THE STRONG MORITA EQUIVALENCE FOR INCLUSIONS OF C^⊠ -ALGEBRAS AND CONDITIONAL EXPECTATIONS FOR EQUIVALENCE BIMODULES

THE STRONG MORITA EQUIVALENCE FOR INCLUSIONS OF C^*-ALGEBRAS AND CONDITIONAL EXPECTATIONS FOR EQUIVALENCE BIMODULES

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Abstract. We shall introduce the notions of the strong Morita equivalence for unital inclusions of unital C^*-algebras and conditional expectations from an equivalence bimodule onto its closed subspace with respect to conditional expectations from unital C^*-algebras onto their unital C^*-subalgebras. Also, we shall study their basic properties.

Keywords: inclusions of C^*-algebras, the strong Morita equivalence, equivalence bimodules, conditional expectations.

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1. Introduction

In the previous paper [16], following Jansen and Waldmann [9], we introduced the notion of the strong Morita equivalence for coactions of a finite dimensional C^*-Hopf algebra on unital C^*-algebras. Modifying this notion, we shall introduce the notion of the strong Morita equivalence for unital inclusions of unital C^*-algebras. Also, we shall introduce the notion of conditional expectations from an equivalence bimodule onto its closed subspace with respect to conditional expectations from unital C^*-algebras onto their unital C^*-subalgebras. Furthermore, we shall study their basic properties.

To specify, let A and B be unital C^*-algebras and H a finite dimensional C^*-Hopf algebra. Let H^0 be its dual C^*-Hopf algebra. Let ρ and σ be coactions of H^0 on A and B, respectively. Then we can obtain the unital inclusions $A \subset A \rtimes_\rho H$ and $B \subset B \rtimes_\sigma H$ and the canonical conditional expectations E_1^ρ and E_1^σ from $A \rtimes_\rho H$ and $B \rtimes_\sigma H$ onto A and B, respectively. We suppose that ρ and σ are strongly Morita equivalent. Then there are an $A-B$-equivalence bimodule X and a coaction λ of H^0 on X with respect to (A, B, ρ, σ). Let E^λ be the linear map from $X \rtimes_\lambda H$ onto X defined by

$$E_1^\lambda(x \rtimes_\lambda h) = \tau(h)x$$

for any $x \in X$, $h \in H$, where τ is the Haar trace on H.

In Section 2, we give the notion of the strong Morita equivalence for unital inclusions of unital C^*-algebras so that $A \subset A \rtimes_\rho H$ and $B \subset B \rtimes_\sigma H$ are strongly Morita equivalent. We also give the notion of conditional expectations from an equivalence bimodule onto its closed subspace with respect to conditional expectations from unital C^*-algebras onto their unital C^*-subalgebras so that E^λ is a conditional expectation from $X \rtimes_\lambda H$ onto X with respect to E^A and E^B.

In Sections 3, 4 and 5, we study the properties of conditional expectations from an equivalence bimodule onto its closed subspace with respect to conditional expectations from unital C^*-algebras onto their unital C^*-subalgebras. In Sections 6, 7 and 8, we give the upward and downward basic constructions for a conditional expectation from an equivalence bimodule onto its closed subspace and a duality result which are similar to the ordinary basic constructions for conditional expectations from unital C^*-algebras onto their unital C^*-subalgebras. Furthermore, in Section 9, we study a relationship between the upward basic construction and the downward basic construction for the conditional expectation from an equivalence bimodule onto its closed subspace. Finally In Section 10, we show that the strong Morita equivalence for unital inclusions of unital C^*-algebras preserves their paragroups.

Let A and B be C^*-algebras and X an $A-B$-bimodule. Then we denote its left A-action and right B-action on X by $a\cdot x$ and $x\cdot b$ for any $a\in A$, $b\in B$ and $x\in X$. For a C^*-algebra A, we denote by $M_n(A)$ the $n\times n$-matrix algebra over A and I_n denotes the unit element in $M_n(\mathbb{C})$. We identify $M_n(A)$ with $A\otimes M_n(\mathbb{C})$.

2. The strong Morita equivalence and basic properties

We begin this section with the following definition: Let A, B, C and D be C^*-algebras.

Definition 1. Inclusions of C^*-algebras $A\subset C$ and $B\subset D$ with $\overline{AC}=C$ and $\overline{BD}=D$ are *strongly Morita equivalent* if there are a $C-D$-equivalence bimodule Y and its closed subspace X satisfying the following conditions:
(1) $a\cdot x\in X$, ${}_C\langle x, y\rangle\in A$ for any $a\in A$, $x, y\in X$ and $\overline{{}_C\langle X, X\rangle}=A$, $\overline{{}_C\langle Y, X\rangle}=C$,
(2) $x\cdot b\in X$, $\langle x, y\rangle_B\in B$ for any $b\in B$, $x, y\in X$ and $\overline{\langle X, X\rangle_D}=B$, $\overline{\langle Y, X\rangle_D}=D$.
Then we say that the inclusion $A\subset C$ are strongly Morita equivalent to the inclusion $B\subset D$ with respect to the $C-D$-equivalent bimodule Y and its closed subspace X. We note that X can be regarded as an $A-B$-equivalence bimodule.

Remark 2.1. (1) If Y is a $C-D$-equivalence bimodule, $\overline{C\cdot Y}=\overline{Y\cdot D}=Y$ by Brown, Mingo and Shen [5, Proposition 1.7].
(2) If strongly Morita equivalent inclusions $A\subset C$ and $B\subset D$ are unital inclusions of unital C^*-algebras, we do not need to take the closure in Definition 1.

Proposition 2.2. *The strong Morita equivalence for inclusions of C^*-algebras is an equivalence relation.*

Proof. It suffices to show the transitivity since the other conditions clearly hold. Let $A\subset C$ and $B\subset D$ and $K\subset L$ be inclusions of C^*-algebras. We suppose that $A\subset C$ is strongly Morita equivalent to $B\subset D$ with respect to a $C-D$-equivalence bimodule Y and its closed subspace X and that $B\subset D$ is strongly Morita equivalent to $K\subset L$ with respect to a $D-L$-equivalence bimodule W and its closed subspace Z. We consider the closed subspace of $Y\otimes_D W$ spanned by the set

$$\{x\otimes z\in Y\otimes_D W\,|\, x\in X, \quad z\in Z\}.$$

We denote it by $X\otimes_D Z$. For any $x_1, x_2\in X$, $z_1, z_2\in Z$ and $a\in A$, $k\in K$,

$$\begin{aligned}
a\cdot(x_1\otimes z_1)&=(a\cdot x_1)\otimes z_1\in X\otimes_D Z,\\
(x_1\otimes z_1)\cdot k&=x_1\otimes(z_1\cdot k)\in X\otimes_D Z,\\
{}_C\langle x_1\otimes z_1\,, x_2\otimes z_2\rangle&={}_C\langle x_1\cdot{}_D\langle z_1,\, z_2\rangle,\, x_2\rangle={}_C\langle x_1\cdot{}_B\langle z_1,\, z_2\rangle,\, x_2\rangle\\
&={}_A\langle x_1\cdot{}_B\langle z_1,\, z_2\rangle,\, x_2\rangle\in A,\\
\langle x_1\otimes z_1\,, x_2\otimes z_2\rangle_L&=\langle z_1,\, \langle x_1,\, x_2\rangle_D\cdot z_2\rangle_L=\langle z_1,\, \langle x_1,\, x_2\rangle_B\cdot z_2\rangle_L\\
&=\langle z_1,\, \langle x_1,\, x_2\rangle_B\cdot z_2\rangle_K\in K.
\end{aligned}$$

Also, by Definition 1 and Remark 2.1,

$$
\begin{aligned}
\overline{{}_C\langle X\otimes_D Z,\, X\otimes_D Z\rangle} &= \overline{{}_C\langle X\cdot {}_B\langle Z,\, Z\rangle,\, X\rangle} = \overline{{}_A\langle X\cdot B,\, X\rangle} = \overline{{}_A\langle X,\, X\rangle} = A, \\
\overline{\langle X\otimes_D Z,\, X\otimes_D Z\rangle_L} &= \overline{\langle Z,\, \langle X,\, X\rangle_B\cdot Z\rangle_L} = \overline{\langle Z,\, B\cdot Z\rangle_K} = \overline{\langle Z,\, Z\rangle_K} = K, \\
\overline{{}_C\langle Y\otimes_D W,\, X\otimes_D Z\rangle} &= \overline{{}_C\langle Y\cdot {}_D\langle W,\, Z\rangle,\, X\rangle} = \overline{{}_C\langle Y\cdot D,\, X\rangle} = \overline{{}_C\langle Y,\, X\rangle} = C, \\
\overline{\langle Y\otimes_D W,\, X\otimes_D Z\rangle_L} &= \overline{\langle W,\, \langle Y,\, X\rangle_D\cdot Z\rangle_L} = \overline{\langle W,\, D\cdot Z\rangle_L} = \overline{\langle D\cdot W,\, Z\rangle_L} \\
&= \overline{\langle W,\, Z\rangle_L} = L.
\end{aligned}
$$

Therefore, we obtain the conclusion. □

Let $A\subset C$ and $B\subset D$ be unital inclusions of unital C^*-algebras which are strongly Morita equivalent with respect to a $C-D$-equivalence bimodule Y and its closed subspace X. Let E^A and E^B be conditional expectations from C and D onto A and B, respectively. Let E^X be a linear map from Y onto X.

Definition 2. With above notations, we say that E^X is a *conditional expectation* from Y onto X with respect to E^A and E^B if E^X satisfies the following conditions:
(1) $E^X(c\cdot x) = E^A(c)\cdot x$ for any $c\in C$, $x\in X$,
(2) $E^X(a\cdot y) = a\cdot E^X(y)$ for any $a\in A$, $y\in Y$,
(3) $E^A({}_C\langle y, x\rangle) = {}_C\langle E^X(y), x\rangle$ for any $x\in X$, $y\in Y$,
(4) $E^X(x\cdot d) = x\cdot E^B(d)$ for any $d\in D$ $x\in X$,
(5) $E^X(y\cdot b) = E^X(y)\cdot b$ for any $b\in B$, $y\in Y$,
(6) $E^B(\langle y, x\rangle_D) = \langle E^X(y), x\rangle_D$ for any $x\in X$, $y\in Y$.

By Definition 1, we can see that $E^A({}_C\langle y, x\rangle) = {}_A\langle E^X(y), x\rangle$ for any $x\in X$, $y\in Y$ and that $E^B(\langle y, x\rangle_D) = \langle E^X(y), x\rangle_B$ for any $x\in X$, $y\in Y$.

Let $A\subset C$ and $B\subset D$ be unital inclusions of unital C^*-algebras, which are strongly Morita equivalent with respect to a $C-D$-equivalence bimodule Y and its closed subspace X. By Kajiwara and Watatani [11, Lemma 1.7 and Corollary 1.28], there are elements $x_1, \dots, x_n\in X$ such that $\sum_{i=1}^n \langle x_i, x_i\rangle_B = 1$. We consider X^n as an $M_n(A)-B$-equivalence bimodule in the evident way and let $\overline{x} = (x_1, x_2, \dots, x_n)\in X^n$. Then $\langle\overline{x}, \overline{x}\rangle_B = 1$. Let $p = {}_{M_n(A)}\langle\overline{x}, \overline{x}\rangle$ and $z = {}_{M_n(A)}\langle\overline{x}, \overline{x}\rangle\cdot\overline{x}$. Also, let Ψ_B be the map from B to $M_n(A)$ defined by

$$\Psi_B(b) = {}_{M_n(A)}\langle z\cdot b,\, z\rangle = [{}_A\langle x_i b,\, x_j\rangle]_{ij=1}^n$$

for any $b\in B$. Then p is a full projection in $M_n(A)$, that is, $M_n(A)pM_n(A) = M_n(A)$ and Ψ_B is an isomorphism of B onto $pM_n(A)p$ by the proof of Rieffel [22, Proposition 2.1]. We repeat the above discussions for the $C-D$-equivalence bimodule Y in the following way: We note that

$$\sum_{i=1}^n \langle x_i\,, x_i\rangle_D = \sum_{i=1}^n \langle x_i,\, x_i\rangle_B = 1.$$

We consider Y^n as an $M_n(C)-D$-equivalence bimodule in the evident way. Then $\overline{x} = (x_1, \dots, x_n)\in Y^n$ and

$$
\begin{aligned}
p &= {}_{M_n(A)}\langle\overline{x},\, \overline{x}\rangle = {}_{M_n(C)}\langle\overline{x},\, \overline{x}\rangle \in M_n(C), \\
z &= {}_{M_n(A)}\langle\overline{x},\, \overline{x}\rangle\cdot\overline{x} = {}_{M_n(C)}\langle\overline{x},\, \overline{x}\rangle\cdot\overline{x}\in Y^n.
\end{aligned}
$$

Let Ψ_D be the map from D to $M_n(C)$ defined by

$$\Psi_D(d) = {}_{M_n(C)}\langle z\cdot d,\, z\rangle$$

for any $d\in D$. By the proof of [22, Proposition 2.1] p is a full projection in $M_n(C)$, that is, $M_n(C)pM_n(C) = M_n(C)$ and Ψ_D is an isomorphism of D onto $pM_n(C)p$.

Also, we see that $\Psi_B = \Psi_D|_B$ by the definitions of Ψ_B and Ψ_D. Let Ψ_X be the map from X to $M_n(A)$ defined by

$$\Psi_X(x) = \begin{bmatrix} {}_A\langle x, x_1\rangle & {}_A\langle x, x_2\rangle & \dots & {}_A\langle x, x_n\rangle \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{n\times n}$$

for any $x \in X$. Let $f = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{n\times n}$.

Lemma 2.3. *With the above notations, Ψ_X is a bijective linear map from X onto $(1 \otimes f)M_n(A)p$.*

Proof. It is clear that Ψ_X is linear and that $(1\otimes f)\Psi_X(x) = \Psi_X(x)$ for any $x \in X$. We note that $p = [{}_A\langle x_i, x_j\rangle]_{i,j=1}^n$. Then for any $x \in X$

$$\Psi_X(x)p = \begin{bmatrix} \sum_{i=1}^n {}_A\langle x, x_i\rangle\, {}_A\langle x_i, x_1\rangle & \dots & \sum_{i=1}^n {}_A\langle x, x_i\rangle\, {}_A\langle x_i, x_n\rangle \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}_{n\times n} .$$

Here for $j = 1, 2, \dots, n$

$$\sum_{i=1}^n {}_A\langle x, x_i\rangle\, {}_A\langle x_i, x_j\rangle = \sum_{i=1}^n {}_A\langle {}_A\langle x, x_i\rangle \cdot x_i, x_j\rangle = \sum_{i=1}^n {}_A\langle x \cdot \langle x_i, x_i\rangle_B, x_j\rangle = {}_A\langle x, x_j\rangle.$$

Thus we can see that $\Psi_X(x)p = \Psi_X(x)$ for any $x \in X$. Hence Ψ_X is the linear map from X to $(1 \otimes f)M_n(A)p$. Let $y \in (1 \otimes f)M_n(A)p$. Then we can write that

$$y = \begin{bmatrix} y_1 & \dots & y_n \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} p = \begin{bmatrix} \sum_{i=1}^n y_i\, {}_A\langle x_i, x_1\rangle & \dots & \sum_{i=1}^n y_i\, {}_A\langle x_i, x_n\rangle \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix},$$

where $y_1, \dots, y_n \in A$. Modifying Remark after [11, Lemma 1.11], let χ be the linear map from $(1 \otimes f)M_n(A)p$ to X defined by

$$\chi(y) = \sum_{ij=1}^n y_i\, {}_A\langle x_i, x_j\rangle \cdot x_j.$$

Then since $\sum_{j=1}^n \langle x_j, x_j\rangle_B = 1$,

$$\begin{aligned} &(\Psi_X \circ \chi)(y) \\ &= \begin{bmatrix} {}_A\langle \sum_{ij=1}^n y_i\, {}_A\langle x_i, x_j\rangle \cdot x_j, x_1\rangle & \dots & {}_A\langle \sum_{ij=1}^n y_i\, {}_A\langle x_i, x_j\rangle \cdot x_j, x_n\rangle \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \\ &= \begin{bmatrix} {}_A\langle \sum_{ij=1}^n y_i \cdot x_i \cdot \langle x_j, x_j\rangle_B, x_1\rangle & \dots & {}_A\langle \sum_{ij=1}^n y_i \cdot x_i \cdot \langle x_j, x_j\rangle_B, x_n\rangle \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \\ &= y. \end{aligned}$$

Also,

$$(\chi \circ \Psi_X)(x) = \sum_{ij=1}^{n} {}_A\langle x, x_i\rangle \, {}_A\langle x_i, x_j\rangle \cdot x_j = \sum_{ij=1}^{n} {}_A\langle x, x_i\rangle \cdot x_i \cdot \langle x_j, x_j\rangle_B$$
$$= \sum_{i=1}^{n} {}_A\langle x, x_i\rangle \cdot x_i = \sum_{i=1}^{n} x \cdot \langle x_i, x_i\rangle_B = x.$$

Thus we obtain the conclusion. □

Lemma 2.4. *With the above notations, Ψ_X satisfies the following:*
(1) $\Psi_X(a \cdot x) = a \cdot \Psi_X(x)$ for any $a \in A$, $x \in X$,
(2) $\Psi_X(x \cdot b) = \Psi_X(x) \cdot \Psi_B(b)$ for any $b \in B, x \in X$,
(3) ${}_A\langle \Psi_X(x), \Psi_X(y)\rangle = {}_A\langle x, y\rangle$ for any $x, y \in X$,
where we identify A with $(1 \otimes f)M_n(A)(1 \otimes f) = A \otimes f$,
(4) $\langle \Psi_X(x), \Psi_X(y)\rangle_{pM_n(A)p} = \Psi_B(\langle x, y\rangle_B)$ for any $x, y \in X$.

Proof. (1) Let $a \in A$ and $x \in X$. Then

$$\Psi_X(a \cdot x) = \begin{bmatrix} {}_A\langle a \cdot x, x_1\rangle & \dots & {}_A\langle a \cdot x, x_n\rangle \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} = a \cdot \Psi_X(x).$$

Hence we obtain (1).
(2) Let $b \in B$ and $x \in X$. Then

$$\Psi_X(x) \cdot \Psi_B(b) = \begin{bmatrix} {}_A\langle x, x_1\rangle & \dots & {}_A\langle x, x_n\rangle \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}_{n\times n} [{}_A\langle x_i \cdot b, x_j\rangle]_{ij=1}^{n}$$
$$= \begin{bmatrix} \sum_{i=1}^{n} {}_A\langle x, x_i\rangle \, {}_A\langle x_i \cdot b, x_1\rangle & \dots & \sum_{i=1}^{n} {}_A\langle x, x_i\rangle \, {}_A\langle x_i \cdot b, x_n\rangle \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}_{n\times n}.$$

Here for $j = 1, 2, \dots, n$,

$$\sum_{i=1}^{n} {}_A\langle x, x_i\rangle \, {}_A\langle x_i \cdot b, x_j\rangle = \sum_{i=1}^{n} {}_A\langle x \cdot \langle x_i, x_i\rangle_B b, x_j\rangle = {}_A\langle x \cdot b, x_j\rangle.$$

Thus we obtain (2).
(3) Let $x, y \in X$. Then since we identify A with $A \otimes f$,

$${}_A\langle \Psi_X(x), \Psi_X(y)\rangle = \sum_{i=1}^{n} {}_A\langle x, x_i\rangle \, {}_A\langle y, x_i\rangle^* = \sum_{i=1}^{n} {}_A\langle x, x_i\rangle \, {}_A\langle x_i, y\rangle$$
$$= \sum_{i=1}^{n} {}_A\langle \, {}_A\langle x, x_i\rangle \cdot x_i, y\rangle = \sum_{i=1}^{n} {}_A\langle x \cdot \langle x_i, x_i\rangle_B, y\rangle = {}_A\langle x, y\rangle.$$

Hence we obtain (3).
(4) Let $x, y \in X$. Then

$$\langle \Psi_X(x), \Psi_X(y)\rangle_{pM_n(A)p} = \Psi_X(x)^* \Psi_X(y) = [{}_A\langle x, x_i\rangle^* \, {}_A\langle y, x_j\rangle]_{ij=1}^{n}.$$

On the other hand,

$$\begin{aligned}\Psi_B(\langle x, y\rangle_B) &= [{}_A\langle x_i \cdot \langle x, y\rangle_B, x_j\rangle]_{ij=1}^n = [{}_A\langle {}_A\langle x_i, x\rangle \cdot y, x_j\rangle]_{ij}^n \\ &= [{}_A\langle x_i, x\rangle {}_A\langle y, x_j\rangle]_{ij=1}^n.\end{aligned}$$

Hence we obtain (4). □

Let Ψ_Y be the map from Y to $M_n(C)$ defined by

$$\Psi_Y(x) = \begin{bmatrix} {}_C\langle x, x_1\rangle & \dots & {}_C\langle x, x_n\rangle \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}_{n\times n}$$

for any $x \in Y$.

Corollary 2.5. *With the above notations, Ψ_Y is a bijective linear map from Y onto $(1\otimes f)M_n(C)p$ satisfying the following:*
(1) $\Psi_Y(c\cdot x) = c\cdot\Psi_Y(x)$ for any $c \in C$, $x \in Y$,
(2) $\Psi_Y(x\cdot d) = \Psi_Y(x)\cdot\Psi_D(d)$ for any $d \in D$, $x \in Y$,
(3) ${}_C\langle\Psi_Y(x), \Psi_Y(y)\rangle = {}_C\langle x, y\rangle$ for any $x, y \in Y$,
where we identify C with $(1\otimes f)M_n(C)(1\otimes f) = C\otimes f$,
(4) $\langle\Psi_Y(x), \Psi_Y(y)\rangle_{pM_n(C)p} = \Psi_D(\langle x, y\rangle_D)$ for any $x, y \in Y$,
(5) $\Psi_X = \Psi_Y|_X$.

Proof. It is clear that $\Psi_X = \Psi_Y|_X$ by the definitions of Ψ_X and Ψ_Y. By Lemmas 2.3 and 2.4, we obtain the others. □

Let $A \subset C$ and $B \subset D$ be unital inclusions of unital C^*-algebras. We suppose that $A \subset C$ and $B \subset D$ are strongly Morita equivalent with respect to a $C-D$-equivalence bimodule Y and its closed subspace X. Then by Lemmas 2.3, 2.4 and Corollary 2.5, we may assume that

$$B = pM_n(A)p, \quad D = pM_n(C)p, \quad Y = (1\otimes f)M_n(C)p, \quad X = (1\otimes f)M_n(A)p,$$

where p is a projection in $M_n(A)$ satisfying that $M_n(A)pM_n(A) = M_n(A)$, that is, p is a full in $M_n(A)$ and n is a positive integer. We regard X and Y as an $A - pM_n(A)p$-equivalence bimodule and a $C - pM_n(C)p$-equivalence bimodule in the usual way.

We consider the following: Let $A \subset C$ be a unital inclusion of unital C^*-algebras and p a full projection in $M_n(A)$. Then the inclusion $pM_n(A)p \subset pM_n(C)p$ is strongly Morita equivalent to $A \subset C$ with respect to the $C - pM_n(C)p$-equivalence bimodule $(1\otimes f)M_n(C)p$ and its closed subspace $(1\otimes f)M_n(A)p$. Let E^A be a conditional expectation of Watatani index-finite type from C onto A. We denote by $\mathrm{Ind}_W(E^A)$ the Watatani index of E^A. We note that $\mathrm{Ind}_W(E^A) \in C\cap C'$. Let $\{(u_i, u_i^*)\}_{i=1}^N$ be a quasi-basis for E^A. Then $\{(u_i\otimes I_n,\, u_i^*\otimes I_n)\}_{i=1}^N$ is a quasi-basis for $E^A\otimes\mathrm{id}$, the conditional expextation from $M_n(C)$ onto $M_n(A)$. Since p is a full projection in $M_n(A)$, there is elements $a_1, \dots, a_K$, $b_1, \cdots, b_K$ in $M_n(A)$ such that $\sum_{i=1}^K a_i p b_i = 1_{M_n(A)}$. Let E_p^A be the conditional expectation from $pM_n(C)p$ onto $pM_n(A)p$ defined by

$$E_p^A(x) = (E^A\otimes\mathrm{id})(x)$$

for any $x \in pM_n(A)p$. Then by routine computations, we can see that

$$\{(p(u_i\otimes I_n)a_j p,\, pb_j(u_i^*\otimes I_n)p)\}_{i=1,\dots,N,\, j=1,\dots,K}$$

is a quasi-basis for E_p^A. Furthermore,

$$\begin{aligned}\mathrm{Ind}_W(E_p^A) &= \sum_{i,j} p(u_i \otimes I_n)a_j p b_j(u_i^* \otimes I_n)p = \sum_i p(u_i u_i^* \otimes I_n)p \\ &= p(\mathrm{Ind}_W(E^A) \otimes I_n)p = (\mathrm{Ind}_W(E^A) \otimes I_n)p.\end{aligned}$$

Let F be the linear map from $(1 \otimes f)M_n(C)p$ onto $(1 \otimes f)M_n(A)p$ defined by

$$F((1 \otimes f)xp) = (E^A \otimes \mathrm{id})((1 \otimes f)xp) = (1 \otimes f)(E^A \otimes \mathrm{id})(x)p$$

for any $x \in M_n(C)$.

Lemma 2.6. *With the above notations, F is a conditional expectation from $(1 \otimes f)M_n(C)p$ onto $(1 \otimes f)M_n(A)p$ with respect to E^A and E_p^A.*

Proof. It suffices to show that F satisfies Conditions (1)-(6) in Definition 2.
(1) For any $c \in C$, $x \in M_n(A)$,

$$\begin{aligned}F(c \cdot (1 \otimes f)xp) &= F((c \otimes f)xp) = F((1 \otimes f)(c \otimes I_n)xp) \\ &= (1 \otimes f)(E^A \otimes \mathrm{id})((c \otimes I_n)x)p = (1 \otimes f)(E^A(c) \otimes I_n)xp \\ &= E^A(c) \cdot (1 \otimes f)xp.\end{aligned}$$

Thus we obtain Condition (1) in Definition 2.
(2) For any $a \in A$, $y \in M_n(C)$,

$$\begin{aligned}F(a \cdot (1 \otimes f)yp) &= F((1 \otimes f)(a \otimes I_n)yp) = (1 \otimes f)(E^A \otimes \mathrm{id})((a \otimes I_n)y)p \\ &= a \cdot (1 \otimes f)(E^A \otimes \mathrm{id})(y)p = a \cdot F((1 \otimes f)yp).\end{aligned}$$

Thus we obtain Condition (2) in Definition 2.
(3) For any $x \in M_n(A), y \in M_n(C)$,

$$\begin{aligned}{}_C\langle F((1 \otimes f)yp),\, (1 \otimes f)xp\rangle &= {}_C\langle (1 \otimes f)(E^A \otimes \mathrm{id})(y)p,\, (1 \otimes f)xp\rangle \\ &= (1 \otimes f)(E^A \otimes \mathrm{id})(y)px^*(1 \otimes f) \\ &= (E^A \otimes \mathrm{id})((1 \otimes f)ypx^*(1 \otimes f)) \\ &= (E^A \otimes \mathrm{id})({}_C\langle (1 \otimes f)yp,\, (1 \otimes f)xp\rangle)\end{aligned}$$

since we identify C with $(1 \otimes f)M_n(C)(1 \otimes f) = C \otimes f$. Thus we obtain Condition (3) in Definition 2.
(4) For any $y \in M_n(C)$, $x \in M_n(A)$,

$$\begin{aligned}F((1 \otimes f)xp \cdot pyp) &= F((1 \otimes f)xpyp) = (1 \otimes f)(E^A \otimes \mathrm{id})(xpy)p \\ &= (1 \otimes f)xp(E^A \otimes \mathrm{id})(y)p = (1 \otimes f)xp \cdot E_p^A(pyp).\end{aligned}$$

Thus we obtain Condition (4) in Definition 2.
(5) For any $x \in M_n(A)$, $y \in M_n(C)$,

$$\begin{aligned}F((1 \otimes f)yp \cdot pxp) &= F((1 \otimes f)ypxp) = (1 \otimes f)(E^A \otimes \mathrm{id})(ypx)p \\ &= (1 \otimes f)(E^A \otimes \mathrm{id})(y)p \cdot pxp = F((1 \otimes f)yp) \cdot pxp.\end{aligned}$$

Thus we obtain Condition (5) in Definition 2.
(6) For any $x \in M_n(A), y \in M_n(C)$,

$$\begin{aligned}\langle F((1 \otimes f)yp),\, (1 \otimes f)xp\rangle_{pM_n(C)p} &= p(E^A \otimes \mathrm{id})(y)^*(1 \otimes f)xp \\ &= p(E^A \otimes \mathrm{id})(y^*(1 \otimes f)x)p \\ &= E_p^A(\langle (1 \otimes f)yp,\, (1 \otimes f)xp\rangle_{pM_n(C)p}).\end{aligned}$$

Thus we obtain Condition (6) in Definition 2. Therefore, we obtain the conclusion. $\square$

Theorem 2.7. *Let $A \subset C$ and $B \subset D$ be unital inclusions of unital C^*-algebras, which are strongly Morita equivalent with respect to a $C-D$-equivalence bimodule Y and its closed subspace X. If there is a conditional expectation E^A of Watatani index-finite type from C onto A, then there are a conditional expectation E^B of Watatani index-finite type from D onto B and a conditional expectation E^X from Y onto X with respect to E^A and E^B. Also, if there is a conditional expectation E^B of Watatani index-finite type from D onto B, then we have the same result as above.*

Proof. This is immediate by Lemmas 2.3, 2.4, 2.6 and Corollary 2.5. □

3. One-sided conditional expectations on full Hilbert C^*-modules

Let $B \subset D$ be a unital inclusion of unital C^*-algebras and let Y be a full right Hilbert D-module and X its closed subspace satisfying the following:
(1) $x \cdot b \in X$, $\langle x, y\rangle_D \in B$ for any $b \in B$, $x, y \in X$,
(2) $\overline{\langle X, X\rangle_D} = B$, $\overline{\langle Y, X\rangle_D} = D$,
(3) There is a finite set $\{x_i\}_{i=1}^n \subset X$ such that for any $y \in Y$

$$\sum_{i=1}^n x_i \cdot \langle x_i, y\rangle_D = y.$$

We note that Y is of finite type and that X can be regarded as a full right Hilbert B-module of finite type in the sense of Kajiwara and Watatani [11]. Let $\mathbb{B}_D(Y)$ be the C^*-algebra of all right D-linear operators on Y for which has a right adjoint D-linear operator on Y. Let $C = \mathbb{B}_D(Y)$. For any $x, y \in Y$, let $\theta_{x,y}^Y$ be the rank-one operator on Y defined by

$$\theta_{x,y}^Y(z) = x \cdot \langle y, z\rangle_D$$

for any $z \in Y$. Then $\theta_{x,y}^Y$ is a right D-module operator. Hence $\theta_{x,y}^Y \in C$ for any $x, y \in Y$. Since D is unital, by [11, Lemma 1.7], C is the C^*-algebra of all linear spans of such $\theta_{x,y}^Y$. Let A_0 be the linear spans of the set $\{\theta_{x,y}^Y \,|\, x, y \in X\}$. By the assumptions, $\sum_{i=1}^n \theta_{x_i,x_i}^Y = 1_Y$. Hence A_0 is a $*$-algebra. Let A be the closure of A_0 in $\mathbb{B}_D(Y)$. Then A is a unital C^*-subalgebra of C. Let $\mathbb{B}_B(X)$ be the C^*-algebra defined in the same way as above. Let π be the map from $\mathbb{B}_B(X)$ to A defined by $\pi(\theta_{x,y}^X) = \theta_{x,y}^Y$, where $x, y \in X$ and $\theta_{x,y}^X$ is the rank-one operator on X defined as above. Then clearly π is injective and $\pi(\mathbb{B}_B(X)) = A_0$. Thus A_0 is closed and $A_0 = A$.

Lemma 3.1. *With the above notations and assumptions, the inclusion $A \subset C$ is unital and strongly Morita equivalent to the unital inclusion $B \subset D$ with respect to Y and its closed subspace X.*

Proof. By the above discussions, the inclusion $A \subset C$ is unital. Clearly A and B are strongly Morita equivalent with respect to X and C and D are strongly Morita equivalent with respect to Y. For any $x, y, z \in Y$,

$$\theta_{x,y}^Y(z) = x \cdot \langle y, z\rangle_D = x \cdot \langle \sum_{i=1}^n x_i \cdot \langle x_i, y\rangle_D \,,\, z\rangle_D = \sum_{i=1}^n x \cdot \langle y, x_i\rangle_D \, \langle x_i, z\rangle_D$$
$$= \sum_{i=1}^n \theta_{[x \cdot \langle y, x_i\rangle_D],\, x_i}^Y(z).$$

Since $x_i \in X$, $[x \cdot \langle y, x_i\rangle_D] \in Y$ for $i = 1, 2, \ldots, n$, $\theta_{x,y}^Y \in {}_C\langle Y, X\rangle$ for any $x, y \in Y$. Thus ${}_C\langle Y, X\rangle = C$. Therefore, $A \subset C$ is strongly Morita equivalent to $B \subset D$ with respect to a $C-D$-equivalence bimodule Y and its closed subspace X. □

Furthermore, we suppose that there is a conditional expectation E^B of Watatani index-finite type from D onto B.

Definition 3. Let E^X be a linear map from Y onto X. We say that E^X is a *right conditional expectation* from Y onto X with respect to E^B if E^X satisfies the following conditions:
(1) $E^X(x\cdot d)=x\cdot E^B(d)$ for any $d\in D$, $x\in X$,
(2) $E^X(y\cdot b)=E^X(y)\cdot b$ for any $b\in B$, $y\in Y$,
(3) $E^B(\langle y,x\rangle_D)=\langle E^X(y),x\rangle_D$ for any $x\in X$, $y\in Y$.

Remark 3.2. (i) By Definition 3, we can see that $E^B(\langle y,x\rangle_D)=\langle E^X(y),x\rangle_B$ for any $x\in X$, $y\in Y$.
(ii) E^X is a projection of norm one from Y onto X. Indeed, by Raeburn and William [21, the proof of Lemma 2.8], for any $y\in Y$,

$$\begin{aligned}||E^X(y)||&=\sup\{||\langle E^X(y),z\rangle_B||\mid ||z||\leq 1,\ z\in X\}\\&=\sup\{||E^B(\langle y,z\rangle_D)||\mid ||z||\leq 1, z\in X\}\\&\leq\sup\{||y||\,||z||\mid ||z||\leq 1,\ z\in X\}\\&=||y||.\end{aligned}$$

Since $E^X(x)=x$ for any $x\in X$, E^X is a projection of norm one from Y onto X.

Lemma 3.3. *With the same assumptions as in Lemma 3.1, we suppose that there is a conditional expectation E^B of Watatani index-finite type from D onto B. Then there is a right conditional expectation E^X from Y onto X with respect to E^B.*

Proof. Let E^X be the linear map from Y to X defined by

$$\langle E^X(y),x\rangle_B=E^B(\langle y,x\rangle_D)$$

for any $x\in X$, $y\in Y$. We show that Conditions (1), (2) in Definition 3 hold. Indeed, for any $x,y\in X$, $d\in D$,

$$\langle y,E^X(x\cdot d)\rangle_B=E^B(\langle y,x\cdot d\rangle_D)=E^B(\langle y,x\rangle_D d)=\langle y,x\rangle_B E^B(d)=\langle y,x\cdot E^B(d)\rangle_B.$$

Hence $E^X(x\cdot d)=x\cdot E^B(d)$ for any $x\in X$, $d\in D$. For any $b\in B$, $y\in Y$, $x\in X$,

$$\begin{aligned}\langle x,E^X(y\cdot b)\rangle_B&=E^B(\langle x,y\cdot b\rangle_D)=E^B(\langle x,y\rangle_D b)=E^B(\langle x,y\rangle_D)b\\&=\langle x,E^X(y)\rangle_B b=\langle x,E^X(y)\cdot b\rangle_B.\end{aligned}$$

Hence $E^X(y\cdot b)=E^X(y)\cdot b$ for any $y\in Y$, $b\in B$. □

Lemma 3.4. *Let $A\subset C$ and $B\subset D$ be unital inclusions of unital C^*-algebras, which are strongly Morita equivalent with respect to a $C-D$-equivalence bimodule Y and its closed subspace X. Let E^B be a conditional expectation of Watatani index-finite type from D onto B and E^X a right conditional expectation from Y onto X with respect to E^B. Then for any $a\in A$, $y\in Y$, $E^X(a\cdot y)=a\cdot E^X(y)$.*

Proof. Since X is full with the left A-valued inner product, it suffices to show that

$$E^X({}_A\langle x,z\rangle\cdot y)={}_A\langle x,z\rangle\cdot E^X(y)$$

for any $x,z\in X$, $y\in Y$. Indeed,

$$\begin{aligned}E^X({}_A\langle x,z\rangle\cdot y)&=E^X(x\cdot\langle z,y\rangle_D)=x\cdot E^B(\langle z,y\rangle_D)=x\cdot\langle z,E^X(y)\rangle_B\\&={}_A\langle x,z\rangle\cdot E^X(y).\end{aligned}$$

□

Proposition 3.5. *With the same assumptions as in Lemma 3.4, there is a conditional expectation E^A from C onto A such that E^X is a conditional expectation from Y onto X with respect to E^A and E^B.*

Proof. Let E^A be the linear map from C onto A defined by

$$E^A(c)\cdot x = E^X(c\cdot x)$$

for any $c\in C$, $x\in X$. First, we note that Conditions in Definition 2 except Condition (3) hold by the assumptions and Lemma 3.4. We show that Condition (3) in Definition 2 holds. Indeed fot any $x, z\in X$, $y\in Y$,

$$E^A({}_C\langle y, x\rangle)\cdot z = E^X({}_C\langle y, x\rangle\cdot z) = E^X(y\cdot\langle x, z\rangle_B) = E^X(y)\cdot\langle x, z\rangle_B = {}_C\langle E^X(y), x\rangle\cdot z.$$

Hence for any $x\in X$, $y\in Y$, $E^A({}_C\langle y,\, x\rangle) = {}_C\langle E^X(y),\, x\rangle$. Next, we show that E^A is a conditional expectation from C onto A. For any $a\in A$, $x\in X$,

$$E^A(a)\cdot x = E^X(a\cdot x) = a\cdot E^X(x) = a\cdot x$$

by Lemma 3.4. Hence $E^A(a) = a$ for any $a\in A$. For any $c\in C$, $x\in X$,

$$||E^A(c)\cdot x|| = ||E^X(c\cdot x)|| \le ||c\cdot x|| \le ||c||\,||x||$$

by Remark 3.2 (ii). Hence $||E^A|| = 1$ since $E^A(a) = a$ for any $a\in A$. Thus E^A is a projection of norm one from C onto A. It follows by Tomiyama [25, Theorem 1] that E^A is a conditional expectation from C onto A. Therefore, we obtain the conclusion. □

Let $B\subset D$ be a unital inclusion of unital C^*-algebras and let Y be a full right Hilbert D-module and X its closed subspace satisfying Conditions (1)-(3) in the beginning of this section. We suppose that there is a conditional expectation E^B of Watatani index-finite type from D onto B. Let $C = \mathbb{B}_D(Y)$ and let A be the C^*-subalgebra, the linear spans of the set $\{\theta^Y_{x,y} \,|\, x, y\in X\}$. Then by Lemmas 3.1, 3.3, 3.4 and Proposition 3.5, there are a conditional expectation E^X from Y onto X and a conditional expectation E^A from C onto A such that E^X is a conditional expectation from Y onto X with respect to E^A and E^B. We note that a conditional expectation E^A is depend only on E^B and E^X by Condition (3) in Definition 2. Hence by Theorem 2.7, E^A is of Watatani index-finite type. Thus we obtain the following corollary:

Corollary 3.6. *With the same notations as in Proposition 3.5, a conditional expectation E^A from C onto A defined in Proposition 3.5 is of Watatani index-finite type.*

Combining the above results, we obtain the following:

Theorem 3.7. *Let $B\subset D$ be a unital inclusion of unital C^*-algebras and let Y be a full right Hilbert D-module and X its closed subspace satisfying Conditions (1)-(3) in the beginning of this section. Let E^B be a conditional expectation of Watatani index-finite type from D onto B. Let $C = \mathbb{B}_D(Y)$ and let A be the C^*-subalgebra, the linear spans of the set $\{\theta^Y_{x,y} \,|\, x, y\in X\}$. Then there are a conditional expectation E^A of Watatani index-finite type from C onto A and a conditional expectation E^X from Y onto X with respect to E^A and E^B.*

Remark 3.8. (i) In the same way as in Definition 3, we can define a left conditional expectation in the following situation: Let $A\subset C$ be a unital inclusion of unital C^*-algebras and let Y be a full left Hilbert C-module and X its closed subspace satisfying that
(1) $a\cdot x\in X$, ${}_C\langle x, y\rangle\in A$ for any $a\in A$, $x, y\in X$,
(2) $\overline{{}_C\langle X, X\rangle} = A$, $\overline{{}_C\langle Y, X\rangle} = C$,
(3) There is a finite set $\{x_i\}_{i=1}^n\subset Y$ such that for any $y\in Y$

$$\sum_{i=1}^n {}_C\langle y, x_i\rangle\cdot x_i = y.$$

We note that Y is of finite type and that X can be regarded as a full left Hilbert A-module of finite type in the sense of Kajiwara and Watatani [11].
(ii) A conditional expectation from an equivalence onto its closed subspace in Definition 2 is a left and right conditional expectation.
(iii) We have the results on a left conditional expectation similar to the above.

4. Examples

In this section, we shall give two examples of conditional expectations from equivalence bimodules onto their closed subspaces.

First, let A and B be unital C^*-algebras which are strongly Morita equivalent with respect to an $A-B$-equivalence bimodule X. Let H be a finite dimensional C^*-Hopf algebra with its dual C^*-Hopf algebra H^0. Let ρ and σ be coactions of H^0 on A and B, respectively. We suppose that ρ and σ are strongly Morita equivalent with respect to a coaction λ of H^0 on X, respectively, that is, $(A, B, X, \rho, \sigma, \lambda, H^0)$ is a covariant system (See [16]). We use the same notations as in [16]. Let

$$C = A \rtimes_\rho H, \quad D = B \rtimes_\sigma H$$

be crossed products of C^*-algebras A and B by the actions of the finite dimensional C^*-Hopf algebra H induced by ρ and σ, respectively. Also, let $Y = X \rtimes_\lambda H$ be the crossed product of an $A-B$-equivalence bimodule X by the action of H induced by λ. Then by [16, Corollary 4.7], Y is a $C-D$-equivalence bimodule and C and D are strongly Morita equivalent with respect to Y. We can see that the unital inclusion $A \subset C$ and $B \subset D$ are strongly Morita equivalent with respect to Y and its closed subspace X by easy computations. Indeed, it suffices to show that ${}_C\langle X, Y\rangle = C$ and $\langle X, Y\rangle_D = D$ since the other conditions in Definition 1 clearly hold. For any $x, y \in X$, $h \in H$,

$$\begin{aligned} {}_C\langle x \rtimes_\lambda 1\,, (1 \rtimes_\rho h)^*(y \rtimes_\lambda 1)\rangle &= ((1 \rtimes_\rho h)^*\,{}_C\langle y \rtimes_\lambda 1, x \rtimes_\rho 1\rangle)^* \\ &= {}_C\langle x \rtimes_\lambda 1,\, y \rtimes_\lambda 1\rangle(1 \rtimes_\rho h) = {}_A\langle x, y\rangle \rtimes_\rho h. \end{aligned}$$

Hence ${}_C\langle X,\, Y\rangle = C$. Also,

$$\langle x \rtimes_\lambda 1\,, y \rtimes_\lambda h\rangle_D = \langle x, y\rangle_B \rtimes_\sigma h.$$

Thus $\langle X, Y\rangle_D = D$.

Let E_1^ρ and E_1^σ be the canonical conditional expectations from $A \rtimes_\rho H$ and $B \rtimes_\sigma H$ onto A and B defined by

$$E_1^\rho(a \rtimes_\rho h) = \tau(h)a, \quad E_1^\sigma(b \rtimes_\sigma h) = \tau(h)b$$

for any $a \in A$, $b \in B$, $h \in H$, respectively, where τ is the Haar trace on H. Let E_1^λ be the linear map from $X \rtimes_\lambda H$ onto X defined by

$$E_1^\lambda(x \rtimes_\lambda h) = \tau(h)x$$

for any $x \in X$, $h \in H$.

Proposition 4.1. *With the above notations, E_1^λ is a conditional expectation from $X \rtimes_\lambda H$ onto X with respect to E^A and E^B.*

Proof. Let X, Y and E_1^λ be as above. We claim that E_1^ρ, E_1^σ and E_1^λ satisfy Conditions (1)-(6) in Definition 2. Indeed, we compute the following:
(1) For any $a \in A$, $x \in X$, $h \in H$,

$$\begin{aligned} E_1^\lambda((a \rtimes_\rho h) \cdot (x \rtimes_\lambda 1)) &= E_1^\lambda(a \cdot [h_{(1)} \cdot_\lambda x] \rtimes_\lambda h_{(2)}) \\ &= a \cdot x\tau(h) \rtimes_\lambda 1 = E_1^\rho(a \rtimes_\rho h) \cdot (x \rtimes_\lambda 1). \end{aligned}$$

(2) For any $a \in A$, $x \in X$, $h \in H$,

$$E_1^\lambda((a \rtimes_\rho 1) \cdot (x \rtimes_\lambda h)) = E_1^\lambda(a \cdot x \rtimes_\lambda h) = \tau(h)a \cdot x \rtimes_\lambda 1 = (a \rtimes_\rho 1) \cdot E_1^\lambda(x \rtimes_\lambda h).$$

(3) For any $x, y \in X$, $h \in H$,

$$\begin{aligned}
E_1^{\rho}({}_C\langle y \rtimes_\lambda h,\, x \rtimes_\lambda 1\rangle) &= E_1^{\rho}({}_A\langle y,\, [S(h_{(1)})^* \cdot_\lambda x]\rangle \rtimes_\rho h_{(2)}) \\
&= {}_A\langle y,\, [S(h_{(1)})^* \cdot_\lambda x]\rangle \tau(h_{(2)}) \\
&= {}_A\langle y,\, \overline{\tau(h)}x\rangle = {}_A\langle E_1^{\lambda}(y \rtimes_\lambda h),\, x\rangle.
\end{aligned}$$

(4) For any $b \in B$, $x \in X$, $h \in H$,

$$E_1^{\lambda}((x \rtimes_\lambda 1) \cdot (b \rtimes_\sigma h)) = E_1^{\lambda}(x \cdot b \rtimes_\lambda h) = \tau(h)(x \cdot b \rtimes_\lambda 1) = (x \rtimes_\lambda 1) \cdot E_1^{\sigma}(b \rtimes_\sigma h).$$

(5) For any $b \in B$, $x \in X$, $h \in H$,

$$\begin{aligned}
E_1^{\lambda}((x \rtimes_\lambda h) \cdot (b \rtimes_\sigma 1)) &= E_1^{\lambda}(x \cdot [h_{(1)} \cdot_\sigma b] \rtimes_\lambda h_{(2)}) = x \cdot b\tau(h) \rtimes_\lambda 1 \\
&= E_1^{\lambda}(x \rtimes_\lambda h) \cdot (b \rtimes_\sigma 1).
\end{aligned}$$

(6) For any $x, y \in X$, $h \in H$,

$$\begin{aligned}
E_1^{\sigma}(\langle y \rtimes_\lambda h,\, x \rtimes_\lambda 1\rangle_D) &= E_1^{\sigma}([h_{(1)}^* \cdot_\sigma \langle y,\, x\rangle_B] \rtimes_\sigma h_{(2)}^*) \\
&= \tau(h^*)\langle y,\, x\rangle_B = \langle E_1^{\lambda}(y \rtimes_\lambda h),\, x \rtimes_\lambda 1\rangle_B.
\end{aligned}$$

Therefore, we obtain the conclusion. □

We shall give another example. Let $A \subset B$ be a unital inclusion of unital C^*-algebras and let F be a conditional expectation of Watatani index-finite type from B onto A. Let f be the Jones projection and B_1 the C^*-basic construction for F. Let F_1 be its dual conditional expectation from B_1 onto B. Let f_1 be the Jones projection and B_2 the C^*-basic construction for F_1. Let F_2 be the dual conditional expectation of F_1 from B_2 onto B_1. Then A is strongly Morita equivalent to B_1 and B is strongly Morita equivalent to B_2 by Watatani [26]. Since F and F_1 are of Watatani index-finite type, B and B_1 can be equivalence bimodules, that is, B can be regarded as a $B_1 - A$-equivalence bimodule as follows: For any $a \in A$, $x, y, z \in B$,

$${}_{B_1}\langle x,\, y\rangle = xfy^*, \quad \langle x,\, y\rangle_A = F(x^*y), \qquad xfy \cdot z = xF(yz), \quad x \cdot a = xa.$$

Also, B_1 can be regarded as a $B_2 - B$-equivalence bimodule as follows: For any $b \in B$, $x, y, z \in B_1$,

$${}_{B_2}\langle x,\, y\rangle = xf_1y^*, \quad \langle x,\, y\rangle_B = F_1(x^*y), \qquad xf_1y \cdot z = xF_1(yz), \quad x \cdot b = xb.$$

We denote by $\mathrm{Ind}_W(F)$ the Watatani index of a conditional expectation F from B onto A. Also, let $\{(w_i,\, w_i^*)\}_{i=1}^n$ be a quasi-basis for F_1.

Lemma 4.2. *With the above notations, we suppose that* $\mathrm{Ind}_W(F) \in A$. *Then the inclusions* $A \subset B$ *and* $B_1 \subset B_2$ *are strongly Morita equivalent.*

Proof. Let θ be the linear map from B to B_1 defined by

$$\theta(x) = \mathrm{Ind}_W(F)^{\frac{1}{2}}xf$$

for any $x \in B$. Then for any $a \in A$, $x, y, z \in B$,

$$\theta(xfy \cdot z \cdot a) = \theta(xF(yz)a) = \mathrm{Ind}_W(F)^{\frac{1}{2}}xF(yz)af = \mathrm{Ind}_W(F)^{\frac{1}{2}}xF(yz)fa.$$

On the other hand, since $\mathrm{Ind}_W(F) \in A \cap B'$,

$$\begin{aligned}
xfy \cdot \theta(z) \cdot a &= xfy \cdot \mathrm{Ind}_W(F)^{\frac{1}{2}}zf \cdot a = \sum_{i=1}^n xfyw_if_1w_i^* \cdot \mathrm{Ind}_W(F)^{\frac{1}{2}}zf \cdot a \\
&= xfy\mathrm{Ind}_W(F)^{\frac{1}{2}}zfa = xF(y\mathrm{Ind}_W(F)^{\frac{1}{2}}z)fa = \mathrm{Ind}_W(F)^{\frac{1}{2}}xF(yz)fa.
\end{aligned}$$

Thus θ is a $B_1 - A$-bimodule map. Furthermore, for any $x, y \in B$,

$$\begin{aligned}\langle \theta(x), \theta(y) \rangle_B &= F_1(\theta(x)^* \theta(y)) = F_1((\mathrm{Ind}_W(F)^{\frac{1}{2}} xf)^* (\mathrm{Ind}_W(F)^{\frac{1}{2}} yf)) \\ &= \mathrm{Ind}_W(F) F_1(fx^* yf) = \mathrm{Ind}_W(F) F_1(F(x^* y) f) = F(x^* y) \\ &= \langle x, y \rangle_A, \\ {}_{B_2}\langle \theta(x), \theta(y) \rangle &= \theta(x) f_1 \theta(y)^* = \mathrm{Ind}_W(F) xff_1 fy^* = xfy^* = {}_{B_1}\langle x, y \rangle\end{aligned}$$

by [26, Lemma 2.3.5]. Thus we regard B as a closed subspace of the $B_2 - B$-equivalence bimodule B_1 by the map θ. In order to obtain the conclusion, it suffices to show that ${}_{B_2}\langle B, B_1 \rangle = B_2$ and $\langle B, B_1 \rangle_B = B$ since the other conditions in Definition 1 clearly hold. Let $x, y, z \in B$. Then

$${}_{B_2}\langle x, yfz \rangle = {}_{B_2}\langle \theta(x), yfz \rangle = {}_{B_2}\langle \mathrm{Ind}_W(F)^{\frac{1}{2}} xf, yfz \rangle = \mathrm{Ind}_W(F)^{\frac{1}{2}} xff_1 z^* fy.$$

Since $f_1 z^* = z^* f_1$, ${}_{B_2}\langle B, B_1 \rangle = B_2$. Also,

$$\begin{aligned}\langle x, yfz \rangle_B &= \langle \theta(x), yfz \rangle_B = \langle \mathrm{Ind}_W(F)^{\frac{1}{2}} xf, yfz \rangle_B = F_1(\mathrm{Ind}_W(F)^{\frac{1}{2}} fx^* yfz) \\ &= F_1(\mathrm{Ind}_W(F)^{\frac{1}{2}} F(x^* y) fz) = \mathrm{Ind}_W(F)^{-\frac{1}{2}} F(x^* y) z.\end{aligned}$$

Hence $\langle B, B_1 \rangle_B = B$. Therefore, we obtain the conclusion. □

Proposition 4.3. *With the above notations, we regard B as a closed subspace of B_2 by the linear map θ defined in Lemma 4.2 and we suppose that $\mathrm{Ind}_W(F) \in A$. Then there is a conditional expectation G from B_1 onto B with respect to F and F_2.*

Proof. Let G be the linear map from B_1 onto B defined by

$$G(xfy) = xF(y)f = \theta(\mathrm{Ind}_W(F)^{-\frac{1}{2}} xF(y))$$

for any $x, y \in B$, where we identify $\theta(\mathrm{Ind}_W(F)^{-\frac{1}{2}} xF(y))$ with $\mathrm{Ind}_W(F)^{-\frac{1}{2}} xF(y)$. By routine computations, we can see that G satisfies Conditions (1)-(6) in Definition 2. Indeed, we compute the following:
(1) For any $x_1 = afb$, $y_1 = a_1 fb_1 \in B_1$, $a, b, a_1, b_1 \in B$ and $z \in B$,

$$\begin{aligned}G(x_1 f_1 y_1 \cdot \theta(z)) &= G(x_1 f_1 y_1 \cdot \mathrm{Ind}_W(F)^{\frac{1}{2}} zf) = G(x_1 F_1(y_1 \mathrm{Ind}_W(F)^{\frac{1}{2}} zf)) \\ &= G(afbF_1(a_1 fb_1 \mathrm{Ind}_W(F)^{\frac{1}{2}} zf)) \\ &= G(\mathrm{Ind}_W(F)^{\frac{1}{2}} afbF_1(a_1 F(b_1 z) f)) \\ &= \mathrm{Ind}_W(F)^{-\frac{1}{2}} aF(ba_1 F(b_1 z)) f \\ &= \mathrm{Ind}_W(F)^{-\frac{1}{2}} aF(ba_1) F(b_1 z) f.\end{aligned}$$

On the other hand,

$$\begin{aligned}F_2(x_1 f_1 y_1) \cdot z &= \mathrm{Ind}_W(F)^{-1} x_1 y_1 \cdot z = \mathrm{Ind}_W(F)^{-1} afba_1 fb_1 \cdot z \\ &= \mathrm{Ind}_W(F)^{-1} aF(ba_1) fb_1 \cdot z = \mathrm{Ind}_W(F)^{-1} aF(ba_1) F(b_1 z).\end{aligned}$$

Since we identify $\theta(\mathrm{Ind}_W(F)^{-1} aF(ba_1) F(b_1 z))$ with $\mathrm{Ind}_W(F)^{-\frac{1}{2}} aF(ba_1) F(b_1 z) f$, we can see that G satisfies Condition (1) in Definition 2.
(2) For any $a, b, x, y \in B$,

$$G(afb \cdot xfy) = G(afbxfy) = G(aF(bx) fy) = \theta(\mathrm{Ind}_W(F)^{-\frac{1}{2}} aF(bx) F(y)).$$

On the other hand,

$$\begin{aligned}afb \cdot G(xfy) &= afb \cdot \mathrm{Ind}_W(F)^{-\frac{1}{2}} xF(y) = aF(b \mathrm{Ind}_W(F)^{-\frac{1}{2}} xF(y)) \\ &= \mathrm{Ind}_W(F)^{-\frac{1}{2}} aF(bx) F(y).\end{aligned}$$

Thus G satisfies Condition (2) in Definition 2.
(3) For any $x, y, z \in B$,

$$ {}_{B_2}\langle G(xfy),\, \theta(z)\rangle = {}_{B_2}\langle xF(y)f,\, \mathrm{Ind}_W(F)^{\frac{1}{2}}zf\rangle = \mathrm{Ind}_W(F)^{-\frac{1}{2}}xF(y)fz^*. $$

On the other hand,

$$ \begin{aligned} F_2({}_{B_2}\langle xfy,\, \theta(z)\rangle) &= F_2({}_{B_2}\langle xfy,\, \mathrm{Ind}_W(F)^{\frac{1}{2}}zf\rangle) = F_2(xfyf_1fz^*\mathrm{Ind}_W(F)^{\frac{1}{2}}) \\ &= \mathrm{Ind}_W(F)^{-\frac{1}{2}}xfyfz^* = \mathrm{Ind}_W(F)^{-\frac{1}{2}}xF(y)fz^*. \end{aligned} $$

Thus G satisfies Condition (3) in Definition 2.
(4) For any $b, z \in B$,

$$ G(\theta(z)\cdot b) = G(\mathrm{Ind}_W(F)^{\frac{1}{2}}zf\cdot b) = G(\mathrm{Ind}_W(F)^{\frac{1}{2}}zfb) = \mathrm{Ind}_W(F)^{\frac{1}{2}}zF(b)f. $$

On the other hand,

$$ \theta(z)\cdot F(b) = \mathrm{Ind}_W(F)^{\frac{1}{2}}zfF(b) = \mathrm{Ind}_W(F)^{\frac{1}{2}}zF(b)f. $$

Thus G satisfies Condition (3) in Definition 2.
(5) For any $a \in A$, $x, y \in B$,

$$ G(a\cdot xfy) = G(axfy) = axF(y)f = a\cdot G(xfy). $$

Thus G satisfies Condition (5) in Definition 2.
(6) For any $x, y, z \in B$,

$$ \begin{aligned} F(\langle xfy,\, \theta(z)\rangle_B) &= F(F_1(y^*fx^*\mathrm{Ind}_W(F)^{\frac{1}{2}}zf)) = F(F_1(y^*F(x^*z)\mathrm{Ind}_W(F)^{\frac{1}{2}}f)) \\ &= \mathrm{Ind}_W(F)^{-\frac{1}{2}}F(y^*F(x^*z)) = \mathrm{Ind}_W(F)^{-\frac{1}{2}}F(y^*)F(x^*z). \end{aligned} $$

On the other hand,

$$ \langle G(xfy),\, \theta(z)\rangle_B = \langle xF(y)f,\, \mathrm{Ind}_W(F)^{\frac{1}{2}}zf\rangle_B = \mathrm{Ind}_W(F)^{-\frac{1}{2}}F(y^*)F(x^*z). $$

Thus G satisfies Condition (6) in Definition 2. Therefore, we obtain the conclusion. $\square$

5. Linking algebras and conditional expectations

Let $A \subset C$ and $B \subset D$ be unital inclusions of unital C^*-algebras, which are strongly Morita equivalent with respect to a $C-D$-equivalence bimodule Y and its closed subspace X. We regard Y and X as a full right Hilbert D-module and its closed subspace, respectively. Then Y and X satisfy the conditions at the beginning of Section 3. We also note that the full right Hilbert D-module $Y \oplus D$ and its closed subspace $X \oplus B$ satisfy Conditions at the beginning of Section 3. Let $L_X = \mathbb{B}_B(X \oplus B)$ and $L_Y = \mathbb{B}_D(Y \oplus D)$. By Raeburn and Williams [21, Corollary 3.21], L_X and L_Y are isomorphic to the linking algebras induced by equivalence bimodules X and Y, respectively. We denote the linking algebras by the same symbols L_X and L_Y, respectively. In the same way as in the proof of Brown, Green and Rieffel [4, Theorem 1.1], we obtain the following proposition:

Proposition 5.1. *Let $A \subset C$ and $B \subset D$ be unital inclusions of unital C^*-algebras. Then the inclusions $A \subset C$ and $B \subset D$ are strongly Morita equivalent if and only if there is a unital inclusion of unital C^*-algebras $K \subset L$ and projections in K satisfying that*
(1) $pKp \cong A$, $pLp \cong C$,
(2) $qKq \cong B$, $qLq \cong D$,
(3) $KpK = KqK = K$, $LpL = LqL = L$, $p + q = 1_L$.

We suppose that there is a conditional expectation E^B of Watatani index-finite type from D onto B. By Lemma 3.3, there is a right conditional expectation E^X from Y onto X with respect to E^B.

Lemma 5.2. *The linear map $E^X \oplus E^B$ is a right conditional expectation from $Y \oplus D$ onto $X \oplus B$ with respect to E^B.*

Proof. We show that Conditions (1)-(3) in Definition 3 hold.
(1) For any $x \in X$, $b \in B$, $d \in D$,

$$(E^X \oplus E^B)((x\oplus b)\cdot d) = (E^X \oplus E^B)((x\cdot d)\oplus bd) = x\cdot E^B(d)\oplus bE^B(d) = (x\oplus b)\cdot E^B(d).$$

(2) For any $b \in B$, $y \in Y$, $d \in D$,

$$(E^X \oplus E^B)((y \oplus d) \cdot b) = (E^X \oplus E^B)((y \cdot b) \oplus db) = (E^X(y) \oplus d) \cdot b.$$

(3) For any $x \in X$, $b \in B$, $y \in Y$, $d \in D$,

$$\begin{aligned}
\langle (E^X \oplus E^B)(y \oplus d),\, x \oplus b\rangle_D &= \langle E^X(y) \oplus E^B(d),\, x \oplus b\rangle_D \\
&= \langle E^X(y),\, x\rangle_D + E^B(d)^* b \\
&= E^B(\langle y,\, x\rangle_D) + E^B(d^* b) \\
&= E^B(\langle y \oplus d,\, x \oplus b\rangle_D).
\end{aligned}$$

Therefore, Conditions (1)-(3) in Definition 3 hold. □

By Proposition 3.5 and Corollary 3.6, there is a conditional expectation E^{L_X} of Watatani index-finite type from L_Y onto L_X such that $E^X \oplus E^B$ is a conditional expectation from $Y \oplus D$ onto $X \oplus B$ with respect to E^{L_X} and E^B. Since we identify L_X and L_Y with the linking algebras induced by equivalence bimodules X and Y, respectively, we obtain the following proposition:

Proposition 5.3. *With the above notations, we can write*

$$E^{L_X}(\begin{bmatrix} c & x \\ \widetilde{y} & d \end{bmatrix}) = \begin{bmatrix} E^A(c) & E^X(x) \\ \widetilde{E^X(y)} & E^B(d) \end{bmatrix}$$

for any element $\begin{bmatrix} c & x \\ \widetilde{y} & d \end{bmatrix} \in L_Y$, where for any $z \in X$, we denote by $\widetilde{z}$ its corresponding element in $\widetilde{X}$, the dual Hilbert C^*-bimodule of X.

Proof. Let $\theta_{y\oplus d, z\oplus f}$ be the rank-one operator on $Y \oplus D$ induced by $y \oplus d, z \oplus f \in Y \oplus D$. Then by Definition 2, for any $x \oplus b \in X \oplus B$,

$$\begin{aligned}
E^{L_X}(\theta_{y\oplus d,z\oplus f}) \cdot (x \oplus b) &= (E^X \oplus E^B)(\theta_{y\oplus d,z\oplus f}(x \oplus b)) \\
&= (E^X \oplus E^B)(y \oplus d \cdot \langle z \oplus f,\, x \oplus d\rangle_D) \\
&= (E^X \oplus E^B)(y \oplus d \cdot (\langle z, x\rangle_D + f^* b)) \\
&= E^X(y \cdot (\langle z, x\rangle_D + f^* b)) \oplus E^B(d(\langle z, x\rangle_D + f^* b)).
\end{aligned}$$

On the other hand, since we identify L_X and L_Y with the linking algebras induced by X and Y, respectively, by the proof of [21, Corollary 3.21], we regard $\theta_{y\oplus d,z\oplus f}$ as an element $\begin{bmatrix} {}_C\langle y, z\rangle & y \cdot f^* \\ \widetilde{z \cdot d^*} & df^* \end{bmatrix}$. Then

$$\begin{aligned}
\begin{bmatrix} E^A({}_C\langle y, z\rangle) & E^X(y \cdot f^*) \\ \widetilde{E^X(z \cdot d^*)} & E^B(df^*) \end{bmatrix} \begin{bmatrix} x \\ b \end{bmatrix} &= \begin{bmatrix} E^A({}_C\langle y, z\rangle) \cdot x + E^X(y \cdot f^*) \cdot b \\ \langle E^X(z \cdot d^*),\, x\rangle_D + E^B(df^*)b \end{bmatrix} \\
&= \begin{bmatrix} E^X({}_C\langle y,\, z\rangle \cdot x + y \cdot f^* b) \\ E^B(\langle z \cdot d^*,\, x\rangle_D + df^* b) \end{bmatrix} \\
&= E^{L_X}(\theta_{y\oplus d,z\oplus f}) \cdot (x \oplus b).
\end{aligned}$$

Therefore, we obtain the conclusion. □

Lemma 5.4. *With the above notations, let* $\{(u_i, u_i^*)\}_{i=1}^n$ *and* $\{(v_j, v_j^*)\}_{j=1}^m$ *be any quasi-bases for* E^A *and* E^B*, respectively. Then for any* $y \in Y$*,*

$$y = \sum_{j=1}^m E^X(y \cdot v_j) \cdot v_j^* = \sum_{i=1}^n u_i \cdot E^X(u_i^* \cdot y).$$

Proof. By the discussions in Section 2, we may assume the following:

$$B = pM_k(A)p, \quad D = pM_k(C)p, \quad X = (1 \otimes f)M_k(A)p, \quad Y = (1 \otimes f)M_k(C)p,$$

where k is a positive integer, $f = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{k \times k}$ and p is a full projection in $M_k(A)$. Furthermore, we regard X and Y as an $A - pM_k(A)p$-equivalence bimodule and a $C - pM_k(C)p$-equivalence bimodule in the usual way. Also, we can suppose that

$$E^B = (E^A \otimes \mathrm{id}_{M_k(\mathbb{C})})|_{pM_k(C)p}, \quad E^X = (E^A \otimes \mathrm{id}_{M_k(\mathbb{C})})|_{(1 \otimes f)M_k(C)p},$$

respectively. Let $\{(u_i, u_i^*)\}_{i=1}^n$ be any quasi-basis for E^A. For any $c \in C$, $h \in M_k(\mathbb{C})$,

$$\begin{aligned}
\sum_{i=1}^n u_i \cdot E^X(u_i^* \cdot (1 \otimes f)(c \otimes h)p) &= \sum_{i=1}^n u_i \cdot (E^A \otimes \mathrm{id}_{M_k(\mathbb{C})})((u_i^* \otimes f)(c \otimes h)p) \\
&= \sum_{i=1}^n u_i \cdot (E^A(u_i^* c) \otimes fh)p \\
&= \sum_{i=1}^n (u_i E^A(u_i^* c) \otimes fh)p \\
&= \sum_{i=1}^n (c \otimes fh)p = (1 \otimes f)(c \otimes h)p.
\end{aligned}$$

Replacing the left hand side by the right hand side, in the similar way to the above, we can obtain the other equation. □

Lemma 5.5. *With the above notations, for any* $y \in Y$*,*

$$\mathrm{Ind}_W(E^A) \cdot y = y \cdot \mathrm{Ind}_W(E^B).$$

Proof. By Lemma 5.4, for any $y \in Y$,

$$\sum_{i,j} u_i \cdot E^X(u_i^* \cdot y \cdot v_j) \cdot v_j^* = \sum_j y \cdot v_j v_j^* = y \cdot \mathrm{Ind}_W(E^B).$$

Similarly

$$\sum_{i,j} u_i \cdot E^X(u_i^* \cdot y \cdot v_j) \cdot v_j^* = \mathrm{Ind}_W(E^A) \cdot y.$$

Hence, we obtain the conclusion. □

Corollary 5.6. *With the above notations,*

$$\{(\begin{bmatrix} u_i & 0 \\ 0 & v_j \end{bmatrix}, \begin{bmatrix} u_i & 0 \\ 0 & v_j \end{bmatrix}^*) \,|\, i = 1, 2, \dots, n, \; j = 1, 2, \dots, m\}$$

is a quasi-basis for E^{L_X} *and* $\mathrm{Ind}_W(E^{L_X}) = \begin{bmatrix} \mathrm{Ind}_W(E^A) & 0 \\ 0 & \mathrm{Ind}_W(E^B) \end{bmatrix}$.

Proof. By Lemma 5.4 and routine computations, we can see that

$$\{(\begin{bmatrix} u_i & 0 \\ 0 & v_j \end{bmatrix}, \begin{bmatrix} u_i & 0 \\ 0 & v_j \end{bmatrix}^*) \,|\, i=1,2,\dots,n,\ j=1,2,\dots,m\}$$

is a quasi-basis for E^{L_X}. Hence by the definition of Watatani index, we can see that $\mathrm{Ind}_W(E^{L_X}) = \begin{bmatrix} \mathrm{Ind}_W(E^A) & 0 \\ 0 & \mathrm{Ind}_W(E^B) \end{bmatrix}$. □

6. The upward basic construction

Let $A \subset C$ and $B \subset D$ be unital inclusions of unital C^*-algebras, which are strongly Morita equivalent with respect to a $C-D$-equivalence bimodule Y and its closed subspace X. We suppose that there are conditional expectations E^A and E^B from C and D onto A and B, which are of Watatani index-finite type, respectively. Also, we suppose that there is a conditional expectation E^X from Y onto X with respect to E^A and E^B. Let e_A and e_B be the Jones projections for E^A and E^B, respectively and let C_1 and D_1 be the C^*-basic constructions for E^A and E^B, respectively. We regard C and D as a $C_1 - A$-equivalence bimodule and a $D_1 - B$-equivalence bimodule in the same way as in Section 4. Let

$$Y_1 = C \otimes_A X \otimes_B \widetilde{D},$$

where $\widetilde{D}$ is the dual equivalence bimodule of D, a $B - D_1$-equivalence bimodule. Clearly Y_1 is a $C_1 - D_1$-equivalence bimodule. Let E^Y be the linear map from Y_1 to Y defined by

$$E^Y(c \otimes x \otimes \widetilde{d}) = \mathrm{Ind}_W(E^A)^{-1} c \cdot x \cdot d^*$$

for any $c \in C$, $d \in D$, $x \in X$. Then E^Y is well-defined, clearly. For any $y \in Y$,

$$E^Y(\sum_{i=1}^n u_i \otimes E^X(u_i^* \cdot y) \otimes \widetilde{1}) = \sum_{i=1}^n \mathrm{Ind}_W(E^A)^{-1} u_i \cdot E^X(u_i^* \cdot y) = \mathrm{Ind}_W(E^A)^{-1} \cdot y$$

by Lemma 5.4. Hence E^Y is surjective. Also, we note that

$$E^Y(c \otimes x \otimes \widetilde{d}) = \mathrm{Ind}_W(E^A)^{-1} c \cdot x \cdot d^* = c \cdot x \cdot d^* \mathrm{Ind}_W(E^B)^{-1}$$

for any $c \in C$, $d \in D$, $x \in X$ by Lemma 5.5. Let ϕ be the linear map from Y to Y_1 defined by

$$\phi(y) = \sum_{i,j} u_i \otimes E^X(u_i^* \cdot y \cdot v_j) \otimes \widetilde{v_j}$$

for any $y \in Y$.

Lemma 6.1. *With the above notations, we have the following conditions: For any $c \in C$, $d \in D$, $y, z \in Y$,*
(1) $\phi(c \cdot y) = c \cdot \phi(y)$,
(2) $\phi(y \cdot d) = \phi(y) \cdot d$,
(3) ${}_{C_1}\langle \phi(y), \phi(z) \rangle = {}_C\langle y, z \rangle$,
(4) $\langle \phi(y), \phi(z) \rangle_{D_1} = \langle y, z \rangle_D$.

Proof. Let $c \in C$, $d \in D$, $y, z \in Y$. Then

$$\begin{aligned}\phi(c \cdot y) &= \sum_{i,j} u_i \otimes E^X(u_i^* c \cdot y \cdot v_j) \otimes \widetilde{v_j} = \sum_{i,j,k} u_i \otimes E^X(E^A(u_i^* c u_k) u_k^* \cdot y \cdot v_j) \otimes \widetilde{v_j} \\ &= \sum_{i,j,k} u_i E^A(u_i^* c u_k) \otimes E^X(u_k^* \cdot y \cdot v_j) \otimes \widetilde{v_j} = \sum_{j,k} c u_k \otimes E^X(u_k^* \cdot y \cdot v_j) \otimes \widetilde{v_j} \\ &= c \cdot \phi(y).\end{aligned}$$

Hence we obtain Condition (1). In the similar way to the above, we can obtain Condition (2). Next we show Conditions (3) and (4).

$$
\begin{aligned}
{}_{C_1}\langle \phi(y),\, \phi(z)\rangle &= \sum_{i,j,k,l} {}_{C_1}\langle u_i \otimes E^X(u_i^* \cdot y \cdot v_j) \otimes \widetilde{v_j},\, u_k \otimes E^X(u_k^* \cdot z \cdot v_l) \otimes \widetilde{v_l}\rangle \\
&= \sum_{i,j,k,l} {}_{C_1}\langle u_i {}_A\langle E^X(u_i^* \cdot y \cdot v_j) \otimes \widetilde{v_j},\, E^X(u_k^* \cdot z \cdot v_l) \otimes \widetilde{v_l}\rangle,\, u_k\rangle \\
&= \sum_{i,j,k,l} u_i {}_A\langle E^X(u_i^* \cdot y \cdot v_j) \otimes \widetilde{v_j},\, E^X(u_k^* \cdot z \cdot v_l) \otimes \widetilde{v_l}\rangle e_A u_k^* \\
&= \sum_{i,j,k,l} u_i {}_A\langle E^X(u_i^* \cdot y \cdot v_j) \cdot \langle v_j,\, v_l\rangle_B,\, E^X(u_k^* \cdot z \cdot v_l)\rangle e_A u_k^* \\
&= \sum_{i,j,k,l} u_i {}_A\langle E^X(u_i^* \cdot y \cdot v_j) \cdot E^B(v_j^* v_l),\, E^X(u_k^* \cdot z \cdot v_l)\rangle e_A u_k^* \\
&= \sum_{i,j,k,l} u_i {}_A\langle E^X(u_i^* \cdot y \cdot v_j E^B(v_j^* v_l)),\, E^X(u_k^* \cdot z \cdot v_l)\rangle e_A u_k^* \\
&= \sum_{i,k,l} u_i {}_A\langle E^X(u_i^* \cdot y \cdot v_l),\, E^X(u_k^* \cdot z \cdot v_l)\rangle e_A u_k^* \\
&= \sum_{i,k,l} u_i E^A({}_C\langle u_i^* \cdot y \cdot v_l,\, E^X(u_k^* \cdot z \cdot v_l)\rangle) e_A u_k^* \\
&= \sum_{i,k,l} u_i E^A(u_i^* {}_C\langle y \cdot v_l,\, E^X(u_k^* \cdot z \cdot v_l)\rangle) e_A u_k^* \\
&= \sum_{k,l} {}_C\langle y \cdot v_l,\, E^X(u_k^* \cdot z \cdot v_l)\rangle e_A u_k^* \\
&= \sum_{k,l} {}_C\langle y,\, E^X(u_k^* \cdot z \cdot v_l) \cdot v_l^*\rangle e_A u_k^* \\
&= \sum_{k} {}_C\langle y, u_k^* \cdot z\rangle e_A u_k^* \\
&= \sum_{k} {}_C\langle y, z\rangle u_k e_A u_k^* \\
&= {}_C\langle y, z\rangle .
\end{aligned}
$$

Hence we obtain Condition (3). Similarly we obtain Condition (4). □

By the above lemma, we can identify Y with a closed subspace of Y_1 satisfying Conditions (1), (2) in Definition 1 except the conditions that ${}_C\langle Y_1, Y\rangle = C$ and $\langle Y_1, Y\rangle_D = D$.

Lemma 6.2. *With the above, we identify Y with a closed subspace of Y_1 by the linear map ϕ. Then ${}_{C_1}\langle Y_1, Y\rangle = C_1$ and $\langle Y_1, Y\rangle_{D_1} = D_1$.*

Proof. Let $c\otimes x\otimes \widetilde{d}\in Y_1$ and $y\in Y$. Since $\phi(y)=\sum_{i,j}u_i\otimes E^X(u_i^*\cdot y\cdot v_j)\otimes\widetilde{v_j}$,

$$\begin{aligned}
{}_{C_1}\langle c\otimes x\otimes\widetilde{d},\phi(y)\rangle &= \sum_{i,j}{}_{C_1}\langle c\otimes x\otimes\widetilde{d},\, u_i\otimes E^X(u_i^*\cdot y\cdot v_j)\otimes\widetilde{v_j}\rangle \\
&= \sum_{i,j}{}_{C_1}\langle c\cdot{}_A\langle x\otimes\widetilde{d}\,,\, E^X(u_i^*\cdot y\cdot v_j)\otimes\widetilde{v_j}\rangle,\, u_i\rangle \\
&= \sum_{i,j}{}_{C_1}\langle c\cdot{}_A\langle x\cdot E^B(d^*v_j),\, E^X(u_i^*\cdot y\cdot v_j)\rangle,\, u_i\rangle \\
&= \sum_{i,j}c_A\langle x\cdot E^B(d^*v_j),\, E^X(u_i^*\cdot y\cdot v_j)\rangle e_A u_i^* \\
&= \sum_{i,j}ce_A\,{}_A\langle x\cdot E^B(d^*v_j),\, E^X(u_i^*\cdot y\cdot v_j)\rangle u_i^* \\
&= \sum_{i,j}ce_A\,{}_C\langle x\cdot E^B(d^*v_j),\, u_i\cdot E(u_i^*\cdot y\cdot v_j)\rangle \\
&= \sum_{j}ce_A\,{}_C\langle x\cdot E^B(d^*v_j),\, y\cdot v_j\rangle \\
&= \sum_{j}ce_A\,{}_C\langle x\cdot E^B(d^*v_j)v_j^*,\,,y\rangle \\
&= ce_A\,{}_C\langle x\cdot d^*,\, y\rangle = ce_A\,{}_C\langle x, y\cdot d\rangle.
\end{aligned}$$

Since ${}_C\langle X,Y\rangle = C$, we obtain that ${}_{C_1}\langle Y_1,Y\rangle = C_1$. Also, since $\langle X,Y\rangle_D = D$, we obtain that $\langle Y_1,Y\rangle_{D_1}=D_1$ in the same way as above. ☐

By Lemmas 6.1 and 6.2, we obtain the following corollary:

Corollary 6.3. *With the above notations, the inclusions $C\subset C_1$ and $D\subset D_1$ are strongly Morita equivalent with respect to the C_1-D_1-equivalence bimodule Y_1 and its closed subspace Y.*

Let E^C and E^D be the dual conditional expectations of E^A and E^B, respectively.

Lemma 6.4. *With the above notations, E^Y is a conditional expectation from Y_1 onto Y with respect to E^C and E^D.*

Proof. We show that Conditions (1)-(6) in Definition 2 hold. We note that we identify Y with $\phi(Y)\subset Y_1$.
(1) For any $c_1,c_2\in C$, $y\in Y$,

$$\begin{aligned}
E^Y(c_1e_Ac_2\cdot y) &= \sum_{i,j}E^Y(c_1e_Ac_2\cdot u_i\otimes E^X(u_i^*\cdot y\cdot v_j)\otimes\widetilde{v_j}) \\
&= \sum_{i,j}E^Y(c_1E^A(c_2u_i)\otimes E^X(u_i^*\cdot y\cdot v_j)\otimes\widetilde{v_j}) \\
&= \sum_{i,j}\mathrm{Ind}_W(E^A)^{-1}c_1E^A(c_2u_i)\cdot E^X(u_i^*\cdot y\cdot v_j)\cdot v_j^* \\
&= \mathrm{Ind}_W(E^A)^{-1}c_1c_2\cdot y = E^C(c_1e_Ac_2)\cdot y.
\end{aligned}$$

(2) For any $c_1,c_2\in C$, $x\in X$, $d\in D$,

$$\begin{aligned}
E^Y(c_1\cdot c_2\otimes x\otimes\widetilde{d}) &= E^Y(c_1c_2\otimes x\otimes\widetilde{d}) = \mathrm{Ind}_W(E^A)^{-1}c_1c_2\cdot x\cdot d^* \\
&= c_1\cdot E^Y(c_2\otimes x\otimes\widetilde{d}).
\end{aligned}$$

(3) By the proof of Lemma 6.2, for any $c \in C$, $d \in D$, $x \in X$, $y \in Y$,

$$\begin{aligned} E^C({}_{C_1}\langle c \otimes x \otimes \widetilde{d},\, y\rangle) &= \mathrm{Ind}_W(E^A)^{-1} c\ {}_C\langle x \cdot d^*,\, y\rangle \\ &= \mathrm{Ind}_W(E^A)^{-1}{}_C\langle c \cdot x \cdot d^*,\, y\rangle = {}_{C_1}\langle E^Y(c \otimes x \otimes \widetilde{d}),\, y\rangle. \end{aligned}$$

(4) By Lemma 5.5, we can see that

$$E^Y(y \cdot d_1 e_B d_2) = y \cdot E^D(d_1 e_B d_2)$$

for any $d_1, d_2 \in D$, $y \in Y$ in the same way as in the proof of Condition (1).
(5) In the same way as in the proof of Condition (2), we can see that

$$E^Y(c \otimes x \otimes \widetilde{d_1} \cdot d_2) = E^Y(c \otimes x \otimes \widetilde{d_1}) \cdot d_2$$

for any $c \in C$, $d_1, d_2 \in D$, $x \in X$.
(6) By Lemma 5.5 we can see that

$$E^B(\langle c \otimes x \otimes \widetilde{d}, y\rangle_{D_1} = \langle E^Y c \otimes x \otimes \widetilde{d}),\, y\rangle_{D_1}.$$

for any $c \in C$, $d \in D$, $x \in X$, $y \in Y$. Therefore we obtain the conclusion. □

Definition 4. In the above situation, Y_1 is called the *upward basic construction* of Y for E^X. Also, E^Y is called the *dual* conditional expectation of E^X.

Remark 6.5. The linear map ϕ from Y to Y_1 defined in the above is independent of the choice of quasi-bases $\{(u_i, u_i^*)\}$ and $\{(v_j, v_j^*)\}$ for E^A and E^B, respectively. Indeed, let $\{(w_i, w_i^*)\}$ and $\{(z_j, z_j^*)\}$ be another pair of quasi-bases for E^A and E^B, respectively. Then for any $y \in Y$,

$$\begin{aligned} \sum_{i,j} w_i \otimes E^X(w_i^* \cdot y \cdot z_j) \otimes \widetilde{z_j} &= \sum_{i,j,k,l} u_k E^A(u_k^* w_i) \otimes E^X(w_i^* \cdot y \cdot z_j) \otimes [v_l E^B(v_l^* z_j)]\widetilde{\ } \\ &= \sum_{i,j,k,l} u_k \otimes E^X(E^A(u_k^* w_i) w_i^* \cdot y \cdot z_j) \otimes E^B(z_j^* v_l) \cdot \widetilde{v_l} \\ &= \sum_{j,k,l} u_k \otimes E^X(u_k^* \cdot y \cdot z_j E^B(z_j^* v_l)) \otimes \widetilde{v_l} \\ &= \sum_{k.l} u_k \otimes E^X(u_k^* \cdot y \cdot v_l) \otimes \widetilde{v_l} = \phi(y). \end{aligned}$$

Next, we shall show that the upward basic construction for equivalence bimodules is unique in a certain sense.

Let $A \subset C$ and $B \subset D$ be unital inclusions of unital C^*-algebras as above. Also, let E^A, E^B, E^X and C_1, D_1 be as above.

Lemma 6.6. *With the above notations,* $\mathrm{Ind}_W(E^A) \in A$ *if and only if* $\mathrm{Ind}_W(E^B) \in B$.

Proof. We assume that $\mathrm{Ind}_W(E^A) \in A$. By the discussions before Lemma 2.6, we may assume that

$$B = pM_k(A)p, \quad D = pM_k(C)p, \quad E^B(E^A \otimes \mathrm{id}_{M_k(\mathbb{C})})|_{pM_k(C)p},$$

where $k \in \mathbb{N}$ and p is a projection in $M_k(A)$ satisfying that $M_k(A)pM_k(A) = M_k(A)$ and $M_k(C)pM_k(C) = M_k(C)$. Then by the discussions before Lemma 2.6,

$$\mathrm{Ind}_W(E^B) = (\mathrm{Ind}_W(E^A) \otimes I_k)p.$$

Since $\mathrm{Ind}_W(E^A) \in A$, $\mathrm{Ind}_W(E^B) \in pM_k(A)p = B$. Thus, we obtain the conclusion. □

Let W be a $C_1 - D_1$-equivalence bimodule. We suppose that $\mathrm{Ind}_W(E^A) \in A$. Then $\mathrm{Ind}_W(E^B) \in B$ by Lemma 6.6. Also, we suppose that Y is included in W as its closed subspace and that the inclusions $C \subset C_1$ and $D \subset D_1$ are strongly Morita equivalent with respect to W and its closed subspace Y. Furthermore, we suppose that there is a conditional expectation F^Y from W onto Y with respect to E^C and E^D satisfying that

$$F^Y(e_A \cdot y \cdot e_B) = \mathrm{Ind}_W(E^A)^{-1} \cdot E^X(y) \qquad (*)$$

for any $y \in Y$, where e_A and e_B are the Jones projections for E^A and E^B, respectively. We note that in Lemma 6.9, we shall show that the conditional expectation E^Y from Y_1 onto Y with respect to E^C and E^D satisfies that

$$E^Y(e_A \cdot y \cdot e_B) = \mathrm{Ind}_W(E^A)^{-1} \cdot E^X(y)$$

for any $y \in Y$. We show that there is a $C_1 - D_1$-equivalence bimodule isomorphism θ from W onto Y_1 such that

$$F^Y = E^Y \circ \theta.$$

Let $\{(u_i, u_i^*)\}_{i=1}^n$ and $\{(v_j, v_j^*)\}_{j=1}^m$ be quasi-bases for E^A and E^B, respectively and let $\{(w_i, w_i^*)\}_{i=1}^n$ and $\{(z_j, z_j^*)\}_{j=1}^m$ be their dual quasi-bases for E^C and E^D defined by

$$\begin{aligned} w_i &= u_i e_A \mathrm{Ind}_W(E^A)^{\frac{1}{2}}, (i = 1, 2, \dots, n), \\ z_j &= v_j e_B \mathrm{Ind}_W(E^B)^{\frac{1}{2}}, (j = 1, 2 \dots, m), \end{aligned}$$

respectively. Let θ be the map from W to Y_1 defined by

$$\begin{aligned} \theta(y) &= \mathrm{Ind}_W(E^A) \sum_{i,j} u_i \otimes E^X(F^Y(e_A u_i^* \cdot y \cdot v_j e_B)) \otimes \widetilde{v_j} \\ &= \sum_{i,j} u_i \otimes E^X(F^Y(e_A u_i^* \cdot y \cdot v_j e_B)) \otimes \widetilde{v_j} \cdot \mathrm{Ind}_W(E^B). \end{aligned}$$

for any $y \in W$. Clearly θ is a linear map from W to Y_1.

Lemma 6.7. *With the above notations, for any $c_1, c_2 \in C$, $d_1, d_2 \in D$ and $y \in W$,*

$$\theta(c_1 e_A c_2 \cdot y) = c_1 e_A c_2 \cdot \theta(y), \quad \theta(y \cdot d_1 e_B d_2) = \theta(y) \cdot d_1 e_B d_2.$$

Proof. For any $c_1, c_2 \in C$ and $y \in W$,

$$\begin{aligned} \theta(c_1 e_A c_2 \cdot y) &= \mathrm{Ind}_W(E^A) \sum_{i,j} u_i \otimes E^X(F^Y(E^A(u_i^* c_1) e_A c_2 \cdot y \cdot v_j e_B)) \otimes \widetilde{v_j} \\ &= \mathrm{Ind}_W(E^A) \sum_{i,j} u_i E^A(u_i^* c_1) \otimes E^X(F^Y(e_A c_2 \cdot y \cdot v_j e_B)) \otimes \widetilde{v_j} \\ &= \mathrm{Ind}_W(E^A) \sum_{i,j} c_1 \otimes E^X(F^Y(e_A E^A(c_2 u_i) u_i^* \cdot y \cdot v_j e_B)) \otimes \widetilde{v_j} \\ &= \mathrm{Ind}_W(E^A) \sum_{i,j} c_1 e_A c_2 \cdot u_i \otimes E^X(F^Y(e_A u_i^* \cdot y \cdot v_j e_B)) \otimes \widetilde{v_j} \\ &= c_1 e_A c_2 \cdot \theta(y). \end{aligned}$$

Similarly we can see that $\theta(y \cdot d_1 e_B d_2) = \theta(y) \cdot d_1 e_B d_2$ for any $d_1, d_2 \in D$ and $y \in W$. Therefore, we obtain the conclusion. $\square$

Lemma 6.8. *With the above notations, θ is surjective.*

Proof. By Lemma 6.7 and Condition $(*)$, for any $c \in C$, $d \in D$ and $x \in X$

$$\begin{aligned}\theta(ce_A \cdot x \cdot e_B d^*) &= ce_A \cdot \theta(x) \cdot e_B d^* \\ &= \sum_{i,j} ce_A \cdot u_i \otimes E^X(u_i^* \cdot x \cdot v_j) \otimes \widetilde{v_j} \cdot e_B d^* \\ &= \sum_{i,j} c \otimes E^X(E^A(u_i)u_i^* \cdot x \cdot v_j E^B(v_j^*)) \otimes \widetilde{d} = c \otimes x \otimes \widetilde{d}.\end{aligned}$$

Hence θ is surjective. $\square$

Next, we show that θ preserves the both-sided inner products.

Lemma 6.9. *For any $y \in Y$,*

$$\begin{aligned}&e_A \cdot y \cdot e_B = e_A \cdot \phi(y) \cdot e_B = e_A \cdot E^X(y) = E^X(y) \cdot e_B, \\ &E^Y(e_A \cdot y \cdot e_B) = \mathrm{Ind}_W(A)^{-1} \cdot E^X(y) = E^X(y) \cdot \mathrm{Ind}_W(B)^{-1}.\end{aligned}$$

Proof. For any $y \in Y$,

$$\begin{aligned}e_A \cdot y \cdot e_B &= e_A \cdot \sum_{i,j} u_i \otimes E^X(u_i^* \cdot y \cdot v_j) \otimes \widetilde{v_j} \cdot e_B \\ &= \sum_{i,j} 1 \otimes E^X(E^A(u_i)u_i^* \cdot y \cdot v_j E^B(v_j^*)) \otimes \widetilde{1} = 1 \otimes E^X(y) \otimes \widetilde{1}.\end{aligned}$$

Also, by the similar computations to the above, for any $y \in Y$

$$e_A \cdot E^X(y) = e_A \cdot \phi(E^X(y)) = E^X(y) \cdot e_B = 1 \otimes E^X(y) \otimes \widetilde{1}.$$

Furthermore,

$$\begin{aligned}E^Y(e_A \cdot y \cdot e_B) &= E^Y(e_A \cdot E^X(y)) = E^C(e_A) \cdot E^X(y) \\ &= \mathrm{Ind}_W(A)^{-1} \cdot E^X(y) = E^X(y) \cdot \mathrm{Ind}_W(B)^{-1}\end{aligned}$$

by Lemma 5.5. Thus, we obtain the conclusion. $\square$

Lemma 6.10. *With the above notations, θ preserves the both-sided inner products.*

Proof. Let $y_1, y_2 \in W$. Then

$$\theta(y_1) = \mathrm{Ind}_W(E^A) \sum_{i,j} u_i \otimes x_1 \otimes \widetilde{v_j}, \quad \theta(y_2) = \mathrm{Ind}_W(E^A) \sum_{i_1,j_1} u_{i_1} \otimes x_2 \otimes \widetilde{v_{j_1}},$$

where

$$x_1 = E^X(F^Y(e_A u_i^* \cdot y_1 \cdot v_j e_B)), \quad x_2 = E^X(F^Y(e_A u_{i_1}^* \cdot y_2 \cdot v_{j_1} e_B)).$$

Hence by Lemma 6.9,

$$
\begin{aligned}
{}_{C_1}\langle\theta(y_1),\theta(y_2)\rangle &= \mathrm{Ind}_W(E^A)^2\sum_{i,j,i_1,j_1}{}_{C_1}\langle u_i\otimes x_1\otimes\widetilde{v_j},\,u_{i_1}\otimes x_2\otimes\widetilde{v_{j_1}}\rangle \\
&= \mathrm{Ind}_W(E^A)^2\sum_{i,j,i_1,j_1}{}_{C_1}\langle u_i\,{}_A\langle x_1\otimes\widetilde{v_j},x_2\otimes\widetilde{v_{j_1}}\rangle,u_{i_1}\rangle \\
&= \mathrm{Ind}_W(E^A)^2\sum_{i,j,i_1,j_1}{}_{C_1}\langle u_i\,{}_A\langle x_1\cdot{}_B\langle\widetilde{v_j},\widetilde{v_{j_1}}\rangle,x_2\rangle,u_{i_1}\rangle \\
&= \mathrm{Ind}_W(E^A)^2\sum_{i,j,i_1,j_1}{}_{C_1}\langle u_i\,{}_A\langle x_1\cdot\langle v_j,v_{j_1}\rangle_B,x_2\rangle,u_{i_1}\rangle \\
&= \mathrm{Ind}_W(E^A)^2\sum_{i,j,i_1,j_1}{}_{C_1}\langle u_i\,{}_A\langle x_1\cdot E^B(v_j^*v_{j_1}),x_2\rangle,u_{i_1}\rangle \\
&= \mathrm{Ind}_W(E^A)^2\sum_{i,j,i_1,j_1}u_ie_A\,{}_A\langle x_1\cdot E^B(v_j^*v_{j_1}),x_2\rangle u_{i_1}^* \\
&= \mathrm{Ind}_W(E^A)^2 \\
&\times\sum_{i,i_1,j_1}u_ie_A\,{}_A\langle E^X(F^Y(e_Au_i^*\cdot y_1\cdot v_{j_1}e_B)),\,E^X(F^Y(e_Au_{i_1}^*\cdot y_2\cdot v_{j_1}e_B))\rangle u_{i_1}^* \\
&= \mathrm{Ind}_W(E^A)^2 \\
&\times\sum_{i,i_1,j_1}u_i\,{}_{C_1}\langle e_A\cdot F^Y(e_Au_i^*\cdot y_1\cdot v_{j_1}e_B)\cdot e_B,\,e_A\cdot F^Y(e_Au_{i_1}^*\cdot y_2\cdot v_{j_1}e_B)\cdot e_B\rangle u_{i_1}^* \\
&= \mathrm{Ind}_W(E^A)^2 \\
&\times\sum_{i,i_1,j_1}{}_{C_1}\langle u_ie_A\cdot F^Y(e_Au_i^*\cdot y_1\cdot v_{j_1}e_B)\cdot e_B,\,u_{i_1}e_A\cdot F^Y(e_Au_{i_1}^*\cdot y_2\cdot v_{j_1}e_B)\cdot e_B\rangle \\
&= \sum_{i,i_1,j_1}{}_{C_1}\langle w_i\cdot F^Y(w_i^*\cdot y_1\cdot v_{j_1}e_B)\cdot e_B\,,\,w_{i_1}\cdot F^Y(w_{i_1}^*\cdot y_2\cdot v_{j_1}e_B)\cdot e_B\rangle \\
&= \sum_{j_1}{}_{C_1}\langle y_1\cdot v_{j_1}e_B,\,y_2\cdot v_{j_1}e_B\rangle=\sum_{j_1}{}_{C_1}\langle y_1\cdot v_{j_1}e_Bv_{j_1}^*,\,y_2\rangle={}_{C_1}\langle y_1,y_2\rangle.
\end{aligned}
$$

Also, by Lemma 6.9, we ca see that $\langle\theta(y_1),\theta(y_2)\rangle_{D_1}=\langle y_1,y_2\rangle_{D_1}$ in the same way as in the above. Therefore, we obtain the conclusion. □

Proposition 6.11. *With the above notations, θ is a C_1-D_1-equivalence bimodule isomorphism from W onto Y_1 such that $F^Y=E^Y\circ\theta$.*

Proof. By Lemmas 6.7, 6.8 and 6.10, we have only to show that $F^Y=E^Y\circ\theta$. For any $y\in W$,

$$
\begin{aligned}
(E^Y\circ\theta)(y) &= \sum_{i,j}u_i\cdot E^X(F^Y(e_Au_i^*\cdot y\cdot v_je_B))\cdot v_j^* \\
&= \mathrm{Ind}_W(E^A)\sum_{i,j}u_i\cdot F^Y(e_A\cdot F^Y(e_Au_i^*\cdot y\cdot v_je_B)\cdot e_B)\cdot v_j^* \\
&= \mathrm{Ind}_W(E^A)\sum_{i,j}F^Y(u_ie_A\cdot F^Y(e_Au_i^*\cdot y\cdot v_je_B)\cdot e_Bv_j^*) \\
&= \mathrm{Ind}_W(E^A)^{-1}\sum_{i,j}F^Y(w_i\cdot F^Y(w_i^*\cdot y\cdot z_j)\cdot z_j^*) \\
&= \mathrm{Ind}_W(E^A)^{-1}\sum_{j}F^Y(y\cdot z_jz_j^*) \\
&= F^Y(y)
\end{aligned}
$$

by Condition $(*)$ and Lemma 5.5. Therefore, we obtain the conclusion. $\square$

Summing up the above discussions, we obtain the following theorem:

Theorem 6.12. *Let $A\subset C$ and $B\subset D$ be unital inclusions of unital C^*-algebras. Let E^A and E^B be conditional expectations from C and D onto A and B of Watatani index-finite type, respectively. Let E^X be a conditional expectation from Y onto X with respect to E^A and E^B. Let C_1 and D_1 be the C^*-basic constructions and e_A and e_B the Jones projections for E^A and E_B, respectively. We suppose that the Watatani index, $\mathrm{Ind}_W(E^A)$ is in A. Let W be a C_1-D_1-equivalence bimodule satisfying that Y is included in W as its closed subspace and that the inclusions $C\subset C_1$ and $D\subset D_1$ are strongly Morita equivalent to with respect to W and its closed subspace Y. Also we suppose that there is a conditional expectation F^Y from W onto Y with respect to E^C and E^D satisfying that*

$$F^Y(e_A\cdot y\cdot e_B)=\mathrm{Ind}_W(E^A)^{-1}\cdot E^X(y)$$

for any $y\in Y$, where E^C and E^D are the dual conditional expectations from C_1 and D_1 onto C and D for E^A and E^B, respectively. Then there is a C_1-D_1-equivalence bimodule isomorphism θ from W onto Y_1 such that $F^Y=E^Y\circ\theta$, where Y_1 is the upward basic construction of Y for E^X and E^Y is the dual conditional expectation of E^X.

7. Duality

In this section, we shall present a certain duality theorem for inclusions of equivalence bimodules.

Let $A\subset C$ and $B\subset D$ be unital inclusions of unital C^*-algebras, which are strongly Morita equivalent with respect to a $C-D$-equivalence bimodule Y and its closed subspace X. Let E^A and E^B be conditional expectations of Watatani index-finite type from C and D onto A and B, respectively. Let E^X be a conditional expectation from Y onto X with respect to E^A and E^B. Let C_1 and D_1 be the C^*-basic constructions for E^A and E^B and e_A and e_B the Jones projections for E^A and E^B, respectively. Let Y_1 be the upward basic construction for E^X and let E^C, E^D and E^Y be the dual conditional expectations from C_1, D_1 and Y_1 onto C, D and Y, respectively. Furthermore, let C_2 and D_2 be the C^*-basic constructions for E^C and E^D, respectively and e_C and e_D the Jones projections for E^C and E^D, respectively. Let Y_2 be the upward basic construction for E^Y and let E^{C_1}, E^{D_1} and E^{Y_1} be the dual conditional expectations from C_2, D_2 and Y_2 onto C_1, D_1 and Y_1, respectively. Let $\{(u_i, u_i^*)\}_{i=1}^k$ and $\{(v_i, v_i^*)\}_{i=1}^{k_1}$ be quasi-bases for E^A and E^B, respectively. We note that we can assume that $k=k_1$.

We suppose that $\mathrm{Ind}_W(E^A)\in A$. Then $\mathrm{Ind}_W(E^B)\in B$ by Lemma 5.5. By Proposition 4.3, the inclusions $C_1\subset C_2$ and $A\subset C$ are strongly Morita equivalent with respect to the C_2-C-equivalence bimodule C_1 and its closed subspace C. Also, there is a conditional expectation G from C_1 onto C with respect to E^C and E^A. Let $p=[E^A(u_i^*u_j)]_{i,j=1}^k$. Then by the discussions in Section 2, p is a full projection in $M_k(A)$. Let Ψ_{C_1} be the map from C_1 to $M_k(A)$ defined by

$$\Psi_{C_1}(c_1e_Ac_1)=[E^A(u_i^*c_1)E^A(c_2u_j)]_{i,j=1}^k$$

for any $c_1, c_2\in C$. Then by the discussions in Section 2, Ψ_{C_1} is an isomorphism of C_1 onto $pM_k(A)p$. Let Ψ_{C_2} be the map from C_2 to $M_k(C)$ defined by

$$\begin{aligned}\Psi_{C_2}(c_1e_Cc_2)&=[E^C(w_i^*c_1)E^C(c_2w_j)]_{i,j=1}^k\\&=[E^C(\mathrm{Ind}_W(E^A)^{\frac12}e_Au_i^*c_1)E^C(\mathrm{Ind}_W(E^A)^{\frac12}c_2u_je_A)]\\&=[\mathrm{Ind}_W(E^A)E^C(e_Au_i^*c_1)E^C(c_2u_je_A)]\end{aligned}$$

for any $c_1, c_2 \in C_1$, where $\{(w_i, w_i^*)\}_{i=1}^k$ is the quasi-basis for E^C defined by $w_i = \mathrm{Ind}_W(E^A)^{\frac{1}{2}} u_i e_A$ for $i = 1, 2, \dots, k$. Then Ψ_{C_2} is also an isomorphism of C_2 onto $pM_k(C)p$. Furthermore, let Φ_C be the map from C to $M_k(A)$ defined by

$$\Phi_C(c) = \begin{bmatrix} E^A(u_1^* c) \\ \vdots \\ E^A(u_k^* c) \end{bmatrix}$$

for any $c \in C$, By the discussions in Section 2, Φ_C is a $C_1 - A$-equivalence bimodule isomorphism of the $C_1 - A$-equivalence bimodule C onto the $pM_k(A)p - A$-equivalence bimodule $pM_k(A)(1 \otimes f)$, where $f = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \in M_k(\mathbb{C})$ and we identify A and C_1 with $A \otimes f$ and $pM_k(A)p$, respectively. Let Φ_{C_1} be the map from C_1 to $M_k(C)$ defined by

$$\Phi_{C_1}(c) = \begin{bmatrix} E^C(w_1^* c) \\ \vdots \\ E^C(w_k^* c) \end{bmatrix}$$

for any $c \in C$. Then by the discussions in Section 2, Φ_{C_1} is a $C_2 - C$-equivalence bimodule isomorphism of the $C_2 - C$-equivalence bimodule C_1 onto the $pM_k(C)p - C$-equivalence bimodule $pM_k(C)(1 \otimes f)$, where $f = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \in M_k(\mathbb{C})$ and we identify C and C_2 with $C \otimes f$ and $pM_k(C)p$, respectively. Thus, the inclusion $C_1 \subset C_2$ can be identified with the inclusion $pM_k(A)p \subset pM_k(C)p$, the $C_1 - A$-equivalence bimodule C can be identified with the $pM_k(A)p - A$-equivalence bimodule $pM_k(A)(1 \otimes f)$ and E^C can be identified with $(E^A \otimes \mathrm{id})|_{pM_k(A)p}$ by the above isomorphisms. Similar results to the above hold, that is, let $q = [E^B(v_i^* v_j)]_{i,j=1}^k$. Then q is a full projection in $M_k(B)$ Then the inclusion $D_1 \subset D_2$ is identified the inclusion $qM_k(B)q \subset qM_k(D)q$, the $D_1 - B$-equivalence bimodule D is identified with $qM_k(B)q - B$-equivalence bimodule $qM_k(B)(1 \otimes f)$ and E^D is identified with $(E^D \otimes \mathrm{id})|_{qM_k(B)q}$ by the following isomorphisms: Let Ψ_{D_1} be the isomorphism of D_1 onto $qM_k(B)q$ defined by

$$\Psi_{D_1}(d_1 e_B d_2) = [E^B(v_i^* d_1) E^B(d_2 v_j)]_{i,j=1}^k,$$

for any $d_1, d_2 \in D$. Let Ψ_{D_2} be the isomorphism of D_2 onto $qM_k(D)q$ defined by

$$\Psi_{D_2}(d_1 e_D d_2) = [E^D(z_i^* d_1) E^D(d_2 z_j)]_{i,j=1}^k$$

for any $d_1, d_2 \in D_1$, where $\{(z_i, z_i^*)\}_{i=1}^k$ is the quasi-basis for E^D defined by $z_i = \mathrm{Ind}_W(B)^{\frac{1}{2}} v_i e_B$ for $i = 1, 2, \dots, k$. Furthermore, let Φ_D be the $D_1 - B$-equivalence bimodule isomorphism of D onto $qM_k(B)(1 \otimes f)$ defined by

$$\Phi_D(d) = \begin{bmatrix} E^B(v_1^* d) \\ \vdots \\ E^B(v_k^* d) \end{bmatrix}$$

for any $d \in D$, where we identify D_1 with $qM_k(B)q$. Let Φ_{D_1} be the $D_2 - D$-equivalence bimodule isomorphism of D_1 onto $qM_k(D)(1 \otimes f)$ defined by

$$\Phi_{D_1}(d) = \begin{bmatrix} E^D(z_1^* d) \\ \vdots \\ E^D(z_k^* d) \end{bmatrix}$$

for any $d \in D_1$, where we identify D_2 with $qM_k(D)q$.

Let Y_1 and Y_2 be the upward basic constructions for E^X and E^Y, respectively. By the definitions of Y_1 and Y_2,

$$Y_1 = C \otimes_A X \otimes_B \widetilde{D}, \quad Y_2 = C_1 \otimes_C Y \otimes_D \widetilde{D_1}.$$

Then

$$Y_1 \cong pM_k(A)(1 \otimes f) \otimes_A X \otimes_B (1 \otimes f)M_k(B)q$$

as $C_1 - D_1$-equivalence bimodules where we identify $pM_k(A)p$ and $qM_k(B)q$ are identified with C_1 and D_1, respectively. We regard $p \cdot M_k(X) \cdot q$ as a $pM_k(A)p - qM_k(B)q$-equivalence bimodule in the usual way. Similarly

$$Y_2 \cong pM_k(C)(1 \otimes f) \otimes_C Y \otimes_D (1 \otimes f)M_k(D)q$$

as $C_2 - D_2$-equivalence bimodules, where we identify $pM_k(C)p$ and $qM_k(D)q$ are identified with C_2 and D_2, respectively.

Lemma 7.1. *With the above notations,*

$$pM_k(A)(1 \otimes f) \otimes_A X \otimes_B (1 \otimes f)M_k(B)q \cong p \cdot M_k(X) \cdot q$$

as $pM_k(A)p - qM_k(B)q$-equivalence bimodules. Hence $Y_1 \cong p \cdot M_k(X) \cdot q$ as $C_1 - D_1$-equivalence bimodules, where we identify $pM_k(A)p$ and $qM_k(B)q$ with C_1 and D_1, respectively.

Proof. We have only to show that

$$pM_k(A)(1 \otimes f) \otimes_A X \otimes_B (1 \otimes f)M_k(B)q \cong p \cdot M_k(X) \cdot q$$

as $pM_k(A)p - qM_k(B)q$-equivalence bimodules. Let Φ be the map from $pM_k(A)(1 \otimes f) \otimes_A X \otimes_B (1 \otimes f)M_k(B)q$ to $p \cdot M_k(X) \cdot q$ defined by

$$\Phi(pa(1 \otimes f) \otimes x \otimes (1 \otimes f)bq) = pa \cdot (x \otimes f) \cdot bq$$

for any $a \in M_k(A)$, $b \in M_k(B)$, $x \in X$. Then it is clear that Φ is well-defined and a $pM_k(A)p - qM_k(B)q$-bimodule. For any $a_1, a_2 \in M_k(A)$, $b_1, b_2 \in M_k(B)$ and $x_1, x_2 \in X$,

$$\begin{aligned}
&{}_{pM_k(A)p}\langle pa_1(1 \otimes f) \otimes x_1 \otimes (1 \otimes f)b_1q, \, pa_2(1 \otimes f) \otimes x_2 \otimes (1 \otimes f)b_2q\rangle \\
&= {}_{pM_k(A)p}\langle pa_1(1 \otimes f) \cdot {}_A\langle x_1 \otimes (1 \otimes f)b_1q, \, x_2 \otimes (1 \otimes f)b_2q\rangle, \, pa_2(1 \otimes f)\rangle \\
&= {}_{pM_k(A)p}\langle pa_1 {}_A\langle x_1 \otimes (1 \otimes f)b_1q, \, x_2 \otimes (1 \otimes f)b_2q\rangle \otimes f, \, pa_2(1 \otimes f)\rangle \\
&= pa_1[{}_A\langle x_1 \otimes (1 \otimes f)b_1q, \, x_2 \otimes (1 \otimes f)b_2q\rangle \otimes f]a_2^* p \\
&= pa_1[{}_A\langle x_1 \cdot {}_B\langle (1 \otimes f)b_1q, \, (1 \otimes f)b_2q\rangle, \, x_2\rangle \otimes f]a_2^* p \\
&= pa_1[{}_A\langle x_1 \cdot (1 \otimes f)b_1qb_2^*(1 \otimes f), \, x_2\rangle \otimes f]a_2^* p.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&{}_{pM_k(A)p}\langle pa_1 \cdot (x_1 \otimes f) \cdot b_1q, \, pa_2 \cdot (x_1 \otimes f) \cdot b_2q\rangle \\
&= pa_1(1 \otimes f)_{M_k(A)}\langle (x_1 \otimes f) \cdot b_1q, \, (x_2 \otimes f) \cdot b_2q\rangle (1 \otimes f)a_2^* p \\
&= pa_1[{}_A\langle x_1 \cdot (1 \otimes f)b_1qb_2^*(1 \otimes f), \, x_2\rangle \otimes f]a_2^* p.
\end{aligned}$$

Hence Φ preserves the left $pM_k(A)p$-valued inner products. Also,

$$\begin{aligned}
&\langle pa_1(1\otimes f)\otimes x_1\otimes(1\otimes f)b_1q,\, pa_2(1\otimes f)\otimes x_2\otimes(1\otimes f)b_2q\rangle_{qM_k(B)q} \\
&=\langle x_1\otimes(1\otimes f)b_1q,\, \langle pa_1(1\otimes f),\, pa_2(1\otimes f)\rangle_A\cdot x_2\otimes(1\otimes f)b_2q\rangle_{qM_k(B)q} \\
&=\langle x_1\otimes(1\otimes f)b_1q,\, (1\otimes f)a_1^*pa_2(1\otimes f)\cdot x_2\otimes(1\otimes f)b_2q\rangle_{qM_k(B)q} \\
&=\langle(1\otimes f)b_1q,\, [\langle x_1,\, (1\otimes f)a_1^*pa_2(1\otimes f)\cdot x_2\rangle_B\otimes f]b_2q\rangle_{qM_k(B)q} \\
&=qb_1^*(1\otimes f)[\langle x_1,\, (1\otimes f)a_1^*pa_2(1\otimes f)\cdot x_2\rangle_B\otimes f]b_2q \\
&=qb_1^*[\langle x_1,\, (1\otimes f)a_1^*pa_2(1\otimes f)\cdot x_2\rangle_B\otimes f]b_2q.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\langle pa_1\cdot(x_1\otimes f)\cdot b_1q,\, pa_2\cdot(x_2\otimes f)\cdot b_2q\rangle_{qM_k(B)q} \\
&=qb_1^*(1\otimes f)\langle pa_1\cdot(x_1\otimes f),\, pa_2\cdot(x_2\otimes f)\rangle_{M_k(B)}(1\otimes f)b_2q \\
&=qb_1^*[\langle x_1,\, (1\otimes f)a_1^*pa_2(1\otimes f)\cdot x_2\rangle_B\otimes f]b_2q.
\end{aligned}$$

Thus Φ preserves the right $qM_k(B)q$-valued inner products. Furthermore, let $\{f_{ij}\}_{i,j=1}^k$ be a system of matrix units of $M_k(\mathbb{C})$. Then since $f=f_{11}$, for any $x\in X$ and $i,j=1,2,\dots,k$,

$$\begin{aligned}
p(1\otimes f_{i1})\otimes x\otimes(1\otimes f_{1j})q &= p(1\otimes f_{i1})(1\otimes f)\otimes x\otimes(1\otimes f)(1\otimes f_{1j})q \\
&\in pM_k(A)(1\otimes f)\otimes_A X\otimes_B(1\otimes f)M_k(B)q.
\end{aligned}$$

Then by the definition of $p\cdot M_k(X)\cdot q$, for $i,j=1,2,\dots,k$,

$$\Phi(p(1\otimes f_{i1})\otimes x\otimes(1\otimes f_{1j})q)=p(1\otimes f_{i1})\cdot(x\otimes f)\cdot(1\otimes f_{1j})q=p\cdot(x\otimes f_{ij})\cdot q.$$

This means that Φ is surjective. Therefore, we obtain the conclusion. $\square$

Corollary 7.2. *With the above notations,*

$$pM_k(C)(1\otimes f)\otimes_C Y\otimes_D(1\otimes f)M_k(D)q\cong p\cdot M_k(Y)\cdot q$$

as $pM_k(C)p-qM_k(D)q$-equivalence bimodules. Hence $Y_2\cong p\cdot M_k(Y)\cdot q$ as C_2-D_2-equivalence bimodules, where we identify $pM_k(C)p$ and $qM_k(D)q$ with C_2 and D_2, respectively.

Proof. This is immediate by Lemma 6.1. $\square$

By the above discussions, we can obtain the C_1-D_1-equivalence bimodule isomorphism $\overline{\Phi_1}$ from Y_2 onto $p\cdot M_k(Y)\cdot q$ defined by

$$\overline{\Phi_1}(c_1\otimes y\otimes\widetilde{d_1})=[E^C(w_i^*c_1)\cdot y\cdot E^D(d_1^*z_j)]_{i,j=1}^k$$

for any $c_1\in C_1$, $d_1\in D_1$, $y\in Y$, where we identify C_1 and D_1 with $pM_k(C)p$ and $qM_k(D)q$ by the isomorphisms defined above, respectively. Also, we can obtain the $C-D$-equivalence bimodule isomorphism $\overline{\Phi}$ from Y_1 onto $p\cdot M_k(X)\cdot q$ defined by

$$\overline{\Phi}(c\otimes x\otimes d)=[E^A(u_i^*c)\cdot x\cdot E^B(d^*v_j)]_{i,j=1}^k$$

for any $c\in C$, $d\in D$, $x\in X$, where we identify C and D with $pM_k(A)p$ and $qM_k(B)q$ by the isomorphisms defined above, respectively.

Let $E^{p\cdot M_k(X)\cdot q}$ be the conditional expectation from $p\cdot M_k(Y)\cdot q$ onto $p\cdot M_k(X)\cdot q$ defined by

$$E^{p\cdot M_k(X)\cdot q}=(E^X\otimes\mathrm{id}_{M_k(\mathbb{C})})|_{p\cdot M_k(Y)\cdot q}$$

with respect to conditional expectations induced by $E^A\otimes\mathrm{id}_{M_k(\mathbb{C})}$ and $E^B\otimes\mathrm{id}_{M_k(\mathbb{C})}$.

Lemma 7.3. *With the above notations, we have*

$$E^{p\cdot M_k(X)\cdot q}\circ\overline{\Phi_1}=\overline{\Phi}\circ E^{Y_1}.$$

Proof. We can prove this lemma by routine computations. Indeed, for any $c_1 \in C_1$, $d_1 \in D_1$, $y \in Y$,

$$
\begin{aligned}
(E^{p\cdot M_k(X)\cdot q} \circ \overline{\Phi_1})(c_1 \otimes y \otimes \widetilde{d_1}) &= E^{p\cdot M_k(X)\cdot q}([E^C(w_i^* c_1) \cdot y \cdot E^D(d_1^* z_j)]_{i,j=1}^k) \\
&= [E^X(E^C(w_i^* c_1) \cdot y \cdot E^D(d_1^* z_j)]_{i,j=1}^k.
\end{aligned}
$$

Let $c_1 = c_2 e_A c_3, \quad c_2, c_3 \in C$ and $d_1 = d_2 e_B d_3, \quad d_2, d_3 \in D$. We note that for any $i, j = 1, 2, \ldots, k$,

$$
w_i = u_i e_A \mathrm{Ind}_W(E^A)^{\frac{1}{2}}, \quad z_j = v_j e_B \mathrm{Ind}_W(E^B)^{\frac{1}{2}}.
$$

Hence

$$
\begin{aligned}
&[E^X(E^C(w_i^* c_1) \cdot y \cdot E^D(d_1^* z_j))]_{i,j=1}^k \\
&= [E^X(E^C(\mathrm{Ind}_W(E^A)^{\frac{1}{2}} e_A u_i^* c_2 e_A c_3) \cdot y \cdot E^D(d_3^* e_B d_2^* v_j e_B \mathrm{Ind}_W(E^B)^{\frac{1}{2}}))]_{ij}^k \\
&= [E^X(\mathrm{Ind}_W(E^A)^{-\frac{1}{2}} E^A(u_i^* c_2) c_3 \cdot y \cdot d_3^* \, E^B(d_2^* v_j) \mathrm{Ind}_W(E^B)^{-\frac{1}{2}})]_{ij=1}^k \\
&= [\mathrm{Ind}_W(E^A)^{-\frac{1}{2}} E^A(u_i^* c_2) \cdot E^X(c_3 \cdot y \cdot d_3^*) \cdot E^B(d_2^* v_j) \mathrm{Ind}_W(E^B)^{-\frac{1}{2}}]_{ij=1}^k \\
&= \mathrm{Ind}_W(E^A)^{-1}[E^A(u_i^* c_2) \cdot E^X(c_3 \cdot y \cdot d_3^*) \cdot E^B(d_2^* v_j)]_{ij=1}^k
\end{aligned}
$$

by Lemma 5.5. On the other hand,

$$
\begin{aligned}
E^{Y_1}(c_1 \otimes y \otimes \widetilde{d_1}) &= \mathrm{Ind}_W(E^A)^{-1} c_1 \cdot y \cdot d_1^* = \mathrm{Ind}_W(E^A)^{-1} c_1 \cdot \phi(y) \cdot d_1^* \\
&= \sum_{i,j} \mathrm{Ind}_W(E^A)^{-1} c_1 \cdot u_i \otimes E^X(u_i^* \cdot y \cdot v_j) \otimes \widetilde{v_j} \cdot d_1^*
\end{aligned}
$$

Since $c_1 = c_2 e_A c_3$ and $d_1 = d_2 e_B d_3$,

$$
E^{Y_1}(c_1 \otimes y \otimes \widetilde{d_1}) = \sum_{i,j} \mathrm{Ind}_W(E^A)^{-1} c_2 E^A(c_3 u_i) \otimes E^X(u_i^* \cdot y \cdot v_j) \otimes [d_2 E^B(d_3 v_j)]\widetilde{}.
$$

Hence

$$
\begin{aligned}
&(\overline{\Phi} \circ E^{Y_1})(c_1 \otimes y \otimes \widetilde{d_1}) \\
&= \sum_{i,j} \mathrm{Ind}_W(E^A)^{-1}[(E^A(u_l^* c_2 E^A(c_3 u_i)) \cdot E^X(u_i^* \cdot y \cdot v_j) \cdot E^B(E^B(v_j^* d_3^*) d_2^* v_m)]_{l,m=1}^k \\
&= \sum_{i,j} \mathrm{Ind}_W(E^A)^{-1}[E^A(u_l^* c_2) E^A(c_3 u_i) \cdot E^X(u_i^* \cdot y \cdot v_j) \cdot E^B(v_j^* d_3^*) E^B(d_2^* v_m)]_{l,m=1}^k \\
&= \sum_{i,j} \mathrm{Ind}_W(E^A)^{-1}[E^A(u_l^* c_2) \cdot E^X(E^A(c_3 u_i) u_i^* \cdot y \cdot v_j E^B(v_j^* d_3^*)) \cdot E^B(d_2^* v_m)]_{l,m=1}^k \\
&= \mathrm{Ind}_W(E^A)^{-1}[E^A(u_l^* c_2) \cdot E^X(c_3 \cdot y \cdot d_3^*) \cdot E^B(d_2^* v_m)]_{l,m=1}^k.
\end{aligned}
$$

Therefore, we obtain the conclusion. $\square$

Theorem 7.4. *Let $A \subset C$ and $B \subset D$ be unital inclusions of unital C^*-algebras, which are strongly Morita equivalent with respect to a $C - D$-equivalence bimodule Y and its closed subspace X. Let E^A and E^B be conditional expectations of Watatani index-finite type from C and D onto A and B, respectively and let E^X be a conditional expectation from Y onto X with respect to E^A and E^B. Let C_1, D_1 and Y_1 be the C^*- basic constructions and the upward basic construction for E^A, E^B and E^X, respectively. Also, let E^C, E^D and E^Y be the dual conditional expectations from C_1, D_1 and Y_1 onto C, D and Y, respectively. Furthermore, in the same way as above, we define the C^*-basic constructions and the upward basic constructions C_2, D_2 and Y_2 for E^C, E^D and E^Y, respectively and we define the* *second dual conditional expectations E^{C_1}, E^{D_1} and E^{Y_1}, respectively. Then there are a positive integer k and full projections $p \in M_k(A)$ and $q \in M_k(B)$ with*

$$pM_k(A)p \cong C_1, \quad qM_k(B)q \cong D_1,$$
$$pM_k(C)p \cong C_2, \quad qM_k(D)q \cong D_2$$

such that there are a $C_1 - D_1$-eqivalence bimodule isomorphism $\overline{\Phi}$ of Y_1 onto $p \cdot M_k(X) \cdot q$ and a $C_2 - D_2$-equivalence bimodule isomorphism $\overline{\Phi_1}$ of Y_2 onto $p \cdot M_k(Y) \cdot q$ satifying that

$$E^{p \cdot M_k(X) \cdot q} \circ \overline{\Phi_1} = \overline{\Phi} \circ E^{Y_1}$$

where $E^{p \cdot M_k(X) \cdot q}$ is the conditional expectation from $p \cdot M_k(Y) \cdot q$ onto $p \cdot M_k(X) \cdot q$ defined by

$$E^{p \cdot M_k(X) \cdot q} = (E^X \otimes \mathrm{id}_{M_k(\mathbb{C})})|_{p \cdot M_k(Y) \cdot q} \, .$$

Proof. This is immediate by Lemmas 6.1, 7.3 and Corollary 7.2. □

8. The downward basic construction

Let $A \subset C$ and $B \subset D$ be unital inclusions of unital C^*-algebras which are strongly Morita equivalent with respect to a $C - D$-equivalence bimodule Y and its closed subspace X. Let E^A and E^B be conditional expectations of Watatani index-finite type from C and D onto A and B, respectively. Let E^X be a conditional expectation from Y onto X with respect to E^A and E^B. We suppose that $\mathrm{Ind}_W(E^A) \in A$. Then by Lemma 6.6, $\mathrm{Ind}_W(E^B) \in B$. Also, we suppose that there are full projections p and q in C and D satisfying that

$$E^A(p) = \mathrm{Ind}_W(E^A)^{-1}, \quad E^B(q) = \mathrm{Ind}_W(E^B)^{-1},$$

respectively. Then by [19, Proposition 2.6], we obtain the following: Let $P = \{p\}' \cap A$ and let E^P be the conditional expectation from A onto P defined by

$$E^P(a) = \mathrm{Ind}_W(E^A)E^A(pap)$$

for any $a \in A$. Similarly, let $Q = \{q\}' \cap B$ and let E^Q be the conditional expectation from B onto Q defined by

$$E^Q(b) = \mathrm{Ind}_W(E^B)E^B(qbq)$$

for any $b \in B$. Then $\mathrm{Ind}_W(E^P) = \mathrm{Ind}_W(E^A) \in P \cap C'$ and $\mathrm{Ind}_W(E^Q) = \mathrm{Ind}_W(E^B) \in Q \cap D'$. Furthermore, we can see that

$$ApA = C, \quad BqB = D,$$
$$pap = E^P(a), \quad qbq = E^Q(b)$$

for any $a \in A$ and $b \in B$. Also, the unital inclusions $A \subset C$ and $B \subset D$ can be regarded as the C^*-basic constructions of the unital inclusions $P \subset A$ and $Q \subset B$, respectively. In this section, we shall show that the unital inclusions $P \subset A$ and $Q \subset B$ are strongly Morita equivalent and that there is a conditional expectation from X onto its closed subspace with respect to E^P and E^Q.

Let $Z = \{x \in X \mid p \cdot x = x \cdot q\}$. Then Z is a closed subspace of X.

Lemma 8.1. *With the above notations, Z is a Hilbert $P - Q$-bimodule in the sense of Brown, Mingo and Shen* [5].

Proof. This lemma can be proved by routine computations. Indeed, for any $a \in P$, $x \in Z$,

$$p \cdot (a \cdot x) = pa \cdot x = a \cdot (p \cdot x) = a \cdot (x \cdot q) = (a \cdot x) \cdot q.$$

Hence $a \cdot x \in Z$ for any $a \in P$, $x \in Z$. Similarly for any $b \in Q$, $x \in Z$, $x \cdot b \in Z$. For any $x, y \in Z$,

$$p \cdot {}_A\langle x, y\rangle = {}_C\langle p \cdot x, y\rangle = {}_C\langle x \cdot q, y\rangle = {}_C\langle x, p \cdot y\rangle = {}_A\langle x, y\rangle \cdot p.$$

Hence ${}_A\langle x, y\rangle \in P$ for any $x, y \in Z$. Similarly for any $x, y \in Z$, $\langle x, y\rangle_A \in Q$. Since Z is a closed subspace of the $A-B$-equivalence bimodule X, Z is a Hilbert $P-Q$-bimodule in the sense of Brown, Mingo and Shen [5]. $\square$

Let E^Z be the linear map from X to Z defined by

$$E^Z(x) = \mathrm{Ind}_W(E^A) \cdot E^X(p \cdot x \cdot q)$$

for any $x \in X$. We note that

$$E^Z(x) = E^X(p \cdot x \cdot q) \cdot \mathrm{Ind}_W(E^B)$$

for any $x \in X$ by Lemma 5.5.

Lemma 8.2. *With the above notations, E^Z satisfies Conditions (1)-(6) in Definition 2.*

Proof. For any $a \in A$, $z \in Z$,

$$\begin{aligned} E^Z(a \cdot z) &= \mathrm{Ind}_W(E^A) \cdot E^X(p \cdot (a \cdot z) \cdot q) = \mathrm{Ind}_W(E^A) \cdot E^X(pa \cdot z \cdot q) \\ &= \mathrm{Ind}_W(E^A) \cdot E^X(pap \cdot z) = \mathrm{Ind}_W(E^A) E^A(pap) \cdot z = E^P(a) \cdot z. \end{aligned}$$

Hence E^Z satisfies Condition (1) in Definition 2. Similarly E^Z satisfies Condition (4) in Definition 2. For any $b \in Q$, $x \in X$,

$$\begin{aligned} E^Z(x \cdot b) &= \mathrm{Ind}_W(E^A) \cdot E^X(p \cdot (x \cdot b) \cdot q) = \mathrm{Ind}_W(E^A) \cdot E^X(p \cdot x \cdot qb) \\ &= \mathrm{Ind}_W(E^A) \cdot E^X(p \cdot x \cdot q) \cdot b = E^Z(x) \cdot b. \end{aligned}$$

Hence E^Z satisfies Condition (5) in Definition 2. Similarly E^Z satisfies Condition (2) in Definition 2. For any $x \in X$, $z \in Z$,

$$\begin{aligned} {}_P\langle E^Z(x),\, z\rangle &= {}_A\langle \mathrm{Ind}_W(E^A) \cdot E^X(p \cdot x \cdot q),\, z\rangle = \mathrm{Ind}_W(E^A)\,{}_A\langle E^X(p \cdot x \cdot q),\, z\rangle \\ &= \mathrm{Ind}_W(E^A) E^A({}_A\langle p \cdot x \cdot q,\, z\rangle) = \mathrm{Ind}_W(E^A) E^A(p_A\langle x,\, z \cdot q\rangle) \\ &= \mathrm{Ind}_W(E^A) E^A(p_A\langle x,\, p \cdot z\rangle) = \mathrm{Ind}_W(E^A) E^A(p_A\langle x, z\rangle p) \\ &= E^P({}_A\langle x, z\rangle). \end{aligned}$$

Hence E^Z satisfies Condition (3) in Definition 2. Also, in the same way as above, by Lemma 5.5, we can see that E^Z satisfies Condition (6) in Definition 2. $\square$

Lemma 8.3. *With the above notations, ${}_A\langle X, Z\rangle = A$, $\langle X, Z\rangle_B = B$.*

Proof. Since E^Z is surjective by Lemma 8.2,

$$\begin{aligned} {}_A\langle X, Z\rangle &= {}_A\langle X,\, E^Z(X)\rangle = {}_A\langle X,\, \mathrm{Ind}_W(E^A) \cdot E^X(p \cdot X \cdot q)\rangle \\ &= {}_A\langle X,\, E^X(p \cdot X \cdot q)\rangle \mathrm{Ind}_W(E^A) = E^A({}_C\langle X,\, p \cdot X \cdot q\rangle) \mathrm{Ind}_W(E^A) \\ &= E^A({}_C\langle X,\, X \cdot q\rangle p) \mathrm{Ind}_W(E^A). \end{aligned}$$

Since $X \cdot B = X$ by [5, Proposition 1.7] and $BqB = D$,

$$\begin{aligned} {}_A\langle X,\, Z\rangle &= E^A({}_C\langle X \cdot B,\, X \cdot Bq\rangle p) \mathrm{Ind}_W(E^A) = E^A({}_C\langle X,\, X \cdot BqB\rangle p) \mathrm{Ind}_W(E^A) \\ &= E^A({}_C\langle X,\, X \cdot D\rangle p) \mathrm{Ind}_W(E^A). \end{aligned}$$

Since $B \subset D$, $X = X \cdot B \subset X \cdot D$ by [5, Proposition 1.7]. Hence

$$\begin{aligned} {}_A\langle X,\, Z\rangle &\supset E^A({}_C\langle X,\, X\rangle p) \mathrm{Ind}_W(E^A) = E^A({}_A\langle X,\, X\rangle p) \mathrm{Ind}_W(E^A) \\ &= E^A(Ap) \mathrm{Ind}_W(E^A) = A. \end{aligned}$$

Since ${}_A\langle X,\, Z\rangle \subset A$, we obtain that ${}_A\langle X,\, Z\rangle = A$. Similarly we obtain that $\langle X,\, Z\rangle_B = B$. Therefore we obtain the conclusion. $\square$

Corollary 8.4. *With the above notations, Z is a $P-Q$-equivalence bimodule and E^Z is a conditional expectation from X onto Z with respect to E^P and E^Q.*

Proof. First, we show that Z is a $P-Q$-equivalence bimodule. By Lemma 8.1, we have only to show that Z is full with the both sided inner products. Since E^Z is surjective by Lemma 8.2,

$$\begin{aligned} {}_P\langle Z, Z\rangle &= {}_P\langle E^Z(X), E^Z(X)\rangle = E^P({}_A\langle X, E^Z(X)\rangle) = E^P({}_A\langle X, Z\rangle) \\ &= E^P(A) = P \end{aligned}$$

by Lemma 8.3. Similarly $\langle Z, Z\rangle_Q = Q$. Thus, Z is a $P-Q$-equivalence bimodule. Hence E^Z is a conditional expectation from X onto Z with respect to E^P and E^Q. □

Proposition 8.5. *With the above notations, unital inclusions $P \subset A$ and $Q \subset B$ are strongly Morita equivalent with respect to the $P-Q$- equivalence bimodule X and its closed subspace Z and there is a conditional expectation from X onto Z with respect to E^P and E^Q.*

Proof. This is immediate by Lemmas 8.1, 8.2 and Corollary 8.4. □

Definition 5. In the above situation, Z is called the *downward basic construction* of X for E^X. Also, E^Z is called the *pre-dual* conditional expectation of E^X.

9. Relation between the upward basic construction and the downward basic construction

Let $A \subset C$ and $B \subset D$ be unital inclusions of unital C^*-algebras, which are strongly Morita equivalent with respect to a $C-D$-equivalence bimodule Y and its closed subspace X. Let E^A and E^B be conditional expectations of Watatani index-finite type from C and D onto A and B, respectively. Let E^X be a conditional expectation from Y onto X with respect to E^A and E^B. We suppose that $\mathrm{Ind}_W(E^A) \in A$ and $\mathrm{Ind}_W(E^B) \in B$. Let e_A and e_B be the Jones' projections for E^A and E^B, respectively. Then by [26, Lemma 2.1.1],

$$A = \{a \in C \,|\, e_A a = a e_A\}, \quad B = \{b \in D \,|\, e_B b = b e_B\},$$

respectively. Let C_1 and D_1 be the C^*-basic constructions for E^A and E^B, respectively and let E^C and E^D be the dual conditional expectations from C_1 and D_1 onto C and D, respectively. Then e_A and e_B are full projections in C_1 and D_1, respectively by [26, Lemma 2.1.6] and

$$\mathrm{Ind}_W(E^C) = \mathrm{Ind}_W(E^A) \in A, \quad \mathrm{Ind}_W(E^D) = \mathrm{Ind}_W(E^B) \in B,$$

respectively. Furthermore,

$$\begin{aligned} E^A(x) &= \mathrm{Ind}_W(E^C)E^C(e_A x e_A) \quad \text{for any } x \in C, \\ E^B(x) &= \mathrm{Ind}_W(E^D)E^D(e_B x e_B) \quad \text{for any } x \in D, \end{aligned}$$

respectively. Let Y_1 be the upward basic construction for E^X and E^Y the dual conditional expectation of E^X from Y_1 onto Y. We recall that Y can be regarded as a closed subspace of Y_1 by the linear map ϕ from Y to Y_1 defined by

$$\phi(y) = \sum_{i,j} u_i \otimes E^X(u_i^* \cdot y \cdot v_j) \otimes \widetilde{v_j},$$

for any $y \in Y$, where $\{(u_i, u_i^*)\}$ and $\{(v_j, v_j^*)\}$ are quasi-bases for E^A and E^B, respectively and

$$Y_1 = C \otimes_A X \otimes_B \widetilde{D}.$$

Let

$$Z = \{y \in Y \,|\, e_A \cdot \phi(y) = \phi(y) \cdot e_B\}.$$

By the discussions in Section 8, Z is a closed subspace of Y and Z is an $A-B$-equivalence bimodule.

Lemma 9.1. *With the above notations, $Z = X$*

Proof. For any $x \in X$,

$$\begin{aligned} e_A \cdot \phi(x) &= \sum_{i,j} e_A \cdot u_i \otimes E^X(u_i^* \cdot x \cdot v_j) \otimes \widetilde{v_j} \\ &= \sum_{i,j} 1 \otimes E^X(E^A(u_i)u_i^* \cdot x \cdot v_j) \otimes \widetilde{v_j} \\ &= \sum_j 1 \otimes E^X(x \cdot v_j) \otimes \widetilde{v_j} = \sum_j 1 \otimes x \cdot E^B(v_j) \otimes \widetilde{v_j} \\ &= \sum_j 1 \otimes x \otimes [v_j E^B(v_j^*)]\widetilde{\ } = 1 \otimes x \otimes \widetilde{1}. \end{aligned}$$

Similarly, $\phi(x) \cdot e_B = 1 \otimes x \otimes \widetilde{1}$. Hence $x \in Z$. Thus $X \subset Z$. Also, let $y \in Z$. Since $e_A \cdot \phi(y) = \phi(y) \cdot e_B$,

$$e_A \cdot \phi(y) = e_A^2 \cdot \phi(y) = e_A \cdot \phi(y) \cdot e_B.$$

Also, since

$$e_A \cdot \phi(y) = \sum_j 1 \otimes E^X(y \cdot v_j) \otimes \widetilde{v_j} \quad \text{and} \quad e_A \cdot \phi(y) \cdot e_B = 1 \otimes E^X(y) \otimes \widetilde{1},$$

we see that

$$\sum_j 1 \otimes E^X(y \cdot v_j) \otimes \widetilde{v_j} = 1 \otimes E^X(y) \otimes \widetilde{1}.$$

Using the conditional expectation E^Y,

$$\mathrm{Ind}_W(E^A)^{-1} \cdot E^X(y) = \sum_j \mathrm{Ind}_W(E^A)^{-1} \cdot E^X(y \cdot v_j) \cdot v_j^* = \mathrm{Ind}_W(E^A)^{-1} \cdot y$$

by Lemma 5.4. Thus $E^X(y) = y$, that is, $y \in X$. Therefore, we obtain the conclusion. $\square$

By Lemmas 6.9 and 9.1, we obtain the following:

Proposition 9.2. *With the above notations, X can be regarded as the downward basic construction for E^Y and E^X can be regarded as the pre-dual conditional expectation of E^Y.*

Next, let p and q be full projections in C and D satisfying that

$$E^A(p) = \mathrm{Ind}_W(E^A)^{-1}, \quad E^B(q) = \mathrm{Ind}_W(E^B)^{-1},$$

respectively. Let P, Q, E^P, E^Q and Z, E^Z be as in Section 8. We shall show that Y is the upward basic construction for E^Z and that E^X is the dual conditional expectation of E^Z. By Section 8, we can see that

$$\mathrm{Ind}_W(E^P) = \mathrm{Ind}_W(E^A) \in P \cap C', \quad \mathrm{Ind}_W(E^Q) = \mathrm{Ind}_W(E^B) \in Q \cap D'.$$

Also, we can see that

$$E^Z(x) = \mathrm{Ind}_W(E^A) \cdot E^X(p \cdot x \cdot q).$$

Furthermore, we can regard C and D as the C^*-basic constructions for E^P and E^Q, respectively by [19, Proposition 2.6]. We can also regard p and q as the Jones projections in C and D, respectively. Hence by Proposition 6.11, we obtain the following proposition:

Proposition 9.3. *With the above notations, Y can be regarded as the upward basic construction for E^Z and E^X can be regarded as the dual conditional expectation of E^Z.*

10. The strong Morita equivalence and the paragroups

In this section, we show that the strong Morita equivalence for unital inclusions of unital C^*-algebras preserves their paragroups. We begin this section with the following easy lemmas:

Lemma 10.1. *Let $A \subset C$ and $B \subset D$ be unital inclusions of unital C^*-algebras, which are strongly Morita equivalent with respect to a $C-D$-equivalence bimodule Y and its closed subspace X. Then $C \cdot X = X \cdot D = Y$.*

Proof. Since X is an $A-B$-equivalence bimodule and $A \subset C$ is a unital inclusion, there are elements $x_1, x_2, \dots x_n \in X$ such that $\sum_{i=1}^n \langle x_i, x_i\rangle_B = 1_D$. Then for any $y \in Y$,

$$y = y \cdot 1_D = \sum_{i=1}^n y \cdot \langle x_i, x_i\rangle_B = \sum_{i=1}^n {}_C\langle y, x_i\rangle \cdot x_i.$$

Hence we can see that $C \cdot X = Y$. Similarly we obtain that $X \cdot D = Y$. □

Let $A \subset C$ and $B \subset D$ be as above. Let $C \subset C_1$ and $D \subset D_1$ be unital inclusion of unital C^*-algebras, which are strongly Morita equivalent with respect to a $C_1 - D_1$-equivalence bimodule Y_1 and its closed subspace Y. We note that $X \subset Y \subset Y_1$.

Lemma 10.2. *With the above notations, the inclusion $A \subset C_1$ and $B \subset D_1$ are strongly Morita equivalent with respect to the $C_1 - D_1$-equivalence bimodule Y_1 and its closed subspace X.*

Proof. It suffices to show that

$${}_{C_1}\langle Y_1, X\rangle = C_1 \quad \langle Y_1, X\rangle_{D_1} = D_1.$$

Indeed, by [5, Proposition 1.7] and Lemma 10.1,

$$\begin{aligned} {}_{C_1}\langle Y_1, X\rangle &= {}_{C_1}\langle Y_1 \cdot D_1, X\rangle = {}_{C_1}\langle Y_1, X \cdot D_1\rangle = {}_{C_1}\langle Y_1, X \cdot DD_1\rangle \\ &= {}_{C_1}\langle Y_1, Y \cdot D_1\rangle = {}_{C_1}\langle Y_1, Y_1\rangle = C_1. \end{aligned}$$

Similarly, we can prove that $\langle Y_1, X\rangle_{D_1} = D_1$. □

Let $A \subset C$ and $B \subset D$ be unital inclusions of unital C^*-algebras, which are strongly Morita equivalent with respect to a $C-D$-equivalence bimodule Y and its closed subspace X. Then by Lemmas 2.3, 2.4 and Corollary 2.5, we may assume that

$$B = pM_n(A)p,\ D = pM_n(C)p,\ Y = (1 \otimes f)M_n(C)p,\ X = (1 \otimes f)M_n(A)p,$$

where p is a full projection in $M_n(A)$ and n is a positive integer. We regard X and Y as an $A - pM_n(A)p$-equivalence bimodule and a $C - pM_n(C)p$-equivalence bimodule in the usual way.

Lemma 10.3. *With the above notations, we suppose that unital inclusions of unital C^*-algebras $A \subset C$ and $B \subset D$ are strongly Morita equivalent. Then the relative commutants $A' \cap C$ and $B' \cap D$ are isomorphic.*

Proof. By the above discussions, we have only to show that

$$A' \cap C \cong (pM_n(A)p)' \cap pM_n(C)p,$$

where p is a projection in $M_n(A)$ satisfying the above. By routine computations, we can see that

$$M_n(A)' \cap M_n(C) = \{c \otimes I_n \mid c \in A' \cap C\}.$$

Hence we can see that $A' \cap C \cong M_n(A)' \cap M_n(C)$. Next, we claim that $M_n(A)' \cap M_n(C) \cong (M_n(A) \cap M_n(C))p$. Indeed, let π be the map from $M_n(A)' \cap M_n(C)$

onto $(M_n(A)' \cap M_n(C))p$ defined by $\pi(x) = px$ for any $x \in M_n(A)' \cap M_n(C)$. Since p is a projection in $M_n(A)$, π is a homomorphism of $M_n(A)' \cap M_n(C)$ onto $(M_n(A)' \cap M_n(C))p$. We suppose that $xp = 0$ for an element $x \in M_n(A)' \cap M_n(C)$. Since p is full in $M_n(A)$, there are elements $z_1, \ldots, z_m \in M_n(A)$ such that

$$\sum_{i=1}^{m} z_i p z_i^* = 1_{M_n(A)}.$$

Then

$$0 = \sum_{i=1}^{m} z_i x p z_i^* = \sum_{i=1}^{m} x z_i p z_i^* = x.$$

Hence π is injective. Thus π is an isomorphism of $M_n(A)' \cap M_n(C)$ onto $(M_n(A)' \cap M_n(C))p$. Finally we show that

$$(pM_n(A)p)' \cap pM_n(C)p = (M_n(A)' \cap M_n(C))p.$$

Indeed, by easy computations, we can see that

$$pM_n(A)p)' \cap pM_n(C)p \supset (M_n(A)' \cap M_n(C))p.$$

We prove the inverse inclusion. Let $y \in (pM_n(A)p)' \cap pM_n(C)p$. Let $w = \sum_{i=1}^{m} z_i y z_i^*$. Then for any $x \in M_n(A)$,

$$\begin{aligned} wx &= \sum_{i,j=1}^{m} z_i y z_i^* x z_j p z_j^* = \sum_{i,j=1}^{m} z_i y p z_i^* x z_j p z_j^* = \sum_{i,j}^{m} z_i p z_i^* x z_j p y z_j^* \\ &= \sum_{j=1}^{m} x z_j p y z_j^* = \sum_{j=1}^{m} x z_j y z_j^* = xw. \end{aligned}$$

Hence $w \in M_n(A)' \cap M_n(C)$. On the other hand,

$$wp = pw = \sum_{i=1}^{m} p z_i y z_i^* = \sum_{i=1}^{m} p z_i p y z_i^* = \sum_{i=1}^{m} y p z_i p z_i^* = yp = y.$$

Thus $y \in (M_n(A)' \cap M_n(C))p$. Hence

$$(pM_n(A)p)' \cap pM_n(C)p = (M_n(A)' \cap M_n(C))p.$$

Therefore, we obtain the conclusion. □

Let $A \subset C$ and $B \subset D$ be as above. We suppose that there is a conditional expectation E^A of Watatani index-finite type from C onto A. Then by Section 2, there are a conditional expectation of Watatani index-finite type from D onto B and a conditional expectation E^X from Y onto X with respect to E^A and E^B. For any $n \in \mathbb{N}$, let C_n and D_n be the n-th C^*-basic constructions for conditional expectations E^A and E^B, respectively. Then by Corollary 6.3, the inclusions $C_{n-1} \subset C_n$ and $D_{n-1} \subset D_n$ are strongly Morita equivalent for any $n \in \mathbb{N}$, where $C_0 = C$ and $D_0 = D$. Thus, by Lemma 10.2, $A \subset C_n$ and $B \subset D_n$ are strongly Morita equivalent for any $n \in \mathbb{N}$.

Theorem 10.4. *Let $A \subset C$ and $B \subset D$ be unital inclusions of unital C^*-algebras, which are strongly Morita equivalent. We suppose that there is a conditional expectation of Watatani index-finite type from C onto A. Then the paragroups of $A \subset C$ and $B \subset D$ are isomorphic.*

Proof. This is immediate by the above discussions and Lemma 10.3. □

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