text{id} \otimes \Delta^0)(\lambda(u_\gamma)((\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1})(a)) \\ &= (\text{id} \otimes \Delta^0)(\lambda(u_\gamma))((\text{id} \otimes \Delta^0) \circ (\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1})(a). \end{aligned}$$

Thus

$$(\text{id} \otimes \Delta^0)(\lambda(a)) = (\text{id} \otimes \Delta^0)(u)((\text{id} \otimes \Delta^0) \circ (\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1})(a).$$

By Equation (6)

$$[(\rho \otimes \text{id})(u)(u \otimes 1^0) - ((\text{id} \otimes \Delta^0)(u))((\text{id} \otimes \Delta^0) \circ (\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1})(a)] = 0$$

for any $a \in A$. Therefore,

$$(\rho \otimes \text{id})(u)(u \otimes 1^0) = (\text{id} \otimes \Delta^0)(u).$$

\square

Remark 6.3. By Lemma 6.2, we can see that the coaction $(\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1}$ of H^0 on A is exterior equivalent to ρ .

Conversely, let u be a unitary element in $M(A \otimes H^0)$ satisfying that

$$\rho = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \circ \alpha^{-1}, \quad (\underline{\rho} \otimes \text{id})(u)(u \otimes 1^0) = (\text{id} \otimes \Delta^0)(u).$$

Let λ_u be the linear map from X_α to $X_\alpha \otimes H^0$ defined by

$$\lambda_u(x) = \rho(x)u$$

for any $x \in X_\alpha$. Then by routine computations, we can see that λ_u is a coaction of H^0 on X_α with respect to (A, A, ρ, ρ) .

Proposition 6.4. *With the above notations, the following conditions are equivalent:*

- (1) $[X_\alpha] \in \text{Im} f_\rho$,
- (2) *There is a unitary element $u \in M(A \otimes H^0)$ such that*

$$\rho = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1}, \quad (\underline{\rho} \otimes \text{id})(u)(u \otimes 1^0) = (\text{id} \otimes \Delta^0)(u).$$

Proof. This is immediate by Lemma 6.2 and the above discussion. \square

Let u be a unitary element in $M(A \otimes H^0)$ satisfying Condition (2) in Proposition 6.4. Let λ_u be as above. We call λ_u the coaction of H^0 on X_α with respect to (A, A, ρ, ρ) induced by u .

Let $\alpha, \beta \in \text{Aut}(A)$ satisfying that there are unitary elements $u, v \in M(A \otimes H^0)$ such that

$$\begin{aligned} \rho &= \text{Ad}(u) \circ (\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1}, & (\underline{\rho} \otimes \text{id})(u)(u \otimes 1^0) &= (\text{id} \otimes \Delta^0)(u), \\ \rho &= \text{Ad}(v) \circ (\beta \otimes \text{id}) \circ \rho \circ \beta^{-1}, & (\underline{\rho} \otimes \text{id})(v)(v \otimes 1^0) &= (\text{id} \otimes \Delta^0)(v). \end{aligned}$$

Lemma 6.5. *With the above notations, we have the following:*

$$(\underline{\rho} \otimes \text{id})(u(\underline{\alpha} \otimes \text{id})(v))(u(\underline{\alpha} \otimes \text{id})(v) \otimes 1^0) = (\text{id} \otimes \Delta^0)(u(\underline{\alpha} \otimes \text{id})(v)).$$

Proof. By routine computations, we can see that

$$((\alpha \circ \beta) \otimes \text{id}) \circ \rho \circ (\alpha \circ \beta)^{-1} = \text{Ad}((\underline{\alpha} \otimes \text{id})(v^*)) \circ \text{Ad}(u^*) \circ \rho.$$

Thus we obtain that

$$\rho = \text{Ad}(u(\underline{\alpha} \otimes \text{id})(v)) \circ ((\alpha \circ \beta) \otimes \text{id}) \circ \rho \circ (\alpha \circ \beta)^{-1}.$$

Since $\rho \circ \alpha = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \circ \rho$,

$$(\underline{\rho} \otimes \text{id})((\underline{\alpha} \otimes \text{id})(v)) = (u \otimes 1^0)(\underline{\alpha} \otimes \text{id} \otimes \text{id})((\underline{\rho} \otimes \text{id})(v))(u \otimes 1^0)^*.$$

Hence by routine computations, we can see that

$$(\underline{\rho} \otimes \text{id})(u(\underline{\alpha} \otimes \text{id})(v))(u(\underline{\alpha} \otimes \text{id})(v) \otimes 1^0) = (\text{id} \otimes \Delta^0)(u(\underline{\alpha} \otimes \text{id})(v)).$$

\square

Let α, β and u, v be as above. Let λ_u and λ_v be coactions of H^0 on X_α and X_β with respect to (A, A, ρ, ρ) induced by u and v , respectively. Let $u\sharp v = u(\underline{\alpha} \otimes \text{id})(v) \in M(A \otimes H^0)$. By Lemma 6.5, we can define the coaction $\lambda_{u\sharp v}$ of H^0 on $X_{\alpha \circ \beta}$ with respect to (A, A, ρ, ρ) induced by $u\sharp v$. By easy computations, we can see that $X_\alpha \otimes_A X_\beta$ is isomorphic to $X_{\alpha \circ \beta}$ by an $A - A$ -equivalence bimodule isomorphism π

$$\pi : X_\alpha \otimes_A X_\beta \rightarrow X_{\alpha \circ \beta} : x \otimes y \mapsto x\alpha(y).$$

We identify $X_\alpha \otimes_A X_\beta$ with $X_{\alpha \circ \beta}$ by the above $A - A$ -equivalence bimodule isomorphism π .

Lemma 6.6. *With the above notations, for $[X_\alpha, \lambda_u], [X_\beta, \lambda_v] \in \text{Pic}_H^\rho(A)$,*

$$[X_\alpha, \lambda_u][X_\beta, \lambda_v] = [X_{\alpha \circ \beta}, \lambda_{u\sharp v}] \in \text{Pic}_H^\rho(A),$$

where $u\sharp v = u(\underline{\alpha} \otimes \text{id})(v) \in M(A \otimes H^0)$.

Proof. By the definition of the product in $\text{Pic}_H^\rho(A)$,

$$[X_\alpha, \lambda_u][X_\beta, \lambda_v] = [X_\alpha \otimes_A X_\beta, \lambda_u \otimes \lambda_v].$$

Hence it suffices to show that

$$\pi(h \cdot_{\lambda_u \otimes \lambda_v} x \otimes y) = h \cdot_{\lambda_{u\#v}} \pi(x \otimes y)$$

for any $x \in X_\alpha, y \in X_\beta$ and $h \in H$. For any $x \in X_\alpha, y \in X_\beta$ and $h \in H$,

$$\begin{aligned} \pi(h \cdot_{\lambda_u \otimes \lambda_v} x \otimes y) &= \pi([h_{(1)} \cdot_{\lambda_u} x] \otimes [h_{(2)} \cdot_{\lambda_v} y]) \\ &= \pi([h_{(1)} \cdot_\rho x] \widehat{u}(h_{(2)}) \otimes [h_{(3)} \cdot_\rho y] \widehat{v}(h_{(4)})) \\ &= [h_{(1)} \cdot_\rho x] \widehat{u}(h_{(2)}) \alpha([h_{(3)} \cdot_\rho y] \widehat{v}(h_{(4)})). \end{aligned}$$

Since $\rho \circ \alpha = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \circ \rho$,

$$\begin{aligned} \pi(h \cdot_{\lambda_u \otimes \lambda_v} x \otimes y) &= [h_{(1)} \cdot_\rho x] [h_{(2)} \cdot_\rho \alpha(y)] \widehat{u}(h_{(3)}) \alpha(\widehat{v}(h_{(4)})) \\ &= [h_{(1)} \cdot_\rho x \alpha(y)] (u(\underline{\alpha} \otimes \text{id})(v)) \widehat{h}(h_{(2)}) \\ &= h \cdot_{\lambda_{u\#v}} x \alpha(y). \end{aligned}$$

Therefore, we obtain the conclusions. \square

Corollary 6.7. *With the above notations, for any $[X_\alpha, \lambda_u] \in \text{Pic}_H^\rho(A)$,*

$$[X_\alpha, \lambda_u]^{-1} = [X_{\alpha^{-1}}, \lambda_{(\underline{\alpha}^{-1} \otimes \text{id})(u^*)}] \in \text{Pic}_H^\rho(A).$$

Proof. This is immediate by Lemma 6.6 and routine computations. \square

For any $\alpha \in \text{Aut}(A)$, let $U_\alpha^\rho(M(A \otimes H^0))$ be the set of all unitary elements $u \in M(A \otimes H^0)$ satisfying that

$$\rho = \text{Ad}(u) \circ (\alpha \otimes \text{id}) \circ \rho \circ \alpha^{-1}, \quad (\underline{\rho} \otimes \text{id})(u)(u \otimes 1^0) = (\text{id} \otimes \Delta^0)(u).$$

Lemma 6.8. *With the above notations, for any $\alpha \in \text{Aut}(A)$, we have the following:*

- (1) *For any $u \in U_{\text{id}}^\rho(M(A \otimes H^0))$ and $v \in U_\alpha^\rho(M(A \otimes H^0))$, $uv \in U_\alpha^\rho(M(A \otimes H^0))$,*
- (2) *For any $u, v \in U_\alpha^\rho(M(A \otimes H^0))$, $uv^* \in U_{\text{id}}^\rho(M(A \otimes H^0))$.*

Proof. (1) This is immediate by Lemma 6.6.

(2) By Corollary 6.7, $(\alpha^{-1} \otimes \text{id})(v^*) \in U_{\alpha^{-1}}^\rho(M(A \otimes H^0))$. Hence $uv^* \in U_{\text{id}}^\rho(M(A \otimes H^0))$. \square

Lemma 6.9. *Let $u \in U_{\text{id}}^\rho(M(A \otimes H^0))$. Then the following conditions are equivalent:*

- (1) $[{}_A A_A, \lambda_u] = [{}_A A_A, \rho]$ in $\text{Pic}_H^\rho(A)$,
- (2) *There is a unitary element $w \in M(A) \cap A'$ such that $u = (w^* \otimes 1^0) \underline{\rho}(w)$.*

Proof. We suppose Condition (1). Then there is an $A - A$ -equivalence bimodule automorphism π of ${}_A A_A$ such that

$$\rho(\pi(x)) = (\pi \otimes \text{id})(\lambda_u(x)) = (\pi \otimes \text{id})(\rho(x)u)$$

for any $x \in {}_A A_A$. We note that $\pi \in {}_A \mathbf{B}_A({}_A A_A)$ and that

$${}_A \mathbf{B}_A({}_A A_A) \cong A' \cap \mathbf{B}_A(A_A) \cong A' \cap M(A).$$

Hence there is a unitary element $w \in A' \cap M(A)$ such that $\pi(x) = wx$ for any $x \in A$. Thus for any $x \in A$

$$\rho(wx) = (w \otimes 1^0) \rho(x)u.$$

Therefore $u = (w^* \otimes 1^0)\underline{\rho}(w)$. Next we suppose Condition (2). Let π be the $A - A$ -equivalence bimodule automorphism of ${}_A A_A$ defined by $\pi(x) = wx$ for any $x \in {}_A A_A$. Then for any $x \in {}_A A_A$

$$\begin{aligned}\rho(\pi(x)) &= \rho(wx) = \rho(xw) = \rho(x)\underline{\rho}(w) = \rho(x)(w \otimes 1^0)u = (w \otimes 1^0)\rho(x)u \\ &= (\pi \otimes \text{id})(\lambda_u(x)).\end{aligned}$$

Thus we obtain Condition (1). \square

Corollary 6.10. *Let $\alpha \in \text{Aut}(A)$ and $u, v \in \text{U}_{\text{id}}^{\rho}(M(A \otimes H^0))$. Then the following conditions are equivalent:*

- (1) $[X_{\alpha}, \lambda_u] = [X_{\alpha}, \lambda_v]$ in $\text{Pic}_H^{\rho}(A)$,
- (2) *There is a unitary element $w \in M(A) \cap A'$ such that $u = (w^* \otimes 1^0)\underline{\rho}(w)v$.*

Proof. We suppose Condition (1). By Lemma 6.6 and Corollary 6.7, we can see that $[{}_A A_A, \lambda_{uv^*}] = [{}_A A_A, \rho]$ in $\text{Pic}_H^{\rho}(A)$. Thus by Lemma 6.9 there is a unitary element in $w \in M(A) \cap A'$ such that $uv^* = (w^* \otimes 1^0)\underline{\rho}(w)$. Hence we obtain Condition (2). Conversely we suppose Condition (2). Then there is a unitary element $w \in M(A) \cap A'$ such that $uv^* = (w^* \otimes 1^0)\underline{\rho}(w)$. Hence $[{}_A A_A, \lambda_{uv^*}] = [{}_A A_A, \rho]$ in $\text{Pic}_H^{\rho}(A)$. Since $[X_{\alpha}, \lambda_u][X_{\alpha}, \lambda_v]^{-1} = [{}_A A_A, \lambda_{uv^*}]$ in $\text{Pic}_H^{\rho}(A)$ by Lemma 6.6 and Corollary 6.7, $[X_{\alpha}, \lambda_u] = [X_{\alpha}, \lambda_v]$ in $\text{Pic}_H^{\rho}(A)$. \square

We shall compute $\text{Ker}f_{\rho}$, the kernel of f_{ρ} . Let $[X, \lambda] \in \text{Pic}_H^{\rho}(A)$. Then by Proposition 6.4, we can see that $[X] = [{}_A A_A]$ in $\text{Pic}(A)$ if and only if there is a unitary element $u \in \text{U}_{\text{id}}^{\rho}(M(A \otimes H^0))$ such that $[X, \lambda] = [{}_A A_A, \lambda_u]$ in $\text{Pic}_H^{\rho}(A)$. Furthermore by Corollary 6.10, $[{}_A A_A, \lambda_u] = [{}_A A_A, \lambda_v]$ in $\text{Pic}_H^{\rho}(A)$ if and only if there is a unitary element $w \in M(A) \cap A'$ such that $u = (w^* \otimes 1^0)\underline{\rho}(w)v$, where $u, v \in \text{U}_{\text{id}}^{\rho}(M(A \otimes H^0))$. We define an equivalence relation in $\text{U}_{\text{id}}^{\rho}(M(A \otimes H^0))$ as follows: Let $u, v \in \text{U}_{\text{id}}^{\rho}(M(A \otimes H^0))$, written $u \sim v$ if there is a unitary element $w \in M(A) \cap A'$ such that

$$u = (w^* \otimes 1^0)\underline{\rho}(w)v.$$

Let $\text{U}_{\text{id}}^{\rho}(M(A \otimes H^0))/\sim$ be the set of all equivalence classes in $\text{U}_{\text{id}}^{\rho}(M(A \otimes H^0))$. We denote by $[u]$ the equivalence class of $u \in \text{U}_{\text{id}}^{\rho}(M(A \otimes H^0))$. By Lemma 6.8, $\text{U}_{\text{id}}^{\rho}(M(A \otimes H^0))$ is a group. Hence $\text{U}_{\text{id}}^{\rho}(M(A \otimes H^0))/\sim$ is a group by easy computations.

Proposition 6.11. *With the above notations, $\text{Ker}f_{\rho} \cong \text{U}_{\text{id}}^{\rho}(M(A \otimes H^0))/\sim$ as groups.*

Proof. Let π be a map from $\text{U}_{\text{id}}^{\rho}(M(A \otimes H^0))/\sim$ to $\text{Ker}f_{\rho}$ defined by

$$\pi([u]) = [{}_A A_A, \lambda_u]$$

for any $u \in \text{U}_{\text{id}}^{\rho}(M(A \otimes H^0))$. By the above discussions, we can see that π is well-defined and bijective. For any $u, v \in \text{U}_{\text{id}}^{\rho}(M(A \otimes H^0))$

$$\pi([u])\pi([v]) = [{}_A A_A, \lambda_u][{}_A A_A, \lambda_v] = [{}_A A_A, \lambda_{uv}] = \pi([uv])$$

by Lemma 6.6. Therefore, we obtain the conclusion. \square

We recall that there is a homomorphism Φ of $\text{Aut}_H^{\rho^s}(A^s)$ to $\text{Pic}_H^{\rho^s}(A^s)$ defined by

$$\Phi(\alpha) = [X_{\alpha}, \lambda_{\alpha}]$$

for any $\alpha \in \text{Aut}_H^{\rho^s}(A^s)$, where λ_{α} is a coaction of H^0 on X_{α} induced by ρ^s (See Section 5). Then the following results hold:

Lemma 6.12. *With the above notations, for any $\alpha \in \text{Aut}_H^{\rho^s}(A^s)$*

$$(f_{\rho^s} \circ \Phi)(\alpha) = [X_\alpha]$$

in $\text{Pic}(A^s)$. Furthermore, if $\widehat{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ in $(A \rtimes_\rho H) \otimes H$, then

$$\text{Im} f_{\rho^s} = \{[X_\alpha] \in \text{Pic}(A^s) \mid \alpha \in \text{Aut}_H^{\rho^s}(A^s)\}.$$

Proof. This is immediate by easy computations. \square

Let G be a subgroup of $\text{Pic}(A^s)$ defined by

$$G = \{[X_\alpha] \in \text{Pic}(A^s) \mid \alpha \in \text{Aut}_H^{\rho^s}(A^s)\}.$$

Theorem 6.13. *Let H be a finite dimensional C^* -Hopf algebra with its dual C^* -algebra H^0 . Let A be a unital C^* -algebra and ρ a coaction of H^0 on A with $\widehat{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ in $(A \rtimes_\rho H) \otimes H$. Let $A^s = A \otimes \mathbf{K}$ and ρ^s the coaction of H^0 on A^s induced by ρ . Let $\text{U}_{\text{id}}^{\rho^s}(M(A^s \otimes H^0))$ be the group of all unitary elements $u \in M(A^s \otimes H^0)$ satisfying that*

$$\rho^s = \text{Ad}(u) \circ \rho^s, \quad (\underline{\rho}^s \otimes \text{id})(u)(u \otimes 1^0) = (\text{id} \otimes \Delta^0)(u).$$

Then we have the following exact sequence:

$$1 \longrightarrow \text{U}_{\text{id}}^{\rho^s}(M(A^s \otimes H^0)) / \sim \longrightarrow \text{Pic}_H^{\rho^s}(A^s) \longrightarrow G \longrightarrow 1,$$

where “ \sim ” is the equivalence relation in $\text{U}_{\text{id}}^{\rho^s}(M(A^s \otimes H^0))$ defined in this section.

Proof. This is immediate by Proposition 6.11 and Lemma 6.12. \square

Let A be a UHF-algebra of type N^∞ , where $N = \dim H$. Let ρ be the coaction of H^0 on A defined in [16, Section 7], which has the Rohlin property. We note that $\widehat{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ in $(A \rtimes_\rho H) \otimes H$ by [16, Definition 5.1].

Corollary 6.14. *With the above notations, we have the following exact sequence:*

$$1 \longrightarrow \text{U}_{\text{id}}^{\rho^s}(M(A^s \otimes H^0)) \longrightarrow \text{Pic}_H^{\rho^s}(A^s) \longrightarrow G \longrightarrow 1.$$

Proof. Since A^s is simple, $M(A^s) \cap (A^s)' = \mathbf{C}1$ by Pedersen [21, Corollary 4.4.8]. Therefore by Theorem 6.13, we obtain the conclusion. \square

7. EQUIVARIANT PICARD GROUPS AND CROSSED PRODUCTS

Let (ρ, u) be a twisted coaction of H^0 on a unital C^* -algebra A . Let f be a map from $\text{Pic}_H^{\rho, u}(A)$ to $\text{Pic}_{H^0}^{\widehat{\rho}}(A \rtimes_{\rho, u} H)$ defined by

$$f([X, \lambda]) = [X \rtimes_\lambda H, \widehat{\lambda}]$$

for any $[X, \lambda] \in \text{Pic}_H^{\rho, u}(A)$. In this section, we shall show that f is an isomorphism of $\text{Pic}_H^{\rho, u}(A)$ onto $\text{Pic}_{H^0}^{\widehat{\rho}}(A \rtimes_{\rho, u} H)$. We can see that f is well-defined in a straightforward way. We show that f is a homomorphism of $\text{Pic}_H^{\rho, u}(A)$ to $\text{Pic}_{H^0}^{\widehat{\rho}}(A \rtimes_{\rho, u} H)$. Let A, B and C be unital C^* -algebras and $(\rho, u), (\sigma, v)$ and (γ, w) be twisted coactions of H^0 on A, B and C , respectively. Let λ be a twisted coaction of H^0 on an A – B -equivalence bimodule X with respect to $(A, B, \rho, u, \sigma, v)$. Also, let μ be a twisted coaction of H^0 on a B – C -equivalence bimodule Y with respect to $(B, C, \sigma, v, \gamma, w)$. Let Φ be a linear map from $(X \otimes_B Y) \rtimes_{\lambda \otimes \mu} H$ to $(X \rtimes_\lambda H) \otimes_{B \rtimes_{\sigma, v} H} (Y \rtimes_\mu H)$ defined by

$$\Phi(x \otimes y \rtimes_{\lambda \otimes \mu} h) = (x \rtimes_\lambda 1) \otimes (y \rtimes_\mu h)$$

for any $x \in X, y \in Y$ and $h \in H$. By routine computations, Φ is well-defined. We note that $(X \rtimes_\lambda H) \otimes_{B \rtimes_{\sigma, v} H} (Y \rtimes_\mu H)$ is consisting of finite sums of elements in the form $(x \rtimes_\lambda 1) \otimes (y \rtimes_\mu h)$ by the definition of $(X \rtimes_\lambda H) \otimes_{B \rtimes_{\sigma, v} H} (Y \rtimes_\mu H)$, where

$x \in X, y \in Y$ and $h \in H$. Hence we can see that Φ is bijective and its inverse map Φ^{-1} is :

$$(X \rtimes_{\lambda} H) \otimes_{B \rtimes_{\sigma, u} H} (Y \rtimes_{\mu} H) \rightarrow (X \otimes_B Y) \rtimes_{\lambda \otimes \mu} H : (x \rtimes_{\lambda} 1) \otimes (y \rtimes_{\mu} h) \mapsto x \otimes y \rtimes_{\lambda \otimes \mu} h.$$

Furthermore, we have the following lemmas:

Lemma 7.1. *With the above notations,*

$$\begin{aligned} A \rtimes_{\rho, u} H \langle \Phi(x \otimes y \rtimes_{\lambda \otimes \mu} h), \Phi(z \otimes r \rtimes_{\lambda \otimes \mu} l) \rangle &= A \rtimes_{\rho, u} H \langle x \otimes y \rtimes_{\lambda \otimes \mu} h, z \otimes r \rtimes_{\lambda \otimes \mu} l \rangle, \\ \langle \Phi(x \otimes y \rtimes_{\lambda \otimes \mu} h), \Phi(z \otimes r \rtimes_{\lambda \otimes \mu} l) \rangle_{C \rtimes_{\gamma, w} H} &= \langle x \otimes y \rtimes_{\lambda \otimes \mu} h, z \otimes r \rtimes_{\lambda \otimes \mu} l \rangle_{C \rtimes_{\gamma, w} H} \end{aligned}$$

for any $x, z \in X, y, r \in Y$ and $h, l \in H$.

Proof. We can prove this lemma by routine computations. Indeed,

$$\begin{aligned} &A \rtimes_{\rho, u} H \langle \Phi(x \otimes y \rtimes_{\lambda \otimes \mu} h), \Phi(z \otimes r \rtimes_{\lambda \otimes \mu} l) \rangle \\ &= A \rtimes_{\rho, u} H \langle (x \rtimes_{\lambda} 1) \otimes (y \rtimes_{\mu} h), (z \rtimes_{\lambda} 1) \otimes (r \rtimes_{\mu} l) \rangle \\ &= A \rtimes_{\rho, u} H \langle (x \rtimes_{\lambda} 1)_{B \rtimes_{\sigma, y} H} \langle y \rtimes_{\mu} h, r \rtimes_{\mu} l \rangle, z \rtimes_{\lambda} 1 \rangle \\ &= A \rtimes_{\rho, u} H \langle (x \rtimes_{\lambda} 1)_{(B \langle y, [S(h_{(2)} l_{(3)}^*)^* \cdot \mu r] \widehat{w}(S(h_{(1)} l_{(2)}^*)^*, l_{(1)}))} \rtimes_{\sigma, v} h_{(3)} l_{(4)}^*)}, z \rtimes_{\lambda} 1 \rangle \\ &= A \rtimes_{\rho, u} H \langle x_B \langle y, [S(h_{(2)} l_{(3)}^*)^* \cdot \mu r] \widehat{w}(S(h_{(1)} l_{(2)}^*)^*, l_{(1)})) \rtimes_{\lambda} h_{(3)} l_{(4)}^*, z \rtimes_{\lambda} 1 \rangle \\ &= A \langle x_B \langle y, [S(h_{(2)} l_{(3)}^*)^* \cdot \mu r] \widehat{w}(S(h_{(1)} l_{(2)}^*)^*, l_{(1)})) \rangle, [S(h_{(3)} l_{(4)}^*)^* \cdot \lambda z] \rangle \rtimes_{\rho, u} h_{(4)} l_{(5)}^*. \end{aligned}$$

On the other hand,

$$\begin{aligned} &A \rtimes_{\rho, u} H \langle x \otimes y \rtimes_{\lambda \otimes \mu} h, z \otimes r \rtimes_{\lambda \otimes \mu} l \rangle \\ &= A \langle x \otimes y, [S(h_{(2)} l_{(3)}^*)^* \cdot \lambda \otimes \mu z \otimes r] \widehat{w}(S(h_{(1)} l_{(2)}^*)^*, l_{(1)}) \rangle \rtimes_{\rho, u} h_{(3)} l_{(4)}^* \\ &= A \langle x \otimes y, [S(h_{(3)} l_{(4)}^*)^* \cdot \lambda z] \otimes [S(h_{(2)} l_{(3)}^*)^* \cdot \mu r] \widehat{w}(S(h_{(1)} l_{(2)}^*)^*, l_{(1)}) \rangle \rtimes_{\rho, u} h_{(4)} l_{(5)}^* \\ &= A \langle x_B \langle y, [S(h_{(2)} l_{(3)}^*)^* \cdot \mu r] \widehat{w}(S(h_{(1)} l_{(2)}^*)^*, l_{(1)}) \rangle, [S(h_{(3)} l_{(4)}^*)^* \cdot \lambda z] \rangle \rtimes_{\rho, u} h_{(4)} l_{(5)}^*. \end{aligned}$$

Thus we obtain that

$$A \rtimes_{\rho, u} H \langle \Phi(x \otimes y \rtimes_{\lambda \otimes \mu} h), \Phi(z \otimes r \rtimes_{\lambda \otimes \mu} l) \rangle = A \rtimes_{\rho, u} H \langle x \otimes y \rtimes_{\lambda \otimes \mu} h, z \otimes r \rtimes_{\lambda \otimes \mu} l \rangle.$$

Similarly we obtain that

$$\langle \Phi(x \otimes y \rtimes_{\lambda \otimes \mu} h), \Phi(z \otimes r \rtimes_{\lambda \otimes \mu} l) \rangle_{C \rtimes_{\gamma, w} H} = \langle x \otimes y \rtimes_{\lambda \otimes \mu} h, z \otimes r \rtimes_{\lambda \otimes \mu} l \rangle_{C \rtimes_{\gamma, w} H}.$$

□

Lemma 7.2. *With the above notations, Φ is an $A \rtimes_{\rho, u} H - C \rtimes_{\sigma, w} H$ -equivalence bimodule isomorphism of $(X \otimes_B Y) \rtimes_{\lambda \otimes \mu} H$ onto $(X \rtimes_{\lambda} H) \otimes_{B \rtimes_{\sigma, v} H} (Y \rtimes_{\mu} H)$ satisfying that*

$$\Phi(\phi \cdot \widehat{\lambda \otimes \mu} (x \otimes y \rtimes_{\lambda \otimes \mu} h)) = \phi \cdot \widehat{\lambda \otimes \mu} \Phi(x \otimes y \rtimes_{\lambda \otimes \mu} h)$$

for any $x \in X, y \in Y, h \in H$ and $\phi \in H^0$.

Proof. By Lemma 7.1 and the remark after Jensen and Thomsen [12, Definition 1.1.18], we see that Φ is an $A \rtimes_{\rho, u} H - C \rtimes_{\sigma, w} H$ -equivalence bimodule isomorphism of $(X \otimes_B Y) \rtimes_{\lambda \otimes \mu} H$ onto $(X \rtimes_{\lambda} H) \otimes_{B \rtimes_{\sigma, v} H} (Y \rtimes_{\mu} H)$. Furthermore, for any $x \in X, y \in Y, h \in H$ and $\phi \in H^0$,

$$\begin{aligned} \Phi(\phi \cdot \widehat{\lambda \otimes \mu} (x \otimes y \rtimes_{\lambda \otimes \mu} h)) &= \Phi(x \otimes y \rtimes_{\lambda \otimes \mu} h_{(1)} \phi(h_{(2)})) \\ &= (x \rtimes_{\lambda} 1) \otimes (y \rtimes_{\mu} h_{(1)}) \phi(h_{(2)}) \\ &= [\phi_{(1)} \cdot \widehat{\lambda} (x \rtimes_{\lambda} 1)] \otimes [\phi_{(2)} \cdot \widehat{\mu} (y \rtimes_{\mu} h)] \\ &= \phi \cdot \widehat{\lambda \otimes \mu} \Phi(x \otimes y \rtimes_{\lambda \otimes \mu} h). \end{aligned}$$

Therefore, we obtain the conclusion. □

Corollary 7.3. *Let f be a map from $\text{Pic}_H^{\rho,u}(A)$ to $\text{Pic}_{H^0}^{\widehat{\rho}}(A \rtimes_{\rho,u} H)$ defined by $f([X, \lambda]) = [X \rtimes_{\lambda} H, \widehat{\lambda}]$ for any $[X, \lambda] \in \text{Pic}_H^{\rho,u}(A)$. Then f is a homomorphism of $\text{Pic}_H^{\rho,u}(A)$ to $\text{Pic}_{H^0}^{\widehat{\rho}}(A \rtimes_{\rho,u} H)$.*

Proof. This is immediate by Lemma 7.2. \square

Next, we construct the inverse homomorphism of f of $\text{Pic}_{H^0}^{\widehat{\rho}}(A \rtimes_{\rho} H)$ to $\text{Pic}_H^{\rho,u}(A)$. First, we note the following: Let (α, v) and (β, z) be twisted coactions of H^0 on unital C^* -algebras A and B , respectively. We suppose that there is an isomorphism Φ of B onto A such that $(\Phi \otimes \text{id}) \circ \beta = \alpha \circ \Phi$ and $v = (\Phi \otimes \text{id})(z)$. Let $(X, \lambda) \in \text{Equi}_H^{\alpha,v}(A)$. We construct an element $(X_{\Phi}, \lambda_{\Phi})$ in $\text{Equi}_H^{\beta,z}(B)$ from $(X, \lambda) \in \text{Equi}_H^{\alpha,v}(A)$ and Φ as follows: Let $X_{\Phi} = X$ as vector spaces. For any $x, y \in X_{\Phi}$ and $b \in B$,

$$\begin{aligned} b \cdot x &= \Phi(b)x, & x \cdot b &= x\Phi(b) \\ B\langle x, y \rangle &= \Phi^{-1}(A\langle x, y \rangle), & \langle x, y \rangle_B &= \Phi^{-1}(\langle x, y \rangle_A). \end{aligned}$$

We regard λ as a linear map from X_{Φ} to $X_{\Phi} \otimes H^0$. We denote it by λ_{Φ} . Then $(X_{\Phi}, \lambda_{\Phi})$ is an element in $\text{Equi}_H^{\beta,z}(B)$. By easy computations, the map

$$\text{Pic}_H^{\alpha,v}(A) \rightarrow \text{Pic}_H^{\beta,z}(B) : [X, \lambda] \mapsto [X_{\Phi}, \lambda_{\Phi}]$$

is well-defined and it is an isomorphism of $\text{Pic}_H^{\alpha,v}(A)$ onto $\text{Pic}_H^{\beta,z}(B)$. By Corollary 7.3, there is the homomorphism \widehat{f} of $\text{Pic}_{H^0}^{\widehat{\rho}}(A \rtimes_{\rho,u} H)$ to $\text{Pic}_H^{\widehat{\rho}}(A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0)$ defined by

$$\widehat{f}([Y, \mu]) = [Y \rtimes_{\mu} H^0, \widehat{\mu}]$$

for any $[Y, \mu] \in \text{Pic}_{H^0}^{\widehat{\rho}}(A \rtimes_{\rho,u} H)$. By Proposition 2.8, there are an isomorphism Ψ_A of $A \otimes M_N(\mathbf{C})$ onto $A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0$ and a unitary element $U \in (A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0) \otimes H^0$ such that

$$\begin{aligned} \text{Ad}(U) \circ \widehat{\rho} &= (\Psi_A \otimes \text{id}_{H^0}) \circ (\rho \otimes \text{id}_{M_N(\mathbf{C})}) \circ \Psi_A^{-1}, \\ (\Psi_A \otimes \text{id}_{H^0} \otimes \text{id}_{H^0})(u \otimes I_N) &= (U \otimes 1^0)(\widehat{\rho} \otimes \text{id}_{H^0})(U)(\text{id} \otimes \Delta^0)(U^*). \end{aligned}$$

Let $\overline{\rho} = (\Psi_A^{-1} \otimes \text{id}_{H^0}) \circ \widehat{\rho} \circ \Psi_A$. By the above discussions, there is the isomorphism g_1 of $\text{Pic}_H^{\widehat{\rho}}(A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0)$ onto $\text{Pic}_H^{\overline{\rho}}(A \otimes M_N(\mathbf{C}))$ defined by

$$g_1([X, \lambda]) = [X_{\Psi_A}, \lambda_{\Psi_A}]$$

for any $[X, \lambda] \in \text{Pic}_H^{\widehat{\rho}}(A \rtimes_{\rho,u} H \rtimes_{\widehat{\rho}} H^0)$. Furthermore, the coaction $\overline{\rho}$ of H^0 on $A \otimes M_N(\mathbf{C})$ is exterior equivalent to the twisted coaction $(\rho \otimes \text{id}, u \otimes I_N)$. Indeed,

$$\rho \otimes \text{id}_{M_N(\mathbf{C})} = (\Psi_A^{-1} \otimes \text{id}_{H^0}) \circ \text{Ad}(U) \circ \widehat{\rho} \circ \Psi_A = \text{Ad}(U_1) \circ \overline{\rho}.$$

Let $U_1 = (\Psi_A^{-1} \otimes \text{id}_{H^0})(U)$. Since $(\Psi_A^{-1} \otimes \text{id}_{H^0} \otimes \text{id}_{H^0}) \circ (\text{id} \otimes \Delta^0) = (\text{id} \otimes \Delta^0) \circ (\Psi_A^{-1} \otimes \text{id}_{H^0})$,

$$u \otimes I_N = (U_1 \otimes 1^0)(\overline{\rho} \otimes \text{id})(U_1)(\text{id} \otimes \Delta^0)(U_1^*).$$

We also note the following: We consider twisted coactions (α, v) and (β, z) of H^0 on a unital C^* -algebra A . We suppose that (α, v) and (β, z) are exterior equivalent. Then there is a unitary element w in $A \otimes H^0$ such that

$$\beta = \text{Ad}(w) \circ \alpha, \quad z = (w \otimes 1^0)(\rho \otimes \text{id})(w)v(\text{id} \otimes \Delta^0)(w^*)$$

By Lemmas 3.9 and 5.6 and their proofs, there is the isomorphism g_2 of $\text{Pic}_H^{\alpha,v}(A)$ onto $\text{Pic}_H^{\beta,z}(A)$ defined by $g_2([X, \lambda]) = [X, \text{Ad}(w) \circ \lambda]$ for any $[X, \lambda] \in \text{Pic}_H^{\alpha,v}(A)$, where $\text{Ad}(w) \circ \lambda$ means a linear map from X to $X \otimes H^0$ defined by $(\text{Ad}(w) \circ \lambda)(x) = w\lambda(x)w^*$ for any $x \in X$, which is a coaction of H^0 on $X \otimes H^0$ with

respect to $(A, A, \beta, z, \beta, z)$. Since $\bar{\rho}$ and $(\rho \otimes \text{id}, u \otimes I_N)$ are exterior equivalent, by the above discussions, there is the isomorphism g_2 of $\text{Pic}_H^{\bar{\rho}}(A \otimes M_N(\mathbf{C}))$ onto $\text{Pic}_H^{\rho \otimes \text{id}_{M_N(\mathbf{C})}, u \otimes I_N}(A \otimes M_N(\mathbf{C}))$ defined by

$$g_2([X, \lambda]) = [X, \text{Ad}(U_1) \circ \lambda]$$

for any $[X, \lambda] \in \text{Pic}_H^{\bar{\rho}}(A \otimes M_N(\mathbf{C}))$. By easy computations, (ρ, u) is strongly Morita equivalent to $(\rho \otimes \text{id}_{M_N(\mathbf{C})}, u \otimes I_N)$. Hence by Lemma 5.6 and its proof, there is the isomorphism g_3 of $\text{Pic}_H^{\rho, u}(A)$ onto $\text{Pic}_H^{\rho \otimes \text{id}_{M_N(\mathbf{C})}, u \otimes I_N}(A \otimes M_N(\mathbf{C}))$ defined by

$$g_3([X, \lambda]) = [X \otimes M_N(\mathbf{C}), \lambda \otimes \text{id}_{M_N(\mathbf{C})}]$$

for any $[X, \lambda] \in \text{Pic}_H^{\rho, u}(A)$. Let $g = g_3^{-1} \circ g_2 \circ g_1 \circ \hat{f}$. Then g is a homomorphism of $\text{Pic}_{H^0}^{\hat{\rho}}(A \rtimes_{\rho, u} H)$ to $\text{Pic}_H^{\rho, u}(A)$.

Proposition 7.4. *With the above notations, $g \circ f = \text{id}$ on $\text{Pic}_H^{\rho, u}(A)$.*

Proof. Let $[X, \lambda] \in \text{Pic}_H^{\rho, u}(A)$. By the definitions of f, \hat{f}, g_1 and g_2 ,

$$(g_2 \circ g_1 \circ \hat{f} \circ f)([X, \lambda]) = [(X \rtimes_{\lambda} H \rtimes_{\hat{\lambda}} H^0)_{\Psi_A}, \text{Ad}(U_1) \circ (\hat{\lambda})_{\Psi_A}].$$

Let Ψ_X be the linear map from $X \otimes M_N(\mathbf{C})$ to $X \rtimes_{\lambda} H \rtimes_{\hat{\lambda}} H^0$ defined in Proposition 3.6 and we regard Ψ_X as an $A \otimes M_N(\mathbf{C}) - A \otimes M_N(\mathbf{C})$ -equivalence bimodule isomorphism of $X \otimes M_N(\mathbf{C})$ onto $(X \rtimes_{\lambda} H \rtimes_{\hat{\lambda}} H^0)_{\Psi_A}$. Also, since $\text{Ad}(U) \circ \hat{\lambda} = (\Psi_X \otimes \text{id}) \circ (\lambda \otimes \text{id}) \circ \Psi_X^{-1}$ by Proposition 3.6, for any $x \in A \otimes M_N(\mathbf{C})$,

$$\begin{aligned} (\text{Ad}(U_1) \circ (\hat{\lambda})_{\Psi_A})(x) &= U_1 \cdot (\hat{\lambda})_{\Psi_A}(x) \cdot U_1^* = U \hat{\lambda}(x) U^* \\ &= ((\Psi_X \otimes \text{id}) \circ (\lambda \otimes \text{id}) \circ \Psi_X^{-1})(x). \end{aligned}$$

Thus

$$[(X \rtimes_{\lambda} H \rtimes_{\hat{\lambda}} H^0)_{\Psi_A}, \text{Ad}(U_1) \circ (\hat{\lambda})_{\Psi_A}] = [X \otimes M_N(\mathbf{C}), \lambda \otimes \text{id}]$$

in $\text{Pic}_H^{\rho \otimes \text{id}_{M_N(\mathbf{C})}, u \otimes I_N}(A \otimes M_N(\mathbf{C}))$. Since $g_3([X, \lambda]) = [X \otimes M_N(\mathbf{C}), \lambda \otimes \text{id}_{M_N(\mathbf{C})}]$, we obtain the conclusion. \square

Theorem 7.5. *Let (ρ, u) be a twisted coaction of H^0 on a unital C^* -algebra A . Then $\text{Pic}_H^{\rho, u}(A) \cong \text{Pic}_{H^0}^{\hat{\rho}}(A \rtimes_{\rho, u} H)$.*

Proof. Let f, \hat{f}, g_i , ($i = 1, 2, 3$) and g as in the proof of Proposition 7.4. By Proposition 7.4, $g \circ f = \text{id}$ on $\text{Pic}_H^{\rho, u}(A)$. Hence f is injective and g is surjective. Furthermore, we can see that \hat{f} is injective by Proposition 7.4. Since $g = g_3^{-1} \circ g_2 \circ g_1 \circ \hat{f}$ and g_i ($i = 1, 2, 3$) are bijective, g is injective. It follows that g is bijective. Therefore, f is an isomorphism of $\text{Pic}_H^{\rho, u}(A)$ onto $\text{Pic}_{H^0}^{\hat{\rho}}(A \rtimes_{\rho, u} H)$. \square

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