WREATH DETERMINANTS, ZONAL SPHERICAL FUNCTIONS ON SYMMETRIC GROUPS AND THE ALON-TARSI CONJECTURE

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WREATH DETERMINANTS, ZONAL SPHERICAL FUNCTIONS ON SYMMETRIC GROUPS AND THE ALON-TARSI CONJECTURE*

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Abstract

In the article, we give several formulas for a certain zonal spherical function on the symmetric group in terms of polynomial functions on matrices called the alpha-determinant and wreath determinant. We also explain the relation between these objects and the Alon-Tarsi conjecture on the enumeration of Latin squares. In particular, we give an alternative proofs of (i) Glynn's result on a special case of the Alon-Tarsi conjecture, and (ii) the result due to Kumar and Landsberg on the equivalence between a special case of Kumar's conjecture on plethysms and the Alon-Tarsi conjecture. Most of the results given here are already announced in the articles [8, 9].

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1 Introduction

For a given pair of positive integers n and k, let $\omega_{n,k}$ be the function on the symmetric group $\mathfrak{S}_{kn}$ of degree kn defined by

$$\omega_{n,k}(g) = \frac{1}{|\mathcal{K}|} \sum_{y \in \mathcal{K}} \chi^{(k^n)}(gy), \quad g \in \mathfrak{S}_{kn},$$

where $\mathcal{K} = \mathfrak{S}_{(k^n)}$ is a Young subgroup of $\mathfrak{S}_{kn}$ corresponding to the partition $(k^n) = (k, \dots, k) \vdash kn$, and $\chi^{(k^n)}$ is the irreducible character of $\mathfrak{S}_{kn}$ corresponding to the same partition (k^n). This function is biinvariant with respect to $\mathcal{K}$, that is,

$$\omega_{n,k}(ygy') = \omega_{n,k}(g), \qquad \forall g \in \mathfrak{S}_{kn},\ \forall y, y' \in \mathcal{K}.$$

We refer to the function $\omega_{n,k}$ as a zonal spherical on $\mathfrak{S}_{kn}$ with respect to $\mathcal{K}$. Note that in the case where $n = 2$, $\omega_{2,k}$ is indeed a zonal spherical function associated

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to the Gelfand pair $(\mathfrak{S}_{2k}, \mathfrak{S}_k \times \mathfrak{S}_k)$ in the ordinary sense (see, e.g. Macdonald [12, Chapter VII]).

The purpose of the article is to give several formulas for $\omega_{n,k}$ in terms of polynomial functions on matrices called the *alpha-determinant* [13, 14] (Theorem 4.1) and *wreath determinant* [10] (Theorem 4.6). The alpha-determinant is a parametric deformation of the ordinary determinant, which interpolates the determinant and permanent. The wreath-determinant wrdet_k is a polynomial function on the space $\mathrm{Mat}_{n,kn}$ consisting of n by kn matrices, which is defined via the alpha-determinant (see (3.1)), and it has a nice characterization in terms of a suitable $\mathrm{GL}_{kn} \times \mathcal{K}$-action (see (W1)–(W3) in §3). When $k = 1$, the 1-wreath determinant wrdet_1 on $\mathrm{Mat}_n = \mathrm{Mat}_{n,n}$ agrees with the usual determinant. In this sense, our result provides a 'quasi-determinantal' formula for the zonal spherical function $\omega_{n,k}$.

As an application of our formulas, we show that the values of $\omega_{n,k}$ do not vanish when k is equal to $p-1$ for a certain *odd prime number* p. In particular, we observe that the *Alon-Tarsi conjecture on the Latin squares* is true when the size of squares is $p-1$ for an odd prime p. This gives an alternative proof of Glynn's result [5]. We also look at a conjecture on certain *plethysms* due to Kumar and see that the conjecture in a special case is equivalent to the Alon-Tarsi conjecture, which is originally obtained in [11].

Most of the results given here are already announced in the articles [8, 9].

2 Preliminaries

2.1 General conventions

The symmetric group of degree n is denoted by $\mathfrak{S}_n$. For $\sigma \in \mathfrak{S}_n$, $P(\sigma) = (\delta_{i\sigma(j)})$ is the permutation matrix of σ. The set of m by n complex matrices is denoted by $\mathrm{Mat}_{m,n}$, and we write $\mathrm{Mat}_n = \mathrm{Mat}_{n,n}$ for short. The identity matrix of size n is I_n, and $\mathbf{1}_{m,n}$ is the m by n matrix all of whose entries are one. We write $\mathbf{1}_n$ to indicate $\mathbf{1}_{n,n}$. We denote by $A \otimes B$ the Kronecker product of matrices defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix} \in \mathrm{Mat}_{mp,nq}$$

for $A = (a_{ij}) \in \mathrm{Mat}_{m,n}$ and $B \in \mathrm{Mat}_{p,q}$. The general linear group of degree n is GL_n. We always work on the vector spaces and/or algebras over the complex number field $\mathbb{C}$. The cardinality of a set S is denoted by $|S|$.

Let x_{ij} $(1 \le i, j \le n)$ be independent commuting variables, and put $X = (x_{ij})_{1 \le i,j \le n}$. For $M = (m_{ij}) \in \mathrm{Mat}_n$ such that $m_{ij} \in \mathbb{Z}_{\ge 0}$, define

$$x^M := \prod_{i,j} x_{ij}^{m_{ij}}.$$

By this notation, we have

$$\det X = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\, x^{P(\sigma)}$$

for instance. When $p = p(x_{11}, \dots, x_{nn})$ is a polynomial in x_{ij}'s, we denote by $[p]_M$ the coefficient of the monomial x^M in p.

2.2 Double cosets

We fix a pair of positive integers n and k in what follows. Let $\Omega = (\Omega_1, \dots, \Omega_n)$ be a set partition of $\{1, 2, \dots, kn\}$ given by

$$\begin{aligned}\Omega_i &:= \left\{ m \in \mathbb{Z} \,\middle|\, \left\lceil \frac{m}{k} \right\rceil = i \right\} \\ &= \{(i-1)k + r \,|\, r = 1, 2, \dots, k\} \quad (i = 1, \dots, n)\end{aligned}$$

and define

$$\mathcal{K} := \{ g \in \mathfrak{S}_{kn} \,|\, g\Omega_i = \Omega_i \ (i = 1, \dots, n) \}.$$

Notice that $\mathcal{K}$ is isomorphic to the direct product $\mathfrak{S}_k^n = \overbrace{\mathfrak{S}_k \times \cdots \times \mathfrak{S}_k}^{n}$ of the n copies of $\mathfrak{S}_k$. Put

$$m_{ij}(g) := |g\Omega_i \cap \Omega_j| \quad (1 \le i, j \le n), \qquad \mathsf{M}(g) := (m_{ij}(g))_{1 \le i,j \le n}$$

for $g \in \mathfrak{S}_{kn}$, that is, $m_{ij}(g)$ counts the number of elements in Ω_i which are sent into Ω_j by g. For $g, g' \in \mathfrak{S}_{kn}$, we see that

$$\mathcal{K}g\mathcal{K} = \mathcal{K}g'\mathcal{K} \iff \mathsf{M}(g) = \mathsf{M}(g')$$

and

$$|\mathcal{K}g\mathcal{K}| = \frac{|\mathcal{K}|^2}{\mathsf{M}(g)!},$$

where $\mathsf{M}(g)! = \prod_{i,j=1}^n m_{ij}(g)!$. Put

$$\mathcal{M}_{n,k} := \left\{ M = (m_{ij}) \in \mathrm{Mat}_n(\mathbb{Z}_{\ge 0}) \,\middle|\, \sum_{r=1}^n m_{ir} = \sum_{s=1}^n m_{sj} = k \ (1 \le i, j \le n) \right\}.$$

The map

$$\mathcal{K}\backslash\mathfrak{S}_{kn}/\mathcal{K} \ni \mathcal{K}g\mathcal{K} \mapsto \mathsf{M}(g) \in \mathcal{M}_{n,k}$$

is bijective. Thus $\mathcal{M}_{n,k}$ gives a 'coordinate system' for the set $\mathcal{K}\backslash\mathfrak{S}_{kn}/\mathcal{K}$ of double cosets.

2.3 Immanants and zonal spherical functions

For each $\lambda \vdash kn$, define

$$\omega_{\mathcal{K}}^{\lambda}(g) := \frac{1}{|\mathcal{K}|} \sum_{y \in \mathcal{K}} \chi^{\lambda}(gy) \quad (g \in \mathfrak{S}_{kn}), \tag{2.1}$$

where χ^λ is the irreducible character of $\mathfrak{S}_{kn}$ corresponding to λ. These are $\mathcal{K}$-biinvariant functions on $\mathfrak{S}_{kn}$, and hence we refer to these as zonal spherical functions.

Since χ^λ are $\mathbb{Z}$-valued, the functions $\omega_{\mathcal{K}}^\lambda$ are $\mathbb{Q}$-valued. Observe that $\omega_{n,k} = \omega_{\mathcal{K}}^{(k^n)}$. The function $\omega_{\mathcal{K}}^\lambda$ is identically zero unless $\lambda \geq (k^n)$ with respect to the dominance ordering

$$\lambda \geq \mu \iff \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i, \quad \forall i \geq 1$$

on partitions of the same size.

The *immanant* of a matrix $A = (a_{ij}) \in \mathrm{Mat}_N$ associated to $\lambda \vdash N \in \mathbb{Z}_{>0}$ is

$$\mathrm{Imm}^\lambda A = \sum_{\sigma \in \mathfrak{S}_N} \chi^\lambda(\sigma) \prod_{i=1}^{N} a_{i\sigma(i)}. \tag{2.2}$$

Notice that $\mathrm{Imm}^{(1^N)} A = \det A$ and $\mathrm{Imm}^{(N)} A = \mathrm{per}\, A$, where $\mathrm{per}\, A$ is the permanent of A. For later use, we give an expression of the value of $\omega_{\mathcal{K}}^\lambda$ in terms of immanants.

Lemma 2.1. *For any $A = (a_{ij}) \in \mathrm{Mat}_{n,kn}$, we have*

$$\mathrm{Imm}^\lambda(A \otimes \mathbf{1}_{k,1}) = \sum_{\tau \in \mathfrak{S}_{kn}} \omega_{\mathcal{K}}^\lambda(\tau) \prod_{j=1}^{kn} a'_{j\tau(j)}, \tag{2.3}$$

where $a'_{ij} = a_{\lceil i/k \rceil, j}$ is the (i,j)-entry of $A \otimes \mathbf{1}_{k,1}$.

Proof. Since $a'_{y(i)j} = a'_{ij}$ for any $y \in \mathcal{K}$, it follows that

$$\begin{aligned}\mathrm{Imm}^\lambda(A \otimes \mathbf{1}_{k,1}) &= \sum_{\sigma \in \mathfrak{S}_{kn}} \chi^\lambda(\sigma) \prod_{i=1}^{kn} a'_{i\sigma(i)} = \frac{1}{|\mathcal{K}|} \sum_{y \in \mathcal{K}} \sum_{\sigma \in \mathfrak{S}_{kn}} \chi^\lambda(\sigma) \prod_{i=1}^{kn} a'_{y(i)\sigma(i)} \\ &= \frac{1}{|\mathcal{K}|} \sum_{y \in \mathcal{K}} \sum_{\tau \in \mathfrak{S}_{kn}} \chi^\lambda(\tau y) \prod_{j=1}^{kn} a'_{j\tau(j)} = \sum_{\tau \in \mathfrak{S}_{kn}} \omega_{\mathcal{K}}^\lambda(\tau) \prod_{j=1}^{kn} a'_{j\tau(j)}\end{aligned}$$

as desired. □

Lemma 2.2. *Let $\lambda \vdash kn$.*

(i) *For $g \in \mathfrak{S}_{kn}$,*

$$\omega_{\mathcal{K}}^\lambda(g) = \frac{1}{|\mathcal{K}|} \mathrm{Imm}^\lambda((I_n \otimes \mathbf{1}_k) P(g)).$$

(ii) *It holds that*

$$\mathrm{Imm}^\lambda(X \otimes \mathbf{1}_k) = \sum_{\tau \in \mathfrak{S}_{kn}} \omega_{\mathcal{K}}^\lambda(\tau) x^{\mathrm{M}(\tau)}.$$

In particular,

$$\omega_{\mathcal{K}}^\lambda(g) = \frac{\mathrm{M}(g)!}{|\mathcal{K}|^2} \left[\mathrm{Imm}^\lambda(X \otimes \mathbf{1}_k)\right]_{\mathrm{M}(g)}$$

for $g \in \mathfrak{S}_{kn}$.

Proof. We get (i) if we set $A = (I_n \otimes \mathbf{1}_{1,k})P(g)$ with $g \in \mathfrak{S}_{kn}$ in (2.3). If we set $A = X \otimes \mathbf{1}_{1,k}$ in (2.3), then we have (ii) since $a'_{i\tau(i)} = x_{pq}$ when $i \in \Omega_p$ and $\tau(i) \in \Omega_q$ and

$$\sum_{\tau \in \mathfrak{S}_{kn}} \omega^{\lambda}_{\mathcal{K}}(\tau) x^{\mathsf{M}(\tau)} = \sum_{M \in \mathcal{M}_{n,k}} \sum_{\substack{\tau \in \mathfrak{S}_{kn} \\ \mathsf{M}(\tau) = M}} \omega^{\lambda}_{\mathcal{K}}(\tau) x^{M} = \sum_{M \in \mathcal{M}_{n,k}} \frac{|\mathcal{K}|^2}{M!} \omega^{\lambda}_{\mathcal{K}}(g_M) x^{M},$$

where g_M is an arbitrarily chosen element in $\mathfrak{S}_{kn}$ such that $\mathsf{M}(g_M) = M$. □

3 The alpha-determinant and wreath determinant

We recall the definitions and basic facts on the alpha-determinant and wreath determinant. The alpha-determinant is first introduce by Vere-Jones [14] as α-permanent, whose definition is slightly different from ours; here we follow the convention in [13]. For the wreath determinant, see [10] for the detailed information.

First we define a class function $\nu(\cdot)$ on $\mathfrak{S}_N$ by

$$\nu(\sigma) := N - \sum_{i \geq 1} m_i(\sigma) = \sum_{i \geq 2} (i-1) m_i(\sigma)$$

for $\sigma \in \mathfrak{S}_N$ when the cycle type of σ is $1^{m_1(\sigma)} 2^{m_2(\sigma)} \dots N^{m_N(\sigma)}$. Notice that $\nu(\sigma\tau) = \nu(\sigma) + \nu(\tau)$ if σ and τ are disjoint.

Remark 3.1. For each $\sigma \in \mathfrak{S}_N$, $\nu(\sigma)$ is equal to the distance between the identity e and σ on the *Cayley graph* of $\mathfrak{S}_N$ whose generating set consists of all transpositions.

Remark 3.2. The value of $\nu(\sigma)$ for $\sigma \in \mathfrak{S}_N$ is invariant under the standard embedding $\mathfrak{S}_N \hookrightarrow \mathfrak{S}_{N'}$ ($N' > N$) which regards σ as an element in $\mathfrak{S}_{N'}$ leaving $N' - N$ letters $N+1, \dots, N'$ fixed. Namely, it would be natural to regard the function $\nu(\cdot)$ as a class function on the infinite symmetric group $\mathfrak{S}_\infty = \bigcup_{N \geq 1} \mathfrak{S}_N$.

The *alpha-determinant* of an N by N matrix $A = (a_{ij}) \in \mathrm{Mat}_N$ is

$$\det_\alpha A := \sum_{\sigma \in \mathfrak{S}_N} \alpha^{\nu(\sigma)} \prod_{i=1}^{N} a_{i\sigma(i)}.$$

Note that $\det_{-1} A = \det A$ and $\det_1 A = \operatorname{per} A$. The alpha-determinant is multilinear in rows and columns, is invariant under the transposition, and has Laplace expansion formula. We see that

$$\det_\alpha(AP(\sigma)) = \det_\alpha(P(\sigma)A)$$

for any $A \in \mathrm{Mat}_N$ and $\sigma \in \mathfrak{S}_N$ because $\nu(\cdot)$ is a class function on $\mathfrak{S}_N$, but the equation $\det_\alpha(AB) = \det_\alpha(BA)$ does not hold in general. We also note that we have

$$\det_\alpha \begin{pmatrix} A & B \\ O & C \end{pmatrix} = \det_\alpha A \det_\alpha C$$

if A and C are square matrices.

Example 3.3. We have

$$\det_\alpha \mathbf{1}_N = \sum_{\sigma\in\mathfrak{S}_N} \alpha^{\nu(\sigma)} = \prod_{j=1}^{N-1}(1+j\alpha).$$

For an n by kn matrix $A=(a_{ij})\in \mathrm{Mat}_{n,kn}$, the *$k$-wreath determinant* of A is defined by

$$\mathrm{wrdet}_k A := \det_{-1/k}(A\otimes \mathbf{1}_{k,1}). \tag{3.1}$$

Note that the 1-wreath determinant wrdet_1 is the ordinary determinant. The wreath-determinant wrdet_k is characterized as a polynomial function on the space $\mathrm{Mat}_{n,kn}$ by the following three conditions up to a scalar multiple (see [10] for the proof):

(W1) wrdet_k is multilinear in columns.

(W2) $\mathrm{wrdet}_k(QA) = (\det Q)^k\,\mathrm{wrdet}_k(A)$ for $Q\in\mathrm{Mat}_n$ and $A\in\mathrm{Mat}_{n,kn}$.

(W3) $\mathrm{wrdet}_k(AP(\sigma)) = \mathrm{wrdet}_k(A)$ for $\sigma\in\mathcal{K}$ and $A\in\mathrm{Mat}_{n,kn}$. In other words, if $A_i\in\mathrm{Mat}_{n,k}$ $(i=1,2,\dots,n)$, then

$$\mathrm{wrdet}_k(A_1P(\sigma_1)\ A_2P(\sigma_2)\ \dots\ A_nP(\sigma_n)) = \mathrm{wrdet}_k(A_1\ A_2\ \dots\ A_n)$$

for any $\sigma_1,\dots,\sigma_n\in\mathfrak{S}_k$.

In fact, instead of (W3), the k-wreath determinant satisfies a slightly stronger relative invariance

(W3′) $\mathrm{wrdet}_k(AP(g)) = \chi_{n,k}(g)\,\mathrm{wrdet}_k(A)$ for $g\in\mathcal{K}\rtimes\mathfrak{S}_n = \mathfrak{S}_n\wr\mathfrak{S}_k < \mathfrak{S}_{kn}$ and $A\in\mathrm{Mat}_{n,kn}$, where $\chi_{n,k}$ is defined by

$$\chi_{n,k}(g) = (\operatorname{sgn}\tau)^k, \qquad g=(\sigma,\tau)\in\mathcal{K}\rtimes\mathfrak{S}_k. \tag{3.2}$$

(W3′) means that if $A_i\in\mathrm{Mat}_{n,k}$ $(i=1,2,\dots,n)$, then

$$\mathrm{wrdet}_k(A_{\tau(1)}\ A_{\tau(2)}\ \dots\ A_{\tau(n)}) = (\operatorname{sgn}\tau)^k\ \mathrm{wrdet}_k(A_1\ A_2\ \dots\ A_n)$$

for any $\tau\in\mathfrak{S}_n$. This readily follows from (W2) by taking $Q=I_k\otimes P(\tau)$. Here we regard the wreath product $\mathfrak{S}_n\wr\mathfrak{S}_k$ as a subgroup of $\mathfrak{S}_{kn}$ so that we have

$$P(g) = P(\sigma)\cdot(I_k\otimes P(\tau)), \qquad g=(\sigma,\tau)\in\mathfrak{S}_k\wr\mathfrak{S}_n.$$

Remark 3.4. The definition of the wreath determinant is a bit different from the original one in [10], where the k-wreath determinant is defined for the kn by n rectangular matrices.

Example 3.5. We have

$$\begin{aligned}\mathrm{wrdet}_k(I_n\otimes\mathbf{1}_{1,k}) = \det_{-1/k}(I_n\otimes\mathbf{1}_k) &= \det_{-1/k}\begin{pmatrix}\mathbf{1}_k & & & \\ & \mathbf{1}_k & & \\ & & \ddots & \\ & & & \mathbf{1}_k\end{pmatrix}\\ &= (\det_{-1/k}\mathbf{1}_k)^n = \left(\frac{k!}{k^k}\right)^n.\end{aligned} \tag{3.3}$$

More generally, for $A \in \mathrm{Mat}_n$, we have

$$\mathrm{wrdet}_k(A \otimes \mathbf{1}_{1,k}) = \mathrm{wrdet}_k(A \cdot (I_n \otimes \mathbf{1}_{1,k})) = \Big(\frac{k!}{k^k}\Big)^n (\det A)^k.$$

4 Formulas for zonal spherical functions

The alpha-determinant is written as a linear combination of immanants as

$$\det_\alpha A = \frac{1}{N!} \sum_{\lambda \vdash N} f^\lambda f_\lambda(\alpha) \,\mathrm{Imm}^\lambda A, \tag{4.1}$$

where $f^\lambda = \chi^\lambda(e)$, e being the identity permutation, and

$$f_\lambda(\alpha) = \prod_{i=1}^{l(\lambda)} \prod_{j=1}^{\lambda_i} (1 + (j-i)\alpha)$$

is the modified content polynomial for λ. This is immediate from the well-known expansion formula

$$\alpha^{\nu(\cdot)} = \frac{1}{N!} \sum_{\lambda \vdash N} f^\lambda f_\lambda(\alpha) \chi^\lambda. \tag{4.2}$$

Theorem 4.1. *For $g \in \mathfrak{S}_{kn}$, we have*

$$\begin{aligned} \omega_{n,k}(g) &= \frac{k^{kn}}{|\mathcal{K}|} \det_{-1/k}((I_n \otimes \mathbf{1}_k) P(g)) \\ &= \Big(\frac{k^k}{k!}\Big)^n \sum_{y \in \mathcal{K}} \Big(-\frac{1}{k}\Big)^{\nu(gy)}. \end{aligned}$$

Proof. By (4.1) and Lemma 2.2 (i), we have

$$\det_{-1/k}((I_n \otimes \mathbf{1}_k) P(g)) = \frac{|\mathcal{K}|}{(kn)!} \sum_{\lambda \vdash kn} f^\lambda f_\lambda(-1/k) \omega_{\mathcal{K}}^\lambda(g).$$

Since $f_\lambda(-1/k) = 0$ if $\lambda_1 > k$ and $\mathrm{Imm}^\lambda(A \otimes \mathbf{1}_{k,1}) = 0$ unless $\lambda \geq (k^n)$, only the term for $\lambda = (k^n)$ survives in the righthand side of the equation above. By the hook formula for f^λ and the definition of $f_\lambda(\alpha)$, we readily obtain

$$f^{(k^n)} f_{(k^n)}(-1/k) = \frac{(kn)!}{k^{kn}}.$$

This completes the proof of the first equality. The second equality is immediate by the definition of the alpha-determinant. $\square$

Using Theorem 4.1, we obtain the stability of $\omega_{n,k}$ with respect to n as well as the non-vanishingness of $\omega_{n,k}$ when $k+1$ is prime as follows.

Corollary 4.2. *If $m > n$, then $\omega_{m,k}(g) = \omega_{n,k}(g)$ for any $g \in \mathfrak{S}_{kn}$, where we regard $g \in \mathfrak{S}_{kn}$ as an element in $\mathfrak{S}_{km}$ by the standard embedding.*

Proof. We regard $\mathfrak{S}_k^m$ as a direct product $\mathfrak{S}_k^n \times \mathfrak{S}_k^{m-n}$. If $g \in \mathfrak{S}_{kn}$ and $(y_1, y_2) \in \mathfrak{S}_k^n \times \mathfrak{S}_k^{m-n}$, then gy_1 and y_2 are disjoint permutations, and hence it follows that $\nu(gy_1y_2) = \nu(gy_1) + \nu(y_2)$. Thus we have

$$\begin{aligned}\omega_{m,k}(g) &= \Big(\frac{k^k}{k!}\Big)^m \sum_{(y_1,y_2)\in\mathfrak{S}_k^n\times\mathfrak{S}_k^{m-n}} \Big(-\frac{1}{k}\Big)^{\nu(gy_1y_2)} \\ &= \Big(\frac{k^k}{k!}\Big)^m \sum_{y_1\in\mathfrak{S}_k^n} \Big(-\frac{1}{k}\Big)^{\nu(gy_1)} \sum_{y_2\in\mathfrak{S}_k^{m-n}} \Big(-\frac{1}{k}\Big)^{\nu(y_2)} \\ &= \Big(\frac{k^k}{k!}\Big)^n \sum_{y_1\in\mathfrak{S}_k^n} \Big(-\frac{1}{k}\Big)^{\nu(gy_1)} \\ &= \omega_{n,k}(g)\end{aligned}$$

as desired. □

Theorem 4.3. *Let p be an odd prime. The function $\omega_{n,k}$ does not vanish on $\mathfrak{S}_{kn}$ if $k = p-1$.*

Proof. By Theorem 4.1, we have

$$\omega_{n,k}(g) = \Big(\frac{(p-1)^{p-1}}{(p-1)!}\Big)^n \sum_{y\in\mathcal{K}} \Big(-\frac{1}{p-1}\Big)^{\nu(gy)} \equiv \frac{1}{|\mathcal{K}|} \sum_{y\in\mathcal{K}} 1 \equiv 1 \pmod p$$

for any $g \in \mathfrak{S}_{kn}$, which implies the desired nonvanishingness. □

Remark 4.4. In [7], the inverse of Theorem 4.3 is proved. In fact, the authors show that if $n \geq 3$ and $k+1$ is composite, then one can find $M \in \mathcal{M}_{n,k}$ such that $\big[(\det X)^k\big]_M = 0$.

We give a formula for the function $\omega_{n,k}$ in terms of the wreath determinant.

Lemma 4.5. *For $A \in \mathrm{Mat}_{n,kn}$, we have*

$$\mathrm{wrdet}_k A = \frac{1}{k^{kn}} \mathrm{Imm}^{(k^n)}(A \otimes \mathbf{1}_{k,1}).$$

Proof. By the definition of the wreath determinant and the formula (4.1), we have

$$\begin{aligned}\mathrm{wrdet}_k A &= \det\nolimits_{-1/k}(A \otimes \mathbf{1}_{k,1}) \\ &= \frac{1}{(kn)!} \sum_{\lambda\vdash kn} f^\lambda f_\lambda(-1/k)\, \mathrm{Imm}^\lambda(A \otimes \mathbf{1}_{k,1}).\end{aligned}$$

The conclusion follows from a similar discussion as in the proof of Theorem 4.1. □

Theorem 4.6. *For $g \in \mathfrak{S}_{kn}$, we have*

$$\begin{aligned}\omega_{n,k}(g) &= \frac{\mathsf{M}(g)!}{|\mathcal{K}|} \big[(\det X)^k\big]_{\mathsf{M}(g)} \\ &= \frac{\mathrm{wrdet}_k((I_n \otimes \mathbf{1}_{1,k})P(g))}{\mathrm{wrdet}_k(I_n \otimes \mathbf{1}_{1,k})}.\end{aligned}$$

Proof. By Lemma 4.5, we see that

$$\operatorname{wrdet}_k(X \otimes \mathbf{1}_{1,k}) = \frac{1}{k^{kn}} \operatorname{Imm}^{(k^n)}(X \otimes \mathbf{1}_k).$$

On the other hand, by (W2) and (3.3), we have

$$\operatorname{wrdet}_k(X \otimes \mathbf{1}_{1,k}) = (\det X)^k \operatorname{wrdet}_k(I_n \otimes \mathbf{1}_{1,k}) = \Bigl(\frac{k!}{k^k}\Bigr)^n (\det X)^k.$$

Thus it follows that

$$(\det X)^k = \frac{1}{|\mathcal{K}|} \operatorname{Imm}^{(k^n)}(X \otimes \mathbf{1}_k).$$

Hence, by Lemma 2.2 (ii), we have the first equality. The second equality is obtained by Theorem 4.1 and the equation

$$\det_{-1/k}((I_n \otimes \mathbf{1}_k)P(g)) = \operatorname{wrdet}_k((I_n \otimes \mathbf{1}_{1,k})P(g)),$$

which follows from the definition of the wreath determinant. □

As a corollary, we see that the relative invariance of the function $\omega_{n,k}$ with respect to the wreath product $\mathfrak{S}_k \wr \mathfrak{S}_n$.

Corollary 4.7. *For any $g \in \mathfrak{S}_{kn}$ and $h, h' \in \mathfrak{S}_k \wr \mathfrak{S}_n$, we have*

$$\omega_{n,k}(hgh') = \chi_{n,k}(hh')\omega_{n,k}(g).$$

Here $\chi_{n,k}$ is the character of $\mathfrak{S}_k \wr \mathfrak{S}_n$ defined by (3.2). In particular, $\omega_{n,k}$ is $\mathfrak{S}_k \wr \mathfrak{S}_n$-biinvariant if k is even.

Proof. Let $h = (\sigma, \tau), h' = (\sigma', \tau') \in \mathcal{K} \rtimes \mathfrak{S}_n = \mathfrak{S}_k \wr \mathfrak{S}_n$. Since

$$(I_n \otimes \mathbf{1}_{1,k})P(h) = (I_n \otimes \mathbf{1}_{1,k})P(\sigma)(I_k \otimes P(\tau)) = P(\tau)(I_n \otimes \mathbf{1}_{1,k}),$$

we have

$$\begin{aligned}
\omega_{n,k}(hgh') &= \frac{\operatorname{wrdet}_k((I_n \otimes \mathbf{1}_{1,k})P(hgh'))}{\operatorname{wrdet}_k(I_n \otimes \mathbf{1}_{1,k})} \\
&= \frac{\operatorname{wrdet}_k(P(\tau)(I_n \otimes \mathbf{1}_{1,k})P(g)P(h'))}{\operatorname{wrdet}_k(I_n \otimes \mathbf{1}_{1,k})} \\
&= \det P(\tau)^k \chi_{n,k}(h') \frac{\operatorname{wrdet}_k((I_n \otimes \mathbf{1}_{1,k})P(g))}{\operatorname{wrdet}_k(I_n \otimes \mathbf{1}_{1,k})} \\
&= \chi_{n,k}(hh')\omega_{n,k}(g)
\end{aligned}$$

as desired. □

5 Applications

5.1 The Alon-Tarsi conjecture on Latin squares

A *Latin square* of degree n is an n by n matrix whose rows and columns are permutations of $1, 2, \ldots, n$. The set of all Latin squares of degree n is denoted by $\mathrm{LS}(n)$.

Example 5.1. There are twelve Latin squares of degree 3:

$$\begin{aligned}\mathrm{LS}(3) = \Bigg\{ &\begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\3&1&2\\2&3&1\end{pmatrix}, \begin{pmatrix}2&3&1\\1&2&3\\3&1&2\end{pmatrix}, \begin{pmatrix}2&3&1\\3&1&2\\1&2&3\end{pmatrix}, \\ &\begin{pmatrix}3&1&2\\1&2&3\\2&3&1\end{pmatrix}, \begin{pmatrix}3&1&2\\2&3&1\\1&2&3\end{pmatrix}, \begin{pmatrix}1&3&2\\2&1&3\\3&2&1\end{pmatrix}, \begin{pmatrix}1&3&2\\3&2&1\\2&1&3\end{pmatrix}, \\ &\begin{pmatrix}2&1&3\\1&3&2\\3&2&1\end{pmatrix}, \begin{pmatrix}2&1&3\\3&2&1\\1&3&2\end{pmatrix}, \begin{pmatrix}3&2&1\\1&3&2\\2&1&3\end{pmatrix}, \begin{pmatrix}3&2&1\\2&1&3\\1&3&2\end{pmatrix} \Bigg\}.\end{aligned}$$

For $L \in \mathrm{LS}(n)$, we associate $2n$ permutations $r_1, \ldots, r_n, c_1, \ldots, c_n \in \mathfrak{S}_n$ to it by

$$L = \begin{pmatrix} r_1(1) & \ldots & r_1(n) \\ \vdots & \ddots & \vdots \\ r_n(1) & \ldots & r_n(n) \end{pmatrix} = \begin{pmatrix} c_1(1) & \ldots & c_n(1) \\ \vdots & \ddots & \vdots \\ c_1(n) & \ldots & c_n(n) \end{pmatrix}.$$

Then we define

$$\operatorname{sgn} L := \prod_{i=1}^{n} \operatorname{sgn} r_i \prod_{i=1}^{n} \operatorname{sgn} c_i,$$

and we call L *even* (resp. *odd*) if $\operatorname{sgn} L = +1$ (resp. -1). We denote by $\mathrm{els}(n)$ and $\mathrm{ols}(n)$ the numbers of even and odd Latin squares of degree n respectively. Since the map $\mathrm{LS}(n) \ni L \mapsto P(\sigma)L \in \mathrm{LS}(n)$ for a given $\sigma \in \mathfrak{S}_n$ is a bijection and $\operatorname{sgn}(P(\sigma)L) = (\operatorname{sgn} \sigma)^n \operatorname{sgn} L$ for $L \in \mathrm{LS}(n)$, we have $\mathrm{els}(n) = \mathrm{ols}(n)$ when n is odd. When n is even, it is conjectured that the numbers of even and odd Latin squares are always different.

Conjecture 5.2 (Alon-Tarsi conjecture)**.** $\mathrm{els}(n) \neq \mathrm{ols}(n)$ *if n is even.*

This conjecture originally arose from the study of colorings of graphs. Indeed, if the Alon-Tarsi conjecture for even n is true, then we see that the Dinitz conjecture below for n follows [1].

Proposition 5.3 (Dinitz conjecture)**.** *The line graph of the biclique* (*or complete bipartite graph*) $K_{n,n}$ *is n-choosable.*

We remark that the Dinitz conjecture itself is already settled down by Galvin [4]. There are also various statements which are equivalent to or related with the Alon-Tarsi conjecture (see, e.g. [6, 11]). The Alon-Tarsi conjecture is proved to be true in the case where $n = p + 1$ by Drisko [2] and in the case where $n = p - 1$ by Glynn [5], where p is an odd prime; We also refer to [3].

We need another statement which is equivalent to the Alon-Tarsi conjecture. Define

$$\mathsf{L}(n) := \{\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathfrak{S}_n^n \,|\, P(\sigma_1) + \cdots + P(\sigma_n) = \mathbf{1}_n\}.$$

For $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathsf{L}(n)$, the matrix

$$L(\boldsymbol{\sigma}) := \sum_{i=1}^{n} i\, P(\sigma_i)$$

is a Latin square of degree n, and every Latin square is uniquely obtained in this way. A Latin square $L = L(\boldsymbol{\sigma})$ ($\boldsymbol{\sigma} \in \mathsf{L}(n)$) is called *symbol even* (resp. *symbol odd*) if

$$\operatorname{symsgn} L := \prod_{i=1}^{n} \operatorname{sgn} \sigma_i$$

is $+1$ (resp. -1). We denote by $\operatorname{sels}(n)$ and $\operatorname{sols}(n)$ the number of symbol even and symbol odd Latin squares of degree n respectively. It is known that

$$\operatorname{sels}(n) - \operatorname{sols}(n) = (-1)^{n(n-1)/2}(\operatorname{els}(n) - \operatorname{ols}(n))$$

for every n (see, e.g. [5]), so Conjecture 5.2 is equivalent to the

Conjecture 5.4. $\operatorname{sels}(n) \neq \operatorname{sols}(n)$ *if n is even.*

Since

$$\begin{aligned}
[(\det X)^n]_{\mathbf{1}_n} &= \sum_{\substack{\sigma_1, \ldots, \sigma_n \in \mathfrak{S}_n \\ P(\sigma_1) + \cdots + P(\sigma_n) = \mathbf{1}_n}} \prod_{i=1}^{n} (\operatorname{sgn} \sigma_i) \\
&= \sum_{\boldsymbol{\sigma} \in \mathsf{L}(n)} \operatorname{symsgn} L(\boldsymbol{\sigma}) \\
&= \sum_{L \in \mathrm{LS}(n)} \operatorname{symsgn} L = \operatorname{sels}(n) - \operatorname{sols}(n),
\end{aligned}$$

we obtain the following result by Theorem 4.6.

Theorem 5.5. *When n is even, the Alon-Tarsi conjecture on $\mathrm{LS}(n)$ is equivalent to the following assertions.*

(1) $[(\det X)^n]_{\mathbf{1}_n} \neq 0$.

(2) $\operatorname{wrdet}_n((I_n \otimes \mathbf{1}_{1,n})P(g_n)) = \operatorname{wrdet}_n(\overbrace{I_n \ \ldots \ I_n}^{n}) \neq 0$.

(3) $\omega_{n,n}(g_n) \neq 0$.

Here the permutation $g_n \in \mathfrak{S}_{n^2}$ is given by

$$g_n((i-1)n + j) = (j-1)n + i, \qquad 1 \leq i, j \leq n, \tag{5.1}$$

which is a product of $n(n-1)/2$ disjoint transpositions and $\mathsf{M}(g_n) = \mathbf{1}_n$.

Thus, Theorem 5.5 (3) together with Theorem 4.3 gives another proof of the

Corollary 5.6 (Glynn [5])**.** *The Alon-Tarsi conjecture for Latin squares of degree n is true if $n = p - 1$ for an odd prime p.*

5.2 A remark on Kumar's conjecture on plethysms

Let k and n be positive integers as heretofore, and V be a finite dimensional vector space over $\mathbb{C}$ such that $\dim V \geq n$. The symmetric group $\mathfrak{S}_m$ acts on $V^{\otimes m}$ from the right by

$$(v_1 \otimes \cdots \otimes v_m) \cdot \sigma := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)} \quad (\sigma \in \mathfrak{S}_m).$$

This action linearly extends to that of the group algebra $\mathbb{C}\mathfrak{S}_m$. We understand that the symmetric tensor power $S^m(V)$ of V is a subspace of $V^{\otimes m}$ spanned by the vectors of the form

$$v_1 \cdots v_m := v_1 \otimes \cdots \otimes v_m \cdot \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \sigma = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}. \tag{5.2}$$

Set

$$\mathcal{H} = \mathcal{K} \rtimes \mathfrak{S}_n = \mathfrak{S}_k \wr \mathfrak{S}_n, \quad \mathcal{K}' = \mathfrak{S}_n^k, \quad \mathcal{H}' = \mathcal{K}' \rtimes \mathfrak{S}_k = \mathfrak{S}_n \wr \mathfrak{S}_k,$$

and

$$\boldsymbol{e}(G) = \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{C}\mathfrak{S}_{kn}$$

for $G < \mathfrak{S}_{kn}$. We have then

$$S^n(S^k V) = V^{\otimes kn} \cdot \boldsymbol{e}(\mathcal{H}), \qquad S^k(S^n V) = V^{\otimes kn} \cdot \boldsymbol{e}(\mathcal{H}').$$

Define a linear transformation $\tau = \tau_{k,n}$ on $V^{\otimes kn}$ by

$$\tau \colon V^{\otimes kn} \ni \overbrace{\underbrace{v_1^1 \otimes \cdots \otimes v_k^1}_{k} \otimes \cdots \otimes \underbrace{v_1^n \otimes \cdots \otimes v_k^n}_{k}}^{n}$$
$$\longmapsto \overbrace{\underbrace{v_1^1 \otimes \cdots \otimes v_1^n}_{n} \otimes \cdots \otimes \underbrace{v_k^1 \otimes \cdots \otimes v_k^n}_{n}}^{k} \in V^{\otimes kn},$$

or equivalently,

$$\tau(v_1 \otimes v_2 \otimes \cdots \otimes v_{kn}) = (v_1 \otimes v_2 \otimes \cdots \otimes v_{kn}) \cdot g_{n,k},$$

where the permutation $g_{n,k} \in \mathfrak{S}_{kn}$ is defined by

$$g_{n,k}((i-1)n + j) = (j-1)k + i, \qquad 1 \leq i \leq k, \ 1 \leq j \leq n. \tag{5.3}$$

We notice that $g_{n,n}$ equals g_n defined in (5.1). Using this, we define a map $h_{n,k}$ by

$$h_{n,k} := p \circ \tau \circ i \colon S^n(S^k V) \overset{i}{\hookrightarrow} V^{\otimes kn} \xrightarrow{\tau} V^{\otimes kn} \overset{p}{\twoheadrightarrow} S^k(S^n V),$$

where i is the inclusion and p is the natural projection (i.e. multiplication by $\boldsymbol{e}(\mathcal{H}')$ from the right as in (5.2)). Notice that $h_{n,k}(v) = v \cdot g_{n,k} \boldsymbol{e}(\mathcal{H}')$ for $v \in S^n(S^k V)$. This map is clearly a $\mathrm{GL}(V)$-intertwiner between two left $\mathrm{GL}(V)$-modules $S^n(S^k V)$ and $S^k(S^n V)$.

Example 5.7.

$$\begin{aligned}
h_{2,2}((v_1v_2)(v_3v_4)) &= (p\circ\tau)\left(\frac{v_1\otimes v_2+v_2\otimes v_1}{2}\otimes\frac{v_3\otimes v_4+v_4\otimes v_3}{2}\right)\\
&= \frac{1}{4}p\Bigg(v_1\otimes v_3\otimes v_2\otimes v_4+v_2\otimes v_3\otimes v_1\otimes v_4\\
&\qquad +v_1\otimes v_4\otimes v_2\otimes v_3+v_2\otimes v_4\otimes v_1\otimes v_3\Bigg)\\
&= \frac{(v_1v_3)(v_2v_4)+(v_2v_3)(v_1v_4)+(v_1v_4)(v_2v_3)+(v_2v_4)(v_1v_3)}{4}
\end{aligned}$$

Motivated by the Hadamard-Howe conjecture on the maximality of $h_{n,k}$, it is conjectured by Kumar that $\ker h_{n,k}$ does not contain $\mathbf{E}_V^{(k^n)}$, the irreducible $\mathrm{GL}(V)$-module with highest weight $(k^n)=(k,\dots,k)$, if $n\le k$ and k is even (see [11, Conjecture 1.6]). We focus on this problem below.

By the Schur-Weyl duality

$$V^{\otimes kn}=\bigoplus_{\lambda\vdash kn}\mathbf{E}_V^{\lambda}\boxtimes\mathbf{M}_{kn}^{\lambda},$$

where $\mathbf{M}_{kn}^{\lambda}$ is the irreducible $\mathfrak{S}_{kn}$-module corresponding to λ, the multiplicity of $\mathbf{E}_V^{\lambda}$ in $S^n(S^kV)$ as a left $\mathrm{GL}(V)$-module is equal to $\dim(\mathbf{M}_{kn}^{\lambda}\cdot\boldsymbol{e}(\mathcal{H}))$, which is majorated by $\dim(\mathbf{M}_{kn}^{\lambda}\cdot\boldsymbol{e}(\mathcal{K}))=K_{\lambda(k^n)}$, the Kostka number.

Remark 5.8. Similarly, we see that the multiplicity of $\mathbf{E}_V^{\lambda}$ in $S^k(S^nV)$ is majorated by $K_{\lambda(n^k)}$. Especially, if $n>k$, then $S^k(S^nV)$ does not contain $\mathbf{E}_V^{(k^n)}$ since $K_{(k^n)(n^k)}=0$.

Lemma 5.9. *The multiplicity of* $\mathbf{E}_V^{(k^n)}$ *in* $S^n(S^kV)$ *is exactly one if* k *is even.*

Proof. Since we know that the multiplicity $\dim(\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{H}))$ of $\mathbf{E}_V^{(k^n)}$ in $S^n(S^kV)$ is at most one, we should show that it is at least one. Take a nonzero $\mathcal{K}$-invariant vector $w\in\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{K})$, which is unique up to constant multiple since $\dim\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{K})=K_{(k^n)(k^n)}=1$. We see that

$$w\cdot g=\omega_{n,k}(g)w+w^{\perp}(g)\tag{5.4}$$

for $g\in\mathfrak{S}_{kn}$ where $w^{\perp}(g)$ is a certain vector in the orthocomplement of $\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{K})$ in $\mathbf{M}_{kn}^{(k^n)}$ with respect to the invariant inner product on $\mathbf{M}_{kn}^{(k^n)}$. Since k is even, we see that $\omega_{n,k}(g)=1$ for $g\in\mathcal{H}$ by Corollary 4.7. Hence it follows that

$$w\cdot\boldsymbol{e}(\mathcal{H})=w\cdot\boldsymbol{e}(\mathcal{H})\boldsymbol{e}(\mathcal{K})=\frac{1}{|\mathcal{H}|}\sum_{g\in\mathcal{H}}(w\cdot g)\cdot\boldsymbol{e}(\mathcal{K})=w+\frac{1}{|\mathcal{H}|}\sum_{g\in\mathcal{H}}w^{\perp}(g)\cdot\boldsymbol{e}(\mathcal{K})=w.$$

Namely, we have $\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{K})\subset\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{H})$. Thus we see that

$$\dim(\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{H}))\ge\dim(\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{K}))=K_{(k^n)(k^n)}=1$$

as desired. □

Remark 5.10. If k is *odd*, then $w \cdot g = (\operatorname{sgn} \tau) w$ for $w \in \mathbf{M}_{kn}^{(k^n)}$ and $g = (\sigma, \tau) \in \mathcal{H}$. Thus, in this case, we have $\mathbf{M}_{kn}^{(k^n)} \cdot \boldsymbol{e}(\mathcal{H}) = 0$, and hence $S^n(S^k V)$ does not contain $\mathbf{E}_V^{(k^n)}$.

We restrict our attention on the special case where $k = n$ and n is even. We have $\mathcal{K} = \mathcal{K}'$ and $\mathcal{H} = \mathcal{H}'$ in this case. The map $h_{n,n}$ is then a $\mathrm{GL}(V)$-intertwiner from $S^n(S^n V)$ onto itself. Since the multiplicity of $\mathbf{E}_V^{(n^n)}$ in $S^n(S^n V)$ is one, the restriction of $h_{n,n}$ on $\mathbf{E}_V^{(n^n)}$ must be a scalar by Schur's lemma, and the scalar is given by $\omega_{n,n}(g_n)$ by (5.4) since $h_{n,n}(v) = v \cdot g_n \boldsymbol{e}(\mathcal{H})$. Therefore we obtain the

Theorem 5.11. *When n is even, we have*

$$h_{n,n}(v) = \omega_{n,n}(g_n) v$$

if $v \in S^n(S^n V)$ belongs to the (n^n)-isotypic component. In particular, $\ker h_{n,n} \supset \mathbf{E}_V^{(n^n)}$ *if and only if* $\omega_{n,n}(g_n) = 0$.

As a corollary, we obtain the

Corollary 5.12 ([11, Theorem 1.9 (b)])**.** *The Alon-Tarsi conjecture on* $\mathrm{LS}(n)$ *is equivalent to the assertion that* $\ker h_{n,n}$ *does not contain* $\mathbf{E}_V^{(n^n)}$.

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