The bottom line shaking up studying the basic noncommutative geometry, towards understanding group C^☒-algebras and more ahead

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The bottom line shaking up studying the basic noncommutative geometry, towards understanding group C^*-algebras and more ahead

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Abstract

We would like to review and study the basic noncommutative geometry by Khalkhali, but only partially as contained in the first chapter.

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1 Introduction

As a back fire to the past, for a return sparkle to the future, we as beginers would like to review and study the basic noncommutative geometry by Masoud Khalkhali [41], but not totally, to be partially selected, extended, modified, or edited by our taste, in a suitable order. For this, as nothing but a running commentary, we made some considerable effort to read and understand the texts and approximately half the contents thoroughly, to some extent, to be self-contained or not, at the basic level, within time and space limited for publication. The rest untouchable at this moment may be considered in the possible next time if any chance.

This paper is organized as follows. In Section 2, we look at the Space-C^*-algebra and Geometry-Algebra Tables as Dictionary of a couple of types, the first of which is the Space-C^*-algebra Table 2 as a part of Dictionary, well known to C^*-algebra experts. Given in the table 2 are the correspondences of general properties of spaces and C^*-algebras, and the correspondence in their homology and cohomology theories as K-theory or KK-theory, and more structures. We also consider both the geometric part and the algebraic part in the

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Geometry-Algebra Table 3. In Section 3, as given in the Group-Algebra Table 4, we consider group $*$-algebras of discrete or locally compact groups with convolution and involution and their group C^*-algebras, in some details. Moreover, we consider twisted group $*$-algebras and twisted group C^*-algebras as well, and twisted or not crossed product C^*-algebras, but somewhat limited. As well, noncommutative tori as motivated typical and important examples in noncommutative geometry are considered to some limited extent. In Section 4, we consider Hopf algebras equipped with additional co-algebraic structures to-beyond usual algebras and also do quantum (classical) groups as in the quantum Lie theory, also as in the context of noncommutative geometry.

Note that some notations are slightly changed by our taste from the original ones in [41].

First of all, let us look at the following table of the contents of this paper.

It just looks like drawing a (deformed) portrait of a beauty, as does a kid.

Table 1: Contents ★

Do you like this shape? Yes, we do.

Items cited in the references of this paper are only a part of those of [41] related to the contents and some additional items, collected by us. The details may be checked sometime later, probably, $\cdots$.

2 Dictionary looking first

2.1 Space-C^*-algebra and Geometry-Algebra

First of all, let us look at the following table as a part of Dictionary.

Table 2: An overview on spaces and C^*-algebras ★

Space Theory	C^*-algebra Theory
Topological spaces as spectrums	C^*-algebras
Compact, Hausdorff or T_2-spaces Non-compact, locally compact T_2 T_1-spaces, as point closedness T_0-spaces, as primitive unitary eq classes Non-T_0-spaces, as non-prim unitary eq Second countable or not Open or closed subsets, Both Connected components Closure of dense subsets Point or SČ compactifications Covering dimension, more $\cdots$	Unital commutative, up to Morita equivalence Non-unital commutative, up to Morita eq CCR or Liminary, as compact representations GCR or Type I, as extending comp rep Non-type I, as non-extending comp Separable or non-separable Closed ideals or quotients, Direct summands Minimal projections C^*-norm completion of dense $*$-subalgebras Unit or multipliers adjointment Real, or stable ranks, more $\cdots$
Product spaces and topology , Dynamical systems, Minimality, more $\cdots$	Tensor products and C^*-norms Crossed products, Simplicity, more $\cdots$
Topological K-theory (cohomology) Vector bundles, up to stable eq, Winding number, more $\cdots$	C^*-algebraic K-theory (homology) Projective modules, up to K_0-classes, Unitaries, up to K_1-classes, more $\cdots$
Homology theory Inclusion, Excision, more $\cdots$	Cohomology theory (cyclic or not) Extension or K-homology theory (cohomology)
Continuous maps, Unification	$*$-Homomorphisms, KK-theory
Differential structure Smooth structure Spin structure, more $\cdots$	Derivatives Dense smooth $*$-subalgebras Spectral triples, more $\cdots$
Integration as probability	Positive functionals, states, or traces
Borel or measure spaces	W^*(or vN)-algebras (weakly closed)
Classical objects or operations	Some quantum analogues

May recall that a C^*-algebra is defined to an algebra $\mathfrak{A}$ over the complex field $\mathbb{C}$, equipped with involution as an anti-linear $*$-algebra map from $\mathfrak{A}$ to $\mathfrak{A}$ denoted as a^* for $a \in \mathfrak{A}$, and the submultiplicative norm $\|\cdot\|$ satisfying

$\|ab\| \leq \|a\|\|b\|$ for $a, b \in \mathfrak{A}$, and the C^*-norm condition as $\|a^*a\| = \|a\|^2$, so that $\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$ and hence $\|a\| \leq \|a^*\| \leq \|(a^*)^*\|$ with $(a^*)^* = a$, such that $\mathfrak{A}$ is a Banach space with respect to the norm.

Theorem 2.1.1. (Gelfand-Naimark), (cf. [55]). *Any C^*-algebra is isomorphic to a closed $*$-subalgebra of the von Neumann C^*-algebra $\mathbb{B}(H)$ of all bounded operators on a Hilbert space H.*

Any unital commutative C^*-algebra is isomorphic to the C^*-algebra $C(X)$ of all continuous, $\mathbb{C}$-valued functions on a compact Hausdorff space X.

Any commutative C^*-algebra is isomorphic to $C_0(X)$ the C^*-algebra of all continuos, $\mathbb{C}$-valued functions on a locally compact Hausdorff space X vanishing at infinity.

The spectrum of a C^*-algebra $\mathfrak{A}$ is the space $\mathfrak{A}^\wedge$ of equivalence classes of irreducible representations of $\mathfrak{A}$ with the hull-kenel topology. Those class definitions of C^*-algebras, such as being liminary, and of type I, are given by the separation axioms for their spectrums (cf. [29], [55], [57]).

In particular, the spectrum of $C(X)$ is identified with X, which is also the same as the space of maximal ideals of $C(X)$ with the Jacobson topology. Namely, a point x of X is identified with the evaluation map π_x at x on $C(X)$ as a character, defined as $\pi_x(f) = f(x)$ for $f \in C(X)$, and with the kernel of π_x as a maximal ideal of $C(X)$, under the Gelfand transform (cf. [55]).

The category CH of compact Hausdorff spaces with continuous maps is equivalent to the category UCC of unital commutative C^*-algebras with unital $*$-homomorphisms. Namely, the functor $C = C(\cdot)$ is defined by pullback diagram as

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle C(\cdot)}\downarrow & & \downarrow{\scriptstyle C(\cdot)} \\ C(X) & \xleftarrow{\varphi^*} & C(Y) \end{array} \qquad \varphi^*(f) = f \circ \varphi, f \in C(Y).$$

As well, the category LCH of locally compact Hausdorff spaces with continuous proper maps is equivalent to the category CC of commutative C^*-algebras with proper $*$-homomorphisms.

Note that a continuous map $\varphi : X \to Y$ of locally compact spaces is defined to be **proper** if the inverse image $\varphi^{-1}(K)$ is compact for any compact subset K of Y.

As well, a $*$-homomorphism $\rho : \mathfrak{A} \to \mathfrak{B}$ of C^*-algebras is defined to be **proper** if the image of any approximate identity for $\mathfrak{A}$ under ρ is an approximate identity for B. Equivalently, for any nonzero irreducible representation π of $\mathfrak{B}$ as a non-zero class of $\mathfrak{B}^\wedge$, the composite $\pi \circ \rho$ is not zero as in $\mathfrak{A}^\wedge$. Also, ρ is proper if and only if $\rho(\mathfrak{A})\mathfrak{B}$ is dense in $\mathfrak{B}$. Note that an approximate identify for a C^*-algebra $\mathfrak{A}$ is a net of elements e_j of $\mathfrak{A}$ such that $\lim_j ae_j = a$ and $\lim_j e_j a = a$ for any $a \in \mathfrak{A}$ (cf. [55], [57]).

Example 2.1.2. Let X be a locally compact Hausdoff space and $X^+ = X \cup \{\infty\}$ be the one-point compactfication of X by adding ∞. Then $C(X^+)$ is isomorphic to the unitization $C_0(X)^+ = C_0(X) \oplus \mathbb{C}1$ of $C_0(X)$, as a C^*-algebra (cf. [74]). $\triangleleft$

Example 2.1.3. Let $C^b(X)$ be the C^*-algebra of all bounded, continuous functions on a locally compact Hausforff space X. Then $C^b(X)$ is isomorphic to $C(\beta X)$ with βX the Stone-Čech compactification of X, identified with $C^b(X)^\wedge$ the spectrum. Also, $C^b(X)$ is isomorphic to the multiplier C^*-algebra $M(C_0(X))$ as the largest unitization of $C_0(X)$ (cf. [74]). $\triangleleft$

Example 2.1.4. Let X be a locally compact Hausdorff space and U be an open subset of X with K the complement of U in X. Then there is the following short exact sequence of C^*-algebras:

$$0 \to C_0(U) \xrightarrow{\ i\ } C_0(X) \xrightarrow{\ q\ } C_0(K) \to 0$$

with $C_0(U)$ a closed ideal of $C_0(X)$ and $C_0(K)$ as a quotient, where the map i is the canonical inclusion map by defining values on K to be zero, and q is the restriction map to K. $\triangleleft$

Example 2.1.5. Let $X \times Y$ be the product space of locally compact Hausdorff spaces X and Y. Then $C(X \otimes Y)$ is isomorphic to the tensor product C^*-algebra $C(X) \otimes C(Y)$. For some details about the C^*-norms for tensor products of C^*-algebras, may refer to [55] or [68]. $\triangleleft$

Table 3: Functorial correspondences between geometry and algebra ♦

Geometry	Algebra
Affine algebraic varieties (or sets) over an algebraically closed field, in algebraic geometry	Unital, finitely generated, commutative, and reduced algebras (without nilpotent elements)
Affine schemes	Commutative rings
Quasi-coherent sheaves of modules over the spectrum of a commutative ring	Modules over a commutative ring as sections over the sheaves
Compact Riemann surfaces	Algebraic function fields
Sets	Complete atomic Boolean algebras

The correspondences are considered in the next subsections.

2.2 Affine varieties and Commutative reduced algebras

May refer to [35] and [8].

An affine **algebraic** variety, also called an (irreducible) algebraic (sub)set (of $\mathbb{F}$), over an algebraically closed field $\mathbb{F}$ (with the induced Z topology) is a subset V of the affine space $\mathbb{F}^n$ as the set of all n-tuples of elements of $\mathbb{F}$, which is the set of common zeros of a collection $\mathfrak{P}$ of polynomials in n variables over $\mathbb{F}$, that is,

$$V = V(\mathfrak{P}) = Z(\mathfrak{P}) = \{z = (z_1, \cdots, z_n) \in \mathbb{F}^n \,|\, p(z) = 0, p \in \mathfrak{P}\}.$$

Without loss of generality, we may assume that $\mathfrak{P}$ is an ideal of the polynomial ring $\mathbb{F}[x_1, \cdots, x_n]$ in n variables over $\mathbb{F}$.

In fact, any element $f = f(x) = f(x_1, \cdots, x_n) \in \mathbb{F}[x_1, \cdots, x_n]$ is viewed as a function from $\mathbb{F}^n$ to $\mathbb{F}$, by sending $z \in \mathbb{F}^n$ to $f(z) \in \mathbb{F}$. Then define $V(f) = Z(f) = \{z \in \mathbb{F}^n \,|\, f(z) = 0\}$. For any $\mathfrak{P} \subset \mathbb{F}[x_1, \cdots, x_n]$, let $I(\mathfrak{P})$ denote the ideal of $\mathbb{F}[x_1, \cdots, x_n]$ generated by $\mathfrak{P}$. Then it holds that $V(\mathfrak{P}) = V(I(\mathfrak{P}))$. Also, since $\mathbb{F}[x_1, \cdots, x_n]$ is a Noetherian ring, any its ideal, and in particular $V(\mathfrak{P})$, has a finite set of generators.

The set of all $V(\mathfrak{P}) \subset \mathbb{F}^n$ for any $\mathfrak{P} \subset \mathbb{F}[x_1, \cdots, x_n]$ is closed under taking finite unions and arbitrary intersections. As well, the empty set $\emptyset$ and $\mathbb{F}^n$ are assumed to be algebraic sets.

For instance, if $f(z) = z_1 \cdots z_n - 1$ and $g(z) = z_1 \cdots z_n$ for $z = (z_1, \cdots, z_n) \in \mathbb{F}^n$, then $V(\{f, g\}) = V(f) \cap V(g) = \emptyset$. As well, if $h(z) = f(z)g(z)$, then $V(h) = V(f) \cup V(g)$ which is not irreducible, but algebraic. Also, if $f(z) = 0$ for $z \in \mathbb{F}^n$, then $V(f) = \mathbb{F}^n$.

Therefore, the Zariski (**Z**) topology for $\mathbb{F}^n$ is defined by defining open subsets of $\mathbb{F}^n$ to be the complements of algebraic subsets of $\mathbb{F}^n$.

Example 2.2.1. Let $n = 1$. The affine line over $\mathbb{F}$ is $\mathbb{F}$. Every ideal of $\mathbb{F}[x]$ is principal. Thus, every algebraic subset of $\mathbb{F}$ is the set $Z(f)$ of zeros of a single polynomial $f = f(x) \in \mathbb{F}[x]$. Since $\mathbb{F}$ is algebraically closed, every nonzero polynomial $f(x) \in \mathbb{F}[x]$ can be decomposed as $c(x - a_1) \cdots (x - a_l)$ for some $c, a_1, \cdots, a_l \in \mathbb{F}$ and $l \geq 1$. Then $Z(f) = \{a_1, \cdots, a_n\}$. Thus, the set of algebraic subsets of $\mathbb{F}$ is equal to the set of all finite subsets of $\mathbb{F}$, together with the empty set and $\mathbb{F}$. In particular, the Zariski (Z) topology is not Hausdorff. Hence, $\mathbb{F}$ can not be finite.

For an affine algebraic variety $V(\mathfrak{P}) \subset \mathbb{F}^n$, an open subset of $V(\mathfrak{P})$ with the induced topology is said to be a quasi-affine variety.

A **morphism** between affine varieties $V \subset \mathbb{F}^n$ and $W \subset \mathbb{F}^m$ is given by a map $f : V \to W$, which is the restriction of a polynomial map (?) from $\mathbb{F}^n$ to $\mathbb{F}^m$ (cf. The definitions below). Then the category *Aff-Alg-Var=AAV* of affine varieties with morphisms is formed.

Definition 2.2.2. Let Y be a quasi-affine variety of $\mathbb{F}^n$. A function $f : Y \to \mathbb{F}$ is said to be **regular** at a point $y \in Y$ if there is an open subset U of Y, with $y \in U$, and polynomials $g(x), h(x) \in \mathbb{F}[x_1, \cdots, x_n]$ such that $f(z) = \frac{g(z)}{h(z)}$ for $z \in U$, with h nowhere zero on U. Say that f is regular on Y if it is regular at every point of Y.

Definition 2.2.3. Let X, Y be affine, or quasi-affine varieties over $\mathbb{F}$. A **morphism** $\varphi : X \to Y$ is a Z continuous map such that for every open subset $W \subset Y$ and for every regular function $f : W \to \mathbb{F}$, the function $f \circ \varphi : V = \varphi^{-1}(W) \to \mathbb{F}$

is regular. Namely, for some $g, h \in \mathbb{F}[x_1, \cdots, x_m]$ and $g^\sim, h^\sim \in \mathbb{F}[x_1, \cdots, x_n]$ locally existed,

$$\begin{array}{ccc} \mathbb{F}^n \supset X \supset V = \varphi^{-1}(W) & \xrightarrow{\varphi} & W \subset Y \subset \mathbb{F}^m \\ {\scriptstyle f\circ\varphi=\frac{g^\sim}{h^\sim}} \big\downarrow & & \big\downarrow {\scriptstyle f=\frac{g}{h}} \\ \mathbb{F} & = \!= \!= & \mathbb{F}. \end{array}$$

A **reduced** algebra is defined to be an algebra with no nilpotent elements. Namely, for an element x in such a algebra, if $x^n = 0$ for some n, as nilpotenty, then $x = 0$. Consider the category *Comm-Red-Alg=CRA* of unital, finitely generated, commutative, and reduced algebras with unital algebra homomorphisms.

There is the opposite equivalence between the two categories

$$\textit{Aff-Alg-Var} = AAV \cong^{op} \textit{Comm-Red-Alg} = CRA.$$

The equivalence functor associates to an affine variety $V \subset \mathbb{F}^n$ its **coordinate** ring $\mathcal{O}[V]$ defined by

$$\mathcal{O}[V] = \mathrm{Reg}(V, \mathbb{F}) \cong \mathbb{F}[x_1, \cdots, x_n]/I(V) = \mathbb{F}[x_1, \cdots, x_n](V)$$

(corrected), where $\mathcal{O}[V] = \mathrm{Reg}(V, \mathbb{F})$ is the ring of all regular functions on Y, and $I(V)$ is the vanishing ideal of V defined by

$$I(V) = \{p \in \mathbb{F}[x_1, \cdots, x_n] \,|\, p(V) = 0\}.$$

Then, $\mathcal{O}[V]$ is unital, finitely generated,commutative, and reduced. Moreover, given a morphism of varieties $f : V \to W$, its pullback defines a unital algebra homomorphism $f^* : \mathcal{O}[W] \to \mathcal{O}[V]$. Namely,

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle \varphi\circ f \in \mathcal{O}[V]} \big\downarrow & & \big\downarrow {\scriptstyle \varphi \in \mathcal{O}[W]} \\ \mathbb{F} & = \!= \!= & \mathbb{F} \end{array} \qquad \begin{array}{c} \varphi \in \mathrm{Hom}(V, W) \\ {\scriptstyle f^*} \big\downarrow {\scriptstyle \text{pullback}} \\ f^*(\varphi) = \varphi \circ f \in \mathrm{Hom}(\mathcal{O}[W], \mathcal{O}[V]). \end{array}$$

Hence, the contravariant functor $\mathcal{O} : AAV \to CRA$ is defined as $V \mapsto \mathcal{O}[V]$.

A finitely generated, unital commutative algebra A with n generators can be written as a quotient $\mathbb{F}[x_1, \cdots, x_n]/J$ by some ideal J. Moreover, such an algebra A is a reduced algebra, so that it has no nilpotent elements, if and only if the ideal J is a radical ideal in the sense that if $x^n \in J$, then $x \in J$. In this case, as one of the classical forms of the Hilbert Nullstellensatz (HN), A can be recovered as the coordinate ring $\mathcal{O}[V]$ of an affine variety V

It then follows that the coordinate ring functor $\mathcal{O}$ is essentially surjective, as the main step in the opposite equivalence, and moreover, the functor is full and faithful, easily deduced.

As in the Gelfand-Naimark (GN) correspondence, under the Hilbert N (HN) correspondence $\mathcal{O}$, geometric constructions in algebraic geometry can be translated into algebraic terms and vise-versa. For instance, disjoint union and direct product of affine varieties V_1 and V_2 are done

$$\mathcal{O}[V_1 \sqcup V_2] \cong \mathcal{O}[V_1] \oplus \mathcal{O}[V_2], \quad \mathcal{O}[V_1 \times V_2] \cong \mathcal{O}[V_1] \otimes \mathcal{O}[V_2],$$

and V is irreducible if and only if $\mathcal{O}[V]$ is an integral domain.

Theorem 2.2.4. ([35]). *There is the arrow-reversing equivalence functor between the category AAV of affine algebraic varieties V over $\mathbb{F}$ and the category FID of finitely generated integral domains $\mathcal{O}[V]$ over $\mathbb{F}$.*

Theorem 2.2.5. (Hilbert Nullstellensats [35]). *Let I' be an ideal of $A = \mathbb{F}[x_1, \cdots, x_n]$ and $f \in A$ that vanishes on $Z(I')$. Then there is a positive integer $r > 0$ such that $f^r \in I'$. Namely, f is contained in the radical of I' :*

$$f \in \sqrt{I'} = \{f \in A \,|\, f^r \in I' \text{ for some } r > 0\} = I(Z(I')).$$

Then there is the 1-1 *inclusion reversing, correspondence between algebraic subsets of $\mathbb{F}^n$ and radical ideals of A, so $I = \sqrt{I}$, given as*

$$\begin{aligned}
\mathbb{F}^n \supset V = V(I'(\mathfrak{P})) &\mapsto I(V(I'(\mathfrak{P}))) = \sqrt{I'(\mathfrak{P})} = \sqrt{\sqrt{I'(\mathfrak{P})}} \subset A, \\
\mathbb{F}^n \supset V_2(I_2') \supset V_1(I_1') &\mapsto I(V_2(I_2')) = \sqrt{I_2'} \subset I(V_1(I_1')) = \sqrt{I_1'} \subset A, \\
A \supset I = \sqrt{I} &\mapsto V(I) = Z(I) \subset \mathbb{F}^n, \\
A \supset I_2 \supset I_1 &\mapsto Z(I_2) \subset Z(I_1) \subset \mathbb{F}^n,
\end{aligned}$$

with $Z(I(V)) = \overline{V} = V$ and $I(V(I')) = \sqrt{I'} = I'$. Furthermore, an algebraic subset of $\mathbb{F}^n$ is irreducible if and only if its radical ideal is a prime ideal.

Note that in general, if $f \in \sqrt{I'}$, then $f = f^1 \in \sqrt{I'}$. Hence, $\sqrt{I'} \subset \sqrt{\sqrt{I'}}$. Conversely, if $g \in \sqrt{\sqrt{I'}}$, then $g^r \in \sqrt{I'}$ for some $r > 0$. Thus, $(g^r)^{r'} \in I'$ for some $r' > 0$. Hence, $\sqrt{\sqrt{I'}} \subset \sqrt{I'}$.

Proof. Given is the proof of only the last part. Suppose that V is irreducible. If $fg \in I(V)$, then $fg(V) = 0$ and $Z(fg) = Z(f) \cup Z(g) \subset \mathbb{F}^n$. Hence $V = (V \cap Z(f)) \cup (V \cap Z(g))$ as a union of closed subsets of V. Since V is irreducible, we have either $V = V \cap Z(f)$ or $V = V \cap Z(g)$, and thus $V \subset Z(f)$ or $V \subset Z(g)$. Hence either $f(V) = 0$ or $g(V) = 0$, and thus $f \in I(V)$ or $g \in I(V)$.

Conversely, let P be a prime ideal of A, and suppose that $V(P) = V_1 \cap V_2$ as a union of closed subsets. Then $P = I(V_1 \cap V_2) = I(V_1) \cap I(V_2)$, so that either $P = I(V_1)$ or $P = I(V_2)$. Therefore, $Z(P) = V_1$ or V_2. □

There are also various equivalent ways of characterizing smooth (or non-singular) varieties in terms of their coordinate rings.

Unlike the GN correspondence, the HN correspondence does not seem to indicate what is the notion of a noncommutaive affine variety, or noncommutative (affine) algebraic geometry, in general. There seems to be a lot to be done remained in this area. But a possible approach has been pursued at least in the smooth case.

As a particularly important characterization of non-singularity, that lends itself to noncommutative generalization, is the following result of Grothendieck (cf. [48]).

Theorem 2.2.6. *A variety V is smooth if and only if its coordinate ring $\mathcal{O}[V]$ has the lifting property with respect to nilpotent extensions, in the sense that for any unital commutative algebra A and a nilpotent ideal I of A, the following map induced by taking the quotient A/I is surjective*:

$$\mathrm{Hom}(\mathcal{O}[V], A) \to \mathrm{Hom}(\mathcal{O}[V], A/I) \to 1.$$

Motivated by that characterization for smoothness of varieties, an algebra B over $\mathbb{C}$, not necessarily commutative, is defined to be NC smooth (or quasi-free) (NCS) if the above lifting property holds by replacing $\mathcal{O}[V]$ with B, for any algebra A, by Cuntz and Quillen [25].

A free algebra, also known as tensor algebra, or algebra of noncommutative polynomials, is smooth in that sense. But commutative algebras which are smooth need not be smooth in that sense. In fact, it is shown that an algebra is NC smooth if and only if it has Hochschild cohomological dimension 1 ([25]). In particular, the algebras of polynomials in more than 1 variables and in general, the coordinate rings of smooth varieties of dimension more than 1 are not NC smooth. Nevertheless, that notion of NCS has played an important role in the development of a version of NC algebraic geometry (cf. [44], [47]).

An alternative approach to NC algebraic geometry is proposed by [1]. As one of the underlying ideas, the projective Nullstellensatz theorem (cf. [35]) characterizes the graded coordinate ring of a projective variety defined as sections of powers of an ample line bundle over the variety. Thus, in this approach, a noncommutative variety is represented by some noncommutative graded ring.

Now recall from [35] that the **projective** n-space $\mathbb{P}^n = \mathbb{P}^n(\mathbb{F})$ is the set of equivalence classes of elements of $(\mathbb{F}^{n+1})^* = \mathbb{F}^{n+1} \setminus 0$ under the equivalence relation given that for $(z_j), (w_j) \in (\mathbb{F}^{n+1})^*$, $(z_j) \sim (w_j)$ if there is $\lambda \in \mathbb{F}^*$ such that $\lambda(z_j) = (\lambda z_j) = (w_j)$.

A **graded** ring is a ring R with a decomposition $R = \oplus_{d\geq 0} R_d$ as a direct sum of abelian groups R_d (of homogeneous elements of degree d) such that for any $d_1, d_2 \geq 0$, $R_{d_1}R_{d_2} \subset R_{d_1+d_2}$. An ideal I of R is homogeneous if $I = \oplus_{d\geq 0}(I \cap R_d)$. An ideal of R is homogeneous if and only if it is generated by homogeneous elements. The set of homogeneous ideals is stable under taking direct sum, direct product, intersection, and radical $\sqrt{\cdot}$. A homogeneous ideal I is prime if for homogeneous elements $f, g \in I$, $fg \in I$ implies $f \in I$ or $g \in I$.

The polynomial ring $R = \mathbb{F}[x_0, x_1, \cdots, x_n]$ becomes a graded ring by taking R_d to be the subspace of R generated by monomials of degree d in $x_0, \cdots, x_n$. Then note that for $f \in R_d$, $f(\lambda z_0, \cdots, \lambda z_n) = \lambda^d f(z_0, \cdots, z_n)$. Hence, the being zero or not of f depends only on the equivalence class $[(z_j)] \in \mathbb{P}^n$. Thus, for $f \in R_d$, there is a function $f^\sim : \mathbb{P}^n \to \{0, 1\}$ defined as

$$f^\sim([(z_j)]) = \begin{cases} 0 & \text{if } f(z_0, \cdots, z_n) = 0, \\ 1 & \text{if } f(z_0, \cdots, z_n) \neq 0. \end{cases}$$

For any subset H of some homogeneous elements of R, define the **zero** set of H to be

$$Z(H) = \{p \in \mathbb{P}^n \mid\ f^\sim(p) = 0 \text{ for any } f \in H\}.$$

If I is a homogeneous ideal of R, then define $Z(I)$ to be $Z(H)$, where H is the set of all homogeneous elements of I. Since R is a Noetherian ring, for any subset H of homogeneous elements of R, there is a finite subset F of H such that $Z(H) = Z(F)$.

There are more theories that parallel to the unprojective theories, to be continued.

2.3 Affine group schemes as functors

May refer to [73].

For any unital commutative ring R, there are corresponding the $n \times n$ special or general linear groups $SL_n(R)$ or $GL_n(R)$ with determinant 1 or non-zero, respectively. In particular, $GL_1(R) = R^*$ is the multiplicative group of R. Also, R is the additive group $R^+ = R$. Let $\mu_n(R) = \{x \in R \mid x^n = 1\}$ be the n-th roots of unity in R, as a group under multiplication. Let $\alpha_p(R) = \{x \in R \mid x^p = 0\}$ as a group under addition.

Let k be a base ring or a field, such as $\mathbb{Z}$ and so on.

Theorem 2.3.1. *Let F be a functor from k-algebras R to sets. If the elements of $F(R)$ correspond to solutions in R of some family of equations, then there is a k-algebra A and a natural correspondence between $F(R)$ and* $\mathrm{Hom}_k(A, R)$. *The converse also holds.*

Proof. Suppose we have some family of polynomial equations $\{p_l\}_{l \in L}$ over k, with respect to some $\{a_j\}_{j \in J}$ of R. Then take the polynomial ring $P = k[\{x_j\}_{j \in J}]$ over k, with each indeterminate x_j as each variable a_j in the equations. Divide it by the ideal I generated by the relations defined as all the equations, to obtain the quotient algebra $A = P/I$.

Let $F(R)$ be given by the solutions of the equations p_l in R. Any k-algebra homomorphism $\varphi : A \to R$ takes general solutions to a solution of R corresponding to an element of $F(R)$. Since φ is determined by sending the indeterminates, we have an injection from $\mathrm{Hom}_k(A, R)$ into $F(R)$. This is actually bijective by generality of solutions.

Any k-algebra B arises in this way from some family of equations. Indeed, let $\{b_j\}_{j \in J}$ be the set of generators of B. There is the ring homomorphism from $P = k[\{x_j\}_{j \in J}]$ onto B by sending x_j to b_j. Choose polynomials generating the kernel I. Then $B \cong A = P/I$. □

Such a functor F is said to be **representable** by A. An affine group scheme over k is defined to be a representable functor from k-algebras to groups.

Example 2.3.2. Let $k = \mathbb{Z}$ and $R = \mathbb{R}$. Let $F(R) = GL_1(R) = \mathbb{R}^* = \{x \in \mathbb{R} \mid x \neq 0\}$ as a group. Then $A = \mathbb{R}[1] \cong \mathbb{R} = R$. And $\mathrm{Hom}_k(A, R) = R$ since any element of which is determined by sending 1 to an element of R. Therefore, identified with are $F(R)$ and $\mathrm{Hom}_k(A, R)^{-1}$ of invertible maps.

Example 2.3.3. The determinant map $\det : GL_2 \to GL_1$ of groups as functors is natural in the sense that for any algebra homomorphism $\rho : R \to R'$, the following commutes:

$$\begin{array}{ccccc} R & \xrightarrow{GL_2} & GL_2(R) & \xrightarrow{\det} & GL_1(R) \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle(\rho\circ(\cdot))_{ij}} & & \big\downarrow{\scriptstyle\rho\circ(\cdot)} \\ R' & \xrightarrow{GL_2} & GL_2(R') & \xrightarrow{\det} & GL_1(R'). \end{array}$$

Theorem 2.3.4. (The Yoneda Lemma). *Let F_1 and F_2 be (set-valued) functors from k-algebras R, represented by k-algebras A_1 and A_2, as $F_j(R) = \mathrm{Hom}_k(A_j, R)$ for $j = 1, 2$. The natural maps Φ from F_1 to F_2 correspond to k-algebra homomorphisms φ from A_2 to A_1.*

Proof. Let $\varphi : A_2 \to A_1$ be given. For any $\psi \in F_1(R) = \mathrm{Hom}_k(A_1, R)$, the composition $\psi \circ \varphi$ belongs to $F_2(R) = \mathrm{Hom}_k(A_2, R)$. Then for any homomorphism $\rho : R \to R'$, the following diagram commutes:

$$\begin{array}{ccccc} R & \xrightarrow{F_1} & F_1(R) = \mathrm{Hom}_k(A_1, R) & \xrightarrow{(\cdot)\circ\varphi} & F_2(R) = \mathrm{Hom}_k(A_2, R) \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\rho\circ(\cdot)} & & \big\downarrow{\scriptstyle\rho\circ(\cdot)} \\ R' & \xrightarrow{F_1} & F_1(R') = \mathrm{Hom}_k(A_1, R') & \xrightarrow{(\cdot)\circ\varphi} & F_2(R') = \mathrm{Hom}_k(A_2, R') \end{array}$$

and let $\Phi = (\cdot) \circ \varphi$.

Conversely, let $\Phi : F_1 \to F_2$ be a natural map. Since $F_j(R) = \mathrm{Hom}_k(A_j, R)$, then for any $\rho \in F_1(R) = \mathrm{Hom}_k(A_1, R)$, the diagram

$$\begin{array}{ccc} F_1(A_1) = \mathrm{Hom}_k(A_1, A_1) & \xrightarrow{\rho\circ(\cdot)} & F_1(R) = \mathrm{Hom}_k(A_1, R) \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi} \\ F_2(A_1) = \mathrm{Hom}_k(A_2, A_1) & \xrightarrow{\rho\circ(\cdot)} & F_2(R) = \mathrm{Hom}_k(A_2, R) \end{array}$$

commutes. In particular, let $\varphi = \Phi(\mathrm{id}_{A_1}) : A_2 \to A_1$, where $\mathrm{id}_{A_1} : A_1 \to A_1$ is the identity map. Then for any $\rho \in F_1(R)$, we have

$$\Phi(\rho) = \Phi(\rho \circ \mathrm{id}_{A_1}) = \rho \circ \varphi$$

and hence $\Phi = (\cdot) \circ \varphi$. $\square$

Such a natural functor $\Phi : F_1 \to F_2$ is said to be a natural correspondence if $F_1(R) \to F_2(R)$ is bijective for any R.

Corollary 2.3.5. *A natural functor $\Phi : F_1 \to F_2$ represented by A_1 and A_2 is a natural correspondence if and only if the corresponding $\varphi : A_2 \to A_1$ is an isomorphism.*

2.4 Affine schemes and Commutative rings

Let A be a unital commutative ring. The (prime) **spectrum** of A is defined to be a pair $(\mathrm{Sp}(A), \mathcal{O}_A)$, also called a ringed space, where $\mathrm{Sp}(A)$ also called the spectrum of A, as a set consists of all prime ideals of A, with the Zariski topology, and $\mathcal{O}_A$ is the sheaf of rings on $\mathrm{Sp}(A)$, both defined below.

Note that an ideal I of A is said to be **prime** if $I \neq A$ and for any $a, b \in A$, $ab \in I$ (corrected from A) implies that either $a \in I$ or $b \in I$. Given an ideal I of A, let $V(I)$ denote the set of all prime ideals of A which contain I. The **Zariski** topology on $\mathrm{Sp}(A)$ is defined by assuming that any $V(I)$ is a closed subset of $\mathrm{Sp}(A)$. Indeed, for ideals I, J, I_j of A, we have $V(IJ) = V(I) \cup V(J)$ (corrected from the intersection) and $V(\sum_j I_j) = \cap V(I_j)$. Note that $V(\{0\}) = \mathrm{Sp}(A)$ and $V(A) = \emptyset$.

Check that if $IJ \subset K \in \mathrm{Sp}(A)$, then $I \subset K$ or $J \subset K$. If not so, then IJ is not contained in K. Conversely, note that $IJ \subset I$ and $IJ \subset J$. Check also that if $\sum_j I_j \subset K \in \mathrm{Sp}(A)$, then $I_j \subset K$ for any j. Its converse also holds.

As well, $IJ \subset I \cap J$, but the equality does not hold in general. However, if $A = I + J$, then the equality does hold (cf. [54]).

May check that the space $\mathrm{Sp}(A)$ is compact but non-Housdorff in general.

For each prime ideal P of A, denote by $A_P = A/P^c$ the **localization** of A at P, where P^c is the complement of P in A, which is a multiplicative closed subset of A. For an open subset U of $\mathrm{Sp}(A)$, let $\mathcal{O}_A(U)$ be the set of all continuous sections from U to $\cup_{P \in U} A_P$, where such a section is said to be **continuous** if locally around any point $P \in U$, it is of the form $\frac{f}{g}$ with $g \notin P$. May check that $\mathcal{O}_A$ is a sheaf of commutative rings on $\mathrm{Sp}(A)$.

The spectrum functor Sp is defined by sending A to $(\mathrm{Sp}(A), \mathcal{O}_A)$.

An **affine scheme** is a ringed space $(X, \mathcal{O})$ such that X is homeomorphic to $\mathrm{Sp}(A)$ for a commutative ring A and $\mathcal{O}$ is isomorphic to $\mathcal{O}_A$.

The spectrum functor Sp defines an equivalence between the category AS of affine schemes with continuous maps and the category CR of commutative rings with unital ring homomorphisms, as

$$\mathrm{Sp} : CR \to AS, \quad A \mapsto \mathrm{Sp}(A) = X \text{ and } \mathcal{O}_A = \mathcal{O},$$

so that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \mathrm{Sp}}\downarrow & & \downarrow{\scriptstyle \mathrm{Sp}} \\ \mathrm{Sp}(A) & \xleftarrow{f^*} & \mathrm{Sp}(B) \end{array}$$

where $f^*(Q) = f^{-1}(Q)$ for any $Q \in \mathrm{Sp}(B)$. Note that for a maximal ideal of B, the inverse image $f^{-1}(B)$ is not necessarily maximal. This is the reason to consider the prime spectrum for an arbitrary ring, not the maximal spectrum.

The inverse equivalence for Sp above is given by the global section functor Γ that sends an affine scheme X to the ring ΓX of its global sections.

In the same vein, the categories MR of modules over a ring can be identified with the categories SMS of sheaves of modules over the spectrum of the ring.

Let A be a commutative ring and left M be an A-module. Define a **sheaf** $\mathfrak{M}$ of modules over $\mathrm{Sp}(A)$ as follows. For each prime ideal P of A, let M_P denote the localization of M at P. For any open subset U of $\mathrm{Sp}(A)$, let $\mathfrak{M}(U)$ denote the set of continuous sections from U to $\cup_P M_P$, where such a section has the form of a fraction $\frac{m}{f}$ locally for $m \in M$ and $f \in A_P$. Then M is recovered from $\mathfrak{M}$ by showing that $M \cong \Gamma\mathfrak{M}$ the space of global sections of $\mathfrak{M}$.

Sheaves of $\mathcal{O}_A$-modules on $\mathrm{Sp}(A)$ obtained in that way is said to be **quasi-coherent** sheaves, which are local models for a more general notion of quasi-coherent sheaves on arbitrary schemes.

The functors Sh sending M to $\mathfrak{M}$ and Γ sending $\mathfrak{M}$ to $\Gamma\mathfrak{M}$ define an equivalence of the MR over A and the quasi-coherent SMS of $\mathrm{Sp}(M)$. Namely,

$$Sh : MR \to SMS \quad \text{and} \quad \Gamma : SMS \to MR.$$

Based on this correspondence, given an algebra A, not necessarily commutative, the category of A-modules may be replaced with the categrory of quasi-coherent sheaves over the noncommutative space as $\mathrm{Sp}(A)$. This is a nice idea in the development of the subject of noncommutative algebraic geometry, about which nothing is considered here (cf. [1], [44], [47]).

Recall from [52] the following. Let X be a topological space. A **presheaf** on X is defined to be a system (or functor) F that for any open subset U of X, there is an abelian group $F(U)$ such that for any inclusion $U \subset V$ of open subsets of X, there is a homomorphism $\varphi_{UV} : F(V) \to F(U)$ satisfying that the following commutes:

$$\begin{array}{ccccccccc}
\emptyset & \xleftarrow{\subset} & U & \xrightarrow{\subset} & V & \xrightarrow{\subset} & W & \xrightarrow{\subset} & X \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\
F(\emptyset) = \{0\} & \xleftarrow{\varphi_{\emptyset U}} & F(U) & \xleftarrow{\varphi_{UV}} & F(V) & \xleftarrow{\varphi_{VW}} & F(W) & \xleftarrow{\varphi_{WX}} & F(X)
\end{array}$$

such as $\varphi_{UV} \circ \varphi_{VW} = \varphi_{UW}$, with $\varphi_{UU} : F(U) \to F(U)$ the identity map for any open U of X. A homomorphism ψ of presheaves F and G on X is defined as that for any open $U \subset V$ of X, there are homomorphisms $\varphi(U) : F(U) \to G(U)$ such that the following commutes:

$$\begin{array}{ccc}
F(V) & \xrightarrow{\psi(V)} & G(V) \\
{\scriptstyle \varphi_{UV}}\downarrow & & \downarrow{\scriptstyle \varphi_{UV}} \\
F(U) & \xrightarrow{\psi(U)} & G(U).
\end{array}$$

A presheaf F on X is defined to be a **sheaf** on X if, as the local condition, for any open U of X and its open covering $U \subset \cup_j U_j$ in X, there is $s_j \in F(U_j)$ for each j such that for any i, j, the restricion s_i on $U_i \cap U_j$, that is $\varphi_{U_i \cap U_j, U_i}(s_i)$, is equal to s_j on $U_i \cap U_j$, then there is $s \in F(U)$ uniquely such that s on U_j is equal to s_j for every j.

By replacing the abelian groups such as $F(U)$ with groups or rings or so, sheaves of those are obtained.

Example 2.4.1. Let X be a topological space and G be a group (or ring) with the discrete topology. Then the direct product $X \times G$ becomes a sheaf. This is called a constant or trivial sheaf. Note that $F(U) = G$ for any open $U \subset X$.

Example 2.4.2. Let X be a tological space and Y be a topological abelian group such as $\mathbb{R}^n$ or $\mathbb{C}^n$. For any open $U \subset X$, define $F(U)$ to be $C(U, Y)$ the set (or additive group, or ring for $n = 1$) of continuous maps from U to Y. For open subsets $U \subset V$ of X, let $\varphi_{VU} : C(U, Y) \to C(V, Y)$ be the restriction map. Then the sheaf $C(\cdot, Y)$ of continuous functions over X is obtained, so that the diagram commutes

$$\begin{tikzcd}
\emptyset \arrow[d, "C(\cdot,Y)"] & U \arrow[l, "\subset"'] \arrow[r, "\subset"] \arrow[d, "C(\cdot,Y)"] & V \arrow[r, "\subset"] \arrow[d, "C(\cdot,Y)"] & W \arrow[r, "\subset"] \arrow[d, "C(\cdot,Y)"] & X \arrow[d, "C(\cdot,Y)"'] \\
C(\emptyset, Y) = \{0\} & C(U, Y) \arrow[l, "\varphi_{\emptyset U}"'] & C(V, Y) \arrow[l, "\varphi_{UV}"'] & C(W, Y) \arrow[l, "\varphi_{VW}"'] & C(X, Y). \arrow[l, "\varphi_{WX}"']
\end{tikzcd}$$

2.5 Riemann surfaces and Function fields

It is shown that the category RS_c of compact Riemann surfaces is equivalent to the category FF_a of algebraic function fields. For this correspondence, may refer to [32].

A **Riemann** surface is defined to be a complex manifold of complex dimension 1. A morphism between Riemann surfaces X and Y is given by a holomorphic map $f : X \to Y$.

An algebraic **function** field is defined to be a finite extension of the field $\mathbb{C}(x)$ of rational functions in one variable x. A morphism of function fields is given by an algebra map.

To a compact Riemann surface X, accociated is the field $M(X)$ of meromorphic functions on X. For example, the field $M(S^2)$ of meromorphic functions of the Riemann sphere $S^2 \approx \mathbb{C} \cup \{\infty\}$, with no holes, is the field $\mathbb{C}(x)$ of rational functions.

To a finite extension of $\mathbb{C}(x)$, associated is the compact Riemann surface of the algebraic function $p(z, w) = 0$, where w is a generator of the field over $\mathbb{C}(x)$. This correspondence is essentially due to Riemann. Despite its depth and beauty, this correspondence so far may not be revealed by any way of the noncommutative analogue of complex geometry.

Another possible approach to the complex structures in noncommutative geometry may be based on the notion of a positive Hochschild cocycle on an involutive algebra, as defined in [11]. As an other contribution, noncommutative complex structures motivated by the Dolbeault complex is introduced in [42], and as well, a detailed study of holomorphic structures on noncommutative tori and holomorphic vector bundles on noncommutative 2-tori is carried out in [59].

It is shown in [10] that positive Hochschild cocycles on the algebra of smooth functions on a compact oriented 2-dimensional manifold encode the information for defining a holomorphic structure on the surface, which suggests that the p-H cocycles may be used in a possible framework for holomorphic noncommutative structures. The corresponding problem of characterizing holomophic structures on general dimensional manifolds by using positive Hochschild cocyles may be still open. In the case of noncommutative 2-tori, a positve Hochschild 2-cocycle on the non-com 2-torus can be defined as complex structures. As well, a natural complex structure on the Podlé qunatum 2-sphere is defined in [42]. With this additional structure, the quantum 2-sphere is said to be the quantum projective line as $\mathbb{C}P^1_q$, which resembles the classical Riemann sphere in several suitable ways.

2.6 Sets and Boolean algebras

The set theory can be regarded as the geometrization of logic. There is a duality or correspondence between the category Set of sets with (set) maps and the category Boo of complete atomic Boolean algebras (cf. [2]).

A **Boolean** algebra is defined to be a unital ring B such that any element x of B satisfies the equation $x^2 = x$. A Boolean algebra is necessarily commutative.

Indeed, for any $x, y \in B$, let $c = xy - yx$. Then

$$c = c^2 = (xy - yx)^2 = xyxy - xy^2x - yx^2y + yxyx = xy - xyx - yxy + yx,$$

which implies that $xyx = 2yx - yxy$ and $yxy = 2yx - xyx$. Therefore, we have

$$\begin{aligned} xy &= xyxy = (2yx - yxy)y = 2yxy - yxy = yxy, \\ xy &= xyxy = x(2yx - xyx) = 2xyx - xyx = xyx, \end{aligned}$$

so that $yx = yxyx = y(2yx - yxy) = yxy = xy$.

Define a partial **order** relation $x \leq y$ for $x, y \in B$ if there is an $y' \in B$ such that $x = yy'$.

Check that $x \leq x$ since $x = x1$ with $1 \in B$. If $x \leq y$ and $y \leq z$, then $x = yy'$ and $y = zz'$ for some $y', z' \in B$. Then $x = zz'y'$ with $z'y' \in B$. Hence $x \leq z$. If $x \leq y$ and $y \leq x$, then $x = yy'$ and $y = xx'$ for some $y', x' \in B$. Then

$$x = yy' = y^2y' = yx = xy = x^2x' = xx' = y.$$

An **atom** of a Boolean algebra B is defined to be an element $x \in B$ such that there is no $y \in B$ with $0 < y < x$. A Boolean algebra B is said to be **atomic** if every element $x \in B$ is the supremum of all the atoms $y \in B$ with $y < x$. A Boolean algebra B is said to be **complete** if every subset of B has its supremum and infimum in B. A morphism of complete Boolean algebras is given by a unital ring map which preserves infimums and supremums.

Example 2.6.1. Let S be a set. Let $B = B_S = 2^S = \{f : S \to \{0, 1\} = \mathbb{Z}_2\}$ the set of all functions from S to the two points set. Then $B = 2^S$ is a complete atomic Boolean algebra.

Indeed, for any $f \in B$, we have $f^2 = f$ since $f(s)^2 = f(s)$ for any $s \in S$, because of $0^2 = 0$ and $1^2 = 1$. The constant map $1 = 1(s) = 1$ for any $s \in S$ is the unit for B. It is clear that B is commutative. Define a partial order relation $f \leq g$ for $f, g \in B$ if $f(s) \leq g(s)$ for every $s \in B$. If $f \leq g$ in this sense, then $f = fg$. Hence $f \leq g$ in that sense. Conversely, if $f \leq g$ in that sense so that $f = gh$ for some $h \in B$, then $f = gh \leq 1g = g$ in this sense. An atom of B is given by a characteristic function χ_t at an element $t \in S$ such that $\chi_t(s) = 0$ for $s \neq t$ and $\chi_t(t) = 1$. Also, any $f \in B$ is written as the supremum $\sup_n \sum_{j=1}^n \chi_{t_j}$ with $t_j \in S$ with $f(t_j) = 1$ for some n finite or unbounded. As well, for any subset C of B, $\sup C$ is given as $\sup_n \sum_{j=1}^n \chi_{t_j}$ such that there is $f \in C$ with $f(t_j) = 1$, and $\inf C$ is given as $\sup_n \sum_{j=1}^n \chi_{t_j}$ such that $f(t_j) = 1$ for any $f \in C$. $\triangleleft$

Any map $\varphi : S \to T$ of sets S and T defines a **morphism** of the associated complete atomic Boolean algebras B_S, B_T by pull-back as

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ 2 \big\downarrow B & & 2 \big\downarrow B \\ 2^S = B_S \ni f \circ \varphi & \xleftarrow{\varphi^*} & f \in 2^T = B_T. \end{array}$$

This system is a contravariant functor from the category *Set* to the category *Boo*.

As for the opposite direction, given a Boolean algebra B, define its **spectrum** $B^\wedge$ to be $\mathrm{Hom}_{Boo}(B, \{0,1\})$ the set of all Boolean algebra homomorphisms from B to a two points set as $\{0,1\}$, viewed as the Boolean algebra of two elements 0 and 1. Any Boolean algebra map $\psi : B \to C$ induces a set map $\psi^\wedge : C^\wedge \to B^\wedge$ defined as $\psi^\wedge = \psi^*$ the pull-back of ψ, so that $\psi^\wedge(f) = f \circ \psi$ for $f \in C^\wedge$. Namely,

$$\begin{array}{ccc} B & \xrightarrow{\psi} & C \\ \mathrm{Hom}(\cdot, \mathbb{Z}_2) \big\downarrow & & \big\downarrow \mathrm{Hom}(\cdot, \mathbb{Z}_2) \\ B^\wedge & \xleftarrow{\psi^\wedge = \psi^*} & C^\wedge \end{array}$$

This system is a contravariant functor from the category *Boo* to he category *Set*.

This and that functors give anti-equivalences of the categories, quasi-inverse to each other. Thus, we have a duality between the category of sets as geometric objects and the category of commutative algebras as complete atomic Boolean algebras. This result is a special case of the Stone duality between Boolean algebras and a certain class of topological spaces (cf. [38]).

Example 2.6.2. Let $S = \mathbb{Z}_2$ as a set. Then $B_S = 2^S$ is generated by the characteristic functions χ_0 and χ_1 for $0, 1 \in \mathbb{Z}_2$. Hence, $B_S = \{0, \chi_0, \chi_1, 1 = \chi_0 + \chi_1\}$, whichi is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a ring. Therefore,

$$\begin{aligned} B_{\mathbb{Z}_2}^\wedge &= \mathrm{Hom}(B_{\mathbb{Z}_2}, \mathbb{Z}_2) \cong \mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2) \\ &\cong \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) \times \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2. \quad \triangleleft \end{aligned}$$

There may not be the notion of some kind of noncommutative sets, obtained as quantizing the set theory by noncommutative Boolean algebras if any or not.

3 Group C^*-algebras around

Table 4: An overview on group and algebra with structure ⊚

Group First	Algebra Next
Groups Topological groups Space symmetry	Group algebras Group (Banach or) C^*-algebras Hilbert space symmetry
Group semi-direct products G-actions on spaces Dynamical systems Classical mechanics	Crossed product (C^*-) algebras Bicrossed product algebras C^*-dynamical systems Quantum mechanics
Groups with 2-cocycles Systems with 2-cocycles	Twisted group (C^*-) algebras Twisted crossed product algebras
Groups with duals Classical groups	Co-algebras, Hopf algebras, Quantum (deformed) groups
Pontryagin duality for Topo-Abelian Groups	Takai duality for C^*-crossed products by TAG

Note that group (C^*-) algebras are not complete invariants of groups up to isomorphisms in general, even in the commutative case (cf. [29], [62]). But in the noncommutative case, there are some classification results on the isomorphism problem of groups and their group (C^*-) algebras, in some cases. This problem seems to be still somewhat open and interesting. May solve it. As one of the most important results, it is obtained by Pimsner-Voiculescu [58] that the free groups with no relations are not isomorphic by computing the K-theory groups of the (full or reduced) C^*-algebras of the free groups ([3]).

On the other hand, it is known that the representation theory of groups as the unitary duals of their unitary representations up to unitary equivalences is identified with the representation theory of group C^*-algebras as the spectrums of their irreducible representations up to unitary equivalences (cf. [28], [29], [57]).

3.1 Discrete group $*$-algebras

Let G be a discrete group with e the unit element. Let $\mathbb{C}[G]$ denote the group algebra of G. The **group** $*$-algebra $\mathbb{C}G(= \mathbb{C}[G]$ as an algebra) of G is a unital $*$-algebra defined as follows. As a vector space, $\mathbb{C}G$ is the complex vector space generated by the set of all elements of G as the canonical basis $\{\delta_g \,|\, g \in G\}$, so that any element of $\mathbb{C}G$ is a finite linear combination $\sum_{j=1}^n \alpha_j \delta_{g_j}$ for some $\alpha_j \in \mathbb{C}$, $g_j \in \mathbb{C}$, and $n \in \mathbb{N}$, where each vector $\delta_g \in \mathbb{C}G$ is identified with each element $g \in G$. The multiplication and involution for the basis elements of $\mathbb{C}G$ are defined as

$$\delta_g \delta_h = \delta_{gh}, \quad \delta_g^* = \delta_{g^{-1}}, \quad g, h \in G$$

and these operations are extended to all elements of $\mathbb{C}G$ by linearity.

Indeed, check that

$$(\delta_g \delta_h)^* = (\delta_{gh})^* = \delta_{h^{-1}g^{-1}} = \delta_{h^{-1}}\delta_{g^{-1}} = \delta_h^* \delta_g^*.$$

The unit of $\mathbb{C}G$ is given by δ_e.

Lemma 3.1.1. *The group $*$-algebra $\mathbb{C}G$ is commutative if and only if G is commutative.*

Proof. By definition, element-wise commutativity in $\mathbb{C}G$ is equivalent to that in G, and which extends to that in $\mathbb{C}G$. $\square$

Note that each element of $\mathbb{C}G$ is identified with a (continuous) function from G to $\mathbb{C}$ with finite (or compact) support, where we assume that a discrete group G has the discrete topology, so that any function from G to $\mathbb{C}$ is continuous. We denote by $C_c(G, \mathbb{C})$ the unital $*$-algebra of all (**continuous**) functions from G to $\mathbb{C}$ with finite (or **compact**) support, with convolution product and involution defined as

$$(f_1 * f_2)(g) = \sum_{h,k \in G, hk=g} f_1(h) f_2(k) = \sum_{h \in G} f_1(h) f_2(h^{-1}g)$$

and $f^*(g) = \overline{f(g^{-1})}$ for $f_1, f_2, f \in C_c(G, \mathbb{C})$. We may call it as a $\mathbf{C_c}$ algebra.

Lemma 3.1.2. *Endowed with convolution and involution, the C_c algebra $C_c(G, \mathbb{C})$ is a unital $*$-algebra, where the unit is given by the characteristic function χ_1 at the unit $1 \in G$ defined as $\chi_1(1) = 1$ and $\chi_1(g) = 0$ for any $g \neq 1$ in G.*

Proof. If $f_1, f_2 \in C_c(G, \mathbb{C})$ with finite supports $\mathrm{supp}(f_1), \mathrm{supp}(f_2) \subset G$, then

$$(f_1 * f_2)(g) = \sum_{h \in \mathrm{supp}(f_1)} f_1(h) f_2(h^{-1}g).$$

Moreover, if the intersection $\mathrm{supp}(f_1)^{-1}g \cap \mathrm{supp}(f_2) = \emptyset$, then $(f_1 * f_2)(g) = 0$, where $\mathrm{supp}(f_1)^{-1} = \{g^{-1} \in G \,|\, g \in \mathrm{supp}(f_1)\}$. Therefore, if $(f_1 * f_2)(g) \neq 0$, then $\mathrm{supp}(f_1)^{-1}g \cap \mathrm{supp}(f_2) \neq \emptyset$. In this case, if $w = h^{-1}g \in \mathrm{supp}(f_2)$ for $h \in \mathrm{supp}(f_1)$, then $g = hw \in \mathrm{supp}(f_1)\mathrm{supp}(f_2)$. Thus, we obtain $\mathrm{supp}(f_1 * f_2) \subset \mathrm{supp}(f_1)\mathrm{supp}(f_2)$. Hence $f_1 * f_2 \in C_c(G, \mathbb{C})$.

For $f_1, f_2, f_3 \in C_c(G, \mathbb{C})$, we compute

$$\begin{aligned}(f_1 * (f_2 * f_3))(g) &= \sum_{h \in G} f_1(h)(f_2 * f_3)(h^{-1}g) \\ &= \sum_{h \in G} f_1(h) \sum_{k \in G} f_2(k) f_3(k^{-1}h^{-1}g) \\ &= \sum_{h \in G} f_1(h) \sum_{k \in G} f_2(h^{-1}hk) f_3((hk)^{-1}g) \\ &= \sum_{h \in G} f_1(h) \sum_{hk=u \in G} f_2(h^{-1}u) f_3(u^{-1}g) \\ &= \sum_{u \in G} \left(\sum_{h \in G} f_1(h) f_2(h^{-1}u) \right) f_3(u^{-1}g) \\ &= \sum_{u \in G} (f_1 * f_2)(u) f_3(u^{-1}g) = ((f_1 * f_2) * f_3)(g).\end{aligned}$$

As well, check that for $f \in C_c(G, \mathbb{C})$,

$$f^{**}(g) = \overline{f^*(g^{-1})} = f(g).$$

Moreover,

$$\begin{aligned}(f_1 * f_2)^*(g) &= \overline{(f_1 * f_2)(g^{-1})} \\ &= \sum_{h = g^{-1}k, k \in G} \overline{f_1(h) f_2(h^{-1}g^{-1})} \\ &= \sum_{k \in G} \overline{f_2((k)^{-1}) f_1((k^{-1}g)^{-1})} \\ &= \sum_{k \in G} f_2^*(k) f_1^*(k^{-1}g) = (f_2^* * f_1^*)(g).\end{aligned}$$

Furthermore,

$$\begin{aligned}(\chi_1 * f)(g) &= \sum_{h \in G} \chi_1(h) f(h^{-1}g) = \chi_1(1) f(g) = f(g), \\ (f * \chi_1)(g) &= \sum_{h \in G} f(h) \chi_1(h^{-1}g) = f(g) \chi_1(1) = f(g).\end{aligned}$$

□

Proposition 3.1.3. *The group $*$-algebra $\mathbb{C}G$ is isomorphic to $C_c(G, \mathbb{C})$ as a $*$-algebra, by the Dirac transform, named so by us, so that $\mathbb{C}G$ is identified with $C_c(G, \mathbb{C})$.*

Proof. Define the **Dirac** transform as a $*$-isomorphism Φ from $\mathbb{C}G$ to $C_c(G, \mathbb{C})$ by sending any element $\delta_g \in \mathbb{C}G$ to χ_g the characteristic function at g and by extending by linearity. Check that for any $g_1, g_2, g_3 \in G$,

$$\begin{aligned}
&\Phi(\delta_{g_1}) * \Phi(\delta_{g_2})(g_3) = (\chi_{g_1} * \chi_{g_2})(g_3) \\
&= \sum_{h \in g} \chi_{g_1}(h)\chi_{g_2}(h^{-1}g_3) = \chi_{g_2}(g_1^{-1}g_3) \\
&= \chi_{g_1 g_2}(g_3) = \Phi(\delta_{g_1 g_2})(g_3) = \Phi(\delta_{g_1}\delta_{g_2})(g_3).
\end{aligned}$$

Also,

$$\Phi(\delta_{g_1})^*(g_2) = \chi_{g_1}^*(g_2) = \overline{\chi_{g_1}(g_2^{-1})} = \chi_{g_1^{-1}}(g_2) = \Phi(\delta_{g_1}^*)(g_2).$$

Since if $g \neq h \in G$, then $\chi_g \neq \chi_h$, the injectivity of Φ holds. If any $f \in C_c(G, \mathbb{C})$ has a finite support $\{g_1, \cdots, g_n\} \subset G$, then $f = \sum_{j=1}^n f(g_j)\chi_{g_j}$. Hence $f = \Phi(\sum_{j=1}^n f(g_j)\delta_{g_j})$. Thus, the surjectivity of Φ holds. □

Since the map Φ is a homomorphism sending the pointwise multiplication to the convolution, we may define that Φ is the inverse Fourier transform $\mathfrak{F}^{-1}$ for a discrete group G without considering its dual, or with G as self-dual, and that Φ^{-1} is the **Fourier** transform $\mathfrak{F}$ for G in the same sense. Namely,

$$\mathfrak{F} = \Phi^{-1} : C_c(G, \mathbb{C}) \to \mathbb{C}[G], \quad \mathfrak{F}(f_1 * f_2) = \mathfrak{F}(f_1)\mathfrak{F}(f_2)$$

where for $f = \sum_{j=1}^n f(g_j)\chi_{g_j} \in C_c(G, \mathbb{C})$, we define $\mathfrak{F}(f) = \sum_{j=1}^n f(g_j)\delta_{g_j}$.

Example 3.1.4. Let $G = \mathbb{Z}$ be the group of integers, where for $g, h \in G$, we may identify $gh = g + h$, $g^{-1} = -g \in G = \mathbb{Z}$. Then $\mathbb{C}[G]$ is identified with the algebra $\mathbb{C}[u, u^{-1}]$ of Laurent polynomials generated by an invertible element u with inverse u^{-1}. Namely, for some $n \in \mathbb{N}$, $n_j \in \mathbb{Z}$, $\alpha_j \in \mathbb{C}$,

$$\sum_{j=1}^n \alpha_j \delta_{n_j} = \sum_{j=1}^n \alpha_j u^{n_j} \in \mathbb{C}[G] = \mathbb{C}[u, u^{-1}].$$

Example 3.1.5. Let $G = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ be the finite cyclic group of order n. Then $\mathbb{C}G$ can be identified with the quotient algebra $\mathbb{C}[u]/(u^n - 1)$ of the polynomial algebra $\mathbb{C}[u]$ generated by u and the unit 1.

Any element $\sum_{j=1}^n \alpha_j \delta_{g_j} \in \mathbb{C}[G]$ with $\alpha_j \in \mathbb{C}$, $g_j \in G$ is identified with $\sum_{j=1}^n \alpha_j u^{g_j} \in \mathbb{C}[u]$ (mod $u^n = 1$). It then follows that both $\mathbb{C}[G]$ and $\mathbb{C}[u]$ (mod $u^n = 1$) have pointwise multiplication of the same type. Where is the mess?

There is the magical finite Fourier transform from the quotient to the direct sum of n copies of $\mathbb{C}$ with pointwise multiplication:

$$\mathfrak{F} : \mathbb{C}[u]/(u^n - 1) \to \oplus^n \mathbb{C} = \mathbb{C}^n$$

untangling the mess by multiplication in the quotient algebra (which looks like a mess). The magical $\mathfrak{F}$ is defined by sending u to $(1, \zeta, \zeta^2, \cdots, \zeta^{n-1})$ and extending properly to an algebra isomorphism, where $\zeta \in \mathbb{C}$ denotes a primitive n-th root of unity.

Note that $\mathfrak{F}(u)^n = (1, \cdots, 1) \in \mathbb{C}^n$.

In particular, if $n = 2$, then $\mathbb{C}\mathbb{Z}_2 = \mathbb{C}1 + \mathbb{C}u$ (mod $u^2 = 1$). For $\alpha, \beta \in \mathbb{C}$ and $(\rho, \gamma) \in \mathbb{C}^n$, with ζ primitive with $\zeta^2 = 1$, suppose that

$$\alpha(1,1) + \beta(1,\zeta) = (\alpha + \beta, \alpha + \beta\zeta) = (\rho, \gamma) \in \mathbb{C}^n.$$

By solving this system of the equations with respect to α, β, we obtain $\beta = \frac{\rho-\gamma}{1-\zeta}$, $\alpha = \rho - \frac{\rho-\gamma}{1-\zeta} \in \mathbb{C}$. Thus the Fourier $\mathfrak{F}$ in this case is certainly onto $\mathbb{C}^2$.

Similarly, in the general case, the map $\mathfrak{F}$ is shown to be onto $\mathbb{C}^n$ by solving the system of the algebraic n linear equations, as solved in Linear Algebra.

Moreover, endowed with the supremum or maximum norm on $\mathbb{C}^n$, the $*$-algebra $\mathbb{C}^n$ becomes a C^*-algebra with the C^*-norm condition. As well, the C^*-norm on $\mathbb{C}G$ is induced by the isomorphism $\mathfrak{F} : \mathbb{C}[G] \to \mathbb{C}^n$. Namely, for $f \in \mathbb{C}[G]$, define $\|f\| = \|\mathfrak{F}(f)\|$, so that

$$\|f^*f\| = \|\mathfrak{F}(f^*f)\| = \|\mathfrak{F}(f)^*\mathfrak{F}(f)\| = \|\mathfrak{F}(f)\|^2 = \|f\|^2.$$

Furthermore, elements of $\mathbb{C}^n$ are identified with diagonal matrices in the complex $n \times n$ matrix algebra $M_n(\mathbb{C})$, and $\mathbb{C}^n$ acts on the complex Euclidean or Hilbert space $\mathbb{C}^n$ by matrix multiplication. In this case, the operator norm for $\mathbb{C}^n \subset M_n(\mathbb{C})$ on the diagonal is the same as the supremum norm for $\mathbb{C}^n$.

Anyhow, in this case, $\mathbb{C}G$ becomes a commutative C^*-algebra, so that by the Gelfand transform, $\mathbb{C}G$ is isomorphic the C^*-algebra $C((\mathbb{C}G)^\wedge)$ of all continuous, complex-valued functions on the spectrum $(\mathbb{C}G)^\wedge$, which is identified with the dual group $G^\wedge$ of G of all characters χ_j on G, with $G^\wedge \cong G$, defined as $\chi_j(g) = \zeta^j \in \mathbb{T}$ the 1-torus for the generator $g \in G$ identified with $\zeta \in \mathbb{T}$. $\lhd$

Example 3.1.6. Let G be a finite group. The representation theory of finite groups implies that as a golden tower,

$$\mathbb{C}G = \mathbb{C}[G] \cong \oplus_{\mathfrak{C}} M_{n_j}(\mathbb{C}) \text{ over the set } \mathfrak{C} \text{ of conjugacy classes of } G$$

for some $n_j \geq 1$.

3.2 Twisted discrete group $*$-algebras

Definition 3.2.1. Let G be a (discrete) group and $\mathbb{T}$ be the one torus group. Let $\sigma : G \times G \to \mathbb{T}$ be a 2-**cocycle** (or multiplier) for G satisfying the cocycle condition (CC)

$$\sigma(g_1, g_2)\sigma(g_1g_2, g_3) = \sigma(g_1, g_2g_3)\sigma(g_2, g_3)$$

for any $g_1, g_2, g_3 \in G$, and moreover (added), for any $g \in G$ and the unit $1 \in G$,

$$\sigma(g, 1) = \sigma(1, g) = 1 \in \mathbb{T}$$

(cf. [56]).

If $\sigma \equiv 1$, then σ is said to be trivial.

We denote by $\mathbb{C}(G,\sigma)$ the **twisted** group $*$-algebra of all complex-valued (continuous) functions on G with finite support, endowed with σ-twisted convolution as multiplication (corrected) and σ-twisted involution as involution, defined as:

$$(f_1 * f_2)(g) = \sum_{h\in G} f_1(h)f_2(h^{-1}g)\sigma(h,h^{-1}g),$$
$$f^*(g) = \overline{\sigma(g,g^{-1})f(g^{-1})},$$

for $f_1, f_2, f \in (G,\sigma)$, where the overline means the complex conjugate.

Note that if $\sigma = 1$, then $\mathbb{C}(G,\sigma) = C_c(G,\mathbb{C}) \cong \mathbb{C}G$.

We may call $\mathbb{C}(G,\sigma)$ a σ-C_c algebra.

Lemma 3.2.2. *For a 2-cocycle (or multiplier) $\sigma : G \times G \to \mathbb{T}$ for a (discrete) group G, it holds that for any $g \in G$, $\sigma(g,g^{-1}) = \sigma(g^{-1},g)$, so that*

$$\sigma(g,g^{-1})\overline{\sigma(g^{-1},g)} = 1.$$

Proof. In the (CC) above, set $g_1 = g$, $g_2 = g^{-1}$, and $g_3 = g$. □

Lemma 3.2.3. *The twisted group $*$-algebra $\mathbb{C}(G,\sigma)$ of a discrete (G,σ) is a unital involutive algebra under the σ-twisted convolution and involution, where the unit is given by the characteristic function χ_1 at $1 \in G$.*

Proof. If $f_1, f_2 \in \mathbb{C}(G,\sigma)$ with finite support $\mathrm{supp}(f_1), \mathrm{supp}(f_2) \subset G$, then

$$(f_1 * f_2)(g) = \sum_{h\in \mathrm{supp}(f_1)} f_1(h)f_2(h^{-1}g)\sigma(h,h^{-1}g).$$

Moreover, if the intersection $\mathrm{supp}(f_1)^{-1}g \cap \mathrm{supp}(f_2) = \emptyset$, then $(f_1 * f_2)(g) = 0$, where $\mathrm{supp}(f_1)^{-1} = \{g^{-1} \in G \,|\, g \in \mathrm{supp}(f_1)\}$. Therefore, if $(f_1 * f_2)(g) \neq 0$, then $\mathrm{supp}(f_1)^{-1}g \cap \mathrm{supp}(f_2) \neq \emptyset$. In this case, $\mathrm{supp}(f_1)^{-1}g \subset \mathrm{supp}(f_2)$. Thus, we obtain $\mathrm{supp}(f_1 * f_2) \subset \mathrm{supp}(f_1)\mathrm{supp}(f_2)$. Hence $f_1 * f_2 \in \mathbb{C}(G,\sigma)$.

For $f_1, f_2, f_3 \in \mathbb{C}(G,\sigma)$, we compute

$$
\begin{aligned}
&(f_1 * (f_2 * f_3))(g) = \sum_{h\in G} f_1(h)(f_2 * f_3)(h^{-1}g)\sigma(h,h^{-1}g),\\
&= \sum_{h\in G} f_1(h) \sum_{k\in G} f_2(k)f_3(k^{-1}h^{-1}g)\sigma(k,k^{-1}h^{-1}g)\sigma(h,h^{-1}g)\\
&= \sum_{h\in G} f_1(h) \sum_{k\in G} f_2(h^{-1}hk)f_3((hk)^{-1}g)\sigma(k,(hk)^{-1}g)\sigma(h,h^{-1}g)\\
&= \sum_{h\in G} f_1(h) \sum_{hk=u\in G} f_2(h^{-1}u)f_3(u^{-1}g)\sigma(h^{-1}u,u^{-1}g)\sigma(h,(h^{-1}u)(u^{-1}g))\\
&= \sum_{u\in G} \left(\sum_{h\in G} f_1(h)f_2(h^{-1}u)\sigma(h,h^{-1}u) \right) f_3(u^{-1}g)\sigma(u,u^{-1}g)\\
&= \sum_{u\in G} (f_1 * f_2)(u)f_3(u^{-1}g)\sigma(u,u^{-1}g) = ((f_1 * f_2) * f_3)(g)
\end{aligned}
$$

with, thanks to the cocycle condition,

$$\begin{aligned}&\sigma(h^{-1}u, u^{-1}g)\sigma(h, (h^{-1}u)(u^{-1}g)) = \sigma(h, (h^{-1}u)(u^{-1}g))\sigma(h^{-1}u, u^{-1}g)\\ &= \sigma(h, h^{-1}u)\sigma(hh^{-1}u, u^{-1}g) = \sigma(h, h^{-1}u)\sigma(u, u^{-1}g).\end{aligned}$$

As well, check that for $f \in \mathbb{C}(G, \sigma)$,

$$\begin{aligned}f^{**}(g) &= \overline{\sigma(g, g^{-1})f^*(g^{-1})}\\ &= \overline{\sigma(g, g^{-1})}\sigma(g^{-1}, g)f(g) = f(g).\end{aligned}$$

Moreover,

$$\begin{aligned}&(f_1 * f_2)^*(g) = \overline{\sigma(g, g^{-1})(f_1 * f_2)(g^{-1})}\\ &= \overline{\sigma(g, g^{-1})} \sum_{h=g^{-1}k, k\in G} \overline{f_1(h)f_2(h^{-1}g^{-1})\sigma(h, h^{-1}g^{-1})}\\ &= \overline{\sigma(g, g^{-1})} \sum_{k\in G} \overline{f_2((k)^{-1})f_1((k^{-1}g)^{-1})\sigma(g^{-1}k, k^{-1})}\\ &= \sum_{k\in G} \overline{\sigma(k^{-1}, g)f_2((k)^{-1})f_1((k^{-1}g)^{-1})\sigma(k^{-1}g, g^{-1})\sigma(g^{-1}k, k^{-1})}\\ &= \sum_{k\in G} \overline{\sigma(k, k^{-1})f_2(k^{-1})\sigma(k^{-1}g, (k^{-1}g)^{-1})f_1((k^{-1}g)^{-1})}\sigma(k, k^{-1}g)\\ &= (f_2^* * f_1^*)(g)\end{aligned}$$

where

$$\begin{aligned}&\sigma(g, g^{-1}) = \sigma(k^{-1}, 1)\sigma(g, g^{-1}) = \sigma(k^{-1}, g)\sigma(k^{-1}g, g^{-1}),\\ &\sigma(k^{-1}g, g^{-1}k) = \sigma(k^{-1}g, g^{-1}k)\sigma(1, k^{-1}) = \sigma(k^{-1}g, g^{-1})\sigma(g^{-1}k, k^{-1}),\end{aligned}$$

and

$$\sigma(k^{-1}, k) = \sigma(k^{-1}, k)\sigma(1, k^{-1}g) = \sigma(k^{-1}, g)\sigma(k, k^{-1}g)$$

with $\overline{\sigma(k, k^{-1})}\sigma(k^{-1}, k) = 1$, and hence

$$\overline{\sigma(k, k^{-1})}\sigma(k, k^{-1}g) = \overline{\sigma(k^{-1}, g)}.$$

Furthermore,

$$\begin{aligned}(\chi_1 * f)(g) &= \sum_{h\in G} \chi_1(h)f(h^{-1}g)\sigma(h, h^{-1}g) = \chi_1(1)f(g)\sigma(1, g) = f(g),\\ (f * \chi_1)(g) &= \sum_{h\in G} f(h)\chi_1(h^{-1}g)\sigma(h, h^{-1}g) = f(g)\chi_1(1)\sigma(g, 1) = f(g).\end{aligned}$$

□

Example 3.2.4. Let $\theta \in \mathbb{R}$ and let $G = \mathbb{Z}^2$. Define the **Khalkhali** 2-cocycle $c_\theta : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$ on $\mathbb{Z}^2$ as

$$c_\theta((m, n), (m', n')) = \exp(2\pi i\theta(mn' - nm')).$$

Check that

$$
\begin{aligned}
&c_\theta((m_1,n_1),(m_2,n_2))c_\theta((m_1+m_2,n_1+n_2),(m_3,n_3))\\
&=\exp(2\pi i\theta(m_1n_2-n_1m_2))\exp(2\pi i\theta((m_1+m_2)n_3-(n_1+n_2)m_3))\\
&=\exp(2\pi i\theta(m_1(n_2+n_3)+m_2(-n_1+n_3)-m_3(n_1+n_2))),\\
&c_\theta((m_1,n_1),(m_2+m_3,n_2+n_3))c_\theta((m_2,n_2),(m_3,n_3))\\
&=\exp(2\pi i\theta(m_1(n_2+n_3)-n_1(m_2+m_3)))\exp(2\pi i\theta(m_2n_3-n_2m_3))\\
&=\exp(2\pi i\theta(m_1(n_2+n_3)+m_2(-n_1+n_3)-m_3(n_1+n_2)))
\end{aligned}
$$

coincide. The corresponding twisted group $*$-algebra $\mathbb{C}(\mathbb{Z}^2,c_\theta)$ (with multiplication corrected) is isomorphic to the (universal) $*$-algebra generated by two unitaries U and V with the commutation relation $VU=e^{2\pi i\theta(-2)}UV$ (corrected). Indeed, U and V may be identified with the characteristic functions $\chi_{(1,0)}$ and $\chi_{(0,1)}$ at the generators $(1,0),(0,1)\in\mathbb{Z}^2$ respectively. Then compute that

$$
\begin{aligned}
&(\chi_{(1,0)}*\chi_{(0,1)})((m,n))\\
&=\sum_{(k,l)\in\mathbb{Z}^2}\chi_{(1,0)}(k,l)\chi_{(0,1)}(m-k,n-l)c_\theta((k,l),(m-k,n-l))\\
&=\chi_{(0,1)}(m-1,n)c_\theta((1,0),(m-1,n))\\
&=\exp(2\pi i\theta n)\chi_{(1,1)}(m,n)=\exp(2\pi i\theta)\chi_{(1,1)}(m,n),\\
&(\chi_{(0,1)}*\chi_{(1,0)})((m,n))\\
&=\sum_{(k,l)\in\mathbb{Z}^2}\chi_{(0,1)}(k,l)\chi_{(1,0)}(m-k,n-l)c_\theta((k,l),(m-k,n-l))\\
&=\chi_{(1,0)}(m,n-1)c_\theta((0,1),(m,n-1))\\
&=\exp(2\pi i\theta(-m))\chi_{(1,1)}(m,n)=\exp(2\pi i\theta(-1))\chi_{(1,1)}(m,n).
\end{aligned}
$$

Therefore, it does follow that

$$
\chi_{(0,1)}*\chi_{(1,0)}((m,n))=\exp(2\pi i\theta(-2))(\chi_{(1,0)}*\chi_{(0,1)})((m,n)). \quad \triangleleft
$$

Example 3.2.5. ([56]). Let $\theta\in\mathbb{R}$ and let $G=\mathbb{Z}^2$. Define the **Packer** 2-cocycle $w_\theta:\mathbb{Z}^2\times\mathbb{Z}^2\to\mathbb{T}$ on $\mathbb{Z}^2$ as

$$
w_\theta((m,n),(m',n'))=\exp(2\pi i\theta nm').
$$

Check that

$$
\begin{aligned}
&w_\theta((m_1,n_1),(m_2,n_2))w_\theta((m_1+m_2,n_1+n_2),(m_3,n_3))\\
&=\exp(2\pi i\theta n_1m_2)\exp(2\pi i\theta(n_1+n_2)m_3)\\
&=\exp(2\pi i\theta(m_2n_1+m_3(n_1+n_2))),\\
&w_\theta((m_1,n_1),(m_2+m_3,n_2+n_3))w_\theta((m_2,n_2),(m_3,n_3))\\
&=\exp(2\pi i\theta n_1(m_2+m_3))\exp(2\pi i\theta n_2m_3)\\
&=\exp(2\pi i\theta(m_2n_1+m_3(n_1+n_2)))
\end{aligned}
$$

coincide. The corresponding twisted group $*$-algebra $\mathbb{C}(\mathbb{Z}^2, w_\theta)$ is isomorphic to the (universal) $*$-algebra generated by two unitaries U and V with the commutation relation $VU = e^{2\pi i\theta}UV$. Indeed, U and V may be identified with the characteristic functions $\chi_{(1,0)}$ and $\chi_{(0,1)}$ at the generators $(1,0),(0,1) \in \mathbb{Z}^2$ respectively. Then compute that

$$
\begin{aligned}
&(\chi_{(1,0)} * \chi_{(0,1)})((m,n)) \\
&= \sum_{(k,l)\in\mathbb{Z}^2} \chi_{(1,0)}(k,l)\chi_{(0,1)}(m-k,n-l)w_\theta((k,l),(m-k,n-l)) \\
&= \chi_{(0,1)}(m-1,n)w_\theta((1,0),(m-1,n)) = \chi_{(1,1)}(m,n), \\
&(\chi_{(0,1)} * \chi_{(1,0)})((m,n)) \\
&= \sum_{(k,l)\in\mathbb{Z}^2} \chi_{(0,1)}(k,l)\chi_{(1,0)}(m-k,n-l)w_\theta((k,l),(m-k,n-l)) \\
&= \chi_{(1,0)}(m,n-1)w_\theta((0,1),(m,n-1)) \\
&= \exp(2\pi i\theta m)\chi_{(1,1)}(m,n) = \exp(2\pi i\theta)\chi_{(1,1)}(m,n).
\end{aligned}
$$

Therefore, it does follow that

$$
\chi_{(0,1)} * \chi_{(1,0)}((m,n)) = \exp(2\pi i\theta)(\chi_{(1,0)} * \chi_{(0,1)})((m,n)). \quad \triangleleft
$$

Example 3.2.6. Let $\mathbb{Z}_q$ be the cyclic group of order $q \geq 2$ and let $G = \mathbb{Z}_q \times \mathbb{Z}_q$. For a rational number $\theta = \frac{p}{q}$ with p, q relatively prime, define the Khalkhali 2-cocycle c_θ in the same manner as in the case of $G = \mathbb{Z}^2$. Then $\mathbb{C}(G, c_\theta)$ is isomorphic to the (universal) $*$-algebra generated by two unitaries U and V satisfying $U^q = 1$, $V^q = 1$, and $VU = e^{2\pi i\theta(-2)}UV$ (corrected in our sense). It is shown that this algebra is isomorphic to the matrix algebra $M_q(\mathbb{C})$ (if q is relatively prime with 2). Therefore, $M_q(\mathbb{C})$ is not a group algebra, but is a twisted group $*$-algebra! It also is a crossed product C^*-algebra $\mathbb{C}^q \rtimes_T \mathbb{Z}_q$ with T the translation action.

Note that

$$
\begin{aligned}
&(\chi_{(1,0)} * \chi_{(j,0)})((m,n)) \\
&= \sum_{(k,l)\in\mathbb{Z}^2} \chi_{(1,0)}(k,l)\chi_{(j,0)}(m-k,n-l)c_\theta((k,l),(m-k,n-l)) \\
&= \chi_{(j,0)}(m-1,n)c_\theta((1,0),(m-1,n)) \\
&= \exp(2\pi i\theta n)\chi_{(j+1,0)}(m,n) = \chi_{(j+1,0)}(m,n), \\
&(\chi_{(0,1)} * \chi_{(0,j)})((m,n)) \\
&= \sum_{(k,l)\in\mathbb{Z}^2} \chi_{(0,1)}(k,l)\chi_{(0,j)}(m-k,n-l)c_\theta((k,l),(m-k,n-l)) \\
&= \chi_{(0,j)}(m,n-1)c_\theta((0,1),(m,n-1)) \\
&= \exp(2\pi i\theta(-m))\chi_{(0,j+1)}(m,n) = \chi_{(0,j+1)}(m,n).
\end{aligned}
$$

As well, in the case of the Packer 2-cocycle w_θ on $\mathbb{Z}_q$,

$$\begin{aligned}
&(\chi_{(1,0)} * \chi_{(j,0)})((m,n)) \\
&= \sum_{(k,l)\in\mathbb{Z}^2} \chi_{(1,0)}(k,l)\chi_{(j,0)}(m-k,n-l)w_\theta((k,l),(m-k,n-l)) \\
&= \chi_{(j,0)}(m-1,n)w_\theta((1,0),(m-1,n)) \\
&= \exp(2\pi i\theta 0)\chi_{(j+1,0)}(m,n) = \chi_{(j+1,0)}(m,n), \\
&(\chi_{(0,1)} * \chi_{(0,j)})((m,n)) \\
&= \sum_{(k,l)\in\mathbb{Z}^2} \chi_{(0,1)}(k,l)\chi_{(0,j)}(m-k,n-l)w_\theta((k,l),(m-k,n-l)) \\
&= \chi_{(0,j)}(m,n-1)w_\theta((0,1),(m,n-1)) \\
&= \exp(2\pi i\theta m)\chi_{(0,j+1)}(m,n) = \chi_{(0,j+1)}(m,n).
\end{aligned}$$

Moreover, $M_q(\mathbb{C})$ is generated by the two (Voiculescu) unitaries defined as

$$V = \begin{pmatrix} e^{2\pi i\frac{1}{q}} = z & & & 0 \\ & e^{2\pi i\frac{2}{q}} = z^2 & & \\ & & \ddots & \\ 0 & & & 1 = z^q \end{pmatrix}, \quad U = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Then $VV^* = 1_q = V^*V$, $UU^t = 1_q = U^tU$, $U^q = 1_q$ and $V^q = 1_q$, where 1_q is the unit matrix of $M_q(\mathbb{C})$. In this case, compute that

$$VU = \begin{pmatrix} 0 & & & z \\ z^2 & 0 & & \\ & \ddots & \ddots & \\ & & z^q & 0 \end{pmatrix} = z\begin{pmatrix} 0 & & & 1 \\ z & 0 & & \\ & \ddots & \ddots & \\ & & z^{q-1} & 0 \end{pmatrix} = zUV.$$

Also, the translation action T with order q (as well on the diagonal $\mathbb{C}^q$) is given by

$$UVU^t = \begin{pmatrix} 1 = z^q & & & \\ & z & & \\ & & \ddots & \\ & & & z^{q-1} \end{pmatrix} \equiv T(U)$$

Furthermore, the C^*-algebra $C^*(V)$ generated by V is isomorphic to the C^*-algebra $C(\mathrm{sp}(V))$ of all continuous, complex-valued functions on the spectrum $\mathrm{sp}(V)$ of V, by the Gelfand transform. Since $\mathrm{sp}(V)$ consists of the set $\{z, z^2, \cdots, z^q\}$ of distinct q points, so that $C^*(V)$ is isomorphic to $\mathbb{C}^q$. $\lhd$

3.3 Twisted or not group C^*-algebras

Let G be a discrete group and let $H = L^2(G)$ denote the Hilbert space of all square summable, complex-valued functions ξ on G, so that

$$\sum_{g\in G} |\xi(g)|^2 = \sum_{g\in G} \xi(g)\overline{\xi(g)} = \langle \xi, \xi\rangle = \|\xi\|_2^2 < \infty.$$

There is the canonical orthonormal basis for $L^2(G)$ consisting of delta functions as δ_g at $g \in G$, also in $C_c(G, \mathbb{C}) = \mathbb{C}G$. The **left regular** representation of G is defined to be the unitary representation $\lambda : G \to \mathbb{B}(L^2(G))$ defined by $(\lambda_g \xi)(h) = \xi(g^{-1}h)$ for $g, h \in G$, which extends by linearity to an injective $*$-algebra homomorphism $\lambda : \mathbb{C}G \to \mathbb{B}(H)$, where $\mathbb{B}(H)$ denotes the C^*-algebra of all bounded operators on the (or a) Hilbert H. Then the **reduced** group C^*-algebra of G, denoted as $C_r^*(G)$, is defined to be the C^*-operator norm closure of $\lambda(\mathbb{C}G)$ in $\mathbb{B}(H)$. Since G is a discrete group, it is a unital C^*-algebra. There is the canonical linear functional $\tau : C_r^*(G) \to \mathbb{C}$ defined as $\tau(a) = \langle a\delta_e, \delta_e\rangle$ for $a \in C_r^*(G)$, which is a positive and faithful trace, so that $\tau(a^*a) > 0$ if $a \neq 0$, and $\tau(ab) = \tau(ba)$ for any $a, b \in C_r^*(G)$. May refer to [27] for the faithfulness of τ on $C_r^*(G)$.

The **universal** representation of G or $\mathbb{C}G$ is defined to be the direct product representation $\oplus_\pi \pi : G, \mathbb{C}G \to \mathbb{B}(\oplus_\pi H_\pi)$, where π runs over all irreducible unitary representations of G, or corresponding irreducible $*$-representations of $\mathbb{C}G$, and $\oplus_\pi H_\pi = H_{uv}$ means the direct product Hilbert space of the representation Hilbert spaces H_π of π. Then the **full** group C^*-algebra of G is defined to be the C^*-algebra completion of $\oplus_\pi \pi(\mathbb{C}G)$ under the C^*-norm

$$\| \oplus_\pi \pi(f)\| = \sup_\pi \|\pi(f)\|, \quad f \in \mathbb{C}G,$$

where the equivalence class $[\pi]$ of π may run over $G^\wedge$ or $(\mathbb{C}G)^\wedge$ the respective spectrums. Note that for any π and any $f \in \mathbb{C}G \subset L^1(G)$ the Banach $*$-algebra of all summable, complex-valued functions on G, with the 1-norm as,

$$\|\pi(f)\| \leq \sum_j |f(g_j)|\|\pi(g_j)\| = \sum_j |f(g_j)| = \|f\|_1,$$

so that $\| \oplus_\pi \pi(f)\| \leq \|f\|_1$.

There is the canonical surjective C^*-algebra homomorphism Φ from $C^*(G)$ to $C_r^*(G)$, induced by the continuous identity map $\mathbb{C}G$ to $\mathbb{C}G$ with the respective full and reduced norms. This quotient map Φ is a $*$-isomorphism if and only if G is amenable (cf. [4] or [57]). For example, either abelian, nilpotent, or solvable groups, and compact groups are amenable. On the other hand, the free, non-abelian groups are non-amenable.

Example 3.3.1. Let G be an abelian discrete group. Then $C^*(G) = C_r^*(G)$ is a unital commutative C^*-algebra. By the Gelfand-Naimark theorem, $C^*(G)$ is isomorphic to $C(G^\wedge)$ the C^*-algebra of all continuous, complex-valued functions on $G^\wedge = \mathrm{Hom}(G, \mathbb{T})$ the group of characters of G, known as the Pontryagin dual of G. In this case, the canonical trace on $C^*(G)$ is identified with

If $G = \mathbb{Z}^n$, then $C^*(G) \cong C(\mathbb{T}^n)$ where $\mathbb{T}^n$ is the real n-dimensional torus.

For a locally compact topological group G, there are associated two C^*-algebras, in general, denoted as $C^*(G)$ the full group C^*-algebra of G and $C_r^*(G)$ the reduced group C^*-algebra of G. These C^*-algebras are defined to be the C^*-algebra completions of the convolution group $*$-algebra $C_c(G, \mathbb{C})$ of all continuous, complex valued functions on G with compact support, respective, under the different norms induced by the universal representation and the left regular representation of G as well as $C_c(G, \mathbb{C})$. As the universal properties, the space as the unitary dual $G^\wedge$ of G of equivalence classes of unitary irreducible representations of G corresponds to the spectrum $C^*(G)^\wedge$ of $C^*(G)$ of equivalence classes of irreducible $*$-representations of $C^*(G)$, in the sense that the representation theory of G is identified with that of $C^*(G)$. Also, any unitary representation of G which are equivalent to a sub-representation of the left regular representation of G is identified with that of $C_r^*(G)$ with the same property. By universality, there is an onto quotient $*$-homomorphism from $C^*(G)$ to $C_r^*(G)$, which is a $*$-isomorphism if and only if G is amenable (cf. [57]).

More precisely, let G be a locally compact topological group with μ a left Haar measure on G. Denote by $L^1(G, \mu) = L^1(G)$ the Banach $*$-algebra of all μ-integrable, measure functions on G up to measure zero equivalence, with convolution product defined as

$$(f * g)(t) = \int_G f(s)g(s^{-1}t)d\mu(s), \quad t \in G, f, g \in L^1(G, \mu)$$

and involution defined as

$$f^*(t) = \Delta_G(t^{-1})\overline{f(t^{-1})}, \quad t \in G, f \in L^1(G, \mu)$$

where $\Delta_G : G \to \mathbb{R}^*$ is the modular character of G.

The left regular $*$-representation λ of $L^1(G)$ on the Hilbert space $L^2(G)$ of all square integrable measurable functions on G is defined by

$$(\lambda(f)\xi)(t) = \int_G f(s)\xi(s^{-1}t)d\mu(s), \quad f \in L^1(G), t \in G, \xi \in L^2(G).$$

The reduced group C^*-algebra $C_r^*(G)$ of G is defined to be the C^*-algebra generated by all $\lambda(f)$ for $f \in L^1(G)$ in $\mathbb{B}(L^2(G))$.

As in the discrete case, the universal representation $\oplus_\pi \pi$ from $L^1(G)$ to $\mathbb{B}(\oplus_\pi H_\pi)$ is defined similarly, where the equivalence class $[\pi]$ of π runs over $L^1(G)^\wedge$ the spectrum of $L^1(G)$, identified with $G^\wedge$ that of G. As well, the universal C^*-norm is defined as

$$\|f\| = \sup_\pi \|\pi(f)\|, \quad f \in L^1(G).$$

Then the full group C^*-algebra $C^*(G)$ of G is defined to be the operator norm closure of $(\oplus_\pi \pi)(L^1(G))$.

Note that in these definitions, $L^1(G)$ may be replaced by $C_c(G,\mathbb{C})$ of all continuous functions on G with compact support.

We may check that

Lemma 3.3.2. *Endowed with convolution and involution, the C_c algebra $C_c(G,\mathbb{C})$ of a non-finite, compact or non-compact, locally compact group G is a non-unital, $*$-algebra, where a suitable unit can be adjoined and may be given by the point measure δ_1 at the unit $1 \in G$, by assuming that $\delta_1 * f = f = f * \delta_1$ for any $f \in C_c(G,\mathbb{C})$. The same holds for $L^1(G,\mathbb{C})$.*

Proof. If $f_1, f_2 \in C_c(G,\mathbb{C})$ with compact supports $\mathrm{supp}(f_1), \mathrm{supp}(f_2) \subset G$, then

$$(f_1 * f_2)(g) = \int_{\mathrm{supp}(f_1)} f_1(h) f_2(h^{-1}g) d\mu(h).$$

Moreover, if the intersection $\mathrm{supp}(f_1)^{-1}g \cap \mathrm{supp}(f_2) = \emptyset$, then $(f_1 * f_2)(g) = 0$, where $\mathrm{supp}(f_1)^{-1} = \{g^{-1} \in G \,|\, g \in \mathrm{supp}(f_1)\}$. Therefore, if $(f_1 * f_2)(g) \neq 0$, then $\mathrm{supp}(f_1)^{-1}g \cap \mathrm{supp}(f_2) \neq \emptyset$. In this case, if $w = h^{-1}g \in \mathrm{supp}(f_2)$ for $h \in \mathrm{supp}(f_1)$, then $g = hw \in \mathrm{supp}(f_1)\mathrm{supp}(f_2)$. Thus, we obtain $\mathrm{supp}(f_1 * f_2) \subset \mathrm{supp}(f_1)\mathrm{supp}(f_2)$. Hence $f_1 * f_2 \in C_c(G,\mathbb{C})$.

For $f_1, f_2, f_3 \in C_c(G,\mathbb{C})$, we compute

$$\begin{aligned}
(f_1 * (f_2 * f_3))(g) &= \int_G f_1(h)(f_2 * f_3)(h^{-1}g) d\mu(h) \\
&= \int_G f_1(h) d\mu(h) \int_G f_2(k) f_3(k^{-1}h^{-1}g) d\mu(k) \\
&= \int_G f_1(h) d\mu(h) \int_G f_2(h^{-1}hk) f_3((hk)^{-1}g) d\mu(hk) \\
&= \int_G f_1(h) d\mu(h) \int_{hk=u\in G} f_2(h^{-1}u) f_3(u^{-1}g) d\mu(u) \\
&= \int_{u\in G} \left(\int_{h\in G} f_1(h) f_2(h^{-1}u) d\mu(h) \right) f_3(u^{-1}g) d\mu(u) \\
&= \int_{u\in G} (f_1 * f_2)(u) f_3(u^{-1}g) d\mu(u) = ((f_1 * f_2) * f_3)(g).
\end{aligned}$$

As well, check that for $f \in C_c(G,\mathbb{C})$,

$$f^{**}(g) = \Delta_G(g^{-1})\overline{f^*(g^{-1})} = \Delta_G(g^{-1})\Delta_G(g) f(g) = f(g).$$

Moreover,

$$\begin{aligned}
&(f_1 * f_2)^*(g) = \Delta_G(g^{-1})\overline{(f_1 * f_2)(g^{-1})} \\
&= \Delta_G(k^{-1}k)\Delta_G(g^{-1}) \int_{h=g^{-1}k, k\in G} \overline{f_1(h) f_2(h^{-1}g^{-1})} d\mu(h) \\
&= \int_{k\in G} \Delta_G(k^{-1})\overline{f_2((k)^{-1})}\Delta_G((k^{-1}g)^{-1})\overline{f_1((k^{-1}g)^{-1})} d\mu(k) \\
&= \int_{k\in G} f_2^*(k) f_1^*(k^{-1}g) d\mu(k) = (f_2^* * f_1^*)(g).
\end{aligned}$$

Furthermore, suppose that χ is the unit of $C_c(G, \mathbb{C})$. Note that if so, the unit extends to that of $L^1(G, \mathbb{C})$ by L^1-density of $C_c(G, \mathbb{C})$. For convenience, we show that if $L^1(G, \mathbb{C})$ has the unit χ extended, then a contradiction is deduced in the following. Then, for any $f \in L^1(G, \mathbb{C})$,

$$(\chi * f)(g) = \int_G \chi(h) f(h^{-1}g) d\mu(h) = f(g),$$
$$(f * \chi)(g) = \int_G f(h) \chi(h^{-1}g) d\mu(h) = f(g).$$

Then χ is not the zero function on G, so that there is $g_0 \in G$ such that $\chi(g_0) \neq 0$. Since $\chi * \chi = \chi$, then $\|\chi\|_1 \leq \|\chi\|_1^2$, so that $\|\chi\|_1 \leq 1$. If we take $f \in L^1(G, \mathbb{C})$ such that $f(g_0) = 1$ and $0 \leq f(g) \leq \frac{1}{2}$ for almost $g \in U$ a compact neighbourhood containging 1 and g_0, and $f = 0$ on the complement U^c. It then follows that

$$1 = |f(g_0)| \leq \int_G |\chi(h) f(h^{-1}g)| d\mu(h) \leq \frac{1}{2} \int_U |\chi(h)| d\mu(h) \leq \frac{1}{2}.$$

□

Twisted group C^*-algebras

We denote by $C^*(G, \sigma)$ the **twisted** full group C^*-algebra of a discrete (G, σ), that is defined to be the universal C^*-algebra completion of the twisted group $*$-algebra $\mathbb{C}(G, \sigma)$ of complex-valued functions on G with finite support, or of the twisted L^1 $*$-algebra $L^1(G, \sigma)$ of all summable, complex-valued functions on G, with σ-twisted convolution as multiplication (corrected) and σ-twisted involution as involution, as mentioned above. Similarly, the twisted **reduced** group C^*-algebra of (G, σ), denoted as $C_r^*(G, \sigma)$ is defined to be the reduced C^*-algebra completion of either $\mathbb{C}(G, \sigma)$ or $L^1(G, \sigma)$ under the left regular representation.

Definition 3.3.3. Let G be a locally compact group with a left Haar measure μ. A 2-**cocycle** or multiplier on G is defined to be a measurable function $\sigma : G \times G \to \mathbb{T}$ satisfying the cocycle condition (CC)

$$\sigma(g_1, g_2)\sigma(g_1 g_2, g_3) = \sigma(g_1, g_2 g_3)\sigma(g_2, g_3)$$

for any $g_1, g_2, g_3 \in G$, and moreover (added), for any $g \in G$ and the unit $1 \in G$,

$$\sigma(g, 1) = \sigma(1, g) = 1 \in \mathbb{T}$$

(cf. [56]). If $\sigma \equiv 1$, then σ is said to be trivial. We denote by $C_c(G, \sigma)$ the twisted group $*$-algebra of all complex-valued (continuous) functions on G with compact support, endowed with σ-twisted convolution as multiplication (corrected) and σ-twisted involution as involution, defined as:

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g) \sigma(h, h^{-1}g) d\mu(h)$$
$$f^*(g) = \Delta_G(g^{-1}) \overline{\sigma(g, g^{-1}) f(g^{-1})},$$

for $f_1, f_2, f \in C_c(G, \sigma)$, where the overline means the complex conjugate. The twisted full group C^*-algebra $C^*(G, \sigma)$ of a locally compact group (G, σ) is defined to be the universal C^*-algebra completion of the twisted group $*$-algebra $C_c(G, \sigma)$ or $L^1(G, \sigma)$ of all integrable, complex-valued functions on G with the same operations. Similarly, the twisted reduced group C^*-algebra $C_r^*(G, \sigma)$ of (G, σ) is defined to be the reduced C^*-algebra completion of either $C_c(G, \sigma)$ or $L^1(G, \sigma)$ under the left regular representation.

3.4 Twisted or not crossed product C^*-algebras

Note that the full group C^*-algebra $C^*(G)$ of either a discrete group or a locally compact group G is viewed as a crossed product C^*-algebra $\mathbb{C} \rtimes_1 G$ with the trivial action 1 of G on $\mathbb{C}$. The full crossed product C^*-algebra $\mathfrak{A} \rtimes_\alpha G$ of a C^*-algebra $\mathfrak{A}$ by an action α of G on $\mathfrak{A}$ by automorphisms is defined similarly and extendedly by replacing $\mathbb{C}$ of $\mathbb{C} \rtimes_1 G$ with $\mathfrak{A}$ and 1 with α.

In particular, for $f, f_1, f_2 \in C_c(G, \mathfrak{A})$ the C_c algebra as a dense $*$-subalgebra of $\mathfrak{A} \rtimes_\alpha G$ of all C_c-functions on G, define the convolution and involution involving the action α as

$$(f_1 * f_2)(g) = \int_G f_1(h)\alpha_h(f_2)(h^{-1}g)d\mu(h),$$
$$f^*(g) = \Delta_G(g^{-1})\overline{\alpha_g(f)(g^{-1})}, \quad g \in G.$$

More details may be considered in the next time. May refer to [57].

Note also that the full twisted group C^*-algebra $C^*(G, \sigma)$ of either a discrete group or a locally compact group G is viewed as a twisted crossed product C^*-algebra $\mathbb{C} \rtimes_1^\sigma G$ with the trivial action 1 of G on $\mathbb{C}$. The full twisted crossed product C^*-algebra $\mathfrak{A} \rtimes_\alpha^\sigma G$ of a C^*-algebra $\mathfrak{A}$ by an action α of G on $\mathfrak{A}$ by automorphisms is defined similarly and extendedly by replacing $\mathbb{C}$ of $\mathbb{C} \rtimes_1^\sigma G$ with $\mathfrak{A}$ and 1 with α.

In particular, for $f, f_1, f_2 \in C_c(G, \sigma, \mathfrak{A})$ the C_c algebra as a dense $*$-subalgebra of $\mathfrak{A} \rtimes_\alpha^\sigma G$ of all C_c-functions on G, define the convolution and involution involving the action α and the twist by σ as

$$(f_1 * f_2)(g) = \int_G f_1(h)\alpha_h(f_2)(h^{-1}g)\sigma(h, h^{-1}g)d\mu(h),$$
$$f^*(g) = \Delta_G(g^{-1})\overline{\sigma(g, g^{-1})\alpha_g(f)(g^{-1})}, \quad g \in G.$$

More details may be left to be considered in the future. May refer to [56] and more.

Briefly consider only the following examples and the beautiful stars.

Example 3.4.1. Let $G = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ denote the cyclic group of order n. Define an action α of $\mathbb{Z}_n$ on $X = \mathbb{Z}_n$ as a space such that the generator of G acts as the translation by 1 mod n. The action α on X induces the action on $C(X)$ by the same symbol α. Moreover, $C(X)$ is isomorphic to the group C^*-algebra $C^*(\mathbb{Z}_n)$ generated by a unitary V such that $V^n = 1$. The (full) crossed product C^*-algebra $C(X) \rtimes_\alpha G$ is then generated by the unitary V of $C(X)$ and the unitary U corresponding to the generator of G such that $UVU^* = \lambda^{-1}V$, with $\lambda = e^{2\pi i \frac{1}{n}}$. It is shown that

$$C(\mathbb{Z}_n) \rtimes_\alpha \mathbb{Z}_n \cong M_n(\mathbb{C}).$$

For more details, see the next subsection, it is shown in which that $M_n(\mathbb{C})$ is generated by two unital matrices u and v satisfying $u^n = 1$, $v^n = 1$, and the same commutation relation, so that the isomorphism is obtained by sending U to u and V to v respectively. $\triangleleft$

Example 3.4.2. Let G be a locally compact abelian group. Then G acts on G by the left translation as α and as well on $C_0(G)$ the C^*-algebra of all continuous, complex-valued functions on G vanishing at infinity. Then we have

$$C_0(G) \rtimes_\alpha G \cong \mathbb{K}(L^2(G))$$

where $\mathbb{K}(L^2(G))$ denotes the C^*-algebra of all compact operators on the Hilbert space $L^2(G)$ of measurable, square integrable functions on G. $\triangleleft$

What's more. For a crossed product C^*-algebra $\mathfrak{A} \rtimes_\alpha G$ of a C^*-algebra $\mathfrak{A}$ by an action α of an abelian group G on $\mathfrak{A}$ by automorphisms, there is the **dual** action $\alpha^\wedge$ of the dual group $G^\wedge$ of G on $\mathfrak{A} \rtimes_\alpha G$ defined as that $\alpha^\wedge_\chi(f)(g) = \chi(g)f(g)$ for $g \in G$, $\chi \in G^\wedge$ and $f \in \mathfrak{A} \rtimes G$ as a function from G to $\mathfrak{A}$, integrable or continuous with compact support.

★ The Takai duality theorem [67] states the following isomorphism

$$(\mathfrak{A} \rtimes_\alpha G) \rtimes_{\alpha^\wedge} G^\wedge \cong \mathfrak{A} \otimes \mathbb{K}(L^2(G))$$

(cf. [3], [57]). This theorem also says that the double crossed product of $\mathfrak{A}$ as the left hand side is stably isomorphic, or (strongly) Morita equivalent to $\mathfrak{A}$.

★ The Pontryagin duality theorem (cf. [52]) states that

$$G \cong (G^\wedge)^\wedge$$

for G a topological abelian group and $G^\wedge$ the dual TAG of all characters of G.

It then looks like that the T-duality above is slightly or essentially different from the P-duality in a context or sense.

Example 3.4.3. Let G be a locally compact topological group acting continuously on a locally compact Hausdorff space X. Define an action α of G on $C_0(X)$ the C^*-algebra of all continuous complex-valued functions on X vanishing at infinity by $(\alpha_g f)(x) = f(g^{-1}x)$ for $g \in G$, $x \in X$. Then the **tansformation** group C^*-algebra is defined to be the corresponding crossed product C^*-algebra $C_0(X) \rtimes_\alpha G$ (cf. [56]). Note that such an action α of G on X defines a topological dynamical system (X, G, α). Investigating the corresponding relation between the dynamical systems (X, G, α) and the transformation group crossed products $C_0(X) \rtimes_\alpha G$ is still a big industry, developing.

A particular, important case is considered in the next subsection, where $G = \mathbb{Z}$ of integers, $X = \mathbb{T}$ the 1-torus, and the action α is given by the rotation by angle $\theta \in [0, 2\pi]$, the crossed product for which is the noncommutative 2-torus denoted as $\mathbb{T}^2_\theta$ (cf. [61], [70]). $\triangleleft$

3.5 Quantum mechanics and Noncommutative tori

Quantum mechanics as mathematics

The canonical commutation relation (**CCR**) in quantum mechanics (**QM**) is defined to the equation (with notation changed)

$$\mathcal{M}P - P\mathcal{M} = \frac{h}{2\pi i}1,$$

where $\mathcal{M}$ is the momentum operator and P is the position operator, which are realized by unbounded self-adjoint operators on a Hilbert space as follows.

Let $L^2(\mathbb{R})$ be the **Hilbert** space of all square summable, measurable, complex-valued functions on the real line $\mathbb{R}$. Let $C^\infty_{c,p}(\mathbb{R})$ be the space of all piece-wise smooth functions on $\mathbb{R}$ with compact support. For $f \in L^2(\mathbb{R})$, define the **position** operator $P = M_x$ as the multiplication operator $(Pf)(x) = xf(x) = M_x f(x)$ for $x \in \mathbb{R}$. For $f \in L^2(\mathbb{R}) \cap C^\infty_{c,p}(\mathbb{R})$, define the **momentum** operator $\mathcal{M} = \frac{h}{2\pi i}D$ as the differential operator $(\mathcal{M}f)(x) = \frac{h}{2\pi i}\frac{df}{dx}(x) = \frac{h}{2\pi i}(Df)(x)$ for $x \in \mathbb{R}$, which extends to $L^2(\mathbb{R})$ by L^2-density of $C^\infty_{c,p}(\mathbb{R})$ in $L^2(\mathbb{R})$, where the case of one-sided derivatives at jumps in $\mathbb{R}$ is omitted. It then follows that for $f \in L^2(\mathbb{R}) \cap C^\infty_{c,p}(\mathbb{R})$,

$$(P\mathcal{M} - \mathcal{M}P)f(x) = x\frac{h}{2\pi i}\frac{df}{dx}(x) - \frac{h}{2\pi i}\frac{d}{dx}(xf(x)) = -\frac{h}{2\pi i}f(x),$$

which extends to $L^2(\mathbb{R})$. As well, for $f, g \in L^2(\mathbb{R})$, by the definitions of the L^2 inner product and the adjoint P^* of P,

$$\langle P^* f, g\rangle = \langle f, Pg\rangle = \int_{\mathbb{R}} f(x)\overline{xg(x)}dx = \int_{\mathbb{R}} xf(x)\overline{g(x)}dx = \langle Pf, g\rangle,$$

and hence $P = P^*$ self-adjoint. Also, for $f, g \in L^2(\mathbb{R}) \cap C^\infty_c(\mathbb{R})$,

$$\langle \mathcal{M}^* f, g\rangle = \langle f, \mathcal{M}g\rangle = \int_{\mathbb{R}} f(x)\overline{\frac{h}{2\pi i}\frac{dg}{dx}(x)}dx = \int_{\mathbb{R}} \frac{h}{2\pi i}\frac{df}{dx}(x)\overline{g(x)}dx = \langle \mathcal{M}f, g\rangle,$$

by integration by parts, and hence $\mathcal{M} = \mathcal{M}^*$, extended to $L^2(\mathbb{R})$.

Moreover, we define a continuous function f on $\mathbb{R}$ as, for $x \in \mathbb{R}$, $f(x) = \frac{1}{|x|}$ with $|x| \geq 1$ and $f(x) = 1$ with $|x| < 1$. Then $f \in L^2(\mathbb{R})$ with L^2 norm 2, because

$$\|f\|_2^2 \equiv \int_{\mathbb{R}} |f(x)|^2 dx = 2 + 2\int_1^\infty \frac{1}{x^2}dx = 2 + 2[-\frac{1}{x}]^\infty_{x=1} = 4.$$

But

$$\|Pf\|_2^2 = \int_{\mathbb{R}} |xf(x)|^2 dx = 1 + 2\int_1^\infty 1dx = 1 + 2[x]_{x=1}^\infty = \infty.$$

Therefore, the operator P is not bounded, i.e., unbounded.

Furthermore, we define piece-wise smooth functions f_n on $\mathbb{R}$ as, for $0 \neq n \in \mathbb{N}$ and $x \in \mathbb{R}$, $f_n(x) = |x|^n$ with $|x| \leq 1$ and $f_n(x) = 0$ with $|x| > 1$. Then

$$\|f_n\|_2^2 = 2\int_0^1 x^{2n} dx = 2[\frac{1}{2n+1}x^{2n+1}]_{x=0}^1 = \frac{2}{2n+1} < 1.$$

But

$$\|\mathcal{M}f_n\|_2^2 = 2\int_0^1 n^2 x^{2(n-1)} dx = 2n^2[\frac{1}{2n-1}x^{2n-1}]_{x=0}^1 = \frac{2n^2}{2n-1},$$

which goes to ∞, as $n \to \infty$. Hence, the operator $\mathcal{M}$ is unbounded.

Let $i = \sqrt{-1}$. Define the one parameter family of unitaries U_t for $t \in \mathbb{R}$ generated by $\mathcal{M}$:

$$\begin{aligned} U_t &= \int_{\mathbb{R}} e^{it\lambda} dE_\lambda = \int_{\mathbb{R}} \sum_{k=0}^\infty \frac{(it)^k}{k!} \lambda^k dE_\lambda \\ &= \sum_{k=0}^\infty \frac{(it)^k}{k!} \int_{\mathbb{R}} \lambda^k dE_\lambda = \sum_{k=0}^\infty \frac{1}{k!}(it)^k \mathcal{M}^k = \exp(it\mathcal{M}), \\ &\quad \text{with } U_t^* = \int_{\mathbb{R}} \overline{e^{it\lambda}} dE_\lambda = U_{-t}, \text{ where } \mathcal{M} = \int_{\mathbb{R}} \lambda\, dE_\lambda \end{aligned}$$

the spectral resolution for $\mathcal{M}$ self-adjoint and unbounded. Similarly, define $V_s = \exp(isP)$ for $s \in \mathbb{R}$. It then follows by using the CCR that

$$\begin{aligned} V_s U_t &= \exp(isP)\exp(it\mathcal{M}) = \sum_{k=0}^\infty \frac{(is)^k}{k!} P^k \sum_{l=0}^\infty \frac{(it)^l}{l!} \mathcal{M}^l \\ &= \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(is)^k}{k!} \frac{(it)^l}{l!} P^k \mathcal{M}^l \\ &= 1 + isP + it\mathcal{M} + \frac{(is)^2}{2!}P^2 + (is)(it)(\mathcal{M}P - \frac{h}{2\pi i}1) + \frac{(it)^2}{2!}\mathcal{M}^2 \\ &\quad + \frac{(is)^3}{3!}P^3 + \frac{(is)^2(it)}{2!}(\mathcal{M}P^2 - 2\frac{h}{2\pi i}P) + \frac{(is)(it)^2}{2!}(\mathcal{M}^2 P - 2\frac{h}{2\pi i}\mathcal{M}) \\ &\quad + \frac{(it)^3}{3!}\mathcal{M}^3 + \cdots \\ &= e^{-st\frac{h}{2\pi}i} \exp(it\mathcal{M})\exp(isP) = e^{-st\hbar i} \exp(it\mathcal{M})\exp(isP) \end{aligned}$$

with $\hbar = \frac{h}{2\pi}$ (corrected).

If we start with $\mathcal{M} = hi\frac{d}{dx}$ and $P = M_x$ with $[\mathcal{M}, P] = hi1$, then the exponential is given by $e^{sthi} = e^{2\pi i\hbar st}$. In this case, we may set $\hbar st = \theta \in \mathbb{R}$ and define the C^*-algebra $\mathcal{Q}_\theta$ generated by those unitaries $\exp(isP)$ and $\exp(it\mathcal{M})$ with $\theta = \hbar st$ in the C^*-algebra $\mathbb{B}(L^2(\mathbb{R}))$ of all bounded operators on $L^2(\mathbb{R})$, which may be called as the quantum **mechanics** C^*-algebra.

Note as well (cf. [36]) that for any Borel function f on $\mathbb{R}$, the spectral integral $\int_{\mathbb{R}} f dE$ for f with respect to the spectral measure $E = (E_t)_{t\in\mathbb{R}}$ on a Hilbert space H is defined by the strong limit

$$\int_{\mathbb{R}} f dE\xi = \lim_{n\to\infty} \int_{\mathbb{R}} \chi_{[-n,n]} f dE\xi$$

for any $\xi \in H$, where $\chi_{[-n,n]}$ is the characteristic function on the closed interval $[-n, n]$ for $n \in \mathbb{N}$, and the spectral measure $E = (E_t)_{t\in\mathbb{R}}$ is defined by a one-parameter family of projections $E_t = E((-\infty, t])$ for $t \in \mathbb{R}$ on H, such that the following conditions are satisfied:

Operator Monotonicity $E_s \leq E_t$ for $s < t$;

Right Continuity in the strong sense $E_s\xi = \lim_{t\to s+0} E_t\xi$ for any $\xi \in H$;

Zero $0 = \lim_{t\to-\infty} E_t\xi$; **I**dentity $1 = \lim_{t\to\infty} E_t\xi$, for any $\xi \in H$.

In particular, it then follows that for a finite or infinite partition of $\mathbb{R}$ with (t_j) points of partition,

$$\begin{aligned}
\left(\int_{\mathbb{R}} f dE\right)^2 &= \left(\lim_{n\to\infty} \int_{\mathbb{R}} \chi_{[-n,n]} f dE\right)^2 \\
&= \left(\lim_{n\to\infty} \lim_{k\to\infty} \sum_{j=-k}^{k>0} \chi_{[-n,n]}(t_j) f(t_j)(E_{t_j} - E_{t_{j-1}})\right)^2 \\
&= \lim_{n\to\infty} \lim_{k\to\infty} \sum_{j=-k}^{k>0} \chi_{[-n,n]}(t_j) f(t_j)^2 (E_{t_j} - E_{t_{j-1}}) = \int_{\mathbb{R}} f^2 dE.
\end{aligned}$$

In Fourier Analysis (cf. [72]), the following commutative diagram holds:

$$\begin{array}{ccc}
L^1(\mathbb{R}) \cap D^1(\mathbb{R}) \cap D^{-1}(L^1(\mathbb{R}) \cap D^1(\mathbb{R})) & \xrightarrow{D = \frac{d}{dt}} & L^1(\mathbb{R}) \cap D^1(\mathbb{R}) \\
\wedge \downarrow & & \downarrow \wedge \\
C_0(\mathbb{R}) & \xrightarrow[M_{iw}]{} & C_0(\mathbb{R})
\end{array}$$

where that $Df \in L^1(\mathbb{R}) \cap D^1(\mathbb{R})$ is assumed for a differentiable and summable function $f \in L^1(\mathbb{R}) \cap D^1(\mathbb{R})$ on $\mathbb{R}$, and where the Fourier transform $\wedge$ is defined to be $f^\wedge(w) = \int_{\mathbb{R}} f(t) e^{-iwt} dt \in C_0(\mathbb{R})$ the C^*-algebra of all bounded continuous functions on $\mathbb{R}$ vanishing at infinity, for $f \in L^1(\mathbb{R})$ the Banach $*$-algebra of all integrable measurable functions on $\mathbb{R}$ with convolution and involution, and $D^1(\mathbb{R})$ is the algebra of all differential functions on $\mathbb{R}$, and $D^{-1}(\cdot)$ means the inverse image by the differential operator D. Moreover, the commutative diagram may be restricted to $L^2(\mathbb{R})$ at four corners and all the restricted corners are extended to $L^2(\mathbb{R})$ at four corners by taking L^2 closure. Furthermore, the commutative diagram may pass to $L^2(S^1)$ by taking quotients by $\mathbb{R}$ (mod 1). What's more. the following commutative diagram holds:

$$\begin{array}{ccc} L^1(\mathbb{R}) \cap D^1(\mathbb{R}) \cap M_t^{-1}(L^1(\mathbb{R})) & \xrightarrow[M_t]{} & L^1(\mathbb{R}) \\ \downarrow \wedge & & \wedge \downarrow \\ C_0(\mathbb{R}) \cap D^1(\mathbb{R}) & \xrightarrow{iD=i\frac{d}{dw}} & C_0(\mathbb{R}) \end{array}$$

where $M_t^{-1}(\cdot)$ means the inverse image by the multiplication operator M_t. Moreover, the commutative diagram is restricted and extended to $L^2(\mathbb{R})$ at four corners, passing to $L^2(S^1)$ as well.

Example 3.5.1. (Noncommutative tori). There is the connection between quantum mechanics and the noncommutative 2-torus, as defined in the following. The noncommutative **2-torus** $\mathbb{T}^2_\theta$ is defined to be the universal unital C^*-algebra generated by two unitaries U and V subject to the commutation relation $VU = e^{2\pi i\theta}UV$ with $e^{2\pi i\theta} = \lambda \in \mathbb{T}$ the 1-torus for $\theta \in \mathbb{R}$, so that both the spectrums of U and V become $\mathbb{T}$. The **universality** in this case means that for any unital C^*-algebra $\mathfrak{B}$ generated by two unitaries with the same relation, there is a unital C^*-algebra homomorphism from $\mathbb{T}^2_\theta$ onto $\mathfrak{B}$. Note that the spectrum of a unitary operator is a compact (or closed) subset of the 1-torus $\mathbb{T}$.

In particular, if $\theta = 0$ (mod 1), then $\mathbb{T}^2_0$ is isomorphic to the C^*-algebra $C(\mathbb{T}^2)$ of all continuous complex-valued functions on the 2-torus, isomorphic to the C^*-tensor product $C(\mathbb{T}) \otimes C(\mathbb{T})$ of $C(\mathbb{T})$.

Consider the unitary operators $Q = M_{e^{2\pi ix}}$ and T_θ on the Hilbert space $L^2(S^1)$ on the real 1-dimensional sphere S^1 identified with $\mathbb{R}$ (mod 1) homeomorphically, defined respectively by the multiplication operator and the translation operator

$$(Qf)(x) = e^{2\pi ix}f(x) \quad \text{and} \quad (T_\theta f)(x) = f(x+\theta)$$

for $f \in L^2(S^1)$ and $x \in \mathbb{R}$ (mod 1). It then follows that for $f \in L^2(S^1)$,

$$\begin{aligned}(T_\theta Qf)(x) &= (Qf)(x+\theta) = e^{2\pi i(x+\theta)}f(x+\theta) \\ &= e^{2\pi i\theta}e^{2\pi ix}(T_\theta f)(x) = \lambda(QT_\theta f)(x).\end{aligned}$$

As well, for $f, g \in L^2(S^1)$,

$$\langle Q^*f, g\rangle = \langle f, Qg\rangle = \int_{\mathbb{R}} f(t)\overline{e^{2\pi ix}g(x)}dx = \int_{\mathbb{R}} e^{-2\pi ix}f(x)\overline{g(x)}dx = \langle M_{e^{-2\pi ix}}f, g\rangle,$$

$$\langle T_\theta^* f, g\rangle = \langle f, T_\theta g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x+\theta)}dx = \int_{\mathbb{R}} f(y-\theta)\overline{g(y)}dy = \langle T_{-\theta}f, g\rangle,$$

so that $Q^* = M_{e^{-2\pi ix}}$ and $T_\theta^* = T_{-\theta}$, and moreover, $Q^*Q = QQ^* = 1$ and $T_\theta^*T_\theta = T_\theta T_\theta^* = 1$ the identity operator. Define the **rotation** C^*-algebra $\mathfrak{A}_\theta$ as the C^*-algebra generated by these unitaries Q and T_θ in the C^*-algebra $\mathbb{B}(L^2(S^1))$ of all bounded operators on $L^2(S^1)$. The rotation C^*-algebra $\mathfrak{A}_\theta$ is viewed as a faithful representation of the noncommutative 2-torus $\mathbb{T}^2_\theta$ on $L^2(S^1)$. As well, the quantum mechanics C^*-algebra $\mathcal{Q}_\theta$ is another faithful representation of $\mathbb{T}^2_\theta$. Namely, $\mathbb{T}^2_\theta \cong \mathfrak{A}_\theta$ and $\mathbb{T}^2_\theta \cong \mathcal{Q}_\theta$ as a C^*-algebra. Such isomorphisms are obtained by knowing the spectrum of each unitary generator equal to the 1-torus $\mathbb{T}$.

In Fourier Analysis (cf. [72]), the following dual (inner and **outer**) commutative diagram holds:

$$\begin{tikzcd} L^1(\mathbb{R}) \arrow[r, "\mathbf{T_a}", "M_{e^{iat}}"'] \arrow[d, "\wedge"', "\wedge"] & L^1(\mathbb{R}) \arrow[d, "\wedge"', "\wedge"] \\ C_0(\mathbb{R}) \arrow[r, "T_a", "\mathbf{M_{e^{-iaw}}}"'] & C_0(\mathbb{R}) \end{tikzcd}$$

which may be restricted to $L^2(\mathbb{R})$ at four corners and all the restricted corners are extended to $L^2(\mathbb{R})$. The commutative diagram may pass to $L^2(S^1)$ by taking quotients by $\mathbb{R}$ (mod 1). $\lhd$

Denote by $\mathcal{O}(\mathbb{T}^2_\theta)$ the unital $*$-algebra $\mathbb{C}[U, V : \emptyset]/(VU - \lambda UV)$ generated by unitaries U and V subject to the relation $VU = \lambda UV$, as the coordinate ring of an algebraic noncommutative torus, as well as a dense subalgebra of $\mathbb{T}^2_\theta$, where $\mathbb{C}[U, V : \emptyset]$ is the free algebra generated by U and V with no relation and $(VU - \lambda UV)$ is the two-sided ideal of $\mathbb{C}[U, V : \emptyset]$ generated by $VU - \lambda UV$.

Let $e_n(x) = e^{2\pi i n x}$ for $x \in \mathbb{R}$ (mod 1) and $n \in \mathbb{Z}$. Then $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(S^1)$. A positive, faithful **trace** $\tau : \mathbb{T}^2_\theta \to \mathbb{C}$ is defined as $\tau(a) = \langle ae_0, e_0\rangle$ for $a \in \mathbb{T}^2_\theta$, such that $\tau(ab) = \tau(ba)$ for $a, b \in \mathbb{T}^2_\theta$ and $\tau(a^*a) > 0$ if $a \neq 0$ (cf. [60]). By using the relations $Ue_n = e_{n+1}$ and $Ve_n = e^{2\pi i\theta}e_n$, check that for finite sums $\sum_{m,n} a_{m,n}U^mV^n \in \mathcal{O}(\mathbb{T}^2_\theta)$ with $a_{m,n} \in \mathbb{C}$,

$$\begin{aligned} \tau(\sum_{m,n} a_{m,n}U^mV^n) &= \sum_{m,n} a_{m,n}\tau(U^mV^n) \\ &= \sum_{m,n} a_{m,n}\langle e_m, e_0\rangle = \sum_n a_{0,n}\int_{S^1} dx = \sum_n a_{0,n}. \end{aligned}$$

The definition for τ should be corrected **as** $\tau(a) = \langle a\delta_0, \delta_0\rangle$, where δ_0 means the Dirac point measure at zero $(0, 0)$, with respect to $(m, n) \in \mathbb{Z}^2$, so that $\tau(\sum_{m,n} a_{m,n}U^mV^n) = a_{0,0}$, with the inner product for $l^2(\mathbb{Z}^2)$.

Since $e^{2\pi i(\theta+n)} = e^{2\pi i\theta}$ for any $n \in \mathbb{Z}$, we have $\mathbb{T}^2_{\theta+n} \cong \mathbb{T}^2_\theta$.

Since the relation $VU = \lambda UV$ with $\lambda = e^{2\pi i\theta}$ is converted to $UV = \overline{\lambda}VU$, exchanging the unitary generators implies that $\mathbb{T}^2_\theta \cong \mathbb{T}^2_{-\theta} \cong \mathbb{T}^2_{1-\theta}$. Thus may restrict the range of θ to the interval $[0, \frac{1}{2}]$. It is known that the noncommutative tori $\mathbb{T}^2_\theta$ for $\theta \in [0, \frac{1}{2}]$ are mutually non-isomorphic.

• If θ is irrational, $\mathbb{T}^2_\theta$ is a simple C^*-algebra, i.e., without no proper closed two-sided ideals, so that it has no finite dimensional representations (cf. [60]).

Proof. (Added). We use the fact later checked that $\mathbb{T}^2_\theta$ is viewed as the crossed product C^*-algebra $C(\mathbb{T}) \rtimes_{\alpha_\theta} \mathbb{Z}$ of $C(\mathbb{T}) \cong C^*(U)$ the C^*-algebra generated by

U, by the action α_θ of $\mathbb{Z}$, defined as

$$\alpha_\theta(U) = VUV^* = \lambda U = M_\lambda U = e^{2\pi i\theta}z = e^{2\pi i(x+\theta)} \in C(\mathbb{T}),$$

where U is identified with the coordinate function $z = e^{2\pi ix}$ on $\mathbb{T} \cong \mathbb{R}$ (mod 1), so that $\alpha_\theta \in \mathrm{Aut}(\mathbb{T}^2_\theta)$ the automorphism group of $\mathbb{T}^2_\theta$.

In general, let X be a compact Hausdorff space and G a discrete group, and let $C(X) \rtimes_\alpha G$ be a crossed product C^*-algebra of $C(X)$ by the action α of G. Then any (non-trivial or not) closed two-sided ideal $\mathfrak{I}$ of $C(X) \rtimes_\alpha G$ bijectively corresponds to a non-tirival α-invariant closed subset of X or $C(X)$ by taking the corresponding quotient by $\mathfrak{I}$.

It then easily follows that if θ is irrational, then the unique α_θ-invariant closed subset of $\mathbb{T}$ is just $\mathbb{T}$. $\square$

• If θ is a rational number $\frac{p}{q}$, with p and q relatively prime and $q > 0$, then $\mathbb{T}^2_\theta$ has finite dimensional representations. Indeed,

Proof. (Added). Use the above proof. If $\theta = \frac{p}{q}$, then $\alpha_\theta^q = M_{e^{2\pi ip}} = M_1$. Therefore, there is a quotient homomorphism from $\mathbb{T}^2_\theta$ to $\mathbb{C}^q \rtimes_{\alpha_\theta} \mathbb{Z}$. Moreover, the crossed product $\mathbb{C}^q \rtimes_{\alpha_\theta} \mathbb{Z}$ contains $\mathbb{C}^q \rtimes_{\alpha_\theta} \mathbb{Z}_q$ as a quotient C^*-algebra, which is isomorphic to $M_q(\mathbb{C})$ (cf. [3]). $\square$

What's more (cf. [33]).

Proposition 3.5.2. *If θ is a rational number $\frac{p}{q}$, with p and q relatively prime and $q > 0$, then $\mathbb{T}^2_\theta$ is isomorphic to the algebra $C(\mathbb{T}^2, \mathrm{End}(E))$ of continuous sections of the endomorphism bundle of a flat rank q complex vector bundle E on the 2-torus $\mathbb{T}^2$.*

Proof. The required bundle E over $\mathbb{T}^2$ is obtained as a quotient of the trivial bundle $\mathbb{T}^2 \times \mathbb{C}^q$ by a free action of the direct product group $G = \mathbb{Z}_q \times \mathbb{Z}_q$ of the cyclic group $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ of order q.

Consider the unitary $q \times q$ matrices of $M_q(\mathbb{C})$ with $\lambda = e^{2\pi i\frac{p}{q}}$:

$$u = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda^{q-1} \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

which satisfy $uu^* = 1_q = u^*u$ and $vv^t = 1_q = v^tv$ and the relations $u^q = 1$, $v^q = 1$, and

$$uv = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ \lambda & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \lambda^{q-1} & 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 & \cdots & \lambda^{q-1} \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \lambda^{q-2} & 0 \end{pmatrix} = \lambda vu$$

(corrected). Define a pair (β, γ) of commuting automorphisms of order q of the trivial vector bundle $\mathbb{T}^2 \times \mathbb{C}^q$ by sending $(z_1, z_2, \xi) \in \mathbb{T} \times \mathbb{T} \times \mathbb{C}^q$ to $(\lambda z_1, z_2, u\xi)$ and $(z_1, \lambda z_2, v\xi)$ respectively, and hence define an action of $\mathbb{Z}_q \times \mathbb{Z}_q$. Indeed,

$$\begin{aligned} \beta^q(z_1, z_2, \xi) &= (\lambda^q z_1, z_2, u^q \xi) = (z_1, z_2, \xi), \\ \gamma^q(z_1, z_2, \xi) &= (z_1, \lambda^q z_2, v^q \xi) = (z_1, z_2, \xi), \\ \gamma\beta(z_1, z_2, \xi) &= (\lambda z_1, \lambda z_2, vu\xi) \\ \beta\gamma(z_1, z_2, \xi) &= (\lambda z_1, \lambda z_2, uv\xi) = (\lambda z_1, \lambda z_2, \lambda vu\xi) \end{aligned}$$

(and hence the definition implies a pair of non-commuting automorphisms of order q).

That action is free. Moreover, the quotient of the base space is again the torus. In this way, a flat bundle E over $\mathbb{T}^2$ is obtained.

By definition, the space of sections of $\mathrm{End}(E)$ is the fixed point algebra of the induced action of G on $C(\mathbb{T}^2, M_q(\mathbb{C})) \cong C(\mathbb{T}^2) \otimes M_q(\mathbb{C})$.

Using the matrix units for $M_q(\mathbb{C})$, we can write a section of this bundle as $s = \sum_{i,j=1}^q f_{ij}(z_1, z_2) \otimes u^i v^j$ with $f_{ij}(z_1, z_2) \in C(\mathbb{T}^2)$.

It is shown that such a section is G-invariant if and only if the coefficients of s have the form $f_{ij}(z_1^q, z_2^q)$.

Within $\mathbb{T}^2_\theta$ with $\theta = \frac{p}{q}$, we have $U^q V = V U^q$ and $V^q U = U V^q$, and hence U^q and V^q belong to the center of $\mathbb{T}^2_\theta$ Any element of $\mathbb{T}^2_\theta$ has a unique expression as $S = \sum_{i,j=1}^q f_{ij}(U^q, V^q) U^i V^j$ with $f_{ij} \in C(\mathbb{T}^2)$.

The required isomorphism is defined by sending such S to the corresponding s such as above. □

It follows from the proof above that the closed subalgebra generated by U^q and V^q is in fact the center $Z(\mathbb{T}^2_{\frac{p}{q}})$ of $\mathbb{T}^2_{\frac{p}{q}}$, so that $Z(\mathbb{T}^2_{\frac{p}{q}}) \cong C(\mathbb{T}^2)$.

The dense $*$-subalgebra $\mathfrak{T}^2_\theta$ of $\mathbb{T}^2_\theta$ for $\theta \in \mathbb{R}$, called as the (**smooth**) algebra of smooth functions on the noncommutative 2-torus, is defined by $a \in \mathfrak{T}^2_\theta$ if $a = \sum_{(m,n)\in\mathbb{Z}^2} a_{m,n} U^m V^n$, where the complex sequence $(a_{m,n})$ over $\mathbb{Z}^2$ belongs to the Schwartz space $\mathfrak{S}(\mathbb{Z}^2)$ of rapidly decreasing sequences over $\mathbb{Z}^2$, such that

$$\sup_{m,n\in\mathbb{Z}} (1 + m^2 + n^2)^k |a_{m,n}| < \infty, \quad k \in \mathbb{N}.$$

Note that in the case of $\theta = 0$, for $f \in C(\mathbb{T}^2)$, the inverse Fourier transform $f^\vee$ of f belongs to $\mathfrak{S}(\mathbb{Z}^2)$ if and only if f belong to $C^\infty(\mathbb{T}^2)$ the algebra of smooth functions on $\mathbb{T}^2$. May prove it, but no time at this moment.

Note that the (classical) **Fourier** transform $\wedge : L^1(\mathbb{Z}^2) \to C(\mathbb{T}^2)$ is defined as

$$g^\wedge(z_1, z_2) = \sum_{m,n\in\mathbb{Z}} g(m,n) z_1^m z_2^n$$

for $g \in L^1(\mathbb{Z}^2)$ the Banach $*$-algebra of integrable functions on $\mathbb{Z}^2$ with convolution and involution. The Fourier transform extends to the C^*-algebra isomorphism from the group C^*-algebra $C^*(\mathbb{Z}^2)$ to $C(\mathbb{T}^2)$. As well, the inverse Fourier transform is defined as

$$f^{\vee}(m,n) = \int_{\mathbb{T}^2} f(z_1,z_2) z_1^m z_2^n dz_1 dz_2, \quad f \in C(\mathbb{T}^2).$$

May prove that the function $f^{\vee}$ defined so belongs to $C^*(\mathbb{Z}^2)$.

There are the following inclusions with settings as

$$\mathcal{O}(\mathbb{T}^2_\theta) = P(\mathbb{T}^2)_\theta \subset \mathfrak{T}^2_\theta = C^\infty(\mathbb{T}^2)_\theta \subset \mathbb{T}^2_\theta = C(\mathbb{T}^2)_\theta$$

resembling those of algebras of algebraic functions or polynomials of coordinates, of smooth functions, and of continuous functions on $\mathbb{T}^2$, at $\theta = 0$.

It is shown that if $\theta = \frac{p}{q}$ a rational, then $\mathfrak{T}^2_{\frac{p}{q}}$ is isomorphic to the space $C^\infty(\mathbb{T}^2, \mathrm{End}(E))$ of smooth sections of the bundle $\mathrm{End}(E)$ over $\mathbb{T}^2$.

A **derivation** on a complex algebra A is defined to be a $\mathbb{C}$-linear map $\delta : A \to A$ such that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$ as the product formula of differentiation of functions. A derivation on A is determined by values of generators of A under δ (and in fact, as well as those of their products assigned by computation, necessary to define the derivation).

A $*$-derivation on an involutive algebra A over $\mathbb{C}$ is a derivation δ on A such that $\delta(a^*) = \delta(a)^*$ for any $a \in A$.

In particular, if δ is a derivation on a unital algebra with the unit 1, then $\delta(1) = \delta(1)1 + 1\delta(1) = 2\delta(1)$. Therefore, $\delta(1) = 0$.

Example 3.5.3. First define (involutive) linear maps $\delta_1, \delta_2 : \mathfrak{T}^2_\theta \to \mathfrak{T}^2_\theta$ as $*$-derivations by $\delta_1(U) = 2\pi i U$ and $\delta_1(V) = 0$ and $\delta_2(U) = 0$ and $\delta_2(V) = 2\pi i V$. Next define as that

$$\begin{aligned}
\delta_1(U^2) &= \delta_1(U)U + U\delta_1(U) = 2(2\pi i)U^2, \delta_1(V^2) = 0, \\
\delta_1(UV) &= \delta_1(U)V + U\delta_1(V) = (2\pi i U)V = 2\pi i UV, \\
\delta_1(VU) &= \delta_1(\lambda UV) = \lambda 2\pi i UV = 2\pi i VU, \\
\delta_2(U^2) &= 0, \quad \delta_2(V^2) = 2(2\pi i)V^2, \\
\delta_2(UV) &= \delta_2(U)V + U\delta_2(V) = U(2\pi i V) = 2\pi i UV, \\
\delta_2(VU) &= 2\pi i VU.
\end{aligned}$$

Also define as that

$$\begin{aligned}
\delta_1(U^3) &= \delta_1(U^2)U + U^2\delta_1(U) = 3(2\pi i)U^3, \delta_1(V^3) = 0, \\
\delta_1(U^2V) &= \delta_1(U^2)V + U\delta_1(V) = 2(2\pi i)U^2V, \\
\delta_1(VU^2) &= V\delta_1(U^2) = 2(2\pi i)VU^2, \\
\delta_1(UV^2) &= \delta_1(U)V^2 + U\delta_1(V^2) = 2\pi i UV^2, \\
\delta_1(V^2U) &= V^2\delta_1(U) = 2\pi i V^2U, \\
\delta_2(U^3) &= 0, \quad \delta_2(V^3) = 3(2\pi i)V^3, \\
\delta_2(U^2V) &= \delta_2(U^2)V + U^2\delta_2(V) = 2\pi i U^2V, \\
\delta_2(UV^2) &= \delta_2(U)V^2 + U\delta_2(V^2) = 2(2\pi i)UV^2, \\
\delta_2(VU^2) &= 2\pi i VU^2, \quad \delta_2(V^2U) = 2(2\pi i)V^2U.
\end{aligned}$$

Inductively, assigning the values of the products $U^m V^n$ for $m, n \in \mathbb{Z}$ under δ_1 and δ_2, by that $U^* = U^{-1}$, $V^* = V^{-1}$, and $\delta_j(a^*) = \delta_j(a)^*$ for $a \in \mathfrak{T}_\theta^2$, $j = 1, 2$, we obtain that with $a_{m,n} \in \mathbb{C}$ for $m, n \in \mathbb{Z}$, but finitely many, or not, with $(a_{m,n}) \in \mathfrak{S}(\mathbb{Z}^2)$,

$$\delta_1\left(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n\right) = 2\pi i \sum_{m>0, n\in\mathbb{Z}} m a_{m,n} U^m V^n - 2\pi i \sum_{m<0, n\in\mathbb{Z}} m a_{m,n} U^m V^n,$$

$$\delta_2\left(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n\right) = 2\pi i \sum_{m\in\mathbb{Z}, n>0} n a_{m,n} U^m V^n - 2\pi i \sum_{m\in\mathbb{Z}, n<0} n a_{m,n} U^m V^n$$

(corrected).

The trace τ on $\mathbb{T}_\theta^2$ defined as $\tau(\sum_{m,n} a_{m,n}) = a_{0,0}$ has the invariance property as a noncommutative analogue of the invariance property of the Haar measure for the torus, as that $\tau(\delta_j(a)) = 0$ for any $a \in \mathfrak{T}_\theta^2$ and $j = 1, 2$.

Indeed, it is shown by computation above that $\delta_j(a)$ has the zero component $a_{0,0} = 0$ at $(0, 0)$ because of killing the constants as $\delta_j(1) = 0$ as the usual differentiation for functions.

Those $*$-derivations δ_j on $\mathfrak{T}_\theta^2$ generate commuting one-parameter group of $*$-automorphisms $\alpha_j(t)$ of $\mathbb{T}_\theta^2$ for $t \in \mathbb{R}$.

Namely, define $\alpha_j(t) = \exp(t\delta_j)$ on $\mathfrak{T}_\theta^2$. Check that

$$\begin{aligned}
&\delta_2(\delta_1(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n)) \\
&= -4\pi^2 \sum_{m>0,n>0} mn a_{m,n} U^m V^n + 4\pi^2 \sum_{m>0,n<0} mn a_{m,n} U^m V^n \\
&\quad + 4\pi^2 \sum_{m<0,n>0} mn a_{m,n} U^m V^n - 4\pi^2 \sum_{m<0,n<0} mn a_{m,n} U^m V^n, \\
&= \delta_1(\delta_2(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n)),
\end{aligned}$$

and hence δ_1 commutes with δ_2 on $\mathfrak{T}_\theta^2$, so that $\alpha_1(t)$ commutes with $\alpha_2(t)$ on $\mathfrak{T}_\theta^2$, and moreover, since $\alpha_j(t)$ are unitaries, they extends to those on $\mathbb{T}_\theta^2$ by continuity.

As well, a continuous action α of the 2-torus $\mathbb{T}^2$ on $\mathbb{T}_\theta^2$ is defined as $\alpha_{z_1,z_2} U = z_1 U$ and $\alpha_{z_1,z_2} V = z_2 V$ for $(z_1, z_2) \in \mathbb{T}^2$.

Note that by definition, $\exp(t\delta_1)(U) = e^{2\pi i t} U$, $\alpha_1(t)V = 0$, and $\alpha_2(t)U = 0$, $\exp(t\delta_2)V = e^{2\pi i t} V$.

3.6 Vector bundles, projective modules, and projections

Vector bundles look like buildings or houses on the grounds as spaces,
while their continuous sections do like the roofs!
So, projective modules just like the black shadows, or windows.

It is shown by Swan [65] that the category VB of complex vector bundles on a compact Hausdorff space X is equivalent to the category PM of finitely generated, projective modules over the algebra $C(X)$ of continuous, complex-valued functions on X. Also, there are similar results for real vector bundles and quaternionic vector bundles by [65].

More earlier, it is shown by Serre [63] that algebraic vector bundles over an affine algebraic variety are characterized as finite projective modules over the coordinate ring of the variety.

Therefore, a finite projective module E over a non-commutative algebra A may be defined to be assumed as a noncommutative vector bundle over A represented as a noncommutative space.

Recall that a right module P over a unital algebra A is defined to be **projective** if there is a right A-module Q such that $P \oplus Q$ is a free A-module as A^n for some finite integer $n \geq 1$ or $n = \infty$. Equivalently, every A-module surjection from P to R (or R to P corrected) splits as a right A-module map. There is certainly another definition for projectivity of modules, but omitted. If for some $n \in \mathbb{N}$, there is a surjection from A^n to P, then P is said to be **finite** or finitely generated. Thus, a finite projective A-module is just a direct summand of A^n for some $n \in \mathbb{N}$.

Given a vector bundle $\pi : E \to X$ as a projection, with fibers as the inverse images $\pi^{-1}(x)$ for $x \in X$ as vector spaces of ranks as locally constants, let $\Gamma(E)$ be the set of all continuous (global) sections $s : X \to E$, so that the composites $\pi \circ s = \mathrm{id}_X$ the identity map on X. Then $\Gamma(E)$ with fiberwise scalar multiplication and addition is a $C(X)$-module.

For a (fiberwise) bundle map $\rho : E \to F$ of vector bundles over X, define a module map $\Gamma(\rho) : \Gamma(E) \to \Gamma(F)$ by $\Gamma(\rho)(s)(x) = \rho(s(x))$ for $s \in \Gamma(E)$ and $x \in X$. Namely,

$$\begin{array}{ccc} E = \cup_{x\in X}\pi^{-1}(x) & \xrightarrow{\rho} & F = \cup_{x\in X}\pi^{-1}(x) \\ \pi \big\downarrow s\uparrow & & \pi \big\downarrow \rho\circ s\uparrow \\ X & = & X. \end{array}$$

Thus, defined is $\Gamma : E \to \Gamma(E)$ the global section functor from the category VB of vector bundles over X with continuous bundle maps to the category Mod of $C(X)$-modules with module maps.

It is shown that Γ defines an equivalence between the categories.

Proof. Note that for any $\pi : E \to X$, there is a vector bundle F over X such that $E \oplus F \cong X \times \mathbb{C}^n$ a trivial bundle for some n. Therefore, $\Gamma(E) \oplus \Gamma(F) \cong C(X)^n$.

Let P be a finite projective $C(X)$-module, so that there is a $C(X)$-module Q such that $P \oplus Q \cong C(X)^n$. Then there is a corresponding idempotent $p \in M_n(C(X)) \cong C(X, M_n(\mathbb{C}))$ such that $pC(X)^n = P$. Since $C(X)^n \cong X \otimes \mathbb{C}^n$, define a vector bundle E over X as the image of p as an $n \times n$ projection-valued function on X, and as a subbundle of the trivial bundle $X \times \mathbb{C}^n$. Namely,

$$E = \cup_{x\in X} p(x)\mathbb{C}^n \ \underset{\leftarrow s}{\xrightarrow{\pi}} \ X.$$

It then follows that $\Gamma(E) \cong P$. $\square$

In general, let A be a unital algebra and let $M_n(A)$ denote the algebra of $n \times n$ matrices with entries in A. Then A^n is regarded as a right (or left) A-module. Then $M_n(A)$ is identified with $\mathrm{End}(A^n)$ of endomorphisms of A^n.

Let $p \in M_n(A)$ be an idempotent, so that $p^2 = p$. The left multiplication $M_p : A^n \to A^n$ by p defined as $M_p\xi = p\xi$ for $\xi \in A^n$ becomes a right A-module map. Let $P = M_pA^n$ be the image of M_p and $Q = (1 - M_p)A^n$ be the kernel of M_p with 1 the identity map on A^n. Then $P \oplus Q \cong A^n$. Hence P, Q are finite projective modules.

Conversely, if P is a finite projective right A-module, then there is a module Q such that $P \oplus Q \cong A^n$. Let $\Phi : A^n \to A^n$ be the corresponding right A-module map defined as the projection to P in A^n, with Φ the identity map on P and the zero map on Q. Then $P = \Phi(A^n)$ with $\Phi^2 = \Phi$. Hence Φ is identified with an idempotent of $M_n(A)$.

Suppose that $P \oplus Q \cong A^n$ as well as $P \oplus R \cong A^m$ for a finite projective module P, with $p \in M_n(A)$ and $p' \in M_m(A)$ respective corresponding idempotents or projections. Then there are maps $u \in \mathrm{Hom}(A^n, A^m)$ and $v \in \mathrm{Hom}(A^m, A - n)$ defined as the compositions in the top and bottom lines of the diagram

$$\begin{array}{ccccc} u : A^n \cong P \oplus Q & \longrightarrow & P & \longrightarrow & P \oplus R \cong A^m \\ \| & & \| & & \| \\ A^n \cong P \oplus Q & \longleftarrow & P & \longleftarrow & P \oplus R \cong A^m : v \end{array}$$

so that $v \circ u = p \in M_n(A)$ and $u \circ v = p' \in M_m(A)$.

In general, two such projections satisfying the above relations are said to be **Murray-von Neumann** equivalent. Conversely, Murray-von Neumann equivalent projections define isomorphic finite projective modules.

Example 3.6.1. (The Hopf line bundle on the 2-sphere S^2). It is also known as the magnetic monopole bundle. It is discovered independently by Hopf and Dirac in 1931, motivated by the different considerations. Let $\sigma_1, \sigma_2, \sigma_3 \in M_2(\mathbb{C})$ such that the respective canonical anti-commutation relations (CaCR) hold as

$$[\sigma_i, \sigma_j]_+ \equiv \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}1_2, \quad i, j = 1, 2, 3,$$

where δ_{ij} is the Kronecker symbol and 1_2 is the 2×2 identity matrix. As a canonical choice, we may take the Pauli spin matrices as, with $i^2 = -1$,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\diamond$ Indeed, with 0_2 the 2×2 zero matrix, we compute

$$\sigma_1^2 = 1_2 = \sigma_2^2 = \sigma_3^2,$$
$$\sigma_1\sigma_2 + \sigma_2\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 0_2,$$
$$\sigma_2\sigma_3 + \sigma_3\sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 0_2,$$
$$\sigma_1\sigma_3 + \sigma_3\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0_2. \quad \square$$

Define a function $f : S^2 \to M_2(\mathbb{C})$ by

$$f(x) = f(x_1, x_2, x_3) = \sum_{j=1}^{3} x_j \sigma_j, \quad x = (x_1, x_2, x_3) \in S^2, \sum_{j=1}^{3} x_j^2 = 1.$$

Then, for any $x \in S^2$,

$$\begin{aligned} f^2(x) &= (\sum_{j=1}^{3} x_j \sigma_j)(\sum_{k=1}^{3} x_k \sigma_k) \\ &= \sum_{j=1}^{3} x_j^2 \sigma_j^2 + \sum_{1 \leq j < k \leq 3} x_j x_k [\sigma_j, \sigma_k]_+ = 1_2. \end{aligned}$$

Define

$$p(x) = \frac{1}{2}(1 + f(x)) = \frac{1}{2} \begin{pmatrix} 1 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & 1 - x_3 \end{pmatrix}$$

for $x \in S^2$, with $1 = 1(x) = 1_2$, so that the function $p(x)^2 = p(x)$ is a (self-adjoint) idempotent of $C(S^2, M_2(\mathbb{C}))$ the C^*-algebra of all continuous, $M_2(\mathbb{C})$-valued functions on S^2, isomorphic to $C(S^2) \otimes M_2(\mathbb{C}) \cong M_2(C(S^2))$, with $p(x) = p^*(x)$ the transposed complex conjugate. Then defined from the CaCR is the corresponding complex vector bundle V over S^2, where the fiber V_x at $x \in S^2$ is the complex 1-dimensional subspace of $\mathbb{C}^2$ given as the image $p(x)\mathbb{C}^2$ of $p(x)$. Since the trace $\mathrm{tr}(p(x)) = 1$, equal to the rank $\mathrm{rk}(p(x))$, for any $x \in S^2$, the bundle is a complex line (or 1-dimensional) bundle over S^2.

$\diamond$ Namely,

$$\begin{tikzcd} V = \cup_{x \in S^2} V_x \arrow[d] & \mathbb{C} \cong V_x = p(x)\mathbb{C}^2 \arrow[l] \\ S^2 \ni x \arrow[r, equal] & S^2 \ni x \arrow[u, "p"'] \end{tikzcd}$$

Note as well that the trace $\mathrm{tr}(f(x)) = 0$ for any $x \in S^2$. $\square$

It can be shown that the line bundle V is associated to the Hopf fibration

$$\begin{tikzcd} S^3 \approx S^1(\mathbb{C}) \arrow[d] & S^1 \approx S^0(\mathbb{C}) \arrow[l] \\ S^2 \approx \mathbb{P}^1(\mathbb{C}) & \end{tikzcd}$$

◇ Recall from [77] that the **Hopf** mapping $h_{\mathbb{C}}$ from $S^1(\mathbb{C}) \approx S^3$ to S^2 is defined as

$$h_{\mathbb{C}}(z,w) = (|z|^2, \overline{z}w) \in \mathbb{R} \times \mathbb{C}, \quad z,w \in \mathbb{C}, |z|^2 + |w|^2 = 1,$$

with

$$|z|^2(1-|z|^2) = |zw|^2 = |\overline{z}w|^2, \quad 0 < |z| \leq 1.$$

Hence the fiber as the inverse image by $h_{\mathbb{C}}$ at any $(t, \overline{z}w) \in (0,1) \times \mathbb{C}$ is homeomorphic to S^1, given as $|z|^2 = t$. There is a homeomorphism between the image $h_{\mathbb{C}}(S^3)$ and the complex projective line $\mathbb{P}^1(\mathbb{C})$, defined as

$$(|z|^2, \overline{z}w) \mapsto \begin{pmatrix} |z|^2 & z\overline{w} \\ \overline{z}w & 1-|z|^2 \end{pmatrix} \equiv p(z,w) \in M_2(\mathbb{C}),$$

which is a projection with trace 1, and as well $\mathbb{P}^1(\mathbb{C}) \approx S^2$. In the end, the complex line bundle V is associated to the Hopf fibration over $\mathbb{C}$, in the sense that

$$S^3/S^1 \approx S^1(\mathbb{C})/S^0(\mathbb{C}) \approx S^2 \approx \mathbb{P}^1(\mathbb{C}). \quad \square$$

As well, the complex line bundle V is just the pull back of the canonical line bundle over $\mathbb{P}^1(\mathbb{C})$.

◇ Namely,

$$\begin{array}{ccc} V = \cup_{x \in S^2} V_x & \longrightarrow & \mathbb{P}^1(\mathbb{C}) \times \mathbb{C} \\ \Big\downarrow & & \Big\downarrow \\ S^2 & \longrightarrow & \mathbb{P}^1(\mathbb{C}) \end{array} \qquad \square$$

Example 3.6.2. The example above can be generalized to the higer even dimensional spheres S^{2n}. Construct matrices $\sigma_1, \cdots, \sigma_{2n+1}$ in $M_{2^n}(\mathbb{C})$ satisfying the Clifford algebra relations (cf. [40])

$$[\sigma_i, \sigma_j]_+ = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}1_{2^n}, \quad 1 \leq i,j \leq 2n+1.$$

Define the $2^n \times 2^n$ matrix-valued function f on the $2n$-dimensional sphere S^{2n} by

$$f(x) = f(x_1, \cdots, x_{2n+1}) = \sum_{j=1}^{2n+1} x_j\sigma_j, \quad x \in S^{2n}, \sum_{j=1}^{2n+1} x_j = 1.$$

Similarly as in the example above, it holds that $f^2(x) = 1_{2^n}$ for any $x \in S^{2n}$, so that $p(x) = \frac{1}{2}(1_{2^n} + f(x))$ is an idempotent of $M_{2^n}(C(S^{2n}))$, which defines a vector bundle over S^{2n}.

◇ Recall from [64] the following. The complex **Clifford** algebra of $\mathbb{R}^{2n}$, denoted as $Cl^*(\mathbb{R}^{2n})$, is generated by the unit 1 and (basis) elements of the $\mathbb{R}^{2n}$ over $\mathbb{C}$, with the relations $x^2 = \|x\|^2 1 = \langle x,x \rangle 1$ for $x \in \mathbb{R}^{2n}$. As a C^*-algebra, $Cl^*(\mathbb{R}^{2n})$ is isomorphic to $M_{2^n}(\mathbb{C})$. As a note, $Cl^*(\mathbb{R}^{2n+1})$ is isomorphic to the direct sum $M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C}) \cong \mathbb{C}^2 \otimes M_{2^n}(\mathbb{C})$ as a C^*-algebra. $\square$

$\diamond$ We now construct the matrices $\sigma_j^\sim$ for $1 \leq j \leq 2n+1$ by using σ_1, σ_2, and σ_1 in the case of $n=1$ as follows. Consider the case of $n=2$. We define

$$\sigma_j^\sim = \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix} \equiv \sigma_j \oslash \sigma_j \in M_4(\mathbb{C}), \quad j = 1, 2, 3.$$

Moreover, define

$$\sigma_4^\sim = \begin{pmatrix} 0_2 & i1_2 \\ -i1_2 & 0_2 \end{pmatrix} \quad \text{and} \quad \sigma_5^\sim = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix} \quad \text{in } M_4(\mathbb{C}).$$

The direct computation implies that $[\sigma_i^\sim, \sigma_j^\sim]_+ = 2\delta_{ij}1_4$ for $1 \leq i, j \leq 5$. As above, construct $f^\sim(x)$ by using $\sigma_j^\sim$ for $1 \leq j \leq 5$. Then the trace $\operatorname{tr}(f^\sim(x)) = 0$ for any $x \in S^4$. Hence, $p_2(x) = \frac{1}{2}(1_4 + f^\sim(x))$ is a self-adjoint idempotent of $M_4(\mathbb{C})$ with trace 2. Therefore, there is a complex 2-dimensional vector bundle over S^4 associated to the projection $p_2(x)$. $\square$

Example 3.6.3. (The Hopf line bundle on the quantum spheres). The **Podleś** quantum sphere S_q^2 is defined to be the C^*-algebra generated by the elements a, a^*, and $b = b^*$ subject to the relations

$$\begin{aligned} &aa^* + q^{-4}b^2 = 1, \quad a^*a + b^2 = 1, \\ &ab = q^{-2}ba, \quad a^*b = q^2ba^*. \end{aligned}$$

The quantum analogue of the Dirac or Hopf monopole line bundle over S^2 is given by the following idempotent in $M_2(S_q^2)$:

$$e_q = \frac{1}{2}\begin{pmatrix} 1 + q^{-2}b & qa \\ q^{-1}a^* & 1 - b \end{pmatrix}$$

(cf. [34] and also [6]). Check that

$$\begin{aligned} e_q^2 &= \frac{1}{4}\begin{pmatrix} (1+q^{-2}b)^2 + aa^* & qa + q^{-1}ba + qa - qab \\ q^{-1}a^* + q^{-3}a^*b + q^{-1}a^* - q^{-1}ba^* & a^*a + (1-b)^2 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 2(1+q^{-2}b) & 2qa \\ 2q^{-1}a^* & 2(1-b) \end{pmatrix} = e_q. \end{aligned}$$

Similar to the commutative case of S^2, for any $n \in \mathbb{Z}$, there is a quantum line bundle with topological charge n over S_q^2. Refer to [34] for its explicit description in terms of projections.

There is a noncommutative analogue of the Hopf 2-plane bundle over the 4-sphere S^4, associated to the principal $SU(2)$-bundle $S^7 \to S^4$ with fiber $SU(2)$. May refer to the survey [46] as well as references therein for its description.

Example 3.6.4. (Projective modules on noncommutative tori). Suppose that θ is rational. Then the noncommutative torus $\mathbb{T}_\theta^2 \cong \mathfrak{A}_\theta$ is isomorphic to the C^*-algebra of continuous sections of a C^*-algebra bundle of matrix algebras over the 2-torus $\mathbb{T}^2$.

★ The theorem of Swan implies that finite projective modules on $\mathfrak{A}_\theta$ corresponds to vector bundles on $\mathbb{T}^2$, up to isomorphism.

It follows that if $\theta \notin \mathbb{Z}$ rational, then $\mathbb{T}^2_\theta$ contains non-trivial projections as matrix projection-valued, continuous sections of some constant ranks. Note that for $\theta = n \in \mathbb{Z}$, $\mathfrak{A}_n \cong C(\mathbb{T}^2)$ has no non-trivial idempotent since $\mathbb{T}^2$ is connected.

Example 3.6.5. For $\theta \notin \mathbb{Z}$ irrational, it is shown that there are non-trivial projections of $\mathfrak{A}_\theta$ named as the **Powers-Rieffel** projections (cf. [33]). Let $0 < \theta \leq \frac{1}{2}$. Define by functional calculus,

$$p = U^{-1} f_{-1}(V) + f_0(V) + f_1(V)U, \quad f_{-1}, f_0, f_1 \in C^\infty(\mathbb{R}/\mathbb{Z})$$

(or $C(\mathbb{R}/\mathbb{Z})$) with $f_{-1} = \overline{f_1}$, such that for $t \in \mathbb{R}$ mod $\mathbb{Z}$, $f_1(t)f_1(t-\theta) = 0$,

$$f_1(t)f_0(t \mp \theta) = (1 - f_0(t))f_1(t), \quad \text{and} \quad f_0(t)(1 - f_0(t)) = |f_1(t)|^2 + |f_1(t \pm \theta)|^2$$

to obtain that $p = p^* = p^2$ (partly corrected), as in [61], [74]. Note that in the following computation we need to assume $-$ and $+$ in those respective signs $\mp$ and $\pm$ in the equations, but we need to assume $+$ and $-$ to have a concrete example as given in [74].

Indeed, note that $(f_1(V)U)^* = U^{-1}f_{-1}(V)$ and $f_0(V) = f_0(V)^*$ by the positivity $f_0(t) = |f_0(t)|^2 + |f_1(t)|^2 + |f_1(t+\theta)|^2 \geq 0$. Compute that

$$\begin{aligned} p^2 &= U^* f_{-1}(V)U^* f_{-1}(V) + U^*(f_{-1}f_0)(V) + U^*|f_1|^2(V)U \\ &\quad + f_0(V)U^* f_{-1}(V) + f_0^2(V) + (f_0 f_1)(V)U \\ &\quad + f_1(V)UU^* f_{-1}(V) + f_1(V)Uf_0(V) + f_1(V)Uf_1(V)U \end{aligned}$$

with

$$\begin{aligned} &U^*|f_1|^2(V)U + f_0^2(V) + f_1(V)UU^* f_{-1}(V) \\ &= |f_1|^2(\lambda V) + |f_0|^2(V) + |f_1|^2(V) = f_0(V), \end{aligned}$$

and

$$\begin{aligned} &f_1(V)Uf_1(V)U = f_1(V)f_1(UVU^*)U^2 = f_1(V)f_1(\overline{\lambda}V)U^2 = 0, \\ &U^* f_{-1}(V)U^* f_{-1}(V) = [f_1(V)Uf_1(V)U]^* = 0, \end{aligned}$$

and moreover,

$$\begin{aligned} &(f_0 f_1)(V)U + f_1(V)Uf_0(V) = (f_0 f_1)(V)U + f_1(V)Uf_0(V)U^*U \\ &= (f_0 f_1)(V)U + f_1(V)f_0(\overline{\lambda}V)U = f_1(V)U, \end{aligned}$$

and

$$\begin{aligned} &U^*(f_{-1}f_0)(V) + f_0(V)U^* f_{-1}(V) = U^*(f_0 f_{-1})(V) + U^*Uf_0(V)U^* f_{-1}(V) \\ &= U^*(f_0(V) + f_0(\overline{\lambda}V))f_{-1}(V) = U^*(f_0(V) + 1 - f_0(V))f_{-1}(V) = U^* f_{-1}(V) \end{aligned}$$

since $\overline{f_1(t)} f_0(t-\theta) = (1 - f_0(t))\overline{f_1(t)}$.

There are certainly some such solutions f_0 and f_1 satisfying the equations (cf. [74]). It is shown by [33] that

$$\tau(p) = \int_0^1 f_0(t)dt = \int_0^\theta f_0(t)dt + \int_0^\theta (1 - f_0(t))dt = \theta.$$

By following [74] we may define as that f_0 has support equal to the interval $[0, \theta + \delta]$ for some $0 < \delta < \frac{\theta}{2}$ and with values in $[0, 1]$ and 1 on $[\delta, \theta]$ and with integral on $[0, 1]$ equal to θ and that $f_1(t) = \sqrt{f_0(t)(1 - f_0(t))}$ but with support equal to only the interval $[0, \delta]$. In this case, the signs $+$ and $-$ are required as mentioned above. This failure is in fact corrected as that we do define similarly $f_1(t) = \sqrt{f_0(t)(1 - f_0(t))}$ with support equal to only the interval $[\theta, \theta + \delta]$, instead of $[0, \delta]$. May write a picture to check it.

Let $E = \mathfrak{S}(\mathbb{R})$ be the **Schwartz** space of rapidly decreasing smooth functions on $\mathbb{R}$, where a function $f \in E$ is rapidly decreasing if for any $n, k \in \mathbb{N}$, there is a constant $C_{n,k}$ such that $|f^{(n)}(x)|(1 + x^2)^k < C_{n,k}$ for all $x \in \mathbb{R}$, where $f^{(n)}$ is the n-th derivative of f.

Define a left $\mathfrak{T}_\theta^2$-module structure on E by

$$(Uf)(x) = f(x - \theta) \quad \text{and} \quad (Vf)(x) = e^{2\pi i x} f(x), \quad f \in E, x \in \mathbb{R}.$$

It is shown by [9] that E is finitely generated and projective as such a module.

For $j = 1, 2$, let E_j be a left $\mathfrak{T}_{\theta_j}^2$-module, on which the generators U_j and V_j of $\mathfrak{T}_{\theta_j}^2$ act as above. Define a left action of $\mathfrak{T}_{\theta_1+\theta_2}^2$ on $E_1 \otimes E_2$ as

$$U(\xi_1 \otimes \xi_2) = U_1\xi_1 \otimes U_2\xi_2 \quad \text{and} \quad V(\xi_1 \otimes \xi_2) = V_1\xi_1 \otimes V_2\xi_2.$$

For each $p, q \in \mathbb{N}$, relatively prime, using the $q \times q$ matrices u and v defined before, define a finite dimensional representation of $\mathfrak{T}_{\frac{p}{q}}^2$ on the vector space $\mathbb{C}^q = E'_{p,q}$ (corrected). Now take $\theta_1 = \theta - \frac{p}{q}$ and $\theta_2 = \frac{p}{q}$ to obtain a sequence of $\mathfrak{T}_\theta^2$-modules as $E_{\theta,p,q} = E_{\theta_1} \otimes E'_{p,q}$ with $E_{\theta_2} = E'_{p,q}$.

There is also an equivalent definition for $E_{p,q}$ ([9], [24]). Let $E_{p,q} = \mathfrak{S}(\mathbb{R} \times \mathbb{Z}_q)$, where $\mathbb{Z}_q$ is the cyclic group of order q. Define an $\mathfrak{T}_\theta^2$-module structure on $E_{p,q}$ by

$$(Uf)(x, j) = f(x + \theta - \frac{p}{q}, j - 1) \quad \text{and} \quad (Vf)(x, j) = e^{2\pi i(x - i\frac{p}{q})} f(x, j)$$

for $f = f(x, j) \in \mathfrak{S}(\mathbb{R} \times \mathbb{Z}_q)$. It is shown that if $p - q\theta \neq 0$, then the module $E_{p,q}$ is finitely generated and projective. In particular, if θ is irrational, then the same holds. $\triangleleft$

For more examples of noncommutative vector bundles, may refer to [12], [33], [46].

4 Hopf algebras and Quantum groups hybrid

What is Hopf? It's a motivated question. Solved as below.

4.1 Hopf algebras

Example 4.1.1. Let G be a finite group and $\mathfrak{H} = C(G)$ denote the commutative algebra of all (bounded and continuous) complex-valued functions on G. The group structure on G is defined by the multiplication, inversion, and unit or elements maps as

$$\begin{aligned} p : G \times G \to G, &\quad p(g,h) = gh, \\ i : G \to G, &\quad i(g) = g^{-1}, \\ u : \{g\} \to G, &\quad u(g) = g \in G. \end{aligned}$$

These maps are assumed to satisfy the compatible axioms such as associativity $(gh)k = g(hk) \in G$, inverse $g^{-1}g = gg^{-1} = 1 \in G$, and so on:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{p \times \mathrm{id}} & G \times G \\ {\scriptstyle \mathrm{id}\times p}\downarrow & & \downarrow{\scriptstyle p} \\ G \times G & \xrightarrow{p} & G \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{(i,\mathrm{id})} & G \times G \\ {\scriptstyle (\mathrm{id},i)}\downarrow & & \downarrow{\scriptstyle p} \\ G \times G & \xrightarrow{p} & G \end{array}$$

These group maps are dualized into the algebra homomorphisms as

$$\begin{aligned} \Delta = p^* : \mathfrak{H} \to \mathfrak{H} \otimes \mathfrak{H}, &\quad p^*(f) = f \circ p, \\ S = i^* : \mathfrak{H} \to \mathfrak{H}, &\quad i^*(f) = f \circ i, \\ \varepsilon = u^* : \mathfrak{H} \to \mathbb{C}, &\quad u^*(f) = f \circ u, \quad u = 1 \end{aligned}$$

called respectively, the **co-multiplication**, **antipode**, and **co-unit** for $\mathfrak{H}$. Note that the algebraic tensor product $C(G) \otimes C(G)$ is identified with $C(G \times G)$, since G is finite. Indeed, $f_1 \otimes f_2$ is mapped to $(f_1 \times f_2)(g_1, g_2) = f_1(g_1)f_2(g_2)$ injectively, and surjectively but only when G is finite.

Define the **multiplication** and the **unit** or **constant** map for $C(G)$ as

$$\begin{aligned} m : C(G) \otimes C(G) \to C(G), &\quad m(f_1 \otimes f_2)(g) = f_1(g)f_2(g), \quad g \in G \\ \eta : \mathbb{C} \to C(G), &\quad \eta(1) = \chi_G = 1_G \end{aligned}$$

where $\chi_G(g) = 1_G(g) = 1 \in \mathbb{C}$ for $g \in G$. Then the group associativity, group inverse, and unit or elements maps for groups are dualized into the following algebra co-associativity, antipode, and counit axioms for $\mathfrak{H}$ as commutaive diagrams:

$$\begin{array}{ccc} \mathfrak{H} & \xrightarrow{\Delta} & \mathfrak{H} \otimes \mathfrak{H} \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta\otimes\mathrm{id}} \\ \mathfrak{H} \otimes \mathfrak{H} & \xrightarrow{\mathrm{id}\otimes\Delta} & \otimes^3 \mathfrak{H} \end{array}$$

where $(\Delta \otimes \mathrm{id})\Delta f = (\Delta \otimes \mathrm{id})(f \circ p) = f \circ p \circ (p, \mathrm{id})$ and $(\mathrm{id} \otimes \Delta)\Delta f = (\mathrm{id} \otimes \Delta)(f \circ p) = f \circ p \circ (\mathrm{id}, p)$ are identified in $\otimes^3 \mathfrak{H}$, because $(f \circ p \circ (p, \mathrm{id}))(g_1, g_2, g_3) =$

$(f \circ p)(g_1 g_2, g_3) = f((g_1 g_2) g_3)$ and $(f \circ p \circ (\mathrm{id}, p))(g_1, g_2, g_3) = (f \circ p)(g_1, (g_2 g_3)) = f(g_1(g_2 g_3))$ for $g_1, g_2, g_3 \in G$, and the next (corrected, as added with Δ)

$$\begin{array}{ccccc} \mathfrak{H} & \xrightarrow{\Delta} & \mathfrak{H} \otimes \mathfrak{H} & \xrightarrow{S \otimes \mathrm{id}} & \mathfrak{H} \otimes \mathfrak{H} \\ {\scriptstyle\Delta}\downarrow & & & & \downarrow{\scriptstyle m} \\ \mathfrak{H} \otimes \mathfrak{H} & \xrightarrow{\mathrm{id} \otimes S} & \mathfrak{H} \otimes \mathfrak{H} & \xrightarrow{m} & \mathfrak{H} \end{array}$$

where $m(S \otimes \mathrm{id})\Delta f = m(f \circ p \circ (i, \mathrm{id}))$ and $m(\mathrm{id} \otimes S)\Delta f = m(f \circ p \circ (\mathrm{id}, i))$ are identified, because $(f \circ p \circ (i, \mathrm{id}))(g_1, g_2) = (f \circ p)(g_1^{-1}, g_2) = f(g_1^{-1} g_2)$ and $(f \circ p \circ (\mathrm{id}, i))(g_1, g_2) = f(g_1 g_2^{-1})$, so that for $g \in G$,

$$m(S \otimes \mathrm{id})\Delta f(g) = f(g^{-1} g) = f(1) = f(g g^{-1}) = m(\mathrm{id} \otimes S)\Delta f(g)$$

with $(\eta \circ \varepsilon)(f) = \eta(f(1)) = f(1)\chi_G = f(1)1_G = f(1) \in \mathfrak{H}$, and moreover,

$$\begin{array}{ccc} \mathfrak{H} & \xrightarrow{\Delta} & \mathfrak{H} \otimes \mathfrak{H} \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \varepsilon \otimes \mathrm{id}} \\ \mathfrak{H} \otimes \mathfrak{H} & \xrightarrow{\mathrm{id} \otimes \varepsilon} & \mathfrak{H} \end{array}$$

where $(\varepsilon \otimes \mathrm{id})\Delta f = f \circ p \circ (1, \mathrm{id})$ and $(\mathrm{id} \otimes \varepsilon)\Delta f = f \circ p \circ (\mathrm{id}, 1)$, so that

$$(\varepsilon \otimes \mathrm{id})\Delta f(g) = f(1g) = f(g) = f(g1) = (\mathrm{id} \otimes \varepsilon)\Delta f(g), \quad g \in G$$

with $\mathrm{id}(f) = f \in \mathfrak{H}$. It follows from these commutative diagrams above that the unital commutative algebra $\mathfrak{H} = C(G)$, equipped with the comultiplication Δ, antipode S, and counit ε, becomes a unital commutative **Hopf** algebra. $\lhd$

Definition 4.1.2. A unital algebra $\mathfrak{H}$ with the usual multiplication $m : \mathfrak{H} \otimes \mathfrak{H} \to \mathfrak{H}$, defined as $m(f_1 \otimes f_2) = f_1 f_2 \in \mathfrak{H}$, and the constant inclusion map $\eta : \mathbb{C} \to \mathfrak{H}$, defined as $\eta(\lambda) = \lambda 1 \in \mathfrak{H}$, is said to be a **Hopf** algebra if there are unital algebra homomorphisms $\Delta : \mathfrak{H} \to \mathfrak{H} \otimes \mathfrak{H}$ and $\varepsilon : \mathfrak{H} \to \mathbb{C}$, and a linear map $S : \mathfrak{H} \to \mathfrak{H}$, called respectively the **comultiplication**, the **counit**, and the **antipode** of $\mathfrak{H}$, such that the following axioms as above are satisfied

$$\begin{aligned} (\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta : \mathfrak{H} \to \otimes^3 \mathfrak{H}, \\ m(S \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes S)\Delta = \eta \circ \varepsilon : \mathfrak{H} \to \mathfrak{H}, \\ (\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id} : \mathfrak{H} \to \mathfrak{H}. \end{aligned}$$

If the existence of an antipode as S is not assumed, then $\mathfrak{H}$ is said to be a **bialgebra**. A Hopf algebra is said to be **commutative** if it is commutative as an algebra. A Hopf algebra $\mathfrak{H}$ is called **cocommutative** if $\sigma\Delta = \Delta$, where $\sigma : \mathfrak{H} \otimes \mathfrak{H} \to \otimes^2 \mathfrak{H}$ is the flip map defined as $\sigma(f_1 \otimes f_2) = f_2 \otimes f_1$.

Remark. Such a vector space $\mathfrak{H}$ together with Δ linear and ε is said to be a coalgebra (cf. [52]).

Remark. If G is only a finite monoid, then $C(G)$ is a bialgebra. Also, $\mathfrak{H} = C(G)$ is cocommutative if and only if G is a commutative group.

Proof. If G is finite and has a multiplication p without inverse, but with the unit 1, or is a finite, unital semi-group, then $C(G)$ is a unital algebra by the usual operations, with the multiplication m and η, as well as $\Delta = p^*$ and $\varepsilon = 1^*$, as the bialgebra structure. For instance, $G = \{0, 1\}$, with $0+0 = 0 = 1_G$, $0+1 = 1$ but $1+1 = 1$, so that there is no inverse for 1. Then $C(G) = \mathbb{C}^2$ as an algebra.

Suppose now that G is a commutative group. Then, for $f \in C(G)$,

$$\sigma\Delta f(g_1, g_2) = \Delta f(g_2, g_1) = f(g_2 g_1) = f(g_1 g_2) = \Delta f(g_1, g_2), \quad g_1, g_2 \in G.$$

Hence $\sigma\Delta = \Delta$. Conversely, if G is non-commutative, then there are $g_1, g_2 \in G$ such that $g_1 g_2 \neq g_2 g_1$. Let $\chi_{g_1 g_2} \in C(G)$ be the characteristic function at $g_1 g_2$. It then follows from the same computation as above that $\sigma\Delta\chi_{g_1 g_2}(g_1, g_2) = 0$ but $\Delta\chi_{g_1 g_2}(g_1, g_2) = 1$. Thus, $\sigma\Delta\chi_{g_1 g_2} \neq \Delta\chi_{g_1 g_2}$, so that $\sigma\Delta \neq \Delta$. $\square$

Example 4.1.3. Let G be a discrete group and let $\mathfrak{H} = \mathbb{C}[G]$ the group algebra of G, consisting of finite formal linear combinations as $\sum_{j=1}^n a_j g_j$ with $g_j \in G$, $a_j \in \mathbb{C}$. Then $\mathfrak{H}$ becomes a linear space over $\mathbb{C}$ and a unital algebra under the multiplication induced by the multiplication of G as

$$\sum_{j=1}^n a_j g_j \sum_{k=1}^n b_k h_k = \sum_{j,k=1}^n a_j b_k g_j h_k, \quad a_j, b_k \in \mathbb{C}, g_j, h_k \in G$$

and with the same unit as the unit 1 of G. For $g \in G$, define

$$\Delta(g) = g \otimes g \in \mathfrak{H} \otimes \mathfrak{H}, \quad S(g) = g^{-1} \in \mathfrak{H}, \quad \varepsilon(g) = 1 \in \mathbb{C}$$

and extend which by linearity to $\mathfrak{H}$. Note that for $g_1, g_2 \in G$,

$$\Delta(g_1 g_2) = g_1 g_2 \otimes g_1 g_2 = (g_1 \otimes g_1)(g_2 \otimes g_2) = \Delta(g_1)\Delta(g_2)$$

and as well $\varepsilon(g_1 g_2) = 1 = \varepsilon(g_1)\varepsilon(g_2)$. Then $\mathfrak{H}$ equipped with (Δ, S, ε) as before is a cocommutative Hopf algebra, and which is commutative if and only if G is commutative.

Indeed, for $g \in G$,

$$\begin{aligned}
(\Delta \otimes \mathrm{id})\Delta(g) &= g \otimes g \otimes g = (\mathrm{id} \otimes \Delta)\Delta g, \\
m(S \otimes \mathrm{id})\Delta(g) &= m(g^{-1} \otimes g) = g^{-1} g = 1, \\
m(\mathrm{id} \otimes S)\Delta(g) &= m(g \otimes g^{-1}) = g g^{-1} = 1, \\
(\varepsilon \otimes \mathrm{id})\Delta(g) &= 1g = g = g1 = (\mathrm{id} \otimes \varepsilon)\Delta(g).
\end{aligned}$$

Check the cocommutativity now.

$$\sigma\Delta(g) = \sigma(g \otimes g) = g \otimes g = \Delta(g), \quad g \in G,$$

which implies that $\sigma\Delta = \Delta$ on $\mathfrak{H}$.

Note that the group G can be recovered from the algebra $\mathfrak{H}$ as a subset of the set of group-like, non-zero elements h of $\mathfrak{H}$ defined as $\Delta(h) = h \otimes h$. For instance, $\Delta(\alpha g) = \alpha g \otimes \alpha g$ for $\alpha \in \mathbb{C}$, $g \in G$, but for $g_1 \neq g_2 \in G$ and $\alpha\beta \neq 0$,

$$\Delta(\alpha g_1 + \beta g_2) = \alpha g_1 \otimes \alpha g_1 + \beta g_2 \otimes \beta g_2 \neq (\alpha g_1 + \beta g_2) \otimes (\alpha g_1 + \beta g_2)$$

Note also that if G is finite, there are two associated Hopf algebras $C(G)$ and $\mathfrak{H} = \mathbb{C}[G]$. These algebras are dual to each other in some deep sense. In fact, by the Fourier transform or the Gelfand transform, $\mathbb{C}[G] = \mathbb{C}G \cong C_c(G, \mathbb{C})$ but with convolution as product, checked above, is isomorphic to $C(G^\wedge)$, where $G^\wedge$ is the dual group of G, which is identified with G in this case. $\triangleleft$

Example 4.1.4. Let $\mathfrak{g}$ be a Lie algebra with Lie bracket product $[\cdot,\cdot]$ and let $\mathfrak{H} = U(\mathfrak{g})$ be the universal **enveloping** algebra of $\mathfrak{g}$, which is defined to be the quotient $T(\mathfrak{g})/I(\mathfrak{g})$ of the tensor algebra

$$T(\mathfrak{g}) = \oplus_{n\geq 0} \otimes^n \mathfrak{g}, \quad \otimes^n \mathfrak{g} = \mathfrak{g} \otimes \cdots \otimes \mathfrak{g} \quad (n \text{ fold}), \quad \otimes^1 \mathfrak{g} = \mathfrak{g}, \quad \otimes^0 \mathfrak{g} = \mathbb{C}1$$

of $\mathfrak{g}$ by the two-sided ideal $I(\mathfrak{g})$ generated by elements $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in \mathfrak{g}$, so that $x \otimes y - y \otimes x$ may be identified with $xy - yx$ in the quotient in some case of $\mathfrak{g}$ an algebra with $[x, y] = xy - yx$. Then $\mathfrak{H} = U(\mathfrak{g})$ is a unital associative algebra. Indeed, for $f_1, f_2 \in T(\mathfrak{g})$,

$$\begin{aligned}(f_1 + I(\mathfrak{g})) \otimes (f_2 + I(\mathfrak{g})) &= f_1 \otimes f_2 + f_1 \otimes I(\mathfrak{g}) + I(\mathfrak{g}) \otimes f_2 + I(\mathfrak{g}) \otimes I(\mathfrak{g}) \\ &= f_1 \otimes f_2 + I(\mathfrak{g}).\end{aligned}$$

The canonical map $i : \mathfrak{g} \to U(\mathfrak{g})$ is universal in the sense that for any other associative algebra A, any linear map $\varphi : \mathfrak{g} \to A$ satisfying $\varphi([x, y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$ for any $x, y \in \mathfrak{g}$ uniquely factorises through the map i, with $\varphi^\sim$ defined as

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{i} & U(\mathfrak{g}) \\ \Vert & & \downarrow \varphi^\sim \\ \mathfrak{g} & \xrightarrow{\varphi} & A \end{array} \qquad \varphi^\sim([[x, y]]) = \varphi^\sim([x \otimes y - y \otimes x]) = \varphi([x, y])$$

and then $\varphi^\sim(x + I(\mathfrak{g})) = \varphi(x)$ for $x \in \mathfrak{g}$, which extends to $\mathfrak{H}$. By using the universal property of $U(\mathfrak{g})$ as well as the pair $U(\mathfrak{g}), i)$, there are uniquely determined algebra homomorphisms $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, $\varepsilon : U(\mathfrak{g}) \to \mathbb{C}$ and an anti-algebra map $S : U(\mathfrak{g}) \to U(\mathfrak{g})$, defined as

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad \varepsilon(1) = 1 \in \mathbb{C}, \quad S(x) = -x, \quad S(1) = 1 \in \mathfrak{H}$$

for any $x \in \mathfrak{g}$, viewed in $U(\mathfrak{g})$.

Note that when $\mathfrak{g}$ is an algebra, for $x, y \in \mathfrak{g}$,

$$\begin{aligned}\Delta(xy) &= xy \otimes 1 + 1 \otimes xy, \\ \Delta x \Delta y &= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y)\end{aligned}$$

and subtracting the lower side from the upper implies $-(x \otimes y + y \otimes x)$, and hence Δ is not an algebra homomorphism mod $I(\mathfrak{g})$ in general, but a Lie algebra homomorphism mod $I(\mathfrak{g})$. Indeed, for $x, y \in \mathfrak{g}$, mod $I(\mathfrak{g})$,

$$\begin{aligned}\Delta[x, y] &= \Delta(x \otimes y - y \otimes x) \\ &= \Delta x \otimes \Delta y - \Delta y \otimes \Delta x = [\Delta x, \Delta y] \in \otimes^4 \mathfrak{g}\end{aligned}$$

if defined so, as extending Δ. As another definition extending Δ,

$$\begin{aligned}
&\Delta(x \otimes y) - \Delta(y \otimes x) \\
&= x \otimes y \otimes 1 \otimes 1 + 1 \otimes 1 \otimes x \otimes y - y \otimes x \otimes 1 \otimes 1 - 1 \otimes 1 \otimes y \otimes x \\
&= (x \otimes y - y \otimes x) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes (x \otimes y - y \otimes x) = \Delta[x, y] \in \otimes^4 \mathfrak{g}
\end{aligned}$$

mod $I(\mathfrak{g})$, but which is not equal to

$$\begin{aligned}
&\Delta x \otimes \Delta y - \Delta y \otimes \Delta x \\
&= (x \otimes 1 + 1 \otimes x) \otimes (y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y) \otimes (x \otimes 1 + 1 \otimes x) \\
&= x \otimes 1 \otimes y \otimes 1 + x \otimes 1 \otimes 1 \otimes y + 1 \otimes x \otimes y \otimes 1 - 1 \otimes x \otimes 1 \otimes y \\
&- y \otimes 1 \otimes x \otimes 1 - y \otimes 1 \otimes 1 \otimes x - 1 \otimes y \otimes x \otimes 1 - 1 \otimes y \otimes 1 \otimes x.
\end{aligned}$$

It is checked that $(U(\mathfrak{g}), \Delta, \varepsilon, S)$ is a cocommutative Hopf algebra. The $\mathfrak{h}$ is commutative if and only if $\mathfrak{g}$ is an abelian Lie algebra. In this case, $U(\mathfrak{g})$ is the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$.

Indeed, for $x \in \mathfrak{g}$,

$$\begin{aligned}
(\Delta \otimes \mathrm{id})\Delta x &= (\Delta \otimes \mathrm{id})(x \otimes 1 + 1 \otimes x) \\
&= x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 2 \otimes 1 \otimes x, \\
(\mathrm{id} \otimes \Delta)\Delta x &= (\mathrm{id} \otimes \Delta)(x \otimes 1 + 1 \otimes x) \\
&= 2x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x,
\end{aligned}$$

and subtracting the lower side from the upper yields $-x \otimes 1 \otimes 1 + 1 \otimes 1 \otimes x$, and note that

$$x \otimes (1 \otimes 1) - (1 \otimes 1) \otimes x - [x, 1 \otimes 1] \in I(\mathfrak{g})$$

with $[x, 1 \otimes 1] = [x \otimes 1, 1 \otimes 1] = 0$, so that the upper side and the lower side above are identified mod $I(\mathfrak{g})$. In fact, note as well that the Lie algebra generated by $x \otimes 1$ and $1 \otimes 1$ is abelian, so that $[x \otimes 1, 1 \otimes 1] = 0$.

The second holds even for $T(\mathfrak{g})$ that for $x \in \mathfrak{g}$,

$$\begin{aligned}
m(S \otimes \mathrm{id})\Delta x &= m(S \otimes \mathrm{id})(x \otimes 1 + 1 \otimes x) \\
&= m(-x \otimes 1 + 1 \otimes x) = -x1 + 1x = -x + x = 0, \\
m(\mathrm{id} \otimes S)\Delta x &= m(\mathrm{id} \otimes S)(x \otimes 1 + 1 \otimes x) \\
&= m(x \otimes 1 + 1 \otimes (-x)) = x1 + 1(-x) = x - x = 0,
\end{aligned}$$

as well as $(\eta \circ \varepsilon)(x) = \eta(0) = 0$. The third also holds that for $x \in \mathfrak{g}$,

$$\begin{aligned}
(\varepsilon \otimes \mathrm{id})\Delta x &= (\varepsilon \otimes \mathrm{id})(x \otimes 1 + 1 \otimes x) = 0 \cdot 1 + 1x = x, \\
(\mathrm{id} \otimes \varepsilon)\Delta x &= (\mathrm{id} \otimes \varepsilon)(x \otimes 1 + 1 \otimes x) = x1 + 1 \cdot 0 = x.
\end{aligned}$$

But the third notion may be replaced with

$$m((\varepsilon^{\sim} \otimes \mathrm{id})\Delta - (\mathrm{id} \otimes \varepsilon^{\sim})\Delta) = 0 : \mathfrak{H} \to \mathfrak{H},$$

where assumed is $\varepsilon^{\sim} : \mathfrak{H} \to \mathbb{C}1$ as in $\mathbb{C}1 \subset \mathfrak{H}$. In fact, $\varepsilon \otimes \mathrm{id}$ is used in the sense of $m \circ (\varepsilon^{\sim} \otimes \mathrm{id})$.

Finally, check the cocommutativity as

$$\sigma\Delta x = \sigma(x \otimes 1 + 1 \otimes x) = 1 \otimes x + x \otimes 1 = \Delta x, \quad x \in \mathfrak{g}. \quad \triangleleft$$

Let $\mathfrak{H}$ be a Hopf algebra. A group-**like** element of $\mathfrak{H}$ is defined to be a nonzero element $h \in \mathfrak{H}$ such that $\Delta h = h \otimes h$. For such $h \in \mathfrak{H}$,

$$\begin{aligned} hS(h) &= m(\mathrm{id} \otimes S)\Delta h = m(S \otimes \mathrm{id})\Delta h = S(h)h \\ &= (\eta \circ \varepsilon)(h) = \varepsilon(h)1_{\mathfrak{H}} \in \mathfrak{H} \end{aligned}$$

(corrected). Thus, if $\varepsilon(h) \neq 0$, as in the case of $\varepsilon(h) = 1$, then a group-like element h is invertible with inverse $\varepsilon(h)^{-1}S(h)$. It then follows that the set $GL_{\Delta}(\mathfrak{H})$ of all (invertible) group-like elements of $\mathfrak{H}$ forms a subgroup of the multiplicative group $GL(\mathfrak{H})$ of invertible elements of $\mathfrak{H}$. For example, if $\mathfrak{H} = \mathbb{C}G$, then $GL_{\Delta}(\mathfrak{H}) = G$ (up to multiplication by $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$). A **primitive** element of a Hopf algebra $\mathfrak{H}$ is defined to be an element $h \in \mathfrak{H}$ such that $\Delta h = 1 \otimes h + h \otimes 1$. Define the bracket $[x, y] = xy - yx$ for $x, y \in \mathfrak{H}$. Then the bracket of two primitive elements of $\mathfrak{H}$ is again a primitive element.

Indeed, check that

$$\begin{aligned} &\Delta[x, y] = \Delta(xy - yx) = \Delta x \Delta y - \Delta y \Delta x \\ &= (1 \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1) - (1 \otimes y + y \otimes 1)(1 \otimes x + x \otimes 1) \\ &= 1 \otimes xy + y \otimes x + x \otimes y + xy \otimes 1 - 1 \otimes yx - x \otimes y - y \otimes x - yx \otimes 1 \\ &= 1 \otimes (xy - yx) + (xy - yx) \otimes 1 = 1 \otimes [x, y] + [x, y] \otimes 1. \end{aligned}$$

It follows that the set of primitive elements of $\mathfrak{H}$ forms a Lie algebra $\mathfrak{p}(\mathfrak{H})$. If $\mathfrak{H} = U(\mathfrak{g})$, then $\mathfrak{g}$ is contained in $\mathfrak{p}(\mathfrak{H})$, and it is shown by using the Poincaré-Birkhoff-Witt (**PBW**) theorem that $\mathfrak{g} = \mathfrak{p}(U(\mathfrak{g}))$ (cf. [43]). It says that $U(\mathfrak{g})$ is viewed as a linear space generated by monomials $x_1^{n_1} \otimes \cdots \otimes x_k^{n_k}$ for $n_1, \cdots, n_k \geq 0$, where $\{x_1, \cdots, x_k\}$ is a basis for $\mathfrak{g}$ as a linear space, and $1 = x_1^0 \otimes \cdots \otimes x_k^0$ (cf. [52]). For instance,

$$\begin{aligned} &\Delta(x_1 \otimes x_2) = \Delta x_1 \otimes \Delta x_2 = (1 \otimes x_1 + x_1 \otimes 1) \otimes (1 \otimes x_2 + x_2 \otimes 1) \\ &= 1 \otimes x_1 \otimes 1 \otimes x_2 + 1 \otimes x_1 \otimes x_2 \otimes 1 + x_1 \otimes 1 \otimes 1 \otimes x_2 + x_1 \otimes 1 \otimes x_2 \otimes 1 \\ &\neq x_1 \otimes x_2 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes x_1 \otimes x_2. \end{aligned}$$

Example 4.1.5. Let G be a compact topological group and let $C(G)$ denote the algebra of continuous, complex-valued functions on G. If G is not finite, $C(G)$ can not become a Hopf algebra. The problem is in defining the coproduct Δ as the dual of the multiplication of G, caused by the fact that the algebraic tensor product $C(G) \otimes C(G)$ is only dense in $C(G \times G)$ with the uniform norm, and these are different if G is infinite. Basically, there are two methods to deal with this problem. Either restrict to an appropriate dense subalgebra of $C(G)$, to define the coproduct on that subalgebra, or broaden the notion of

Hopf algebras by allowing completed topological (such as C^* or W^*) tensor products, as apposed to algebraic ones. In general, some algebraic difficulties or information disappear in C^* or W^*-completions, considerably. Those two approaches are essentially equivalent, in the sense of making the similar theory. Eventually, it is led to the Woronowicz theory of compact quantum groups [75].

A continuous function $f : G \to \mathbb{C}$ is said to be **representative** if the set of all left translations of f by elements of G forms a finite dimensional subspace of $C(G)$. It is shown that f is representative if and only it appears as a **matrix** entry of a finite dimensional complex representation of G.

In fact, if $\pi : G \to GL_n(\mathbb{C})$ is a representation of G, then since $\pi(gh) = \pi(g)\pi(h)$ with $\pi(g) = (\pi(g)_{ij})_{i,j=1}^n$, we have $\pi(g^{-1}h)_{ij} = \sum_{k=1}^n \pi(g^{-1})_{ik}\pi(h)_{kj} \in \mathbb{C}$. It follows that the matrix entries $\pi(h)_{ij}$ as functions for $h \in G$ are representative. Namely, the corresponding subspace is generated by the functions $\pi_{kj}(h)$, $1 \leq k \leq n$. Conversely, for a representative function f on G, its finite dimensional subspace S_f of $C(G)$ is invariant under the left regular representation λ of G on $C(G)$. Thus, restricting λ to the subspace S_f yields a finite dimensional representation of G.

Indeed, assume that S_f is generated by $f = f_1, \cdots, f_k$ with $k = \dim S_f$, and that for any $g \in G$, $\lambda_g f_j = \alpha_{1j}f_1 + \cdots + \alpha_{kj}f_k \in S_f$ with $\alpha_{1j}, \cdots, \alpha_{kj} \in \mathbb{C}$, $1 \leq j \leq k$, where the coefficients are dependent upon g. Then obtained is a k-dimensional representation of G, defined as

$$G \ni h \mapsto \begin{pmatrix} \alpha_{11}f_1(h) & \cdots & \alpha_{1k}f_1(h) \\ \vdots & \ddots & \vdots \\ \alpha_{k1}f_k(h) & \cdots & \alpha_{kk}f_k(h) \end{pmatrix} \in GL_k(\mathbb{C}).$$

Let $\mathfrak{H} = RF(G)$ denote the linear space generated by representative functions on G. Then $\mathfrak{H}$ is a subalgebra of $C(G)$, which is closed under complex conjugation. Moreover, the **Peter-Weyl** theorem implies that $RF(G)$ is a dense $*$-subalgebra of $C(G)$ with respect to the supremum norm (cf. [5]).

Indeed, if $f_1, f_2 \in \mathfrak{H}$, and S_{f_j} for $j = 1, 2$ is generated by $f_{j1}, \cdots, f_{jk_j}$, with $\dim S_{f_j} = k_j$, then $S_{f_1 f_2}$ is generated by $f_{1s_1}f_{2s_2}$ for $1 \leq s_j \leq k_j$, $j = 1, 2$.

★ The theorem of Peter and Weyl states that $RF(G)$ is dense in $C(G)$ as well as in $L^2(G)$, and that irreducible characters $\mathrm{tr}_\pi \circ \pi$ of G for π irreducible representations of G, with tr_π the canonical trace on the representation space of π, generate a dense subspace of the space of continuous class functions of G, such as $\varphi \in C(G)$ satisfying $\varphi(gxg^{-1}) = \varphi(x)$ for any $g, x \in G$.

Now let $p : G \times G \to G$ denote the product as multiplication of G and let $p^* : C(G) \to C(G \times G)$ denote the dual map of p, defined as $p^*f(x, y) = (f \circ p)(x, y) = f(xy)$ for $x, y \in G$. It is checked that if f is a representative function on G, then $p^*f \in RF(G) \otimes RF(G) \subset C(G \times G)$ (cf. [5] and [33]).

Indeed, suppose that $f(x) = \pi(x)_{ij}$ for $x \in G$ and for some k-dimensional representation π of G and some $1 \leq i, j \leq k$. Since $\pi(xy) = \pi(x)\pi(y) \in GL_k(\mathbb{C})$

for $x, y \in G$, we obtain that

$$f(xy) = \pi(xy)_{ij} = \sum_{l=1}^{k} \pi(x)_{il}\pi(y)_{lj} = \sum_{l=1}^{k} \pi(x)_{il} \otimes \pi(y)_{lj} \in RF(G) \otimes RF(G).$$

The Hopf algebra structure for $\mathfrak{H} = RF(G)$ is defined by the formulas

$$\Delta f = p^* f, \quad \varepsilon(f) = f(1), \quad (Sf)(g) = f(g^{-1})$$

Alternatively, may describe $RF(G)$ as the linear space generated by matrix coefficients such as $\pi(x)_{ij}$, of isomorphism classes of irreducible, finite dimensional, complex representations π of G. In this case, the coproduct is defined as

$$\Delta(\pi_{ij}) = \sum_{l=1}^{k} \pi_{il} \otimes \pi_{lj}, \quad \dim \pi = k.$$

As well, $\varepsilon(\pi_{ij}) = \pi(1)_{ij} = \delta_{ij}1$, and $(S\pi_{ij})(g) = \pi(g^{-1})_{ij}$.

The algebra $\mathfrak{H}$ is finitely generated as an algebra if and only if G is a compact Lie group. $\triangleleft$

Example 4.1.6. Let $G = U(1) = \mathbb{T}$ be the group of complex numbers of absolute value 1. Irreducible representations of G are all 1-dimensional, and $G^\wedge$ of which is identified with $\mathbb{Z}$, given as $\varphi_n(z) = z^n$ for $n \in \mathbb{Z}$ and $z \in \mathbb{T}$. It is shown that $\mathfrak{H} = RF(G)$ is the Laurent polynomial algebra $\mathbb{C}[u, u^{-1}] \subset C(G)$, with u a unitary so that $uu^* = u^*u = 1$, and with comultiplication, counit, and antipode, given as $\Delta(u^n) = u^n \otimes u^n$, $\varepsilon(u^n) = 1$, and $S(u^n) = u^{-n}$, for $n \in \mathbb{Z}$.

Indeed, for any $n \in \mathbb{Z}$ and $z, w \in \mathbb{T}$, $\varphi_n(wz) = w^n z^n$, as a function for $z \in \mathbb{T}$ in $\mathbb{C}\varphi_n$.

Example 4.1.7. Let $G = SU(2)$ be the group of unitary 2 by 2 complex matrices with determinant 1, which is identified with the real 3-dimensional sphere S^3, but defined as $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$ by complex coordinates. Let α and β denote the coordinate functions on $\mathbb{C}^2$ defined as $\alpha(z_1, z_2) = z_1$ and $\beta(z_1, z_2) = z_2$, which satisfy the relation $\alpha\alpha^* + \beta\beta^* = 1$ on $S^3 \subset \mathbb{C}^2$. It is shown that the algebra $C(SU(2)) = C(S^3)$ is the universal unital commutative C^*-algebra $\mathfrak{A}$ generated by two generators α and β by the same notation, with the same relation $\alpha\alpha^* + \beta\beta^* = 1$. This relation is equivalent to say that

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad \text{is a unitary matrix over } \mathfrak{A}.$$

Indeed,

$$UU^* = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} \alpha^* & -\beta \\ \beta^* & \alpha \end{pmatrix} = \begin{pmatrix} \alpha\alpha^* + \beta\beta^* & -\alpha\beta + \beta\alpha \\ -\beta^*\alpha^* + \alpha^*\beta^* & \beta^*\beta + \alpha^*\alpha \end{pmatrix}$$
$$U^*U = \begin{pmatrix} \alpha^* & -\beta \\ \beta^* & \alpha \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha^*\alpha + \beta\beta^* & \alpha^*\beta - \beta\alpha^* \\ \beta^*\alpha - \alpha\beta^* & \beta^*\beta + \alpha\alpha^* \end{pmatrix}.$$

For those to become the identity 2×2 matrix, it is required that α, β are normal, α, α^* commute with β, β^*, and $\alpha\alpha^* + \beta\beta^* = 1$.

All irreducible unitary representations of $SU(2)$ are given by tensor products of the fundamental representation whose matrix is U ([5]).

Recall from [5] the following. There is the standard linear isomorphism of $\mathbb{C}^2$ and $\mathbb{H}$ the quaternion algebra, given by

$$(z_1, z_2) \in \mathbb{C}^2 \mapsto z_1 + z_2 j = \begin{pmatrix} z_1 & 0 \\ 0 & z_1^* \end{pmatrix} + \begin{pmatrix} z_2 & 0 \\ 0 & z_2^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix} \in \mathbb{H}.$$

The quaternion group $Sp(1)$ of $\mathbb{H}$ is the group of unit quaternions, identified with $SU(2)$. Thus, the unitary matrix U over $\mathfrak{A}$ is viewed as a $SU(2)$-valued, continuous function $U = U(z_1, z_2)$ on $SU(2) = S^3$. As well, the standard or fundamental representation M of $SU(2)$ is defined to be the (left) matrix multiplication on $\mathbb{C}^2$, as $M_g\xi = g\xi$ for $g \in SU(2)$, $\xi \in \mathbb{C}^2$. This representation M is irreducible. Because if not, there is a complex 1-dimensional subspace of $\mathbb{C}^2$ invariant under the corresponding action, but which is impossible, since $SU(2)$ involves the rotation matrices. Irreducible unitary representations of $SU(2)$ are given by the trivial representation, the fundamental M, and the symmetric sub-representations $\otimes_s^n M$ of tensor products $\otimes^n M$ of M with the representation space V_n with dimension $n+1$, contained in $\otimes^n \mathbb{C}^2$, and identified with the space of homogeneous polynomials of degree n in two variables z_1 and z_2, contained in $\mathbb{C}[z_1, z_2]$. For instance, $V_1 = \mathbb{C}^2 \cong \mathbb{C}z_1 \oplus \mathbb{Z}z_2$ and

$$\begin{aligned} V_2 &= \mathbb{C}(z_1 \otimes z_1) \oplus \mathbb{C}(z_1 \otimes z_2 - z_2 \otimes z_1) \oplus \mathbb{C}(z_2 \otimes z_2) \\ &\cong \mathbb{C}z_1^2 \oplus \mathbb{C}z_1 z_2 \oplus \mathbb{C}z_2^2 \cong \mathbb{C}^3. \end{aligned}$$

As well, the exterior 2-power $\Lambda^2\mathbb{C}^2 = \mathbb{C}(z_1 \otimes z_2)$, so that $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong V_2 \oplus \Lambda^2\mathbb{C}^2$.

It is then shown that $\mathfrak{H} = RF(SU(2))$ is the $*$-subalgebra of $C(SU(2))$ generated by α and β. The coproduct, counit, and antipode for $\mathfrak{H}$ are uniquely induced from those on the equivalent generator U as

$$\Delta U = U \otimes^{\sim} U, \quad \text{and} \quad \varepsilon(U) = 1, \quad S(U) = U^*$$

in $SU(2, \mathfrak{H})$ the $SU(2)$ over $\mathfrak{H}$, so that $S(\alpha) = \alpha^*$, $S(\beta) = -\beta$, $S(\beta^*) = -\beta^*$, and $S(\alpha^*) = \alpha$, where

$$\begin{aligned} U \otimes^{\sim} U &= \begin{pmatrix} \alpha \otimes \alpha + \beta \otimes (-\beta^*) & \alpha \otimes \beta + \beta \otimes (\alpha^*) \\ (-\beta^*) \otimes \alpha + \alpha^* \otimes (-\beta^*) & (-\beta^*) \otimes \beta + \alpha^* \otimes \alpha^* \end{pmatrix} \\ &= \begin{pmatrix} \Delta(\alpha) & \Delta(\beta) \\ \Delta(-\beta^*) & \Delta(\alpha^*) \end{pmatrix} = \Delta(U), \end{aligned}$$

so that $\Delta(\alpha^*) = \Delta(\alpha)^*$ and $\Delta(\beta^*) = \Delta(\beta)^*$ in $\mathfrak{H} \otimes \mathfrak{H}$.

Example 4.1.8. An affine algebraic group, say over $\mathbb{C}$, is an affine algebraic variety G such that G is a group, and the multiplication map $p : G \times G \to G$ and the inversion map $i : G \to G$ are morphisms of varieties. The coordinate ring $\mathfrak{H} = \mathcal{O}[G]$ of an affine algebraic group G is a commutative Hopf algebra, involving the maps Δ, ε, and S, defined as the duals of the multiplication, the unit, and the inversion of G, similar to the case of finite or compact groups.

Example 4.1.9. Let $G = GL_n(\mathbb{C})$ be the general linear group of all invertible $n \times n$ matrices over $\mathbb{C}$. As an algebra, $\mathfrak{H} = \mathcal{O}[GL_n(\mathbb{C})]$ is generated by pairwise commuting elements x_{ij} and D for $i, j = 1, \cdots, n$, with the relation $\det(x_{ij})D = 1$. The coproduct, counit, and antipode of $\mathfrak{H}$ are given by

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \quad \Delta(D) = D \otimes D, \quad \varepsilon(x_{ij}) = \delta_{ij},$$
$$\varepsilon(D) = 1, \quad S(x_{ij}) = D\mathrm{Adj}(x_{ij}), \quad S(D) = D^{-1}.$$

These formulas are obtained by dualizing the usual linear algebra formulas for the matrix multiplication, the identity matrix, and the adjoint formula for the inverse.

Example 4.1.10. More generally, an affine group scheme over a commutative ring R is a commutative Hopf algebra over R.

The language of representable functors à la Grothendieck is cast to the above case as follows (cf. [73]).

Given such a Hopf algebra $\mathfrak{H}$, for any (unital) commutative algebra A over R, the set $G = \mathrm{Hom}(\mathfrak{H}, A)$ of algebra maps from $\mathfrak{H}$ to A is a group under the convolution product. The **convolution** product $f_1 * f_2$ of any two linear maps $f_1, f_2 : \mathfrak{H} \to A$ is defined as the composition

$$\mathfrak{H} \xrightarrow[\text{coproduct}]{\Delta} \mathfrak{H} \otimes \mathfrak{H} \xrightarrow{f_1 \otimes f_2} A \otimes A \xrightarrow[\text{product}]{m} A,$$

or equivalently, by

$$\begin{aligned}(f_1 * f_2)(h) &= \sum_{\Delta(h) = \sum_j h_{1j} \otimes h_{2j}} f_1(h_{1j}) f_2(h_{2j}) \\ &= \sum_{\Delta(h) = \sum_j h_{1j} \otimes h_{2j}} m(f_1 \otimes f_2)(h_{1j} \otimes h_{2j}) = m(f_1 \otimes f_2)\Delta h.\end{aligned}$$

Check that for $h, h' \in \mathfrak{H}$ and $f_1, f_2 \in G$,

$$\begin{aligned}&(f_1 * f_2)(hh') = m(f_1 \otimes f_2)\Delta(hh') = m(f_1 \otimes f_2)(\Delta(h)\Delta(h')) \\ &= \sum_{j,k} m(f_1 \otimes f_2)(h_{1j}h'_{1k} \otimes h_{2j}h'_{2k}) = \sum_{j,k} f_1(h_{1j}) f_1(h'_{1k}) f_2(h_{2j}) f_2(h'_{2k}) \\ &= \sum_{j,k} f_1(h_{1j}) f_2(h_{2j}) f_1(h'_{1k}) f_2(h'_{2k}) \quad \text{(because of } A \text{ commutative)} \\ &= \sum_{j,k} m(f_1 \otimes f_2)(h_{1j} \otimes h_{2j}) m(f_1 \otimes f_2)(h'_{1k} \otimes h'_{2k}) \\ &= [m(f_1 \otimes f_2)\Delta h][m(f_1 \otimes f_2)(\Delta h')] = (f_1 * f_2)(h)(f_1 * f_2)(h'),\end{aligned}$$

where $\Delta(h) = \sum_j h_{1j} \otimes h_{2j}$ and $\Delta(h') = \sum_k h'_{1k} \otimes h'_{2k}$. This formula seems to be not extended to the noncommutative case in general.

By the way, what is the unit for G ? If A is unital, then G contains $1_{\mathfrak{H}}$ as the unit function on $\mathfrak{H}$. If $\Delta h = h \otimes h$, then $(f * 1)(h) = m(f(h) \otimes 1) = f(h)$. Thus, $1_{\mathfrak{H}}$ is the unit for G. As for the inverse for $f \in G$, any $f \in G$ is invertible? Possibly, the right definition as G in this case of Δ should be $G^{-1} = \mathrm{Hom}(\mathfrak{H}, A)^{-1}$, which denotes the group of all invertible elements of G. Then, for $f, f^{-1} \in G^{-1}$ with $f^{-1}(h) = (f(h))^{-1} \in A^{-1}$ for any $h \in \mathfrak{H}$, where A^{-1} denotes the group of all invertible elements of A, we have $(f * f^{-1})(h) = m(f(h) \otimes f^{-1}(h)) = 1 \in A^{-1}$.

Then define a functor F from the category of commutative algebras over R to the category of groups, as

$$F : \textit{Comm-Alg}_R \to \textit{Grp}, \quad A \mapsto F(A) = G^{-1} = \mathrm{Hom}(\mathfrak{H}, A)^{-1}.$$

This functor F is representable, in the sense of being represented by $\mathfrak{H}$.

Conversely, let $F' : \textit{Comm-Alg}_R \to \textit{Grp}$ be a representatible functor represented by a unital commutative algebra $\mathfrak{K}$, as $F'(A) = \mathrm{Hom}(\mathfrak{K}, A)^{-1}$. Then $\mathfrak{K} \otimes \mathfrak{K}$ represents $F' \otimes F'$.

Indeed, $\mathrm{Hom}(\mathfrak{K} \otimes \mathfrak{K}, A) \cong \mathrm{Hom}(\mathfrak{K}, A) \otimes \mathrm{Hom}(\mathfrak{K}, A)$. For $f_1, f_2 \in \mathrm{Hom}(\mathfrak{K}, A)$, defined is $f_1 \otimes f_2 \in \mathrm{Hom}(\otimes^2 \mathfrak{K}, A)$. Conversely, any element of $\mathrm{Hom}(\otimes^2 \mathfrak{K}, A)$ is determined by values of simple tensors, which corresponds to some element of $\otimes^2 \mathrm{Hom}(\mathfrak{K}, A)$. But $\mathrm{Hom}(\mathfrak{K} \otimes \mathfrak{K}, A)^{-1}$ may not be equal to $\mathrm{Hom}(\mathfrak{K}, A)^{-1} \otimes \mathrm{Hom}(\mathfrak{K}, A)^{-1}$. However, that contains it, so represented by $\mathfrak{K} \otimes \mathfrak{K}$ in this sense.

Applying the Yoneda lemma we obtain maps $\Delta : \mathfrak{K} \to \mathfrak{K} \otimes \mathfrak{K}$, $\varepsilon : \mathfrak{K} \to \mathbb{C}$, and $S : \mathfrak{K} \to \mathfrak{K}$ satisfying the axioms, so that $\mathfrak{K}$ becomes a Hopf algebra. Thus, the equivalence between $\textit{Comm-Alg}_R$ and $\textit{Grp}$ is obtained.

Example 4.1.11. Consider the functor μ_n from the category of commutative algebras A over R to the category of groups, by sending A to the group of its n-th roots of unity. This functor is representable by the Hopf algebra $\mathfrak{H} = R[x]/(x^n - 1)$ as the quotient of the polynomial algebra $R[x]$ by the relation $x^n = 1$. Its coproduct, counit, and antipode are given respectively by $\Delta(x) = x \otimes x$, $\varepsilon(x) = 1$, and $S(x) = x^{n-1}$.

Note that $x^n = x^{n-1}x = 1$. Thus, $x^{-1} = x^{n-1}$.

In general, an algebraic group, such as GL_n or SL_n, is an affine group scheme, represented by its coordinate ring. Refer to [73].

Example 4.1.12. Let $\mathfrak{H}$ be a finite dimensional Hopf algebra and let $\mathfrak{H}^* = \mathrm{Hom}(\mathfrak{H}, \mathbb{C})$ denote the linear dual of $\mathfrak{H}$. By dualizing the algebra and co-operations of $\mathfrak{H}$, the following maps are obtained (with Δ^* corrected):

$$\begin{aligned}
&m^* = \Delta' : \mathfrak{H}^* \to \mathfrak{H}^* \otimes \mathfrak{H}^*, \quad \varphi \mapsto \varphi \circ m, \\
&\eta^* = \varepsilon' : \mathfrak{H}^* \to \mathbb{C}, \quad \varphi \mapsto \varphi \circ \eta(1) = \varphi(1), \\
&\Delta^* = m' : \mathfrak{H}^* \otimes \mathfrak{H}^* \to \mathfrak{H}^*, \quad \varphi_1 \otimes \varphi_2 \mapsto m \circ (\varphi_1 \otimes \varphi_2) \circ \Delta = (\varphi_1 \otimes \varphi_2) \circ \Delta, \\
&\varepsilon^* = \eta' : \mathbb{C} \to \mathfrak{H}^*, \quad 1 \mapsto \varepsilon, \\
&S^* = S' : \mathfrak{H}^* \to \mathfrak{H}^*, \quad \varphi \mapsto \varphi \circ S.
\end{aligned}$$

With these dashed operations as undashed, $\mathfrak{H}^*$ becomes a Hopf algebra, called the **dual** of $\mathfrak{H}$. Namely,

$$\begin{aligned}(\Delta' \otimes \mathrm{id})\Delta' = (\mathrm{id} \otimes \Delta')\Delta' : \mathfrak{H}^* \to \otimes^3 \mathfrak{H}^*,\\ m'(S' \otimes \mathrm{id})\Delta' = m'(\mathrm{id} \otimes S')\Delta' = \eta' \circ \varepsilon' : \mathfrak{H}^* \to \mathfrak{H}^*,\\ (\varepsilon' \otimes \mathrm{id})\Delta' = (\mathrm{id} \otimes \varepsilon')\Delta' = \mathrm{id} : \mathfrak{H}^* \to \mathfrak{H}^*.\end{aligned}$$

Indeed, check that for $\varphi \in \mathfrak{H}^*$ and $x, y, z \in \mathfrak{H}$,

$$\begin{aligned}(\Delta' \otimes \mathrm{id})\Delta'\varphi(x, y, z) = \varphi \circ m \circ (m \otimes \mathrm{id})(x, y, z) = \varphi((xy)z),\\ (\mathrm{id} \otimes \Delta')\Delta'\varphi(x, y, z) = \varphi \circ m \circ (\mathrm{id} \otimes m)(x, y, z) = \varphi(x(yz)),\end{aligned}$$

both certainly equal, and

$$\begin{aligned}m'(S' \otimes \mathrm{id})\Delta'\varphi(x) = \varphi \circ m \circ (S \otimes \mathrm{id}) \circ \Delta(x),\\ m'(\mathrm{id} \otimes S')\Delta'\varphi(x) = \varphi \circ m \circ (\mathrm{id} \otimes S) \circ \Delta(x),\end{aligned}$$

which seems to be different in general, and

$$(\eta' \circ \varepsilon')\varphi = \eta'(\varphi(1)) = \varphi(1)\varepsilon \in \mathfrak{H}^*,$$

so that it is necessary to have that for any $x \in \mathfrak{H}$,

$$m \circ (S \otimes \mathrm{id}) \circ \Delta(x) = 1 = m \circ (\mathrm{id} \otimes S) \circ \Delta(x) \in \mathfrak{H},$$

and moreover,

$$\begin{aligned}(\varepsilon' \otimes \mathrm{id})\Delta'\varphi(x) = \varphi \circ m(1, x) = \varphi(x),\\ (\mathrm{id} \otimes \varepsilon')\Delta'\varphi(x) = \varphi \circ m(x, 1) = \varphi(x) = \mathrm{id}(\varphi)(x),\end{aligned}$$

with $x = 1x = x1$ identified.

Note that $\mathfrak{H}$ is commutative if and only if $\mathfrak{H}^*$ is cocommutative, and that $\mathfrak{H}$ is cocommutative if and only if $\mathfrak{H}^*$ is commutative.

Indeed, if $\mathfrak{H}$ is commutative, then for any $x, y \in \mathfrak{H}$ and $\varphi \in \mathfrak{H}^*$,

$$\tau'\Delta'\varphi(x, y) = \varphi \circ m \circ \tau(x, y) = \varphi(yx) = \varphi(xy) = \Delta'\varphi(x, y)$$

with $\tau' = \tau^* : \otimes^2 \mathfrak{H}^* \to \otimes^2 \mathfrak{H}^*$ defined as $\tau^*(\varphi_1 \otimes \varphi_2) = (\varphi_1 \otimes \varphi_2) \circ \tau$, and hence $\tau'\Delta' = \Delta' : \mathfrak{H}^* \to \mathfrak{H}^* \otimes \mathfrak{H}^*$. Conversely, if $\mathfrak{H}^*$ is cocommutative, then for any $x, y \in \mathfrak{H}$, the equation $\varphi(yx) = \varphi(xy)$ holds for any $\varphi \in \mathfrak{H}^*$. It then implies that $yx = xy \in \mathfrak{H}$.

Also, if $\mathfrak{H}$ is cocommutative, then

$$\begin{aligned}m'(\varphi_1 \otimes \varphi_2)(x) &= m \circ (\varphi_1 \otimes \varphi_2) \circ \Delta(x) = m \circ (\varphi_1 \otimes \varphi_2) \circ \tau\Delta(x)\\ &= m \circ (\varphi_2 \otimes \varphi_1) \circ \Delta(x) = m'(\varphi_2 \otimes \varphi_1)(x).\end{aligned}$$

Note that the multiplication of $\varphi_1, \varphi_2 \in \mathfrak{H}^*$ is defined to be $m'(\varphi_1 \otimes \varphi_2) \in \mathfrak{H}^*$ (cf. [52]). $\triangleleft$

Example 4.1.13. The second dual $\mathfrak{H}^{**} = (\mathfrak{H}^*)^*$ of a finite dimensional Hopf algebra $\mathfrak{H}^*$ is identified with $\mathfrak{H}$ as a Hopf algebra, where any element of $\mathfrak{H}^{**}$ is identified with the evaluation map at some element of $\mathfrak{H}$ on $\mathfrak{H}^*$. By dualizing the algebra $\mathfrak{H}^*$ and co-operations of $\mathfrak{H}^*$, the following maps are obtained

$$
\begin{aligned}
&(m')^* = \Delta'' : \mathfrak{H}^{**} \to \mathfrak{H}^{**} \otimes \mathfrak{H}^{**}, \quad \psi \mapsto \psi \circ m', \\
&(\eta')^* = \varepsilon'' : \mathfrak{H}^{**} \to \mathbb{C}, \quad \psi \mapsto \psi \circ \eta'(1) = \psi(\varepsilon), \\
&(\Delta')^* = m'' : \mathfrak{H}^{**} \otimes \mathfrak{H}^{**} \to \mathfrak{H}^{**}, \quad \psi_1 \otimes \psi_2 \mapsto m' \circ (\psi_1 \otimes \psi_2) \circ \Delta' = (\psi_1 \otimes \psi_2) \circ \Delta', \\
&(\varepsilon')^* = \eta'' : \mathbb{C} \to \mathfrak{H}^{**}, \quad 1 \mapsto \varepsilon', \\
&(S')^* = S'' : \mathfrak{H}^{**} \to \mathfrak{H}^{**}, \quad \psi \mapsto \psi \circ S'.
\end{aligned}
$$

Let $p, p_1, p_2 \in \mathfrak{H}$ be corresponding to $\psi, \psi_1, \psi_2 \in \mathfrak{H}^{**}$ respectively. Then for $\varphi, \varphi_1, \varphi_2 \in \mathfrak{H}^*$,

$$\Delta'' p(\varphi_1 \otimes \varphi_2) = p \circ m'(\varphi_1 \otimes \varphi_2) = m(\varphi_1 \otimes \varphi_2)\Delta(p),$$

with $\Delta'' p$ identified with $\Delta(p)$, and

$$\varepsilon''(p) = p(\varepsilon) = \varepsilon(p) = 1,$$

and

$$m''(p_1 \otimes p_2)(\varphi) = (p_1 \otimes p_2) \circ \Delta' \varphi = (p_1 \otimes p_2) \circ (\varphi \circ m) = \varphi(p_1 p_2)$$

with $m''(p_1 \otimes p_2)$ identified with $p_1 p_2 \in \mathfrak{H}$, and

$$\eta''(1)\varphi = \varepsilon'(\varphi) = \varphi(1)$$

with $\eta''(1)$ identified with $1 \in \mathfrak{H}$, and

$$S''(p)(\varphi) = (p \circ S')(\varphi) = p(\varphi \circ S) = \varphi(S(p))$$

with $S''(p)$ identified with $S(p) \in \mathfrak{H}$. $\triangleleft$

Example 4.1.14. For a finite group G, we have $(\mathbb{C}G)^* \cong C(G)$ with $\mathfrak{H} = \mathbb{C}G$. Indeed, for any $g \in G$, the characteristice function δ_g at g in $C(G)$ is identified with the element δ_g^* of $\mathfrak{H}^*$ defined as $\delta_g^*(\sum_j \alpha_j g_j) = \alpha_0 \in \mathbb{C}$ with $g_0 = g \in G$, since

$$\delta_g(\sum_j \alpha_j g_j + \sum_k \beta_k g_k) = \alpha_0 + \beta_0 = \delta_g(\sum \alpha_j g_j) + \delta_g(\sum_k \beta_k g_k). \quad \triangleleft$$

Note that the linear dual of an infinite dimensional, Hopf algebra $\mathfrak{H}$ may is not a Hopf algebra. The main problem is that we obtain the dualized product as a coproduct $m^* = \Delta' : \mathfrak{H}^* \to (\mathfrak{H} \otimes \mathfrak{H})^*$ defined as $m^*(\varphi) = \varphi \circ m$, but $\mathfrak{H}^* \otimes \mathfrak{H}^*$ is only a proper subspace of $(\mathfrak{H} \otimes \mathfrak{H})^*$.

Note that

Proposition 4.1.15. *The dual $\mathfrak{H}^*$ of a coalgebra $\mathfrak{H}$ as a linear space with Δ linear and ε is always an algebra by $m' = \Delta^*$.*

Proof. Check that for $\varphi_1, \varphi_2, \varphi_3 \in \mathfrak{H}^*$

$$\begin{aligned}
&m'(m'(\varphi_1 \otimes \varphi_2) \otimes \varphi_3) = m \circ (m'(\varphi_1 \otimes \varphi_2) \otimes \varphi_3) \circ \Delta \\
&= m \circ ([m \circ (\varphi_1 \otimes \varphi_2) \circ \Delta] \otimes \varphi_3) \circ \Delta = m \circ ([m \circ (\varphi_1 \otimes \varphi_2)] \otimes \varphi_3)(\Delta \otimes \mathrm{id}) \circ \Delta \\
&= m \circ (\varphi_1 \otimes [m \circ (\varphi_2 \otimes \varphi_3)])(\mathrm{id} \otimes \Delta) \circ \Delta = m'(\varphi_1 \otimes m'(\varphi_2 \otimes \varphi_3)),
\end{aligned}$$

which shows the associativity for $m' = \Delta^*$. □

Remark. This seems to be the very reason as the role of Δ on $\mathfrak{H}$, in a sense.

To avoid the problem in the case of dimension ∞, as one way, we may consider the restricted duals $\mathfrak{H}^\circ$ of Hopf algebras $\mathfrak{H}$, which are always Hopf algebras ([26] and [66]). The main idea is to consider continuous linear functionals on $\mathfrak{H}$ with respect to the linearly compact topology on $\mathfrak{H}$, instead of all linear functionals on $\mathfrak{H}$. But the dual restricted may be too small to deal with, though.

Remark. The finite **dual** A° of an algebra A is defined to be the subspace of the dual A^* of all φ, for which the kernel of φ contains an ideal I of A such that A/I is finite dimensional. Then A° is a coalgebra as in A^*. There is a 1-1 correspondence between algebra homomorphisms from an algebra A to the dual algebra $\mathfrak{H}^*$ of a coalgebra $\mathfrak{H}$ and coalgebra homomorphisms from $\mathfrak{H}$ to A° (cf. [52]). Namely, $\mathrm{Hom}(A, \mathfrak{H}^*) \cong \mathrm{Hom}(\mathfrak{H}, A^\circ)$.

A better way to have the Hopf duality to cover the infinite dimensional case is given by the Hopf pairing. A **Hopf pairing** between two Hopf algebras $\mathfrak{H}_1$ and $\mathfrak{H}_2$ is given by a bilinear map

$$\langle \cdot, \cdot \rangle : \mathfrak{H}_1 \otimes \mathfrak{H}_2 \to \mathbb{C}, \quad h_1 \otimes h_2 \mapsto \langle h_1, h_2 \rangle$$

satisfying the following relations that for $h, h_1, h_2 \in \mathfrak{H}_1$ and $g, g_1, g_2 \in \mathfrak{H}_2$,

$$\begin{aligned}
\langle h_1 h_2, g \rangle &= \sum_k \langle h_1, g_{1k} \rangle \langle h_2, g_{2k} \rangle, \quad \text{with } \Delta(g) = \sum_k g_{1k} \otimes g_{2k}, \\
\langle h, g_1 g_2 \rangle &= \sum_j \langle h_{1j}, g_1 \rangle \langle h_{2j}, g_2 \rangle, \quad \text{with } \Delta(h) = \sum_j h_{1j} \otimes h_{2j},
\end{aligned}$$

and $\langle h, 1 \rangle = \varepsilon(h)$ and $\langle 1, g \rangle = \varepsilon(g)$.

Example 4.1.16. Let $\mathfrak{H} = U(\mathfrak{g})$ be the enveloping Hopf algebra of the Lie algebra $\mathfrak{g}$ of a Lie group G and let $\mathfrak{K} = RF(G)$ be the Hopf algebra of representable functions on G. There is a canonical non-degenerate pairing from $\mathfrak{H} \otimes \mathfrak{K}$ to $\mathbb{C}$ defined by

$$\langle X_1 \otimes \cdots \otimes X_n, f \rangle = X_1(\cdots (X_n([\Delta \otimes (\otimes^{n-2}\mathrm{id})] \cdots (\Delta \otimes \mathrm{id})(\Delta f) \cdots)) \cdots)$$

(corrected), where $X(f) = \frac{d}{dt} f(\exp(tX))|_{t=0}$ for $X \in \mathfrak{g}$ and $f \in \mathfrak{K}$ (cf. [33]).

Indeed, check that for $f, g \in \mathfrak{K}$ and f identified with f_{ij} an $n \times n$ matrix component of a finite dimensional representation of G,

$$\begin{aligned}
&\langle X_1 \otimes X_2, f\rangle = X_1(X_2(\Delta(f))) = X_1(X_2(f \circ p)) \\
&= X_1(X_2(\sum_{l=1}^{n} f_{il} \otimes f_{lj})) = \sum_{l=1}^{n} \langle X_1, f_{il}\rangle\langle X_2, f_{lj}\rangle, \\
&\langle X, fg\rangle = \langle X, f\rangle g(1) + f(1)\langle X, g\rangle \\
&= (X \otimes 1 + 1 \otimes X)(f \otimes g) = (\Delta X)(f \otimes g).
\end{aligned}$$

Also, $\langle X, 1\rangle = 0 = \varepsilon(X)$ and $\langle 1, f\rangle = 1 = f(1) = \varepsilon(f)$, possibly when $f = f_{jj}$. $\triangleleft$

We shall see that there is an analogous pairing between compact quantum groups of classical Lie groups and their associated enveloping algebras, soon later below.

As a question, is every cocommutative Hopf algebra (CHA) a universal enveloping algebra (UEA)? The answer is negative because, for example, group algebras (GA) are also cocommutative, as seen above. There are two major structure theorems which settle this equestion over an algebraically closed field of characteristic zero.

Theorem 4.1.17. (Kostant and, independently, Cartier [66], [7]). *Any cocommutative Hopf algebra $\mathfrak{H}$ over an algebraically closed field $\mathbb{F}$ of characteristic zero is isomorphic as a Hopf algebra to a crossed product algebra $\mathfrak{K} = U(P(\mathfrak{H})) \rtimes G(\mathfrak{H})$, where $P(\mathfrak{H})$ is the Lie algebra of primitive elements of $\mathfrak{H}$, and $G(\mathfrak{H})$ is the group of all group-like elements of $\mathfrak{H}$, and $G(\mathfrak{H})$ acts on $P(\mathfrak{H})$ by inner automorphisms $\mathrm{Ad(g)} = \mathrm{g} \cdot \mathrm{g}^{-1}$, and the coalgebra structure of $\mathfrak{K}$ is given simply by the tensor product of two two coalgebras UEA $U(P(\mathfrak{H}))$ and GA $\mathbb{F}G(\mathfrak{H})$. Namely, CHA =UEA$\rtimes_{Ad}$GA.*

To state the next theorem, let $\mathfrak{H}$ be a Hopf algebra over a not necessarily algebraically closed, field k of characteristic zero, and let $\mathfrak{I}$ denote the kernel of the counit map ε on $\mathfrak{H}$. Let $\Delta_r : \mathfrak{I} \to \mathfrak{I} \otimes \mathfrak{I}$ denote the **reduced** coproduct. By definition, $\Delta_r(h) = \Delta(h) - 1 \otimes h - h \otimes 1$.

Note that $\Delta_r(h) = 0$ if and only if h is a primitive element of $\mathfrak{H}$.

Let $\mathfrak{I}_n \subset \mathfrak{I} = \mathfrak{I}_0$ denote the kernel of the iterated coproduct $\Delta_r^{n+1} : I \to \otimes^{n+1}\mathfrak{I}$. Then the increasing sequence $(\mathfrak{I}_n)_{n\in\mathbb{N}}$ of subspaces of $\mathfrak{H}$ is obtained and said to be the **coradical** filtration of $\mathfrak{H}$. It is a **Hopf** algebra filtration in the sense that $\mathfrak{I}_i\mathfrak{I}_j \subset \mathfrak{I}_{i+j}$ and $\Delta_r(\mathfrak{I}_n) \subset \sum_{i,j\in\mathbb{N}, i+j=n} \mathfrak{I}_i \otimes \mathfrak{I}_j$. A Hopf algebra $\mathfrak{H}$ is said to be **connected** or **conilpotent** if its coradical filtration $(\mathfrak{I}_j)_{j\in\mathbb{N}}$ is exhaustive as $\cup_{j\in\mathbb{N}}\mathfrak{I}_j = \mathfrak{I}_0$. Equivalently, for any $h \in \mathfrak{I}$, it holds that $\Delta_r^n(h) = 0$ for some n.

Theorem 4.1.18. (Cartier-Milnor-Moore). *A cocommutative Hopf algebra over a field of characteristic zero is isomorphic as a Hopf algebra to the envelipoing algebra of a Lie algebra if and only if the CHA is connected.*

The Lie algebra $\mathfrak{g}$ in question is the Lie algebra of primitive elements h of $\mathfrak{H}$, so that $\Delta(h) = h \otimes 1 + 1 \otimes h$.

As a typical application, let $\mathfrak{H} = \otimes_{j \geq 0} \mathfrak{H}_j$ be a graded cocommutative Hopf algebra. Then $\mathfrak{H}$ is connected if and only if $\mathfrak{H}_0 = \mathbb{F}$. It then implies that if $\mathfrak{H}$ is connected, then $\mathfrak{H} = U(\mathfrak{g})$ an EA.

4.2 Quantum groups

Example 4.2.1. As a prototypical example of a compact quantum group, the Woronowicz quantum group (WQG) $SU_q(2)$ for $0 < q \leq 1$ is defined to the unital (universal) C^*-algebra, denoted as $C(SU_q(2))$, generated by two elements α and β subject to the relations $\beta\beta^* = \beta^*\beta$, $\alpha\beta = q\beta\alpha$, $\alpha\beta^* = q\beta^*\alpha$, and

$$\alpha\alpha^* + q^2\beta^*\beta = \alpha^*\alpha + \beta^*\beta = 1 \in C(SU_q(2)).$$

Note that these relations are equivalent to say that

$$\begin{aligned} U_q &= \begin{pmatrix} \alpha & q\beta \\ -\beta^* & \alpha^* \end{pmatrix} \in M_2(C(SU_q(2))) \quad \text{is unitary, so that} \\ U_q U_q^* &= \begin{pmatrix} \alpha\alpha^* + q^2\beta\beta^* & -\alpha\beta + q\beta\alpha \\ -\beta^*\alpha^* + q\alpha^*\beta^* & \beta^*\beta + \alpha^*\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ U_q^* U_q &= \begin{pmatrix} \alpha^*\alpha + \beta\beta^* & q\alpha^*\beta - \beta\alpha^* \\ q\beta^*\alpha - \alpha\beta^* & q^2\beta^*\beta + \alpha\alpha^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

which imply $\beta\beta^* = \beta^*\beta$.

This Woronowicz C^*-algebra is not a Hopf algebra in the sense given above, but as with compact topological groups, it has a dense subalgebra to become Hopf. Let $\mathcal{O}(SU_q(2))$ denote the dense $*$-subalgebra of $C(SU_q(2))$ generated by α and β. This is the analogue of the algebra $LRF(SU(2))$ of representative functions of $SU(2)$. It follows that $\mathcal{O}(SU_q(2))$ is a Hopf algebra with coproduct, counit, and antipode defined by

$$\Delta U_q = U_q \otimes^{\sim} U_q, \quad \text{and} \quad \varepsilon(U_q) = 1, \quad S(U_q) = U_q^*$$

so that $S(\alpha) = \alpha^*$, $S(q\beta) = -\beta$, $S(\beta^*) = -q\beta^*$, and $S(\alpha^*) = \alpha$, where

$$\begin{aligned} U_q \otimes^{\sim} U_q &= \begin{pmatrix} \alpha \otimes \alpha + q\beta \otimes (-\beta^*) & \alpha \otimes q\beta + q\beta \otimes (\alpha^*) \\ (-\beta^*) \otimes \alpha + \alpha^* \otimes (-\beta^*) & (-\beta^*) \otimes q\beta + \alpha^* \otimes \alpha^* \end{pmatrix} \\ &= \begin{pmatrix} \Delta(\alpha) & \Delta(q\beta) \\ \Delta(-\beta^*) & \Delta(\alpha^*) \end{pmatrix} = \Delta(U_q) \end{aligned}$$

(corrected), so that $\Delta(\alpha^*) = \Delta(\alpha)^*$ and $\Delta(\beta^*) = \Delta(\beta)^*$. Note that the coproduct Δ is only defined in the dense $\mathcal{O}(SU_q(2))$ of matrix elements of the quantum group, and its extension to $C(SU_q(2))$ is the completed tensor product

$$\Delta : C(SU_q(2)) \to C(SU_q(2)) \otimes C(SU_q(2)),$$

with unvisible part, because of completion.

In particular, at $q = 1$, it holds that $C(SU_1(2)) = C(SU(2))$ the commutative C^*-algebra of continuous functions on $SU(2)$.

May refer to [45] for a survey of compact and locally compact quantum groups. $\triangleleft$

Example 4.2.2. The quantum enveloping algebra $U_q(su(2))$ is defined to be an algebra over $\mathbb{C}$ generated by elements e, f, and k, subject to the relations $kk^{-1} = k^{-1}k = 1$ and $kek^{-1} = qe$, $kfk^{-1} = \frac{1}{q}f$, and $[f,e] = \frac{1}{q-q^{-1}}(k^2 - k^{-2})$ (cf. [43]). Define the coproduct, the antipode, and the counit of $U_q(su(2))$ by

$$\begin{aligned}
&\Delta(k) = k \otimes k, && S(k) = k^{-1}, && \varepsilon(k) = 1,\\
&\Delta(e) = e \otimes k + k^{-1} \otimes e, && S(e) = -qe, && \varepsilon(e) = 0,\\
&\Delta(f) = f \otimes k + k^{-1} \otimes f, && S(f) = -q^{-1}f, && \varepsilon(f) = 0.
\end{aligned}$$

Recall from [5] that the Lie algebra $su(2)$ of $SU(2)$ consists of skew-Hermitian 2×2 matrices over $\mathbb{C}$ with trace zero. A basis of $su(2)$ over $\mathbb{R}$ is given by

$$iH = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with $(iH)^* = -iH$, $Y^* = -Y$, and $Z^* = -Z$, and with trace zero, and with det 1, satisfying the relations $Y(iH) = -(iH)Y = Z$, $(iH)Z = -Z(iH) = Y$, $ZY = -YZ = iH$, so that $[iH, Y] = -2Z$, $[iH, Z] = 2Y$, and $[Y, Z] = -2iH$. Then $(iH)Y(iH)^{-1} = -Y$, $(iH)Z(iH)^{-1} = -Z$, and $(iH)^2 - (iH)^{-2} = 0$.

There is a Hopf pairing $\langle \cdot, \cdot \rangle : U_q(su(2)) \otimes \mathcal{O}(SU_q(2)) \to \mathbb{C}$ given by $\langle k, \alpha \rangle = \frac{1}{\sqrt{q}} = \langle k^{-1}, \alpha^* \rangle$, $\langle k^{-1}, \alpha \rangle = \sqrt{q} = \langle k, \alpha^* \rangle$, $\langle e, \beta^* \rangle = -1$, $\langle f, \beta \rangle = \frac{1}{q}$, and with the pairing for all other couples of generators zero. $\triangleleft$

4.3 Symmetry in Noncommutative Geometry

Let $\mathfrak{H}$ be a Hopf algebra with Δ, ε, and S. A unital algebra A is said to be a left $\mathfrak{H}$-**module** algebra if A is a left $\mathfrak{H}$-module by a map $\rho : \mathfrak{H} \otimes A \to A$, and if the multiplication and the unit map of A are morphisms of $\mathfrak{H}$-modules. Namely,

$$h(ab) = \sum_j h_{1j}(a)h_{2j}(b), \quad \Delta h = \sum_j h_{1j} \otimes h_{2j}, \quad h \in \mathfrak{H}, a, b \in A,$$

and $h1 = \varepsilon(h)1 \in A$, and $1(a) = a$ (added). Namely, it looks like that

$$\begin{array}{ccc}
\mathfrak{H} \otimes A \otimes A & \xrightarrow{\mathrm{id}_{\mathfrak{H}} \otimes m} & \mathfrak{H} \otimes A \\
\Delta \big\downarrow & & \big\downarrow \rho \\
\mathfrak{H} \otimes A \otimes \mathfrak{H} \otimes A & \xrightarrow{\rho \otimes \rho} & A
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathfrak{H} \otimes A & \xrightarrow{\rho} & A \\
\varepsilon \otimes \mathrm{id} \big\downarrow & & \big\downarrow \mathrm{id}_A \\
\mathbb{C} \otimes A & = & A
\end{array}$$

(which are correct?)

Group-like elements $h \in \mathfrak{H}$ as $\Delta h = h \otimes h$ act as unit preserving algebra automorphisms of an $\mathfrak{H}$-module algebra A.

Indeed, $h(ab) = h(a)h(b)$. Thus, $h(\cdot)$ is an algebra homomorphism of A. In particular, $h(a) = h(a1) = h(a)h(1)$. Note that h is invertible in $\mathfrak{H}$ with inverse $S(h)$.

Primitive elements $h \in \mathfrak{H}$ as $\Delta h = 1 \otimes h + h \otimes 1$ act as derivations of A. Indeed, $h(ab) = ah(b) + h(a)b$.

Example 4.3.1. For $\mathfrak{H} = \mathbb{C}G$ the group Hopf algebra of a discrete group G, with Δ, ε, and S, an $\mathfrak{H}$-module algebra structure on a unital algebra A is given by an action of G by unit preserving algebra automorphisms of A.

Indeed, for any $g \in G$ and $a \in A$, we have $g(ab) = g(a)g(b)$ since $\Delta g = g \otimes g$. Thus, $g(\cdot)$ is an algebra homomorphism of A. In particular, $g(a) = g(a1) = g(a)g(1)$. Moreover, $g^{-1}(g(a)) = (g^{-1}g)(a) = 1_G(a) = a$.

Similarly, there is a 1-1 correspondence between $U(\mathfrak{g})$-module algebra structures on A and Lie actions of the Lie algebra $\mathfrak{g}$ on A by derivations.

Indeed, for any $X \in \mathfrak{g}$, with $\Delta X = X \otimes 1 + 1 \otimes X \in U(\mathfrak{g})$, we have $X(ab) = X(a)b + aX(b)$. $\lhd$

Example 4.3.2. Recall that the Podleś quantum sphere S_q^2 is the $*$- or C^*-algebra generated by elements a, a^* and $b = b^*$ subject to the relations $aa^* + q^{-4}b^2 = 1$, $a^*a + b^2 = 1$, $ab = q^{-2}ba$, and $a^*b = q^2ba^*$.

Define a $U_q(su(2))$-module algebra structure on S_q^2 as

$$\begin{aligned}
&ka = qa, && ka^* = q^{-1}a^*, && kb = b,\\
&ea = 0, && ea^* = -q^{\frac{3}{2}}(1+q^{-2})b, && eb = q^{\frac{5}{2}}a,\\
&fa = q^{-\frac{7}{2}}(1+q^2)b, && fa^* = 0, && fb = -q^{-\frac{1}{2}}a^*
\end{aligned}$$

for the generators $k, e, f \in U_q(su(2))$ with $\Delta k = k \otimes k$, $\Delta e = e \otimes k + k^{-1} \otimes e$, and $\Delta f = f \otimes k + k^{-1} \otimes f$.

Recall also that the quantum analogue of the Dirac or Hopf monopole line bundle over S^2 is given by the idempotent $e_q \in M_2(S_q^2)$ defined as

$$e_q = \frac{1}{2}\begin{pmatrix} 1+q^{-2}b & qa \\ q^{-1}a^* & 1-b \end{pmatrix}.$$

This noncommutative line bundle is equivariant with respect to the $U_q(su(2))$-module action, as follows. Consider the 2-dimensional standard representation of $U_q(su(2))$ on $\mathbb{C}^2$ by sending the generators k, e, f respectively to

$$\begin{pmatrix} \sqrt{q}^{-1} & 0 \\ 0 & \sqrt{q} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

as identified. We then obtain an action of $U_q(su(2))$ on $M_2(S_q^2) \cong M_2(\mathbb{C}) \otimes S_q^2$ as the tensor product of modules by the formula: for $m \in M_2(\mathbb{C})$ and $a \in S_q^2$,

$$h(m \otimes a) = \sum_j h_{1j}(m)h_{2j}(a), \quad h, \Delta h = \sum_j h_{1j} \otimes h_{2j} \in U_q(su(2)).$$

It holds that $h(e_q) = \varepsilon(h)e_q$ for any $h \in U_q(su(2))$.

For instance, with $\Delta k = k \otimes k$,

$$\begin{aligned}
k(e_q) &= \begin{pmatrix} \sqrt{q}^{-1} & 0 \\ 0 & \sqrt{q} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes k(1 + q^{-2}b) + k \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \otimes k(qa) \\
&\quad + k \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} \otimes k(q^{-1}a^*) + k \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \otimes k(1-b) \\
&= \frac{1}{2} \begin{pmatrix} \sqrt{q}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \otimes (k + q^{-2}b) + \begin{pmatrix} 0 & \frac{\sqrt{q}^{-1}}{2} \\ 0 & 0 \end{pmatrix} \otimes q^2 a \\
&\quad + \begin{pmatrix} 0 & 0 \\ \frac{\sqrt{q}}{2} & 0 \end{pmatrix} \otimes q^{-2}a^* + \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sqrt{q}}{2} \end{pmatrix} \otimes (k-b),
\end{aligned}$$

which should be equal to $\varepsilon(k)e_q = e_q$? It seems that to have the above equation claimed, we may take $\Delta k = k^{-1} \otimes k$ instead, in a way. $\lhd$

Let $\mathfrak{H}$ be a Hopf algebra. A left either **corepresentation**, **comodule**, or **coaction** of $\mathfrak{H}$ is a vector space M with a map $\rho : M \to \mathfrak{H} \otimes M$ such that $(\Delta \otimes \mathrm{id}_M)\rho = (\mathrm{id}_{\mathfrak{H}} \otimes \rho)\rho$ and $(\varepsilon \otimes \mathrm{id}_M)\rho = \mathrm{id}_M$. Namely, the diagrams commute:

$$\begin{array}{ccc} M & \xrightarrow{\rho} & \mathfrak{H} \otimes M \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\Delta\otimes\mathrm{id}_M} \\ \mathfrak{H} \otimes M & \xrightarrow{\mathrm{id}_{\mathfrak{H}}\otimes\rho} & \mathfrak{H} \otimes \mathfrak{H} \otimes M \end{array} \quad \text{and} \quad \begin{array}{ccc} M & \xrightarrow{\rho} & \mathfrak{H} \otimes M \\ \downarrow{\scriptstyle\mathrm{id}_M} & & \downarrow{\scriptstyle\varepsilon\otimes\mathrm{id}_M} \\ M & = & \mathbb{C} \otimes M. \end{array}$$

These conditions are dual to the axioms for a module over an algebra. An algebra A is said to be a left $\mathfrak{H}$-comodule algebra if A is a left $\mathfrak{H}$-comodule by such a map $\rho : A \to \mathfrak{H} \otimes A$ as $M = A$, to be a morphism of algebras.

Example 4.3.3. A Hopf algebra $\mathfrak{H}$ becomes a left $\mathfrak{H}$-comodule algebra by the coproduct $\Delta : \mathfrak{H} \to \mathfrak{H} \otimes \mathfrak{H}$ as ρ. This is the analogue of the left action of a group G on G as left translations.

For compact quantum groups such as $SU_q(2)$ and their algebraic analogues like $SL_q(2)$, coactions are defined more naturally. Formally, they are obtained by dualizing and and quantizing group actions as maps $G \times X \to X$ for classical groups G and spaces X. $\lhd$

Example 4.3.4. For q a nonzero element of $\mathbb{C}$, the algebra $A = \mathbb{C}_q[x, y : \emptyset]$ of coordinates on the quantum q-plane is defined to be the quotient algebra $\mathbb{C}[x, y : \emptyset]/(yx - qxy)$, where $\mathbb{C}[x, y : \emptyset]$ is the free algebra with two generators x and y, and $(yx - qxy)$ is the two-sided ideal generated by $yx - qxy$. If $q \neq 1$, then $A = \mathbb{C}_q[x, y : \emptyset]$ is noncommutative.

There is the unique $SL_q(2)$-comodule algebra structure $\rho : A \to SL_q(2) \otimes A$ on the quantum q-plane A defined as

$$\rho \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix},$$

so that $\rho(x) = a \otimes x + b \otimes y$ and $\rho(y) = c \otimes x + d \otimes y$, and the defininig relation $\rho(y)\rho(x) = q\rho(x)\rho(y)$ holds.

Check that

$$\begin{aligned}
&q\rho(y)\rho(x) = q(c \otimes x + d \otimes y)(a \otimes x + b \otimes y) \\
&= qca \otimes x^2 + qcb \otimes xy + qda \otimes yx + qdb \otimes y^2, \\
&\rho(x)\rho(y) = (a \otimes x + b \otimes y)(c \otimes x + d \otimes y) \\
&= ac \otimes x^2 + ad \otimes xy + bc \otimes yx + bd \otimes y^2,
\end{aligned}$$

with, by definition, $(ab = qba)$, $ac = qca$, $bd = qdb$, $(cd = qdc)$, $bc = cb$ and $yx = qxy$, $ad = 1 + qbc$ and $da = 1 + q^{-1}bc$ so that

$$qda \otimes yx = (q1 + bc) \otimes qxy = (q^2 1 + (ad - 1)) \otimes xy$$

(cf. [76]). It then follows that $q^2 = 1$, to have $\rho(y)\rho(x) = q^{-1}\rho(x)\rho(y) = q\rho(x)\rho(y)$ consequently.

Similarly, with $0 < q \leq 1$, if we suppose that the same holds in the case of $SU_q(2)$, defined as

$$\rho \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & q\beta \\ -\beta^* & \alpha^* \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix},$$

so that $\rho(x) = \alpha \otimes x + q\beta \otimes y$ and $\rho(y) = -\beta^* \otimes x + \alpha^* \otimes y$, and then

$$\begin{aligned}
&q\rho(y)\rho(x) = q(-\beta^* \otimes x + \alpha^* \otimes y)(\alpha \otimes x + q\beta \otimes y) \\
&= -q\beta^*\alpha \otimes x^2 - q^2\beta^*\beta \otimes xy + q\alpha^*\alpha \otimes yx + q^2\alpha^*\beta \otimes y^2, \\
&\rho(x)\rho(y) = (\alpha \otimes x + q\beta \otimes y)(-\beta^* \otimes x + \alpha^* \otimes y) \\
&= -\alpha\beta^* \otimes x^2 + \alpha\alpha^* \otimes xy - q\beta\beta^* \otimes yx + q\beta\alpha^* \otimes y^2,
\end{aligned}$$

with, by definition, $\alpha\beta^* = q\beta^*\alpha$, $qxy = yx$ and $\beta\beta^* = \beta^*\beta$, $q\alpha^*\beta = \beta\alpha^*$,

$$q\alpha^*\alpha \otimes yx = q(1 - \beta^*\beta) \otimes qxy = (q^2 1 - (1 - \alpha\alpha^*)) \otimes xy.$$

It then follows $q = 1$, to have $\rho(y)\rho(x) = q^{-1}\rho(x)\rho(y) = q\rho(x)\rho(y)$ consequently. $\lhd$

Example 4.3.5. As a non-significant example of a noncommutative (NC) and non-cocommutative (NcC) Hopf algebra, we may start with a noncommutative Hopf algebra U such as the universal enveloping algebras of Lie algebra, and with a non-cocommutative Hopf algebra F such as the algebra of representative functions on a compact group, and make the tensor product Hopf algebra $\mathfrak{H} = F \otimes U$, which is neither commutative nor cocommutative. But a variation of this method provides interesting examples explained below. Another source of interesting examples of NC and NcC Hopf algebras is given by the theory of quantum groups. $\lhd$

The idea is to deform the algebra and coalgebra structures in such a tensor product $F \otimes U$ via an action of U on F and a coaction of F on U, through the crossed product construction. Describe the crossed product construction as below, which is of independent interest as well.

Let $\mathfrak{H}$ be a Hopf algebra and A be a left $\mathfrak{H}$-module algebra. The underlying vector space of the **crossed product** algebra $A \rtimes \mathfrak{H}$ is $A \otimes \mathfrak{H}$, and its product is defined by

$$(a \otimes g)(b \otimes h) = \sum_j a(g_{1j}b) \otimes g_{2j}h, \quad a, b \in A, g, h \in \mathfrak{H},$$

with $\Delta g = \sum_j g_{1j} \otimes g_{2j} \in \mathfrak{H} \otimes \mathfrak{H}$.

May check that $1 \otimes 1 \in A \otimes \mathfrak{H}$ is the unit of $A \rtimes \mathfrak{H}$. We have, with $\Delta 1 = 1 \otimes 1$,

$$\begin{aligned} &(1 \otimes 1)(b \otimes h) = 1(1b) \otimes 1h = b \otimes h, \quad \text{and} \\ &(a \otimes g)(1 \otimes 1) = \sum_j a(g_{1j}1) \otimes g_{2j}1 = (a \otimes 1) \sum_j g_{1j}1 \otimes g_{2j}1, \end{aligned}$$

which may be identified with $a \otimes g$ (in general?), as just in the case of $\Delta g = g \otimes g$ with $g1 = 1 \in A$ and $g1 = g \in \mathfrak{H}$.

Also, $A \rtimes \mathfrak{H}$ is an associative unital algebra. In fact,

$$\begin{aligned} &((a \otimes g)(b \otimes h))(c \otimes l) = \sum_j (a(g_{1j}b) \otimes g_{2j}h)(c \otimes l) \\ &= \sum_j \sum_k a(g_{1j}b)(g_{2j}h)_{1k}(c) \otimes (g_{2j}h)_{2k}l \quad \Delta(g_{2j}h) = \sum_k (g_{2j}h)_{1k} \otimes (g_{2j}h)_{2k}, \\ &(a \otimes g)((b \otimes h)(c \otimes l)) = (a \otimes g) \sum_k b(h_{1k}c) \otimes h_{2k}l \quad \Delta h = \sum_k h_{1k} \otimes h_{2k} \\ &= \sum_j \sum_k a(g_{1j}b(h_{1k}c)) \otimes g_{2j}h_{2k}l, \end{aligned}$$

both of which should be equal. As a possible case, $(g_{2j}h)_{1k}$ may be identified with h_{1k}, and $(g_{2j}h)_{2k}$ with $g_{2j}h_{2k}$.

The above construction deforms multiplication of algebras.

Example 4.3.6. Let $\mathfrak{H} = \mathbb{C}G$ be the group Hopf algebra of a discrete group and let $\mathfrak{H}$ act on an algebra A by automoprhisms of A. Then the algebra $A \rtimes \mathfrak{H}$ is isomorphic to the crossed product algebra $A \rtimes G$.

Indeed, with $\Delta g = g \otimes g$ for $g \in G$,

$$(a \otimes g)(b \otimes h) = ag(b) \otimes gh.$$

In particular, with $g(\cdot) \in \mathrm{Aut}(A)$,

$$(1 \otimes g)(b \otimes 1)(1 \otimes g^{-1}) = (g(b) \otimes g)(1 \otimes g^{-1}) = g(b)g(1) \otimes gg^{-1} = g(b) \otimes 1,$$

which corresponds to $gbg^{-1} = \mathrm{Ad}_g(b) = g(b)$ as a covariance condition. Note also that $A \rtimes G$ contains $\mathfrak{H} = \mathbb{C}G$ as a subalgebra and is generated by A and $\mathfrak{H}$. $\triangleleft$

Example 4.3.7. Let a Lie algebra $\mathfrak{g}$ act by derivations on a commutative algebra A. Then the crossed product algebra $A \rtimes U(\mathfrak{g})$ is viewed as a subalgebra of the algebra of differential operators on A generated by elements of $\mathfrak{g}$ as derivations on A and those of A as multiplication operators on A as coefficients. For instance, if $A = \mathbb{C}[x]$ as the algebra of polynomials with real variable x, and if $\mathfrak{g} = \mathbb{C}$ acts by the differential operator $\frac{d}{dx}$ on A. Then $A \rtimes U(\mathfrak{g})$ is the **Weyl** algebra of differential operators on the line $\mathbb{R}$ with polynomial coefficients. $\lhd$

Let D be a right $\mathfrak{H}$-comodule coalgebra with coaction $D \to D \otimes \mathfrak{H}$ by sending $d \in D$ to $\sum_k d'_{0k} \otimes d'_{1k} \in D \otimes \mathfrak{H}$. The underlying linear space of the crossed product **coalgebra** $\mathfrak{H} \ltimes D$ is $\mathfrak{H} \otimes D$, and its coproduct $\Delta : \mathfrak{H} \ltimes D \to \otimes^2(\mathfrak{H} \ltimes D)$ is defined by, with $\Delta h = \sum_j h_{1j} \otimes h_{2j} \in \otimes^2 \mathfrak{H}$ and $\Delta d = \sum_k d_{1k} \otimes d_{2k} \in \otimes^2 D$ and $D \ni d_{1k} \mapsto \sum_l (d_{1k})'_{0l} \otimes (d_{1k})'_{1l} \in D \otimes \mathfrak{H}$,

$$\Delta(h \otimes d) = \sum_j \sum_k \sum_l h_{1j} \otimes (d_{1k})'_{0l} \otimes h_{2j}(d_{1k})'_{1l} \otimes d_{2k} \in \mathfrak{H} \otimes D \otimes \mathfrak{H} \otimes D$$

(modified), and the counit is defined by $\varepsilon(h \otimes d) = \varepsilon(d)\varepsilon(h)$.

The above construction deforms comultiplication of coalgebras.

Example 4.3.8. If $\Delta h = h \otimes h$ for $h \in \mathfrak{H}$ and $\Delta d = d \otimes d$ for $d \in D$, then

$$\Delta(h \otimes d) = \sum_l h \otimes d'_{0l} \otimes h d'_{1l} \otimes d, \quad d \mapsto \sum_l d'_{0l} \otimes d'_{1l} \in D \otimes \mathfrak{H}.$$

In addition, if d is mapped to $d \otimes 1 \in D \otimes \mathfrak{H}$, then $\Delta(h \otimes d) = (h \otimes d) \otimes (h \otimes d)$. $\lhd$

The idea of obtaining a simultaneous deformation of multiplication and comultiplication of a Hopf algebra by applying both the above constructions simultaneously, going back to G. I. Kac in the 1960s in the context of Kac-von Neumann Hopf algebras, is now generalized to the notion of bicrossed product of matched pairs of Hopf algebras, due to Shahn Majid [49] for more extensive discussions and references. There are several variations of this construction, one of which is the most relevant following for the structure of the Connes-Moscovici Hopf algebra, and as another special case of which, the Drinfeld double of a finite dimensional Hopf algebra ([49], [39]).

Let U and F be two Hopf algebras. Assume that F is a left U-module algebra and U is a right F-comodule coalgebra via $\rho : U \to U \otimes F$. We say that (U, F) is a **matched** pair if the action and coaction satisfy the compatibility conditions: for $u, v \in U$ and $f \in F$, with $\Delta f = \sum_j f_{1j} \otimes f_{2j}$, $\Delta u = \sum_k u_{1k} \otimes u_{2k}$,

$$\Delta(u(f)) = \sum_j \sum_k \sum_l (u_{1k})'_{0l} f_{1j} \otimes (u_{1k})'_{1l}(u_{2k}(f_{2j})),$$

$$\rho(uv) = \sum_k \sum_l \sum_s (u_{1k})'_{0l} v'_{0s} \otimes (u_{1k})'_{01}(u_{2k}(v'_{1l})), \quad \rho(1) = 1 \otimes 1,$$

$$\sum_k \sum_l (u_{2k})'_{0l} \otimes (u_{1k}(f))(u_{2k})_{1l} = \sum_k \sum_l (u_{1k})'_{0l} \otimes (u_{1k})'_{1l}(u_{2k}(f))$$

(transformed in our sense), and $\varepsilon(u(f)) = \varepsilon(u)\varepsilon(f)$. Given a matched pair (U, F), define its **bicrossed** product Hopf algebra $F \rtimes^2 U$ to be $F \otimes U$ with both the crossed product algebra structure and the crossed coproduct coalgebra structure, and with its antipode S defined as

$$S(f \otimes u) = \sum_l (1 \otimes S(u'_{0l}))(S(fu'_{1l}) \otimes 1).$$

As a remarkable fact, the bicrossed product $F \rtimes^2 U$ becomes a Hopf algebra, thanks to the above compatibility conditions. May check it, but not now.

Example 4.3.9. The first and simplest example of the bicrossed product construction is given as follows. Let G be a finite group, with a **factorization** $G = G_1 G_2$ in the sense that G_1, G_2 are subgroups of G such that $G_1 \cap G_2 = \{1\}$ and $G_1 G_2 = G$. For $g \in G$, denote by $g = g_1 g_2$ the factorization of g with $g_1 \in G_1$ and $g_2 \in G_2$. Define a left action of G_1 on G_2 by $g : G_2 \ni h \mapsto (gh)_2 \in G_2$ for $g \in G_1$ and $h \in G_2$. Define also a right action of G_2 on G_1 by $h : G_1 \ni g \mapsto (gh)_1 \in G_1$. Then $F = F(G_2)$ as $\mathbb{C}G_2$ is a left $U = \mathbb{C}G_1$-module algebra by the left action of G_1 on G_2, and $U = \mathbb{C}G_1$ is a right F-comodule coalgebra, with the coaction as the dual of the map $F(G_1) \otimes \mathbb{C}G_2 \to F(G_1)$ induced by the right action of G_2 on G_1. May find the details of this example in [49] and [20].

Example 4.3.10. An important example in noncommutative geometry and its applications to transverse geometry and number theory is the family of Connes-Moscovici Hopf algebras $\mathfrak{H}_n$ for $n \geq 1$ ([20], [21], [22]). The CM Hopf algebras are defined as deformations of the group $G = Dif(\mathbb{R}^n)$ of diffeomorphisms of $\mathbb{R}^n$ and also viewed as deformations of the Lie algebra $\mathfrak{a}_n$ of formal vector fields over $\mathbb{R}^n$. These algebra $\mathfrak{H}_n$ appear as quantum symmetries of transverse frame bundles of codimension n foliations, for the first time. Briefly consider the case of $n = 1$ in the following. The main feature of $\mathfrak{H}_1$ stem from the fact that the group G has a factorization of the form $G = G_1 G_2$, where G_1 is the subgroup of G of diffeomorphisms φ such that $\varphi(0) = 0$ and $\varphi'(0) = 1$, and G_2 is the $ax + b$ group of affine diffeomorphisms. Let F denote the Hopf algebra of polynomial functions on the pro-unipotent group G_1, which can be also defined as the continuous dual of the enveloping algebra of the Lie algebra of G_1. The algebra F is a commutative Hopf algebra generated by the Connes-Moscovici coordinate functions δ_n defined by

$$\delta_n(\varphi) = \frac{d^n}{dt^n} \log(\varphi'(t))|_{t=0}, \quad n = 1, 2, \cdots,.$$

Let U be the universal enveloping Hopf algebra of the Lie algebra $\mathfrak{g}_2$ of the $ax + b$ group G_2, with generators X and Y with relation $[X, Y] = X$.

The factorization $G = G_1 G_2$ defines a matched pair (U, F) of Hopf algebras. More precisely, the Hopf algebra F has a right U-module algebra structure defined as $\delta_n(X) = -\delta_{n+1}$ and $\delta_n(Y) = -n\delta_n$. On the other hand, the Hopf algebra U has a left F-comodule coalgebra structure by sending X to $1 \otimes X +$

$\delta_1 \otimes X$ and $Y \mapsto 1 \otimes Y$. May check that (U, F) is a matched pair of Hopf algebras to obtain the resulting bicrossed product Hopf algebra $F \rtimes^2 U$. This is the Connes-Moscovici Hopf algebra $\mathfrak{H}_1$ (cf. [20]).

Therefore, $\mathfrak{H}_1$ is also defined to be the universal Hopf algebra with Δ and S, generated by the generators X, Y, and δ_n $(n \geq 1)$ with relations $[X, Y] = X$, $[X, \delta_n] = \delta_{n+1}$, $[Y, \delta_n] = n\delta_n$, and $[\delta_k, \delta_l] = 0$ for integers $n, k, j \geq 1$, where

$$\begin{aligned} \Delta X &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y, \quad S(X) = -X + \delta_1 Y, \\ \Delta Y &= Y \otimes 1 + 1 \otimes Y, \qquad\qquad S(Y) = -Y, \\ \Delta \delta_1 &= \delta_1 \otimes 1 + 1 \otimes \delta_1, \qquad\qquad S(\delta_1) = -\delta_1. \quad \triangleleft \end{aligned}$$

Another interesting interaction between Hopf algebras and noncommutative geometry is given by the work of Connes and Kreimer in renormalization schemes of quantum field theory. May refer to [13], [14], [15], [16], [17], and [18].

An important feature of $\mathfrak{H}_1$ as the reason of being is that it acts as quantum symmetries of various objects of interest in noncommutative geometry, such as the noncommutative spaces of leaves of codimension 1 foliations and the noncommutative spaces of modular forms modulo the actions of Hecke correspondences.

Example 4.3.11. Let M be a 1-dimensional manifold and $A = C_0^\infty(F^+M)$ denote the algebra of smooth functions with compact support on the bundle F^+M of positively oriented frames on M. Given a discrete subgroup Γ of $Dif^+(M)$ of orientation preserving diffeomorphisms of M, there is a natural prolongation of the action of Γ to F^+M by

$$\gamma(y, y_1) = (\gamma y, \gamma'(y)y_1), \quad \gamma \in \Gamma, y \in M, (y, y_1) \in F^+M.$$

Let $A \rtimes \Gamma$ denote the corresponding crossed product algebra. Then the elements of $A \rtimes \Gamma$ consist of finite linear combinations over $\mathbb{C}$ of terms fu_γ^* for $f \in A$ and u_γ the unitary corresponding to $\gamma \in \Gamma$. The product $(fu_\gamma^*)(gu_{\gamma'}^*)$ is defined by $f(\gamma g)u_{\gamma'\gamma}^*$.

Indeed, if $u_\gamma^* g u_\gamma = \gamma g$ (if correct, but usually $u_\gamma g u_\gamma^* = \gamma g$ used), we have

$$(fu_\gamma^*)(gu_{\gamma'}^*) = f(\gamma g)u_\gamma^* u_{\gamma'}^* = f(\gamma g)u_{\gamma'\gamma}^*.$$

There is an action of $\mathfrak{H}_1$ on $A \rtimes \Gamma$ defined as

$$\begin{aligned} X(fu_\gamma^*) &= y\frac{\partial f}{\partial y}u_\gamma^*, \quad Y(fu_\gamma^*) = y_1\frac{\partial f}{\partial y_1}u_\gamma^*, \\ \delta_n(fu_\gamma^*) &= y^n \frac{d^n}{dy^n}\left(\log\frac{d\gamma}{dy}\right) fu_\gamma^* \end{aligned}$$

(partially corrected in the sense that y as a scalar, y_1 as a row vector and $\frac{\partial f}{\partial y_1}$ as a column vector), (cf. [20]).

Once given those formulas, it can be checked that by a somewhat computation that $A \rtimes \Gamma$ becomes an $\mathfrak{H}_1$-module algebra. In the original application, M is given as a transversal for a codimension 1 foliation and thus $\mathfrak{H}_1$ acts via transverse differential operators (cf. [20]). $\triangleleft$

Remark. The theory of Hopf algebras and Hopf spaces in algebraic topology is invented by H. Hopf by computing the rational cohomology of compact connected Lie groups [37]. The cohomology ring of such a Lie group is a Hopf algebra, which is isomorphic to an exterior algebra with odd generators. The Cartier-Milnor-Moore theorem characterizes connected cocommutative Hopf algebras as enveloping algebras of Lie algebras ([7], [53]). A purely algebraic theory on Hopf algebras is created as the first book by Sweedler [66]. All classical Lie groups and Lie algebras are quantized as the quantum group by Drinfled [30], with the work of Faddeev-Reshetikhin-Takhtajan and Jimbo. The theory of quantum integral systems and quantum inverse scattering methods is developed by the Leningrad and Japanese school in the early 1980s.

After the Pontryagin duality theorem for locally compact abelian groups, it is extended to the case of noncommutative groups, such as the Tannaka-Krein duality theorem as an important first step. It is sharpened by Grothendieck, Deligne, and independently Doplicher and Roberts. Note that the dual of a noncommutative group, in any sense, can not be a group, so that the category of groups is naturally extended and considered to a larger category which is closed under duality and is equivalent to the second dual, as the case of locally compact abelian groups.

The Hopf von Neumann algebras of G. I. Kac and Vainerman are considered in the noncommutatvie measure theory of von Neumann algebras [31]. The theory of compact quantum groups is initiated as an important step by S. L. Woronowicz (cf. [75]). The theory of locally compact quantum groups is developed by Kustermans and Vaes in the category of C^*-algebras [45]. May refer to [7], [39], [43], [49], [50], [51], [66], and [69] for the general theory of Hopf algebras and quantum groups.

Hopf algebras and noncommutative geometry interact in the paper of Connes and Moscovici on transverse index theory [19], and for further developments, see [20], [21], and [22]. The noncommutative and non-cocommutative Hopf algebra in that paper has the quantum symmetries of the noncommutative space of codimension 1 foliations. The same Hopf algebra acts on the noncommutative space of modular Hecke algebras [23]. For a survey of Hopf algebras in noncommutative geometry, may consult [33], [71].

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