GAUSS DECOMPOSITION AND q-DIFFERENCE EQUATIONS FOR JACKSON INTEGRALS OF SYMMETRIC SELBERG TYPE

Ryukyu Math. J., 36(2023), 1-47

GAUSS DECOMPOSITION AND q-DIFFERENCE EQUATIONS FOR JACKSON INTEGRALS OF SYMMETRIC SELBERG TYPE

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Abstract. We provide explicit expressions for two types of first order q-difference systems for the Jackson integral of symmetric Selberg type. One is the q-difference system known to be the q-KZ equation and the other is the q-difference system for parameters different from the q-KZ equation. We use a basis of the systems introduced by Matsuo in his study of the q-KZ equation. As a result, the similarity of these two systems is discussed by concrete calculations. Intermediate calculations are made use of the *Riemann-Hilbelt method for q-difference equation from connection matrix* established by Aomoto.

0. Introduction

Let $q=e^{2\pi\sqrt{-1}\tau}$, $\operatorname{Im}\tau>0$, be the elliptic modulus. Let $\Phi(t)$ be a q-multiplicative function on the algebraic torus $(\mathbb{C}^*)^n$ defined by

$$\Phi(t)=\Phi_{n,m}(t):=t_1^{\alpha_1}\cdots t_n^{\alpha_n}\prod_{j=1}^{n}\prod_{k=1}^{m}\frac{(t_j/x_k)_\infty}{(t_jq^{\beta_k}/x_k)_\infty}\prod_{1\le i<j\le n}\frac{(q^{1-\gamma}t_j/t_i)_\infty}{(q^{\gamma}t_j/t_i)_\infty}, \tag{0.1}$$

where $\alpha_j=\tilde{\alpha}-n+j-2(j-1)\gamma$. In the papers [3, 7], Aomoto and Aomoto–Kato introduced the notion of the *symmetric part* $H^n((\mathbb{C}^*)^n,\Phi,\nabla)_{sym}$ of q-analog of the twisted de Rham cohomology attached to $\Phi(t)$, whose dimension is $\binom{n+m-1}{m-1}$ if parameters $\tilde{\alpha}$, β_k, x_k and γ are generic. If we take a basis $\{\psi_l(t)\}_{l\in L}$ of the cohomology $H^n((\mathbb{C}^*)^n,\Phi,\nabla)_{sym}$, we can construct a solution of a system of holonomic q-difference equations on $(\mathbb{C}^*)^m$ by using the Jackson integrals $\widetilde{\psi}_l=\int\Phi\psi_l\varpi$ (see (1.2) for the definition of Jackson integral). When we denote the q-shift $x_j\to qx_j$ for a value x_j by T_j the q-difference system is expressed by a suitable matrix $K_j(x)$ of rank $\binom{n+m-1}{m-1}$ given by $T_j(\widetilde{\psi}_l)_{l\in L}=(\widetilde{\psi}_l)_{l\in L}K_j(x)$.

In the papers [21, 22], Matsuo claimed that by taking a suitable basis the Jackson integrals for it give a solution of the quantized Knizhnik–Zamolodchikov difference equations for the matrix coefficients of the product of intertwining operators R_{ij} called R-matrices for the quantum affine group $U_q(\widehat{sl}_2)$ in the sense of Frenkel and Reshetikhin [16], i.e., the matrix $K_j(x)$ is expressed as

$$K_j(x)=R_{j,j+1}(\tfrac{x_j}{x_{j+1}})\cdots R_{j,m}(\tfrac{x_j}{x_m})\ D_j\ R_{j,1}(\tfrac{qx_j}{x_1})\cdots R_{j,j-1}(\tfrac{qx_j}{x_{j-1}})\qquad(1\le j\le m),$$

where D_j is some diagonal matrix. Varchenko [26] made Matsuo's work complete. According to the result of [22, 26], the system of q-difference equations for the Jackson integrals $\widetilde{\psi}_l(t)$ with respect to the q-shift $x_j\to qx_j$ eventually reduces to that of the cases $m=2$ because R-matrix R_{ij} is decomposed into a direct sum of these of the

Received November 30, 2023.

case $m=2$. Therefore we consider the problem of finding an explicit expression of the matrix K_j or R_{ij} for $m=2$.

On the other hand, this problem has already been studied by Mimachi [23, 24]. He introduced one of expressions of the matrix $K_j(x)$ by using values of certain Schur polynomials and evaluated it explicitly when $n=1,2$ and 3 (see [24]). Aomoto and Kato [13] also gave another approach to express it in terms of the Gauss matrix decomposition [17] and evaluated it explicitly in the case where $n=1$ and 2, but in the form $K_j(0)K_j(x)^{-1}$. They used a method which they call the Riemann–Hilbert problem for q-difference equation from connection matrix [5, 6]. It is a surprising method because, under some assumptions, the matrix which represents q-difference system is exactly determined only from the information of connection matrix between asymptotic behaviors of its fundamental solutions. However, for evaluating the matrix $K_j(x)$, we do not need such method if we can evaluate the R-matrix R_{ij}. Actually, the explicit form of the R-matrix as the Gauss decomposition is not so difficult to find out. This is one of the aims of the present paper and will lead us to Theorems 1.6 and 1.7 in Section 1. (However we will confirm in Subsection 4.2 that the explicit form of the matrix $K_j(x)$ is also obtained from Aomoto's Riemann–Hilbert method.)

When $n=1$ the Jackson integral [1] associated with $\Phi_{n,2}(t)$ is equivalent to Heine's hypergeometric series

$$ {}_2\varphi_1(q^\alpha,q^\beta;q^\gamma;x)=\sum_{\nu=0}^\infty \frac{(q^\alpha)_\nu(q^\beta)_\nu}{(q^\gamma)_\nu(q)_\nu}x^\nu=\frac{1}{1-q}\frac{(q^\alpha)_\infty(q^\beta)_\infty}{(q^\gamma)_\infty(q)_\infty}\int_0^1 t^X\frac{(q^\gamma t)_\infty(qt)_\infty}{(q^\alpha t)_\infty(q^\beta t)_\infty}\frac{d_qt}{t}, $$

where $x=q^X$. Heine's hypergeometric series satisfies the following transformation formula [18, p.13, Eq.(1.4.1)]

$$ {}_2\varphi_1(q^\alpha,q^\beta;q^\gamma;x)=\frac{(q^\alpha)_\infty(q^\beta x)_\infty}{(q^\gamma)_\infty(x)_\infty}{}_2\varphi_1(x,q^{\gamma-\alpha};q^\beta x;q^\alpha). \tag{0.2} $$

One of the reason the transformation (0.2) holds is the equality

$$ (q^B)_\infty\int_0^1 t^A\frac{(tq^{X+C})_\infty(qt)_\infty}{(tq^X)_\infty(tq^B)_\infty}\frac{d_qt}{t}=(q^C)_\infty\int_0^1 t^X\frac{(tq^{A+B})_\infty(qt)_\infty}{(tq^A)_\infty(tq^C)_\infty}\frac{d_qt}{t}, $$

which changes q^A and q^X. It seems interesting to treat the q-difference system with respect to the parameter $q^{\tilde\alpha}$ as that of x. We state the q-difference system for the same basis as Matsuo's case with respect to the parameter shift $T_{\tilde\alpha}:\tilde\alpha\to\tilde\alpha+1$ for $m=2$:

$$ T_{\tilde\alpha}(\widetilde{\psi}_l)_{l\in L}=(\widetilde{\psi}_l)_{l\in L}A(q^{\tilde\alpha}), \tag{0.3} $$

where $A(q^{\tilde\alpha})$ is a suitable rational matrix of rank $n+1$. As we see in Theorem 5.2 in Section 5, although we do not know each element of the matrix $A(q^{\tilde\alpha})$, we have its Gauss decomposition. The Gauss decomposition of the matrix $A(q^{\tilde\alpha})$ is very similar to that of the matrix $K_j(x)$ (see Theorem 5.2 in Section 5 and compare it with Theorem 1.6). Furthermore, the R-matrix R_{ij} is determined from asymptotic behaviors of certain special solutions for the q-difference system (0.3) through (3.1), (3.3), (3.4), (D.16) and (D.17). In particular, the upper and lower triangular matrices of the Gauss matrix decomposition of R_{ij} are determined from the matrices $A(0)$ or $A(\infty)$ via (2.1), (2.2), (3.6), (3.7) and (D.12) (see Remark D.4 in Appendix D and Examples in Appendix E).

In Section 2, we review the Riemann–Hilbert problem for q-difference equation from connection matrix. In order to evaluate $A(q^{\tilde{\alpha}})$, we use Aomoto's Riemann–Hilbert method for it because $A(q^{\tilde{\alpha}})$ is determined only from the data of the principal connection matrix G which has been studied in [4, 8, 9, 10, 11, 12, 14]. The principal connection matrix G can also be deduced from the hypergeometric pairing studied by Tarasov and Varchenko [25], which is related to this problem.

Remark. This note was written in 1997 when the author was a graduate student at Nagoya University. At that time, the Internet was not yet popular enough, and there were page restrictions on paper publication. Although he compiled it in notebook form, the intermediate calculations were too long, so he did not publish it and only gave printed copies to a limited number of people involved. Therefore, there were several papers [6, 13] in the bibliographic list at that time that had the title of this note. On the other hand, time has passed, and recently the author has found that the main results (Theorems 1.6 and 5.2) of this note can be derived relatively simply by a method different from the method in this note, so he has published another proof of these theorems in [19]. Furthermore, he heard from Prof. Y. Yamada that there was an application of these theorems, and the results of [19] were cited in the recent paper [15], so he decided to publish this note here. Lastly it should also be noted that in [15] the similarity between the matrices $A(q^{\tilde{\alpha}})$ and $K_j(x)$ is explained as consequence of the so-called *base-fiber duality* of the gauge theory.

Contents

1. Quantized Knizhnik–Zamolodchikov difference equations and R-matrix.

1.1. **Notation.** Let $q=e^{2\pi\sqrt{-1}\tau}$, $\mathrm{Im}\,\tau>0$, be the elliptic modulus. For an arbitrary number $c\in\mathbb{C}$, $0<|c|<1$, we use the notations

$$(x;c)_\infty:=\prod_{\nu=0}^{\infty}(1-c^\nu x),\quad (x;c)_\nu:=\frac{(x;c)_\infty}{(xc^\nu;c)_\infty},\quad {}_r(x;c)_s=\frac{(x;c)_r}{(x;c)_{r-s}(x;c)_s}.$$

If $c=q$, we simply write $(x)_\infty:=(x;q)_\infty$ and $(x)_\nu:=(x;q)_\nu$. Let $\vartheta(x)$ be the *Jacobi elliptic theta function* defined by

$$\vartheta(x):=(x)_\infty(q/x)_\infty(q)_\infty.$$

We also use the notations

$$\vartheta(x)_r:=\vartheta(x)\cdot\vartheta(xq^\gamma)\cdots\vartheta(xq^{(r-1)\gamma})\quad\text{and}\quad {}_r\vartheta(x)_s:=\frac{\vartheta(x)_r}{\vartheta(x)_{r-s}\cdot\vartheta(x)_s},$$

which have the relations

$$\lim_{q\to0}\vartheta(x)_r=(x;q^\gamma)_r\quad\text{and}\quad\lim_{q\to0}{}_r\vartheta(x)_s={}_r(x;q^\gamma)_s,$$

if we fix q^γ as a single character c such as a number that does not depend on q. Let $\Phi(t)=\Phi_{n,m}(t)$ be the same function as (0.1). The function $\Phi(t)$ satisfies a quasi-symmetric property with respect to the symmetric group $\mathfrak{S}_n$ of nth order such that

$$\sigma\Phi(t)=U_\sigma(t)\ \Phi(t),\quad \sigma\in\mathfrak{S}_n,\tag{1.1}$$

with a q-periodic function $U_\sigma(t)$ as

$$U_\sigma(t):=\prod_{\substack{i<j\\ \sigma^{-1}(i)>\sigma^{-1}(j)}}\Big(\frac{t_j}{t_i}\Big)^{2\gamma-1}\frac{\vartheta(q^\gamma t_j/t_i)}{\vartheta(q^{1-\gamma}t_j/t_i)}=\mathrm{sgn}\,\sigma\prod_{\substack{i<j\\ \sigma^{-1}(i)>\sigma^{-1}(j)}}q^{-\gamma}\Big(\frac{t_j}{t_i}\Big)^{2\gamma}\frac{\vartheta(q^\gamma t_j/t_i)}{\vartheta(q^{-\gamma}t_j/t_i)}$$

where $\{U_\sigma(t)\}_{\sigma\in\mathfrak{S}_n}$ satisfies the *one cocycle* condition $U_{\sigma\sigma'}(t)=U_\sigma(t)\cdot\sigma U_{\sigma'}(t)$.

Definition 1.1. For an arbitrary point $\xi=(\xi_1,\dots,\xi_n)\in(\mathbb{C}^*)^n$ and a function $f(t)$ of $t=(t_1,\dots,t_n)\in(\mathbb{C}^*)^n$, we define the *Jackson integral of $f(t)$ over the lattice $\langle\xi\rangle$* as follows:

$$\int_{\langle\xi\rangle}f(t)\,\frac{d_qt_1}{t_1}\wedge\cdots\wedge\frac{d_qt_n}{t_n}:=(1-q)^n\sum_{(\nu_1,\dots,\nu_n)\in\mathbb{Z}^n}f(q^{\nu_1}\xi_1,\dots,q^{\nu_n}\xi_n),\tag{1.2}$$

where

$$\langle\xi\rangle:=\{(\xi_1q^{\nu_1},\dots,\xi_nq^{\nu_n})\in(\mathbb{C}^*)^n\,;\,\nu_i\in\mathbb{Z}\ (i=1,\dots,n)\}.$$

We simply write $\varpi=\frac{d_qt_1}{t_1}\wedge\cdots\wedge\frac{d_qt_n}{t_n}$. For any function $\varphi(t)$ we use $\widetilde{\varphi}$ for the Jackson integral defined as

$$\widetilde{\varphi}=\widetilde{\varphi}(\xi):=\int_{\langle\xi\rangle}\Phi(t)\varphi(t)\varpi.$$

1.2. **Matsuo's basis and q-KZ equation.** Let L denote a set of multi-indeces as

$$L = \{(l_1, \dots, l_m) \in \mathbb{Z}_{\geq 0};\ l_1 + \cdots + l_m = n\}.$$

For $l = (l_1, \dots, l_m) \in L$, let $\psi_l(t)$ be the rational functions introduced by Matsuo in [22] as follows:

$$\begin{aligned}\psi_l(t) &= \psi_{(l_1,\dots,l_m)}\begin{bmatrix} x_1, \dots, x_m \\ \beta_1, \dots, \beta_m \end{bmatrix}(t_1, \dots, t_n) \\ &= \mathcal{A}\left[\prod_{j=1}^{m}\left[\prod_{k=1}^{l_j+\cdots+l_m} \frac{1 - q^{\beta_{j-1}} t_k/x_{j-1}}{1 - t_k/x_j}\right] \cdot \prod_{1\leq i<j\leq n} (t_i - q^{-\gamma} t_j)\right]\end{aligned}$$

where $q^{\beta_0} = 0$ and $\mathcal{A}$ is an alternating sum such that

$$\mathcal{A}g(t) := \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sgn}\sigma \cdot \sigma g(t).$$

Let T_j denote the q-shift operator defined by

$$T_j F(x_1, \dots, x_m) = F(x_1, \dots, qx_j, \dots, x_m).$$

We can consider a system of linear ordinary q-difference equations for a vector

$$\boldsymbol{y}(x_1, \dots, x_m) = (\widetilde{\psi}_l)_{l\in L}$$

in a tensor coordinate $(z_l)_{l\in L}$ as follows:

$$T_j \boldsymbol{y}(x_1, \dots, x_m) = \boldsymbol{y}(x_1, \dots, x_m) K_j(x_1, \dots, x_m) \qquad (1 \leq j \leq m),$$

where $K_j(x_1, \dots, x_m)$ is a suitable matrix function of order $\#L = \binom{n+m-1}{m-1}$.

Theorem 1.2 (Matsuo [22], Varchenko [26])**.** *The matrix $K_j(x_1, \dots, x_m)$ is expressed as*

$$\begin{aligned}&K_j(x_1, \dots, x_m) \\ &\quad = R_{j,j+1}(\tfrac{x_j}{x_{j+1}}) R_{j,j+2}(\tfrac{x_j}{x_{j+2}}) \cdots R_{j,m}(\tfrac{x_j}{x_m})\ D_j\ R_{j,1}(\tfrac{qx_j}{x_1}) R_{j,2}(\tfrac{qx_j}{x_2}) \cdots R_{j,j-1}(\tfrac{qx_j}{x_{j-1}})\end{aligned}$$

where D_j is a diagonal matrix defined by

$$(z_l)_{l\in L} D_j := (q^{(\tilde{\alpha}-(n-1)\gamma)l_j} z_l)_{l\in L}$$

and $R_{i,j}(\frac{x_i}{x_j}) = \bigoplus_{\nu=1}^{n} \big[R^{(\nu)}_{i,j}(\frac{x_i}{x_j})\big]^{\binom{\nu+m-1}{\nu}}$ changes the basis $\{\psi_l\}_{l\in L}$ according to the following rule:

$$\left(\psi_{(l_{\sigma(1)},\dots,l_{\sigma(i)},l_{\sigma(i+1)},\dots,l_{\sigma(m)})}\begin{bmatrix} x_{\sigma(1)}, \dots, x_{\sigma(i)}, x_{\sigma(i+1)}, \dots, x_{\sigma(m)} \\ \beta_{\sigma(1)}, \dots, \beta_{\sigma(i)}, \beta_{\sigma(i+1)}, \dots, \beta_{\sigma(m)} \end{bmatrix}(t)\right)_{l_{\sigma(i)}+l_{\sigma(i+1)}=\nu} R^{(\nu)}_{\sigma(i),\sigma(i+1)}(\tfrac{x_{\sigma(i)}}{x_{\sigma(i+1)}})$$

$$= \left(\psi_{(l_{\sigma(1)},\dots,l_{\sigma(i+1)},l_{\sigma(i)},\dots,l_{\sigma(m)})}\begin{bmatrix} x_{\sigma(1)}, \dots, x_{\sigma(i+1)}, x_{\sigma(i)}, \dots, x_{\sigma(m)} \\ \beta_{\sigma(1)}, \dots, \beta_{\sigma(i+1)}, \beta_{\sigma(i)}, \dots, \beta_{\sigma(m)} \end{bmatrix}(t)\right)_{l_{\sigma(i+1)}+l_{\sigma(i)}=\nu}$$

for $\sigma \in \mathfrak{S}_m$.

Remark 1.3. The matrix $R^{(\nu)}_{i,j}(\frac{x_i}{x_j})$ of rank $\nu+1$ coincides with the matrix $R_{i,j}(\frac{x_i}{x_j})$ for $\Phi_{\nu,2}(t)$ (see [26, Theorem 3.5.10]). Therefore, in order to get an explicit expression of q-difference equations for $\Phi_{n,m}(t)$ it suffices to know the matrix $R_{i,j}(\frac{x_i}{x_j})$ in the case $m=2$. From now on we will consider $m=2$.

1.3. **Gauss decomposition of R-matrix.** When $m=2$ we have

$$\Phi(t)=\Phi_{n,2}(t):=t_1^{\alpha_1}\cdots t_n^{\alpha_n}\prod_{j=1}^{n}\frac{(t_j/x_1)_\infty}{(t_jq^{\beta_1}/x_1)_\infty}\frac{(t_j/x_2)_\infty}{(t_jq^{\beta_2}/x_2)_\infty}\prod_{1\le i<j\le n}\frac{(q^{1-\gamma}t_j/t_i)_\infty}{(q^{\gamma}t_j/t_i)_\infty}$$

where $\alpha_j=\tilde{\alpha}-n+j-2(j-1)\gamma$. Let τ be the operation which exchanges x_1, β_1 for x_2, β_2 respectively. We set

$$\psi_s(t):=\psi_{(s,n-s)}\begin{bmatrix}x_1,x_2\\ \beta_1,\beta_2\end{bmatrix}(t),\quad \varphi_s(t):=\tau\psi_{(s,n-s)}=\psi_{(s,n-s)}\begin{bmatrix}x_2,x_1\\ \beta_2,\beta_1\end{bmatrix}(t) \tag{1.3}$$

for $0\le s\le n$. These two bases are connected by the matrix $R_{i,j}(\frac{x_i}{x_j})$ via

$$(\psi_n(t),\psi_{n-1}(t),\dots,\psi_0(t))R_{1,2}(\tfrac{x_1}{x_2})=(\varphi_0(t),\varphi_1(t),\dots,\varphi_n(t))$$

and

$$(\varphi_0(t),\varphi_1(t),\dots,\varphi_n(t))R_{2,1}(\tfrac{x_2}{x_1})=(\psi_n(t),\psi_{n-1}(t),\dots,\psi_0(t)),$$

so that

$$R_{1,2}(\tfrac{x_1}{x_2})R_{2,1}(\tfrac{x_2}{x_1})=I\quad\text{and}\quad R_{1,2}(\tfrac{x_1}{x_2})=J\ \tau R_{2,1}(\tfrac{x_2}{x_1})\ J, \tag{1.4}$$

where I is the identity matrix and J is the matrix $(\delta_{i,n-j})_{i,j=0}^n$. The q-KZ equations are

$$T_1(\widetilde{\psi}_n,\widetilde{\psi}_{n-1},\dots,\widetilde{\psi}_0)=(\widetilde{\psi}_n,\widetilde{\psi}_{n-1},\dots,\widetilde{\psi}_0)K_1(x_1,x_2), \tag{1.5}$$

$$T_2(\widetilde{\psi}_n,\widetilde{\psi}_{n-1},\dots,\widetilde{\psi}_0)=(\widetilde{\psi}_n,\widetilde{\psi}_{n-1},\dots,\widetilde{\psi}_0)K_2(x_1,x_2). \tag{1.6}$$

Theorem 1.4 (Matsuo [22])**.** *The matrices $K_1(x_1,x_2)$ and $K_2(x_1,x_2)$ are expressed as*

$$K_1(x_1,x_2)=R_{1,2}(\tfrac{x_1}{x_2})D_1\quad\textit{and}\quad K_2(x_1,x_2)=D_2R_{2,1}(\tfrac{qx_2}{x_1}),$$

where $D_1=\mathrm{diag}[q^{(\tilde{\alpha}-(n-1)\gamma)(n-s)}]_{s=0}^n$ and $D_2=JD_1J$.

By (1.3), the expression (1.5) is equal to the following:

$$\begin{aligned}T_2(\widetilde{\varphi}_0,\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n)&=\tau T_1(\widetilde{\psi}_n,\widetilde{\psi}_{n-1},\dots,\widetilde{\psi}_0)J=\tau(\widetilde{\psi}_n,\widetilde{\psi}_{n-1},\dots,\widetilde{\psi}_0)\tau K_1(x_1,x_2)J\\&=(\widetilde{\varphi}_0,\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n)\ J\tau R_{1,2}(\tfrac{x_1}{x_2})J\ JD_1J\\&=(\widetilde{\varphi}_0,\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n)R_{2,1}(\tfrac{x_2}{x_1})D_2.\end{aligned} \tag{1.7}$$

Remark 1.5. The expressions (1.5) and (1.6) are essentially the same because, by using (1.7), the equation (1.6) is deduced from (1.5).

Therefore we now take a basis $\{\varphi_s(t); 0\le s\le n\}$ with

$$\varphi_s(t)=\mathcal{A}\left[\prod_{k=1}^{n}\frac{1}{1-t_k/x_2}\prod_{k=1}^{n-s}\frac{1-q^{\beta_2}t_k/x_2}{1-t_k/x_1}\prod_{1\le i<j\le n}(t_i-q^{-\gamma}t_j)\right] \tag{1.8}$$

and a q-difference system

$$T_2(\widetilde{\varphi}_0,\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n)=(\widetilde{\varphi}_0,\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n)K(x_1,x_2). \tag{1.9}$$

From (1.7), we have $K(x_1,x_2)=R_{2,1}(\frac{x_2}{x_1})D_2$ and $D_2=\mathrm{diag}[q^{(\tilde{\alpha}-(n-1)\gamma)s}]_{s=0}^n$.

Theorem 1.6. *The matrix $R_{2,1}(\frac{x_2}{x_1})$ admits the following Gauss decomposition:*

$$R_{2,1}(\tfrac{x_2}{x_1}) = U_R \cdot D_R \cdot L_R,$$

where $U_R = (u_{R,rs})$, $s \geq r$, is an upper triangular matrix,

$$u_{R,rs} = (-1)^{s-r}(\tfrac{x_2}{x_1}q^{-\beta_2})^{(s-r)}q^{-(s-r)(s+r-1)\gamma/2} \cdot \frac{(q^\gamma;q^\gamma)_s}{(q^\gamma;q^\gamma)_{s-r}\cdot(q^\gamma;q^\gamma)_r} \cdot \frac{(q^{\beta_1+(n-s)\gamma};q^\gamma)_{s-r}}{(\frac{x_2}{x_1}q^{\beta_1-\beta_2+(n-2s+1)\gamma};q^\gamma)_{s-r}},$$

$D_R = \mathrm{diag}[d_{R,0},\ldots,d_{R,n}]$,

$$d_{R,r} = q^{-(n-r)\beta_2-r(n-r)\gamma}\cdot\frac{(\frac{x_2}{x_1}q^{\beta_1};q^\gamma)_{n-r}}{(\frac{x_2}{x_1}q^{-\beta_2-(r-1)\gamma};q^\gamma)_r}\cdot\frac{(\frac{x_2}{x_1}q^{\beta_1-\beta_2+(n-2r+1)\gamma};q^\gamma)_r}{(\frac{x_2}{x_1}q^{\beta_1-\beta_2-r\gamma};q^\gamma)_{n-r}},$$

and $L_R = (l_{R,rs})$, $r \geq s$, is a lower triangular matrix,

$$l_{R,rs} = (-1)^{r-s}q^{\beta_2(s-r)}q^{-(r-s)(r+s-1)\gamma/2} \cdot\frac{(q^\gamma;q^\gamma)_{n-s}}{(q^\gamma;q^\gamma)_{r-s}\cdot(q^\gamma;q^\gamma)_{n-r}}\cdot\frac{(q^{\beta_2+s\gamma};q^\gamma)_{r-s}}{(\frac{x_2}{x_1}q^{\beta_1-\beta_2+(n-2r+1)\gamma};q^\gamma)_{r-s}}.$$

We have another Gauss decomposition expression.

Theorem 1.7. *The matrix $R_{2,1}(\frac{x_2}{x_1})$ admits the following Gauss decomposition:*

$$R_{2,1}(\tfrac{x_2}{x_1}) = L'_R \cdot D'_R \cdot U'_R,$$

where $L'_R = (l'_{R,rs})$, $r \geq s$, is a lower triangular matrix,

$$l'_{R,rs} = (-1)^{r-s}q^{-(r-s)(r+s-1)\gamma/2}\cdot\frac{(q^\gamma;q^\gamma)_{n-s}}{(q^\gamma;q^\gamma)_{r-s}\cdot(q^\gamma;q^\gamma)_{n-r}}\cdot\frac{(q^{\beta_2+s\gamma};q^\gamma)_{r-s}}{(\frac{x_2}{x_1}q^{-(n-2s-1)\gamma};q^\gamma)_{r-s}},$$

$D'_R = \mathrm{diag}[d'_{R,0},\ldots,d'_{R,n}]$,

$$d'_{R,r} = q^{-(n-r)(\beta_2+r\gamma)}\cdot\frac{(\frac{x_2}{x_1}q^{\beta_1};q^\gamma)_r}{(\frac{x_2}{x_1}q^{-\beta_2-(n-r-1)\gamma};q^\gamma)_{n-r}}\cdot\frac{(\frac{x_2}{x_1}q^{-(n-2r-1)\gamma};q^\gamma)_{n-r}}{(\frac{x_2}{x_1}q^{-(n-r)\gamma};q^\gamma)_r},$$

and $U'_R = (u'_{R,rs})$, $s \geq r$ is an upper triangular matrix,

$$u'_{R,rs} = q^{-(s-r)r\gamma}\cdot\frac{(q^\gamma;q^\gamma)_s}{(q^\gamma;q^\gamma)_{s-r}\cdot(q^\gamma;q^\gamma)_r}\cdot\frac{(q^{\beta_1+(n-s)\gamma};q^\gamma)_{s-r}}{(\frac{x_1}{x_2}q^{(n-r-s)\gamma};q^\gamma)_{s-r}}.$$

Proof. See Appendix D. □

Remark 1.8. We can also derive Theorem 1.6 by using the method of the Riemann–Hilbert problem for q-difference equation (1.9) from a connection matrix (see Subsection 4.2).

Remark 1.9. From (1.4), we have two kind of expressions

$$R_{1,2}(\tfrac{x_1}{x_2}) = J\tau U_R J\cdot J\tau D_R J\cdot J\tau L_R J = J\tau L'_R J\cdot J\tau D'_R J\cdot J\tau U'_R J. \tag{1.10}$$

Remark 1.10. From (1.4) and (1.10), we have

$$L_R^{-1} = J\tau U_R J,\quad D_R^{-1} = J\tau D_R J,\quad U_R^{-1} = J\tau L_R J,$$

and

$${U'_R}^{-1} = J\tau L'_R J,\quad {D'_R}^{-1} = J\tau D'_R J,\quad {L'_R}^{-1} = J\tau U'_R J.$$

Corollary 1.11. *The matrix $K(x_1, x_2)$ that represents the q-difference system* (1.9) *is given by*

$$K(x_1, x_2) = U_R \cdot D_R \cdot L_R \cdot \mathrm{diag}[q^{(\tilde{\alpha}-(n-1)\gamma)s}]_{s=0}^{n} \qquad (1.11)$$
$$= L'_R \cdot D'_R \cdot U'_R \cdot \mathrm{diag}[q^{(\tilde{\alpha}-(n-1)\gamma)s}]_{s=0}^{n}.$$

Corollary 1.12. *The determinant of the matrices $R_{2,1}(\frac{x_2}{x_1})$ and $K(x)$ are given by the following expressions:*

$$\det R_{2,1}(\tfrac{x_2}{x_1}) = \prod_{r=0}^{n} q^{-(n-r)(\beta_2+r\gamma)} \frac{(\frac{x_2}{x_1} q^{\beta_1}; q^\gamma)_r}{(\frac{x_2}{x_1} q^{-\beta_2-(r-1)\gamma}; q^\gamma)_r},$$
$$\det K(x_1, x_2) = q^{(\tilde{\alpha}-(n-1)\gamma)n(n+1)/2} \det R_{2,1}(\tfrac{x_2}{x_1}). \qquad (1.12)$$

Proof. By Theorem 1.6 or Theorem 1.7, we have

$$\det R_{2,1}(\tfrac{x_2}{x_1}) = \det D_R \ \ (\text{or } \ = \det D'_R).$$

The result now follows from the following identity:

$$\prod_{r=0}^{n} \frac{(\frac{x_2}{x_1} q^{-(n-2r-1)\gamma}; q^\gamma)_{n-r}}{(\frac{x_2}{x_1} q^{-(n-r)\gamma}; q^\gamma)_r} = \prod_{r=0}^{n} \frac{(\frac{x_2}{x_1} q^{\beta_1-\beta_2+(n-2r+1)\gamma}; q^\gamma)_r}{(\frac{x_2}{x_1} q^{\beta_1-\beta_2-r\gamma}; q^\gamma)_{n-r}} = 1.$$

□

Remark 1.13. Eq. (1.12) was conjectured by Mimachi in [24] and another proof for it was given by Aomoto and Kato in [13].

2. Review of Riemann–Hilbert problem for q-difference equation from connection matrices

We recall the notion of the Riemann–Hilbert problem for q-difference equation following [2, 5, 6, 13].

We consider a linear ordinary q-difference equation for a vector function $\boldsymbol{y}(z) = (y_0(z), \ldots, y_n(z))$, $z \in \mathbb{C}^*$, satisfying

$$\boldsymbol{y}(qz) = \boldsymbol{y}(z) A(z),$$

where $A(z)$ is a suitable rational matrix function of order $n+1$. We now assume the following conditions for the matrix $A(z)$;

- P1) The matrix $A(z)$ is holomorphic at $z = 0$ and $z = \infty$.
- P2) $A(0) = \lim_{z\to 0} A(z)$ and $A(\infty) = \lim_{z\to\infty} A(z)$ are diagonalizable, i.e., there exist matrices C^+, C^- and diagonal matrices $D^+ = \mathrm{diag}[\mu_0, \ldots, \mu_n]$, $D^- = \mathrm{diag}[\mu_0^*, \ldots, \mu_n^*]$ such that

$$A(0) = (C^+)^{-1} q^{D^+} C^+, \qquad (2.1)$$
$$A(\infty) = (C^-)^{-1} q^{D^-} C^-, \qquad (2.2)$$

where $q^{D^+} = \mathrm{diag}[q^{\mu_0}, \ldots, q^{\mu_n}]$ and $q^{D^-} = \mathrm{diag}[q^{\mu_0^*}, \ldots, q^{\mu_n^*}]$.

P3) The diagonal elements of D^+ and D^- satisfy the following *non-resonance* condition:[1]

$$q^{\mu_i-\mu_j} \neq q^{\pm 1}, q^{\pm 2}, \dots,$$
$$q^{\mu_i^*-\mu_j^*} \neq q^{\pm 1}, q^{\pm 2}, \cdots .$$

P4) The matrix $A(z)$ does not depend on q.

Under the conditions P1), P2) and P3), from the classical theorem due to G. D. Birkhoff (see [5, 6]), we know that there exists a unique solution $Y_0(z)$ of the equation

$$Y(qz) = Y(z)A(z) \tag{2.3}$$

such that $Y_0(z)$ satisfies the asymptotic behavior

$$Y_0(z) \sim (C^+)^{-1} z^{D^+} C^+ \quad \text{at} \quad z = 0,$$

and we also know that there exists a unique solution $Y_\infty(z)$ of the equation (2.3) such that $Y_\infty(z)$ satisfies the asymptotic behavior

$$Y_\infty(z) \sim (C^-)^{-1} z^{D^-} C^- \quad \text{at} \quad z = \infty.$$

We call $Y_0(z)$ and $Y_\infty(z)$ the *fundamental solutions* of (2.3) at $z = 0$ and $z = \infty$ respectively. The connection matrix $P(z)$ between the fundamental solutions $Y_0(z)$ and $Y_\infty(z)$ is defined by

$$P(z) := Y_0(z) Y_\infty(z)^{-1}.$$

For an arbitrary matrix $X(q)$ depending on q, we denote the limit for $q \to 0$ as

$$(X)_0 = \lim_{q \to 0} X(q).$$

Theorem 2.1 (Aomoto's lemma)**.** *In addition to* P1), P2) *and* P3)*, under the condition* P4)*, the following limit formula holds:*

$$\big(P(z)\big)_0 = A(0)A(z)^{-1}. \tag{2.4}$$

Proof. See [5, 6]. □

This theorem will play a crucial role in calculating $A(z)$ explicitly in the following sections.

Remark 2.2. In the condition P2), if we can choose the matrices $C^+ = \left(c^+_{rs}\right)^n_{r,s=0}$ and $C^- = \left(c^-_{rs}\right)^n_{r,s=0}$ as triangular matrices, the unipotent matrices $\mathrm{diag}[(c^+_{rr})^{-1}]^n_{r=0} C^+$ and $\mathrm{diag}[(c^-_{rr})^{-1}]^n_{r=0} C^-$ also satisfy P2). Thus we assume that the matrices C^+ and C^- are unipotent if they are triangular.

[1] P3) is a condition that should be referred to as H2) in the reference [5], but the symbol H2) does not appear in [5] due to a typo. It should be noted that equations (2) and (3) in [5] actually correspond to the condition H2).

3. Characteristic cycles and fundamental solutions

Let F_r^{n-r}, $0 \le r \le n$, denote the partition of the set $\{1,\dots,n\}$ into subsets $\{1,\dots,n-r\}$ and $\{n-r+1,\dots,n\}$. Let $\xi_{F_r^{n-r}}=(\xi_1,\dots,\xi_n)$, $\eta_{F_r^{n-r}}=(\eta_1,\dots,\eta_n)$, $\zeta_{F_r^{n-r}}=(\zeta_1,\dots,\zeta_n)$ and $\delta_{F_r^{n-r}}=(\delta_1,\dots,\delta_n)$ be the four points in $(\mathbb{C}^*)^n$ defined by

$$\xi_{F_r^{n-r}}:\begin{cases}\xi_1=x_1, & \xi_2=x_1q^{\gamma}, \quad \dots, \quad \xi_{n-r}=x_1q^{(n-r-1)\gamma},\\ \xi_{n-r+1}=x_2, & \xi_{n-r+2}=x_2q^{\gamma}, \quad \dots, \quad \xi_n=x_2q^{(r-1)\gamma},\end{cases}$$

$$\eta_{F_r^{n-r}}:\begin{cases}\eta_1=x_1q^{-\beta_1}, & \eta_2=x_1q^{-\beta_1-\gamma}, \quad \dots, \quad \eta_{n-r}=x_1q^{-\beta_1-(n-r-1)\gamma},\\ \eta_{n-r+1}=x_2q^{-\beta_2}, & \eta_{n-r+2}=x_2q^{-\beta_2-\gamma}, \quad \dots, \quad \eta_n=x_2q^{-\beta_2-(r-1)\gamma},\end{cases}$$

$$\zeta_{F_r^{n-r}}:\begin{cases}\zeta_1=x_1q^{-\beta_1}, & \zeta_2=x_1q^{-\beta_1-\gamma}, \quad \dots, \quad \zeta_{n-r}=x_1q^{-\beta_1-(n-r-1)\gamma},\\ \zeta_{n-r+1}=x_2, & \zeta_{n-r+2}=x_2q^{\gamma}, \quad \dots, \quad \zeta_n=x_2q^{(r-1)\gamma},\end{cases}$$

$$\delta_{F_r^{n-r}}:\begin{cases}\delta_1=x_1, & \delta_2=x_1q^{\gamma}, \quad \dots, \quad \delta_{n-r}=x_1q^{(n-r-1)\gamma},\\ \delta_{n-r+1}=x_2q^{-\beta_2}, & \delta_{n-r+2}=x_2q^{-\beta_2-\gamma}, \quad \dots, \quad \delta_n=x_2q^{-\beta_2-(r-1)\gamma}.\end{cases}$$

We call the lattices $\langle\xi_{F_r^{n-r}}\rangle$, $\langle\eta_{F_r^{n-r}}\rangle$, $\langle\zeta_{F_r^{n-r}}\rangle$ and $\langle\delta_{F_r^{n-r}}\rangle$, $0\le r\le n$, the *characteristic cycles*. Since the ordinary Jackson integrals over the cycles $\langle\eta_{F_r^{n-r}}\rangle$, $\langle\zeta_{F_r^{n-r}}\rangle$ and $\langle\delta_{F_r^{n-r}}\rangle$ diverge, we have to define the *regularized Jackson integrals* for them as follows:

$$\int_{\langle\eta_{F_r^{n-r}}\rangle}\Phi(t)\varphi(t)\varpi := (1-q)^n\sum_{(\nu_1,\dots,\nu_n)\in\mathbb{Z}^n}\operatorname*{Res}_{\substack{t_1=\eta_1q^{\nu_1},\\ \dots\dots,\\ t_n=\eta_nq^{\nu_n}}}\Phi(t_1,\dots,t_n)\varphi(t_1,\dots,t_n)\,\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_n}{t_n},$$

$$\int_{\langle\zeta_{F_r^{n-r}}\rangle}\Phi(t)\varphi(t)\varpi := (1-q)^n\sum_{(\nu_1,\dots,\nu_n)\in\mathbb{Z}^n}\operatorname*{Res}_{\substack{t_1=\zeta_1q^{\nu_1},\\ \dots\dots,\\ t_{n-r}=\zeta_{n-r}q^{\nu_{n-r}}}}\Phi(t_1,\dots,t_{n-r},q^{\nu_{n-r}}\xi_{n-r},\dots,q^{\nu_n}\zeta_n)$$
$$\cdot\,\varphi(t_1,\dots,t_{n-r},q^{\nu_{n-r}}\xi_{n-r},\dots,q^{\nu_n}\zeta_n)\,\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_{n-r}}{t_{n-r}},$$

and

$$\int_{\langle\delta_{F_r^{n-r}}\rangle}\Phi(t)\varphi(t)\varpi := (1-q)^n\sum_{(\nu_1,\dots,\nu_n)\in\mathbb{Z}^n}\operatorname*{Res}_{\substack{t_{n-r+1}=\delta_{n-r+1}q^{\nu_{n-r+1}},\\ \dots\dots,\\ t_n=\delta_nq^{\nu_n}}}\Phi(q^{\nu_1}\delta_1,\dots,q^{\nu_{n-r}}\delta_{n-r},t_{n-r+1},\dots,t_n)$$
$$\cdot\,\varphi(q^{\nu_1}\delta_1,\dots,q^{\nu_{n-r}}\delta_{n-r},t_{n-r+1},\dots,t_n)\,\frac{dt_{n-r+1}}{t_{n-r+1}}\wedge\cdots\wedge\frac{dt_n}{t_n}.$$

Let $\varphi_s(t)$ be the function defined by (1.8) in Section 1. We consider a linear ordinary q-difference equation for a vector function $\boldsymbol{y}(q^{\tilde{\alpha}})=(\widetilde{\varphi}_0,\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n)$ satisfying

$$\boldsymbol{y}(q^{\tilde{\alpha}+1})=\boldsymbol{y}(q^{\tilde{\alpha}})A(q^{\tilde{\alpha}}).$$

where $A(q^{\tilde{\alpha}})$ is a suitable rational matrix function of order $n+1$. We define the following two matrices

$$Y_\xi := \left(\int_{\langle \xi_{F_r^{n-r}} \rangle} \Phi(t)\varphi_s(t)\varpi \right)_{r,s=0}^{n}, \quad Y_\eta := \left(\int_{\langle \eta_{F_r^{n-r}} \rangle} \Phi(t)\varphi_s(t)\varpi \right)_{r,s=0}^{n}, \tag{3.1}$$

which are solutions of the matrix equation

$$Y(q^{\tilde{\alpha}+1}) = Y(q^{\tilde{\alpha}})A(q^{\tilde{\alpha}}). \tag{3.2}$$

We set $(y)^{\tilde{\alpha}} := y_1^{\tilde{\alpha}} \cdots y_n^{\tilde{\alpha}}$ for $y = (y_1, \ldots, y_n)$. The solutions Y_ξ and Y_η have the following asymptotic behaviors:

$$Y_\xi \sim (q^{\tilde{\alpha}})^{D_A^+} C_A^+ \quad \text{at} \quad \tilde{\alpha} \to +\infty, \tag{3.3}$$

$$Y_\eta \sim (q^{\tilde{\alpha}})^{D_A^-} C_A^- \quad \text{at} \quad \tilde{\alpha} \to -\infty \tag{3.4}$$

where

$$(q^{\tilde{\alpha}})^{D_A^+} = \mathrm{diag}[(\xi_{F_r^{n-r}})^{\tilde{\alpha}}]_{r=0}^n, \quad (q^{\tilde{\alpha}})^{D_A^-} = \mathrm{diag}[(\eta_{F_r^{n-r}})^{\tilde{\alpha}}]_{r=0}^n,$$

and $C_A^+ = (c_{A,rs}^+)_{r,s=0}^n$ and $C_A^- = (c_{A,rs}^-)_{r,s=0}^n$ are matrices not depending on $q^{\tilde{\alpha}}$ defined by

$$c_{A,rs}^+ := (1-q)^n \frac{\Phi(\xi_{F_r^{n-r}})\varphi_s(\xi_{F_r^{n-r}})}{(\xi_{F_r^{n-r}})^{\tilde{\alpha}}}, \quad c_{A,rs}^- := (1-q)^n \operatorname*{Res}_{t=\eta_{F_r^{n-r}}} \frac{\Phi(t)\varphi_s(t)}{(t)^{\tilde{\alpha}}} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}. \tag{3.5}$$

Since

$$q^{D_A^+} = \mathrm{diag}[x_1^{n-r} x_2^r \, q^{[r(r-1)+(n-r)(n-r-1)]\gamma/2}]_{r=0}^n,$$

$$q^{D_A^-} = \mathrm{diag}[x_1^{n-r} x_2^r \, q^{-(n-r)\beta_1 - r\beta_2 - [r(r-1)-(n-r)(n-r-1)]\gamma/2}]_{r=0}^n,$$

the condition P3) is satisfied for generic parameters x_1, x_2, β_1, β_2 and γ.

Proposition 3.1. *The matrix C_A^+ is lower triangular. The nonzero elements of the matrix $(C_A^+)_0$ are the following:*

$$(c_{A,rr}^+)_0 = q^{-[r(r-1)+(n-r)(n-r-1)]\gamma/2} \cdot \frac{(\frac{x_2}{x_1} q^{-(n-r)\gamma}; q^\gamma)_r \cdot (\frac{x_2}{x_1} q^{-(n-r-1)\gamma}; q^\gamma)_r}{(q^{\beta_1}; q^\gamma)_{n-r} \cdot (\frac{x_2}{x_1} q^{\beta_1}; q^\gamma)_r \cdot (q^{\beta_2}; q^\gamma)_r \cdot (\frac{x_2}{x_1} q^\gamma; q^\gamma)_r},$$

$$\left(\frac{c_{A,rs}^+}{c_{A,rr}^+}\right)_0 = \frac{c_{A,rs}^+}{c_{A,rr}^+} = q^{-(r-s)(n-r)\gamma} \frac{(q^\gamma; q^\gamma)_{n-s}}{(q^\gamma; q^\gamma)_{r-s}(q^\gamma; q^\gamma)_{n-r}} \frac{(q^{\beta_2+s\gamma}; q^\gamma)_{r-s}}{(\frac{x_2}{x_1} q^{(r+s-n)\gamma}; q^\gamma)_{r-s}} \quad \textit{for} \quad r \geq s.$$

Proof. See Appendix B. □

Proposition 3.2. *The matrix C_A^- is upper triangular. The non-zero elements of the matrix $(C_A^-)_0$ are the following:*

$$(c^-_{A,rr})_0 = (-1)^n \frac{(q^{-\gamma};q^{-\gamma})_{n-r}\cdot(q^{-\gamma};q^{-\gamma})_r}{(1-q^{-\gamma})^n}\cdot\frac{(\frac{x_2}{x_1}q^{-\beta_2-(r-1)\gamma};q^\gamma)_r}{(\frac{x_2}{x_1}q^{\beta_1-\beta_2-(r-1)\gamma};q^\gamma)_r}$$
$$\cdot\frac{1}{{}_n(\frac{x_2}{x_1}q^{\beta_1-\beta_2-(r-1)\gamma};q^\gamma)_r\cdot{}_n(\frac{x_2}{x_1}q^{\beta_1-\beta_2-r\gamma};q^\gamma)_r},$$

$$\left(\frac{c^-_{A,rs}}{c^-_{A,rr}}\right)_0 = \frac{c^-_{A,rs}}{c^-_{A,rr}}$$
$$= (\tfrac{x_2}{x_1}q^{-\beta_2-r\gamma})^{s-r}\frac{(q^{\beta_1+(n-s)\gamma};q^\gamma)_{s-r}}{(\frac{x_2}{x_1}q^{\beta_1-\beta_2+(n-s-r)\gamma};q^\gamma)_{s-r}}\cdot\frac{(q^\gamma;q^\gamma)_s}{(q^\gamma;q^\gamma)_r\cdot(q^\gamma;q^\gamma)_{s-r}}\quad \textit{for}\quad r\le s.$$

Proof. See Appendix C. □

If we define

$$Y_0 := (C^+_A)^{-1}\,Y_\xi,\quad Y_\infty := (C^-_A)^{-1}\,Y_\eta,$$

then the matrices $Y_0 = Y_0(q^{\tilde\alpha})$ and $Y_\infty = Y_\infty(q^{\tilde\alpha})$ are also solutions of the equation (3.2) and satisfy the following asymptotic behaviors:

$$Y_0(q^{\tilde\alpha}) \sim (C^+_A)^{-1}(q^{\tilde\alpha})^{D^+_A}C^+_A \quad \text{at} \quad \tilde\alpha\to+\infty,$$
$$Y_\infty(q^{\tilde\alpha}) \sim (C^-_A)^{-1}(q^{\tilde\alpha})^{D^-_A}C^-_A \quad \text{at} \quad \tilde\alpha\to-\infty.$$

This implies that the matrix $A(q^{\tilde\alpha})$ satisfies the condition P1) and P2) for the unipotent matrices

$$C^+ = \left(\frac{c^+_{A,rs}}{c^+_{A,rr}}\right)^n_{r,s=0},\quad C^- = \left(\frac{c^-_{A,rs}}{c^-_{A,rr}}\right)^n_{r,s=0}$$

in (2.1) and (2.2), i.e.,

$$A(0) = (C^+_A)^{-1}q^{D^+_A}C^+_A = (C^+)^{-1}q^{D^+_A}C^+, \tag{3.6}$$
$$A(\infty) = (C^-_A)^{-1}q^{D^-_A}C^-_A = (C^-)^{-1}q^{D^-_A}C^-. \tag{3.7}$$

Let $G = G(\tilde\alpha,\beta_1,\beta_2,x_1,x_2)$ be the connection matrix between Y_ξ and Y_η defined by

$$G := Y_\xi\,Y_\eta^{-1}, \tag{3.8}$$

which coincides with what Aomoto–Kato called the *principal connection matrix* (see Section 4), and then

$$Y_0(q^{\tilde\alpha})Y_\infty(q^{\tilde\alpha})^{-1} = (C^+_A)^{-1}\,Y_\xi\,Y_\eta^{-1}\,C^-_A = (C^+_A)^{-1}\,G\,C^-_A. \tag{3.9}$$

Since the conditions P1), P2) and P3) are satisfied, by (3.9) and Aomoto's lemma (2.4), we have

$$A(0)A(q^{\tilde\alpha})^{-1} = \Big((C^+_A)^{-1}\,G\,C^-_A\Big)_0$$

if the condition P4) holds for $A(q^{\tilde\alpha})$. In Appendix §A, we will see that the condition P4) holds for $A(q^{\tilde\alpha})$. From (3.6), we have

$$A(q^{\tilde\alpha}) = \Big((C^-_A)^{-1}\,G^{-1}\,q^{D^+_A}\,C^+_A\Big)_0. \tag{3.10}$$

4.1. **Principal connection matrix.** The set $\{\langle \xi_{F_r^{n-r}}\rangle; 0 \le r \le n\}$ makes up a basis of the dual space of $H^n(\bar{X}, \Phi, \nabla)_{sym}$ (see the definition in [12]). For an arbitrary $\xi \in \mathbb{C}^*$, $\langle \xi \rangle$ is expressed uniquely as a linear combination of $\langle \xi_{F_r^{n-r}}\rangle$, $0 \le r \le n$, in such a way that

$$\langle \xi \rangle = \sum_{r=0}^{n} c_r \cdot \langle \xi_{F_r^{n-r}} \rangle \tag{4.1}$$

for some *pseudo-constants* c_r in the sense that they do not change under the displacement $\tilde{\alpha} \mapsto \tilde{\alpha}+1$, $\beta_k \mapsto \beta_k + 1$, $\gamma \mapsto \gamma + 1$ and $x \mapsto qx$. We denote the coefficient c_r by $\big(\langle \xi \rangle; \langle \xi_{F_r^{n-r}} \rangle\big)_\Phi$. Namely, (4.1) means that

$$\int_{\langle \xi \rangle} \Phi(t)\varphi(t)\varpi = \sum_{r=0}^{n} \big(\langle \xi \rangle; \langle \xi_{F_r^{n-r}} \rangle\big)_\Phi \cdot \int_{\langle \xi_{F_r^{n-r}} \rangle} \Phi(t)\varphi(t)\varpi. \tag{4.2}$$

For the other bases $\{\langle \eta_{F_r^{n-r}}\rangle; 0 \le r \le n\}$, $\{\langle \zeta_{F_r^{n-r}}\rangle; 0 \le r \le n\}$ and $\{\langle \delta_{F_r^{n-r}}\rangle; 0 \le r \le n\}$, we define the coefficients $\big(\langle \xi \rangle; \langle \eta_{F_r^{n-r}} \rangle\big)_\Phi$, $\big(\langle \xi \rangle; \langle \zeta_{F_r^{n-r}} \rangle\big)_\Phi$ and $\big(\langle \xi \rangle; \langle \delta_{F_r^{n-r}} \rangle\big)_\Phi$ in the same manner as above. By (4.2) and (3.8), the elements of the principal connection matrix $G = (g_{rs})_{r,s=0}^n$ are written as follows:

$$g_{rs} = \big(\langle \xi_{F_r^{n-r}} \rangle; \langle \eta_{F_s^{n-s}} \rangle\big)_\Phi = \sum_{i=0}^{n} \big(\langle \xi_{F_r^{n-r}} \rangle; \langle \zeta_{F_i^{n-i}} \rangle\big)_\Phi \cdot \big(\langle \zeta_{F_i^{n-i}} \rangle; \langle \eta_{F_s^{n-s}} \rangle\big)_\Phi.$$

Theorem 4.1 (Gauss decomposition [12])**.** *The principal connection matrix G admits the following Gauss decomposition:*

$$G = (H_{\langle \zeta;\xi \rangle})^{-1}\ H_{\langle \zeta;\eta \rangle}, \tag{4.3}$$

where the matrices $H_{\langle \zeta;\xi \rangle} = \big(h_{rs}^{++}\big)_{r,s=0}^n$ and $H_{\langle \zeta;\eta \rangle} = \big(h_{rs}^{+-}\big)_{r,s=0}^n$ are defined by

$$h_{rs}^{++} := \big(\langle \zeta_{F_r^{n-r}} \rangle; \langle \xi_{F_s^{n-s}} \rangle\big)_\Phi, \quad h_{rs}^{+-} := \big(\langle \zeta_{F_r^{n-r}} \rangle; \langle \eta_{F_s^{n-s}} \rangle\big)_\Phi.$$

The matrices $H_{\langle \zeta;\xi \rangle}$ and $H_{\langle \zeta;\eta \rangle}$ become an upper triangular matrix and a lower one respectively. An arbitrary element of the matrices $H_{\langle \zeta;\xi \rangle}$, $H_{\langle \zeta;\eta \rangle}$, $(H_{\langle \zeta;\xi \rangle})^{-1}$ and $(H_{\langle \zeta;\eta \rangle})^{-1}$ is expressed in a theta product form.

Proof. See Theorem 8.3 in [12]. □

Since we have already known the explicit form of $H_{\langle \zeta;\xi \rangle}$ and $H_{\langle \zeta;\eta \rangle}$ (see Theorem 8.1, 8.2 in [12] and Lemma 13 in [13]), in particular, we have

$$\begin{aligned}(h_{rr}^{++})_0 = (-1)^{n-r} &\frac{(q^\gamma;q^\gamma)_{n-r} \cdot (q^{\beta_1};q^\gamma)_{n-r} \cdot (\frac{x_1}{x_2}q^{\beta_2};q^\gamma)_{n-r}}{(1-q^\gamma)^{n-r} \cdot (\frac{x_1}{x_2};q^\gamma)_{n-r}} \\ &\cdot \frac{(\frac{x_1}{x_2}q^{-\beta_1-(n-r-1)\gamma};q^\gamma)_{n-r} \cdot (q^{\tilde{\alpha}+\beta_2-2(n-r-1)\gamma};q^\gamma)_{n-r}}{(\frac{x_1}{x_2}q^{\beta_2-\beta_1-(n-r-1)\gamma};q^\gamma)_{n-r} \cdot (q^{\tilde{\alpha}+\beta_2+\beta_1-(n-r-1)\gamma};q^\gamma)_{n-r}} \\ &\cdot \frac{{}_n(\frac{x_1}{x_2}q^{-(r-1)\gamma};q^\gamma)_r \cdot {}_n(\frac{x_1}{x_2}q^{-r\gamma};q^\gamma)_r}{{}_n(\frac{x_1}{x_2}q^{-(n-r-1)\gamma};q^\gamma)_r \cdot {}_n(\frac{x_1}{x_2}q^{-(n-r)\gamma};q^\gamma)_r},\end{aligned} \tag{4.4}$$

$$\left(\frac{h^{++}_{rs}}{h^{++}_{rr}}\right)_0 = q^{-r(s-r)\gamma}\frac{{}_n(\frac{x_1}{x_2}q^{-(n-r-1)\gamma};q^\gamma)_r\cdot{}_n(\frac{x_1}{x_2}q^{-(n-r)\gamma};q^\gamma)_r\cdot{}_n(\frac{x_1}{x_2}q^{-(s-1)\gamma};q^\gamma)_s}{{}_n(\frac{x_1}{x_2}q^{-(n-s-1)\gamma};q^\gamma)_s\cdot{}_n(\frac{x_1}{x_2}q^{-(n-s)\gamma};q^\gamma)_s\cdot{}_n(\frac{x_1}{x_2}q^{-(r-1)\gamma};q^\gamma)_r}$$
$$\cdot\frac{(\frac{x_1}{x_2}q^{-r\gamma};q^\gamma)_{n-r}}{(\frac{x_1}{x_2}q^{-s\gamma};q^\gamma)_{n-s}}\cdot\frac{(q^{\beta_2+r\gamma};q^\gamma)_{s-r}}{(\frac{x_1}{x_2}q^{\beta_2+(n-s)\gamma};q^\gamma)_{s-r}}$$
$$\cdot\frac{(\frac{x_2}{x_1}q^{-\tilde\alpha-\beta_2+(n-1)\gamma};q^\gamma)_{s-r}\cdot{}_s(q^\gamma;q^\gamma)_r}{(q^{-\tilde\alpha-\beta_2+(2n-r-s-1)\gamma};q^\gamma)_{s-r}\cdot(\frac{x_2}{x_1}q^{(r+s-n)\gamma};q^\gamma)_{s-r}}\quad\text{for}\quad s\ge r,\tag{4.5}$$

and

$$(h^{+-}_{rr})_0 = (-1)^r\frac{(1-q^\gamma)^r\cdot(q^{\tilde\alpha+\beta_2-(2n-r-1)\gamma};q^\gamma)_r}{(q^\gamma;q^\gamma)_r\cdot(q^{\tilde\alpha-2(n-1)\gamma};q^\gamma)_r\cdot(q^{\beta_2};q^\gamma)_r},\tag{4.6}$$

$$\left(\frac{h^{+-}_{rs}}{h^{+-}_{rr}}\right)_0 = q^{-s(r-s)\gamma}\frac{(q^{\tilde\alpha+\beta_1-(n-1)\gamma}\frac{x_2}{x_1};q^\gamma)_{r-s}\cdot(q^{\tilde\alpha+\beta_2-(2n-r-1)\gamma};q^\gamma)_s}{(q^{\tilde\alpha+\beta_2-(2n-r-1)\gamma};q^\gamma)_r}$$
$$\cdot\frac{(q^{\beta_2};q^\gamma)_r\cdot(q^{\beta_1-\beta_2+(n-r-s)\gamma}\frac{x_2}{x_1};q^\gamma)_s}{(q^{\beta_2};q^\gamma)_s\cdot(q^{\beta_1+(n-r)\gamma}\frac{x_2}{x_1};q^\gamma)_{r-s}}{}_r(q^\gamma;q^\gamma)_s\quad\text{for}\quad r\ge s.\tag{4.7}$$

Let τ be the operation which exchanges x_1, β_1 for x_2, β_2.

Theorem 4.2 (Quasi-symmetry of second kind [12])**.** *Under the action of τ, the principal connection matrix $G = G(\tilde\alpha,\beta_1,\beta_2,x_1,x_2)$ changes as follows:*

$$\begin{aligned}\tau G(\tilde\alpha,\beta_1,\beta_2,x_1,x_2) &= G(\tilde\alpha,\beta_2,\beta_1,x_2,x_1)\\ &= S(x_2/x_1)\ G(\tilde\alpha,\beta_1,\beta_2,x_1,x_2)\ {}^tS(q^{\beta_2-\beta_1}x_1/x_2),\end{aligned}\tag{4.8}$$

where we put $S(x) := \big(a_{r,s}(x)\,\delta_{r,s-n}\big)_{r,s=0}^n$ and

$$a_{r,n-r}(x) := x^{2r(n-r)\gamma}q^{-r(n-r)\gamma+r(n-r)(n-2r)\gamma^2}\cdot{}_n\vartheta(xq^{-(r-1)\gamma})_r\cdot{}_n\vartheta(xq^{-r\gamma})_r.$$

Proof. See Theorem 5.2 in [12]. □

Proposition 4.3. *The principal connection matrix G admits the following Gauss decomposition:*

$$G = (H_{\langle\delta;\xi\rangle})^{-1}\ H_{\langle\delta;\eta\rangle}$$

where $H_{\langle\delta;\xi\rangle}$ and $H_{\langle\delta;\eta\rangle}$ are a lower triangular matrix and an upper one respectively defined by

$$\begin{aligned}H_{\langle\delta;\xi\rangle} &:= S'(x_2/x_1)\ \tau H_{\langle\zeta;\xi\rangle}\ {}^tS(x_2/x_1)\\ H_{\langle\delta;\eta\rangle} &:= S'(x_2/x_1)\ \tau H_{\langle\zeta;\eta\rangle}\ {}^tS(q^{\beta_1-\beta_2}x_2/x_1),\end{aligned}$$

where $S'(x) := \big(a_{r,s}(xq^{-\beta_2-(s-1)\gamma})\cdot\delta_{r,s-n}\big)_{r,s=0}^n$.

Proof. From (4.8), we have $\tau G = S(x_2/x_1)\ G\ {}^tS(q^{\beta_2-\beta_1}x_1/x_2)$. Then,

$$G = \tau\tau G = \tau S(x_2/x_1)\,\tau G\,\tau\,{}^tS(q^{\beta_2-\beta_1}x_1/x_2) = S(x_1/x_2)\,\tau G\,{}^tS(q^{\beta_1-\beta_2}x_2/x_1).\tag{4.9}$$

On the other hand, by (4.3), we have

$$\begin{aligned}&(H_{\langle\delta;\xi\rangle})^{-1}\ H_{\langle\delta;\eta\rangle}\\ &= \Big(S'(x_2/x_1)\ \tau H_{\langle\zeta;\xi\rangle}\ {}^tS(x_2/x_1)\Big)^{-1}S'(x_2/x_1)\ \tau H_{\langle\zeta;\eta\rangle}\ {}^tS(q^{\beta_1-\beta_2}x_2/x_1)\end{aligned}$$

$$
\begin{aligned}
&= {}^tS(x_2/x_1)^{-1}\ (\tau H_{\langle\zeta;\xi\rangle})^{-1}\ S'(x_2/x_1)^{-1}\ S'(x_2/x_1)\ \tau H_{\langle\zeta;\eta\rangle}\ {}^tS(q^{\beta_1-\beta_2}x_2/x_1)\\
&= S(x_1/x_2)\ \tau(H_{\langle\zeta;\xi\rangle})^{-1}\ \tau H_{\langle\zeta;\eta\rangle}\ {}^tS(q^{\beta_1-\beta_2}x_2/x_1)\\
&= S(x_1/x_2)\ \tau G\ {}^tS(q^{\beta_1-\beta_2}x_2/x_1). \qquad (4.10)
\end{aligned}
$$

The proposition now follows from (4.9) and (4.10). □

From Proposition 4.3, the elements of the matrices $H_{\langle\delta;\eta\rangle} = \big(h^{--}_{rs}\big)^n_{r,s=0}$ and $H_{\langle\delta;\xi\rangle} = \big(h^{-+}_{rs}\big)^n_{r,s=0}$ are written as follows:

$$
h^{--}_{rs} = \frac{\tau h^{+-}_{n-r,n-s}\cdot a_{s,n-s}(q^{\beta_1-\beta_2}x_2/x_1)}{a_{n-r,r}(q^{-\beta_2-(r-1)\gamma}x_2/x_1)},\ h^{-+}_{rs} = \frac{\tau h^{++}_{n-r,n-s}\cdot a_{s,n-s}(x_2/x_1)}{a_{n-r,r}(q^{-\beta_2-(r-1)\gamma}x_2/x_1)}. \qquad (4.11)
$$

Remark 4.4. In [12], it is proved that the element h^{--}_{rs} is equal to $\big(\langle\delta_{F^{n-r}_r}\rangle;\langle\eta_{F^{n-s}_s}\rangle\big)_\Phi$. Since the elements of the matrix $G=(g_{rs})^n_{r,s=0}$ are expressed as

$$
g_{rs} = \big(\langle\xi_{F^{n-r}_r}\rangle;\langle\eta_{F^{n-s}_s}\rangle\big)_\Phi = \sum_{i=0}^n \big(\langle\xi_{F^{n-r}_r}\rangle;\langle\delta_{F^{n-i}_i}\rangle\big)_\Phi\cdot\big(\langle\delta_{F^{n-i}_i}\rangle;\langle\eta_{F^{n-s}_s}\rangle\big)_\Phi,
$$

we finally have $h^{-+}_{rs} = \big(\langle\delta_{F^{n-r}_r}\rangle;\langle\xi_{F^{n-s}_s}\rangle\big)_\Phi$.

4.2. **A remark on Aomoto–Kato case.** In [13], Aomoto and Kato have also studied the q-difference system (1.9) from the viewpoint of the Riemann–Hilbert method. In this section we will show how to derive Corollary 1.11 by using it.

Since the functions $\Phi_{n,2}(t)$ and $\varphi_s(t)$ are depending on x_1 and x_2, we denote $\Phi_{n,2}(t)$ and $\varphi_s(t)$ by $\Phi(x_1,x_2;t)$ and $\varphi_s(x_1,x_2;t)$ respectively. By the transformation

$$
\int \Phi(x_1,x_2;t)\varphi_s(x_1,x_2;t)\varpi = x_1^{\alpha_1+\cdots+\alpha_n}\int \Phi(1,x_2/x_1;t)\varphi_s(1,x_2/x_1;t)\varpi,
$$

it suffices to consider the q-difference system (1.9) in the case $x_1=1$ and $x_2=x$. For a vector function $\boldsymbol{y}(x) := (\widetilde{\varphi}_0,\widetilde{\varphi}_1,\ldots,\widetilde{\varphi}_n)$ where

$$
\varphi_s(t) = \mathcal{A}\left[\prod_{k=1}^n \frac{1}{1-t_k/x}\prod_{k=1}^{n-s}\frac{1-q^{\beta_2}t_k/x}{1-t_k}\prod_{1\le i<j\le n}(t_i-q^{-\gamma}t_j)\right],
$$

the q-difference system (1.9) is written as

$$
\boldsymbol{y}(qx) = \boldsymbol{y}(x)K(x).
$$

where $K(x) := K(1,x)$. We define the following two matrices

$$
Y_\zeta := \left(\int_{\langle\zeta_{F^{n-r}_r}\rangle}\Phi(t)\varphi_s(t)\varpi\right)^n_{r,s=0},\quad Y_\delta := \left(\int_{\langle\delta_{F^{n-r}_r}\rangle}\Phi(t)\varphi_s(t)\varpi\right)^n_{r,s=0},
$$

where $\zeta_{F^{n-r}_r}$ and $\delta_{F^{n-r}_r}$ is defined in Section 3, and these matrices are solutions of the matrix equation

$$
Y(qx) = Y(x)K(x). \qquad (4.12)
$$

Aomoto and Kato studied in [13] asymptotic behaviors of the solutions Y_ζ and Y_δ:

$$
\begin{aligned}
Y_\zeta &\sim V_+(x)\ x^{D^+_K}C^+_K \quad \text{at}\quad x\to 0,\\
Y_\delta &\sim V_-(x)\ x^{D^-_K}C^-_K \quad \text{at}\quad x\to \infty.
\end{aligned}
$$

where $V_+(x)$ and $V_-(x)$ are pseudo-constant diagonal matrices evaluated in [12] as

$$V_+(x) = \mathrm{diag}[v_r(x)]_{r=0}^n, \quad V_-(x) = \mathrm{diag}[v_r^*(x)]_{r=0}^n,$$

and D_K^+ and D_K^- are diagonal matrices

$$D_K^+ = \mathrm{diag}[r\tilde{\alpha} - (n-r)\beta_2 - r(2n-r-1)\gamma]_{r=0}^n,$$

$$D_K^- = \mathrm{diag}[r\tilde{\alpha} + r\beta_1 - r(r-1)\gamma]_{r=0}^n.$$

The matrices $C_K^+ = (c_{K,rs}^+)_{r,s=0}^n$ and $C_K^- = (c_{K,rs}^-)_{r,s=0}^n$ are a lower triangular matrix and an upper one respectively.

Lemma 4.5 (Aomoto–Kato [13]). *The matrices $(V_+(x))_0$, $(V_-(x))_0$, $(C_K^+)_0$ and $(C_K^-)_0$ are evaluated as follows:*

$$(v_r)_0 = q^{-(n-r)\beta_2} \cdot \frac{(xq^{\beta_1};q^\gamma)_{n-r}}{(xq^{\beta_1-\beta_2};q^\gamma)_{n-r}},$$

$$(v_r^*)_0 = \frac{(-1)^{r(n-r)}q^{r(n-r)\gamma} \cdot (xq^{-\beta_2-(r-1)\gamma};q^\gamma)_r}{(xq^{-\beta_2-(r-1)\gamma};q^\gamma)_r \cdot {}_n(xq^{-\beta_2-(n-2)\gamma};q^\gamma)_r \cdot {}_n(xq^{-\beta_2-(n-1)\gamma};q^\gamma)_r},$$

$$(c_{K,rr}^+)_0 = (-1)^{n-r}q^{(n-r)\beta_2-(n-r)(n-r-1)\gamma/2-r(r-1)\gamma/2} \cdot \frac{(q^\gamma;q^\gamma)_{n-r} \cdot (q^{\tilde{\alpha}+\beta_2-(2n-r-1)\gamma};q^\gamma)_r}{(1-q^\gamma)^{n-r} \cdot (q^{\beta_2};q^\gamma)_r \cdot (q^{\tilde{\alpha}-2(n-1)\gamma};q^\gamma)_r}$$

$$\left(\frac{c_{K,rs}^+}{c_{K,rr}^+}\right)_0 = \frac{c_{K,rs}^+}{c_{K,rr}^+} = q^{(n-r)(s-r)\gamma}\frac{(q^\gamma;q^\gamma)_{n-s}}{(q^\gamma;q^\gamma)_{r-s} \cdot (q^\gamma;q^\gamma)_{n-r}}\frac{(q^{\beta_2+s\gamma};q^\gamma)_{r-s}}{(q^{\tilde{\alpha}+\beta_2-(2n-r-s-1)\gamma};q^\gamma)_{r-s}},$$

$$(c_{K,rr}^-)_0 = (-1)^{r(n-r+1)}q^{-n(n-1)\gamma/2}\frac{(q^\gamma;q^\gamma)_r \cdot (q^{\tilde{\alpha}+\beta_1-(n+r-1)\gamma};q^\gamma)_{n-r}}{(1-q^\gamma)^r \cdot (q^{\beta_1};q^\gamma)_{n-r} \cdot (q^{\tilde{\alpha}-2(n-1)\gamma};q^\gamma)_{n-r}},$$

$$\left(\frac{c_{K,rs}^-}{c_{K,rr}^-}\right)_0 = \frac{c_{K,rs}^-}{c_{K,rr}^-} = q^{(s-r)[\tilde{\alpha}-(n-1)\gamma]+r(r-s)\gamma} \cdot \frac{(q^\gamma;q^\gamma)_s}{(q^\gamma;q^\gamma)_{s-r} \cdot (q^\gamma;q^\gamma)_r}\frac{(q^{\beta_1+(n-s)\gamma};q^\gamma)_{s-r}}{(q^{\tilde{\alpha}+\beta_1-(r+s-1)\gamma};q^\gamma)_{s-r}}. \tag{4.13}$$

If we define

$$Y_0' := (C_K^+)^{-1}\ (V_+(x))^{-1}\ Y_\zeta, \quad Y_\infty' := (C_K^-)^{-1}\ (V_-(x))^{-1}\ Y_\delta,$$

then the matrices $Y_0' = Y_0'(x)$ and $Y_\infty' = Y_\infty'(x)$ are fundamental solutions of the equation (4.12) and satisfy

$$Y_0'(x) \sim (C_K^+)^{-1}x^{D_K^+}C_K^+ \quad \text{at} \quad x \to 0,$$

$$Y_\infty'(x) \sim (C_K^-)^{-1}x^{D_K^-}C_K^- \quad \text{at} \quad x \to \infty.$$

Let $\widehat{P}$ be the connection matrix between Y_ζ and Y_δ defined by

$$\widehat{P} := Y_\zeta\ Y_\delta^{-1},$$

In [12], $\widehat{P}$ is expressed in the form of Gauss decomposition as

$$\widehat{P} := H_{\langle\zeta;\eta\rangle}\ (H_{\langle\delta;\eta\rangle})^{-1}. \tag{4.14}$$

By Aomoto's lemma (2.4), we have

$$K(0)K(x)^{-1} = \big(Y_0'(x)Y_\infty'(x)^{-1}\big)_0 = \big((C_K^+)^{-1}\ (V_+(x))^{-1}\ \widehat{P}\ V_-(x)\ C_K^-\big)_0.$$

From (4.14) and $K(0) = (C_K^+)^{-1} q^{D_K^+} C_K^+$, we have

$$K(x) = \Big((C_K^-)^{-1}\ (V_-(x))^{-1}\ H_{\langle\delta;\eta\rangle}\ H^{-1}_{\langle\zeta;\eta\rangle}\ V_+(x)\ q^{D_K^+}\ C_K^+\Big)_0.$$

Hence, by using (4.6), (4.7), (4.11), Lemma 4.5 and Lemma 5.7 in Section 5, we can evaluate $K(x)$ as (1.11) in Corollary 1.11.

5. Main result for q-difference equations with parameter shift $\tilde\alpha \to \tilde\alpha + 1$

From (3.10) and Proposition 4.3, it follows that

$$A(q^{\tilde\alpha}) = \Big((C_A^-)^{-1}\ (H_{\langle\delta;\eta\rangle})^{-1}\ H_{\langle\delta;\xi\rangle}\ q^{D_A^+}\ C_A^+\Big)_0. \tag{5.1}$$

Since the matrices C_A^- and $H_{\langle\delta;\eta\rangle}$ are upper triangular and the matrices C_A^+ and $H_{\langle\delta;\xi\rangle}$ are lower triangular, we can decompose the matrix $A(q^{\tilde\alpha})$ as the product of lower and upper triangular matrices in the following form:

$$\begin{pmatrix} 1 & u_{01} & u_{02} & \cdots & u_{0n} \\ & 1 & u_{12} & \cdots & u_{1n} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & u_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} d_0 & & & & \\ & d_1 & & & \\ & & \ddots & & \\ & & & d_{n-1} & \\ & & & & d_n \end{pmatrix} \begin{pmatrix} 1 & & & & \\ l_{10} & 1 & & & \\ \vdots & \ddots & \ddots & & \\ l_{n-1,0} & \cdots & l_{n-1,n-1} & 1 & \\ l_{n,0} & \cdots & l_{n,n-2} & l_{n,n-1} & 1 \end{pmatrix}.$$

This expression is unique and we denote by U_A and L_A the above left and right matrices respectively, so that

$$A(q^{\tilde\alpha}) = U_A\ \mathrm{diag}[d_0, \dots, d_n]\ L_A. \tag{5.2}$$

Theorem 5.1. *The elements of U_A, $\mathrm{diag}[d_0, \dots, d_n]$ and L_A are expressed as follows:*

$$u_{rs} = (-1)^{s-r} q^{(s-r)[\tilde\alpha-(n-1)\gamma]-(s-r)(s+r-1)\gamma/2}$$
$$\cdot \frac{(q^\gamma;q^\gamma)_s}{(q^\gamma;q^\gamma)_{s-r}\cdot(q^\gamma;q^\gamma)_r} \frac{(q^{\beta_1+(n-s)\gamma};q^\gamma)_{s-r}}{(q^{\tilde\alpha+\beta_1-2(s-1)\gamma};q^\gamma)_{s-r}} \quad \textit{for} \quad r \le s,$$

$$d_r = q^{\mu_r} \frac{(q^{\tilde\alpha+\beta_1-2(r-1)\gamma};q^\gamma)_r\cdot(q^{\tilde\alpha-2(n-1)\gamma};q^\gamma)_{n-r}}{(q^{\tilde\alpha+\beta_1+\beta_2-(r-1)\gamma};q^\gamma)_r\cdot(q^{\tilde\alpha+\beta_1-(n-1+r)\gamma};q^\gamma)_{n-r}},$$
$$\textit{where } q^{\mu_r} = x_1^{n-r} x_2^r\ q^{[r(r-1)+(n-r)(n-r-1)]\gamma/2},$$

$$l_{rs} = (-1)^{r-s}(x_2/x_1)^{s-r} q^{-(r-s)(r+s-1)\gamma/2}$$
$$\cdot \frac{(q^\gamma;q^\gamma)_{n-s}}{(q^\gamma;q^\gamma)_{r-s}\cdot(q^\gamma;q^\gamma)_{n-r}} \frac{(q^{\beta_2+s\gamma};q^\gamma)_{r-s}}{(q^{\tilde\alpha+\beta_1-2(r-1)\gamma};q^\gamma)_{r-s}} \quad \textit{for} \quad r \ge s.$$

We prove Theorem 5.1 in the following sections. Moreover, separating the matrix depending on x_1 and x_2, we have

Theorem 5.2. *The matrix $A(q^{\tilde{\alpha}})$ is expressed as follows:*

$$A(q^{\tilde{\alpha}}) = \bar{U}_A\ \bar{D}_A\ \bar{L}_A\ \mathrm{diag}[\ x_1^{n-r}x_2^r\ ]_{r=0}^n,$$

where $\bar{U}_A = (\bar{u}_{rs})$ is an upper triangular matrix given by

$$\bar{u}_{rs} = (-1)^{s-r} q^{(s-r)[\tilde{\alpha}-(n-1)\gamma]-(s-r)(s+r-1)\gamma/2}$$

$$\cdot \frac{(q^\gamma;q^\gamma)_s}{(q^\gamma;q^\gamma)_{s-r}\cdot(q^\gamma;q^\gamma)_r} \frac{(q^{\beta_1+(n-s)\gamma};q^\gamma)_{s-r}}{(q^{\tilde{\alpha}+\beta_1-2(s-1)\gamma};q^\gamma)_{s-r}} \quad r \le s,$$

$\bar{D}_A = \mathrm{diag}[\bar{d}_0, \dots, \bar{d}_n]$,

$$\bar{d}_r = q^{[r(r-1)+(n-r)(n-r-1)]\gamma/2} \frac{(q^{\tilde{\alpha}+\beta_1-2(r-1)\gamma};q^\gamma)_r \cdot (q^{\tilde{\alpha}-2(n-1)\gamma};q^\gamma)_{n-r}}{(q^{\tilde{\alpha}+\beta_1+\beta_2-(r-1)\gamma};q^\gamma)_r \cdot (q^{\tilde{\alpha}+\beta_1-(n-1+r)\gamma};q^\gamma)_{n-r}},$$

$\bar{L}_A = (\bar{l}_{rs})$ is a lower triangular matrix given by

$$\bar{l}_{rs} = (-1)^{r-s} q^{-(r-s)(r+s-1)\gamma/2}$$

$$\cdot \frac{(q^\gamma;q^\gamma)_{n-s}}{(q^\gamma;q^\gamma)_{r-s}\cdot(q^\gamma;q^\gamma)_{n-r}} \frac{(q^{\beta_2+s\gamma};q^\gamma)_{r-s}}{(q^{\tilde{\alpha}+\beta_1-2(r-1)\gamma};q^\gamma)_{r-s}} \quad r \ge s.$$

Remark 5.3. If we compare Theorem 5.2 with Corollary 1.11, we find that the matrices $A(q^{\tilde{\alpha}})$ and $K(x_1, x_2)$ are very similar to each other, especially the substitution of q^α into $(x_1/x_2)q^{-\beta_2+(n-1)\gamma}$ transforms $\bar{U}_A$ and $\bar{L}_A$ into U_R and L_R respectively.

Remark 5.4. The elements of $U_A^{-1} = \left(u^*_{rs}\right)_{r,s=0}^n$ coincides with the value $c^-_{K,rs}/c^-_{K,rr}$ of the coefficient matrix C_K^- (Compare (4.13) in Lemma 4.5 with (5.4.2) in Theorem 5.12).

5.1. **Diagonal elements and determinant of $A(q^{\tilde{\alpha}})$.** In this section, we first evaluate the diagonal matrix $\mathrm{diag}[d_0, \dots, d_n]$ of $A(q^{\tilde{\alpha}})$ in the expression of (5.2).

Theorem 5.5. *The elements d_r, $0 \le r \le n$, of the diagonal matrix $\mathrm{diag}[d_0, \dots, d_n]$ are evaluated as follows:*

$$d_r = q^{\mu_r} \frac{(q^{\tilde{\alpha}+\beta_1-2(r-1)\gamma};q^\gamma)_r \cdot (q^{\tilde{\alpha}-2(n-1)\gamma};q^\gamma)_{n-r}}{(q^{\tilde{\alpha}+\beta_1+\beta_2-(r-1)\gamma};q^\gamma)_r \cdot (q^{\tilde{\alpha}+\beta_1-(n-1+r)\gamma};q^\gamma)_{n-r}}, \qquad 0 \le r \le n.$$

In particular,

Corollary 5.6. *The determinant of the matrix $A(q^{\tilde{\alpha}})$ is the following:*

$$\det A(q^{\tilde{\alpha}}) = (x_1x_2)^{n(n+1)/2} q^{(n-1)n(n+1)\gamma/3} \prod_{r=0}^n \frac{(q^{\tilde{\alpha}-2(n-1)\gamma};q^\gamma)_r}{(q^{\tilde{\alpha}+\beta_1+\beta_2-(r-1)\gamma};q^\gamma)_r}.$$

Proof. By Theorem 5.5, we have

$$\det A(q^{\tilde{\alpha}}) = q^{\mu_0+\cdots+\mu_n} \prod_{r=0}^n \frac{(q^{\tilde{\alpha}+\beta_1-2(r-1)\gamma};q^\gamma)_r \cdot (q^{\tilde{\alpha}-2(n-1)\gamma};q^\gamma)_{n-r}}{(q^{\tilde{\alpha}+\beta_1+\beta_2-(r-1)\gamma};q^\gamma)_r \cdot (q^{\tilde{\alpha}+\beta_1-(n-1+r)\gamma};q^\gamma)_{n-r}}.$$

The result follows from the following identity: $\displaystyle\prod_{r=0}^n \frac{(q^{\tilde{\alpha}+\beta_1-2(r-1)\gamma};q^\gamma)_r}{(q^{\tilde{\alpha}+\beta_1-(n-1+r)\gamma};q^\gamma)_{n-r}} = 1.$ □

Proof of Theorem 5.5. From (4.4), (4.6) and (4.11), we have the explicit forms of $(h^{--}_{rr})_0$ and $(h^{-+}_{rr})_0$ as follows:

$$(h_{rr}^{--})_0 = (-1)^{n-r} \frac{(1-q^\gamma)^{n-r} \cdot (q^{\tilde{\alpha}+\beta_1-(n+r-1)\gamma}; q^\gamma)_{n-r}}{(q^\gamma; q^\gamma)_{n-r} \cdot (q^{\tilde{\alpha}-2(n-1)\gamma}; q^\gamma)_{n-r} \cdot (q^{\beta_1}; q^\gamma)_r}$$
$$\cdot \frac{{}_n(\frac{x_2}{x_1} q^{\beta_1-\beta_2-(r-1)\gamma}; q^\gamma)_r \cdot {}_n(\frac{x_2}{x_1} q^{\beta_1-\beta_2-r\gamma}; q^\gamma)_r}{{}_n(\frac{x_2}{x_1} q^{-\beta_2-(n-2)\gamma}; q^\gamma)_r \cdot {}_n(\frac{x_2}{x_1} q^{-\beta_2-(n-1)\gamma}; q^\gamma)_r}, \tag{5.3}$$

$$(h_{rr}^{-+})_0 = (-1)^r \cdot \frac{(q^{\beta_2}; q^\gamma)_r \cdot (\frac{x_2}{x_1} q^{\beta_1}; q^\gamma)_r \cdot (\frac{x_2}{x_1} q^{-\beta_2-(r-1)\gamma}; q^\gamma)_r \cdot (q^\gamma; q^\gamma)_r}{(\frac{x_2}{x_1}; q^\gamma)_r \cdot (\frac{x_2}{x_1} q^{\beta_1-\beta_2-(r-1)\gamma}; q^\gamma)_r \cdot (1-q^\gamma)^r}$$
$$\cdot \frac{(q^{\tilde{\alpha}+\beta_1-2(r-1)\gamma}; q^\gamma)_r}{(q^{\tilde{\alpha}+\beta_1+\beta_2-(r-1)\gamma}; q^\gamma)_r} \cdot \frac{{}_n(\frac{x_2}{x_1} q^{-(n-r-1)\gamma}; q^\gamma)_r \cdot {}_n(\frac{x_2}{x_1} q^{-(n-r)\gamma}; q^\gamma)_r}{{}_n(\frac{x_2}{x_1} q^{-\beta_2-(n-2)\gamma}; q^\gamma)_r \cdot {}_n(\frac{x_2}{x_1} q^{-\beta_2-(n-1)\gamma}; q^\gamma)_r}. \tag{5.4}$$

In Propositions 3.1 and 3.2, we derived the following expressions:

$$(c_{A,rr}^{+})_0 = q^{-[r(r-1)+(n-r)(n-r-1)]\gamma/2}$$
$$\cdot \frac{(\frac{x_2}{x_1} q^{-(n-r)\gamma}; q^\gamma)_r \cdot (\frac{x_2}{x_1} q^{-(n-r-1)\gamma}; q^\gamma)_r}{(q^{\beta_1}; q^\gamma)_{n-r} \cdot (\frac{x_2}{x_1} q^{\beta_1}; q^\gamma)_r \cdot (q^{\beta_2}; q^\gamma)_r \cdot (\frac{x_2}{x_1} q^\gamma; q^\gamma)_r}, \tag{5.5}$$

$$(c_{A,rr}^{-})_0 = (-1)^n q^{-[r(r-1)+(n-r)(n-r-1)]\gamma/2} \ \frac{(q^\gamma; q^\gamma)_{n-r} \cdot (q^\gamma; q^\gamma)_r}{(1-q^\gamma)^n}$$
$$\cdot \frac{(\frac{x_2}{x_1} q^{-\beta_2-(r-1)\gamma}; q^\gamma)_r}{(\frac{x_2}{x_1} q^{\beta_1-\beta_2-(r-1)\gamma}; q^\gamma)_r} \cdot \frac{1}{{}_n(\frac{x_2}{x_1} q^{\beta_1-\beta_2-(r-1)\gamma}; q^\gamma)_r \cdot {}_n(\frac{x_2}{x_1} q^{\beta_1-\beta_2-r\gamma}; q^\gamma)_r}. \tag{5.6}$$

Finally, from (5.1) and (5.2), we have

$$d_r = (c_{A,rr}^{-})_0^{-1} \cdot (h_{rr}^{--})_0^{-1} \cdot (h_{rr}^{-+})_0 \cdot (c_{A,rr}^{+})_0 \cdot q^{\mu_r} \tag{5.7}$$

and the result follows from (5.3)–(5.7). □

5.2. **q-Binomial lemmas.** In this section, we prepare two q-binomial lemmas that will be useful in the following sections.

Lemma 5.7. *Let z, y and c be arbitrary numbers $\in \mathbb{C}$. Then*

$$\sum_{j=0}^{k} (-1)^j z^j c^{3j(j-1)/2} \frac{(c;c)_k}{(c;c)_{k-j} \cdot (c;c)_j} \cdot \frac{(yc^{1-j};c)_j}{(zc^{j-1};c)_j} \cdot \frac{(zyc^j;c)_{k-j}}{(zc^{2j};c)_{k-j}} = 1. \tag{5.8}$$

Remark. When $c \to 1$, the above formula reduces to the following well-known combinatorial formula:

$$(1-z)^k = \big((1-zy) - (z-zy)\big)^k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (z-zy)^j (1-zy)^{k-j}.$$

Proof. Multiply both sides of (5.8) by $(z;c)_{2k-1}$. Then we have

$$(z;c)_{2k-1} = \sum_{j=0}^{k} (-1)^j z^j c^{3j(j-1)/2} \frac{(c;c)_k}{(c;c)_{k-j} \cdot (c;c)_j} \cdot (yc^{1-j};c)_j \cdot (zyc^j;c)_{k-j}$$
$$\cdot (z;c)_{j-1} \cdot (1 - zc^{2j-1}) \cdot (zc^{j+k};c)_{k-j-1}. \tag{5.9}$$

We prove (5.9) instead of (5.8). We denote by $g(z)$ and $g_j(z)$ the summation and summand of the right-hand side of (5.9) respectively:

$$
\begin{aligned}
g_j(z) &:= (-1)^j z^j c^{3j(j-1)/2} \frac{(c;c)_k}{(c;c)_{k-j}\cdot(c;c)_j}\cdot(yc^{1-j};c)_j\cdot(zyc^j;c)_{k-j} \\
&\qquad\qquad\qquad\qquad\qquad \cdot(z;c)_{j-1}\cdot(1-zc^{2j-1})\cdot(zc^{j+k};c)_{k-j-1}, \\
g(z) &:= \sum_{j=0}^{k} g_j(z).
\end{aligned}
$$

In order to prove (5.9) by the factor theorem, we show that the polynomial $g(z)$ of degree $2k-1$ equals 1 at $z=0$ and vanishes at $2k-1$ points $z=c^{1-l}$, $1\le l\le 2k-1$. Since $g(0)=1$ is easy to check, it is enough to show the following:

$$
\begin{aligned}
2g(c^{1-l}) &= 2\sum_{j=0}^{l} g_j(c^{1-l}) = \sum_{j=0}^{l}\Big(g_j(c^{1-l})+g_{l-j}(c^{1-l})\Big)=0, \\
2g(c^{2-l-k}) &= 2\sum_{j=l}^{k} g_j(c^{2-l-k}) = \sum_{j=0}^{k-l}\Big(g_{l+j}(c^{2-l-k})+g_{k-j}(c^{2-l-k})\Big)=0,
\end{aligned}
$$

which are confirmed from the following lemma. □

Lemma 5.8. *For $z=c^{1-l}$ or c^{2-l-k}, $1\le l\le k$, it follows that*

$$
\begin{cases} g_j(c^{1-l})+g_{l-j}(c^{1-l})=0 & \text{for } \ 0\le j\le l, \\ \qquad g_j(c^{1-l})=0 & \text{for } \ l<j\le k \end{cases}
$$

and

$$
\begin{cases} g_{l+j}(c^{2-l-k})+g_{k-j}(c^{2-l-k})=0 & \text{for } \ 0\le j\le k-l, \\ \qquad g_j(c^{2-l-k})=0 & \text{for } \ 0\le j< l. \end{cases}
$$

Proof. It is straightforward and left to the reader. □

Lemma 5.9. *Let z and c be arbitrary numbers $\in\mathbb{C}$. Then*

$$
\sum_{j=0}^{k}\frac{(-1)^j c^{-j(2k-j-1)/2}}{(c;c)_j\cdot(c;c)_{k-j}\cdot(zc^{-2(j-1)};c)_j\cdot(zc^{-k-i+1};c)_{k-j}}=0. \tag{5.10}
$$

Remark. When $c\to 1$, the above formula reduces to the following well-known combinatorial formula:

$$
\sum_{j=0}^{k}(-1)^j\binom{k}{j}=0.
$$

Proof. By multiplying both sides of (5.10) by $(zc^{-2k+2};c)_{2k-1}$, we have

$$
\sum_{j=0}^{k}(-1)^j c^{-j(2k-j-1)/2}\frac{(z;c^{-j+2})_{j-1}\cdot(1-zc^{-2j+1})\cdot(zc^{-2k+2};c)_{k-j-1}}{(c;c)_{k-j}\cdot(c;c)_j}=0. \tag{5.11}
$$

We prove (5.11) instead of (5.10). We denote by $\widetilde{g}(z)$ and $\widetilde{g}_j(z)$ the summation and summand of the right-hand side of (5.9) respectively:

$$
\widetilde{g}_j(z) := (-1)^j c^{-j(2k-j-1)/2}\frac{(z;c^{-j+2})_{j-1}\cdot(1-zc^{-2j+1})\cdot(zc^{-2k+2};c)_{k-j-1}}{(c;c)_{k-j}\cdot(c;c)_j},
$$

$$\widetilde{g}(z) := \sum_{j=0}^{k} \widetilde{g}_j(z).$$

In order to prove (5.11) by the factor theorem, we show that the polynomial $\widetilde{g}(z)$ of degree $k-1$ vanishes at k points $z = c^{l-1}$, $1 \le l \le k$, i.e.,

$$2\widetilde{g}(c^{l-1}) = 2\sum_{j=0}^{l} \widetilde{g}_j(c^{l-1}) = \sum_{j=0}^{l} \Big(\widetilde{g}_j(c^{l-1}) + \widetilde{g}_{l-j}(c^{l-1})\Big) = 0, \quad 1 \le l \le k,$$

which follows from the following lemma. □

Lemma 5.10. *For $z = c^{l-1}$, $1 \le l \le k$, it follows that*

$$\widetilde{g}_j(c^{l-1}) + \widetilde{g}_{l-j}(c^{l-1}) = 0 \quad \text{for} \quad 0 \le j \le l,$$
$$\widetilde{g}_j(c^{l-1}) = 0 \qquad \text{for} \quad l < j \le k.$$

Proof. It is straightforward and left to the reader. □

5.3. Evaluation of L_A.

Theorem 5.11. *The elements of the matrix L_A are expressed in a product of binomials as follows:*

$$l_{rs} = (-1)^{r-s}(\tfrac{x_2}{x_1})^{s-r} q^{-(r-s)(r+s-1)\gamma/2} \cdot \frac{(q^\gamma;q^\gamma)_{n-s}}{(q^\gamma;q^\gamma)_{r-s}\cdot(q^\gamma;q^\gamma)_{n-r}} \frac{(q^{\beta_2+s\gamma};q^\gamma)_{r-s}}{(q^{\tilde\alpha+\beta_1-2(r-1)\gamma};q^\gamma)_{r-s}} \qquad \text{for} \quad r \ge s. \tag{5.12}$$

Proof. From (4.5) and (4.11), we have

$$\left(\frac{h^{-+}_{ri}}{h^{-+}_{rr}}\right)_0 = q^{-i(r-i)\gamma} \frac{(\frac{x_2}{x_1}q^\gamma;q^\gamma)_i\cdot(\frac{x_2}{x_1}q^{-(n-r-1)\gamma};q^\gamma)_r}{(\frac{x_2}{x_1}q^\gamma;q^\gamma)_r\cdot(\frac{x_2}{x_1}q^{-(n-i-1)\gamma};q^\gamma)_i} \cdot \frac{(\frac{x_2}{x_1}q^{-(n-r)\gamma};q^\gamma)_r}{(\frac{x_2}{x_1}q^{-(n-i)\gamma};q^\gamma)_i}$$
$$\cdot \frac{(\frac{x_2}{x_1}q^{\beta_1};q^\gamma)_i}{(\frac{x_2}{x_1}q^{\beta_1};q^\gamma)_r} \cdot \frac{(q^{\beta_1};q^\gamma)_{n-i}\cdot(q^\gamma;q^\gamma)_{n-i}}{(q^{\beta_1};q^\gamma)_{n-r}\cdot(q^\gamma;q^\gamma)_{n-r}}$$
$$\cdot \frac{(q^{-\tilde\alpha-\beta_1+(n-1)\gamma}/\frac{x_2}{x_1};q^\gamma)_{r-i}}{(q^{-\tilde\alpha-\beta_1+(r+i-1)\gamma};q^\gamma)_{r-i}\cdot(q^{(n-r-i)\gamma}/\frac{x_2}{x_1};q^\gamma)_{r-i}\cdot(q^\gamma;q^\gamma)_{r-i}}. \tag{5.13}$$

By Proposition 3.1, it follows that

$$\left(\frac{c^+_{A,is}}{c^+_{A,ii}}\right)_0 = q^{-(i-s)(n-i)\gamma} \frac{(q^\gamma;q^\gamma)_{n-s}}{(q^\gamma;q^\gamma)_{i-s}(q^\gamma;q^\gamma)_{n-i}} \frac{(q^{\beta_2+s\gamma};q^\gamma)_{i-s}}{(\frac{x_2}{x_1}q^{(i+s-n)\gamma};q^\gamma)_{i-s}}, \tag{5.14}$$

$$\left(\frac{c^+_{A,ii}}{c^+_{A,rr}}\right)_0 = q^{[-i(i-1)-(n-i)(n-i-1)+r(r-1)+(n-r)(n-r-1)]\gamma/2}$$
$$\cdot \frac{(\frac{x_2}{x_1}q^{-(n-i)\gamma};q^\gamma)_i\cdot(\frac{x_2}{x_1}q^{-(n-i-1)\gamma};q^\gamma)_i}{(\frac{x_2}{x_1}q^{-(n-r)\gamma};q^\gamma)_r\cdot(\frac{x_2}{x_1}q^{-(n-r-1)\gamma};q^\gamma)_r}$$
$$\cdot \frac{(q^{\beta_1};q^\gamma)_{n-r}\cdot(\frac{x_2}{x_1}q^{\beta_1};q^\gamma)_r\cdot(q^{\beta_2};q^\gamma)_r\cdot(\frac{x_2}{x_1}q^\gamma;q^\gamma)_r}{(q^{\beta_1};q^\gamma)_{n-i}\cdot(\frac{x_2}{x_1}q^{\beta_1};q^\gamma)_i\cdot(q^{\beta_2};q^\gamma)_i\cdot(\frac{x_2}{x_1}q^\gamma;q^\gamma)_i}, \tag{5.15}$$

$$\left(\frac{q^{\mu_i}}{q^{\mu_r}}\right)_0 = (\tfrac{x_2}{x_1})^{i-r} q^{[i(i-1)+(n-i)(n-i-1)-r(r-1)-(n-r)(n-r-1)]\gamma/2}. \tag{5.16}$$

Comparing (5.1) with (5.2), the elements l_{rs} of the matrix L_A are expressed as follows:

$$l_{rs} = \sum_{i=s}^{r} \Big(\frac{\overset{-+}{h}_{ri}}{\overset{-+}{h}_{rr}}\Big)_0 \Big(\frac{\overset{+}{c}_{A,is}}{\overset{+}{c}_{A,ii}}\Big)_0 \Big(\frac{\overset{+}{c}_{A,ii}}{\overset{+}{c}_{A,rr}}\Big)_0 \Big(\frac{q^{\mu_i}}{q^{\mu_r}}\Big)_0.$$

Then, from (5.13)–(5.16), we have

$$\begin{aligned} l_{rs} = {} & \frac{(q^\gamma;q^\gamma)_{n-s}}{(q^\gamma;q^\gamma)_{n-r}} (q^{\beta_2+s\gamma};q^\gamma)_{r-s} \\ & \cdot \sum_{i=s}^{r} (-1)^{r-i} (\tfrac{x_2}{x_1})^{i-r} q^{-(r-i)(r+i-1)\gamma/2-(i-s)(n-i)\gamma} \frac{1}{(q^\gamma;q^\gamma)_{r-i}\cdot(q^\gamma;q^\gamma)_{i-s}} \\ & \cdot \frac{(\tfrac{x_2}{x_1}q^{\tilde\alpha+\beta_1-(n+r-i-2)\gamma};q^\gamma)_{r-i}}{(q^{\tilde\alpha+\beta_1-2(r-1)\gamma};q^\gamma)_{r-i}\cdot(\tfrac{x_2}{x_1}q^{(2i-n+1)\gamma};q^\gamma)_{r-i}\cdot(\tfrac{x_2}{x_1}q^{(i+s-n)\gamma};q^\gamma)_{i-s}}. \quad (5.17) \end{aligned}$$

We have to show the following identity to prove that (5.17) coincides with (5.12):

$$\begin{aligned} & \sum_{i=s}^{r} (-1)^{r-i} (\tfrac{x_2}{x_1})^{i-r} q^{-(r-i)(r+i-1)\gamma/2-(i-s)(n-i)\gamma} \frac{1}{(q^\gamma;q^\gamma)_{r-i}\cdot(q^\gamma;q^\gamma)_{i-s}} \\ & \quad \cdot \frac{(\tfrac{x_2}{x_1}q^{\tilde\alpha+\beta_1-(n+r-i-2)\gamma};q^\gamma)_{r-i}}{(q^{\tilde\alpha+\beta_1-2(r-1)\gamma};q^\gamma)_{r-i}\cdot(\tfrac{x_2}{x_1}q^{(2i-n+1)\gamma};q^\gamma)_{r-i}\cdot(\tfrac{x_2}{x_1}q^{(i+s-n)\gamma};q^\gamma)_{i-s}} \\ & = (-1)^{r-s} (\tfrac{x_2}{x_1})^{s-r} q^{-(r-s)(r+s-1)\gamma/2} \frac{1}{(q^\gamma;q^\gamma)_{r-s}\cdot(q^{\tilde\alpha+\beta_1-2(r-1)\gamma};q^\gamma)_{r-s}}. \quad (5.18) \end{aligned}$$

Dividing (5.18) by the right-hand side of (5.18), (5.18) reads as follows:

$$\begin{aligned} 1 = {} & \sum_{i=s}^{r} (-1)^{i-s} (\tfrac{x_2}{x_1} q^{-(n-i)\gamma+(i+s-1)\gamma/2})^{i-s} \frac{(q^\gamma;q^\gamma)_{r-s}}{(q^\gamma;q^\gamma)_{r-i}\cdot(q^\gamma;q^\gamma)_{i-s}} \\ & \cdot \frac{(q^{\tilde\alpha+\beta_1-(r+i-2)\gamma};q^\gamma)_{i-s}}{(\tfrac{x_2}{x_1}q^{(i+s-n)\gamma};q^\gamma)_{i-s}} \frac{(\tfrac{x_2}{x_1}q^{\tilde\alpha+\beta_1-(n+r-i-2)\gamma};q^\gamma)_{r-i}}{(\tfrac{x_2}{x_1}q^{(2i-n+1)\gamma};q^\gamma)_{r-i}}. \quad (5.19) \end{aligned}$$

If we put $j=i-s$, $k=r-s$, $z=\frac{x_2}{x_1}q^{2s-n+j}$, $y=q^{\tilde\alpha+\beta_1-2s-k+1}$ and $c=q^\gamma$ in Lemma 5.7, then we see the right-hand side of (5.19) is equal to 1 and this concludes the proof of the theorem. □

5.4. **Evaluation of U_A.**

Theorem 5.12. *The elements of $U_A = \big(u_{rs}\big)_{r,s=0}^{n}$ and $U_A^{-1} = \big(u^*_{rs}\big)_{r,s=0}^{n}$ are expressed in a product of binomials as follows:*

$$\begin{aligned} u_{rs} = {} & (-1)^{s-r} q^{(s-r)[\tilde\alpha-(n-1)\gamma]-(s-r)(s+r-1)\gamma/2} \\ & \cdot \frac{(q^\gamma;q^\gamma)_s}{(q^\gamma;q^\gamma)_{s-r}\cdot(q^\gamma;q^\gamma)_r} \frac{(q^{\beta_1+(n-s)\gamma};q^\gamma)_{s-r}}{(q^{\tilde\alpha+\beta_1-2(s-1)\gamma};q^\gamma)_{s-r}} \quad s\ge r, \quad (5.20) \end{aligned}$$

$$\begin{aligned} u^*_{rs} = {} & q^{(s-r)[\tilde\alpha-(n-1)\gamma]+r(r-s)\gamma} \\ & \cdot \frac{(q^\gamma;q^\gamma)_s}{(q^\gamma;q^\gamma)_{s-r}\cdot(q^\gamma;q^\gamma)_r} \frac{(q^{\beta_1+(n-s)\gamma};q^\gamma)_{s-r}}{(q^{\tilde\alpha+\beta_1-(r+s-1)\gamma};q^\gamma)_{s-r}} \quad s\ge r. \quad (5.21) \end{aligned}$$

Proof. We show (5.21) first. From (4.11) and (4.7), we have

$$\left(\frac{h_{ri}^{--}}{h_{rr}^{--}}\right)_0 = q^{(i-r)(i+r-n)\gamma}\frac{{}_n\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2-(i-1)\gamma};q^\gamma\right)_i\cdot{}_n\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2-i\gamma};q^\gamma\right)_i}{{}_n\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2-(r-1)\gamma};q^\gamma\right)_r\cdot{}_n\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2-r\gamma};q^\gamma\right)_r}$$
$$\cdot q^{-(n-i)(i-r)\gamma}\frac{\left(q^{\tilde\alpha+\beta_2-(n-1)\gamma}/\frac{x_2}{x_1};q^\gamma\right)_{i-r}}{\left(q^{\beta_2+r\gamma}/\frac{x_2}{x_1};q^\gamma\right)_{i-r}}\cdot{}_{n-r}\left(q^{\beta_2-\beta_1+(r+i-n)\gamma}/\frac{x_2}{x_1};q^\gamma\right)_{n-i}$$
$$\cdot\frac{\left(q^{\alpha+\beta_1-(n+r-1)\gamma};q^\gamma\right)_{n-i}}{\left(q^{\alpha+\beta_1-(n+r-1)\gamma};q^\gamma\right)_{n-r}}\cdot\frac{\left(q^{\beta_1};q^\gamma\right)_{n-r}}{\left(q^{\beta_1};q^\gamma\right)_{n-i}}\cdot{}_{n-r}\left(q^\gamma;q^\gamma\right)_{n-i}$$
$$= q^{(i-r)(i+r-n)\gamma}\frac{{}_n\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2-(i-1)\gamma};q^\gamma\right)_i\cdot{}_n\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2-i\gamma};q^\gamma\right)_i}{{}_n\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2-(r-1)\gamma};q^\gamma\right)_r\cdot{}_n\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2-r\gamma};q^\gamma\right)_r}$$
$$\cdot q^{(\tilde\alpha-(n+r-1)\gamma)(i-r)}\frac{\left(\frac{x_2}{x_1}q^{-\tilde\alpha-\beta_2+(n+r-i)\gamma};q^\gamma\right)_{i-r}}{\left(\frac{x_2}{x_1}q^{-\beta_2-(i-1)\gamma};q^\gamma\right)_{i-r}}\cdot\frac{\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2-(i-1)\gamma};q^\gamma\right)_{i-r}}{\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2+(n-2i+1)\gamma};q^\gamma\right)_{i-r}}$$
$$\cdot\frac{\left(q^{\beta_1+(n-i)\gamma};q^\gamma\right)_{i-r}}{\left(q^{\alpha+\beta_1-(i+r-1)\gamma};q^\gamma\right)_{i-r}}\cdot\frac{\left(q^\gamma;q^\gamma\right)_{n-r}}{\left(q^\gamma;q^\gamma\right)_{i-r}\cdot\left(q^\gamma;q^\gamma\right)_{n-i}}. \tag{5.22}$$

By Proposition 3.2, we have

$$\left(\frac{c_{A,is}^{-}}{c_{A,ii}^{-}}\right)_0 = \left(\frac{x_2}{x_1}q^{-\beta_2-i\gamma}\right)^{s-i}\frac{\left(q^{\beta_1+(n-s)\gamma};q^\gamma\right)_{s-i}}{\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2+(n-s-i)\gamma};q^\gamma\right)_{s-i}}\cdot\frac{\left(q^\gamma;q^\gamma\right)_s}{\left(q^\gamma;q^\gamma\right)_i\cdot\left(q^\gamma;q^\gamma\right)_{s-i}}, \tag{5.23}$$

$$\left(\frac{c_{A,ii}^{-}}{c_{A,rr}^{-}}\right)_0 = q^{-(i-r)(i+r-n)\gamma}\frac{\left(q^\gamma;q^\gamma\right)_{n-i}\cdot\left(q^\gamma;q^\gamma\right)_i}{\left(q^\gamma;q^\gamma\right)_{n-r}\cdot\left(q^\gamma;q^\gamma\right)_r}$$
$$\cdot\frac{\left(\frac{x_2}{x_1}q^{-\beta_2-(i-1)\gamma};q^\gamma\right)_i}{\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2-(i-1)\gamma};q^\gamma\right)_i}\cdot\frac{\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2-(r-1)\gamma};q^\gamma\right)_r}{\left(\frac{x_2}{x_1}q^{-\beta_2-(r-1)\gamma};q^\gamma\right)_r}$$
$$\cdot\frac{{}_n\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2-(r-1)\gamma};q^\gamma\right)_r\cdot{}_n\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2-r\gamma};q^\gamma\right)_r}{{}_n\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2-(i-1)\gamma};q^\gamma\right)_i\cdot{}_n\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2-i\gamma};q^\gamma\right)_i}. \tag{5.24}$$

Comparing (5.1) with (5.2), using (5.22)–(5.24), the elements u_{rs}^* of the matrix U_A^{-1} are expressed as follows:

$$u_{rs}^* = \sum_{i=r}^{s}\left(\frac{h_{ri}^{--}\,c_{A,is}^{-}}{h_{rr}^{--}\,c_{A,rr}^{-}}\right)_0 = \sum_{i=r}^{s}\left(\frac{h_{ri}^{--}}{h_{rr}^{--}}\right)_0\left(\frac{c_{A,is}^{-}}{c_{A,ii}^{-}}\right)_0\left(\frac{c_{A,ii}^{-}}{c_{A,rr}^{-}}\right)_0$$
$$= q^{[\tilde\alpha-(n-1-r)\gamma](s-r)}\frac{\left(q^\gamma;q^\gamma\right)_s}{\left(q^\gamma;q^\gamma\right)_{s-r}\cdot\left(q^\gamma;q^\gamma\right)_r}\frac{\left(q^{\beta_1+(n-s)\gamma};q^\gamma\right)_{s-r}}{\left(q^{\tilde\alpha+\beta_1-(r+s-1)\gamma};q^\gamma\right)_{s-r}}$$
$$\cdot\sum_{i=r}^{s}\left(\frac{x_2}{x_1}q^{-\tilde\alpha-\beta_2+(n+r-i-1)\gamma}\right)^{s-i}\frac{\left(q^{\tilde\alpha+\beta_1-(r+s-1)\gamma};q^\gamma\right)_{s-i}}{\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2+(n-s-i)\gamma};q^\gamma\right)_{s-i}}$$
$$\cdot\frac{\left(\frac{x_2}{x_1}q^{-\tilde\alpha-\beta_2+(n+r-i)\gamma};q^\gamma\right)_{i-r}}{\left(\frac{x_2}{x_1}q^{\beta_1-\beta_2+(n-2i+1)\gamma};q^\gamma\right)_{i-r}}\cdot{}_{s-r}\left(q^\gamma;q^\gamma\right)_{s-i}.$$

Thus the statement (5.21) in Theorem 5.12 follows from the following identity:

$$1=\sum_{i=r}^{s}\left(\tfrac{x_2}{x_1}q^{-\tilde{\alpha}-\beta_2+(n+r-i-1)\gamma}\right)^{s-i}\cdot\frac{(q^{\tilde{\alpha}+\beta_1-(r+s-1)\gamma};q^{\gamma})_{s-i}}{(\tfrac{x_2}{x_1}q^{\beta_1-\beta_2+(n-s-i)\gamma};q^{\gamma})_{s-i}}$$
$$\cdot\frac{(\tfrac{x_2}{x_1}q^{-\tilde{\alpha}-\beta_2+(n+r-i)\gamma};q^{\gamma})_{i-r}}{(\tfrac{x_2}{x_1}q^{\beta_1-\beta_2+(n-2i+1)\gamma};q^{\gamma})_{i-r}}\cdot{}_{s-r}(q^{\gamma};q^{\gamma})_{s-i}. \tag{5.25}$$

We put $i=r+j$, $s=r+k$, $y=q^{-\tilde{\alpha}-\beta_1+(2r+k-1)\gamma}$, $z=q^{\beta_1-\beta_2+(n-2r-2k+1)\gamma}$ and $c=q^{\gamma}$ in the right-hand side of (5.25). Then, we have

The right-hand side of (5.25)

$$=\sum_{j=0}^{k}(zyc^{k-j-1})^{k-j}\frac{(y^{-1};c)_{k-j}}{(zc^{k-j-1};c)_{k-j}}\cdot\frac{(zyc^{k-j};c)_j}{(zc^{2(k-j)};c)_j}\cdot{}_k(c;c)_j$$
$$=\sum_{j=0}^{k}(-1)^{k-j}z^{k-j}c^{3(k-j)(k-j-1)/2}\frac{(yc^{1-(k-j)};c)_{k-j}}{(zc^{k-j-1};c)_{k-j}}\cdot\frac{(zyc^{k-j};c)_j}{(zc^{2(k-j)};c)_j}\cdot{}_k(c;c)_j$$
$$=\sum_{j=0}^{k}(-1)^{j}z^{j}c^{3j(j-1)/2}\frac{(yc^{1-j};c)_{j}}{(zc^{j-1};c)_{j}}\cdot\frac{(zyc^{j};c)_{k-j}}{(zc^{2j};c)_{k-j}}\cdot{}_k(c;c)_j. \tag{5.26}$$

By Lemma 5.7, we have already known that (5.26) is equal to 1. Hence (5.21) follows.

Next we show (5.20) of Theorem 5.12. What we want to prove is the following:

$$\sum_{j=r}^{s}\left[(-1)^{j-r}q^{(j-r)[\tilde{\alpha}-(n-1)\gamma]-(j-r)(j+r-1)\gamma/2}\cdot\frac{(q^{\gamma};q^{\gamma})_j}{(q^{\gamma};q^{\gamma})_{j-r}\cdot(q^{\gamma};q^{\gamma})_r}\frac{(q^{\beta_1+(n-j)\gamma};q^{\gamma})_{j-r}}{(q^{\tilde{\alpha}+\beta_1-2(j-1)\gamma};q^{\gamma})_{j-r}}\right]\cdot u^*_{js}$$
$$=\begin{cases}1 & (r=s)\\ 0 & (r\neq s)\end{cases}. \tag{5.27}$$

The left-hand side of (5.27) is equal to

$$\sum_{j=r}^{s}\left[(-1)^{j-r}q^{(j-r)[\tilde{\alpha}-(n-1)\gamma]-(j-r)(j+r-1)\gamma/2}\cdot\frac{(q^{\gamma};q^{\gamma})_j}{(q^{\gamma};q^{\gamma})_{j-r}\cdot(q^{\gamma};q^{\gamma})_r}\frac{(q^{\beta_1+(n-j)\gamma};q^{\gamma})_{j-r}}{(q^{\tilde{\alpha}+\beta_1-2(j-1)\gamma};q^{\gamma})_{j-r}}\right]$$
$$\cdot\left[q^{(s-j)[\tilde{\alpha}-(n-1)\gamma]+j(j-s)\gamma}\frac{(q^{\gamma};q^{\gamma})_s}{(q^{\gamma};q^{\gamma})_{s-j}\cdot(q^{\gamma};q^{\gamma})_j}\frac{(q^{\beta_1+(n-s)\gamma};q^{\gamma})_{s-j}}{(q^{\tilde{\alpha}+\beta_1-(j+s-1)\gamma};q^{\gamma})_{s-j}}\right]$$
$$=\sum_{j=r}^{s}(-1)^{j-r}q^{(s-r)[\tilde{\alpha}-(n-1)\gamma]+j(j-s)\gamma-(j-r)(j+r-1)\gamma/2}\cdot\frac{(q^{\gamma};q^{\gamma})_s}{(q^{\gamma};q^{\gamma})_r\cdot(q^{\gamma};q^{\gamma})_{j-r}\cdot(q^{\gamma};q^{\gamma})_{s-j}}$$
$$\cdot\frac{(q^{\beta_1+(n-s)\gamma};q^{\gamma})_{s-r}}{(q^{\tilde{\alpha}+\beta_1-2(j-1)\gamma};q^{\gamma})_{j-r}\cdot(q^{\tilde{\alpha}+\beta_1-(j+s-1)\gamma};q^{\gamma})_{s-j}}. \tag{5.28}$$

If $r=s$, it is easy to see that (5.28) is equal to 1. We assume $r\neq s$ and put $k:=s-r$. (5.28) is equal to

$$q^{k[\alpha-(n+r-1)\gamma]}\frac{(q^\gamma;q^\gamma)_{r+k}\cdot(q^{\beta_1+(n-s)\gamma};q^\gamma)_k}{(q^\gamma;q^\gamma)_r}$$

$$\cdot\left[\sum_{i=0}^{k}\frac{(-1)^i q^{-i(2k-i-1)\gamma/2}}{(q^\gamma;q^\gamma)_i\cdot(q^\gamma;q^\gamma)_{k-i}\cdot(q^{\tilde{\alpha}+\beta_1-2(i+r-1)\gamma};q^\gamma)_i\cdot(q^{\tilde{\alpha}+\beta_1-(i+2r+k-1)\gamma};q^\gamma)_{k-i}}\right]. \tag{5.29}$$

If we put $z=q^{\alpha+\beta_1-2r\gamma}$ and $c=q^\gamma$ in Lemma 5.8, then we see (5.29) is equal to 0. Therefore the proof is complete. □

Appendix A. The condition P4) of Riemann–Hilbert problem for $A(q^{\tilde{\alpha}})$

Proposition A.1. *The matrix $A(q^{\tilde{\alpha}})$ depends only on x, $q^{\tilde{\alpha}}$, q^{β_1}, q^{β_2}, q^γ and it satisfies the condition* P4)*, i.e., it does not depend on q.*

This proposition was suggested to me by Prof. K. Aomoto. Before proving Proposition A.1, we prove five lemmas.

For $\chi=(\nu_1,\ldots,\nu_n)\in\mathbb{Z}^n$, we denote by T^χ the shift operator

$$T^\chi f(t):=f(q^{\nu_1}t_1,\ldots,q^{\nu_n}t_n)$$

for a function $f(t)$ of $t\in(\mathbb{C}^*)^n$. The function $b_\chi(t)$ called the *b-function* is defined by

$$b_\chi(t):=\frac{T^\chi\Phi(t)}{\Phi(t)}. \tag{A.1}$$

Let ∇_χ be the *covariant q-difference operator* defined by

$$\nabla_\chi\varphi(t):=\varphi(t)-b_\chi(t)\cdot T^\chi\varphi(t)$$

for a rational function $\varphi(t)$.

Lemma A.2. *The following equation holds for any $\chi\in\mathbb{Z}^n$:*

$$\int_{\langle\xi\rangle}\Phi(t)\cdot\nabla_\chi\varphi(t)\ \varpi=0.$$

Proof. By definition of the Jackson integral, it follows that

$$\int_{\langle\xi\rangle}\Phi_R(t)\varphi(t)\ \varpi=\int_{\langle\xi\rangle}T^\chi(\Phi_R\ \varphi)(t)\ \varpi.$$

Therefore, from (A.1), we have $\displaystyle\int_{\langle\xi\rangle}\Phi(t)\varphi(t)\ \varpi=\int_{\langle\xi\rangle}\Phi(t)\cdot\big(b_\chi(t)\cdot T^\chi\varphi(t)\big)\ \varpi.$ □

The following lemma is easily deduced from Lemma A.2 and its proof is left to the reader.

Lemma A.3. *The following equation holds for any $\chi\in\mathbb{Z}^n$:*

$$\int_{\langle\xi\rangle}\Phi(t)\cdot\mathcal{A}\big(\nabla_\chi\varphi(t)\big)\ \varpi=0.$$

We set

$$D(t):=\prod_{1\le i<j\le n}(t_i-\ t_j).$$

The following useful lemma was proved by Kadell in [20]:

Lemma A.4 (Kadell's lemma)**.** *Let $Q \in \mathbb{C}$ be an arbitrary number and J be a subset of $\{1, 2, \ldots, n\}$. Then*

$$\mathcal{A}\Big\{\prod_{j\in J} t_j \prod_{1\le i<j\le n} (t_i - Q\ t_j)\Big\} = Q^{e(J)} \cdot \frac{(Q;Q)_{|J|}\cdot (Q;Q)_{n-|J|}}{(1-Q)^n} \cdot e_{|J|}(t)\ D(t), \quad \text{(A.2)}$$

where $|J| := \#J$, $e(J) := \#\{(i,j)\,|\, 1 \le i < j \le n,\ i \notin J,\ j \in J\}$ and $e_k(t)$ is the k-th elementary symmetric polynomial in variables $t_1, \ldots, t_n$.

Proof. See [20]. □

We define $\mathcal{A}_{i,j}(t)$ by the following:

$$\mathcal{A}_{i,j}(t) := D(t)\cdot \mathcal{S}(t_1^2 t_2^2 \cdots t_i^2 \cdot t_{i+1}t_{i+2}\cdots t_j), \quad 0 \le i \le j \le n$$

where $\mathcal{S}$ is a symmetric sum such that $\mathcal{S}g(t) = \sum_{\sigma\in\mathfrak{S}_n} \sigma g(t)$.

In order to make the notations clear, let us write down some elements of $\mathcal{A}_{i,j}(t)$ explicitly.

$$\begin{aligned} \mathcal{A}_{0,r}(t) &= e_r(t)\cdot D(t), \quad 0\le r\le n, & \text{(A.3)}\\ \mathcal{A}_{1,1}(t) &= (t_1^2+t_2^2+\cdots+t_n^2)\cdot D(t), \\ &\ \vdots \\ \mathcal{A}_{n,n}(t) &= t_1^2\cdots t_n^2\cdot D(t), \\ \mathcal{A}_{r,n}(t) &= e_n(t)\cdot \mathcal{A}_{0,r}(t), \quad 0\le r\le n. & \text{(A.4)} \end{aligned}$$

We define $\varphi_{r,s}(t)$ by the following:

$$\varphi_{r,s}(t) := \frac{\mathcal{A}_{r,s}(t)}{\prod_{i=1}^n (1-t_i/x_1)(1-t_i/x_2)}.$$

Lemma A.5. *$\varphi_s(t)$ is expressed as a linear combination of $\varphi_{0,r}(t)$, $0 \le r \le n$:*

$$\varphi_s(t) = \sum_{r=0}^n u_{rs}\ \varphi_{0,r}(t),$$

where the coefficient u_{rs} does not depend on q and $q^{\tilde\alpha}$.

Proof. It is straightforward from Kadell's lemma (A.2). □

Lemma A.6. *$\varphi_{s,n}(t)$, $0 \le s \le n$, is cohomologous to a linear combination of $\varphi_{0,r}(t)$, $0 \le r \le n$:*

$$\int \Phi(t)\varphi_{s,n}(t)\varpi = \sum_{r=0}^n c_{s,r} \int \Phi(t)\varphi_{0,r}(t)\varpi,$$

where the coefficients $c_{s,r}, 0 \le s \le r \le n$, do not depend on q.

Proof. Taking $\chi = (1, 0, \ldots, 0) \in \mathbb{Z}^n$, the b-function (A.1) is

$$b_1(t) := \frac{T_1\Phi(t)}{\Phi(t)} = q^{\tilde\alpha}\frac{1-t_1q^{\beta_1}/x_1}{1-t_1/x_1}\cdot\frac{1-t_1q^{\beta_2}/x_2}{1-t_1/x_2}\cdot\prod_{j=2}^n \frac{t_1-q^{-\gamma}t_j}{t_1-q^{\gamma-1}t_j}.$$

We put

$$\psi(t) = \frac{(t_2\cdots t_r)^2\cdot t_{r+1}\cdots t_s}{\prod_{j=2}^n(1-t_j/x_1)(1-t_j/x_2)} \prod_{1\le i<j\le n}(t_i - q^\gamma t_j).$$

Then $\nabla_1\psi(t) := \psi(t) - b_1(t)\cdot T_1\psi(t)$ is written as

$$\nabla_1\psi(t) = \frac{F(t)}{\prod_{j=1}^{n}(1-t_j/x_1)(1-t_j/x_2)}, \tag{A.5}$$

where

$$F(t) = (t_2\cdots t_r)^2\cdot t_{r+1}\cdots t_s\Bigg[(1-t_1/x_1)(1-t_1/x_2)\prod_{1\le i<j\le n}(t_i - q^{\gamma}t_j)$$
$$- q^{\tilde{\alpha}}(1-t_1q^{\beta_1}/x_1)(1-t_1q^{\beta_2}/x_2)\prod_{j=2}^{n}(t_1-q^{-\gamma}t_j)\prod_{2\le i<j\le n}(t_i-q^{\gamma}t_j)\Bigg].$$

We can expand $\prod_{1\le i<j\le n}(t_i - Q_{ij}t_j)$ as follows:

$$\prod_{1\le i<j\le n}(t_i - Q_{ij}t_j) = \sum_{\sigma\in\mathfrak{S}_n} Q_\sigma\cdot t_{\sigma(1)}^{\,n-1}\,t_{\sigma(2)}^{\,n-2}\cdots t_{\sigma(n-1)}$$

like as

$$\prod_{1\le i<j\le n}(t_i - t_j) = \sum_{\sigma\in\mathfrak{S}_n}\mathrm{sgn}\sigma\cdot t_{\sigma(1)}^{\,n-1}\,t_{\sigma(2)}^{\,n-2}\cdots t_{\sigma(n-1)} = \mathcal{A}(t_1^{\,n-1}\,t_2^{\,n-2}\cdots t_{n-1}).$$

Thus monomials which appear in $F(t)$ are the following:

$$\begin{aligned}
&t_1^2t_2^2\cdots t_r^2\cdot t_{r+1}\cdots t_s\cdot t_{\sigma(1)}^{\,n-1}\,t_{\sigma(2)}^{\,n-2}\cdots t_{\sigma(n-1)},\\
&t_1t_2^2\cdots t_r^2\cdot t_{r+1}\cdots t_s\cdot t_{\sigma(1)}^{\,n-1}\,t_{\sigma(2)}^{\,n-2}\cdots t_{\sigma(n-1)},\\
&t_2^2\cdots t_r^2\cdot t_{r+1}\cdots t_s\cdot t_{\sigma(1)}^{\,n-1}\,t_{\sigma(2)}^{\,n-2}\cdots t_{\sigma(n-1)},\quad \sigma\in\mathfrak{S}_n.
\end{aligned}\tag{A.6}$$

We now define a ordering $\prec$ for $l=(l_1,\dots,l_n)$ and $m=(m_1,\dots,m_n)$ as follows:

$$l\prec m \quad \text{if and only if} \quad l_1+\cdots+l_i\le m_1+\cdots+m_i \quad \text{for all} \quad i\ge 1.$$

We set $t^l := t_1^{l_1}\cdots t_n^{l_n}$ and also define $t^l\prec t^m$ by $l\prec m$. The monomial

$$t_1^2t_2^2\cdots t_r^2\cdot t_{r+1}\cdots t_s\cdot t_1^{\,n-1}\,t_2^{\,n-2}\cdots t_{n-1}$$

has maximum order with respect to $\prec$ in (A.6). And the monomials appearing in

$$\mathcal{A}(t_1^2t_2^2\cdots t_r^2\cdot t_{r+1}\cdots t_s\cdot t_1^{\,n-1}\,t_2^{\,n-2}\cdots t_{n-1})$$

are included in those in $\mathcal{A}_{rs}$. Thus $\mathcal{A}F(t)$ can be expressed as a linear combination of $\mathcal{A}_{r,s}(t)$ and $\mathcal{A}_{i,j}(t)$, $(i,j)\prec(r,s)$ as follows:

$$\begin{aligned}
\mathcal{A}F(t) &= \frac{1-q^{\tilde{\alpha}+\beta_1+\beta_2}}{x_1x_2}\mathcal{A}_{r,s}(t) - c'_{r-1,s+1}\,\mathcal{A}_{r-1,s+1}(t) - c'_{r-2,s+2}\,\mathcal{A}_{r-2,s+2}(t)+\cdots\\
&\qquad - c'_{r-1,s}\,\mathcal{A}_{r-1,s}(t) - c'_{r-2,s+1}\,\mathcal{A}_{r-2,s+1}(t)+\cdots\\
&\qquad - c'_{r-1,s-1}\,\mathcal{A}_{r-1,s-1}(t) - c'_{r-2,s+1}\,\mathcal{A}_{r-2,s+1}(t)+\cdots\\
&= \frac{1-q^{\tilde{\alpha}+\beta_1+\beta_2}}{x_1x_2}\mathcal{A}_{r,s}(t) - \sum_{(i,j)\prec(r,s)} c'_{i,j}\,\mathcal{A}_{i,j}(t).
\end{aligned}$$

From Lemma A.3 and (A.5), $\mathcal{A}\Big\{\dfrac{F(t)}{\prod_{j=1}^{n}(1-t_j/x_1)(1-t_j/x_2)}\Big\}$ is cohomologous to 0. This implies that $\varphi_{r,s}(t)$ is cohomologous to a linear combination of $\varphi_{i,j}(t)$'s of lower order:

$$\int \Phi(t)\varphi_{r,s}(t)\varpi = \sum_{(i,j)\prec(r,s)} c''_{i,j} \int \Phi(t)\varphi_{i,j}(t)\varpi. \tag{A.7}$$

The statement of Lemma A.6 follows from a recurrent use of (A.7) and the fact that all $c''_{i,j}$'s do not depend on q. □

Proof of Proposition A.1. From (A.4) and Lemma A.6, we have

$$T_{\tilde{\alpha}}\widetilde{\varphi}_{0,s} = \int \Phi(t)e_n(t)\varphi_{0,s}(t)\varpi = \widetilde{\varphi}_{s,n} = \sum_{r=0}^{n} c_{rs}\widetilde{\varphi}_{0,r}, \tag{A.8}$$

where the coefficients c_{rs} do not depend on q. We set

$$\boldsymbol{y}(q^{\tilde{\alpha}}) = (\widetilde{\varphi}_0, \widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n), \quad \boldsymbol{y}'(q^{\tilde{\alpha}}) = (\widetilde{\varphi}_{00}, \widetilde{\varphi}_{01}, \ldots, \widetilde{\varphi}_{0n}).$$

By Lemma A.5, we have

$$\boldsymbol{y}(q^{\tilde{\alpha}}) = \boldsymbol{y}'(q^{\tilde{\alpha}})U, \quad \boldsymbol{y}(q^{\tilde{\alpha}+1}) = \boldsymbol{y}'(q^{\tilde{\alpha}+1})U$$

where U is the matrix $(u_{rs})_{rs=0}^{n}$ not depending on q. Then we have

$$\boldsymbol{y}'(q^{\tilde{\alpha}+1}) = \boldsymbol{y}'(q^{\tilde{\alpha}})UA(q^{\tilde{\alpha}})U^{-1},$$

and, by (A.8), the matrix $UA(q^{\tilde{\alpha}})U^{-1}$ does not depend on q. Since the matrices $UA(q^{\tilde{\alpha}})U^{-1}$ and U do not depend on q, the condition P4) holds for $A(q^{\tilde{\alpha}})$. □

Appendix B. Proof of Proposition 3.1

We set

$$\begin{aligned}\Phi'(t) &:= \prod_{j=1}^{n} \frac{(qt_j/x_1)_\infty}{(t_jq^{\beta_1}/x_1)_\infty} \frac{(qt_j/x_2)_\infty}{(t_jq^{\beta_2}/x_2)_\infty} \prod_{1\le i<j\le n} \frac{(q^{1-\gamma}t_j/t_i)_\infty}{(q^\gamma t_j/t_i)_\infty} \\ &= \frac{\Phi(t)}{t_1^{\alpha_1}\cdots t_n^{\alpha_n}\prod_{j=0}^{n}(1-t_j/x_1)(1-t_j/x_2)},\end{aligned} \tag{B.1}$$

$$\phi_s(t_1, \ldots, t_n) := \prod_{k=1}^{n-s}(1-q^{\beta_2}t_k/x_2) \prod_{k=n-s+1}^{n}(1-t_k/x_1) \tag{B.2}$$

and

$$D_{(\nu)}^{\gamma}(t_1, \ldots, t_\nu) := \prod_{1\le i<j\le\nu}(t_i - q^{-\gamma}t_j), \quad \text{in particular} \quad D_{(n)}^{0}(t) = D(t), \tag{B.3}$$

so that $\Phi(t)\varphi_s(t) = t_1^{\alpha_1}\cdots t_n^{\alpha_n}\Phi'(t)\mathcal{A}\{\phi_s(t)D_{(n)}^{\gamma}(t)\}$. For simplicity we abbreviate $\xi_{F_r^{n-r}}$ by $\xi = (\xi_1, \ldots, \xi_n)$ only in this section. By the definition (1.2), we have

$$\int_{\langle\xi\rangle} \Phi(t)\varphi_s(t)\varpi = (1-q)^n \sum_{\nu_1=0}^{\infty}\cdots\sum_{\nu_n=0}^{\infty} \Phi(q^{\nu_1}\xi_1, \ldots, q^{\nu_n}\xi_n)\cdot\varphi_s(q^{\nu_1}\xi_1, \ldots, q^{\nu_n}\xi_n),$$

so that

$$\int_{\langle\xi\rangle} \Phi(t)\varphi_s(t)\varpi - (1-q)^n\Phi(\xi)\varphi_s(\xi) = O(\xi_1^{\tilde{\alpha}}\cdots\xi_n^{\tilde{\alpha}}q^{\tilde{\alpha}}), \tag{B.4}$$

because the factor included in the integrand $\Phi(t)\varphi_s(t)$ with respect to $\tilde{\alpha}$ is only $(t_1\cdots t_n)^{\tilde{\alpha}}$. Let $c^+_{A,rs}$ be the constant not depending on $q^{\tilde{\alpha}}$ defined by (3.5), i.e.,

$$c^+_{A,rs}=(1-q)^n\cdot\xi_2^{-2\gamma}\xi_3^{-4\gamma}\cdots\xi_n^{-2(n-1)\gamma}\cdot(\xi_1^{n-1}\xi_2^{n-2}\cdots\xi_{n-1})^{-1}\cdot\Phi'(\xi)\cdot\mathcal{A}\{\phi_s(\xi)D^{\gamma}_{(n)}(\xi)\}.$$

Then $(1-q)^n\Phi(\xi)\varphi_s(\xi)$ is written as $(\xi_1\cdots\xi_n)^{\tilde{\alpha}}\cdot c^+_{A,rs}$. From (B.4), the asymptotic behavior of the matrix Y_ξ at $\tilde{\alpha}\to\infty$ is the following:

$$Y_\xi\sim(q^{\tilde{\alpha}})^{D^+_A}C^+_A,$$

where $(q^{\tilde{\alpha}})^{D^+_A}=(\xi_1\cdots\xi_n)^{\tilde{\alpha}}$ and $C^+_A=\left(c^+_{A,rs}\right)^n_{r,s=0}$.

Before proving Proposition 3.1, we show four lemmas and two propositions.

Lemma B.1. *Let ν_i, $1\le i\le n$, be integers satisfying $\{\nu_1,\nu_2,\dots,\nu_n\}=\{1,2,\dots,n\}$,*

$$\nu_1<\nu_2<\cdots<\nu_{n-r}\quad\text{and}\quad\nu_{n-r+1}<\nu_{n-r+2}<\cdots<\nu_n.$$

For $\sigma\in\mathfrak{S}_n$, we assume

$$\{\sigma(\nu_1),\dots,\sigma(\nu_{n-r})\}=\{1,2,\dots,n-r\},$$
$$\{\sigma(\nu_{n-r+1}),\dots,\sigma(\nu_n)\}=\{n-r+1,\dots,n\}.$$

Then we have

$$n-r=\sigma(\nu_1)>\sigma(\nu_2)>\cdots>\sigma(\nu_{n-r})=1,$$
$$n=\sigma(\nu_{n-r+1})>\sigma(\nu_{n-r+2})>\cdots>\sigma(\nu_n)=n-r+1$$

if and only if

$$\sigma D^{\gamma}_{(n)}(\xi)\ne 0.$$

Proof. We assume that there exist i and j, $i<j$ such that $1\le\sigma(i)<\sigma(j)\le n-r$ or $n-r+1\le\sigma(i)<\sigma(j)\le n$. When the former holds, we define the set $E:=\{(i,j);1\le i<j\le n\text{ and }1\le\sigma(i)<\sigma(j)\le n-r\}$. We take $(i,j)\in E$ such that $\sigma(j)-\sigma(i)$ is minimum. There exists a number k such that

$$\sigma(k)=\sigma(i)+1. \tag{B.5}$$

Then we have

$$\sigma(i)<\sigma(k)\le\sigma(j). \tag{B.6}$$

We now suppose $k\le i$. Then $k<j$ and $\sigma(k)<\sigma(j)$ by (B.6), so that $(k,j)\in E$. By using (B.5), we have

$$\sigma(j)-\sigma(k)=(\sigma(j)-\sigma(i))-1.$$

This contradicts the fact $\sigma(j)-\sigma(i)$ is minimum.

Thus we have $i<k$. By using (B.6), we have $(i,k)\in E$. Since $\sigma(j)-\sigma(i)$ is minimum, it follows that

$$\sigma(k)-\sigma(i)\ge\sigma(j)-\sigma(i). \tag{B.7}$$

From (B.5), (B.6) and (B.7) we have $\sigma(j)=\sigma(k)=\sigma(i)+1$. Then

$$\xi_{\sigma(i)}-q^{-\gamma}\xi_{\sigma(j)}=\xi_{\sigma(i)}-q^{-\gamma}\xi_{\sigma(i)+1}=0$$

because of $q^\gamma\xi_l=\xi_{l+1}$ for $1\le l<n-r$ by definition. Hence $\sigma D^{\gamma}_{(n)}(\xi)=0$. In the case where $n-r+1\le\sigma(i)<\sigma(j)\le n$, we can prove $\sigma D^{\gamma}_{(n)}(\xi)=0$ in the same way as above.

Conversely we assume $\sigma D^{\gamma}_{(n)}(\xi) = 0$. There exist i and j, $i < j$ such that $\xi_{\sigma(i)} - q^{-\gamma}\xi_{\sigma(j)} = 0$. The case $\sigma(i) = n-r$ is in contradiction with $\xi_{\sigma(j)} = q^{\gamma}\xi_{\sigma(i)} = x_1 q^{(n-r)\gamma}$ for generic x_1 and x_2. Thus $\sigma(i) \neq n-r$. Since $q^{\gamma}\xi_l = \xi_{l+1}$ for $l \neq n-r$ by definition, we have $\xi_{\sigma(j)} = q^{\gamma}\xi_{\sigma(i)} = \xi_{\sigma(i)+1}$, so that $\sigma(j) = \sigma(i)+1$. Therefore $1 \leq \sigma(i) < \sigma(j) \leq n-r$ or $n-r+1 \leq \sigma(i) < \sigma(j) \leq n$ □

Lemma B.2. *For $\sigma \in \mathfrak{S}_n$, there exists i, $n-s+1 \leq i$, such that $\sigma(i) = 1$ if and only if*

$$\sigma\phi_s(\xi) = 0.$$

Proof. By recalling that

$$\sigma\phi_s(\xi) = \prod_{i=1}^{n-s}(1 - q^{\beta_2}\xi_{\sigma(i)}/x_2) \prod_{i=n-s+1}^{n}(1 - \xi_{\sigma(i)}/x_1) \quad \text{and} \quad \xi_1 = x_1,$$

the proof easily follows. □

Lemma B.3. *If $r < s$, then $\mathcal{A}\{\phi_s(\xi)D^{\gamma}(\xi)\} = 0$.*

Proof. Suppose that there exists $\sigma \in \mathfrak{S}_n$ such that $\sigma\phi_s(\xi)D^{\gamma}(\xi) \neq 0$. By Lemma B.1, there exist ν_i's, $\nu_1 < \cdots < \nu_{n-r}$ such that $n-r = \sigma(\nu_1) > \sigma(\nu_2) > \cdots > \sigma(\nu_{n-r}) = 1$. By Lemma B.2, we have $\nu_{n-r} \leq n-s$. Thus $1 \leq \nu_1 < \cdots < \nu_{n-r} \leq n-s$. Therefore $n-r \leq n-s$. □

Lemma B.4. *For $r \geq s$, it follows that*

$$\begin{aligned}\mathcal{A}\{\phi_s(\xi)D^{\gamma}_{(n)}(\xi)\} &= \frac{(q^{-\gamma};q^{-\gamma})_{n-s}}{(1-q^{-\gamma})^{n-s}} \cdot \mathrm{sgn}w \cdot w\phi_s(\xi) \\ &\quad \cdot w\left\{D^{0}_{(n-s)}(t_1,\ldots,t_{n-s})\frac{D^{\gamma}_{(n)}(t_1,\ldots,t_n)}{D^{\gamma}_{(n-s)}(t_1,\ldots,t_{n-s})}\right\}\Bigg|_{t=\xi},\end{aligned} \tag{B.8}$$

where $w = w_{rs} \in \mathfrak{S}_n$ is the permutation defined by

$$w(i) = w_{rs}(i) = \begin{cases} i & \text{if} \quad 1 \leq i \leq n-r, \\ i+s & \text{if} \quad n-r+1 \leq i \leq n-s, \\ 2n-r+1-i & \text{if} \quad n-s+1 \leq i \leq n. \end{cases}$$

Proof. We denote by S_{n-s} the subgroup of $\mathfrak{S}_n$ defined by

$$S_{n-s} := \{\sigma \in \mathfrak{S}_n\,;\, \sigma(i) = i \quad \text{for} \quad n-r+1 \leq i \leq n-r+s\} \simeq \mathfrak{S}_{n-s}.$$

By definition, the left-hand side of (B.8) is

$$\mathcal{A}\{\phi_s(\xi)D^{\gamma}_{(n)}(\xi)\} = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}\sigma \cdot \sigma\phi_s(\xi) \cdot \sigma D^{\gamma}(\xi).$$

By Lemmas B.2 and B.3, it is enough to consider the summation when σ runs over the set $S_{n-s}w = \{\sigma \in \mathfrak{S}_n\,;\, \sigma(i) = 2n-r+1-i \quad \text{for} \quad n-s \leq i \leq n\}$. Then we have

$$\begin{aligned}\mathcal{A}\{\phi_s(\xi)D^{\gamma}_{(n)}(\xi)\} &= \sum_{\sigma \in S_{n-s}} \mathrm{sgn}(\sigma w) \cdot \sigma w\phi_s(\xi) \cdot \sigma w D^{\gamma}_{(n)}(\xi) \\ &= \mathrm{sgn}w \cdot w\phi_s(\xi) \sum_{\sigma \in S_{n-s}} \mathrm{sgn}\sigma \cdot \sigma w D^{\gamma}_{(n)}(\xi),\end{aligned}$$

because $\sigma w\phi_s(\xi) = w\phi_s(\xi)$ for $\sigma \in S_{n-s}$. Since the following also holds for $\sigma \in S_{n-s}$:

$$\sigma w \left\{ \frac{D^\gamma_{(n)}(t_1,\ldots,t_n)}{D^\gamma_{(n-s)}(t_1,\ldots,t_{n-s})} \right\} = w \left\{ \frac{D^\gamma_{(n)}(t_1,\ldots,t_n)}{D^\gamma_{(n-s)}(t_1,\ldots,t_{n-s})} \right\},$$

it follows that

$$\begin{aligned} &\mathcal{A}\{\phi_s(\xi)D^\gamma(\xi)\} \\ &= \mathrm{sgn} w \cdot w\phi_s(\xi) \sum_{\sigma\in S_{n-s}} \mathrm{sgn}\sigma \cdot \sigma w \left\{ D^\gamma_{(n-s)}(t_1,\ldots,t_{n-s}) \frac{D^\gamma_{(n)}(t_1,\ldots,t_n)}{D^\gamma_{(n-s)}(t_1,\ldots,t_{n-s})} \right\}\Bigg|_{t=\xi} \\ &= \mathrm{sgn} w \cdot w\phi_s(\xi) \cdot w \left\{ \frac{D^\gamma_{(n)}(t_1,\ldots,t_n)}{D^\gamma_{(n-s)}(t_1,\ldots,t_{n-s})} \right\}\Bigg|_{t=\xi} \cdot \sum_{\sigma\in S_{n-s}} \mathrm{sgn}\sigma \cdot \sigma w D^\gamma_{(n-s)}(\xi_1,\ldots,\xi_{n-s}). \end{aligned} \tag{B.9}$$

From Kadell's lemma (A.2), we have

$$\sum_{\sigma\in S_{n-s}} \mathrm{sgn}\sigma \cdot \sigma \Big(w D^\gamma_{(n-s)}(t_1,\ldots,t_{n-s}) \Big) = \frac{(q^{-\gamma};q^{-\gamma})_{n-s}}{(1-q^{-\gamma})^{n-s}}\ w D^0_{(n-s)}(t_1,\ldots,t_{n-s}). \tag{B.10}$$

The result follows from (B.9) and (B.10). □

Proposition B.5. *If $r < s$, then $c^+_{A,rs} = 0$. For $r \geq s$, $c^+_{A,rs}$ is expressed as follows:*

$$\begin{aligned} c^+_{A,rs} = {} & \xi_2^{-2\gamma}\xi_3^{-4\gamma}\cdots\xi_n^{-2(n-1)\gamma} \cdot (\xi_1^{n-1}\xi_2^{n-2}\cdots\xi_{n-1})^{-1} \cdot \Phi'(\xi) \cdot \frac{(q^{-\gamma};q^{-\gamma})_{n-s}}{(1-q^{-\gamma})^{n-s}} \\ & \cdot \mathrm{sgn} w \cdot w\phi_s(\xi) \cdot w \left\{ D^0_{(n-s)}(t_1,\ldots,t_{n-s}) \frac{D^\gamma_{(n)}(t_1,\ldots,t_n)}{D^\gamma_{(n-s)}(t_1,\ldots,t_{n-s})} \right\}\Bigg|_{t=\xi}. \end{aligned}$$

Proof. Proposition B.5 follows from Lemmas B.3 and B.4. □

Proposition B.6. *We have*

$$\begin{aligned} &(t_1^{n-1}t_2^{n-2}\cdots t_{n-1})^{-1} \cdot \mathrm{sgn} w \cdot w \left\{ D^0_{(n-s)}(t_1,\ldots,t_{n-s}) \frac{D^\gamma_{(n)}(t_1,\ldots,t_n)}{D^\gamma_{(n-s)}(t_1,\ldots,t_{n-s})} \right\} \\ &= q^{-s(2r-s-1)\gamma/2} \prod_{1\leq i<j\leq n} (1 - d_{ij}\frac{t_j}{t_i}), \end{aligned} \tag{B.11}$$

where

$$d_{ij} = \begin{cases} q^{-\gamma} & \text{if} \quad 1 \leq i \leq n-r, \quad n-r+1 \leq j \leq n-r+s, \\ q^\gamma & \text{if} \quad n-r+1 \leq i \leq n-r+s, \quad i < j \leq n, \\ 1 & \text{otherwise.} \end{cases}$$

We denote the right-hand side of (B.11) by $D'_{rs}(t)$.

Proof. Since $\mathrm{sgn} w = (-1)^{s(2r-s-1)/2}$, we have to show the following identity instead of (B.11):

$$w \left\{ D^0_{(n-s)}(t_1,\ldots,t_{n-s}) \frac{D^\gamma_{(n)}(t_1,\ldots,t_n)}{D^\gamma_{(n-s)}(t_1,\ldots,t_{n-s})} \right\} = (-q^{-\gamma})^{s(2r-s-1)/2} \prod_{1\leq i<j\leq n} (t_i - d_{ij}t_j). \tag{B.12}$$

We expand the left-hand side of (B.12) without w:

$$
\begin{aligned}
&D^0_{(n-s)}(t_1,\ldots,t_{n-s})\frac{D^\gamma_{(n)}(t_1,\ldots,t_n)}{D^\gamma_{(n-s)}(t_1,\ldots,t_{n-s})}\\
&= \prod_{1\le i<j\le n-s}(t_i-t_j)\cdot\prod_{i=1}^{n-s}\prod_{j=n-s+1}^{n}(t_i-q^{-\gamma}t_j)\cdot\prod_{n-s+1\le i<j\le n}(t_i-q^{-\gamma}t_j)\\
&= \prod_{1\le i<j\le n-r}(t_i-t_j)\cdot\prod_{i=1}^{n-r}\prod_{j=n-r+1}^{n-s}(t_i-t_j)\cdot\prod_{n-r+1\le i<j\le n-s}(t_i-t_j)\\
&\quad\cdot\prod_{i=1}^{n-r}\prod_{j=n-s+1}^{n}(t_i-q^{-\gamma}t_j)\cdot\prod_{i=n-r+1}^{n-s}\prod_{j=n-s+1}^{n}(t_i-q^{-\gamma}t_j)\cdot\prod_{n-s+1\le i<j\le n}(t_i-q^{-\gamma}t_j).
\end{aligned}
$$

Therefore we can express the left-hand side of (B.12) as follows:

$$
\begin{aligned}
&w\left\{D^0_{(n-s)}(t_1,\ldots,t_{n-s})\frac{D^\gamma_{(n)}(t_1,\ldots,t_n)}{D^\gamma_{(n-s)}(t_1,\ldots,t_{n-s})}\right\}\\
&= \prod_{1\le i<j\le n-r}(t_{w(i)}-t_{w(j)})\cdot\prod_{i=1}^{n-r}\prod_{j=n-r+1}^{n-s}(t_{w(i)}-t_{w(j)})\cdot\prod_{n-r+1\le i<j\le n-s}(t_{w(i)}-t_{w(j)})\\
&\quad\cdot\prod_{i=1}^{n-r}\prod_{j=n-s+1}^{n}(t_{w(i)}-q^{-\gamma}t_{w(j)})\cdot\prod_{i=n-r+1}^{n-s}\prod_{j=n-s+1}^{n}(t_{w(i)}-q^{-\gamma}t_{w(j)})\\
&\quad\cdot\prod_{n-s+1\le i<j\le n}(t_{w(i)}-q^{-\gamma}t_{w(j)})\\
&= \prod_{1\le i<j\le n-r}(t_i-t_j)\cdot\prod_{i=1}^{n-r}\prod_{j=n-r+s+1}^{n}(t_i-t_j)\cdot\prod_{n-r+s+1\le i<j\le n}(t_i-t_j)\\
&\quad\cdot\prod_{i=1}^{n-r}\prod_{j=n-r+1}^{n-r+s}(t_i-q^{-\gamma}t_j)\cdot\left[\prod_{i=n-r+s+1}^{n}\prod_{j=n-r+1}^{n-r+s}(t_i-q^{-\gamma}t_j)\cdot\prod_{n-r+s\ge i>j\ge n-r+1}(t_i-q^{-\gamma}t_j)\right]\\
&= \prod_{1\le i<j\le n-r}(t_i-t_j)\cdot\prod_{i=1}^{n-r}\prod_{j=n-r+s+1}^{n}(t_i-t_j)\cdot\prod_{n-r+s+1\le i<j\le n}(t_i-t_j)\\
&\quad\cdot\prod_{i=1}^{n-r}\prod_{j=n-r+1}^{n-r+s}(t_i-q^{-\gamma}t_j)\cdot\prod_{i=n-r+1}^{n-r+s}\prod_{i<j\le n}(-q^{-\gamma})(t_i-q^{\gamma}t_j).
\end{aligned}
$$

This coincides with the right-hand side of (B.12). □

Proof of Proposition 3.1. From Proposition B.6, it follows that

$$
\frac{c^+_{A,rs}}{c^+_{A,rr}}=\left(\frac{c^+_{A,rs}}{c^+_{A,rr}}\right)_0=\frac{w_{rs}\phi_s(\xi)\cdot D'_{rs}(\xi)\cdot(q^{-\gamma};q^{-\gamma})_{n-s}}{w_{rr}\phi_r(\xi)\cdot D'_{rr}(\xi)\cdot(q^{-\gamma};q^{-\gamma})_{n-r}}\cdot\frac{1}{(1-q^{-\gamma})^{r-s}}. \tag{B.13}
$$

Since, by definition of w_{rs}, $w_{rs}\phi_s(\xi)$ is

$$w_{rs}\phi_s(\xi) = \prod_{i=1}^{n-r}(1-q^{\beta_2}\xi_i/x_2)\prod_{i=n-r+s}^{n}(1-q^{\beta_2}\xi_i/x_2)\prod_{i=n-r+1}^{n-r+s}(1-\xi_i/x_1),$$

we have

$$\frac{w_{rs}\phi_s(\xi)}{w_{rr}\phi_r(\xi)} = \prod_{i=n-r+s+1}^{n}\frac{(1-q^{\beta_2}\xi_i/x_2)}{(1-\xi_i/x_1)} = \frac{(q^{\beta_2+s\gamma};q^\gamma)_{r-s}}{(\frac{x_2}{x_1}q^{s\gamma};q^\gamma)_{r-s}}. \tag{B.14}$$

From (B.11) we have

$$\frac{D'_{rs}(\xi)}{D'_{rr}(\xi)} = q^{-(r-s)(r-s-1)\gamma/2}\prod_{i=1}^{n-r}\prod_{j=n-r+s+1}^{n}\frac{1-\xi_j/\xi_i}{1-q^{-\gamma}\xi_j/\xi_i}\cdot\prod_{n-r+s+1\le i<j\le n}\frac{1-\xi_j/\xi_i}{1-q^{\gamma}\xi_j/\xi_i}. \tag{B.15}$$

We calculate factors in (B.15) as follows:

$$\prod_{i=1}^{n-r}\prod_{j=n-r+s+1}^{n}\frac{1-\xi_j/\xi_i}{1-q^{-\gamma}\xi_j/\xi_i} = \prod_{j=n-r+s+1}^{n}\frac{1-\xi_j/\xi_1}{1-q^{-\gamma}\xi_j/\xi_{n-r}} = \frac{(\frac{x_2}{x_1}q^{s\gamma};q^\gamma)_{r-s}}{(\frac{x_2}{x_1}q^{(s+r-n)\gamma};q^\gamma)_{r-s}} \tag{B.16}$$

and

$$\prod_{n-r+s+1\le i<j\le n}\frac{1-\xi_j/\xi_i}{1-q^{\gamma}\xi_j/\xi_i} = \prod_{n-r+s+1\le i\le n-1}\frac{1-\xi_{i+1}/\xi_i}{1-q^{\gamma}\xi_n/\xi_i} = \frac{(1-q^\gamma)^{r-s}}{(q^\gamma;q^\gamma)_{r-s}}. \tag{B.17}$$

Thus, by using (B.16) and (B.17) for (B.15), we have

$$\begin{aligned}\frac{D'_{rs}(\xi)}{D'_{rr}(\xi)} &= q^{-(r-s)(r-s-1)\gamma/2}\frac{(\frac{x_2}{x_1}q^{s\gamma};q^\gamma)_{r-s}}{(\frac{x_2}{x_1}q^{(s+r-n)\gamma};q^\gamma)_{r-s}}\frac{(1-q^\gamma)^{r-s}}{(q^\gamma;q^\gamma)_{r-s}}\\ &= \frac{(\frac{x_2}{x_1}q^{s\gamma};q^\gamma)_{r-s}}{(\frac{x_2}{x_1}q^{(s+r-n)\gamma};q^\gamma)_{r-s}}\frac{(1-q^{-\gamma})^{r-s}}{(q^{-\gamma};q^{-\gamma})_{r-s}}.\end{aligned} \tag{B.18}$$

Hence, from (B.13), (B.14) and (B.18), it follows that

$$\begin{aligned}\frac{c^+_{A,rs}}{c^+_{A,rr}} = \Big(\frac{c^+_{A,rs}}{c^+_{A,rr}}\Big)_0 &= \frac{(q^{\beta_2+s\gamma};q^\gamma)_{r-s}}{(\frac{x_2}{x_1}q^{(s+r-n)\gamma};q^\gamma)_{r-s}}\frac{(q^{-\gamma};q^{-\gamma})_{n-s}}{(q^{-\gamma};q^{-\gamma})_{r-s}\cdot(q^{-\gamma};q^{-\gamma})_{n-r}}\\ &= q^{-(r-s)(n-r)\gamma}\frac{(q^{\beta_2+s\gamma};q^\gamma)_{r-s}}{(\frac{x_2}{x_1}q^{(s+r-n)\gamma};q^\gamma)_{r-s}}\frac{(q^{\gamma};q^{\gamma})_{n-s}}{(q^{\gamma};q^{\gamma})_{r-s}\cdot(q^{\gamma};q^{\gamma})_{n-r}}.\end{aligned}$$

Next we show the former part of Proposition 3.1. From Propositions B.5 and B.6, we have

$$\begin{aligned}(c^+_{A,rr})_0 &= \big(\Phi'(\xi)\big)_0\ w_{rr}\phi_r(\xi)\ D'_{rr}(\xi)\ \frac{(q^{-\gamma};q^{-\gamma})_{n-r}}{(1-q^{-\gamma})^{n-r}}\\ &= \big(\Phi'(\xi)\big)_0\ w_{rr}\phi_r(\xi)\prod_{1\le i<j\le n}(1-d'_{ij}\xi_j/\xi_i)\cdot q^{-r(r-1)\gamma/2}\frac{(q^{-\gamma};q^{-\gamma})_{n-r}}{(1-q^{-\gamma})^{n-r}}\\ &= q^{-r(r-1)\gamma/2-(n-r)(n-r-1)\gamma/2}\frac{(q^{\gamma};q^{\gamma})_{n-r}}{(1-q^{\gamma})^{n-r}}\\ &\qquad\cdot\big(\Phi'(\xi)\big)_0\ w_{rr}\phi_r(\xi)\prod_{1\le i<j\le n}(1-d'_{ij}\xi_j/\xi_i),\end{aligned} \tag{B.19}$$

where

$$d'_{ij}=\begin{cases}1 & \text{if} \quad 1\le i<j\le n-r,\\ q^{-\gamma} & \text{if} \quad 1\le i\le n-r, \quad n-r+1\le j\le n,\\ q^{\gamma} & \text{if} \quad n-r+1\le i<j\le n.\end{cases}$$

Since $\big(\Phi'(\xi)\big)_0$ and $w_{rr}\phi_r(\xi)$ in (B.19) are calculated as follows:

$$\begin{aligned}\big(\Phi'(\xi)\big)_0&=\left(\prod_{j=1}^n\frac{(q\xi_j/x_1)_\infty}{(\xi_jq^{\beta_1}/x_1)_\infty}\frac{(q\xi_j/x_2)_\infty}{(\xi_jq^{\beta_2}/x_2)_\infty}\prod_{1\le i<j\le n}\frac{(q^{1-\gamma}\xi_j/\xi_i)_\infty}{(q^\gamma\xi_j/\xi_i)_\infty}\right)_0\\ &=\prod_{j=1}^n\frac{1}{(1-\xi_jq^{\beta_1}/x_1)_\infty\cdot(1-\xi_jq^{\beta_2}/x_2)_\infty}\prod_{1\le i<j\le n}\frac{1}{(1-q^\gamma\xi_j/\xi_i)_\infty}\\ &=\frac{1}{(q^{\beta_1};q^\gamma)_{n-r}\cdot(\frac{x_2}{x_1}q^{\beta_1};q^\gamma)_r\cdot(q^{\beta_2}/\frac{x_2}{x_1};q^\gamma)_{n-r}\cdot(q^{\beta_2};q^\gamma)_r}\prod_{1\le i<j\le n}\frac{1}{(1-q^\gamma\xi_j/\xi_i)_\infty}\end{aligned}$$

and

$$w_{rr}\phi_r(\xi)=\prod_{i=1}^{n-r}(1-q^{\beta_2}\xi_i/x_2)\prod_{i=n-r+1}^{n}(1-\xi_i/x_1)=(q^{\beta_2}/\tfrac{x_2}{x_1};q^\gamma)_{n-r}\cdot(\tfrac{x_2}{x_1};q^\gamma)_r,$$

we have

$$\begin{aligned}(c^+_{A,rr})_0&=q^{-r(r-1)\gamma/2-(n-r)(n-r-1)\gamma/2}\frac{(q^\gamma;q^\gamma)_{n-r}}{(1-q^\gamma)^{n-r}}\\ &\quad\cdot\frac{(\frac{x_2}{x_1};q^\gamma)_r}{(q^{\beta_1};q^\gamma)_{n-r}\cdot(\frac{x_2}{x_1}q^{\beta_1};q^\gamma)_r\cdot(q^{\beta_2};q^\gamma)_r}\prod_{1\le i<j\le n}\frac{1-d'_{ij}\xi_j/\xi_i}{1-q^\gamma\xi_j/\xi_i}.\end{aligned}\tag{B.20}$$

A factor $\prod_{1\le i<j\le n}\frac{1-d'_{ij}\xi_j/\xi_i}{1-q^\gamma\xi_j/\xi_i}$ in (B.20) is evaluated as follows:

$$\begin{aligned}\prod_{1\le i<j\le n}\frac{1-d'_{ij}\xi_j/\xi_i}{1-q^\gamma\xi_j/\xi_i}&=\prod_{1\le i<j\le n-r}\frac{1-\xi_j/\xi_i}{1-q^\gamma\xi_j/\xi_i}\prod_{i=1}^{n-r}\prod_{j=n-r+1}^{n}\frac{1-q^{-\gamma}\xi_j/\xi_i}{1-q^\gamma\xi_j/\xi_i}\\ &=\prod_{1\le i<j\le n-r}\frac{1-\xi_j/\xi_i}{1-q^\gamma\xi_j/\xi_i}\cdot\prod_{i=1}^{n-r}\prod_{j=n-r+1}^{n}\frac{1-q^{-\gamma}\xi_j/\xi_i}{1-\xi_j/\xi_i}\cdot\prod_{i=1}^{n-r}\prod_{j=n-r+1}^{n}\frac{1-\xi_j/\xi_i}{1-q^\gamma\xi_j/\xi_i}\\ &=\frac{(1-q^\gamma)^{n-r}}{(q^\gamma;q^\gamma)_{n-r}}\cdot\frac{(\frac{x_2}{x_1}q^{-(n-r)\gamma};q^\gamma)_r}{(\frac{x_2}{x_1};q^\gamma)_r}\cdot\frac{(\frac{x_2}{x_1}q^{-(n-r-1)\gamma};q^\gamma)_r}{(\frac{x_2}{x_1}q^\gamma;q^\gamma)_r}.\end{aligned}\tag{B.21}$$

Therefore the proposition follows from (B.20) and (B.21). □

Appendix C. Proof of Proposition 3.2

In this section, we abbreviate the point $\eta_{F_r^{n-r}}\in(\mathbb{C}^*)^n$ by $\eta=(\eta_1,\dots,\eta_n)$. For the cycle $\langle\eta\rangle$, we have already given the definition of the regularized Jackson integral as follows:

$$\int_{\langle\eta\rangle}\Phi(t)\varphi(t)\varpi:=(1-q)^n\sum_{(\nu_1,\dots,\nu_n)\in\mathbb{Z}^n}\operatorname*{Res}_{\substack{t_1=\eta_1q^{\nu_1},\\ \dots\dots,\\ t_n=\eta_nq^{\nu_n}}}\Phi(t_1,\dots,t_n)\varphi(t_1,\dots,t_n)\frac{dt_1}{t_1}\wedge\dots\wedge\frac{dt_n}{t_n}$$

$$= (1-q)^n \sum_{\nu_1=0}^{\infty} \cdots \sum_{\nu_n=0}^{\infty} \operatorname*{Res}_{\substack{t_1=\eta_1 q^{-\nu_1},\\ \cdots\cdots,\\ t_n=\eta_n q^{-\nu_n}}} \Phi(t)\varphi(t) \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}. \qquad \text{(C.1)}$$

We define $\Phi'(t)$, $\phi_s(t)$ and $D^{\gamma}_{(n)}(t)$ as in (B.1), (B.2) and (B.3) in Appendix B. Since the factor with respect to $\tilde{\alpha}$ in the function $\Phi(t)\varphi_s(t) = t_1^{\alpha_1} \cdots t_n^{\alpha_n} \Phi'(t) \mathcal{A}\{\phi_s(t) D^{\gamma}_{(n)}(t)\}$ is only $(t_1 \cdots t_n)^{\tilde{\alpha}}$, by (C.1), we have

$$\int_{\langle \eta \rangle} \Phi(t)\varphi_s(t)\varpi - (1-q)^n \operatorname*{Res}_{t=\eta} \Phi(t)\varphi_s(t) = O(\eta_1^{\tilde{\alpha}} \cdots \eta_n^{\tilde{\alpha}} q^{-\tilde{\alpha}}). \qquad \text{(C.2)}$$

Let $c^-_{A,rs}$ be the constant not depending on $q^{\tilde{\alpha}}$ defined by (3.5):

$$c^-_{A,rs} = (1-q)^n \cdot \eta_2^{-2\gamma} \eta_3^{-4\gamma} \cdots \eta_n^{-2(n-1)\gamma} (\eta_1^{n-1} \eta_2^{n-2} \cdots \eta_{n-1})^{-1} \cdot \operatorname*{Res}_{t=\eta} \Phi'(t) \cdot \mathcal{A}\{\phi_s(\eta) D^{\gamma}_{(n)}(\eta)\}.$$

Then $(1-q)^n \operatorname{Res}_{t=\eta} \Phi(t)\varphi_s(t)$ is written as $(\eta_1 \cdots \eta_n)^{\tilde{\alpha}} \cdot c^-_{A,rs}$. From (C.2), the asymptotic behavior of the matrix Y_η at $\tilde{\alpha} \to -\infty$ is the following:

$$Y_\eta \sim (q^{\tilde{\alpha}})^{D^-_A} C^-_A,$$

where $(q^{\tilde{\alpha}})^{D^-_A} = (\eta_1 \cdots \eta_n)^{\tilde{\alpha}}$ and $C^-_A = \left(c^-_{A,rs}\right)^n_{r,s=0}$.

Before proving Proposition 3.2, we show four lemmas and a proposition.

Lemma C.1. *Let* ν_i, $1 \le i \le n$, *be* n *integers satisfying* $\{\nu_1, \nu_2 \ldots, \nu_n\} = \{1, 2, \ldots, n\}$,

$$\nu_1 < \nu_2 < \cdots < \nu_{n-r} \quad \text{and} \quad \nu_{n-r+1} < \nu_{n-r+2} < \cdots < \nu_n.$$

For $\sigma \in \mathfrak{S}_n$, *we assume*

$$\{\sigma(\nu_1), \ldots, \sigma(\nu_{n-r})\} = \{1, 2, \ldots, n-r\},$$
$$\{\sigma(\nu_{n-r+1}), \ldots, \sigma(\nu_n)\} = \{n-r+1, \ldots, n\}.$$

Then we have

$$\sigma(\nu_i) = i \quad \text{for} \quad 1 \le i \le n$$

if and only if

$$\sigma D^{\gamma}_{(n)}(\eta) \neq 0.$$

Proof. We can prove this lemma in same way as Lemma B.1. □

Lemma C.2. *For* $\sigma \in \mathfrak{S}_n$, *there exists* i, $i \le n-s$, *such that* $\sigma(i) = n-r+1$ *if and only if*

$$\sigma\phi_s(\eta) = 0.$$

Proof. From

$$\sigma\phi_s(\eta) = \prod_{i=1}^{n-s} (1 - q^{\beta_2}\eta_{\sigma(i)}/x_2) \prod_{i=n-s+1}^{n} (1 - \eta_{\sigma(i)}/x_1),$$

the lemma follows because $\eta_{n-r+1} = x_2 q^{-\beta_2}$. □

Lemma C.3. *If* $r > s$, *then* $\mathcal{A}\{\phi_s(\eta) D^{\gamma}_{(n)}(\eta)\} = 0$.

Proof. Suppose that there exists $\sigma \in \mathfrak{S}_n$ such that $\sigma\{\phi_s(\eta)D^{\gamma}_{(n)}(\eta)\} \neq 0$. By Lemmas C.1 and C.2, we have

$$\sigma^{-1}(n-r+1) < \sigma^{-1}(n-r+1) < \cdots < \sigma^{-1}(n) \leq n$$

and

$$n-s+1 \leq \sigma^{-1}(n-r+1).$$

Therefore $r \leq s$. □

Lemma C.4. *For $r \leq s$, it follows that*

$$\mathcal{A}\{\phi_s(\eta)D^{\gamma}_{(n)}(\eta)\} = \frac{(q^{-\gamma};q^{-\gamma})_s}{(1-q^{-\gamma})^s}\cdot\phi_s(\eta)\cdot\left\{\frac{D^{\gamma}_{(n)}(t_1,\ldots,t_n)}{D^{\gamma}_{(s)}(t_{n-s+1},\ldots,t_n)}D^0_{(s)}(t_{n-s+1},\ldots,t_n)\right\}\bigg|_{t=\eta}.$$

Proof. We can prove Lemma C.4 in the same way as Lemma B.4. □

Proposition C.5. *If $r > s$, then $c^-_{A,rs} = 0$. For $r \leq s$, $c^-_{A,rs}$ is expressed as*

$$c^-_{A,rs} = (1-q)^n\cdot\eta_2^{-2\gamma}\eta_3^{-4\gamma}\cdots\eta_n^{-2(n-1)\gamma}\cdot\operatorname*{Res}_{t=\eta}\Phi'(t)\cdot\phi_s(\eta)\cdot(\eta_1^{n-1}\eta_2^{n-2}\cdots\eta_{n-1})^{-1}$$
$$\cdot\frac{(q^{-\gamma};q^{-\gamma})_s}{(1-q^{-\gamma})^s}\cdot\left\{\frac{D^{\gamma}_{(n)}(t_1,\ldots,t_n)}{D^{\gamma}_{(s)}(t_{n-s+1},\ldots,t_n)}D^0_{(s)}(t_{n-s+1},\ldots,t_n)\right\}\bigg|_{t=\eta}.$$

Proof. Proposition C.5 follows from Lemmas C.3 and C.4. □

Proof of Proposition 3.2. We set

$$D^*_s(t) := \prod_{i=1}^{n-s}\prod_{j=i+1}^{n}(1-q^{-\gamma}t_j/t_i)\prod_{n-s+1\leq i<j\leq n}(1-t_j/t_i),$$

which is equal to

$$(t_1^{n-1}t_2^{n-2}\cdots t_{n-1})^{-1}\cdot\frac{D^{\gamma}_{(n)}(t_1,\ldots,t_n)}{D^{\gamma}_{(s)}(t_{n-s+1},\ldots,t_n)}D^0_{(s)}(t_{n-s+1},\ldots,t_n).$$

In order to evaluate the constant $(c^-_{A,rs}/c^-_{A,rr})_0$ explicitly, we calculate $D^*_s(\eta)/D^*_r(\eta)$ and $\phi_s(\eta)/\phi_r(\eta)$ first:

$$\begin{aligned}\frac{D^*_s(\eta)}{D^*_r(\eta)} &= \prod_{n-s+1\leq i<j\leq n-r}\frac{(1-\eta_j/\eta_i)}{(1-q^{-\gamma}\eta_j/\eta_i)}\cdot\prod_{i=n-s+1}^{n-r}\prod_{j=n-r+1}^{n}\frac{(1-\eta_j/\eta_i)}{(1-q^{-\gamma}\eta_j/\eta_i)}\\ &= \prod_{i=n-s+1}^{n-r}\frac{(1-\eta_{i+1}/\eta_i)}{(1-q^{-\gamma}\eta_{n-r}/\eta_i)}\cdot\prod_{i=n-s+1}^{n-r}\frac{(1-\eta_{n-r+1}/\eta_i)}{(1-q^{-\gamma}\eta_n/\eta_i)}\\ &= \frac{(1-q^{-\gamma})^{s-r}}{(q^{-\gamma};q^{-\gamma})_{s-r}}\cdot\frac{(\frac{x_2}{x_1}q^{\beta_1-\beta_2+(n-s)\gamma};q^{\gamma})_{s-r}}{(\frac{x_2}{x_1}q^{\beta_1-\beta_2+(n-s-r)\gamma};q^{\gamma})_{s-r}},\end{aligned}\tag{C.3}$$

$$\frac{\phi_s(\eta)}{\phi_r(\eta)} = \prod_{i=n-s+1}^{n-r}\frac{(1-\eta_i/x_1)}{(1-q^{\beta_2}\eta_i/x_2)} = (\tfrac{x_2}{x_1}q^{-\beta_2})^{s-r}\prod_{i=n-s+1}^{n-r}\frac{(1-x_1/\eta_i)}{(1-q^{-\beta_2}x_2/\eta_i)}$$

$$= (\tfrac{x_2}{x_1} q^{-\beta_2})^{s-r} \cdot \frac{(q^{\beta_1+(n-s)\gamma}; q^{\gamma})_{s-r}}{(\frac{x_2}{x_1} q^{\beta_1-\beta_2+(n-s)\gamma}; q^{\gamma})_{s-r}}. \tag{C.4}$$

From Proposition C.5, by using (C.3) and (C.4), it follows that

$$\begin{aligned}
\left(\frac{c^-_{A,rs}}{c^-_{A,rr}}\right)_0 &= \frac{c^-_{A,rs}}{c^-_{A,rr}} = \frac{\phi_s(\eta)}{\phi_r(\eta)} \cdot \frac{D^*_s(\eta)}{D^*_r(\eta)} \cdot \frac{(q^{-\gamma}; q^{-\gamma})_s}{(q^{-\gamma}; q^{-\gamma})_r \cdot (1-q^{-\gamma})_r} \\
&= (\tfrac{x_2}{x_1} q^{-\beta_2})^{s-r} \frac{(q^{\beta_1+(n-s)\gamma}; q^{\gamma})_{s-r}}{(\frac{x_2}{x_1} q^{\beta_1-\beta_2+(n-s-r)\gamma}; q^{\gamma})_{s-r}} \cdot \frac{(q^{-\gamma}; q^{-\gamma})_s}{(q^{-\gamma}; q^{-\gamma})_r \cdot (q^{-\gamma}; q^{-\gamma})_{s-r}} \\
&= (\tfrac{x_2}{x_1} q^{-\beta_2-r\gamma})^{s-r} \frac{(q^{\beta_1+(n-s)\gamma}; q^{\gamma})_{s-r}}{(\frac{x_2}{x_1} q^{\beta_1-\beta_2+(n-s-r)\gamma}; q^{\gamma})_{s-r}} \cdot \frac{(q^{\gamma}; q^{\gamma})_s}{(q^{\gamma}; q^{\gamma})_r \cdot (q^{\gamma}; q^{\gamma})_{s-r}}.
\end{aligned}$$

Next we show the former part of Proposition 3.2. By Proposition C.5, we have

$$(c^-_{A,rr})_0 = \Big(\operatorname*{Res}_{t=\eta} \Phi'(t) \cdot \phi_r(\eta)\Big)_0 \cdot D^*_r(\eta) \cdot \frac{(q^{-\gamma}; q^{-\gamma})_r}{(1-q^{-\gamma})^r}. \tag{C.5}$$

An explicit form of $\operatorname{Res}_{t=\eta} \Phi'(t) \cdot \phi_r(\eta)$ is expressed as follows:

$$\begin{aligned}
\operatorname*{Res}_{t=\eta} \Phi'(t)\phi_r(t) &= \frac{(q\eta_1/x_1)_\infty}{-(q)_\infty} \prod_{i=2}^{n-r} \frac{(q\eta_i/x_1)_\infty}{(\eta_i q^{\beta_1}/x_1)_\infty} \prod_{i=n-r+1}^{n} \frac{(\eta_i/x_1)_\infty}{(\eta_i q^{\beta_1}/x_1)_\infty} \\
&\cdot \frac{(q\eta_{n-r+1}/x_2)_\infty}{-(q)_\infty} \prod_{i=1}^{n-r} \frac{(q\eta_i/x_2)_\infty}{(\eta_i q^{1+\beta_2}/x_2)_\infty} \prod_{i=n-r+2}^{n} \frac{(qt_i/x_2)_\infty}{(t_i q^{\beta_2}/x_2)_\infty} \\
&\cdot \frac{(q^{1-\gamma}\eta_{n-r+1}/\eta_{n-r})_\infty}{(q^{\gamma}\eta_{n-r+1}/\eta_{n-r})_\infty} \prod_{i\neq n-r} \frac{(q^{1-\gamma}\eta_{i+1}/\eta_i)_\infty}{-(q)_\infty} \prod_{1\le i<j\le n-1} \frac{(q^{1-\gamma}\eta_{j+1}/\eta_i)_\infty}{(q^{\gamma}\eta_{j+1}/\eta_i)_\infty},
\end{aligned}$$

so that

$$\begin{aligned}
&\Big(\operatorname*{Res}_{t=\eta} \Phi'(t)\phi_r(t)\Big)_0 \\
&= (-1)^n \prod_{i=2}^{n-r} \frac{1}{1-\eta_i q^{\beta_1}/x_1} \prod_{i=n-r+1}^{n} \frac{1-\eta_i/x_1}{1-\eta_i q^{\beta_1}/x_1} \prod_{i=n-r+2}^{n} \frac{1}{1-\eta_i q^{\beta_2}/x_2} \\
&\cdot \prod_{1\le i<j\le n-r-1} \frac{1}{(1-q^{\gamma}t_{j+1}/t_i)} \prod_{i=1}^{n-r} \prod_{j=n-r+1}^{n} \frac{1}{(1-q^{\gamma}t_j/t_i)} \prod_{n-r+1\le i<j\le n-1} \frac{1}{(1-q^{\gamma}t_{j+1}/t_i)} \\
&= (-1)^n \prod_{i=2}^{n-r} \frac{1}{1-\eta_i q^{\beta_1}} \prod_{i=n-r+1}^{n} \frac{1-\eta_i}{1-\eta_i q^{\beta_1}} \prod_{i=n-r+2}^{n} \frac{1}{1-\eta_i q^{\beta_2}/x} \\
&\quad \cdot \prod_{1\le i<j\le n-r-1} \frac{1}{(1-\eta_j/\eta_i)} \prod_{i=1}^{n-r} \prod_{j=n-r+1}^{n} \frac{1}{(1-q^{\gamma}\eta_j/\eta_i)} \prod_{n-r+1\le i<j\le n-1} \frac{1}{(1-\eta_j/\eta_i)}.
\end{aligned} \tag{C.6}$$

We calculate $D^*_r(\eta)$ as

$$D^*_r(\eta) = \prod_{1\le i<j\le n-r} (1-q^{-\gamma}\eta_j/\eta_i) \prod_{i=1}^{n-r} \prod_{j=n-r+1}^{n} (1-q^{-\gamma}\eta_j/\eta_i) \prod_{n-r+1\le i<j\le n} (1-\eta_j/\eta_i). \tag{C.7}$$

From (C.5), (C.6) and (C.7), we have

$$(c^-_{A,rr})_0 = (-1)^n \prod_{i=2}^{n-r} \frac{1}{1-\eta_i q^{\beta_1}} \prod_{i=n-r+1}^{n} \frac{1-\eta_i}{1-\eta_i q^{\beta_1}} \prod_{i=n-r+2}^{n} \frac{1}{1-\eta_i q^{\beta_2}/x}$$
$$\cdot \prod_{1\le i<j\le n-r-1} \frac{1}{(1-\eta_j/\eta_i)} \prod_{i=1}^{n-r} \prod_{j=n-r+1}^{n} \frac{1}{(1-q^{\gamma}\eta_j/\eta_i)} \prod_{n-r+1\le i<j\le n-1} \frac{1}{(1-\eta_j/\eta_i)}$$
$$\cdot \prod_{1\le i<j\le n-r} (1-q^{-\gamma}\eta_j/\eta_i) \prod_{i=1}^{n-r} \prod_{j=n-r+1}^{n} (1-q^{-\gamma}\eta_j/\eta_i) \prod_{n-r+1\le i<j\le n} (1-\eta_j/\eta_i)$$
$$\cdot \frac{(q^{-\gamma};q^{-\gamma})_r}{(1-q^{-\gamma})^r}. \tag{C.8}$$

We calculate the factors appearing in (C.8) as follows:

$$\prod_{i=2}^{n-r} \frac{1}{1-\eta_i q^{\beta_1}/x_1} \prod_{i=n-r+1}^{n} \frac{1-\eta_i/x_1}{1-\eta_i q^{\beta_1}/x_1} \prod_{i=n-r+2}^{n} \frac{1}{1-\eta_i q^{\beta_2}/x_2}$$
$$= \frac{1}{(q^{-\gamma};q^{-\gamma})_{n-r-1}} \cdot \frac{(\frac{x_2}{x_1} q^{-\beta_2-(r-1)\gamma};q^{\gamma})_r}{(\frac{x_2}{x_1} q^{\beta_1-\beta_2-(r-1)\gamma};q^{\gamma})_r} \cdot \frac{1}{(q^{-\gamma};q^{-\gamma})_{r-1}}, \tag{C.9}$$

$$\prod_{1\le i<j\le n-r-1} \frac{1}{(1-\eta_j/\eta_i)} \prod_{1\le i<j\le n-r} (1-q^{-\gamma}\eta_j/\eta_i)$$
$$= \prod_{1\le i<j\le n-r-1} \frac{1}{(1-\eta_j/\eta_i)} \prod_{1\le i<j\le n-r-1} (1-\eta_{j+1}/\eta_i) \prod_{i=1}^{n-r-1} (1-q^{-\gamma}\eta_{n-r}/\eta_i)$$
$$= \prod_{i=1}^{n-r-2} \frac{(1-\eta_{n-r}/\eta_i)}{(1-\eta_{i+1}/\eta_i)} \prod_{i=1}^{n-r-1} (1-q^{-\gamma}\eta_{n-r}/\eta_i)$$
$$= \frac{(q^{-\gamma};q^{-\gamma})_{n-r-1}}{(1-q^{-\gamma})^{n-r-1}} \cdot (q^{-2\gamma};q^{-\gamma})_{n-r-1}$$
$$= (q^{-\gamma};q^{-\gamma})_{n-r-1} \cdot \frac{(q^{-\gamma};q^{-\gamma})_{n-r}}{(1-q^{-\gamma})^{n-r}}, \tag{C.10}$$

$$\prod_{n-r+1\le i<j\le n-1} \frac{1}{(1-\eta_j/\eta_i)} \prod_{n-r+1\le i<j\le n} (1-\eta_j/\eta_i) = \prod_{i=n-r+1}^{n-1} (1-\eta_n/\eta_i)$$
$$= (q^{-\gamma};q^{-\gamma})_{r-1} \tag{C.11}$$

and

$$\prod_{i=1}^{n-r} \prod_{j=n-r+1}^{n} \frac{(1-q^{-\gamma}\eta_j/\eta_i)}{(1-q^{\gamma}\eta_j/\eta_i)} = \prod_{i=1}^{n-r} \frac{(1-\eta_n/\eta_i)(1-q^{-\gamma}\eta_n/\eta_i)}{(1-q^{\gamma}\eta_{n-r+1}/\eta_i)(1-q^{\gamma}\eta_{n-r+1}/\eta_i)}$$
$$= \frac{(\frac{x_2}{x_1} q^{\beta_1-\beta_2-(r-1)\gamma};q^{\gamma})_{n-r} \cdot (\frac{x_2}{x_1} q^{\beta_1-\beta_2-r\gamma};q^{\gamma})_{n-r}}{(\frac{x_2}{x_1} q^{\beta_1-\beta_2};q^{\gamma})_{n-r} \cdot (\frac{x_2}{x_1} q^{\beta_1-\beta_2+\gamma};q^{\gamma})_{n-r}}$$
$$= \frac{1}{{}_n(\frac{x_2}{x_1} q^{\beta_1-\beta_2-(r-1)\gamma};q^{\gamma})_r \cdot {}_n(\frac{x_2}{x_1} q^{\beta_1-\beta_2-r\gamma};q^{\gamma})_r}. \tag{C.12}$$

From (C.8)–(C.12), it finally follows that

$$\begin{aligned}(c^-_{A,rr})_0 &= (-1)^n \frac{1}{(q^{-\gamma};q^{-\gamma})_{n-r-1}} \cdot \frac{(\frac{x_2}{x_1}q^{-\beta_2-(r-1)\gamma};q^\gamma)_r}{(\frac{x_2}{x_1}q^{\beta_1-\beta_2-(r-1)\gamma};q^\gamma)_r} \cdot \frac{1}{(q^{-\gamma};q^{-\gamma})_{r-1}} \\ &\quad \cdot (q^{-\gamma};q^{-\gamma})_{n-r-1} \cdot \frac{(q^{-\gamma};q^{-\gamma})_{n-r}}{(1-q^{-\gamma})^{n-r}} \cdot (q^{-\gamma};q^{-\gamma})_{r-1} \\ &\quad \cdot \frac{1}{{}_n(\frac{x_2}{x_1}q^{\beta_1-\beta_2-(r-1)\gamma};q^\gamma)_r \cdot {}_n(\frac{x_2}{x_1}q^{\beta_1-\beta_2-r\gamma};q^\gamma)_r} \cdot \frac{(q^{-\gamma};q^{-\gamma})_r}{(1-q^{-\gamma})^r} \\ &= (-1)^n \frac{(q^{-\gamma};q^{-\gamma})_{n-r}\cdot(q^{-\gamma};q^{-\gamma})_r}{(1-q^{-\gamma})^n} \cdot \frac{(\frac{x_2}{x_1}q^{-\beta_2-(r-1)\gamma};q^\gamma)_r}{(\frac{x_2}{x_1}q^{\beta_1-\beta_2-(r-1)\gamma};q^\gamma)_r} \\ &\quad \cdot \frac{1}{{}_n(\frac{x_2}{x_1}q^{\beta_1-\beta_2-(r-1)\gamma};q^\gamma)_r \cdot {}_n(\frac{x_2}{x_1}q^{\beta_1-\beta_2-r\gamma};q^\gamma)_r}.\end{aligned}$$

□

Appendix D. Proof of Theorems 1.6 and 1.7

In Section 1, we write

$$(\varphi_0(t),\varphi_1(t),\ldots,\varphi_n(t))R_{2,1}(\tfrac{x_2}{x_1}) = (\psi_n(t),\psi_{n-1}(t),\ldots,\psi_0(t)), \tag{D.1}$$

so that

$$(\widetilde{\varphi}_0(\xi),\widetilde{\varphi}_1(\xi),\ldots,\widetilde{\varphi}_n(\xi))R_{2,1}(\tfrac{x_2}{x_1}) = (\widetilde{\psi}_n(\xi),\widetilde{\psi}_{n-1}(\xi),\ldots,\widetilde{\psi}_0(\xi)) \tag{D.2}$$

for any $\xi \in (\mathbb{C})^*$. We set

$$Y_\xi := \big(\widetilde{\varphi}_s(\xi_{F_r^{n-r}})\big)_{r,s=0}^n, \quad Y'_\xi := \big(\widetilde{\psi}_{n-s}(\xi_{F_r^{n-r}})\big)_{r,s=0}^n,$$
$$Y_\eta := \big(\widetilde{\varphi}_s(\eta_{F_r^{n-r}})\big)_{r,s=0}^n, \quad Y'_\eta := \big(\widetilde{\psi}_{n-s}(\eta_{F_r^{n-r}})\big)_{r,s=0}^n.$$

By the relation (9.1), we have

$$Y_\xi\ R_{2,1}(\tfrac{x_2}{x_1}) = Y'_\xi, \tag{D.3}$$
$$Y_\eta\ R_{2,1}(\tfrac{x_2}{x_1}) = Y'_\eta. \tag{D.4}$$

We set $(y)^{\tilde\alpha} := y_1^{\tilde\alpha}\cdots y_n^{\tilde\alpha}$ if $y=(y_1,\ldots,y_n)$. The matrices Y_ξ, Y'_ξ, Y_η and Y'_η have the following asymptotic behaviors:

$$Y_\xi \sim (q^{\tilde\alpha})^{D_A^+}C_A^+, \quad Y'_\xi \sim (q^{\tilde\alpha})^{D_A^+}C'^+_A \quad \text{at} \quad \tilde\alpha \to +\infty, \tag{D.5}$$
$$Y_\eta \sim (q^{\tilde\alpha})^{D_A^-}C_A^-, \quad Y'_\eta \sim (q^{\tilde\alpha})^{D_A^-}C'^-_A \quad \text{at} \quad \tilde\alpha \to -\infty, \tag{D.6}$$

where

$$(q^{\tilde\alpha})^{D_A^+} = \mathrm{diag}[(\xi_{F_r^{n-r}})^{\tilde\alpha}]_{r=0}^n, \quad (q^{\tilde\alpha})^{D_A^-} = \mathrm{diag}[(\eta_{F_r^{n-r}})^{\tilde\alpha}]_{r=0}^n,$$

and $C_A^+ = (c^+_{A,rs})_{r,s=0}^n$, $C_A^- = (c^-_{A,rs})_{r,s=0}^n$, $C'^+_A = (c'^+_{A,rs})_{r,s=0}^n$ and $C'^-_A = (c'^-_{A,rs})_{r,s=0}^n$ are matrices not depending on $q^{\tilde\alpha}$ defined by

$$c^+_{A,rs} := (1-q)^n \frac{\Phi(\xi_{F_r^{n-r}})\varphi_s(\xi_{F_r^{n-r}})}{(\xi_{F_r^{n-r}})^{\tilde\alpha}}, \quad c^-_{A,rs} := (1-q)^n \operatorname*{Res}_{t=\eta_{F_r^{n-r}}} \frac{\Phi(t)\varphi_s(t)}{(t)^{\tilde\alpha}}\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_n}{t_n},$$

$$c'^+_{A,rs} := (1-q)^n \frac{\Phi(\xi_{F_r^{n-r}})\psi_{n-s}(\xi_{F_r^{n-r}})}{(\xi_{F_r^{n-r}})^{\tilde\alpha}}, \ c'^-_{A,rs} := (1-q)^n \operatorname*{Res}_{t=\eta_{F_r^{n-r}}} \frac{\Phi(t)\psi_{n-s}(t)}{(t)^{\tilde\alpha}}\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_n}{t_n}.$$

From (D.3) and (D.4), it follows

$$C_A^+ \; R_{2,1}(\tfrac{x_2}{x_1}) = C'^+_A,$$
$$C_A^- \; R_{2,1}(\tfrac{x_2}{x_1}) = C'^-_A,$$

so that we have

Proposition D.1. *The matrix $R_{2,1}(\frac{x_2}{x_1})$ is expressed as*

$$R_{2,1}(\tfrac{x_2}{x_1}) = (C_A^+)_0^{-1}(C'^+_A)_0, \tag{D.7}$$
$$R_{2,1}(\tfrac{x_2}{x_1}) = (C_A^-)_0^{-1}(C'^-_A)_0. \tag{D.8}$$

Lemma D.2. *The relations between Y_ξ and Y'_ξ or between Y_η and Y'_η are expressed as*

$$\widetilde{\psi}_s(\xi_{F^r_{n-r}}) = \tau\left[\widetilde{\varphi}_s(\xi_{F_r^{n-r}})\cdot \mathrm{sgn}\sigma_r \cdot U_{\sigma_r}(\xi_{F_r^{n-r}})\right], \tag{D.9}$$
$$\widetilde{\psi}_s(\eta_{F^r_{n-r}}) = \tau\left[\widetilde{\varphi}_s(\eta_{F_r^{n-r}})\cdot \mathrm{sgn}\sigma_r \cdot U_{\sigma_r}(\eta_{F_r^{n-r}})\right], \tag{D.10}$$

where $\sigma_r \in \mathfrak{S}_n$ is

$$\sigma_r = \begin{pmatrix} 1 & 2 & \cdots & r & r+1 & r+2 & \cdots & n \\ n-r+1 & n-r+2 & \cdots & n & 1 & 2 & \cdots & n-r \end{pmatrix}.$$

Proof. By definition, we have

$$\widetilde{\psi}_s(\xi_{F^r_{n-r}}) = \tau\left[\tau\widetilde{\psi}_s(\tau\xi_{F^r_{n-r}})\right], \tag{D.11}$$

where

$$\tau\xi_{F^r_{n-r}} = (x_2, x_2q^\gamma, \ldots, x_2q^{(r-1)\gamma}, x_1, x_1q^\gamma, \ldots, x_1q^{(n-r-1)\gamma}) = \sigma_r^{-1}\xi_{F_r^{n-r}}.$$

The right-hand side of (D.11) without τ is

$$\begin{aligned}
\tau\widetilde{\psi}_s(\tau\xi_{F^r_{n-r}}) &= \widetilde{\varphi}_s(\sigma_r^{-1}\xi_{F_r^{n-r}}) \qquad \text{(by (1.3))}\\
&= \int_{\langle\sigma_r^{-1}\xi_{F_r^{n-r}}\rangle} \Phi(t)\varphi_s(t)\varpi = \int_{\langle\xi_{F_r^{n-r}}\rangle} \Phi(\sigma_r^{-1}t)\varphi_s(\sigma_r^{-1}t)\varpi\\
&= \int_{\langle\xi_{F_r^{n-r}}\rangle} \sigma_r\Phi(t)\cdot\sigma_r\varphi_s(t)\varpi\\
&= U_{\sigma_r}(\xi_{F_r^{n-r}})\cdot\mathrm{sgn}\sigma_r\cdot\int_{\langle\xi_{F_r^{n-r}}\rangle}\Phi(t)\varphi_s(t)\varpi \quad \text{(by (1.1) and (1.8))}\\
&= U_{\sigma_r}(\xi_{F_r^{n-r}})\cdot\mathrm{sgn}\sigma_r\cdot\widetilde{\varphi}_s(\xi_{F_r^{n-r}}).
\end{aligned}$$

Thus we have (D.9). We also have the relation (D.10) in the same way as above. □

Lemma D.3. *The relations between C_A^+ and C'^+_A or between C_A^- and C'^-_A are expressed as*

$$c'^+_{A,n-r,n-s} = \tau\left[c^+_{A,rs}\cdot\mathrm{sgn}\sigma_r\cdot U_{\sigma_r}(\xi_{F_r^{n-r}})\right],$$
$$c'^-_{A,n-r,n-s} = \tau\left[c^-_{A,rs}\cdot\mathrm{sgn}\sigma_r\cdot U_{\sigma_r}(\eta_{F_r^{n-r}})\right].$$

Proof. It is easily deduced from (D.5), (D.6) and Lemma D.2. □

D.1. **Proof of Theorems 1.6 and 1.7.** From Proposition 3.1, we have that the matrix C_A^- is upper triangular. By Lemma D.3 the matrix C'^-_A is lower triangular. Therefore, from the expression (D.8) and Lemma D.3, we have the Gauss decomposition of the matrix $R_{2,1}(\frac{x_2}{x_1})$ as follows:

$$R_{2,1}(\tfrac{x_2}{x_1}) = U_R \cdot D_R \cdot L_R,$$

where $U_R^{-1} = (u^*_{R,rs})_{r,s=0}^n$, $D_R = \mathrm{diag}[d_{R,0}, \dots, d_{R,n}]$ and $L_R = (l_{R,rs})_{r,s=0}^n$ are the matrices such that

$$u^*_{R,rs} = \left(\frac{c^-_{A,rs}}{c^-_{A,rr}}\right)_0, \quad l_{R,rs} = \left(\frac{c'^-_{A,rs}}{c'^-_{A,rr}}\right)_0 = \tau\left(\frac{c^-_{A,n-r,n-s}}{c^-_{A,n-r,n-r}}\right)_0, \tag{D.12}$$

$$d_{R,r} = (c^-_{A,rr})_0^{-1}(c'^-_{A,rr})_0 = (c^-_{A,rr})_0^{-1}\tau\left[(c^-_{A,n-r,n-r})_0 \cdot \mathrm{sgn}\sigma_{n-r} \cdot \left(U_{\sigma_{n-r}}(\eta_{F^r_{n-r}})\right)_0\right].$$

Hence Theorem 1.6 follows from above Gauss decomposition, Proposition 3.2 and

$$\left(U_{\sigma_r}(\eta_{F_r^{n-r}})\right)_0 = (-q^{-\gamma})^{r(n-r)}{}_n(\tfrac{x_2}{x_1}q^{\beta_1-\beta_2-(r-1)\gamma}; q^\gamma)_r \cdot {}_n(\tfrac{x_2}{x_1}q^{\beta_1-\beta_2-r\gamma}; q^\gamma)_r, \tag{D.13}$$

$$\mathrm{sgn}\sigma_r = (-1)^{r(n-r)}. \tag{D.14}$$

By using Proposition 3.1 and

$$\left(U_{\sigma_r}(\xi_{F_r^{n-r}})\right)_0 = (-q^{-\gamma})^{r(n-r)}{}_n(\tfrac{x_2}{x_1}q^{-(n-r-1)\gamma}; q^\gamma)_r \cdot {}_n(\tfrac{x_2}{x_1}q^{-(n-r)\gamma}; q^\gamma)_r, \tag{D.15}$$

we can prove Theorem 1.7 in the same way as Theorem 1.6. □

Remark D.4. From (D.13), (D.14), (D.15) and Lemma D.3, the matrices $(C'^+_A)_0$ and $(C'^-_A)_0$ are written by

$$(C'^+_A)_0 = J\tau\mathrm{diag}[g_0^+, g_1^+, \dots, g_n^+]\tau(C_A^+)_0 J = J\tau\mathrm{diag}[g_0^+, g_1^+, \dots, g_n^+]JJ\tau(C_A^+)_0 J$$

and

$$(C'^-_A)_0 = J\tau\mathrm{diag}[g_0^-, g_1^-, \dots, g_n^-]\tau(C_A^-)_0 J = J\tau\mathrm{diag}[g_0^-, g_1^-, \dots, g_n^-]JJ\tau(C_A^-)_0 J$$

where

$$\begin{aligned} g_r^+ &= q^{-r(n-r)\gamma}{}_n(\tfrac{x_2}{x_1}q^{-(n-r-1)\gamma}; q^\gamma)_r \cdot {}_n(\tfrac{x_2}{x_1}q^{-(n-r)\gamma}; q^\gamma)_r, \\ g_r^- &= q^{-r(n-r)\gamma}{}_n(\tfrac{x_2}{x_1}q^{\beta_1-\beta_2-(r-1)\gamma}; q^\gamma)_r \cdot {}_n(\tfrac{x_2}{x_1}q^{\beta_1-\beta_2-r\gamma}; q^\gamma)_r. \end{aligned}$$

Thus, from Proposition D.1, the R-matrix $R_{2,1}(\frac{x_2}{x_1})$ are given by

$$R_{2,1}(\tfrac{x_2}{x_1}) = (C_A^+)_0^{-1}\ \mathrm{diag}[\tau g_n^+, \tau g_{n-1}^+, \dots, \tau g_0^+]\ J\tau(C_A^+)_0 J, \tag{D.16}$$

$$R_{2,1}(\tfrac{x_2}{x_1}) = (C_A^-)_0^{-1}\ \mathrm{diag}[\tau g_n^-, \tau g_{n-1}^-, \dots, \tau g_0^-]\ J\tau(C_A^-)_0 J. \tag{D.17}$$

Appendix E. Examples.

E.1. **The case when $m = 2$, $n = 1$.** we explain the simplest case when $n = 1$. In Theorem 1.6, the R-matrix $R_{2,1}(\frac{x_2}{x_1})$ is written by

$$R_{2,1}(\tfrac{x_2}{x_1}) = \begin{pmatrix} \dfrac{\frac{x_2}{x_1}-1}{\frac{x_2}{x_1}-q^{\beta_2}} & \dfrac{(1-q^{\beta_2})\frac{x_2}{x_1}}{\frac{x_2}{x_1}-q^{\beta_2}} \\ \dfrac{1-q^{\beta_2}}{\frac{x_2}{x_1}-q^{\beta_2}} & \dfrac{\frac{x_2}{x_1}q^{\beta_1}-q^{\beta_2}}{\frac{x_2}{x_1}-q^{\beta_2}} \end{pmatrix} = U_R\ D_R\ L_R = L'_R\ D'_R\ U'_R,$$

where

$$U_R = \begin{pmatrix} 1 & -\frac{q^{-\beta_2}(1-q^{\beta_1})\frac{x_2}{x_1}}{1-\frac{x_2}{x_1}q^{\beta_1-\beta_2}} \\ 0 & 1 \end{pmatrix}, \quad D_R = \begin{pmatrix} \frac{q^{-\beta_2}(1-\frac{x_2}{x_1}q^{\beta_1})}{1-\frac{x_2}{x_1}q^{\beta_1-\beta_2}} & 0 \\ 0 & \frac{1-\frac{x_2}{x_1}q^{\beta_1-\beta_2}}{1-\frac{x_2}{x_1}q^{-\beta_2}} \end{pmatrix},$$

$$L_R = \begin{pmatrix} 1 & 0 \\ -\frac{(1-q^{\beta_2})q^{-\beta_2}}{1-\frac{x_2}{x_1}q^{\beta_1-\beta_2}} & 1 \end{pmatrix},$$

and

$$L'_R = \begin{pmatrix} 1 & 0 \\ -\frac{1-q^{\beta_2}}{1-\frac{x_2}{x_1}} & 1 \end{pmatrix}, \; D'_R = \begin{pmatrix} \frac{q^{-\beta_2}(1-\frac{x_2}{x_1})}{1-q^{-\beta_2}\frac{x_2}{x_1}} & 0 \\ 0 & \frac{1-q^{\beta_1}\frac{x_2}{x_1}}{1-\frac{x_2}{x_1}} \end{pmatrix}, \; U'_R = \begin{pmatrix} 1 & \frac{1-q^{\beta_1}}{1-\frac{x_1}{x_2}} \\ 0 & 1 \end{pmatrix}.$$

In Theorem 5.1, the Gauss decomposition of the matrix $A(q^{\tilde{\alpha}})$ is given by

$$\begin{aligned} A(q^{\tilde{\alpha}}) &= \begin{pmatrix} 1 & -\frac{q^{\tilde{\alpha}}(1-q^{\beta_1})}{1-q^{\tilde{\alpha}+\beta_1}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{(1-q^{\tilde{\alpha}})x_1}{1-q^{\tilde{\alpha}+\beta_1}} & 0 \\ 0 & \frac{(1-q^{\tilde{\alpha}+\beta_1})x_2}{1-q^{\tilde{\alpha}+\beta_1+\beta_2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{(1-q^{\beta_2})x_1}{(1-q^{\tilde{\alpha}+\beta_1})x_2} & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{(1-q^{\tilde{\alpha}+\beta_2})x_1}{1-q^{\tilde{\alpha}+\beta_1+\beta_2}} & -\frac{q^{\tilde{\alpha}}(1-q^{\beta_1})x_2}{1-q^{\tilde{\alpha}+\beta_1+\beta_2}} \\ -\frac{(1-q^{\beta_2})x_1}{1-q^{\tilde{\alpha}+\beta_1+\beta_2}} & \frac{(1-q^{\tilde{\alpha}+\beta_1})x_2}{1-q^{\tilde{\alpha}+\beta_1+\beta_2}} \end{pmatrix}, \end{aligned}$$

so that we have

$$A(0) = \begin{pmatrix} x_1 & 0 \\ -(1-q^{\beta_2})x_1 & x_2 \end{pmatrix}, \quad A(\infty) = \begin{pmatrix} x_1 q^{-\beta_1} & -(1-q^{-\beta_1})x_2 q^{-\beta_2} \\ 0 & x_2 q^{-\beta_2} \end{pmatrix}.$$

From (3.6) and (3.7), taking the unipotent matrices C^+ and C^- such that

$$C^+ = \begin{pmatrix} 1 & 0 \\ \frac{1-q^{\beta_2}}{1-\frac{x_2}{x_1}} & 1 \end{pmatrix}, \quad C^- = \begin{pmatrix} 1 & \frac{q^{-\beta_2}(1-q^{\beta_1})\frac{x_2}{x_1}}{1-q^{\beta_1-\beta_2}\frac{x_2}{x_1}} \\ 0 & 1 \end{pmatrix},$$

we can diagonalize $A(0)$ and $A(\infty)$ as follows:

$$A(0) = (C^+)^{-1} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} C^+, \quad A(\infty) = (C^-)^{-1} \begin{pmatrix} x_1 q^{-\beta_1} & 0 \\ 0 & x_2 q^{-\beta_2} \end{pmatrix} C^-$$

From Propositions 3.1 and 3.2, the matrices $(C_A^+)_0$ and $(C_A^-)_0$ are written as a product of diagonal and unipotent matrices as follows:

$$(C_A^+)_0 = \mathrm{diag}[(c_{A,00}^+)_0, (c_{A,11}^+)_0]\; C^+, \quad (C_A^-)_0 = \mathrm{diag}[(c_{A,00}^-)_0, (c_{A,11}^-)_0]\; C^-,$$

where

$$\mathrm{diag}[(c_{A,00}^+)_0, (c_{A,11}^+)_0] = \begin{pmatrix} \frac{1}{1-q^{\beta_1}} & 0 \\ 0 & \frac{1-\frac{x_2}{x_1}}{(1-q^{\beta_1}\frac{x_2}{x_1})(1-q^{\beta_2})} \end{pmatrix}$$

and

$$\mathrm{diag}[(c^-_{A,00})_0, (c^-_{A,11})_0] = \begin{pmatrix} -1 & 0 \\ 0 & -\frac{1-q^{-\beta_2}\frac{x_2}{x_1}}{1-q^{\beta_1-\beta_2}\frac{x_2}{x_1}} \end{pmatrix}.$$

From (D.16) and (D.17), the R-matrix $R_{2,1}(\frac{x_2}{x_1})$ is determined from the matrices $(C^+_A)_0$ or $(C^-_A)_0$ as follows:

$$R_{2,1}(\tfrac{x_2}{x_1}) = (C^+_A)_0^{-1} J\tau(C^+_A)_0 J = (C^-_A)_0^{-1} J\tau(C^-_A)_0 J,$$

and it follows that

$$D'_R = \mathrm{diag}[(c^+_{A,00})_0^{-1}\tau(c^+_{A,11})_0,\ (c^+_{A,11})_0^{-1}\tau(c^+_{A,00})_0]$$

and

$$D_R = \mathrm{diag}[(c^-_{A,00})_0^{-1}\tau(c^-_{A,11})_0,\ (c^-_{A,11})_0^{-1}\tau(c^-_{A,00})_0].$$

Thus, in particular, the upper and lower triangular matrices as factors of the Gauss matrix decomposition of $R_{2,1}(\frac{x_2}{x_1})$ are determined from the matrices C^+ and C^- as follows:

$$L'_R = (C^+)^{-1}, \quad U'_R = J\tau C^+ J,$$

and

$$U_R = (C^-)^{-1}, \quad L_R = J\tau C^- J.$$

E.2. **The case when $m = 2$, $n = 2$.** In Theorem 5.1, the Gauss decomposition of the matrix $A(q^{\tilde{\alpha}})$ is given by

$$A(q^{\tilde{\alpha}}) = \begin{pmatrix} 1 & -\frac{q^{\tilde{\alpha}-\gamma}(1-q^{\beta_1+\gamma})}{1-q^{\tilde{\alpha}+\beta_1}} & \frac{q^{2\tilde{\alpha}-3\gamma}(1-q^{\beta_1})(1-q^{\beta_1+\gamma})}{(1-q^{\tilde{\alpha}+\beta_1-2\gamma})(1-q^{\tilde{\alpha}+\beta_1-\gamma})} \\ 0 & 1 & -\frac{q^{\tilde{\alpha}-2\gamma}(1-q^{2\gamma})(1-q^{\beta_1})}{(1-q^{\gamma})(1-q^{\tilde{\alpha}+\beta_1-2\gamma})} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} \frac{x_1^2 q^{\gamma}(1-q^{\tilde{\alpha}-2\gamma})(1-q^{\tilde{\alpha}-\gamma})}{(1-q^{\tilde{\alpha}+\beta_1-\gamma})(1-q^{\tilde{\alpha}+\beta_1})} & 0 & 0 \\ 0 & \frac{x_1x_2(1-q^{\tilde{\alpha}+\beta_1})(1-q^{\tilde{\alpha}-2\gamma})}{(1-q^{\tilde{\alpha}+\beta_1+\beta_2})(1-q^{\tilde{\alpha}+\beta_1-2\gamma})} & 0 \\ 0 & 0 & \frac{x_2^2 q^{\gamma}(1-q^{\tilde{\alpha}+\beta_1-2\gamma})(1-q^{\tilde{\alpha}+\beta_1-\gamma})}{(1-q^{\tilde{\alpha}+\beta_1+\beta_2-\gamma})(1-q^{\tilde{\alpha}+\beta_1+\beta_2})} \end{pmatrix}$$

$$\cdot \begin{pmatrix} 1 & 0 & 0 \\ -\frac{(1-q^{2\gamma})(1-q^{\beta_2})x_1}{(1-q^{\gamma})(1-q^{\tilde{\alpha}+\beta_1})x_2} & 1 & 0 \\ \frac{q^{-\gamma}(1-q^{\beta_2})(1-q^{\beta_2+\gamma})x_1^2}{(1-q^{\tilde{\alpha}+\beta_1-2\gamma})(1-q^{\tilde{\alpha}+\beta_1-\gamma})x_2^2} & -\frac{q^{-\gamma}(1-q^{\beta_2+\gamma})x_1}{(1-q^{\tilde{\alpha}+\beta_1-2\gamma})x_2} & 1 \end{pmatrix},$$

so that we have

$$A(0) = \begin{pmatrix} x_1^2 q^{\gamma} & 0 & 0 \\ 0 & x_1x_2 & 0 \\ 0 & 0 & x_2^2 q^{\gamma} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{(1-q^{2\gamma})(1-q^{\beta_2})x_1}{(1-q^{\gamma})x_2} & 1 & 0 \\ \frac{q^{-\gamma}(1-q^{\beta_2})(1-q^{\beta_2+\gamma})x_1^2}{x_2^2} & -\frac{q^{-\gamma}(1-q^{\beta_2+\gamma})x_1}{x_2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} q^{\gamma}x_1^2 & 0 & 0 \\ -(1+q^{\gamma})(1-q^{\beta_2})x_1^2 & x_1x_2 & 0 \\ (1-q^{\beta_2})(1-q^{\beta_2+\gamma})x_1^2 & -(1-q^{\beta_2+\gamma})x_1x_2 & q^{\gamma}x_2^2 \end{pmatrix}$$

and

$$A(\infty)=\begin{pmatrix} 1 & q^{-\beta_1-\gamma}(1-q^{\beta_1+\gamma}) & q^{-2\beta_1}(1-q^{\beta_1})(1-q^{\beta_1+\gamma}) \\ 0 & 1 & q^{-\beta_1}(1+q^{\gamma})(1-q^{\beta_1}) \\ 0 & 0 & 1 \end{pmatrix}$$
$$\cdot\begin{pmatrix} x_1^2q^{-2\beta_1-\gamma} & 0 & 0 \\ 0 & x_1x_2q^{-\beta_1-\beta_2} & 0 \\ 0 & 0 & x_2^2q^{-2\beta_2-\gamma} \end{pmatrix}$$
$$=\begin{pmatrix} x_1^2q^{-2\beta_1-\gamma} & -x_1x_2q^{-\beta_1-\beta_2}(1-q^{-\beta_1-\gamma}) & x_2^2q^{-2\beta_2}(1-q^{-\beta_1})(1-q^{-\beta_1-\gamma}) \\ 0 & x_1x_2q^{-\beta_1-\beta_2} & -x_2^2q^{-2\beta_2}(1+q^{-\gamma})(1-q^{-\beta_1}) \\ 0 & 0 & x_2^2q^{-2\beta_2-\gamma} \end{pmatrix}.$$

From (3.6) and (3.7), taking the unipotent matrices C^+ and C^- such that

$$C^+=\begin{pmatrix} 1 & 0 & 0 \\ \frac{q^{-\gamma}(1-q^{2\gamma})(1-q^{\beta_2})}{(1-q^{\gamma})(1-\frac{x_2}{x_1}q^{-\gamma})} & 1 & 0 \\ \frac{(1-q^{\beta_2})(1-q^{\beta_2+\gamma})}{(1-\frac{x_2}{x_1})(1-\frac{x_2}{x_1}q^{\gamma})} & \frac{(1-q^{\beta_2+\gamma})}{(1-\frac{x_2}{x_1}q^{\gamma})} & 1 \end{pmatrix},$$

$$C^-=\begin{pmatrix} 1 & \frac{\frac{x_2}{x_1}q^{-\beta_2}(1-q^{\beta_1+\gamma})}{1-\frac{x_2}{x_1}q^{\beta_1-\beta_2+\gamma}} & \frac{(\frac{x_2}{x_1}q^{-\beta_2})^2(1-q^{\beta_1})(1-q^{\beta_1+\gamma})}{(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2})(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2+\gamma})} \\ 0 & 1 & \frac{\frac{x_2}{x_1}q^{-\beta_2-\gamma}(1-q^{2\gamma})(1-q^{\beta_1})}{(1-q^{\gamma})(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2-\gamma})} \\ 0 & 0 & 1 \end{pmatrix},$$

we can diagonalize $A(0)$ and $A(\infty)$ as follows:

$$A(0)=(C^+)^{-1}\begin{pmatrix} x_1^2q^{\gamma} & 0 & 0 \\ 0 & x_1x_2 & 0 \\ 0 & 0 & x_2^2q^{\gamma} \end{pmatrix}C^+,$$

$$A(\infty)=(C^-)^{-1}\begin{pmatrix} x_1^2q^{-2\beta_1-\gamma} & 0 & 0 \\ 0 & x_1x_2q^{-\beta_1-\beta_2} & 0 \\ 0 & 0 & x_2^2q^{-2\beta_2-\gamma} \end{pmatrix}C^-.$$

From Propositions 3.1 and 3.2, the matrices $(C_A^+)_0$ and $(C_A^-)_0$ are written as a product of diagonal and unipotent matrices as follows:

$$(C_A^+)_0=\mathrm{diag}[(c_{A,00}^+)_0,(c_{A,11}^+)_0,(c_{A,22}^+)_0]\ C^+,$$
$$(C_A^-)_0=\mathrm{diag}[(c_{A,00}^-)_0,(c_{A,11}^-)_0,(c_{A,22}^-)_0]\ C^-,$$

where

$$\mathrm{diag}[(c_{A,00}^+)_0,(c_{A,11}^+)_0,(c_{A,22}^+)_0]$$
$$=\begin{pmatrix} \frac{q^{\gamma}}{(1-q^{\beta_1})(1-q^{\beta_1+\gamma})} & 0 & 0 \\ 0 & \frac{(1-\frac{x_2}{x_1}q^{-\gamma})(1-\frac{x_2}{x_1})}{(1-q^{\beta_1})(1-\frac{x_2}{x_1}q^{\beta_1})(1-q^{\beta_2})(1-\frac{x_2}{x_1}q^{\gamma})} & 0 \\ 0 & 0 & \frac{q^{\gamma}(1-\frac{x_2}{x_1})(1-\frac{x_2}{x_1}q^{\gamma})}{(1-\frac{x_2}{x_1}q^{\beta_1})(1-\frac{x_2}{x_1}q^{\beta_1+\gamma})(1-q^{\beta_2})(1-q^{\beta_2+\gamma})} \end{pmatrix},$$

$$\mathrm{diag}[(c^-_{A,00})_0,(c^-_{A,11})_0,(c^-_{A,22})_0]$$
$$=\begin{pmatrix} \frac{1-q^{-2\gamma}}{1-q^{-\gamma}} & 0 & 0\\ 0 & \frac{(1-\frac{x_2}{x_1}q^{-\beta_2})(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2-\gamma})}{(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2})(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2+\gamma})} & 0\\ 0 & 0 & \frac{(1-q^{-2\gamma})(1-\frac{x_2}{x_1}q^{-\beta_2-\gamma})(1-\frac{x_2}{x_1}q^{-\beta_2})}{(1-q^{-\gamma})(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2-\gamma})(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2})} \end{pmatrix}.$$

From (D.16) and (D.17), the R-matrix $R_{2,1}(\frac{x_2}{x_1})$ is determined from the matrices $(C_A^+)_0$ or $(C_A^-)_0$ as follows:

$$\begin{aligned} R_{2,1}(\tfrac{x_2}{x_1}) &= (C_A^+)_0^{-1}\begin{pmatrix}1&0&0\\0&\tau g_1^+&0\\0&0&1\end{pmatrix}J\tau(C_A^+)_0J\\ &= (C_A^-)_0^{-1}\begin{pmatrix}1&0&0\\0&\tau g_1^-&0\\0&0&1\end{pmatrix}J\tau(C_A^-)_0J, \end{aligned}$$

where

$$\tau g_1^+ = \frac{q^{-\gamma}(1-\frac{x_1}{x_2}q^{\gamma})}{1-\frac{x_1}{x_2}q^{-\gamma}},\quad \tau g_1^- = \frac{q^{-\gamma}(1-\frac{x_1}{x_2}q^{\beta_2-\beta_1+\gamma})}{1-\frac{x_1}{x_2}q^{\beta_2-\beta_1-\gamma}}.$$

If we express the Gauss decomposition of $R_{2,1}(\frac{x_2}{x_1})$ by

$$R_{2,1}(\tfrac{x_2}{x_1}) = U_R\ D_R\ L_R = L'_R\ D'_R\ U'_R,$$

then we have

$$D'_R = \mathrm{diag}[(c^+_{A,00})_0^{-1}\tau(c^+_{A,22})_0,\ (c^+_{A,11})_0^{-1}\tau g_1^+\tau(c^+_{A,11})_0,\ (c^+_{A,22})_0^{-1}\tau(c^+_{A,00})_0]$$

and

$$D_R = \mathrm{diag}[(c^-_{A,00})_0^{-1}\tau(c^-_{A,22})_0,\ (c^-_{A,11})_0^{-1}\tau g_1^-\tau(c^-_{A,11})_0,\ (c^-_{A,22})_0^{-1}\tau(c^-_{A,00})_0].$$

In particular, the upper and lower triangular matrices as factors of the Gauss matrix decomposition of $R_{2,1}(\frac{x_2}{x_1})$ are determined from the matrices C^+ and C^- as follows:

$$L'_R = (C^+)^{-1},\quad U'_R = J\tau C^+J,$$

and

$$U_R = (C^-)^{-1},\quad L_R = J\tau C^-J.$$

Therefore the Gauss decomposition of $R_{2,1}(\frac{x_2}{x_1})$ is given by

$$R_{2,1}(\tfrac{x_2}{x_1})$$

$$= \begin{pmatrix} 1 & -\frac{\frac{x_2}{x_1}q^{-\beta_2}(1-q^{\beta_1+\gamma})}{1-\frac{x_2}{x_1}q^{\beta_1-\beta_2+\gamma}} & \frac{(\frac{x_2}{x_1}q^{-\beta_2})^2q^{-\gamma}(1-q^{\beta_1})(1-q^{\beta_1+\gamma})}{(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2-\gamma})(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2})} \\ 0 & 1 & -\frac{\frac{x_2}{x_1}q^{-\beta_2-\gamma}(1-q^{2\gamma})(1-q^{\beta_1})}{(1-q^{\gamma})(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2-\gamma})} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} \frac{q^{-2\beta_2}(1-\frac{x_2}{x_1}q^{\beta_1})(1-\frac{x_2}{x_1}q^{\beta_1+\gamma})}{(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2})(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2+\gamma})} & 0 & 0 \\ 0 & \frac{q^{-\beta_2-\gamma}(1-\frac{x_2}{x_1}q^{\beta_1})(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2+\gamma})}{(1-\frac{x_2}{x_1}q^{-\beta_2})(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2-\gamma})} & 0 \\ 0 & 0 & \frac{(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2-\gamma})(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2})}{(1-\frac{x_2}{x_1}q^{-\beta_2-\gamma})(1-\frac{x_2}{x_1}q^{-\beta_2})} \end{pmatrix}$$

$$\cdot \begin{pmatrix} 1 & 0 & 0 \\ -\frac{q^{-\beta_2}(1-q^{2\gamma})(1-q^{\beta_2})}{(1-q^{\gamma})(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2+\gamma})} & 1 & 0 \\ \frac{q^{-2\beta_2-\gamma}(1-q^{\beta_2})(1-q^{\beta_2+\gamma})}{(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2-\gamma})(1-\frac{x_2}{x_1}q^{\beta_1-\beta_2})} & -\frac{q^{-\beta_2-\gamma}(1-q^{\beta_2+\gamma})x_1}{1-\frac{x_2}{x_1}q^{\beta_1-\beta_2-\gamma}} & 1 \end{pmatrix}.$$

References

[1] G. E. Andrews, q-series, CBMS Regional Conference Series in Math. 66 AMS, 1986.

[2] K. Aomoto, A note on holomorphic q-difference systems: in Algebraic Analysis, Vol.1, eds. M. Kashiwara & T. Kawai, Academic Press, Inc., Boston, MA, 1988, 25–28. ISBN:0-12-400465-2

[3] K. Aomoto, Finiteness of a cohomology associated with certain Jackson integrals, Tohoku Math. J. 43 (1991), 75–101.

[4] K. Aomoto, Gauss matrix decomposition and a Yang-Baxter equation, J. Math. Anal. and Appli. 182 (1994), 127–133.

[5] K. Aomoto, Connection matrices and Riemann–Hilbert problem for q-difference equations: in Structure of Solutions of Differential Equations, eds. M. Morimoto & T. Kawai, World Scientific Publishing Co., Inc., River Edge, NJ, 1996, 51–69. ISBN:981-02-2321-8

[6] K. Aomoto, Connection matrices and Riemann–Hilbert problem for q-difference equations, Gordon and Breach Science Publishers, Amsterdam, 1999, 7–15. ISBN:90-5699-669-X

[7] K. Aomoto and Y. Kato, A q-analogue of de Rham cohomology associated with Jackson integrals: in Special Functions, Proc. of the Hayashibara Forum, eds. M. Kashiwara & T. Miwa, ICM-90 Satell. Conf. Proc. Springer-Verlag, Tokyo, 1991, 30–62. ISBN: 4-431-70085-4

[8] K. Aomoto and Y. Kato, Gauss decomposition of connection matrices and application to Yang–Baxter equation I, Proc. Japan Acad. Ser. A Math. Sci. 69 (1993), 238–242.

[9] K. Aomoto and Y. Kato, Gauss decomposition of connection matrices and application to Yang–Baxter equation II, Proc. Japan Acad. Ser. A Math. Sci. 69 (1993), 341–344.

[10] K. Aomoto and Y. Kato, Connection coefficients for A-type Jackson integral and Yang–Baxter equation, Contemporary Math. 175 (1994), 1–26.

[11] K. Aomoto and Y. Kato, Connection formula of symmetric A-type Jackson integrals, Duke math. J. 74 (1994), 129–143.

[12] K. Aomoto and Y. Kato, Gauss decomposition of connection matrices for symmetric A-type Jackson integrals, Selecta Math. (N.S.) 1 (1995), 623–666.

[13] K. Aomoto and Y. Kato, Derivation of q-difference equation from connection matrix for Selberg type Jackson integrals, J. Differ. Equations Appl. 4 (1998), 247–278.

[14] K. Aomoto, Y. Kato and K. Mimachi, A solution of Yang-Baxter equation as connection coefficients of a holonomic q-difference system, Duke Math. J. 65 (1992), 7–15.

[15] H. Awata, K. Hasegawa, H. Kanno, R. Ohkawa, S. Shakirov, J. Shiraishi and Y. Yamada, Non-stationary difference equation and affine Laumon apace II: – Quantum Knizhnik–Zamolodchikov equation –, arXiv:2309.15364[math.QA].

[16] I. B. Frenkel and N. Y. Reshetikhin, Quantum affine algebras and holonomic q-difference equations, Commun. Math. Phys. 146 (1992), 1–60
[17] F. R. Gantmacher, The theory of matrices, vol.1, Chelsea, New York, 1959.
[18] G. Gasper and M. Rahman, Basic hypergeometric series, 2nd ed., Encyclopedia Math. Appl. 96, Cambridge University Press, Cambridge, 2004, xxvi+428 pp. ISBN: 0-521-83357-4
[19] M. Ito, q-Difference systems for the Jackson integral of symmetric Selberg type, SIGMA Symmetry Integrability Geom. Methods Appl. 16 (2020), Paper No. 113, 31 pages.
[20] K. Kadell, A proof of Askey's conjectured q-analogue of Selberg's integral and a conjecture of Morris, SIAM J. Math. Anal. 19 (1988), 969–986.
[21] A. Matsuo, Jackson integrals of Jordan–Pochhammer type and quantum Knizhnik–Zamolodochikov equations, Commun. Math. Phys. 151 (1993), 263–273.
[22] A. Matsuo, Quantum algebra structure of certain Jackson integrals, Commun. Math. Phys. 157 (1993), 479–498.
[23] K. Mimachi, Connection problem in holonomic q-difference system associated with a Jackson integral of Jordan–Pochhammer type, Nagoya Math. J. 116 (1989), 149–161.
[24] K. Mimachi, Holonomic q-difference system of the first order associated with a Jackson integral of Selberg type, Duke Math. J. 73 (1994), 453–468.
[25] V. Tarasov and A. Varchenko, Geometry of q-hypergeometric functions, quantum affine algebras and elliptic quantum groups, Astérisque (1997), no. 246, vi+135 pp.
[26] A. Varchenko, Quantized Knizhnik–Zamolodochikov equations, quantum Yang-Baxter equation, and difference equations for q-hypergeometric functions, Commun. Math. Phys. 162 (1994), 499–528.

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