The Hochschild homology and cohomology theory for algebras, reviewed, developed and deformed as gardening quantums or phantoms

Ryukyu Math. J., 33(2020), 61-102

The Hochschild homology and cohomology theory for algebras, reviewed, developed and deformed as gardening quantums or phantoms

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Dedicated to Professor Muneo CHO on his seventieth birthday

Abstract

We would like to study the Hochschild homology and cohomology for algebras, to some possible extent of understanding the first level.

Primary 16E40, 16E45, 17B35, 17B63, 17B66, 46L65, 46L80, 46L87
Keywords: Hochschid, homology, cohomology, algebra, quantum, phantom

1 Introduction

We as beginners would like to review and study the Hochschild (H) cohomology and homology theory for algebras, just following Khalkhali [22] in part only. This is a sort of yabu-kogi, but along such a route, where yabu-kogi in Japanese means a mountain climbing without or out of a route. Within the time limited for publication, we made some considerable effort to study this subject in a decorated but incomplete form.

The contents

Received November 30, 2020.

2 Hochschild cohomology

Let A be an algebra over the field $\mathbb{C}$ of complex numbers. Let M be an A-bimodule in the sense that M is a left and right A-module by left and right actions of A on M compatible in the sense that $a(mb) = (am)b$ for any $a, b \in A$ and $m \in M$.

The **Hochschild** cochain complex of A with coefficients in M, denoted as $(C^*(A, M) = C^*(\otimes^* A, M), \delta_*)$, is defined as that $C^0(\otimes^0 A, M) = M$,

$$C^n(A, M) = C^n(\otimes^n A, M) = \mathrm{Hom}(\otimes^n A, M)$$

as the additive group of all linear maps from the n-fold tensor product $\otimes^n A$ to M, for $n \geq 1$, and the differential as the boundary map

$$\delta = \delta_n : C^n(\otimes^n A, M) \to C^{n+1}(\otimes^{n+1} A, M)$$

for $n \geq 0$ is given by $(\delta_0 m)(a) = ma - am = [m, a]$ for $m \in M$ and $a \in A$, and

$$\begin{aligned}
&(\delta_n f)(a_1, \cdots, a_{n+1}) = a_1 f(a_2, \cdots, a_{n+1}) \\
&+ \sum_{j=1}^{n} (-1)^j f(a_1, \cdots, a_j a_{j+1}, \cdots, a_{n+1}) + (-1)^{n+1} f(a_1, \cdots, a_n) a_{n+1}
\end{aligned}$$

for $f \in C^n(\otimes^n A, M)$ for $n \geq 1$ (corrected by replacing $j+1$ to j the power of (-1) in the sum).

$\star$ We may check that

Proposition 2.1. *It holds that the composition $\delta_{n+1} \circ \delta_n = 0$ for $n \geq 0$.*

Proof. $\star$ Indeed, we compute that for $m \in M$ and $a_1, a_2 \in A$,

$$\begin{aligned}
&(\delta_1 \circ \delta_0)(m)(a_1, a_2) = \delta_1([m, \cdot])(a_1, a_2) \\
&= a_1[m, a_2] - [m, a_1 a_2] + [m, a_1] a_2 \\
&= a_1(m a_2 - a_2 m) - m a_1 a_2 + a_1 a_2 m + (m a_1 - a_1 m) a_2 = 0.
\end{aligned}$$

We also compute that for $f \in C^1(A, M)$, $a_1, a_2, a_3 \in A$,

$$\begin{aligned}
&(\delta_2 \circ \delta_1)(f)(a_1, a_2, a_3) = a_1(\delta_1 f)(a_2, a_3) \\
&\quad - (\delta_1 f)(a_1 a_2, a_3) + (\delta_1 f)(a_1, a_2 a_3) - (\delta_1 f)(a_1, a_2) a_3 \\
&= a_1[a_2 f(a_3) - f(a_2 a_3) + f(a_2) a_3] \\
&\quad - [a_1 a_2 f(a_3) - f(a_1 a_2 a_3) + f(a_1 a_2) a_3] \\
&\quad + [a_1 f(a_2 a_3) - f(a_1 a_2 a_3) + f(a_1) a_2 a_3] \\
&\quad - [a_1 f(a_2) a_3 - f(a_1 a_2) a_3 + f(a_1) a_2 a_3] = 0
\end{aligned}$$

by cancellation (transposed). In general, for $f \in C^n(\otimes^n A, M)$,

$$
\begin{aligned}
&(\delta_{n+1} \circ \delta_n)(f)(a_1, \cdots, a_{n+2}) = a_1(\delta_n f)(a_2, \cdots, a_{n+2}) \\
&\quad \sum_{j=1}^{n+1} (-1)^j (\delta_n f)(a_1, \cdots, a_j a_{j+1}, \cdots, a_{n+2}) + (-1)^{n+2} (\delta_n f)(a_1, \cdots, a_{n+1}) a_{n+2} \\
&= a_1[a_2 f(a_3, \cdots, a_{n+2}) + \sum_{k=1}^{n} (-1)^k f(a_2, \cdots, a_{k+1} a_{k+2}, \cdots, a_{n+2}) \\
&\quad + (-1)^{n+1} f(a_2, \cdots, a_{n+1}) a_{n+2}] - [a_1 a_2 f(a_3, \cdots, a_{n+2}) \\
&\quad - f(a_1 a_2 a_3, a_4, \cdots, a_{n+2}) + \sum_{k=2}^{n} (-1)^k f(a_1 a_2, \cdots, a_{k+1} a_{k+2}, \cdots, a_{n+2}) \\
&\quad + (-1)^{n+1} f(a_1 a_2, a_3, \cdots, a_{n+1}) a_{n+2}] \\
&+ \sum_{j=2}^{n+1} (-1)^j [a_1 f(a_2, \cdots, a_j a_{j+1}, \cdots, a_{n+2}) \\
&\quad + \sum_{k=1}^{j-2} (-1)^k f(a_1, \cdots, a_k a_{k+1}, \cdots, a_j a_{j+1}, \cdots, a_{n+2}) \\
&+ (-1)^{j-1} f(a_1, \cdots, a_{j-1} a_j a_{j+1}, \cdots, a_{n+2}) + (-1)^j f(a_1, \cdots, a_j a_{j+1} a_{j+2}, \cdots, a_{n+2}) \\
&\quad + \sum_{k=j+1}^{n} (-1)^k f(a_1, \cdots, a_j a_{j+1}, \cdots, a_{k+1} a_{k+2}, \cdots, a_{n+2}) \\
&\quad + (-1)^{n+1} f(a_1, \cdots, a_j a_{j+1}, \cdots, a_{n+1}) a_{n+2}] \\
&+ (-1)^{n+2} [a_1 f(a_2, \cdots, a_{n+1}) a_{n+2} \\
&\quad + \sum_{l=1}^{n} (-1)^l f(a_1, \cdots, a_l a_{l+1}, \cdots, a_{n+1}) a_{n+2} + (-1)^{n+1} f(a_1, \cdots, a_n) a_{n+1} a_{n+2}]
\end{aligned}
$$

which should be zero by cancellation(transposed)! □

It then follows that the cohomology groups of the chain complex $(C^*(\otimes^* A, M), \delta_*)$ are defined, and the cohomology groups are denoted by $H^n(A, M) = H^n(\otimes^n A, M)$ for $n \geq 0$. Namely,

$$H^n(A, M) = \ker(\delta_n)/\operatorname{im}(\delta_{n-1})$$

the quotient abelian group of the kernel of δ_n by the image of δ_{n-1}. In particular, $H^0(A, M) = \ker(\delta_0)$. The cohomology $H^*(A, M)$ in this sense is said to be the **Hochschild** cohomology of an algebra A with coefficients in an A-bimodule M.

Example 2.2. Let $M = A$, with bimodule structure as $a(b)c = abc$ for $a, b, c \in A$. In this case, the Hochschild complex $C^*(\otimes^* A, A)$ is also said to be the deformation or Gerstenhaber complex of A. The complex plays an important role in deformation theory of associative algebras, pioneered by Gerstenhaber [16], [17]. In particular, it is shown that $H^2(A, A)$ is the space of equivalence classes of infinitesimal deformations of A, and $H^3(A,A)$ is the space of obstructions for deformations of A.

Example 2.3. Let $M = A^* = \mathrm{Hom}(A,\mathbb{C})$ the linear dual space of A, with A-bimodule structure given by $(afb)(c) = f(bca)$ for $a,b,c \in A$ and $f \in A^*$.

$\star$ Note that for $a_1, a_2, b_1, b_2 \in A$,

$$\begin{aligned}
((a_1+a_2)f(b_1+b_2))(c) &= f((b_1+b_2)c(a_1+a_2)) \\
&= f(b_1ca_1 + b_1ca_2 + b_2ca_1 + b_2ca_2) \\
&= (a_1fb_1)(c) + (a_2fb_1)(c) + (a_1fb_2)(c) + (a_2fb_2)(c), \\
(a_1(a_2fb_1)b_2)(c) &= (a_2fb_1)(b_2ca_1) = f(b_1(b_2ca_1)a_2) \\
&= f((b_1b_2)c(a_1a_2)) = ((a_1a_2)f(b_1b_2))(c).
\end{aligned}$$

This bimodule is relevant to the cyclic cohomology theory, as (not) seen later in this chapter, such that the Hochschild cohomology groups $H^n(A, A^*)$ and the cyclic cohomology groups $HC^n(A)$ (not) defined later makes a long exact sequence.

There is the identification

$$C^n(\otimes^n A, A^*) = \mathrm{Hom}(\otimes^n A, A^*) \cong \mathrm{Hom}(\otimes^{n+1} A, \mathbb{C}), \quad f \mapsto \varphi$$

defined by $\varphi(a_0, a_1, \cdots, a_n) = f(a_1, \cdots, a_n)(a_0)$, so that the Hochschild differential δ is transformed to the differential b, given by, for $n \geq 1$,

$$\begin{aligned}
&(b\varphi)(a_0, a_1, \cdots, a_{n+1}) \\
&= \sum_{j=0}^{n} (-1)^j \varphi(a_0, \cdots, a_j a_{j+1}, \cdots, a_{n+1}) + (-1)^{n+1} \varphi(a_{n+1}a_0, a_1, \cdots, a_n)
\end{aligned}$$

but

$$\begin{aligned}
&(b\varphi)(a_0, a_1) = -\varphi(a_0a_1) + \varphi(a_1a_0) \\
&= \sum_{j=0}^{n=0} (-1)^{j-1} \varphi(a_0, \cdots, a_j a_{j+1}, \cdots, a_{n+1}) + (-1)^{n=0} \varphi(a_{n+1}a_0, a_1, \cdots, a_n)
\end{aligned}$$

(corrected) (cf. [12]).

Proof. $\star$ Check that $b_0 = \delta_0 : A^* \to C^1(A, A^*)$, because

$$\begin{aligned}
(\delta_0 f)(a_1)(a_0) &= (fa_1)(a_0) - (a_1 f)(a_0) \\
&= f(a_1a_0) - f(a_0a_1) = (b_0 f)(a_0, a_1)
\end{aligned}$$

for $f = \varphi \in A^* = \mathrm{Hom}(\otimes^0 A, A^*) \cong \mathrm{Hom}(A, \mathbb{C})$.

$\star$ Check that $b_1 = \delta_1 : C^1(A, A^*) \to C^2(\otimes^2 A, A^*)$, because

$$\begin{aligned}
&(\delta_1 f)(a_1, a_2)(a_0) = a_1 f(a_2)(a_0) - f(a_1a_2)(a_0) + f(a_1)a_2(a_0) \\
&= f(a_2)(a_0a_1) - f(a_1a_2)(a_0) + f(a_1)(a_2a_0) \\
&= \varphi(a_0a_1, a_2) - \varphi(a_0, a_1a_2) + \varphi(a_2a_0, a_1) = (b_1\varphi)(a_0, a_1, a_2).
\end{aligned}$$

$\star$ Check that $b_2 = \delta_2 : C^2(\otimes^2 A, A^*) \to C^3(\otimes^3 A, A^*)$, because

$$\begin{aligned}(\delta_1 f)(a_1, a_2, a_3)(a_0) &= a_1 f(a_2, a_3)(a_0) - f(a_1 a_2, a_3)(a_0) \\ &\quad + f(a_1, a_2 a_3)(a_0) - f(a_1, a_2) a_3(a_0) \\ &= f(a_2, a_3)(a_0 a_1) - f(a_1 a_2, a_3)(a_0) \\ &\quad + f(a_1, a_2 a_3)(a_0) - f(a_1, a_2)(a_3 a_0) \\ &= \varphi(a_0 a_1, a_2, a_3) - \varphi(a_0, a_1 a_2, a_3) \\ &\quad + \varphi(a_0, a_1, a_2 a_3) - \varphi(a_3 a_0, a_1, a_2) = (b_2 \varphi)(a_0, a_1, a_2, a_3).\end{aligned}$$

$\square$

We may denote the Hochshild complex $C^*(\otimes^* A, A^*) = C^*(\otimes^{*+1} A)$ simply by $C^*(A)$ and the Hochschild cohomology $H^*(A, A^*)$ by $Hc^*(A)$.

Example 2.4. We consider the case of $n = 0$. We have

$$H^0(A, M) = \ker(\delta_0) = \{m \in M \mid ma = am \text{ for any } a \in A\}.$$

In particular, for $M = A^*$, as checked above,

$$Hc^0(A) = H^0(A, A^*) = \{f \in A^* \mid f(ab) = f(ba) \text{ for any } a, b \in A\}$$

is the space of traces on A, denoted as $\mathrm{Tr}(A)$. Note that $C^0(A, A^*) = A^*$, and for $f \in A^*$, $bf = 0$ if and only if $f(a_0 a_1) = f(a_1 a_0)$ for $a_0, a_1 \in A$.

Example 2.5. We consider the case of $n = 1$. A Hochschild 1-cocycle $f \in C^1(A, M)$ with $\delta_1 f = 0$, so that $f \in Z^1(A, M)$, is a derivation, that is a $\mathbb{C}$-linear map $f : A \to M$ such that

$$f(ab) = af(b) + f(a)b, \quad a, b \in A.$$

Because $(\delta_1 f)(a, b) = af(b) - f(ab) + f(a)b = 0$. A 1-cocycle $f \in Z^1(A, M)$ is a coboundary in $\mathrm{im}(\delta_0) = B^1(A, M)$ if and only if it is an inner derivation, that is, $f(a) = [m, a]$ for $a \in A$. Note that for $a, b \in A$,

$$\begin{aligned}a[m, b] + [m, a]b &= a(mb - bm) + (ma - am)b \\ &= -abm + mab = [m, ab].\end{aligned}$$

Therefore,

$$H^1(A, M) = \frac{Z^1(A, M)}{B^1(A, M)} = \frac{\text{Derivations } A \to M}{\text{Inner derivations}} = \frac{\mathrm{Der}(A, M)}{\mathrm{Inn}(A, M)}.$$

The H^1 group is said to be the space of outer derivations of A with values in M, denoted as $\mathrm{Out}(A, M)$. In particular, for $M = A$, the space $\mathrm{Der}(A, A)$ of derivations of A is viewed as the Lie algebra of noncommutative vector fields on the noncommutative space represented by A.

$\star$ Indeed, for $\delta_1, \delta_2 \in \mathrm{Der}(A, A)$, the Lie bracket of δ_1, δ_2 is defined as $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1 \in \mathrm{Der}(A, A)$. Check that for $a, b \in A$,

$$\begin{aligned}
&[\delta_1, \delta_2](ab) = \delta_1(\delta_2(ab)) - \delta_2(\delta_1(ab)) \\
&= \delta_1(a\delta_2(b) + \delta_2(a)b) - \delta_2(a\delta_1(b) + \delta_1(a)b) \\
&= a\delta_1(\delta_2(b)) + \delta_1(a)\delta_2(b) + \delta_2(a)\delta_1(b) + \delta_1(\delta_2(a))b \\
&\quad - a\delta_2(\delta_1(b)) - \delta_2(a)\delta_1(b) - \delta_1(a)\delta_2(b) - \delta_2(\delta_1(a))b \\
&= a[\delta_1, \delta_2](b) + [\delta_1, \delta_2](a)b.
\end{aligned}$$

Unless A is commutative, $\mathrm{Der}(A, A)$ need not be an A-module.

$\star$ For instance, for $\delta \in \mathrm{Der}(A, A)$, define a left action $c\delta(\cdot) = \delta(\cdot c)$ by $c \in A$. Then, for $a, b \in A$, in general,

$$(c\delta)(ab) = \delta(abc) = a\delta(bc) + \delta(a)bc = a(c\delta)(b) + \delta(a)bc.$$

But if A is commutative, then

$$(c\delta)(ab) = \delta(abc) = \delta(acb) = ac\delta(b) + \delta(ac)b = a(c\delta)(b) + (c\delta)(a)b.$$

Hence, $c\delta \in \mathrm{Der}(A, A)$.

Example 2.6. We consdier the case of $n = 2$. The Hochschild cohomology group $H^2(A, M)$ classifies abelian (or singular) extensions of A by M (cf. [21]). A **singular** extension B of A by M is defined by a short exact sequence of algebras:

$$0 \to M \to B \to A \to 0$$

such that B is unital, M has trivial multiplication as $M^2 = \{0\}$, and the induced A-bimodule structure on M coincides with the original bimodule structure.

$\star$ It says that M is viewed as a nilpotent part as that

$$M \cong \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \quad \text{so that} \quad \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note also that $A \cong B/M = \{b + M \mid b \in B\}$ as cosets. Then

$$(b + M)M(c + M) = (bM + M^2)(c + M) = bMc \subset M.$$

Two such abelian (or singular) extensions B and B' are said to be **isomorphic** if there is a unital algebra map $\rho : B \to B'$ which induces the identity maps on M and A. Such a map ρ existed is necessarily an isomorphism. Namely,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\mathrm{id}"] & B \arrow[r, "p"] \arrow[d, "\rho"', "\cong"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \arrow[d, equal] \\
0 \arrow[r] & M \arrow[r] & B' \arrow[r, "p"] & A \arrow[r] & 0
\end{tikzcd}$$

Let $\mathrm{Ext}_s(A, M)$ denote the set of isomorphisms classes of such singular extensions. Define a natural bijection between as

$$\mathrm{Ext}_s(A, M) \cong H^2(A, M)$$

as follows. Given such a singular extension B of A by M. Let $s : A \to B$ be a linear splitting for the projection p from B onto A as an algebra homomorphism, so that $p \circ s$ is the identity map on A. Let $f : A \otimes A \to M$ be the curvature for s, defined by $f(a,b) = s(ab) - s(a)s(b)$ for any $a, b \in A$ and $(a,b) = a \otimes b \in A \otimes A$ (where $-f$ may be used as the definition of the curvature for s, cf. [21]).

$\star$ Note that

$$p(f(a,b)) = (p \circ s)(ab) - (p \circ s)(a)(p \circ s)(b) = ab - ab = 0,$$

and thus $f(a,b) \in M$.

It then follows that f is a Hochschild 2-cocycle in $Z^2(A,M)$ with $\delta_2 f = 0$ and its class in $H^2(A,M)$ is independent of the choice of the splitting s.

$\star$ Check that for $(a_1, a_2, a_3) = a_1 \otimes a_2 \otimes a_3 \in \otimes^3 A$,

$$\begin{aligned}
&(\delta_2 f)(a_1,a_2,a_3) = a_1 f(a_2,a_3) - f(a_1a_2,a_3) + f(a_1,a_2a_3) - f(a_1,a_2)a_3 \\
&= a_1(s(a_2a_3) - s(a_2)s(a_3)) - (s(a_1a_2a_3) - s(a_1a_2)s(a_3)) \\
&\quad + (s(a_1a_2a_3) - s(a_1)s(a_2a_3)) - (s(a_1a_2) - s(a_1)s(a_2))a_3 = 0
\end{aligned}$$

with $a_1(s(a_2a_3) - s(a_2)s(a_3)) = s(a_1)(s(a_2a_3) - s(a_2)s(a_3))$ and $(s(a_1a_2) - s(a_1)s(a_2))a_3 = (s(a_1a_2) - s(a_1)s(a_2))s(a_3)$, because $a_1 = s(a_1) + M$ and $a_3 = s(a_3) + M$, so that a_1 and a_3 can be replaced with $s(a_1)$ and $s(a_3)$ mod M respectively, and the left and right multiplications on M by A are defined by mod M. Also, the same calculation holds when s is replaced by another splitting s' from A to B. For the corresponding curvatures $f_s = f$ and $f_{s'}$, it should hold that the difference $f_s - f_{s'}$ can be in the image $\delta_1(C^1(A,M))$.

$\star$ Check the following. Note that $p(s(a) - s'(a)) = 0$ for any $a \in A$. Hence $s - s' : A \to M$ is defined and in $C^1(A,M)$. Then

$$\begin{aligned}
&(\delta_1(s - s'))(a_1,a_2) = a_1(s-s')(a_2) - (s-s')(a_1a_2) + (s-s')(a_1)a_2 \\
&= s(a_1)(s-s')(a_2) - (s-s')(a_1a_2) + (s-s')(a_1)s'(a_2) \\
&= (s(a_1)s(a_2) - s(a_1a_2)) + (s'(a_1a_2) - s'(a_1)s'(a_2)),
\end{aligned}$$

which shows that $f_{s'} - f_s = \delta_1(s - s') \in \delta_1(C^1(A,M))$.

Conversely, given a 2-cochain $f : A \otimes A \to M$, define a multiplication on $B = A \oplus M$ the direct sum by

$$(a,m)(a',m') = (aa', am' + ma' - f(a,a'))$$

for $(a,m), (a',m') \in B$ (corrected by multiplying -1 to f). This product on B defines an associative multiplication if and only if f is a 2-cocycle.

$\star$ Check that

$$\begin{aligned}
&((a,m)(a',m'))(a'',m'') = (aa', am' + ma' - f(a,a'))(a'',m'') \\
&= (aa'a'', (aa')m'' + (am' + ma' - f(a,a'))a'' - f(aa',a'')), \\
&(a,m)((a',m')(a'',m'')) = (a,m)(a'a'', a'm'' + m'a'' - f(a',a'')) \\
&= (aa'a'', a(a'm'' + m'a'' - f(a',a'')) + ma'a'' - f(a,a'a'')).
\end{aligned}$$

Associativity of the product implies that

$$f(a,a')a'' + f(aa',a'') = af(a',a'') + f(a,a'a'').$$

Equivalently,

$$(\delta_2 f)(a,a',a'') = af(a',a'') - f(aa',a'') + f(a,a'a'') - f(a,a')a'' = 0.$$

Namely, $f \in Z^2(A,M)$. Conversely, this condition implies the product associativity.

The extension associated to such a 2-cocycle f is given by

$$0 \to M \to B = A \oplus_f M \to A \to 0,$$

with $A \oplus_f M = A \oplus M$ with the multiplication involving f.

$\star$ Note that for $(0,m),(0,m') \in \{0\} \oplus M = M$ in $A \oplus_f M$, we have

$$(0,m)(0,m') = (0, 0m' + m0 - f(0,0)) = (0,0) \in M \subset A \oplus_f M,$$

and thus $M^2 = \{0\}$. Note that since f is linear, then $f(0,0) = 0$.

It may be checked that these constructions give the bijection as inverses to each other.

$\star$ As a summary, a singular extension B of A by M gives a 2-cocycle f_s for a linear splitting $s : A \to B$, up to its cohomology class. It should follow that the isomorphisms class of B implies the same class of f_s by using the following diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\mathrm{id}"] & B \arrow[r, "p", "s:\leftarrow"'] \arrow[d, "\rho"', "\cong"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \arrow[d, equal] \\
0 \arrow[r] & M \arrow[r] & B' \arrow[r, "p", "s':\leftarrow"'] & A \arrow[r] & 0
\end{tikzcd}$$

Indeed, any $s : A \to B$ can be written as $\rho^{-1} \circ s'$ for some $s' : A \to B'$. Moreover, f_s defines the extension $A \oplus_{f_s} M$ as B', to be shown.

Indeed, the map ρ may be defined by sending $b \in B$ to $(\pi(b), b - s(\pi(b))) \in B'$. Check that

$$\begin{aligned}
&(\pi(b), b - s(\pi(b)))(\pi(b'), b' - s(\pi(b'))) \\
&= (\pi(b)\pi(b'), \pi(b)(b' - s(\pi(b'))) + (b - s(\pi(b)))\pi(b') - f_s(\pi(b), \pi(b'))) \\
&= (\pi(bb'), (s(\pi(b)) + (b - s(\pi(b))))(b' - s(\pi(b'))) + (b - s(\pi(b)))s(\pi(b')) \\
&\quad - s(\pi(b)\pi(b')) + s(\pi(b))s(\pi(b'))) \\
&= (\pi(bb'), bb' - s(\pi(bb'))).
\end{aligned}$$

Also, we have $\Phi : A \oplus_{f_s} M \cong A \oplus_{f_{s'}} M$ for $f_{s'} - f_s = \delta_1(s - s')$, by sending (a,m) to $(a, m + (s - s')(a))$. Check that

$$\begin{aligned}
&\Phi(a,m)\Phi(a',m') = (a, m + (s-s')(a))(a', m' + (s-s')(a')) \\
&= (aa', a(m' + (s-s')(a')) + (m + (s-s')(a))a' - f_{s'}(a,a')) \\
&= (aa', am' + ma' + a(s-s')(a') + (s-s')(a)a' - f_{s'}(a,a')) \\
&= (aa', am' + ma' + (s-s')(aa') - f_s(a,a')) = \Phi(aa', am' + ma' - f_s(a,a')).
\end{aligned}$$

Conversely, any $f \in Z^2(A, M)$ gives an abelian (or singular) extension $A \oplus_f M$ of A by M. The cohomology class $[f] \in H^2(A, M)$ defines the isomorphism class of the extension of $A \oplus_f M$, as shown above.

Example 2.7. For $A = \mathbb{C}$, we have, for $n \geq 1$,

$$Hc^0(\mathbb{C}) = H^0(\mathbb{C}, \mathbb{C}^*) \cong \mathbb{C} \quad \text{and} \quad Hc^n(\mathbb{C}) = H^n(\mathbb{C}, \mathbb{C}^*) \cong 0.$$

$\star$ Since $H^0(\mathbb{C}, \mathbb{C}^*)$ is the space $\mathrm{Tr}(\mathbb{C})$ of traces on $\mathbb{C}$, we have $H^{0,*}(\mathbb{C}) \cong \mathbb{C}^* \cong \mathbb{C}$. Because any linear functional from $\mathbb{C}$ to $\mathbb{C}$ as an element of $\mathbb{C}^*$ is given by the multiplication operator M_w by an element w of $\mathbb{C}$ defined by $M_w(z) = wz$ for $z \in \mathbb{C}$, which is a trace on $\mathbb{C}$. Thus, $C^1(\mathbb{C}, \mathbb{C}^*) \cong \mathbb{C}$.

Also, any linear map from $\mathbb{C}$ to $\mathbb{C}^* \cong \mathbb{C}$ is given by M_w for some $w \in \mathbb{C}$, which is not a derivation from $\mathbb{C}$ to $\mathbb{C}^*$ if $w \neq 0$. Thus, $Z^1(\mathbb{C}, \mathbb{C}^*) = \{0\}$, with no inner derivations from $\mathbb{C}$ to $\mathbb{C}^*$. Hence $H^{1,*}(\mathbb{C}) \cong \{0\}$.

As well, $C^2(\mathbb{C}, \mathbb{C}^*) \cong C^1(\mathbb{C}, \mathbb{C}^*) \cong \mathbb{C}$ since $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$. Since $Z^1(\mathbb{C}, \mathbb{C}^*) = \{0\}$, then δ_1 on $C^1(\mathbb{C}, \mathbb{C}^*)$ is injective, so that $Z^2(\mathbb{C}, \mathbb{C}^*) \cong \mathbb{C}$ and the image of δ_2 is zero. Indeed, compute that for $z_1, z_2, z_3 \in \mathbb{C}$,

$$\begin{aligned}
&(\delta_2 L_w)(z_1, z_2, z_3) = z_1 L_w(z_2, z_3) - L_w(z_1 z_2, z_3) + L_w(z_1, z_2 z_3) - L_w(z_1, z_2) z_3 \\
&= z_1(w z_2 z_3) - w(z_1 z_2) z_3 + w z_1(z_2 z_3) - w(z_1 z_2) z_3 = 0.
\end{aligned}$$

Therefore, $H^2(\mathbb{C}, \mathbb{C}^*) \cong \mathbb{C}/\mathbb{C} \cong \{0\}$.

Also, it follows that $\mathrm{Ext}_s(\mathbb{C}, \mathbb{C}^*) \cong \{0\}$, so that $H^2(\mathbb{C}, \mathbb{C}^*) \cong \{0\}$. Because any non-trivial splitting s from $\mathbb{C}$ to the extension $B \cong \mathbb{C} \oplus_{f_s} \mathbb{C}^*$ is an isomorphism, so that f_s becomes the zero map. In this case, B is only the usual direct sum $\mathbb{C} \oplus \mathbb{C}^*$ as the trivial extension of $\mathbb{C}$ by $\mathbb{C}^*$.

The general case on n may be considered similarly. Indeed, $C^n(\mathbb{C}, \mathbb{C}^*) \cong \mathbb{C}$ since $\otimes^n \mathbb{C} \cong \mathbb{C}$. Also, the boundary map δ_{2n} is the zero map on $\mathbb{C}$, but δ_{2n+1} is the isomorphism on $\mathbb{C}$. Therefore,

$$\begin{aligned}
H^{2n}(\mathbb{C}, \mathbb{C}^*) &= \ker(\delta_{2n})/\mathrm{im}(\delta_{2n-1}) \cong \mathbb{C}/\mathbb{C} \cong 0, \\
H^{2n+1}(\mathbb{C}, \mathbb{C}^*) &= \ker(\delta_{2n+1})/\mathrm{im}(\delta_{2n}) \cong 0/0 \cong 0.
\end{aligned}$$

Example 2.8. Let M be a closed (i.e., compact without boundary), smooth, oriented, n-dimensional manifold and let $A = C^\infty(M)$ denote the algebra of complex-valued, smooth functions on M. For $f_0, f_1, \cdots, f_n \in A$, define the $(n+1)$-linear cochain $\varphi : \otimes^{n+1} A \to \mathbb{C}$ by

$$\varphi(f_0, f_1, \cdots, f_n) = \int_M f_0 df_1 \cdots df_n = \int_M f_0 \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n} dx_1 \cdots dx_n.$$

satisfying the following three properties.

(1) Continuous with respect to the natural Fréchet space topology of A.

(2) Becomes a Hochshild cocycle, and (3) be a cyclic cochain.

Only the Hochschild cocycle property as $b\varphi = 0$ is now checked as follows.

$$
\begin{aligned}
&(b\varphi)(f_0, f_1, \cdots, f_{n+1}) = \\
&\sum_{j=0}^{n} (-1)^j \varphi(f_0, \cdots, f_j f_{j+1}, \cdots, f_{n+1}) + (-1)^{n+1} \varphi(f_{n+1} f_0, \cdots, f_n) = \\
&\sum_{j=0}^{n} (-1)^j \int_M f_0 df_1 \cdots d(f_j f_{j+1}) \cdots df_{n+1} + (-1)^{n+1} \int_M f_{n+1} f_0 df_1 \cdots df_n \\
&= 0, \quad f_0, \cdots, f_{n+1} \in A
\end{aligned}
$$

where we use the Leibniz rule for the de Rham differential d and the graded commutativity of the algebra $(\Omega^* M, d)$ of differential forms on M.

$\star$ Note that

$$
d(f_j f_{j+1}) = \frac{\partial (f_j f_{j+1})}{\partial x_j} dx_j = \left(\frac{\partial f_j}{\partial x_j} f_{j+1} + f_j \frac{\partial f_{j+1}}{\partial x_j} \right) dx_j.
$$

In the case of $n = 0$, M is a point set, and $C^\infty(M) = \mathbb{C}1$, so that $\varphi : \mathbb{C} \to \mathbb{C}$ is the constant map. Then, for $f_0, f_1 \in \mathbb{C}$,

$$
(b\varphi)(f_0, f_1) = \varphi(f_0 f_1) - \varphi(f_1 f_0) = f_0 f_1 - f_1 f_0 = 0.
$$

In the case of $n = 1$,

$$
\varphi(f_0, f_1) = \int_M f_0 df_1 = \int_M f_0 \frac{df_1}{dx} dx.
$$

Then, for $f_0, f_1, f_2 \in C^\infty(M)$,

$$
\begin{aligned}
(b\varphi)(f_0, f_1, f_2) &= \varphi(f_0 f_1, f_2) - \varphi(f_0, f_1 f_2) + \varphi(f_2 f_0, f_1) \\
&= \int_M f_0 f_1 \frac{df_2}{dx} dx - \int_M f_0 \frac{d(f_1 f_2)}{dx} dx + \int_M f_2 f_0 \frac{df_1}{dx} dx = 0
\end{aligned}
$$

by using the differential product rule given first at $\star$.

$\star$ Recall from [4] the basic part in the de Rham theory as in the following, with notation slightly changed. Let $\mathbb{R}^n$ be the real n-dimensional Euclidean space with $(x_1, \cdots, x_n)$ as coordinates of $\mathbb{R}^n$, which plays a local chart of M as above. Let $(d\mathbb{R}^n)^+ = \mathbb{R}[1, dx_1, \cdots, dx_n]$ be the unital algebra over $\mathbb{R}$ generated by 1 and $dx_1, \cdots, dx_n$ with relations $dx_i dx_i = 0$ for $1 \leq i \leq n$ and $dx_i dx_j = -dx_j dx_i$ for $i \neq j$.

The algebra $(d\mathbb{R}^n)^+$ has a linear basis consisting of 1, dx_i for $1 \leq i \leq n$, $dx_i dx_j$ for $1 \leq i < j \leq n$, $\cdots$, $dx_{i_1} dx_{i_2} \cdots dx_{i_k}$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, $\cdots$, and $dx_1 \cdots dx_n$. Namely, $(d\mathbb{R}^n)^+$ is a graded algebra, so that

$$
(d\mathbb{R}^n)^+ = \Omega^* d\mathbb{R}^n = \Omega^0 d\mathbb{R} \oplus (\oplus_{p=1}^n \Omega^p d\mathbb{R}^n)
$$

where $\Omega^0 d\mathbb{R} = \mathbb{R}1$ and each $\Omega^p d\mathbb{R}^n$ is the real vector space generated by $dx_{i_1} \cdots dx_{i_p}$ for every $1 \leq i_1 < \cdots < i_p \leq n$.

The algebra of C^∞ differential forms of $\mathbb{R}^n$ is defined as the tensor product algebra over $\mathbb{R}$

$$\Omega^*\mathbb{R}^n = C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \Omega^* d\mathbb{R}^n,$$

where $C^\infty(\mathbb{R}^n)$ is the algebra of complex (or real) valued, smooth functions on $\mathbb{R}^n$. Any form $\omega \in \Omega^*\mathbb{R}^n$ can be uniquely written as

$$\omega = \sum_I f_I dx_I = f_0 1 + \sum_{q=1}^{n} \sum_{1\leq i_1 < \cdots < i_q \leq n} f_{i_1\cdots i_q} dx_{i_1}\cdots dx_{i_q}, \quad f_0, f_{i_1\cdots i_q} \in C^\infty(\mathbb{R}^n),$$

where we set $dx_0 = 1$. Note that the (wedge) product in $\Omega^*\mathbb{R}^n$ is defined by

$$\omega\omega' = \omega \wedge \omega' = \sum_I f_I dx_I \sum_J f_J dx_J = \sum_I \sum_J f_I f_J dx_I dx_J.$$

The algebra $\Omega^*\mathbb{R}^n$ is a graded algebra, so that

$$\Omega^*\mathbb{R}^n = \oplus_{q=0}^n \Omega^q(\mathbb{R}^n) = \oplus_{q=0}^n \Omega^q$$

with $\Omega^p\Omega^q = \Omega^{p+q}$ for $0 \leq p+q \leq n$ and $\Omega^p\Omega^q = \{0\}$ for $n+1 \leq p+q \leq 2n$, where $\Omega^q(\mathbb{R}^n)$ is the space of C^∞ q-forms on $\mathbb{R}^n$ with q as degree. Namely, for $\omega \in \Omega^q(\mathbb{R}^n)$ with $\deg\omega = q$ for $1 \leq q \leq n$,

$$\omega = \sum_{I_q} f_{I_q} dx_{I_q} = \sum_{1\leq i_1 < \cdots < i_q \leq n} f_{i_1\cdots i_q} dx_{i_1}\cdots dx_{i_q}, \quad f_{i_1\cdots i_q} \in C^\infty(\mathbb{R}^n),$$

and $\Omega^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$.

For $\omega \in \Omega^p$, $\omega' \in \Omega^q$, we have

$$\omega \wedge \omega' = (-1)^{pq}\omega' \wedge \omega = (-1)^{\deg\omega \deg\omega'}\omega' \wedge \omega.$$

The differential operator (or exterior differentiation) $d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n)$ for $0 \leq q \leq n-1$ is defined by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i, \quad \text{for } f \in \Omega^0(\mathbb{R}^n),$$

$$d\omega = \sum_{I_q} df_{I_q} dx_{I_q}, \quad \text{for } \omega = \sum_{I_q} f_{I_q} dx_{I_q} \in \Omega^q(\mathbb{R}^n).$$

Example 2.9. In the case of $n = 3$, as $\Omega^* = \Omega^*[1, dx, dy, dz]$, the spaces $\Omega^0 = \Omega^0(\mathbb{R}^3) = C^\infty(\mathbb{R}^3)$ and $\Omega^3 = \Omega^3(\mathbb{R}^3)$ are identified as a real vector space by 0-forms f and 3-forms $f dxdydz$ identified, but $\Omega^0\Omega^0 = \Omega^0 \neq \{0\} = \Omega^3\Omega^3$. Vector fields $F = (f_1, f_2, f_3)$ on $\mathbb{R}^3$ are identified with 1-forms $f_1dx + f_2dy + f_3dz$ in $\Omega^1(\mathbb{R}^3)$, which may be also identified with 2-forms $f_1dydz + f_2dzdx + f_3dxdy$ in $\Omega^2(\mathbb{R}^3)$ with $dzdx = -dxdz$.

Therefore, the differential on 0-forms as functions f is viewed as the gradient $\mathrm{grad}(f)$:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \mathrm{grad}(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

(or its transposed). The differential on 1-forms as vector fields $F = (f_1, f_2, f_3)$ is computed to be equal to the rotation (or curl) $\mathrm{rot}(F)$ of F as

$$\begin{aligned}
d(f_1dx + f_2dy + f_3dz) &= (\frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial z}dz)dx \\
&+ (\frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy + \frac{\partial f_2}{\partial z}dz)dy + (\frac{\partial f_3}{\partial x}dx + \frac{\partial f_3}{\partial y}dy + \frac{\partial f_3}{\partial z}dz)dz \\
&= (\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z})dydz + (\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x})dzdx + (\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y})dxdy \\
&= \mathrm{rot}(F) = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \\
&= \nabla \times F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times F = (\partial_x, \partial_y, \partial_z) \times F \\
&= \left(\begin{vmatrix} \partial_y & f_2 \\ \partial_z & f_3 \end{vmatrix}, \begin{vmatrix} \partial_z & f_3 \\ \partial_x & f_1 \end{vmatrix}, \begin{vmatrix} \partial_x & f_1 \\ \partial_y & f_2 \end{vmatrix}\right)
\end{aligned}$$

where $\nabla \times F$ is the outer (or vector) product of F by the partial differential operator ∇ defined so, defined as the determinant vector.

The differential on 2-forms as vector fields $F = (f_1, f_2, f_3)$ is computed to be equal to the divergence $\mathrm{div}(F)$ of F:

$$\begin{aligned}
d(f_1dydz + f_2dzdx + f_3dxdy) &= (\frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial z}dz)dydz \\
+ (\frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy &+ \frac{\partial f_2}{\partial z}dz)dzdx + (\frac{\partial f_3}{\partial x}dx + \frac{\partial f_3}{\partial y}dy + \frac{\partial f_3}{\partial z}dz)dxdy \\
= (\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z})dxdydz &= \mathrm{div}(f_1, f_2, f_3)dxdydz.
\end{aligned}$$

Proposition 2.10. *The differential $d : \Omega^p(\mathbb{R}^n) \to \Omega^{p+1}(\mathbb{R}^n)$ is an anti- (or graded) derivation as that, for ω with* $\deg \omega$ *and any $\omega' \in \Omega^*\mathbb{R}^n$,*

$$d(\omega \wedge \omega') = (d\omega) \wedge \omega' + (-1)^{\deg \omega} \omega \wedge d\omega'.$$

Proof. On $\Omega^0(\mathbb{R}^n)$, by differential product rule we have

$$\begin{aligned}
d(fg) &= \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(fg)dx_j \\
&= \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}g + f\frac{\partial g}{\partial x_j}\right) dx_j = gdf + fdg.
\end{aligned}$$

For monomials $\omega = f_I dx_I$ and $\omega' = f_J dx_J$, check that

$$\begin{aligned}
d(\omega \wedge \omega') &= d(f_I f_J)dx_I dx_J = f_J df_I dx_I dx_J + f_I df_J dx_I dx_J \\
&= d\omega \wedge \omega' + f_I(-1)^{\deg \omega} dx_I df_J dx_J = d\omega \wedge \omega' + (-1)^{\deg \omega} \omega \wedge d\omega',
\end{aligned}$$

which extends by linearity for ω with degree and any $\omega' \in \Omega^*\mathbb{R}^n$. $\square$

Proposition 2.11. *It holds that the composition $d^2 = d \circ d = 0$.*

Proof. On $\Omega^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$,

$$d^2 f = d\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i\right) = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j dx_i = 0$$

because of symmetry of partial derivatives of f and skew-symmetry of infinitesimals, so that

$$\frac{\partial^2 f}{\partial x_j \partial x_i} dx_j dx_i + \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j = \left(\frac{\partial^2 f}{\partial x_j \partial x_i} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right) dx_j dx_i = 0.$$

On homogeneous simple forms $\omega = f_I dx_I$,

$$\begin{aligned} d^2\omega &= d^2(f_I dx_I) = d(df_I dx_I) = d(df_I \wedge 1dx_I) \\ &= d(df_I) \wedge dx_I - df_I \wedge d(1dx_I) = 0dx_I - df_I \wedge 0dx_I = 0. \end{aligned}$$

□

The complex $(\Omega^*(\mathbb{R}^n) = \oplus_{p=0}^n \Omega^p(\mathbb{R}^n), d)$ is said to be the **de Rham** complex on $\mathbb{R}^n$. Forms of the kernel $Z^p(\mathbb{R}^n)$ and the image $B^{p+1}(\mathbb{R}^n)$ of $d : \Omega^p \to \Omega^{p+1}$ are said to be closed p-forms and exact $(p+1)$-forms, respectively. Since $d^2 = 0$, exact forms are closed forms.

Note that in the case of $n = 2$,

$$d(fdx + gdy) = (f_x dx + f_y dy)dx + (g_x dx + g_y dy)dy = (g_x - f_y)dxdy,$$

and thus $fdx + gdy$ is a closed 1-form if and only if the partial differential equation $g_x - f_y = 0$ holds. Namely, $fdx + gdy = (f, g)$ as a vector field is a solution to the differential equation.

The p-th de Rham cohomology of $\mathbb{R}^n$ is defined to be the quotient vector space

$$H^p(\mathbb{R}^n) = Z^p(\mathbb{R}^n)/B^p(\mathbb{R}^n).$$

Similarly, for any open subset X of $\mathbb{R}^n$, $H^*(X)$ and $\Omega^*(X)$ are defined by replacing $\mathbb{R}^n$ with X.

Lemma 2.12. (Poincaré). *We have*

$$H^p(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & q = 0, \\ 0 & 1 \le q \le n. \end{cases}$$

Example 2.13. In the case of $n = 0$, $\Omega^* = \mathbb{R}1 = \Omega^*(\mathbb{R}^0) = \Omega^0(\mathbb{R}^0)$. Since $d1 = 0$, we have $H^0(\mathbb{R}^0) = Z^0(\mathbb{R}^0) \cong \mathbb{R}$.

In the case of $n = 1$, for $f \in \Omega^0(\mathbb{R}) = C^\infty(\mathbb{R})$, we have $df = \frac{df}{dx}dx = 0$ if and only if f is a constant function on $\mathbb{R}$. Hence $H^0(\mathbb{R}) = Z^0(\mathbb{R}) = \mathbb{R}$. Also, $\Omega^1(\mathbb{R}) = Z^1(\mathbb{R})$ since $d(fdx) = \frac{df}{dx}dx \wedge dx = 0$. Moreover, any closed 1-form

$\omega = f dx$ is exact because the indefinite integral $F(x) = \int_0^x f(t)dt$ of f gives $dF = f(x)dx = \omega$. Therefore, $H^1(\mathbb{R}) \cong \{0\}$.

Consequently, if X is a disjoint union of m open intervals in $\mathbb{R}$, then $H^0(X) \cong \mathbb{R}^m$ and $H^1(X) \cong \{0\}$.

Let $\Omega_p M = \mathrm{Hom}(\Omega^p M, \mathbb{C}) = (\Omega^p M)^*$ denote then the continuous linear dual of the space $\Omega^p M$ of p-forms on M, where the locally convex topology of $\Omega^p M$ is defined by semi-norms given as, for $\omega = \sum_{I_p} f_{I_p} dx_{I_p} \in \Omega^p M$ (locally),

$$\|\omega\|_n = \sup_{|\alpha| \leq n, I_p, x \in M} |\partial^\alpha f_{I_p}(x)|,$$

where the supremum is taken over all partial derivatives ∂^α of total degree at most n of all components f_{I_p} of ω, and over a fixed finite coordinate covering for M. Elements of $\Omega_p M$ are said to be de Rham p-currents on M. In particular, elements of $\Omega_0 M = (\Omega_0 M)^* = C^\infty(M)^*$ are distributions on M.

The de Rham differential $d : \Omega^p M \to \Omega^{p+1} M$ for $0 \leq p \leq n-1$ is continuous in the topology induced by the semi-norms for homogeneous differential forms. Then we obtain the dual differential $d^* : \Omega_{p+1} M \to \Omega_p M$ defined as $d^*(\rho) = \rho \circ d$, and the de Rham complex of currents on M:

$$\Omega_0 M \xleftarrow{d^*} \Omega_1 M \xleftarrow{d^*} \Omega_2 M \xleftarrow{d^*} \cdots \xleftarrow{d^*} \Omega_n M.$$

The homology of this complex is said to be the de Rham homology of M, denoted as $H_*(M) = \oplus_{0 \leq p \leq n} H_p(M)$.

$\star$ Note that $(d^*)^2 \rho = \rho \circ d \circ d = \rho \circ 0 = 0$. Also, for $f \in \Omega^0$,

$$\|df\|_n = \sup_{|\alpha| \leq n, 1 \leq j \leq n, x \in M} |\partial^\alpha \frac{\partial}{\partial x_j} f(x)| = \|f\|_{n+1}.$$

And for $\omega = \sum_{I_p} f_{I_p} dx_{I_p} \in \Omega^p$,

$$\|d\omega\|_n = \sup_{|\alpha| \leq n, 1 \leq j \leq n, x \in M, I_p} |\partial^\alpha \frac{\partial}{\partial x_j} f_{I_p}(x)| = \|\omega\|_{n+1}.$$

May check that for any p-current $\rho \in \Omega_p M = (\Omega^p M)^*$, closed or not, the cochain defined as

$$\varphi_\rho(f_0, f_1, \cdots, f_p) = \rho(f_0 df_1 \cdots df_p) = \langle \rho, f_0 df_1 \cdots df_p \rangle$$

for $f_0, f_1, \cdots, f_p \in \Omega^0 M = C^\infty(M) = A$ is a Hochshild p-cocycle on A.

$\star$ Check that in the case of $p = 0$,

$$(b\varphi_\rho)(f_0, f_1) = \rho(f_0 f_1) - \rho(f_1 f_0) = 0.$$

In the case of $p = 1$, by using the (usual) Leibniz rule,

$$\begin{aligned}(b\varphi_\rho)(f_0, f_1, f_2) &= \rho((f_0 f_1) df_2) - \rho(f_0 d(f_1 f_2)) + \rho((f_2 f_0) df_1) \\ &= \rho((f_0 f_1) df_2) - \rho(f_0 d(f_1) f_2) - \rho(f_0 f_1 d(f_2)) + \rho((f_2 f_0) df_1) = 0.\end{aligned}$$

The general case would be shown by using the graded Leibniz rule.

As well, φ_ρ is continuous in the natural topology of $\otimes^{p+1} A$. Then taking the quotient we obtain a canonical map from $\Omega_p M$ to the continuous Hochschild cohomology $H^{p,*}_{\mathrm{con}}(C^\infty(M))$ of $C^\infty(M)$. This map is an isomorphism by Connes [11].

Example 2.14. Let $W = \mathbb{C}[1, x, \frac{d}{dx}]$ denote the **Weyl** algebra of differential operators on $\mathbb{R}$ with polynomial coefficients, where the product is defined to be the composition of operators. The Weyl algebra W is also the universal unital algebra generated by elements 1, x, and $\frac{d}{dx}$ with relation $\frac{d}{dx}x - x\frac{d}{dx} = 1$.

$\star$ Note that for $f = f(x)$ a differentiable function on $\mathbb{R}$,

$$\left(\frac{d}{dx}x - x\frac{d}{dx}\right) f = \frac{d}{dx}(xf) - x\frac{df}{dx} = f.$$

It then follows that $H^{0,*}(W) = H^0(W, W^*) = \{0\}$. Namely, there are no nonzero traces on W.

$\star$ Suppose that $f \in W^*$ is a trace on W. Then

$$f(1) = f(\frac{d}{dx}x - x\frac{d}{dx}) = f(\frac{d}{dx}x) - f(\frac{d}{dx}x) = 0.$$

Also, for a positive integer n,

$$f(x^n) = f(x^n(\frac{d}{dx}x - x\frac{d}{dx})) = f((x^n\frac{d}{dx})x) - f(x^{n+1}\frac{d}{dx}) = 0.$$

And

$$f(\frac{d^n}{dx^n}) = f(\frac{d^n}{dx^n}(\frac{d}{dx}x - x\frac{d}{dx})) = f(\frac{d^{n+1}}{dx^{n+1}}x) - f((\frac{d^n}{dx^n}x)\frac{d}{dx}) = 0.$$

Moreover,

$$\begin{aligned} f(x^n\frac{d}{dx}) &= f(x^n\frac{d}{dx}(\frac{d}{dx}x - x\frac{d}{dx})) = f(x^{n+1}\frac{d^2}{dx^2}) - f(\frac{d}{dx}x^n\frac{d}{dx}x), \\ f(\frac{d}{dx}x^n) &= f(\frac{d}{dx}x^n(\frac{d}{dx}x - x\frac{d}{dx})) = f(\frac{d}{dx}x^n\frac{d}{dx}x) - f(x^{n+1}\frac{d^2}{dx^2}). \end{aligned}$$

Therefore, by adding both sides, we obtain $2f(x^n\frac{d}{dx}) = 0$. Furthermore,

$$\begin{aligned} f(x\frac{d^n}{dx^n}) &= f(x\frac{d^n}{dx^n}(\frac{d}{dx}x - x\frac{d}{dx})) = f(x^2\frac{d^{n+1}}{dx^{n+1}}) - f(\frac{d^n}{dx^n}x\frac{d}{dx}x), \\ f(\frac{d^n}{dx^n}x) &= f(\frac{d^n}{dx^n}x(\frac{d}{dx}x - x\frac{d}{dx})) = f(\frac{d}{dx}x\frac{d^n}{dx^n}x) - f(x^2\frac{d^{n+1}}{dx^{n+1}}). \end{aligned}$$

Thus, $2f(x\frac{d^n}{dx^n}) = 0$. And in the general case,

$$\begin{aligned}
f(x^m \frac{d^n}{dx^n}) &= f(x^m \frac{d^n}{dx^n}(\frac{d}{dx}x - x\frac{d}{dx})) = f(x^m \frac{d^{n+1}}{dx^{n+1}}x) - f(x^m(\frac{d^n}{dx^n}x)\frac{d}{dx}), \\
f(x^{m-1}\frac{d^n}{dx^n}x) &= f(x^{m-1}\frac{d^n}{dx^n}x(\frac{d}{dx}x - x\frac{d}{dx})) \\
&= f(x^m \frac{d^n}{dx^n}x\frac{d}{dx}) - f(x^{m-1}(\frac{d^n}{dx^n}x^2)\frac{d}{dx}), \\
\vdots &= \vdots \\
f(\frac{d^n}{dx^n}x^m) &= f(\frac{d^n}{dx^n}x^m(\frac{d}{dx}x - x\frac{d}{dx})) = f(x\frac{d^n}{dx^n}x^m\frac{d}{dx}) - f(\frac{d^n}{dx^n}x^{m+1}\frac{d}{dx}).
\end{aligned}$$

By adding both sides of $m+1$ equations, we obtain $(m+1)f(x^m\frac{d^n}{dx^n}) = 0$ by cancellation. It then follows that $f = 0$ on W.

Example 2.15. Any derivation of the Weyl algebra W is inner. Namely, $H^1(W, W) = \{0\}$.

$\star$ Suppose that $f : W \to W$ is a derivation. Then $f(1) = 1f(1) + f(1)1$. Thus, $f(1) = 0$. Hence $f(x\frac{d}{dx}) = f(\frac{d}{dx}x)$. Namely,

$$xf(\frac{d}{dx}) + f(x)\frac{d}{dx} = \frac{d}{dx}f(x) + f(\frac{d}{dx})x.$$

Thus, $[f(x), \frac{d}{dx}] = [f(\frac{d}{dx}), x]$. And then? It is regretful that the proof here is incomplete.

Example 2.16. Any derivation of the algebra $C(X)$ of continuous, complex-valued functions on a compact Hausdorff space X is zero. Indeed, if $f = g^2$ for some $g \in C(X)$ with $g(x) = 0$ for some $x \in X$, then $(\delta f)(x) = 0$ for any derivation δ. Because, $\delta f = 2g\delta(g)$.

$\star$ Let $f \in C(X)$. Let $\mathrm{Re}(f) + i\mathrm{Im}(f)$ be the decomposition of f into the real and imaginary parts. Let $\mathrm{Re}(f)_\pm$ and $\mathrm{Im}(f)_\pm$ be the non-negative and non-positive parts of $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ respectively, defined as $\mathrm{Re}(f)_+(x) = \max\{\mathrm{Re}(f)(x), 0\}$ for $x \in X$ and $\mathrm{Re}(f)_-(x) = \min\{\mathrm{Re}(f)(x), 0\}$ for $x \in X$. Since $\mathrm{Re}(f)_+$ is continuous on X compact, then there is the maximum value M of $\mathrm{Re}(f)_+$ at some point $\alpha \in X$. Let $g = M1 - \mathrm{Re}(f)_+ \geq 0$. Then $g = h^2$ with $h = \sqrt{g}$ and $h(\alpha) = 0$. Hence, for any derivation δ, we have $\delta(g)(\alpha) = 0$. Since δ is linear, $\delta(\mathrm{Re}(f)_+)(\alpha) = 0$. And then?

Example 2.17. Any derivation of the matrix algebra $M_n(\mathbb{C})$ is inner (cf. [14]).

Proposition 2.18. *Let $Z(A)$ denote the center of an algebra A over $\mathbb{C}$. Then the Hochschild groups $H^n(A, M)$ are $Z(A)$-modules.*

Proof. Define a right action of A as well as $Z(A)$ on $C^n(A, M)$ by $(fa)(\cdots) = f(\cdots)a \in M$ for $f \in C^n(A, M)$ and $a \in A$. Then $Z^n(A, M)$ is invariant under the action by $Z(A)$. Indeed, for $f \in Z^n(A, M)$ with $\delta f = 0$,

$$\begin{aligned}
&(\delta(fa))(a_1, \cdots, a_{n+1}) = a_1(fa)(a_2, \cdots, a_{n+1}) \\
&\quad + \sum_{j=1}^{n} (-1)^{j+1}(fa)(a_1, \cdots, a_j a_{j+1}, \cdots, a_{n+1}) + (-1)^{n+1}(fa)(a_1, \cdots, a_n)a_{n+1} \\
&= (\delta f)(a_1, \cdots, a_{n+1})a = 0
\end{aligned}$$

in our sense. Define a right action by $Z(A)$ on $H^n(A, M)$ by $[f]a = [fa] = fa + B^n(A, M)$. If $f, g \in Z^n(A, M)$ with $f - g \in B^n(A, M)$ with $f - g = \delta(h)$ for some $h \in C^{n-1}(A, M)$, then $fa - ga = \delta(h)a = \delta(ha)$ with $ha \in C^{n-1}(A, M)$. Hence, $[fa] = [ga]$. Thus, the right action by $Z(A)$ on that $H^n(A, M)$ is well defined. □

Example 2.19. Let $U \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and let $X = \sum_{j=1}^{n} X_j \frac{\partial}{\partial x_j}$ denote a smooth vector field on U, with $X_j \in C^\infty(U)$ smooth functions, and X may be identified with $(X_1, \cdots, X_n)$, as vector bundles on U with $C^\infty(U)$ as fibers. Define a derivation $\delta_X : C^\infty(U) \to C^\infty(U)$ by $\delta_X(f) = \sum_{j=1}^{n} X_j \frac{\partial f}{\partial x_j}$ for $f \in C^\infty(U)$. Then there is the bijective correspondence between vector fields on U with $C^\infty(U)$ as fibers and derivations of $C^\infty(U)$ (of the form) by sending X to δ_X.

$\star$ For $f, g \in C^\infty(U)$, with $f_{x_j} = \frac{\partial f}{\partial x_j}$, note that

$$\delta_X(fg) = \sum_{j=1}^{n} X_j(f_{x_j} g + f g_{x_j}) = \delta_X(f)g + f\delta_X(g).$$

Any derivation δ of $C^\infty(U)$ has the form δ_X for some X?

The bracket $[X, Y]$ of vector fields X, Y on U corresponds to the commutator of the derivations δ_X, δ_Y, so that $\delta_{[X,Y]} = [\delta_X, \delta_Y]$.

$\star$ For $f \in C^\infty(U)$, compute that

$$\begin{aligned}
[\delta_X, \delta_Y]f &= \delta_X \delta_Y f - \delta_Y \delta_X f \\
&= \delta_X \sum_{k=1}^{n} Y_k \frac{\partial f}{\partial x_k} - \delta_Y \sum_{j=1}^{n} X_j \frac{\partial f}{\partial x_j} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} X_j \frac{\partial}{\partial x_j} \left(Y_k \frac{\partial f}{\partial x_k} \right) - \sum_{k=1}^{n} \sum_{j=1}^{n} Y_k \frac{\partial}{\partial x_k} \left(X_j \frac{\partial f}{\partial x_j} \right) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \{ X_j \frac{\partial}{\partial x_j} \left(Y_k \frac{\partial}{\partial x_k} \right) - Y_k \frac{\partial}{\partial x_k} \left(X_j \frac{\partial}{\partial x_j} \right) \} f
\end{aligned}$$

which is identified with

$$\sum_{j=1}^{n} \sum_{k=1}^{n} (X_j Y_k - Y_k X_j) f = \delta_{[X,Y]} f$$

with $\sum_{j=1}^n \sum_{k=1}^n (X_j Y_k - Y_k X_j) = [X, Y]$.

For any $x \in U$, define a $C^\infty(U)$-module structure on $\mathbb{C}$ such that $f1 = f(x)1 \in \mathbb{C}$ for $f \in C^\infty(U)$. Then the set $\mathrm{Der}(C^\infty(U), \mathbb{C})$ of $\mathbb{C}$-valued derivations of $C^\infty(U)$ is isomorphic to the complex tangent space of U at x. This correspondence is extended to arbitrary smooth manifolds. For more details of some aspects in differential geometry including differential forms and tensor analysis, connection and curvature formalism, and the Chern-Weil theory, may refer to [25] and [31].

3 H cohomology as a derived functor

Let $A^\odot$ denote the opposite algebra of an algebra A, defined as $A^\odot = A$ as a vector space with the opposite multiplication defined by $a \odot b = ba$ for $a, b \in A$. There is a 1 to 1 correspondence between A-bimodules M and left $A \otimes A^\odot$-modules M, so that

$$amb = (a \otimes b)m = a(b \odot m) = b \odot (am), \quad a, b \in A, m \in M.$$

Define a functor from the category of left $A \otimes A^\odot$-modules M to the category of complex vector spaces by sending M to

$$\mathrm{Hom}_{A \otimes A^\odot}(A, M) = \{m \in M \,|\, ma = a \odot m = am, a \in A\} = H^0(A, M).$$

Assume that A is a unital algebra. Note that A is viewed as a left $A \otimes A^\odot$-module in that sense. Consider the Bar resolution for A defined by

$$0 \leftarrow A \otimes A^\odot = B_0(A) \xleftarrow{\;b'\;} B_1(A) \xleftarrow{\;b'\;} B_2(A) \xleftarrow{\;b'\;} \cdots$$

(corrected by replacing A with $A \otimes A^\odot$) where $B_n(A) = (A \otimes A^\odot) \otimes (\otimes^n A)$ for $n \geq 0$ is the free left $A \otimes A^\odot$-module generated by $\otimes^n A$. The bar differential b' is defined by

$$b'(a \otimes b \otimes a_1 \otimes \cdots \otimes a_n) = aa_1 \otimes b \otimes a_2 \otimes \cdots \otimes a_n +$$
$$\sum_{j=1}^{n-1} (-1)^j (a \otimes b \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n) + (-1)^n a \otimes a_n b \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

$\star$ Check that for $a_1 \in A$,

$$b'(a \otimes b \otimes a_1) = aa_1 \otimes b - a \otimes a_1 b \in A \otimes A^\odot,$$

so that the map b' on $B_1(A)$ is onto $A \otimes A^\odot$(?). Because, in particular, note that $B(1 \otimes 1 \otimes a_1) = a_1 \otimes 1 - 1 \otimes a_1$. Also, certainly, we have $b' \circ b' = 0$ on $B_2(A)$ as that

$$\begin{aligned}
&b'(a \otimes b \otimes a_1 \otimes a_2) = aa_1 \otimes b \otimes a_2 - a \otimes b \otimes a_1 a_2 + a \otimes a_2 b \otimes a_1, \\
&(b')^2(a \otimes b \otimes a_1 \otimes a_2) = (aa_1)a_2 \otimes b - a(a_1 a_2) \otimes b + aa_1 \otimes a_2 b \\
&\quad - aa_1 \otimes a_2 b + a \otimes (a_1 a_2) b - a \otimes a_1(a_2 b) = 0.
\end{aligned}$$

Define the operators $s : B_n(A) \to B_{n+1}(A)$ for $n \geq 0$ by

$$s(a \otimes b \otimes a_1 \otimes \cdots \otimes a_n) = 1 \otimes b \otimes a \otimes a_1 \otimes \cdots \otimes a_n.$$

May check that $b's + sb' = \mathrm{id}$ on $B_n(A)$ for $n \geq 1$ and $b's = \mathrm{id}$ on $A \otimes A^{\odot}$. With $b' = 0$ on $A \otimes A^{\odot}$ and $s = 0$ on $\{0\}$, we have $b's + sb' = \mathrm{id}$ on $A \otimes A^{\odot} = B_0(A)$.

$\star$ Check that for $a_1 \in A$,

$$\begin{aligned}
&(b's)(a_1 \otimes 1 - 1 \otimes a_1) = b'(1 \otimes 1 \otimes a_1 - 1 \otimes a_1 \otimes 1) \\
&= a_1 \otimes 1 - 1 \otimes a_1 - 1 \otimes a_1 + 1 \otimes a_1 = \mathrm{id}(a_1 \otimes 1 - 1 \otimes a_1),
\end{aligned}$$

which implies $b's = \mathrm{id}$ on $A \otimes A^{\odot}$. Check also that

$$\begin{aligned}
&(b's)(a \otimes b \otimes a_1) = b'(1 \otimes b \otimes a \otimes a_1) \\
&= a \otimes b \otimes a_1 - 1 \otimes b \otimes aa_1 + 1 \otimes a_1 b \otimes a, \\
&(sb')(a \otimes b \otimes a_1) = s(aa_1 \otimes b - a \otimes a_1 b) \\
&= 1 \otimes b \otimes aa_1 - 1 \otimes a_1 b \otimes a,
\end{aligned}$$

so by adding both sides of which, we obtain $b's + sb' = \mathrm{id}$ on $B_1(A)$.

The equation shows that the complex $(B_*(A), b')$ is acyclic(?), and hence is a free resolution of A as a left $A \otimes A^{\odot}$-module.

For any A-bimodule M, there is an isomorphism of cochain complexes as

$$\mathrm{Hom}_{A \otimes A^{\odot}}(B_*(A), M) \cong (C^*(A, M), \delta),$$

which shows that the Hochschild cohomology is the left derived functor of the Hom functor, so that

$$H^n(A, M) \cong \mathrm{Ext}^n_{A \otimes A^{\odot}}(A, M) = H^n(\mathrm{Hom}_{A \otimes A^{\odot}}(P_*, M)), \quad n \geq 0$$

with P_* any projective resolution for M. Therefore, one may use any projective resolution of A, or any injective resolution of M, as a left $A \otimes A^{\odot}$-module to compute the Hochschild cohomology groups.

Now recall the definition of the Hochschild homology of an algebra A with coefficients in a bimodule M. The Hochschild homology complex of A with coefficients in M is the chain complex $(C_*(A, M), \delta)$, given by $C_0(A, M) = M$ and $C_n(A, M) = M \otimes (\otimes^n A)$ for $n \geq 1$ and the Hochshild boundary $\delta : C_n(A, M) \to C_{n-1}(A, M)$ defined by

$$\begin{aligned}
&\delta(m \otimes a_1 \otimes \cdots \otimes a_n) = ma_1 \otimes a_2 \otimes \cdots \otimes a_n \\
&\quad + \sum_{j=1}^{n-1} (-1)^j m \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_n.
\end{aligned}$$

with the chain

$$C_0(A, M) \xleftarrow{\ \delta\ } C_1(A, M) \xleftarrow{\ \delta\ } \cdots C_{n-1}(A, M) \xleftarrow{\ \delta\ } C_n(A, M) \cdots$$

satisfying $\delta \circ \delta = 0$.

$\star$ May check that $\delta(m \otimes a_1) = ma_1 - a_1 m$, and

$$\begin{aligned}
&(\delta \circ \delta)(m \otimes a_1 \otimes a_2) = \delta(ma_1 \otimes a_2 - m \otimes a_1 a_2 + a_2 m \otimes a_1) \\
&= (ma_1)a_2 - a_2(ma_1) - m(a_1 a_2) + (a_1 a_2)m + (a_2 m)a_1 - a_1(a_2 m) = 0.
\end{aligned}$$

The Hochschild homology of A with coefficients in M is defined to be the homology of the complex $(C_*(A, M), \delta)$, denoted by $H_n(A, M)$ for $n \geq 0$, where note that $H_0(A, M)$ is defined to be the quotient space $M/[A, M]$ since the image $\delta(C_1(A, M))$ by δ is equal to $[A, M]$ the $\mathbb{C}$-linear subspace of M spanned by commutators $[a, m] = am - ma$ for $a \in A$ and $m \in M$.

As a fact, the Hochschild homology $H_*(A, M)$ is the right derived functor of the functor from the category of left $A \otimes A^{\odot}$-modules M to the category of complex vector spaces, as

$$M \mapsto A \otimes_{A \otimes A^{\odot}} M = H_0(A, M),$$

so that

$$H_n(A, M) \cong \mathrm{Tor}_n^{A \otimes A^{\odot}}(A, M) = H_n(A \otimes_{A \otimes A^{\odot}} P_*),$$

with P_* any projective resolution for M. For the proof, we can use the Bar resolution, as done for cohomology.

For an A-bimodule M, let $M^* = \mathrm{Hom}(M, \mathbb{C})$, which is also an A-bimodule by setting $(afb)(m) = f(bma)$ for $a, b \in A$ and $m \in M$, $f \in M^*$.

$\star$ Check that

$$(a_1(a_2 f b_1) b_2)(m) = (a_2 f b_1)(b_2 m a_1) = f(b_1 b_2 m a_1 a_2) = ((a_1 a_2) f (b_1 b_2))(m).$$

There is the natural isomorphism compatible with differentials

$$\mathrm{Hom}(\otimes^n A, M^*) \cong \mathrm{Hom}(M \otimes (\otimes^n A), \mathbb{C}) = (M \otimes (\otimes^n A))^*, \quad n \geq 0.$$

$\star$ Define as sending $\varphi \mapsto \varphi^{\sim}$ that

$$\varphi(a_1, \cdots, a_n)(m) = \varphi^{\sim}(m, a_1, \cdots, a_n)$$

It then follows that the natural isomorphisms as **duality** hold

$$H^n(A, M^*) \cong H_n(A, M)^*, \quad n \geq 0.$$

The Hochschild homology groups $H_*(A, A)$ may be denoted as $Hh_*(A)$. Then the duality becomes the isomorphisms

$$Hc^n(A) \cong Hh_n(A)^*, \quad n \geq 0.$$

Example 3.1. Let $A = \mathbb{C}[x] = \mathbb{C}[x, 1]$ be the algebra of polynomials generated by 1 and x as a variable. There is the following resolution of A as a left $A \otimes A^{\odot}$-module:

$$0 \leftarrow A \xleftarrow{\ \varepsilon\ } A \otimes A^{\odot} \xleftarrow{\ d\ } A \otimes A^{\odot} \otimes \mathbb{C} \leftarrow 0,$$

where the differentials ε and d are the unique $A \otimes A^{\odot}$-linear extensions of the maps defined by $\varepsilon(1 \otimes 1) = 1$ and $d(1 \otimes 1 \otimes 1) = x \otimes 1 - 1 \otimes x$.

$\star$ Note that $(\varepsilon \circ d)(1 \otimes 1 \otimes 1) = x - x = 0$. Also, define

$$d(a \otimes b \otimes 1) = (a \otimes b)(x \otimes 1 - 1 \otimes x) = ax \otimes b - a \otimes xb$$

and $\varepsilon(a \otimes b) = ab$ for $a, b \in A$, so that $(\varepsilon \circ d)(a \otimes b \otimes 1) = 0$. This is the reason for taking $A^{\odot}$ for A noncommutaive. But in this case, $A = \mathbb{C}[x, 1] = A^{\odot}$.

The complex is equivalent to the complex

$$0 \leftarrow \mathbb{C}[x] \xleftarrow{\ \varepsilon\ } \mathbb{C}[x, y] \xleftarrow{\ d\ } \mathbb{C}[x, y] \leftarrow 0,$$

where for $p(x, y) \in \mathbb{C}[x, y]$,

$$(\varepsilon p)(x) = p(x, x), \quad (dp)(x, y) = (x - y)p(x, y).$$

$\star$ Check that

$$(\varepsilon \circ d)p(x) = (x - x)p(x, x) = 0.$$

Note that $d1(x, y) = (x - y)1(x, y) = x - y$. Also, x and y in $\mathbb{C}[x, y]$ correspond to $x \otimes 1$ and $1 \otimes x$ in $\mathbb{C}[x] \otimes \mathbb{C}[x]$ respectively.

$\star$ The operator $s : \mathbb{C}[x] \to \mathbb{C}[x, y]$ may be defined by $s(x) = x$ and $s(1) = 1$. Indeed, $(\varepsilon \circ s)(x) = \varepsilon(x) = x$ and $(\varepsilon \circ s)(1) = \varepsilon(1) = 1$, so that $\varepsilon \circ s$ is the identity map on $\mathbb{C}[x]$. The operator $s : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ may be defined by $(sp)(x, y) = \frac{1}{x-y}(-p(x, x) + p(x, y))$ as a possible choice. Indeed, it holds that

$$((s \circ \varepsilon) + (d \circ s))p(x, y) = p(x, x) + \frac{x - y}{x - y}(-p(x, x) + p(x, y)) = p(x, y).$$

Note also that if $p(x, y) = p_1(x)p_2(y)$ as a simple tensor of polynomials, then $p(x, x) - p(x, y) = p_1(x)(p_2(x) - p_2(y))$ is divided by $x - y$, and this extends by linearity.

By tensoring that resolution with the right $A \otimes A$-module A, obtained is the complex with the zero differential, with ε converted to the identity map, and so omitted

$$0 \longleftarrow \mathbb{C}[x] \xleftarrow{0 = d \otimes \mathrm{id}} \mathbb{C}[x] \longleftarrow 0.$$

And hence

$$Hh_n(\mathbb{C}[x]) \cong \begin{cases} \mathbb{C}[x], & n = 0, 1, \\ 0 & n \geq 2. \end{cases}$$

That complex is an example of a Koszul resolution (cf. [5] for the general theory in the commutative case.)

$\star$ Note that $A = (A \otimes A)A \cong A \otimes A$ in the case. Also,

$$d(1 \otimes 1 \otimes a) = (x \otimes 1)a - (1 \otimes x)a = xa - ax = xa - xa = 0.$$

Example 3.2. Let $A = \mathbb{C}[x_1, \cdots, x_n]$ denote the algebra of polynomials in n variables $x_1, \cdots, x_n$. Let V be an n-dimensional complex vector space. The Koszul resolution of A, as a left $A \otimes A$-module, is defined by

$$0 \leftarrow A \xleftarrow{\ \varepsilon\ } A \otimes A \xleftarrow{\ d\ } (\otimes^2 A) \otimes \Omega^1 \leftarrow \cdots \xleftarrow{\ d\ } (\otimes^2 A) \otimes \Omega^n \leftarrow 0,$$

where $\Omega^j = \wedge^j V$ is the j-th exterior power of V. The differentials ε and d are defined as before. The differential d has the unique extension to a graded derivation of degree -1 on the graded commutative algebra $(\otimes^2 A) \otimes \wedge^j V$. Note that $A \cong S(V)$ the symmetric algebra of the vector space V with $\dim V = n$.

Let $K(S(V))$ denote the Koszul resolution for $S(V) \cong A$ given above. To show the exactness for the resolution, note that

$$K(S(V \oplus W)) \cong K(S(V) \otimes K(S(W)))$$

for vector spaces V and W. For exact two complexes, their tensor product complex is exact. Thus, the exactness of $K(S(V))$ is reduced to the case where V is 1-dimensional. This case is considered in the last example.

As well, the following complex is shown to be the free resolution of $S(V)$, as a left $S(V) \otimes S(V)$-module

$$0 \leftarrow S(V) \xleftarrow{\ \varepsilon\ } S(V^2) \xleftarrow{\ i_X\ } S(V^2) \otimes E_1 \leftarrow \cdots \xleftarrow{\ i_X\ } S(V^2) \otimes E_n \leftarrow 0$$

with $E_k = \wedge^k V$, and i_X is the interior multiplication (contraction) with respect to the vector field $X = \sum_{j=1}^n (x_j - y_j) \frac{\partial}{\partial y_j}$ on $V^2 = V \times V$. May use the Cartan homotoy formula $di_X + i_X d = L_X$ to find a contracting homotopy for i_X.

As in the 1-dimensional case, the differentials in the complex $A \otimes_{A \otimes A} K(S(V))$ are all zero, so that

$$Hh_j(S(V)) = \mathrm{Tor}_j^{S(V) \otimes S(V)}(S(V), S(V)) \cong S(V) \otimes \wedge^j V.$$

The right-hand side is isomorphic to the module of algebraic differential forms on $S(V)$. Namely,

$$Hh_j(S(V)) \cong \Omega^j(S(V)).$$

This is special case of the Hochschild-Kostant-Rosenberg theorem. More generally, if M is a symmetric A-bimodule, the differentials of $M \otimes_{A \otimes A} K(S(V))$ vanish, and hence

$$H_j(S(V), M) \cong M \otimes \wedge^j V, \quad 0 \leq j \leq n$$

and it is zero otherwise.

Example 3.3. Let $A = T(V) = \mathbb{C} \oplus (\oplus_{j=1}^\infty \otimes^j V)$ denote the tensor algebra of a vector space V. There is the complex

$$0 \leftarrow A \xleftarrow{\ \varepsilon\ } A \otimes A^\odot \xleftarrow{\ d\ } A \otimes A^\odot \otimes V \leftarrow 0,$$

with the differentials ε and d induced by $\varepsilon(1 \otimes 1) = 1$ and $d(1 \otimes 1 \otimes v) = v \otimes 1 - 1 \otimes v$ for $v \in V$, which is a free resolution of A as a left $A \otimes A^\odot$-module. It then follows that $H_j(A, M) \cong A \otimes_{A \otimes A^\odot} M$ for $j = 0, 1$ and $H_j(A, M) = 0$ for any $j \geq 2$. Namely, A has Hochschild homological dimension 1 in this sense.

Example 3.4. There is a continuous analogue, of the resolution for $\mathbb{C}[x]$. For $A = C^\infty(S^1)$ the topological algebra of smooth functions on the circle S^1, the topological Koszul resolution is given by

$$0 \leftarrow A \xleftarrow{\ \varepsilon\ } A \otimes A \xleftarrow{\ d\ } A \otimes A \otimes \mathbb{C} \leftarrow 0$$

with the differentials defined similarly as before, where $\otimes$ means the projective tensor product of locally convex spaces. To verify exactness, with $A \otimes A$ identified with $C^\infty(S^1 \times S^1)$, the differentials are converted as that $(\varepsilon f)(x) = f(x,x)$ for $f \in C^\infty(S^1 \times S^1)$ and $x \in S^1$ and $(df)(x,y) = (x-y)f(x,y)$ for $(x,y) \in S^1 \times S^1$. The homotopy formula

$$f(x,y) = f(x,x) - (x-y)\int_0^1 \frac{\partial}{\partial y} f(x, y + t(x-y))dt$$

implies that the kernel of ε is contained in the image of d.

$\star$ Compute that

$$\int_0^1 \frac{\partial}{\partial y} f(x,y+t(x-y))dt = \left[\frac{1}{x-y} f(x, y+t(x-y))\right]_{t=0}^{1}$$
$$= \frac{1}{x-y}(f(x,x) - f(x,y)).$$

That's it! Therefore, if $\varepsilon f = 0$, then

$$f(x,y) = (x-y)\frac{1}{x-y} f(x,y) = (x-y)\int_0^1 -\frac{\partial}{\partial y} f(x, y+t(x-y))dt.$$

Alternatively, may use Fourier series to establish the exactness.

To compute the continuous Tor functor, apply the functor $(\cdot) \otimes_{A\otimes A} A$ to the above complex, so that we have

$$0 \leftarrow C^\infty(S^1) \xleftarrow{d\otimes \mathrm{id}=0} C^\infty(S^1) \leftarrow 0.$$

Therefore, the continuous Hochshild homology for A is

$$Hh_j(C^\infty(S^1)) = \begin{cases} \Omega^j S^1, & j = 0,1, \\ 0 & j \geq 2, \end{cases}$$

where $\Omega^j S^1 \cong C^\infty(S^1)\omega_j$ is the space of differential forms ω_j of degree j on S^1.

Similarly, the computation by using the continuous version of Ext by applying the functor $\mathrm{Hom}_{A\otimes A}(\cdot, A)$ gives

$$Hc^j(C^\infty(S^1)) = \begin{cases} \Omega_j S^1, & j = 0,1, \\ 0 & j \geq 2, \end{cases}$$

where $\Omega_j S^1 = (\Omega^j S^1)^*$ is the continuous dual space of differential forms ω_j on S^1, i.e., the space of j-currents on S^1.

Note that identification between the continuous tensor product $A \otimes A$ and $C^\infty(S^1 \times S^1)$ plays a crucial role in the proof above. But the algebraic tensor product of $C^\infty(S^1)$ is only dense in $C^\infty(S^1 \times S^1)$, and this makes it difficult to have a resolution to compute the algebraic Hochschild groups of $C^\infty(S^1)$, not yet known so far.

Example 3.5. Let A and B be unital algebras (over $\mathbb{C}$). There are chain maps

$$C_*(A \otimes B) \to C_*(A) \otimes C_*(B) \quad \text{and} \quad C_*(A) \otimes C_*(B) \to C_*(A \otimes B)$$

(over $\mathbb{C}$) to construct and induce inverse isomorphisms (cf. [8], [28]). It then follows that

$$Hh_n(A \otimes B) \cong \oplus_{p+q=n} Hh_p(A) \otimes Hh_q(B), \quad n \geq 0.$$

Namely, that is the Künneth relation between the Hochschild homology groups for A, B, and $A \otimes B$ (over $\mathbb{C}$).

In particular,

$$Hh_0(A \otimes B) \cong Hh_0(A) \otimes Hh_0(B).$$

There is a natural map from $Hc^0(A) \otimes Hc^0(B)$ to $Hc^0(A \otimes B)$, but it need not be surjective in general.

If A is commutative, then the multiplication $m : A \otimes A \to A$ is an algebra map and induces an associative and graded commutative product on $Hh_*(A)$.

$\star$ Namely, for $p + q = n$,

$$0 \to Hh_p(A) \otimes Hh_q(A) \to H_n(A \otimes A) \xrightarrow{m_*} H_n(A).$$

Example 3.6. Let A be the universal unital algebra generated by invertible elements u_1 and u_2 with relation $u_1 u_2 = \lambda u_2 u_1$ for some $\lambda \in \mathbb{C}$ not a root of unity with $|\lambda| = 1$ Namely, $A = \mathbb{C}[u_1, u_2]/(u_1 u_2 - \lambda u_2 u_1)$. Let $\Omega^j = \wedge^j V$, where V is a 2-dimensional complex vector space with basis e_1 and e_2. The following complex of left $A \otimes A^\odot$-modules are defined

$$0 \leftarrow A \xleftarrow{\varepsilon} A \otimes A^\odot \xleftarrow{d_0} A \otimes A^\odot \otimes \Omega^1 \xleftarrow{d_1} A \otimes A^\odot \otimes \Omega^2 \leftarrow 0,$$

where ε is the multiplication map and the differentials d_0 and d_1 are given by $d_0(1 \otimes 1 \otimes e_j) = 1 \otimes u_j - u_j \otimes 1$ for $j = 1, 2$, and

$$d_1(1 \otimes 1 \otimes (e_1 \wedge e_2)) = (u_2 \otimes 1 - \lambda \otimes u_2) \otimes e_1 - (\lambda u_1 \otimes 1 - 1 \otimes u_1) \otimes e_2.$$

This is a resolution of A as an $A \otimes A^\odot$-module, to compute $Hh_*(A)$.

$\star$ Only check that

$$\begin{aligned}
(d_0 \circ d_1)(1 \otimes 1 \otimes (e_1 \wedge e_2)) &= d_0((u_2 \otimes 1 - \lambda \otimes u_2) \otimes e_1) \\
&\quad - d_0((\lambda u_1 \otimes 1 - 1 \otimes u_1) \otimes e_2) \\
&= u_2 \otimes u_1 - \lambda \otimes u_2 u_1 - u_1 u_2 \otimes 1 + \lambda u_1 \otimes u_2 \\
&\quad - \lambda u_1 \otimes u_2 + 1 \otimes u_1 u_2 + \lambda u_2 u_1 \otimes 1 - u_2 \otimes u_1 \\
&= -\lambda \otimes u_2 u_1 - u_1 u_2 \otimes 1 + 1 \otimes u_1 u_2 + \lambda u_2 u_1 \otimes 1 = 0?
\end{aligned}$$

It certainly holds that $(\varepsilon \circ d_0) \circ d_1 = 0$. That's it!

Example 3.7. Let $W = \mathbb{C}[1, x, \frac{d}{dx}]$ be the Weyl algebra. There is a length two resolution for W as a left $W \otimes W^{\odot}$-module, to show that $Hh_2(W) \cong \mathbb{C}$ and $Hh_j(W) \cong 0$ for $j \neq 2$. The generating class for $Hh_2(W)$ is represented by the 2-cycle given by

$$1 \otimes p \otimes q - 1 \otimes q \otimes p + 1 \otimes 1 \otimes 1$$

with $q = x$ and $p = \frac{d}{dx}$. As well, for $\otimes^n W$, it is shown that $Hh_{2n}(\otimes^n W) \cong \mathbb{C}$ and $Hh_j(\otimes^n W) \cong 0$ for $j \neq 2n$. In this case, we have

$$Hh_{2n}(\otimes^n W) \cong \otimes^n Hh_2(W),$$

so that the generating class for $Hh_{2n}(\otimes^n W)$ is represented by the n-fold tensor product of that 2-cycle for W, answering the question.

Let M be an A-bimodule. A cochain $f : \otimes^n \to M$ is said to be normalized if $f(a_1, \cdots, a_n) = 0$ whenever $a_j = 1$ for some j. The space of normalized cochains, denoted by $C^*_{nom}(A, M)$ forms a subcomplex of the Hochschild complex $C^*(A, M)$, and the inclusion map from $C^*_{nom}(A, M)$ to $C^*(A, M)$ is a quasi-isomorphism.

Example 3.8. Let $A = \mathbb{C}[x]/(x^2)$ denote the algebra of dual numbers. The normalized Hochschild complex may be used to compute $Hh_*(A)$.

4 Deformation theory

Let A be a unital complex algebra. An increasing filtration on A is defined to be an increasing sequence of subspaces $F^j(A)$ of A such that $1 \in F^0(A)$, $F^j(A) \subset F^{j+1}(A)$ for integers $j \geq 0$, and $\cup_j F^j(A) = A$ and $F^i(A)F^j(A) \subset F^{i+j}(A)$ for any i, j, with $F^{-1}(A) = \{0\}$. A filtered algebra is an algebra with such a filtration. The associated graded algebra of a filtered algebra A is defined to be the graded algebra $\mathrm{Gr}(A) = \oplus_{j \geq 0}(F^j(A)/F^{j-1}(A))$.

Definition 4.1. An almost commutative algebra is a filtered algebra A whose associated graded algebra $\mathrm{Gr}(A)$ is commutative.

Being almost commutative for A is equivalent to the commutator condition $[F^i(A), F^j(A)] \subset F^{i+j-1}(A)$ for any i, j.

$\star$ May check the equivalence above. Let $a \in F^i(A)$ and $b \in F^j(A)$. Then $ab - ba \in F^{i+j}(A)$. Suppose that the commutator condition holds. Then

$$(a + F^{i-1}(A))(b + F^{j-1}(A)) = ab + aF^{j-1}(A) + F^{i-1}(A)b + F^{i-1}(A)F^{j-1}(A),$$
$$(b + F^{j-1}(A))(a + F^{i-1}(A)) = ba + F^{j-1}(A)a + bF^{i-1}(A) + F^{j-1}(A)F^{i-1}(A).$$

By subtracting both sides, $ab - ba \in [F^i(A), F^j(b)] \subset F^{i+j-1}$, $[a, F^{j-1}(A)] \subset F^{i+j-2}$, $[F^{i-1}, b] \subset F^{i+j-2}$, $[F^{i-1}, F^{j-1}] \subset F^{i+j-3}$. Namely, the right hand side is zero mod F^{i+j-1}.

Example 4.2. The Weyl algebras are almost commutative.

More generally, algebras of differential operators on a smooth manifold and universal enveloping algebras are almost commutative.

Let A be an almost commutative algebra. The Lie algebra brakcet $[x, y] = xy - yx$ for $x, y \in A$ induces the Lie algebra bracket on $\mathrm{Gr}(A)$ defined as

$$[x + F^{i-1}, y + F^{j-1}] = [x, y] + F^{i+j-2}$$

for $x \in F^i$ and $y \in F^j$ (corrected).

$\star$ Note that $xF^{j-1} - F^{j-1}x \subset F^{i+j-2}$ and $F^{i-1}y - yF^{i-1} \subset F^{i+j-2}$.

By the almost commutative assumption, $[x, y]$ belongs to $F^{i+j-1}(A)$, and $\mathrm{Gr}(A)$ is a graded Lie algebra with grading shifted by 1. The induced Lie bracket on $\mathrm{Gr}(A)$ is compatible with multiplication in the sense that the map $x + F^i \mapsto [y + F^j, x + F^i]$ is a derivation.

$\star$ Note that

$$\begin{aligned}
&[z + F^k, (x + F^i)(y + F^j)] = [z + F^k, xy + F^{i+j}] \\
&= [z, xy] + F^{k+i+j-2}, \\
&[z + F^k, x + F^i](y + F^j) + (x + F^i)[z + F^k, y + F^j] \\
&= ([z, x] + F^{k+i-2})(y + F^j) + (x + F^i)([z, y] + F^{k+j-2}) \\
&= [z, x]y + x[z, y] + F^{k+i+j-2}.
\end{aligned}$$

Note also that $[z, x]y+x[z, y] = (zx-xz)y+x(zy-yz) = zxy-xzy+xzy-xyz = [z, xy]$, which always! holds.

The algebra $\mathrm{Gr}(A)$ is said to be the semi-classical limit of the almost commutative algebra A. It is also an example of a Poisson algebra.

Note that A and $\mathrm{Gr}(A)$ are isomorphic as a vector space, but they are not as an algebra in general since A need not be commutative but $\mathrm{Gr}(A)$ is always! commutative. The linear isomorphism $q : \mathrm{Gr}(A) \to A$ is regarded as a naive quantization map. We may demands more in the sense that q is a Lie algebra map such that $q([a + F^i, b + F^j]) = [q(a), q(b)]$ for any $a, b \in A$. This is a formulation by the Dirac quantization rule (cf. [14]).

We may think of A as the algebra of quantum observables of a dynamical system acting on a Hilbert space, and think of $\mathrm{Gr}(A)$ as the algebra of classical observables of functions on the phase space. The no-going theorems as the Groenewold-Van Hove theorem states that it is almost never possible to have q to be a Lie map, under reasonable irreducibility conditions (cf. [1], [19]). There is a remedy to have q defined only for a certain class of elements of $\mathrm{Gr}(A)$, or that the required equation holds in an asymptotic sense as that it does when the Planck constant goes to zero. There are several ways to be done in the context of formal deformation quantization (cf. [2], [7], [27]), or of the C^*-algebraic strict deformation quantization (cf. [23], [32], [33]).

Definition 4.3. Let A be a commutative algebra. A Poisson structure on A is a Lie algebra bracket $[a, b]$ for $a, b \in A$ such that for any $a \in A$, the map $A \to A$

defined by $b \mapsto [a,b]$ is a derivation of A. Namely, we have $[a,bc] = [a,b]c+b[a,c]$ for $a,b,c \in A$.

The vector field defined by the derivation $b \mapsto [a,b]$ is said to be the Hamiltonian vector field of the Hamiltonian function as a.

Definition 4.4. A Poisson algebra is defined to be a commutative algebra A with a Poisson structure.

The semi-classical limit $\mathrm{Gr}(A)$ of any almost commutative algebra A is a Poisson algebra. Conversely, is any Poisson algebra the semi-classical limit of an almost commutative algebra? This is the problem of quantization of Poisson algebras, the answer to which for general Poisson algebras is negative.

A few concrete examples of Poisson algebras are given in the following (cf. [7], [9]).

Example 4.5. A Poisson manifold is defined to be a manifold M whose algebra $A = C^\infty(M)$ of smooth functions is a Poisson algebra, in which assume that the bracket is continuous in the Fréchet topology of A, or equivalently, is a bidifferentiable operator. All the Poisson structures on A are given by $[f,g] = \langle df \wedge dg, \pi\rangle$, where $\pi \in C^\infty(\wedge^2 TM)$ is a smooth 2-vector field on M. This bracket satisfies the Leibniz rule in each variable, and it satisfies the Jacobi identity if and only if $[\pi,\pi] = 0$, where the Schouten bracket $[\pi,\pi] \in C^\infty(\wedge^3 TM)$ is defined in local coordinates as

$$[\pi,\pi]_{ijk} = \sum_{l=1}^{n} \left(\pi_{lj} \frac{\partial}{\partial x_l} \pi_{ik} + \pi_{li} \frac{\partial}{\partial x_l} \pi_{kj} + \pi_{lk} \frac{\partial}{\partial x_l} \pi_{ji} \right).$$

The Poisson bracket in local coordinates is given by $[f,g] = \sum_{i,j} \pi_{ij} \frac{\partial}{\partial x_i} f \frac{\partial}{\partial x_j} g$.

$\star$ Check that

$$\begin{aligned} [f,gh] &= \langle df \wedge d(gh), \pi\rangle = \langle df \wedge (gdh + hdg), \pi\rangle \\ &= g\langle df \wedge dh, \pi\rangle + h\langle df \wedge dg, \pi\rangle = g[f,h] + h[f,g] \end{aligned}$$

and that $[g,f] = \langle dg \wedge df, \pi\rangle = -\langle df \wedge dg, \pi\rangle = -[f,g]$. The Jacobi identity in this case is

$$[[f,g],h] + [[g,h],f] + [[h,f],g] = 0.$$

Namely,

$$\begin{aligned} &\langle d\langle df \wedge dg, \pi\rangle \wedge dh, \pi\rangle + \langle d\langle dg \wedge dh, \pi\rangle \wedge df, \pi\rangle + \langle d\langle dh \wedge df, \pi\rangle \wedge dg, \pi\rangle \\ &= \langle\langle df \wedge dg, d\pi\rangle \wedge dh, \pi\rangle + \langle\langle dg \wedge dh, d\pi\rangle \wedge df, \pi\rangle + \langle\langle dh \wedge df, d\pi\rangle \wedge dg, \pi\rangle \\ &= \langle\langle (df \wedge dg) \wedge dh, d\pi\rangle, \pi\rangle + \langle\langle (dg \wedge dh) \wedge df, d\pi\rangle, \pi\rangle + \langle\langle (dh \wedge df) \wedge dg, d\pi\rangle, \pi\rangle \end{aligned}$$

in some possible sense?

Symplectic manifolds are Poisson manifolds as the simplest examples, which correspond to non-degenerate Poisson structures. A symplectic form on a symplectic manifold is a non-degenerate closed 2-form on the manifold. Given a symplectic form ω, the associated Poisson bracket is defined as $[f,g] = \omega(X_f, X_g)$, where the vector field X_f is the symplectic dual of df and is defined by requiring that the equation $df(Y) = \omega(X_f, Y)$ holds for all smooth vector fields Y on M.

Let $C^\infty_{pl}(T^*M)$ be the algebra of smooth functions on T^*M which are polynomial in the cotangent direction. This is a Poisson algebra under the natural symplectic structure of T^*M, and is the semi-classical limit of the algebra of differential operators on M.

Example 4.6. Let A be a unital commutative algebra. Let $\mathfrak{D}^0(A) = A = \mathrm{End}_A(A) \subset \mathrm{End}_{\mathbb{C}}(A)$ denote the set of differential operators of order zero on A, i.e., A-linear maps from A to A.

$\star$ Each element $a \in A$ becomes a right A-linear, left multiplication map L_a on A in the sense that $L_a(b) = ab$ for $b \in A$, so that $L_a(bc) = abc = L_a(b)c$ for $b, c \in A$.

Let $\mathfrak{D}^1(A)$ be the set of all operators D in $\mathrm{End}_{\mathbb{C}}(A)$ such that $[D, a] \in \mathfrak{D}^0(A)$ for any $a \in A$. Inductively, define $\mathfrak{D}^n(A)$ to be the set of all operators D in $\mathrm{End}_{\mathbb{C}}(A)$ such that $[D, a] \in \mathfrak{D}^{n-1}(A)$ for any $a \in A$. Elements of $\mathfrak{D}^n(A)$ are called differential operators of order n on A. The set $\mathfrak{D}(A) = \cup_{n\geq 0}\mathfrak{D}^n(A)$ is a subalgebra of $\mathrm{End}_{\mathbb{C}}(A)$, and is said to be the algebra of differential operators on A. The algebra $\mathfrak{D}(A)$ is an almost commutative algebra under the filtration given by the subspaces $\mathfrak{D}^n(A)$ for $n \geq 0$.

A linear map $D : A \to A$ is a differential operator of order 1 if and only if $D = \delta + a$, where δ is a derivation on A and $a \in A$.

$\star$ For $a, b, c, d \in A$, check that $[\delta + a, b](cd) = ([\delta + a, b](c))d$ in the following.

$$\begin{aligned}
&[\delta + a, b](cd) = ((\delta + a)b - b(\delta + a))(cd) \\
&= \delta(bcd) + abcd - b\delta(cd) - bacd, \\
&([\delta + a, b](c))d = ((\delta + a)(bc) - b(\delta + a)(c))d \\
&= \delta(bc)d + abcd - b\delta(c)d - bacd,
\end{aligned}$$

with

$$\delta(bcd) - b\delta(cd) = \delta(bc)d + bc\delta(d) - b\delta(c)d - bc\delta(d) = \delta(bc)d - b\delta(c)d.$$

For a general unital commutative algebra A, the semi-classical limit $\mathrm{Gr}(\mathfrak{D}(A))$ and its Poisson structure are not easily identified, except for coordinate rings of smooth affine varieties, or algebras of smooth functions on a manifold.

In the case of the algebra $C^\infty(M)$ of smooth functions on a manifold M with $\dim M = n$, a differential operator D of order k on $C^\infty(M)$ is locally given by $D = \sum_{|I|\leq k} a_I(x)\partial^I$ for $x \in M$, where $I = (i_1, \cdots, i_n)$ is a multi-index, and $\partial^I = \partial_{i_1} \cdots \partial_{i_n}$ is a mixed partial derivative. This expression depends on the local coordinates of M, but its leading terms of total degree k have an invariant meaning if we replace ∂_j with $\xi_j \in T^*M$. For $\xi \in T^*_xM$ and $x \in M$, let $\sigma_p(D)(x, \xi) = \sum_{|I|=k} a_I(x)\xi^I$. Then the function $\sigma_p(D) : T^*M \to \mathbb{C}$ is invariantly defined and said to be the principal symbol of D, belonging to $C^\infty_{pl}(T^*M)$. The polynomial algebra $C^\infty_{pl}(T^*M)$ has the canonical Poisson structure as a subalgebra of $C^\infty(T^*M)$ as the Poisson algebra.

Proposition 4.7. *There is a Poisson algebra isomorphism from* $\mathrm{Gr}(\mathfrak{D}(C^\infty(M)))$ *to* $C^\infty_{pl}(T^*M)$ *induced by the principal symbol* σ_p *as a map.*

Refer to [9] for a proof, or may prove it by proving it for Weyl algebras first.

Example 4.8. Let $\mathfrak{D}(\mathbb{C}[x])$ be the Weyl algebra of differential operators on the real line $\mathbb{R}$. Also, $\mathfrak{D}(\mathbb{C}[x])$ is identified with the unital complex Weyl algebra W generated by x and p with $px - xp = 1$, by sending $x \mapsto x$ and $p \mapsto \frac{d}{dx}$. The defining relation may be replaced with the canonical commutation relation $pq - qp = \frac{h}{2\pi i}1$ in Physics, where h is the Planck constant and p and q represent momentum and position operators. There is the merit in this expression that if let h go to zero, then the commutative algebra of polynomials in two variables is obtained as the semi-classical limit. Also, the imaginary i is necessary if we consider p and q as self-adjoint operators. There is the normalized representation from Physics to Math by sending $q \mapsto x$ and $p \mapsto \frac{h}{2\pi i}\frac{d}{dx}$.

$\star$ Check that the commutator $pq - qp = [p, q]$ is represented as

$$\frac{h}{2\pi i}\frac{d}{dx}(xf(x)) - x\frac{h}{2\pi i}\frac{d}{dx}f(x) = \frac{h}{2\pi i}f(x)$$

for $f(x)$ a differentiable function on $\mathbb{R}$, and p and q are extended to unbounded self-adjoint operators on the Hilbert space $L^2(\mathbb{R})$ by continuity and density.

Any element of $W = \mathfrak{D}(\mathbb{C}[x])$ has the unique finite sum expression $\sum_j a_j(x)\frac{d^j}{dx^j}$ as a differential operator with polynomial coefficients $a_j(x)$. The standard filtration for W is given by degree of the differential operators. The principal symbol map defined as $\sigma_p(\sum_{j=0}^n a_j(x)\frac{d^j}{dx^j}) = a_n(x)y^n$ induces an algebra isomorphism from $\mathrm{Gr}(W)$ to $\mathbb{C}[x, y]$. The induced Poisson bracket on $\mathbb{C}[x, y]$ is the classical Poisson bracket of partial derivatives given by

$$[f, g] = f_x g_y - f_y g_x = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = \det \begin{pmatrix} f \\ g \end{pmatrix}'.$$

The n-fold Weyl algebra $\otimes^n W$ is identified with the algebra $\mathfrak{D}(\mathbb{C}[x_1, \cdots, x_n])$ of differential operators on $\mathbb{C}[x_1, \cdots, x_n]$. The n-fold Weyl algebra $\otimes^n W$ can be also defined as the universal algebra generated by $2n$ generators x_i and p_j for $1 \le i, j \le n$ such that $[p_i, x_j] = \delta_{ij}$ and $[p_i, p_j] = [x_i, x_j] = 0$ for $1 \le i, j \le n$. in particular, the Dixmier conjecture about the automorphisms of $\otimes^n W$ is known. The Hochschild and cyclic cohomology of $\otimes^n W$ are computed in [15] (cf. [28]).

Example 4.9. Let $U(\mathfrak{g})$ denote the enveloping algebra of a Lie algebra $\mathfrak{g}$. The algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is defined to be the quotient of the tensor algebra $T(\mathfrak{g})$ of $\mathfrak{g}$ by the two-sided ideal generated by elements $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in \mathfrak{g}$. For an integer $p \ge 0$, let $F^p(U(\mathfrak{g}))$ be the subspace generated by tensors of degree at most p in $U(\mathfrak{g})$. Then $U(\mathfrak{g})$ becomes a filtered algebra as $U(\mathfrak{g}) = \cup_{p\ge 0} F^p(U(\mathfrak{g}))$. The Poincaré-Birkhoff-Witt theorem implies that the associated graded algebra $\mathrm{Gr}(U(\mathfrak{g}))$ is canonically isomorphic to the symmetric algebra $S(\mathfrak{g})$. This algebra isomorphism is induced by the symmetrization map Sy from $S(\mathfrak{g})$ to $\mathrm{Gr}(U(\mathfrak{g}))$ defined by

$$Sy(X_1X_2\cdots X_p) = \frac{1}{p!}\sum_{\sigma\in S_p} X_{\sigma(1)}X_{\sigma(2)}\cdots X_{\sigma(p)}.$$

Note as well that $S(\mathfrak{g})$ is viewed as the algebra of polynomial functions on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$, and it is a Poisson manifold under the bracket defined as $[f,g](X) = [Df(X), Dg(X)]$ for $f,g\in C^\infty(\mathfrak{g}^*)$ and $X\in\mathfrak{g}^*$, where the canonical isomorphism $\mathfrak{g}\cong(\mathfrak{g}^*)^*$ is used to regard the differential $Df(X)\in(\mathfrak{g}^*)^*$ as an element of $\mathfrak{g}$. The induced Poisson structure on polynomial functions of $S(\mathfrak{g})$ coincides with the Poisson structure in $\mathrm{Gr}(U(\mathfrak{g}))$.

Example 4.10. The algebra of formal pseudo-differential operator on the circle S^1 is obtained by the completion of the algebra of differential operators on S^1 together with the formal inverse ∂^{-1} to $\partial = \frac{d}{dz}$ for $z\in S^1$. A formal pseudo-differential operator on S^1 has an expression of the form $\sum_{j=-\infty}^{n} a_j(z)\partial^j$, where each $a_j(z)$ is a Laurent polynomial. The multiplication is determined uniquely by the rules $\partial z - z\partial = 1$ and $\partial\partial^{-1} = \partial^{-1}\partial = 1$. We may denote by Ψ_d the resulting algebra. The Adler-Manin trace on Ψ_d, as a noncommutative residue, is defined by

$$\mathrm{Tr}(\sum_{j=-\infty}^{n} a_j(z)\partial^j) = \mathrm{Res}(a_{-1}(z),0) = \frac{1}{2\pi i}\int_{S^1} a_{-1}(z)dz$$

(cf. [29]). It can be shown that $\Psi_d/[\Psi_d,\Psi_d]$ is 1-dimensional, so that any trace on Ψ_d is a multiple of the AM trace Tr.

Note that we have $[W,W]=W$ for the Weyl algebra W. There is another interesting difference between Ψ_d and W such that Ψ_d has non-inner derivations. There is also a generalization of Ψ_d to algebras of pseudo-differential operators on higher dimensional spaces. The appropriate extension of the AM trace is the noncommutative residue of Wodzicki (cf. [35]). See also [12] for relations with the Dixmier trace and its role in noncommutative Riemann geometry.

Now we may recall the formal deformation quantization theory of associative or Poisson algebras with star products, developed originally by Gerstenhaber, as a closely related notion to the formulation of quantization of almost commutative algebras by their semi-classical limits as Poisson algebras (cf. [2], [6], [30]).

Let A be a noncommutative algebra over $\mathbb{C}$ and let $A[[h]]$ denote the algebra of formal power series $\sum_{j\geq 0}^{\infty} a_jh^j = \sum_{j\geq 0}^{\infty} h^ja_j$ over A with $a_j\in A$ and h as an indefinite parameter convergeble to zero. A formal deformation of A is defined to be an associative $\mathbb{C}[[h]]$-linear multiplication $*_h : A[[h]]\otimes A[[h]]\to A[[h]]$ such that $*_0 : A\otimes A\to A$ is the original multiplication of A. For any $a,b\in A$, define the star product as

$$a *_h b = B_0(a,b) + hB_1(a,b) + h^2B_2(a,b) + \cdots + \cdots$$

where each $B_j : A \otimes A \to A$ is a Hochschild 2-cochain on A with values in A, to satisfy the associativity as

$$(a *_h b) *_h c = a *_h (b *_h c), \quad a, b, c \in A.$$

The initial condition on $*_h$ implies that $B_0(a, b) = ab$ for any $a, b \in A$.

Define the bracket $[\cdot, \cdot]$ on A by $[a, b] = B_1(a, b) - B_1(b, a)$. Equivalently, or more suggestively, may define as

$$[a, b] = \lim_{h \to 0} \frac{1}{h}(a *_h b - b *_h a).$$

$\star$ Note that

$$a *_h b - b *_h a = h(B_1(a, b) - B_1(b, a)) + h^2(B_2(a, b) - B_2(b, a)) + \cdots + \cdots$$

The associativity of the star product implies that $B_1 : A \otimes A \to A$ is a Hochschild 2-cocycle for the Hochschild cohomolog of A with coefficients in A. Namely, the relation for $a, b, c \in A$

$$(\delta B_1)(a, b, c) = aB_1(b, c) - B_1(ab, c) + B_1(a, bc) - B_1(a, b)c = 0$$

is satisfied.

$\star$ Note that we compute

$$\begin{aligned}
&(a *_h b) *_h c = (ab + hB_1(a, b) + \cdots) *_h c \\
&= (ab) *_h c + (hB_1(a, b)) *_h c + \cdots \\
&= (abc + hB_1(ab, c) + \cdots) + (hB_1(a, b)c + \cdots) + \cdots \\
&a *_h (b *_h c) = a *_h (bc + hB_1(b, c) + \cdots) \\
&= a *_h (bc) + a *_h (hB_1(b, c)) + \cdots \\
&= (abc + hB_1(a, bc) + \cdots) + (ahB_1(b, c) + \cdots) + \cdots
\end{aligned}$$

with $ah = ha$, so that $B_1(ab, c) + B_1(a, b)c = B_1(a, bc) + aB_1(b, c)$ is obtained.

The bracket $[\cdot, \cdot]$ by B_1 satisfies the Jacobi identity.

$\star$ Check that for $a, b, c \in A$,

$$\begin{aligned}
&[[a, b], c] + [[b, c], a] + [[c, a], b] \\
&= [B_1(a, b) - B_1(b, a), c] + [B_1(b, c) - B_1(c, b), a] + [B_1(c, a) - B_1(a, c), b] \\
&= B_1(B_1(a, b), c) - B_1(c, B_1(a, b)) - B_1(B_1(b, a), c) + B_1(c, B_1(b, a)) \\
&\quad + B_1(B_1(b, c), a) - B_1(a, B_1(b, c)) - B_1(B_1(c, b), a) + B_1(a, B_1(c, b)) \\
&\quad + B_1(B_1(c, a), b) - B_1(b, B_1(c, a)) - B_1(B_1(a, c), b) + B_1(b, B_1(a, c)).
\end{aligned}$$

Does this hold? At this moment, we notice that $[b, a] = -[a, b]$.

If A is a commutative algebra, then $(A, [\cdot, \cdot])$ is a Poisson algebra. Thus, in general, $(A, [\cdot, \cdot])$ is said to be a noncommutative Poisson algebra.

The bracket by B_1 can be regarded as the infinitesimal direction of the deformation. The deformation problem in commutative Poisson algebras is to find higher order terms B_j for $j \geq 2$, given B_1.

The associativity for the star product is equivalent to an infinite system of equations involving the cochains B_j such that

$$B_0 \circ^{\sim} B_n + B_1 \circ^{\sim} B_{n-1} + \cdots + B_n \circ^{\sim} B_0 = 0, \quad n \geq 0,$$

(in some sense) and equivalently,

$$\delta B_n = \sum_{j=1}^{n-1} B_j \circ^{\sim} B_{n-j},$$

where the Gerstenhaber product $f \circ^{\sim} g$ of 2-cochains $f, g : A \otimes A \to A$ is defined to the 3-cochain given by

$$(f \circ^{\sim} g)(a, b, c) = f(g(a, b), c) - f(a, g(b, c)), \quad a, b, c \in A.$$

Note that a 2-cochain f defines an associative product if and only if $f \circ^{\sim} f = 0$.

$\star$ Namely,

$$(f \circ^{\sim} f)(a, b, c) = f(f(a, b), c) - f(a, f(b, c)) = 0,$$

so that $f(f(a, b), c) = f(a, f(b, c))$.

Also, the Hochschild coboundary δf of a 2-cochain f can be written as $\delta f = -m \circ^{\sim} f - f \circ^{\sim} m$, where $m : A \otimes A \to A$ is the multiplication of A.

$\star$ Check that

$$\begin{aligned}
(\delta f)(a, b, c) &= af(b, c) - f(ab, c) + f(a, bc) - f(a, b)c \\
&= m(a, f(b, c)) - f(m(a, b), c) + f(a, m(b, c)) - m(f(a, b), c) \\
&= m(a, f(b, c)) - m(f(a, b), c) - f(m(a, b), c) + f(a, m(b, c)) \\
&= -(m \circ^{\sim} f)(a, b, c) - (f \circ^{\sim} m)(a, b, c).
\end{aligned}$$

To solve those equations of B_j with $B_0 = m$, by anti-symmetrizing we can assume that B_1 is anti-symmetric, and hence $B_1(\cdot, \cdot) = \frac{1}{2}[\cdot, \cdot]$, with $B_1(a, b) = -B_1(b, a)$! Assume now that the equation with respect to $B_0, B_1, \cdots, B_n$ holds. Then the sum $\sum_{j=1}^{n} B_j \circ^{\sim} B_{n-j}$ is shown to be a cocycle. Thus the equation of δB_{n+1} holds if and only if the sum cocycle is a cobundary, i.e., its class in $H^{n+2}(A, A)$ (corrected from power 3) should vanish. In the upshot, the third Hochschild cohomology $H^3(A, A)$ is said to be the space of obstructions for the deformation quantization problem. In particular, if the $H^3(A, A)$ vanishes, then any Poisson bracket on A can be deformed. This is only a sufficient condition, and is by no means necessary.

In the most interesting examples, we have $H^3(A, A) \neq 0$ for $A = C^\infty(M)$, as example. To see this, we consider the differential graded Lie algebra $(C^*(A, A), [\cdot, \cdot], \delta)$ of continuous Hochschild cochains on A, and the differential graded Lie algebra $(\wedge^* TM, [\cdot, \cdot], 0)$ of poly-vector fields on M with zero differential 0. The bracket in the first is the Gerstenhaver bracket, and in the second is the Schouten bracket of poly-vector fields. It is shown by a theorem of Connes that the anti-symmetrization map α from $(C^\infty(\wedge^* TM), 0)$ to $(C^*(A, A), \delta)$ sending a poly-vector field $X_1 \wedge \cdots \wedge X_k$ to the functional φ_k defined by

$$\varphi_k(f^1, \cdots, f^k) = df^1(X_1) df^2(X_2) \cdots df^k(X_k)$$

is a quasi-isomorphism of differential graded algebras (see the resolution in Lemma 44 in [11]). In particualr, induced is an isomorphism of graded commutative algebras: $\oplus_k H^k(A, A) \cong \oplus_k C^\infty(\wedge^k TM)$. However, the map α is not a morphism of Lie algebras. This is where the real difficulty of deforming a Poisson structure is hidden.

The formality theorem of Kontsevich states that $(C^*(A, A), \delta, [\cdot, \cdot])$ as a differential graded Lie algebra is formal in the sense that it is quasi-isomorphic to its cohomology ([23]). Equivalently, the map α can be perturbed to a morphism of L_∞-algebras by adding an infinite number of terms. It follows that the original deformation problem of Poisson structures can be transfered to $(C^\infty(\wedge^* TM), 0)$ with the differential zero, so unobstructed. There is also a deep structure hidden in the deformation complex of an associative $(C^*(A, A), \delta)$, such as the following as on a surface.

The first structure is given by the cup product. Now let $C^* = C^*(A, A)$. The cup product $\cup : C^p \times C^q \to C^{p+q}$ is defined by

$$(f \cup g)(a_1, \cdots, a_{p+q}) = f(a_1, \cdots, a_p) g(a_{p+1}, \cdots, a_{p+q}).$$

Note that the cup product is associative and that the union product $\cup$ is compatible with the differential δ, and hence it induces an associative graded product on $H^*(A, A)$. Moreover, this product is graded commutative for any algebra A ([16]).

$\star$ Note that for $f \in C^p$, $g \in C^q$, and $h \in C^r$,

$$\begin{aligned}
&((f \cup g) \cup h)(a_1, \cdots, a_{p+q+r}) = (f \cup g)(a_1, \cdots, a_{p+q}) h(a_{p+q+1}, \cdots, a_{p+q+r}) \\
&= f(a_1, \cdots, a_p) g(a_{p+1}, \cdots, a_{p+q}) h(a_{p+q+1}, \cdots, a_{p+q+r}) \\
&= f(a_1, \cdots, a_p)(g \cup h)(a_{p+1}, \cdots, a_{p+q+r}) \\
&= (f \cup (g \cup h))(a_1, \cdots, a_{p+q+r}).
\end{aligned}$$

$\star$ A possible compatibility should be the following doubled commutative diagram:

$$\begin{array}{ccccc}
C^{p+1} \times C^q & \xrightarrow{\cup} & C^{p+q+1} & \xleftarrow{\cup} & C^p \times C^{q+1} \\
{\scriptstyle (\delta, \mathrm{id})}\uparrow & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle (\mathrm{id}, \delta)} \\
C^p \times C^q & \xrightarrow{\cup} & C^{p+q} & \xleftarrow{\cup} & C^p \times C^q
\end{array}$$

in the sense that $\delta \circ \cup = \cup \circ (\delta, \mathrm{id}) + \cup \circ (\mathrm{id}, \delta)$. But a proof is needed.

The second structure on $(C^*(A, A), \delta)$ is given by a graded Lie bracket, based on the non-associative, Gerstenhaber circle product $\circ^\sim : C^p \times C^q \to C^{p+q-1}$

defined by that for $f \in C^p$ and $g \in C^q$,

$$(f \circ^{\sim} g)(a_1, \cdots, a_{p+q-1}) \\ = \sum_{j=1}^{p} (-1)^{(p+j-1)q} f(a_1, \cdots, g(a_j, \cdots, a_{j+q-1}), \cdots, a_{p+q-1}) \quad \text{(corrected)}.$$

In particular, for $p = 2$ and $q = 2$, we have agreed as

$$(f \circ^{\sim} g)(a_1, a_2, a_3) = f(g(a_1, a_2), a_3) - f(a_1, g(a_2, a_3)).$$

$\star$ Check that for $f, g, h \in C^2$,

$$((f \circ^{\sim} g) \circ^{\sim} h)(a_1, \cdots, a_4) = \\ -(f \circ^{\sim} g)(h(a_1, a_2), a_3, a_4) + (f \circ^{\sim} g)(a_1, h(a_2, a_3), a_4) - (f \circ^{\sim} g)(a_1, a_2, h(a_3, a_4)) \\ = -f(g(h(a_1, a_2), a_3), a_4) + f(h(a_1, a_2), g(a_3, a_4)) + f(g(a_1, h(a_2, a_3)), a_4) \\ - f(a_1, g(h(a_2, a_3), a_4)) - f(g(a_1, a_2), h(a_3, a_4)) + f(a_1, g(a_2, h(a_3, a_4))),$$

and

$$(f \circ^{\sim} (g \circ^{\sim} h))(a_1, a_2, a_3, a_4) = \\ - f((g \circ^{\sim} h)(a_1, a_2, a_3), a_4) + f(a_1, (g \circ^{\sim} h)(a_2, a_3, a_4)) \\ = -f(g(h(a_1, a_2), a_3), a_4) + f(g(a_1, h(a_2, a_3)), a_4) \\ + f(a_1, g(h(a_2, a_3), a_4)) - f(a_1, g(a_2, h(a_3, a_4))).$$

Therefore, we obtain that $((f \circ^{\sim} g) \circ^{\sim} h) = (f \circ^{\sim} (g \circ^{\sim} h))$ if and only if the following equation holds:

$$0 = f(h(a_1, a_2), g(a_3, a_4)) \\ - 2f(a_1, g(h(a_2, a_3), a_4)) - f(g(a_1, a_2), h(a_3, a_4)) + 2f(a_1, g(a_2, h(a_3, a_4))).$$

Nevertheless, it can be shown as in [16] that the corresponding graded bracket $[\cdot, \cdot] : C^p \times C^q \to C^{p+q-1}$ defined by $[f, g] = f \circ^{\sim} g - (-1)^{(p-1)(q-1)} g \circ^{\sim} f$ induces a graded Lie algebra structure on the deformation cohomology $H^*(A, A)$, with the Lie algebra grading, now shifted by 1.

What is interesting most is that the cup product and the Lie algebra structure are compatible in the sense that the graded bracket is a graded derivation for the cup product. Namely, $(H^*(A, A), \cup, [\cdot, \cdot])$ becomes a graded Poisson algebra.

The fine structure of the Hochschild cochain complex $(C^*(A, A), \delta)$ such as the existence of higher order products and their homotopies has been the subject studied in recent years. Those higher order products are relatively easily written down in the form of an algebra structure on the Hochschild complex, but it is hard for them to relate to known geometric structures such as moduli spaces of curves, as predicted by the Deligne conjecture ([24]).

As a natural question from the graded Poisson algebra structure on deformation cohomology $H^*(A, A)$, is it the semi-classical limit of a quantum cohomology theory for algebras?

Example 4.11. The dual $\mathfrak{g}^*$ of a finite dimensional Lie algebra $\mathfrak{g}$ is the simplest non-trivial Poisson manifold. Let $U_h(\mathfrak{g}) = T(\mathfrak{g})/I_h$ be the enveloping algebra with the rescaled bracket $h[\cdot,\cdot]$, where the ideal I_h is generated by elements $x \otimes y - y \otimes x - h[x,y]$ for $x, y \in \mathfrak{g}$. By the Poincaré-Birkhoff-Witt theorem, the anti-symmetrization map $\alpha_h : S(\mathfrak{g}) \to U_h(\mathfrak{g})$ is a linear isomorphism. Define the $*$-hproduct on $S(\mathfrak{g})$ by

$$f *_h g = \alpha_h^{-1}(\alpha_h(f)\alpha_h(g)) = \sum_{n=0}^{\infty} h^n B_n(f,g).$$

With some work, it can be shown that B_n are bi-differential operators, and hence the formula extends to a $*$-hproduct on $C^\infty(\mathfrak{g}^*)$.

Example 4.12. Consider the algebra generated by 1, x and y with relation $xy - yx = \frac{h}{i}1$. Let f, g be polynomials in x and y. By iterated application of the Leibniz rule, given is the formula for the product

$$f *_h g = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-ih)^n}{2^n} B_n(f,g),$$

where $B_0(f,g) = fg$, $B_1(f,g) = [f,g]$ is the standard Poisson bracket, and

$$B_n(f,g) = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\partial_x^k \partial_y^{n-k} f)(\partial_x^{n-k} \partial_y^k g), \quad n \geq 2.$$

Note that this formula does make sense for $f, g \in C^\infty(\mathbb{R}^2)$, and it defines a deformation of this smooth algebra with the standard Poisson structure. This can be extended to arbitrary constant Poisson structure on $\mathbb{R}^2$, with $[f,g] = \sum \pi^{ij} \partial_i f \partial_j g$. Then the Weyl-Moyal quantization $*$-product is given by

$$f *_h g = \exp\left(-i\frac{h}{2} \sum \pi^{ij} \partial_i \wedge \partial_j\right)(f,g).$$

Formal power series of formal deformation theory should be convergent. The Rieffel strict deformation quantization of the Poisson algebra $A = C^\infty(M)$ for a Poisson manifold $(M, [\cdot,\cdot])$ is defined to be a family of pre-C^*-algebra structures $(*_h, \|\cdot\|_h)$ on A for $h \geq 0$ such that the family forms a continuous field of pre-C^*-algebras on $[0,\infty)$, so that the function $h \mapsto \|f\|_h$ for any $f \in A$ is continuous, and

$$\|\frac{1}{ih}(f *_h g - g *_h f)\|_h \to [f,g] \quad (h \to 0), \quad f, g \in A.$$

We then have a family of C^*-algebras $\mathfrak{A}_h$ obtained by completing A with respect to the norm $\|\cdot\|_h$ ([33]).

Example 4.13. It is shown by [32] that the family of smooth (and completed) non-commutative 2-tori A_θ (and $\mathfrak{A}_\theta$ repectively) with $\theta \in [0, 2\pi]$ forms a strict deformation quantization of the Poisson algebra $C^\infty(\mathbb{T}^2)$ of smooth functions on the 2-torus $\mathbb{T}^2$. This is viewed as a special case of a more general case. In fact, let α be a smooth action of $\mathbb{R}^n$ on $A = C^\infty(M)$. Let X_j $(1 \leq j \leq n)$ denote the infinitesimal generators for the action α. Any skew-symmetric n by n matrix J defines a Poisson bracket on A by $[f,g] = \sum_{i,j=1}^n J_{ij}X_i(f)X_j(g)$ for $f, g \in A$. For each $h \in \mathbb{R}$, the new product $*_h$ on A is defined by

$$f *_h g = \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_{hJu}(f)\alpha_v(g)e^{2\pi i\langle u,v\rangle}dudv.$$

The involutive structure is defined by conjugation without deformation, and with the $*$-norm ([32]). For $A = C^\infty(\mathbb{T}^2)$ with the natural $\mathbb{R}^2$-action, the smooth non-commutative tori A_θ are defined as a strict deformation quantization of A as a quantum or phantom field.

As a remark, any Poisson manifold does admit a strict deformation quantization? This question may be still open, even for symplectic manifolds. It is shown in [32] that the canonical symplectic structure on the 2-sphere admits no $SO(3)$-invariant strict deformation quantization. As an intriguing idea given in [13], the existence of a strict deformation quantization of a Poisson manifold should be regarded as an integrability condition for formal deformation quantization. There is also an analogy in the case of formal and convergent power series solutions to differential equations around singular points. As a question, a cohomology theory that could capture the difference between these cases may be a possible theory of ambiguity. This idea is realized by Landsman [27] as the example of strict deformation quantization of Poisson manifold dual to Lie algebroids. It is shown that they are integrable precisely when they can be deformed by the C^*-algebra of the Lie groupoid integrating the given Lie algebroid. Note that the correspoding Poisson manifold is integrable if and only if so is the Lie algebroid. On the other hand, all of Hochschild cohomology as well as $H^3(A, A)$ seem to be irrelevant to stirct C^*-deformation quantization, as so far.

Example 4.14. It is shown to that the Weyl algebra W is a simple algebra, that is, it has no non-trivial two-sided ideals, and so is the same for $\otimes^n W$. Also, any automorphism of W is inner?

Example 4.15. It is shown to that there is no linear map q from $\mathbb{C}[x, y]$ to W such that $q(1) = 1$ and $q(f_x g_y - f_y g_x) = [q(f), q(g)]$ for any $f, g \in \mathbb{C}[x, y]$. This is an important special case of the Groenewold-van Hove no-going theorem ([1], [19]).

Example 4.16. The algebra $A = \mathbb{C}[x]/(x^2)$ of dual numbers is a non-smooth algebra. Its algebra of differential operators is described.

Example 4.17. The algebra Ψ of pseudo-differentail operators has non-inner derivations. The $\log\partial = -\sum_{n=1}^\infty \frac{1}{n}(1-\partial)^n$ does not belong to Ψ, but we have $[\log\partial, a] \in \Psi$ for any $a \in \Psi$. Therefore, the map δ defined by $\delta(a) = [\log\partial, a]$ defines a non-inner derivation of Ψ. The corresponding Lie algebra 2-cocycle φ defined by $\varphi(a, b) = \mathrm{tr}(a[\log\partial, b])$ is said to be the Radul cocycle [26]

5 Topological algebras

For applications of Hochschild cohomology and cyclic cohomology to noncommutative geometry, it is crucial to consider topological algebras, together with topological bimodules, topological resolutions, and continuous cochains and chains. For instance, the algebraic Hochschild groups of the algebra of smooth functions on a smooth manifold are not known, and perhaps are hopelss to compute, but its continuous Hochschild homology and cohomology as a topological algebra can be comupted as we recall below. May refer to [11], [12] for some more details. For locally convex topological vector spaces and topological tensor products, may also refer to [34].

There exists no difficulty in defining continuous analogues of Hochschild cohomology and cyclic cohomology groups for Banach or C^*-algebras. We have to simply replace bimodules by Banach or C^*-bimodules, that are bimodules which are complete by norms, with left and right module actions by bounded operators, and also cochains by continuous ones. Since the multiplication in a Banach or C^*-algebra is a continuous operation, all operators such as the Hochschid boundary map and the cyclic operators extend by continuity. However, the resulting Hochschild and cyclic theory for C^*-algebras is almost useless and does vanish in many interesting cases. This is not surprising because the definition of any Hochschild or cyclic cocycle of an algebra of dimension greater than zero involves differentiating elements of the algebra. This is in sharp contrast with topological K-theory for spaces where the Bott periodicity and so on hold, as well as the K-theory for C^*-algebras where their analogues hold.

Remark. It follows from Connes [10] and Haagerup [20] that a C^*-algebra is amenable if and only if it is nuclear.

A C^*-algebra $\mathfrak{A}$ is said to be amenable if the continous $H^n(\mathfrak{A}, M^*) = 0$ for all $n \geq 1$ and for any Banach dual bimodule M^*. In particular, it then holds that the continuous cohomology $HH^n(\mathfrak{A}) = H^n(\mathfrak{A}, \mathfrak{A}^*) = 0$ for all $n \geq 1$. It also follows from the Connes long exact sequence in cyclic cohomology that the cyclic continuous cohomology $HC^{2n}(\mathfrak{A}) = \mathfrak{A}^*$ and $HC^{2n+1}(\mathfrak{A}) = 0$ for all $n \geq 0$ and for any nuclear C^*-algebra $\mathfrak{A}$. The class of nuclear C^*-algebras contains commutative C^*-algebras, the C^*-algebra of compact operators, and the full (and reduced) group C^*-algebras of amenable groups [3].

The right class of topological algebras for Hochshild and cyclic cohomology theory be to be the class of locally convex algebras [11]. An algebra A that is a locally convex topological vector space is said to be a locally convex algebra if the multiplication map from $A \otimes A$ to A is jointly continuous, in the sense that for any continuous semi-norm p on A, there is a continuous semi-norm p' on A such that $p(ab) \leq p'(a)p'(b)$ for any $a, b \in A$ (corrected as making sense).

It may be mentioned that there are topological algebras with a locally convex topology for which the multiplication map is only separately continuous. Such more general class appears rarely in applications. But for the class of Fréchet algebras, there is no distinction between separate and joint continuity of the multiplication map. In fact, many examples of smooth noncommutative spaces in noncommutative geometry are Fréchet algebras.

Example 5.1. Let $A = C^\infty(S^1)$ as a basic example of Fréchet algebras. We may consider the elements of A as smooth periodic functions on the real line with period one. The topology on A is defined by the sequence of norms p_n for $n \in \mathbb{N}$ defined by

$$p_n(f) = \sup_{0 \le k \le n} \|f^{(k)}\|_\infty = \sup_{0 \le k \le n} \sup_{x \in S^1} |f^{(k)}(x)|.$$

for $f \in A$ and $f^{(k)}$ the k-th derivative of f. Equivalently, we may use the sequence of norms q_n defined as $q_n(f) = \sum_{k=0}^n \frac{1}{k!}\|f^{(k)}\|_\infty$. Note that each q_n is submultiplicative in the sense that $q_n(fg) \le q_n(f)q_n(g)$ for $f, g \in A$.

Locally convex algebras with topology induced by a family of submultiplicative semi-norms are known to be projective limits of Banach algebras.

$\star$ Check that

$$\|(fg)^{(k)}\|_\infty = \|\sum_{j=0}^k {}_kC_j f^{(j)} g^{(k-j)}\|_\infty \le \sum_{j=0}^k \frac{k!}{j!(k-j)!}\|f^{(j)}\|_\infty \|g^{(k-j)}\|_\infty.$$

Therefore,

$$\begin{aligned} q_n(fg) &\le \sum_{k=0}^n \sum_{j=0}^k \frac{1}{j!(k-j)!}\|f^{(j)}\|_\infty \|g^{(k-j)}\|_\infty \\ &= \sum_{k=0}^n \sum_{p+q=k, p,q \ge 0} \frac{1}{p!q!}\|f^{(p)}\|_\infty \|g^{(q)}\|_\infty \le q_n(f)q_n(g). \end{aligned}$$

On the other hand, $\|fg\|_\infty \le \|f\|_\infty \|g\|_\infty$, and

$$\|(fg)'\|_\infty = \|f'g + fg'\|_\infty \le \|f'\|_\infty \|g\|_\infty + \|f\|_\infty \|g'\|_\infty.$$

It then follows that $p_1(fg) \le 2p_1(f)p_1(g)$.

In general, we obtain that $p_n(fg) \le C_n p_n(f) p_n(g)$ for some constant $C_n \ge 0$. In such a case, we may define that p_n is submultiplicative with some constant multiple.

Let M be a closed smooth manifold and $A = C^\infty(M)$ of smooth functions on M as a basic example of Fréchet algebras. The topology of A is defined by the sequence of semi-norms p_n defined by

$$p_n(f) = \sup_{|\alpha| \le n, M \subset \cup_j U_j} \|\partial^\alpha f\|_\infty$$

where the supremum is taken over a fixed, finite coordinate cover $\cup_j U_j$ for M, with $\alpha = (\alpha_1, \cdots, \alpha_{\dim M})$ multi-index of non-negative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_{\dim M}$, and $\partial^\alpha = \partial^{\alpha_1} \cdots \partial^{\alpha_{\dim M}}$ of partial derivatives on U_j.

The Leibniz rule for derivatives of products of functions implies that the multiplication map is indeed jointly continuous.

For two locally convex topological vector spaces V_1 and V_2, their projective tensor product is defined to be a locally convex space $V_1 \otimes_p V_2$ with a universal jointly continuous bilinear map from $V_1 \otimes V_2$ to $V_1 \otimes_p V_2$ (cf. [18], [34]). Equivalently, for any locally convex space W, and a jointly continuous bilinear map from $V_1 \times V_2$ to W, there is a continuous linear map from $V_1 \otimes_p V_2$ to W. The topology of $V_1 \otimes_p V_2$ is defined explicitly by the family of semi-norms $p_1 \otimes_p p_2$ for p_1, p_2 continuous semi-norms on V_1, V_2 respectively, and

$$(p_1 \otimes_p p_2)(t) = \inf\{\sum_j p_1(a_j)p_2(b_j) \,|\, t = \sum_j a_j \otimes b_j \in V_1 \otimes V_2\}.$$

Then $V_1 \otimes_p V_2$ is defined to be the completion of $V_1 \otimes V_2$ under the family of semi-norms $p_1 \otimes_p p_2$.

The topology of $C^\infty(M)$ implies that it is nuclear (cf. [18], [34]). Namely, in particular, for any other smooth compact manifold N, the natural map from $C^\infty(M) \otimes_p C^\infty(N)$ to $C^\infty(M \times N)$ is an isomorphism of topological algebras. This plays an important role in computing the continuous Hochschild cohomology of $C^\infty(M)$.

Let A be a locally convex topological algebra. A topological left A-module is defined to be a locally convex vector space M endowed with a continuous left A-module action $A \otimes_p M \to M$. A topological free left A-module is a module of the type $M = A \otimes_p V$ for a locally convex space V. A topological projective module is a topological module which is a direct summand in a free topological module.

For a locally convex algebra A, consider $\mathrm{Hom}(\otimes_p^{n+1} A, \mathbb{C})$ the space of continuous $(n+1)$-linear functionals on A. The algebraic definitions and results with respect to $\mathrm{Hom}(A \otimes (\otimes^n A), \mathbb{C})$ can be extended to this topological setting, to define the continuous Hochschild theory groups of a locally convex algebra A. Dealing with resolutions is needed to be careful. The right class of topological projective or free resolutions is given by those resolutions that admit continuous linear splitting. For comparison theorems for resolutions and independence of cohomology from resolutions, needed are some extra conditions (cf. [11]).

Example 5.2. A locally convex topology on the smooth noncommutative 2-torus $A_\theta = A_\theta^\infty$ generated by unitaries U and V with the relation $VU = \lambda UV$ for $\lambda = e^{2\pi i\theta}$ is defined by the sequence of norms p_k defined by

$$p_k(a) = \sup_{m,n \in \mathbb{Z}} (1 + |n| + |m|)^k |a_{m,n}| \quad \text{for } a = (a_{m,n}) = \sum_{m,n} a_{m,n} U^m V^n \in A_\theta^\infty,$$

where the smoothness is given by finiteness of the norms. It is shown that the multiplication of A_θ is continuous in this topology.

$\star$ For example, $aU = \sum_{m,n} a_{m,n}\lambda^n U^{m+1}V^n = (a_{m-1,n}\lambda^n)$. Thus,

$$\begin{aligned} p_k(aU) &= \sup_{m,n}(1+|n|+|m|)^k |a_{m-1,n}\lambda^n| \\ &= \sup_{m,n}(1+|n|+|m+1|)^k |a_{m,n}| \\ &\le \sup_{m,n}(1+|n|+|m|)^{k'} |a_{m,n}| \le p_{k'}(a)p_k(U) \end{aligned}$$

for some k' larger than k, where $p_k(U) = 2^k$.

Remark. It is regretful from the time and space limited for publication that the next more story core is left unchecked and untouched by us, but possibly expected to be extended in the next time if any chance exists.

References

[1] R. Abraham and J. Marsden, *Foundations of Mechanics*, Second edition, Addison-Wesley, (1987).

[2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowics, and D. Sternheimer, *Deformation theory and quantization*, I. Deformations of symplectic structures. Ann. Physics **111** (1978), no. 1, 61-110; II. Physical applications. Ann. Physics **111** (1978), no. 1, 111-151.

[3] B. Blackadar, *Operator algebras*, Theory of C^*-algebras and von Neumann algebras, EMS 122. OA and NG III, Springer (2006).

[4] Raoul Bott and Loring W. Tu, *Differential Forms in Algebraic Topology*, GTM **82**, Springer (1982).

[5] N. Bourbaki, *Éléments de mathématique*, Algèbre. Chaptire 10, 1980.

[6] F. A. Berezin, *General concept of quantization*, Comm. Math. Phys. **40** (1975), 153-174.

[7] A. Cannas da Silva and A. Weinstein, *Geometric models for noncommutative algebras*, Berkeley Math. Lect. Notes 10, AMS (1999).

[8] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, (1956).

[9] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, (1997).

[10] A. Connes, *On the cohomology of operator algebras*, J. Funct. Anal. **28** (1978), no. 2, 248-253.

[11] A. Connes, *Noncommutative differential geometry*, Chapter I: The Chern character in K-homology; Chapter II: de Rham homology and noncommutative algebra, IHES, Publ. Math. **62** (1985), 41-144.

[12] A. Connes, *Noncommutative geometry*, Academic Press (1994).

[13] A. Connes and M. Marcolli, *A walk in the noncommutative garden*, An inviation to noncommutative geometry, World Scientific, (2008), 1-128.

[14] P. Dirac, *The fundamental equations of quantum mechanics*, Proc. Royal Soc. London A **109** (1926), 642-653.

[15] B. Feigin and B. L. Tsygan, *Additive K-theory*, K-theory, arithmetic and geometry, LNM 1289, Springer (1987), 67-209.

[16] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2) **78** (1963), 267-288.

[17] M. Gerstenbaber and S. Schack, *Algebraic cohomology and deformation theory*, Deformation theory of algebras and structures and applications (1986), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 247, Kluwer Acad. Publ. (1988), 11-264.

[18] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. no. 16 (1955).

[19] V. Guillemin and S. Sternberg, *Variations on a theme by Kepler*, AMS Colloq. Publ. 42, (1990).

[20] U. Haagerup, *All nuclear C^*-algebras are amenable*, Invent. Math. **74** (1983), no. 2, 305-319.

[21] G. Hochschild, *On the cohomology groups of an associative algebra*, Ann. of Math. (2) **46** (1945), 58-67.

[22] Masoud Khalkhali, *Basic Noncommutative Geometry*, Second edition, EMS (2013), 239 pp.

[23] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), no. 3, 157-216.

[24] M. Kontsevich and Y. Soibelman, *Deformations of algebras over operads and Deligne's conjecture*, Math. Phys. Stud. 21, Kluwer, (2000), 255-307

[25] J. L. Koszul, *Lectures on fibre bundles and differential geometry*, Tata Institute of Fundamental Research Lectures on Math, No. 20, (1965).

[26] O. Kravchenko and B. Khesin, *A central extension of the algebra of pseudo-differential symbols*, Funct. Anal. Appl. **25** (1991), no. 2, 152-154.

[27] N. P. LANDSMAN, *Mathematical topics between classical and quantum mechanics*, Springer Monogor. Math. Springer, (1998).

[28] JEAN-LOUIS LODAY, *Cyclic Homology*, Grundlehren Math. 301, Springer (1992).

[29] YU. I. MANIN, *Algebraic aspects of nonlinear differential equations*, Current problems in Mathematics, Vol. 11, (1978), 5-152 (in Russian).

[30] J. E. MOYAL, *Quantum mechanics as a statistical theory*, Proc. Cambridge Philos. Soc. **45** (1949), 99-124.

[31] E. NELSON, *Tensor analysis*, Princeton Univ. Press, (1967).

[32] M. RIEFFEL, *Deformation quantization of Heisenberg manifolds*, Comm. Math. Phys. **122** (1989), no. 4, 531-562.

[33] M. RIEFFEL, *Quantization and C^*-algebras*, C^*-algebras, Contemp. Math. 167, AMS (1994), 66-97.

[34] F. TREVES, *Topological vector spaces, distributions and kernels*, Academic Press, (1967).

[35] M. WODZICKI, *Noncommutative residue*, I. Fundamentals. K-theory, arithmetic and geometry, LNM 1289, Springer, (1987), 320-399.

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