

DEGENERATION OF HYPERGEOMETRIC  
INTEGRALS ASSOCIATED WITH  
HYPERSPHERE ARRANGEMENT

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## DEGENERATION OF HYPERGEOMETRIC INTEGRALS ASSOCIATED WITH HYPERSPHERE ARRANGEMENT

KAZUHIKO AOMOTO AND MASAHIKO ITO

ABSTRACT. We discuss a degeneracy of  $n$ -dimensional hypergeometric integrals associated with certain hypersphere arrangement. We investigate it in two ways when a certain generic condition for spherical arrangement breaks down. This situation is described using the terms of the Cayley–Menger determinants. One of the main results is the degeneracy of the  $n$ th twisted de Rham cohomology, which is expressed using the contiguity relation of the hypergeometric integral. Complementary to this, the other main result is the degeneracy of the  $n$ th twisted homology over the dual local system, which can be described using the asymptotic behavior of the hypergeometric integral.

### 1. INTRODUCTION

For given  $\alpha_{j0} \in \mathbb{R}$  and  $\alpha_j = (\alpha_{j1}, \dots, \alpha_{jn}) \in \mathbb{R}^n$  ( $j = 1, 2, \dots, m$ ), let  $f_j$  be real quadratic polynomials in  $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$  specified by

$$\begin{aligned} f_j(x) &:= (x, x) + 2(\alpha_j, x) + \alpha_{j0} \\ &= |x + \alpha_j|^2 - |\alpha_j|^2 + \alpha_{j0}, \end{aligned}$$

where  $(x, y) := \sum_{\nu=1}^n x_\nu y_\nu$  and  $|x|^2 := (x, x)$  for  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ . Let  $O_j$  be the point  $-\alpha_j \in \mathbb{R}^n$ , which is the center of the hypersphere  $\Re S_j := \{x \in \mathbb{R}^n \mid f_j(x) = 0\}$ . The radius  $r_i > 0$  of  $\Re S_i$  and the distance  $\rho_{jk} > 0$  between  $O_j$  and  $O_k$  are given by

$$r_j^2 = -\alpha_{j0} + |\alpha_j|^2 \quad \text{and} \quad \rho_{jk}^2 = |\alpha_j - \alpha_k|^2,$$

respectively. Here we consider the  $n - 1$  dimensional hypersphere  $f_j(x) = 0$  in  $\mathbb{C}^n$ , i.e., we define  $S_j$  as

$$S_j := \{x \in \mathbb{C}^n \mid f_j(x) = 0\},$$

and define the  $n$  dimensional hypersphere arrangement  $\mathcal{A}$  as  $\mathcal{A} := \bigcup_{1 \leq j \leq m} S_j$ , For  $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$  let  $\Phi(x) = \Phi(x_1, \dots, x_n)$  be multiplicative meromorphic function on  $\mathbb{C}^n$  specified by

$$\Phi(x) := \prod_{j=1}^m f_j(x)^{\lambda_j}.$$

We set  $X := \mathbb{C}^n - \bigcup_{j=1}^m S_j$ . For  $0 \leq r \leq n$ , we denote by  $\Omega^r = \Omega^r(X, \star \bigcup_{j=1}^{n+1} S_j)$  the space of rational  $r$ -forms on  $X$  whose singularities all lie in the set  $\bigcup_{j=1}^{n+1} S_j$ . For the complex

$$\Omega : 0 \rightarrow \Omega^0 \xrightarrow{\nabla} \Omega^1 \xrightarrow{\nabla} \Omega^2 \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \Omega^n \xrightarrow{\nabla} 0,$$

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where  $\nabla : \Omega^r \rightarrow \Omega^{r+1}$  is the covariant derivation given by

$$\nabla\psi := d\psi + d\log\Phi \wedge \psi \quad (\psi \in \Omega^r),$$

the  $r$ th twisted de Rham cohomology  $H_{\nabla}^r(X, \Omega^r)$  is defined by

$$H_{\nabla}^r(X, \Omega^r) := \text{Ker}(\nabla : \Omega^r \rightarrow \Omega^{r+1}) / \text{Im}(\nabla : \Omega^{r-1} \rightarrow \Omega^r).$$

For a rational function  $\varphi(x)$  which is holomorphic on  $X$  we consider the hypergeometric integral

$$I_{n,m}(\varphi\varpi_n, \mathfrak{z}) := \int_{\mathfrak{z}} \Phi(x)\varphi(x)\varpi_n \quad \text{where} \quad \varpi_n := dx_1 \wedge \cdots \wedge dx_n. \quad (1.1)$$

Here  $\varphi(x)\varpi_n \in \Omega^n$  is a representative in the  $n$ th twisted de Rham cohomology  $H_{\nabla}^n(X, \Omega^r)$  and  $\mathfrak{z}$  is an  $n$ -twisted cycle, which is a representative in the  $n$ th twisted homology  $H_n(X, \mathcal{L}^*)$  over the dual local system  $\mathcal{L}^*$  on  $X$ . The hypergeometric integral (1.1) is identified with the pairing between  $H_{\nabla}^n(X, \Omega^r)$  and  $H_n(X, \mathcal{L}^*)$ :

$$H_{\nabla}^n(X, \Omega^r) \times H_n(X, \mathcal{L}^*) \ni \langle \varphi\varpi_n, \mathfrak{z} \rangle \longrightarrow I_{n,m}(\varphi\varpi_n, \mathfrak{z}).$$

(See [2] for more details.)

The differential or difference equations describing the integral (1.1) can be written using, besides the exponents  $\lambda$ , Cayley–Menger determinants concerning the radius  $r_j > 0$  of  $S_j$  and the distance  $\rho_{jk} > 0$  between the centers  $O_j$  of  $S_j$  and  $O_k$  of  $S_k$ . For that purpose, we introduce the definition of Cayley–Menger determinants first.

Consider the  $(m+2) \times (m+2)$  symmetric matrix  $B = (b_{ij})_{i,j=0,\star,1,2,\dots,m}$ , whose entries are given by  $b_{00} = 0$ ,  $b_{\star\star} = 0$ ,  $b_{0\star} = b_{0j} = 1$  ( $1 \leq j \leq m$ ),  $b_{1\star} = r_j^2$ ,  $b_{ij} = \rho_{ij}^2$  ( $1 \leq i, j \leq m$ ), i.e.,

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & r_1^2 & r_2^2 & r_3^2 & \cdots & r_m^2 \\ 1 & r_1^2 & 0 & \rho_{12}^2 & \rho_{13}^2 & \cdots & \rho_{1,m}^2 \\ 1 & r_2^2 & \rho_{21}^2 & 0 & \rho_{23}^2 & \cdots & \rho_{2,m}^2 \\ 1 & r_3^2 & \rho_{31}^2 & \rho_{32}^2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \rho_{m-1,m}^2 \\ 1 & r_m^2 & \rho_{m,1}^2 & \rho_{m,2}^2 & \cdots & \rho_{m,m-1}^2 & 0 \end{pmatrix}.$$

The Cayley–Menger determinants are defined as the minors of the matrix  $B$ . See [4].

**Definition 1.1.** Denote by  $\rho_{\star j} = \rho_{j\star}$  the radius  $r_j$  for  $j \in \{1, 2, \dots, m\}$  or 0 for  $j = \star$ . The determinant

$$B \begin{pmatrix} 0 & J \\ 0 & K \end{pmatrix} = B \begin{pmatrix} 0 & j_1 & \cdots & j_p \\ 0 & k_1 & \cdots & k_p \end{pmatrix} := \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & \rho_{j_1 k_1}^2 & \cdots & \rho_{j_1 k_p}^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_{j_p k_1}^2 & \cdots & \rho_{j_p k_p}^2 \end{vmatrix}$$

is called the *Cayley–Menger determinant*, where  $J = \{j_1, \dots, j_p\}$  and  $K = \{k_1, \dots, k_p\}$  denote two subsets of the indices in  $\{\star, 1, \dots, m\}$ . We simply write  $B(0J)$  instead of

$B\begin{pmatrix} 0 & J \\ 0 & J \end{pmatrix}$ . For instance, we have the following expressions:

$$\begin{aligned} B(0j) &= -1, & B(0\star j) &= 2r_j^2 > 0, & B(0jk) &= 2\rho_{jk}^2 > 0, \\ B(0\star jk) &= -(\rho_{jk} + r_j - r_k)(\rho_{jk} - r_j + r_k)(-\rho_{jk} + r_j + r_k)(\rho_{jk} + r_j + r_k), \\ B(0jkl) &= -(\rho_{jk} + \rho_{jl} - \rho_{kl})(\rho_{jk} - \rho_{jl} + \rho_{kl})(-\rho_{jk} + \rho_{jl} + \rho_{kl})(\rho_{jk} + \rho_{jl} + \rho_{kl}). \end{aligned}$$

We denote by  $N_m$  the set of indices  $\{1, 2, \dots, m\}$ . In this paper we assume the following *genericity conditions*. For an arbitrary non-empty subset of indices  $J \subset N_m$ , we impose that

$$B(0J)(-1)^{|J|} > 0 \quad \text{for } 1 \leq |J| \leq n+1, \quad (1.2)$$

$$B(0\star J)(-1)^{|J|} \neq 0 \quad \text{for } 1 \leq |J| \leq n+1, \quad (1.3)$$

where  $|J|$  denotes the size of  $J$ . Remark that  $B(0J)$  vanishes if  $|J| \geq n+2$ . In the same way  $B(0\star J)$  vanishes if  $|J| \geq n+3$ . The conditions (1.2) and (1.3) give the moduli space of arrangement of  $n$  dimensional real hyperspheres in general position in  $\mathbb{C}^n$ . In Section 2 we introduce several lemmas using the Cayley–Menger determinants for later sections.

We now return to the topic of the hypergeometric integrals (1.1). Singularity of the integral (1.1) (Landau singularity), i.e., degeneration of analytic and topological structure of  $H_{\nabla}^n(X, \Omega')$  and  $H_n(X, \mathcal{L}^*)$  occurs when one of the conditions (1.2) or (1.3) breaks down. In this paper we shall describe the degeneration of  $H_{\nabla}^n(X, \Omega')$  and  $H_n(X, \mathcal{L}^*)$  in terms of the Cayley–Menger determinants attached to the arrangement  $\mathcal{A}$  when the condition (1.3) is not satisfied, more explicitly when one of the principal determinants  $B(0\star J)$  for  $J \subset N_m, 1 \leq |J| \leq n+1$  tends to 0.

We study this degeneration from two perspectives, one is an algebraic point of view for analytic structure of  $H_{\nabla}^n(X, \Omega')$  and the other is a geometric point of view for topological structure of  $H_n(X, \mathcal{L}^*)$ . One of our main results is stated as Theorem 4.4 in Section 4, which shows a degeneration of the  $n$ th twisted cohomology  $H_{\nabla}^n(X, \Omega')$  for  $B(0\star J) = 0$  ( $1 \leq |J| \leq n+1$ ) by using the contiguity relations obtained in [4]. More precisely, for a fixed subset  $J \subset N_m$  with  $1 \leq |J| \leq n+1$ , the  $n$ th cohomology  $H_{\nabla}^n(X, \Omega')$  degenerates to the quotient by its annihilator  $Ann_J$  as

$$H_{\nabla}^n(X, \Omega') \rightarrow H_{\nabla}^n(X, \Omega')/Ann_J \quad (B(0\star J) \rightarrow 0),$$

where  $Ann_J$  consists of all elements in  $H_{\nabla}^n(X, \Omega')$  vanishing for  $B(0\star J) = 0$ . In this case  $Ann_J$  is to be a one dimensional  $\mathbb{C}$ -subspace of  $H_{\nabla}^n(X, \Omega')$  generated by just one  $n$ -form  $\varphi_J$ , which is explicitly given as (4.6) in Theorem 4.4. Thus Sections 3 and 4 are devoted to stating Theorem 4.4. On the other hand, this one dimensional degeneration is also reflected in the  $n$ th twisted homology  $H_n(X, \mathcal{L}^*)$ . In other words,  $H_n(X, \mathcal{L}^*)$  also degenerates by one dimension in the limit of  $B(0\star J) \rightarrow 0$ . This means that a specific cycle of hypergeometric integrals vanishes in the limit  $B(0\star J) \rightarrow 0$ . We thus consider geometrically the situation in which a spherically faced simplex shrinks into a single point as a cycle of hypergeometric integrals. Then we investigate the behavior of the integral  $I_{n,m}(\varpi_n, \mathfrak{z})$  over a vanishing twisted cycle  $\mathfrak{z} = \mathfrak{z}(J)$  as  $B(0\star J)$  ( $J \subset N_m, 1 \leq |J| \leq n+1$ ) tends to 0 in this isotopic deformation. (See [4] for details of twisted cycles attached to hypersphere arrangement  $\mathcal{A}$ .) Our other main theorem describes explicit asymptotic behavior of the integral for vanishing of  $B(0\star J)$ . We state explicit formulae, in terms of Cayley–Menger determinants, for

this theorem as Theorems 5.2 and 6.5, and the cases where  $J \subset N_m$   $|J| = n + 1$  and where  $J \subset N_m$   $1 \leq |J| \leq n$  correspond to these theorems, respectively.

General aspects involved in this kind of integrals have been presented by several authors (see [1, 5, 8, 12] for instance). See also [9] which has recently given a new incentive to study Feynman integrals from the point of view of twisted cohomology.

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## 2. PRELIMINARY LEMMAS

In this section we introduce several lemmas that will be needed in later sections.

**Lemma 2.1.** *For an arbitrary ordered subset  $J$  of size  $n + 1$  in  $N_m$  the following quadratic identity holds :*

$$\begin{aligned} \sum_{j \in J} B(0 \star \partial_j J) f_j^2 - 2 \sum_{\substack{j, k \in J \\ j < k}} B \begin{pmatrix} 0 & j & \partial_j \partial_k J \\ 0 & k & \partial_j \partial_k J \end{pmatrix} f_j f_k \\ + 2 \sum_{j \in J} B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{pmatrix} f_j - B(0 \star J) = 0, \end{aligned} \quad (2.1)$$

where the symbol  $\partial_j J$  is defined as  $\partial_j J = J - \{j\}$  for  $j \in J$ .

**Proof.** Because of symmetry one may assume that  $J = N_{n+1}$ . Remark first that the equality  $B(0 \star N_{n+1}) = 0$  holds if there exists  $x = (x_1, \dots, x_n) \in X$  such that  $f_j(x) = 0$  ( $1 \leq j \leq n + 1$ ). Take an arbitrary  $\xi = (\xi_1, \dots, \xi_n) \in X$  and put

$$\eta = (\eta_1, \dots, \eta_{n+1}) \quad \text{such that} \quad \eta_j = f_j(\xi) \quad (1 \leq j \leq n + 1).$$

This is equivalent to say that

$$f_j(\xi) \Big|_{\alpha_{j0} \rightarrow \alpha_{j0} - \eta_j} = 0 \quad (1 \leq j \leq n + 1)$$

by replacing  $\alpha_{j0}$  with  $\alpha_{j0} - \eta_j$ . Hence  $B(0 \star N_{n+1})$  being equal to the resultant of the  $n + 1$  quadratic polynomials  $f_j$  in  $x$ . See [4, Lemma 4] for detail. See also [6, Chapter 8, §2] and [7] for “general resultant”, we have

$$0 = B(0 \star N_{n+1}) \Big|_{\substack{\alpha_{j0} \rightarrow \alpha_{j0} - \eta_j \\ (j \in N_{n+1})}}. \quad (2.2)$$

On the other hand a direct calculation shows the identity

$$\begin{aligned}
B(0 \star N_{n+1}) \Big|_{\substack{\alpha_{j0} \rightarrow \alpha_{j0} - \eta_j \\ (j \in N_{n+1})}} &= - \sum_{j \in N_{n+1}} B(0 \partial_j N_{n+1}) \eta_j^2 \\
&+ 2 \sum_{\substack{j, k \in N_{n+1} \\ j < k}} B \begin{pmatrix} 0 & j & \partial_j \partial_k N_{n+1} \\ 0 & k & \partial_j \partial_k N_{n+1} \end{pmatrix} \eta_j \eta_k - 2 \sum_{j \in N_{n+1}} B \begin{pmatrix} 0 & \star & \partial_j N_{n+1} \\ 0 & j & \partial_j N_{n+1} \end{pmatrix} \eta_j \\
&+ B(0 \star N_{n+1}). \tag{2.3}
\end{aligned}$$

From (2.2) and (2.3) the equality (2.1) is obtained.  $\square$

The following two lemmas are an immediate consequence of [4, Lemma 11].

**Lemma 2.2.** *For an arbitrary ordered subset  $J = \{j_1, \dots, j_{n+1}\} \subset N_m$  of size  $n+1$  set*

$$\delta_J := \begin{vmatrix} 1 & \alpha_{j_1 1} & \dots & \alpha_{j_1 n} \\ \vdots & \vdots & & \vdots \\ 1 & \alpha_{j_{n+1} 1} & \dots & \alpha_{j_{n+1} n} \end{vmatrix}.$$

Then we have

$$B(0 J) = 2^n (-1)^{n+1} \delta_J^2, \tag{2.4}$$

and

$$B \begin{pmatrix} 0 & j & K \\ 0 & k & K \end{pmatrix} = 2^n (-1)^{n+1} \delta_{jK} \delta_{kK}$$

for  $K \subset N_m$  satisfying  $|K| = n$  and  $j, k \notin K$ .

**Lemma 2.3.**

(i) *For an arbitrary ordered subset  $J = \{j_1, \dots, j_{n+2}\}$  of size  $n+2$  in  $N_m$  set*

$$\Delta_J := \begin{vmatrix} 1 & \alpha_{j_1 0} & \alpha_{j_1 1} & \dots & \alpha_{j_1 n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_{j_{n+2} 0} & \alpha_{j_{n+2} 1} & \dots & \alpha_{j_{n+2} n} \end{vmatrix}.$$

Then the following identities hold:

$$\begin{aligned}
B(0 J) &= 0, \\
B(0 \star J) &= (-1)^n 2^n \Delta_J^2. \tag{2.5}
\end{aligned}$$

(ii) *For arbitrary subset  $J$  of size  $n+1$*

$$B \begin{pmatrix} 0 & \star & J \\ 0 & k & J \end{pmatrix} = (-1)^{n+1} 2^n \delta_J \Delta_{kJ} \quad \text{for } k \notin J. \tag{2.6}$$

(iii) *For an arbitrary subset  $J$  of size  $n+3$ , we have*

$$B(0 \star J) = 0.$$

**Lemma 2.4.** *For an arbitrary (ordered) subset  $J = \{j_1, \dots, j_{n+2}\}$  of size  $n+2$ , we have*

$$B(0 \star J) - \sum_{j \in J} B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{pmatrix} f_j = 0. \tag{2.7}$$

This is equivalent to

$$\Delta_J - \sum_{\nu=1}^{n+2} (-1)^\nu \delta_{j_1 \dots j_{\nu-1} j_{\nu+1} \dots j_{n+2}} f_{j_\nu} = 0.$$

Due to the Jacobi identity for determinants we have the following.

**Lemma 2.5.** *For  $J \subset N_m$  we have*

$$B(0 \ J)B(0 \star \partial_j J) - \left\{ B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{pmatrix} \right\}^2 = B(0 \ \partial_j J)B(0 \star J) \quad (j \in J), \quad (2.8)$$

$$B(0 \ kJ)B(0 \star J) - \left\{ B \begin{pmatrix} 0 & \star & J \\ 0 & k & J \end{pmatrix} \right\}^2 = B(0 \ J)B(0 \star kJ) \quad (k \notin J). \quad (2.9)$$

### 3. FUNDAMENTAL IDENTITIES AMONG ADMISSIBLE FORMS SPANNING $H_{\nabla}^n(X, \Omega)$

The linear space  $\mathbb{R}^m$  of exponents  $\lambda = (\lambda_1, \dots, \lambda_m)$  ( $\lambda_j \in \mathbb{R}$ ) has a standard basis  $\mathbf{e}_j$  ( $1 \leq j \leq m$ ) such that

$$\lambda = \sum_{j=1}^m \lambda_j \mathbf{e}_j.$$

We simply use the following symbols:

$$\mathbf{e}_J := \sum_{j \in J} \mathbf{e}_j, \quad \lambda_J := \sum_{j \in J} \lambda_j$$

for  $J \subset N_m$ , and in particular,

$$\lambda_\infty := \lambda_{N_m} = \sum_{j=1}^m \lambda_j.$$

The system of  $n$ -differential forms

$$F_J := \frac{\varpi_n}{\prod_{j \in J} f_j} \quad (J \subset N_m \text{ unordered subset})$$

(we call these forms ‘‘admissible forms’’) span the  $n$ -dimensional twisted cohomology  $H_{\nabla}^n(X, \Omega)$ , which is of dimension  $\kappa := \sum_{\nu=1}^n \binom{m}{\nu} + \binom{m-1}{n}$  (see [4, Proposition 8]). We denote by  $\mathfrak{M}_g$  the multiplication operator:

$$\mathfrak{M}_g : \varphi \rightarrow g\varphi$$

for a function  $\varphi$  by a rational function  $g$  on  $X$ . The shift operators  $T_{\pm \mathbf{e}_j} : \lambda \rightarrow \lambda \pm \mathbf{e}_j$  give rise to the action on  $H_{\nabla}^n(X, \Omega)$ :

$$T_{\pm \mathbf{e}_j} : \varphi \varpi_n \longrightarrow \mathfrak{M}_{f_j^{\pm 1}} \varphi \varpi_n$$

for  $\varphi \varpi_n \in H_{\nabla}^n(X, \Omega)$  which does not depend on  $\lambda$ . For each unordered  $J$  ( $|J| = p$ ,  $1 \leq p$ ) define

$$\begin{aligned} W_0(j) \varpi_1 &:= F_j \quad (J = \{j\}), \quad (p = 1) \\ W_0(J) \varpi_n &:= B(0 \star J) F_J - \sum_{j \in J} B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{pmatrix} F_{\partial_j J} \quad (p \geq 2). \end{aligned}$$

Then the inversion formula is expressed as follows:

$$B(0 \star J)F_J = \sum_{\substack{K \subset J \\ |K| \geq 1}} \beta_{K,J} W_0(K) \varpi_n$$

such that

$$\beta_{K,J} := \sum_{K=L_0 \subsetneq L_1 \subsetneq \dots \subsetneq L_p=J} \prod_{\nu=1}^p \frac{B\left(\begin{array}{ccc} 0 & \star & L_{\nu-1} \\ 0 & l_\nu & L_{\nu-1} \end{array}\right)}{B(0 \star L_{\nu-1})},$$

where

$$L_\nu = L_{\nu-1} \cup \{l_\nu\} \quad (1 \leq \nu \leq p).$$

It has been proved in [4] that for unordered  $J \subset N_m$  satisfying  $|J| \geq n+2$  we have

$$W_0(J) \varpi_n = 0. \quad (3.1)$$

Moreover for  $|J| = n+2$  the following identity holds (see [4, Section 6]):

$$0 = W_0(\partial_j J) \varpi_n - \sum_{\substack{k \in J \\ k \neq j}} \frac{B\left(\begin{array}{ccc} 0 & j & \partial_j \partial_k J \\ 0 & k & \partial_j \partial_k J \end{array}\right)}{B(0 \partial_k J)} W_0(\partial_k J) \varpi_n. \quad (3.2)$$

The equation (3.1) for  $|J| = n+2$  is equivalent to the partial fraction expansion:

$$B(0 \star J)F_J - \sum_{j \in J} B\left(\begin{array}{ccc} 0 & j & \partial_j J \\ 0 & \star & \partial_j J \end{array}\right) F_{\partial_j J} = 0. \quad (3.3)$$

On the other hand, from (3.2) we have the following.

**Proposition 3.1.** *Fix unordered  $J \subset N_m$  such that  $|J| = n+2$ . The following fundamental relation holds true among  $F_{\partial_j J}$  ( $j \in J$ ),  $F_{\partial_j \partial_k J}$  ( $j, k \in J, j \neq k$ ):*

$$0 = \sum_{j \in J} \frac{B(0 \star \partial_j J)}{B\left(\begin{array}{ccc} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{array}\right)} F_{\partial_j J} + \sum_{\substack{j, k \in J \\ j < k}} \frac{B(0 \partial_j \partial_k J)}{B\left(\begin{array}{ccc} 0 & j & \partial_j \partial_k J \\ 0 & k & \partial_k \partial_k J \end{array}\right)} F_{\partial_j \partial_k J}, \quad (3.4)$$

which is symmetric with respect to any permutation of the components of  $J$ . The right-hand side of (3.4) will be denoted by  $E_J$ .

**Proof.** Because of symmetry one may assume that  $J = N_{n+2} = \{1, 2, \dots, n+2\}$ . Eq. (3.2) means the following identity:

$$0 = W_0(12 \dots n+1) \varpi_n - \sum_{j=1}^{n+1} \frac{B\left(\begin{array}{ccc} 0 & n+2 & \partial_j \partial_{n+2} J \\ 0 & j & \partial_j \partial_{n+2} J \end{array}\right)}{B(0 \partial_j J)} W_0(\partial_j J) \varpi_n. \quad (3.5)$$

Denote by  $T$  the right-hand side of (3.5) divided by the factor  $B\left(\begin{array}{ccc} 0 & \star & \partial_{n+2} J \\ 0 & n+2 & \partial_{n+2} J \end{array}\right)$ . Then  $T$  equals a linear combination of  $F_{\partial_j J}$  ( $j \in J$ ) and  $F_{\partial_j \partial_k J}$  ( $j, k \in J, j \neq k$ ) which is expressible in the form

$$T = \sum_{j \in J} [T : F_{\partial_j J}] F_{\partial_j J} + \sum_{\substack{j, k \in J \\ j < k}} [T : F_{\partial_j \partial_k J}] F_{\partial_j \partial_k J}. \quad (3.6)$$

We want to show that  $T$  coincides with the right-hand side of (3.4). Since  $T$  is expressed in symmetric form under any permutation of the components  $\{1, 2, \dots, n+1\}$  it is sufficient to describe the coefficients  $[T : F_{\partial_1 J}]$ ,  $[T : F_{\partial_{n+2} J}]$  and  $[T : F_{\partial_1 \partial_2 J}]$ ,  $[T : F_{\partial_1 \partial_{n+2} J}]$ . Indeed, we have

$$[T : F_{\partial_{n+2} J}] = \frac{B(0 \star \partial_{n+2} J)}{B \begin{pmatrix} 0 & \star & \partial_{n+2} J \\ 0 & n+2 & \partial_{n+2} J \end{pmatrix}}, \quad (3.7)$$

which is nothing to prove. Next from Lemmas 2.2, 2.3 and (3.5) we have

$$\begin{aligned} [T : F_{\partial_1 J}] &= - \frac{B(0 \star \partial_1 J) B \begin{pmatrix} 0 & n+2 & \partial_1 \partial_{n+2} J \\ 0 & 1 & \partial_1 \partial_{n+2} J \end{pmatrix}}{B(0 \partial_1 J) B \begin{pmatrix} 0 & \star & \partial_{n+2} J \\ 0 & n+2 & \partial_{n+2} J \end{pmatrix}} \\ &= (-1)^n \frac{B(0 \star \partial_1 J) \delta_{2\dots n+2} \delta_{12\dots n+1}}{B(0 \partial_1 J) \Delta_{12\dots n+2} \delta_{1\dots n+1}} = (-1)^n \frac{B(0 \star \partial_1 J) \delta_{2\dots n+2}}{B(0 \partial_1 J) \Delta_{1\dots n+1} n+2} \\ &= - \frac{B(0 \star \partial_1 J)}{2^n \delta_{2\dots n+1} n+2 \Delta_{12\dots n+1} n+2} = \frac{B(0 \star \partial_1 J)}{B \begin{pmatrix} 0 & \star & \partial_1 J \\ 0 & 1 & \partial_1 J \end{pmatrix}}. \end{aligned} \quad (3.8)$$

In the third step from (3.5) we have

$$[T : F_{\partial_1 \partial_{n+2} J}] = \frac{-v_{11} v_{22} + v_{21} v_{12}}{v_0 v_{22}},$$

where

$$\begin{aligned} v_0 &:= B \begin{pmatrix} 0 & \star & \partial_{n+2} J \\ 0 & n+2 & \partial_{n+2} J \end{pmatrix} = (-1)^n 2^n \Delta_{12\dots n+2} \delta_{1\dots n+1}, \\ v_{11} &:= B \begin{pmatrix} 0 & \star & \partial_1 \partial_{n+2} J \\ 0 & 1 & \partial_1 \partial_{n+2} J \end{pmatrix}, \\ v_{21} &:= B \begin{pmatrix} 0 & \star & \partial_1 \partial_{n+2} J \\ 0 & n+2 & \partial_1 \partial_{n+2} J \end{pmatrix}, \\ v_{12} &:= B \begin{pmatrix} 0 & n+2 & \partial_1 \partial_{n+2} J \\ 0 & 1 & \partial_1 \partial_{n+2} J \end{pmatrix} = -2^n \delta_{12\dots n+1} \delta_{2\dots n+2}, \\ v_{22} &:= B(0 \partial_1 J) = (-1)^{n+1} 2^n \delta_{2\dots n+2}^2. \end{aligned}$$

The Jacobi identity concerning  $-v_{11} v_{22} + v_{21} v_{12}$  implies

$$\begin{aligned} [T : F_{\partial_1 \partial_{n+2} J}] &= - \frac{B \begin{pmatrix} 0 & \star & \partial_1 J \\ 0 & 1 & \partial_1 J \end{pmatrix} B(0 \partial_1 \partial_{n+2} J)}{v_0 v_{22}} \\ &= (-1)^n \frac{B(0 \partial_1 \partial_{n+2} J) \Delta_{12\dots n+2} \delta_{2\dots n+2}}{B(0 \partial_1 J) \Delta_{12\dots n+2} \delta_{1\dots n+1}} = - \frac{B(0 \partial_1 \partial_{n+2} J)}{2^n \delta_{1\dots n+1} \delta_{2\dots n+2}} \\ &= \frac{B(0 \partial_1 \partial_{n+2} J)}{B \begin{pmatrix} 0 & 1 & \partial_1 \partial_{n+2} J \\ 0 & n+2 & \partial_1 \partial_{n+2} J \end{pmatrix}}, \end{aligned} \quad (3.9)$$

since we have

$$\delta_{1\dots n+1} \delta_{2\dots n+2} = -\frac{1}{2^n} B \begin{pmatrix} 0 & 1 & \partial_1 \partial_{n+2} J \\ 0 & n+2 & \partial_1 \partial_{n+2} J \end{pmatrix}.$$

In the final step from (3.5), we have

$$[T : F_{\partial_1 \partial_2 J}] = \frac{1}{v_0} \left( \frac{v_{31} v_{12}}{v_{22}} + \frac{v_{11} v_{32}}{v_{33}} \right), \quad (3.10)$$

where

$$\begin{aligned} v_{31} &:= B \begin{pmatrix} 0 & \star & \partial_1 \partial_2 J \\ 0 & 2 & \partial_1 \partial_2 J \end{pmatrix}, \\ v_{32} &:= B \begin{pmatrix} 0 & n+2 & \partial_2 \partial_{n+2} J \\ 0 & 2 & \partial_2 \partial_{n+2} J \end{pmatrix} = 2^n \delta_{13\dots n+2} \delta_{12\dots n+1}, \\ v_{33} &:= B(0 \ \partial_2 J) = (-1)^{n+1} 2^n \delta_{13\dots n+2}^2. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \frac{v_{31} v_{12}}{v_{22}} + \frac{v_{11} v_{32}}{v_{33}} &= (-1)^n \delta_{12\dots n+1} \left( \frac{v_{31}}{\delta_{2\dots n+2}} - \frac{v_{11}}{\delta_{13\dots n+2}} \right) \\ &= (-1)^n \frac{\delta_{12\dots n+1}}{\delta_{2\dots n+2} \delta_{13\dots n+2}^2} (\delta_{13\dots n+2}^2 v_{31} - \delta_{2\dots n+2} \delta_{13\dots n+2} v_{11}) \\ &= -\frac{1}{2^n} \frac{\delta_{12\dots n+1}}{\delta_{2\dots n+2} \delta_{13\dots n+2}^2} \left( B(0 \ \partial_2 J) v_{31} - B \begin{pmatrix} 0 & 1 & \partial_1 \partial_2 J \\ 0 & 2 & \partial_1 \partial_2 J \end{pmatrix} v_{11} \right) \\ &= -\frac{1}{2^n} \frac{\delta_{12\dots n+1}}{\delta_{2\dots n+2} \delta_{13\dots n+2}^2} B(0 \ \partial_1 \partial_2 J) B \begin{pmatrix} 0 & \star & \partial_2 J \\ 0 & 2 & \partial_2 J \end{pmatrix} \quad (3.11) \end{aligned}$$

in view of the Jacobi identity. From (3.10) and (3.11) we therefore obtain

$$\begin{aligned} [T : F_{\partial_1 \partial_2 J}] &= -\frac{1}{2^n} \frac{\delta_{12\dots n+1}}{\delta_{2\dots n+2} \delta_{13\dots n+2}^2} \cdot \frac{B(0 \ \partial_1 \partial_2 J) B \begin{pmatrix} 0 & \star & \partial_2 J \\ 0 & 2 & \partial_2 J \end{pmatrix}}{v_0} \\ &= \frac{(-1)^{n+1}}{2^n} \frac{1}{\delta_{23\dots n+2} \delta_{13\dots n+2}} B(0 \ \partial_1 \partial_2 J) = \frac{B(0 \ \partial_1 \partial_2 J)}{B \begin{pmatrix} 0 & 1 & \partial_1 \partial_2 J \\ 0 & 2 & \partial_1 \partial_2 J \end{pmatrix}}, \quad (3.12) \end{aligned}$$

since we have

$$B \begin{pmatrix} 0 & \star & \partial_2 J \\ 0 & 2 & \partial_2 J \end{pmatrix} = 2^n \delta_{13\dots n+2} \Delta_{12\dots n+2}.$$

Applying (3.7), (3.8), (3.9) and (3.12) to (3.6), Proposition 3.1 has been proved.  $\square$

The system of equations  $E_J$  ( $J \subset N_m, |J| = n+2$ ) are not linearly independent. There exist a finite number of linear relations:

**Lemma 3.2.** *Fix an ordered set of indices  $J = \{j_1, \dots, j_{n+3}\} \subset N_m$  such that  $|J| = n+3$ ,  $E_{\partial_j J}$  ( $j \in J$ ) are identically linearly related as follows:*

$$\sum_{\nu=1}^{n+3} (-1)^{\nu-1} \Delta_{\partial_{j_\nu} J} E_{\partial_{j_\nu} J} = 0, \quad (3.13)$$

where  $\partial_{j_\nu} J$  means the deletion of the  $\nu$ -th component  $j_\nu$  of  $J$ , i.e.,

$$\partial_{j_\nu} J = \{j_1, \dots, j_{\nu-1}, j_{\nu+1}, \dots, j_{n+3}\}.$$

**Proof.** Eq. (3.13) can be deduced from the definition of  $E_{j_\nu}$  in view of Lemmas 2.2 and 2.3.  $\square$

**Remark.** Eq. (3.13) is alternating with respect to permutations of the components of  $J$ .

We set the following ordering

$$\mathcal{O} : f_1 \prec f_2 \prec \dots \prec f_{n+1} \prec f_{n+2} \prec \dots \prec f_m$$

for the quadratic polynomials  $f_j$ . This ordering induces in a natural way the lexicographic ordering for logarithmic  $n$ -forms  $\frac{df_{j_1}}{f_{j_1}} \wedge \dots \wedge \frac{df_{j_n}}{f_{j_n}}$  ( $1 \leq j_1 < \dots < j_n \leq m$ ).

As a consequence of Proposition 3.1 and Lemma 3.2 the following statement holds true:

**Lemma 3.3.** *The set*

$$\{F_J \mid J \subset N_m, 1 \leq |J| \leq n\} \cup \{F_K \mid \{1\} \subset K \subset N_m, |K| = n+1\} \quad (3.14)$$

*consisting of  $n$ -forms with respect to the ordering  $\mathcal{O}$  forms a basis of  $H_{\nabla}^n(X, \Omega)$ .*

The set (3.14) is called the *NBC (non broken circuit) basis* in the context of [10, 11]. For details the reader is referred to [10, 11] and [4, Proposition 8 and Remark in §6].

**Examples.** For  $n = 1$  and  $m \geq 2$  the set

$$\{F_j \mid 1 \leq j \leq m\} \cup \{F_{1j} \mid 2 \leq j \leq m\}$$

forms the NBC basis of  $H_{\nabla}^1(X, \Omega)$ . For  $n = 2$  and  $m \geq 3$  the set

$$\{F_j \mid 1 \leq j \leq m\} \cup \{F_{jk} \mid 1 \leq j < k \leq m\} \cup \{F_{1jk} \mid 2 \leq j < k \leq m\}$$

forms the NBC basis of  $H_{\nabla}^2(X, \Omega)$ .

#### 4. DEGENERACY OF $H_{\nabla}^n(X, \Omega)$ FOR THE VANISHING OF $B(0 \star J)$

In this section we want to explain how  $H_{\nabla}^n(X, \Omega)$  degenerates when one of the determinants  $B(0 \star J)$  ( $J \subset N_m$ , ( $|J| \leq n+1$ )) vanishes. For this purpose we apply some contiguity formulae obtained in [4, Theorem A in Appendix].

**Definition 4.1.** Let  $\{j\}$  or  $J$  be an unordered subset of  $N_m$  which is of size 1 or  $|J| = p$  ( $p \geq 2$ ). Denote

$$F_J^{(j)} := (\lambda_j - 1)T_{-e_j} F_J = (\lambda_j - 1)\mathfrak{M}_{f_j^{-1}} F_J \quad (j \in J),$$

$$\widehat{F}_J := \sum_{J \in J} F_J^{(j)},$$

$$W_0^{(j)}(J)\varpi := (\lambda_j - 1)T_{-e_j} W_0(J)\varpi = (\lambda_j - 1)\mathfrak{M}_{f_j^{-1}} W_0(J)\varpi_n \quad (j \in J),$$

$$\widehat{W}_0(J)\varpi := \sum_{j \in J} W_0^{(j)}(J)\varpi.$$

Then we have the following:

**Lemma 4.2.** (i)

$$\begin{aligned} \widehat{W}_0(j)\varpi_1 &= W_0^{(j)}(j)\varpi_1 \\ &\sim -(\lambda_\infty + \lambda_j + n - 2)F_j - \sum_{\substack{k \in N_m \\ k \neq j}} \lambda_k \left( F_k + B \begin{pmatrix} 0 & \star & k \\ 0 & \star & j \end{pmatrix} F_{kj} \right) \quad (J = \{j\}). \end{aligned} \quad (4.1)$$

$$\begin{aligned} \text{(ii)} \quad \widehat{W}_0(J)\varpi_n &\sim (\lambda_\infty + n - p)B(0J)F_J + \sum_{k \in J^c} \lambda_k \sum_{l \in J} B \begin{pmatrix} 0 & \star & \partial_l J \\ 0 & l & \partial_l J \end{pmatrix} F_{k\partial_l J} \\ &\quad - \sum_{k \in J^c} \sum_{l \in J} \lambda_k B \begin{pmatrix} 0 & \star & k & \partial_l J \\ 0 & \star & l & \partial_l J \end{pmatrix} F_{kJ} \quad (2 \leq |J| \leq n+1), \end{aligned} \quad (4.2)$$

where  $J^c$  is the complement of  $J$  defined as  $J^c = N_m - J$ .

**Remark.** In the case  $|J| = n+2$ ,  $F_J$  is linearly related with  $F_K$  ( $1 \leq |K| \leq n+1$ ) by the equality (3.3) such that  $W_0(J)\varpi_n$  vanishes identically.

**Proof.** Due to [4, Theorem A] one sees the following:

$$\begin{aligned} W_0^{(j)}(j)\varpi_1 &\sim -(\lambda_\infty + \lambda_j + n - 2)F_j \\ &\quad - \sum_{\substack{k \in N_m \\ k \neq j}} \lambda_k \left( F_k + B \begin{pmatrix} 0 & \star & k \\ 0 & \star & j \end{pmatrix} F_{kj} \right) \quad (J = \{j\}), \end{aligned} \quad (4.3)$$

$$W_0^{(j)}(J)\varpi_n \sim (\lambda_\infty + n - p)Z_J^j + \sum_{k \in J^c} \lambda_k Z_{J,k}^j \quad (|J| = p \geq 2), \quad (4.4)$$

where

$$\begin{aligned} Z_J^j &:= B(0 \partial_j J)F_{\partial_j J} - \sum_{\nu \in \partial_j J} B \begin{pmatrix} 0 & j & \partial_j \partial_\nu J \\ 0 & \nu & \partial_j \partial_\nu J \end{pmatrix} F_{\partial_\nu J} + B \begin{pmatrix} 0 & \star & \partial_\nu J \\ 0 & j & \partial_j J \end{pmatrix} F_J, \\ Z_{J,k}^j &:= B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & k & \partial_j J \end{pmatrix} F_{k\partial_j J} - \sum_{\nu \in \partial_j J} B \begin{pmatrix} 0 & \star & j & \partial_j \partial_\nu J \\ 0 & k & \nu & \partial_j \partial_\nu J \end{pmatrix} F_{k\partial_\nu J} \\ &\quad - B \begin{pmatrix} 0 & \star & k & \partial_j J \\ 0 & \star & j & \partial_j J \end{pmatrix} F_{kJ} \quad (k \in J^c). \end{aligned}$$

These correspond to the contiguity relations, i.e., holonomic linear difference equations relative to the exponents  $\lambda$ , for the integral  $I(\lambda) := I_{n,m}(\varpi_n, \mathfrak{z})$ :

$$I_{n,m}(W_0^{(j)}(J)\varpi_n, \mathfrak{z}) = (\lambda_\infty + n - p)I_{n,m}(Z_J^j, \mathfrak{z}) + \sum_{k \in J^c} I_{n,m}(Z_{J,k}^j, \mathfrak{z}),$$

more explicitly

$$\begin{aligned} I_{n,m}(W_0^{(j)}(j)\varpi_n, \mathfrak{z}) &= (\lambda_j - 1)B(0 \star j)I(\lambda - 2\mathbf{e}_j) \\ &= -(\lambda_\infty + \lambda_j + n - 2)I(\lambda - \mathbf{e}_j) - \sum_{\substack{k \in N_m \\ k \neq j}} \lambda_k \{I(\lambda - \mathbf{e}_k) \\ &\quad + B \begin{pmatrix} 0 & \star & k \\ 0 & \star & j \end{pmatrix} I(\lambda - \mathbf{e}_j - \mathbf{e}_k)\} \quad (J = \{j\}), \end{aligned}$$

and

$$\begin{aligned}
& I_{n,m}(W_0^{(j)}(J)\varpi_n, \mathfrak{z}) \\
&= (\lambda_j - 1) \left\{ B(0 \star J) I(\lambda - \mathbf{e}_j - \mathbf{e}_J) - \sum_{k \in \partial_j J} I(\lambda - \mathbf{e}_{\partial_k J} - \mathbf{e}_j) B \begin{pmatrix} 0 & \star & \partial_k J \\ 0 & k & \partial_k J \end{pmatrix} \right\} \\
&= (\lambda_\infty + n - p) I_{n,m}(Z_J^j, \mathfrak{z}) + \sum_{k \notin J} I_{n,m}(Z_{J,k}^j, \mathfrak{z}) \quad (j \in J),
\end{aligned}$$

where

$$\begin{aligned}
I_{n,m}(Z_J^j, \mathfrak{z}) &= B(0 \partial_j J) I(\lambda - \mathbf{e}_{\partial_j J}) - \sum_{l \in \partial_j J} B \begin{pmatrix} 0 & j & \partial_l \partial_j J \\ 0 & l & \partial_l \partial_j J \end{pmatrix} I(\lambda - \mathbf{e}_{\partial_l J}) \\
&\quad + B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{pmatrix} I(\lambda - \mathbf{e}_J), \\
I_{n,m}(Z_{J,k}^j, \mathfrak{z}) &= B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & k & \partial_j J \end{pmatrix} I(\lambda - \mathbf{e}_k - \mathbf{e}_{\partial_j J}) \\
&\quad - \sum_{l \in \partial_j J} B \begin{pmatrix} 0 & \star & j & \partial_l \partial_j J \\ 0 & k & l & \partial_l \partial_j J \end{pmatrix} I(\lambda - \mathbf{e}_k - \mathbf{e}_{\partial_l J}) \\
&\quad - B \begin{pmatrix} 0 & \star & k & \partial_j J \\ 0 & \star & j & \partial_j J \end{pmatrix} I(\lambda - \mathbf{e}_k - \mathbf{e}_J) \quad (n+1 \geq |J| \geq 2).
\end{aligned}$$

Hence (4.1) and (4.2) can be expressed as

$$\widehat{W}_0(J)\varpi_n = a_J F_J + \sum_{j \in J} a_j F_{\partial_j J} + \sum_{k \in J^c} a_k F_{k J} + \sum_{\substack{k \in J^c \\ j \in J}} a_{k,j} F_{k, \partial_j J},$$

by using the constants  $a_J, a_j, a_k, a_{k,j}$ .

$$a_J := (\lambda_\infty + n - p) B(0 J),$$

$$a_j := (\lambda_\infty + n - p) B(0 \partial_j J) - \sum_{l \in J} (\lambda_\infty + n - p) B \begin{pmatrix} 0 & l & \partial_l \partial_j J \\ 0 & j & \partial_l \partial_j J \end{pmatrix} = 0,$$

because of the identity

$$\sum_{l \in \partial_j J} B \begin{pmatrix} 0 & l & \partial_l \partial_j J \\ 0 & j & \partial_l \partial_j J \end{pmatrix} = B(0 \partial_j J) \quad (j \in J).$$

Furthermore

$$\begin{aligned}
a_k &:= -\lambda_k \sum_{l \in J} B \begin{pmatrix} 0 & \star & k & \partial_l J \\ 0 & \star & l & \partial_l J \end{pmatrix}, \\
a_{k,j} &:= \lambda_k \left( B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & k & \partial_j J \end{pmatrix} - \sum_{l \in \partial_j J} B \begin{pmatrix} 0 & \star & l & \partial_l \partial_j J \\ 0 & k & j & \partial_l \partial_j J \end{pmatrix} \right) \\
&= \lambda_k B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{pmatrix},
\end{aligned}$$

because of the identity

$$\sum_{l \in \partial_j J} B \begin{pmatrix} 0 & \star & l & \partial_l \partial_j J \\ 0 & k & j & \partial_l \partial_j J \end{pmatrix} = B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & k & \partial_j J \end{pmatrix} - B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{pmatrix}.$$

The proof is now complete.  $\square$

By applying the operation “hat”:  $F_J \rightarrow \widehat{F}_J$  to the identities (4.3) and (4.4) we see that the following recurrence formula for  $\widehat{F}_J$  with respect to the shift  $\lambda \rightarrow \lambda - \mathbf{e}_j$  ( $1 \leq j \leq m$ ) holds true.

**Proposition 4.3.** *Fix a subset  $J$  of  $N_m$  with  $1 \leq |J| \leq n+1$ . Then we have*

$$B(0 \star J) \widehat{F}_J \sim V_J + \sum_{j \in J} B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{pmatrix} \widehat{F}_{\partial_j J}, \quad (4.5)$$

where

$$V_J := V_J^0 + V_J^1 = \widehat{W}_0(J) \varpi_n + \sum_{j \in J} (\lambda_j - 1) B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{pmatrix} F_J$$

such that

$$\begin{aligned} V_J^0 &:= \left( (\lambda_\infty + n - p - 1) B(0J) + \sum_{j \in J} \lambda_j B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{pmatrix} \right) F_J \\ &\quad + \sum_{k \in J^c} \lambda_k \sum_{j \in J} B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{pmatrix} F_{k \partial_j J}, \\ V_J^1 &:= - \sum_{k \in J^c} \lambda_k \left( B(0 \star J) - B \begin{pmatrix} 0 & k & J \\ 0 & \star & J \end{pmatrix} \right) F_{kJ}. \end{aligned}$$

According to the weight filtration, (4.5) can be rewritten as

$$B(0 \star J) \widehat{F}_J \sim \overline{V}_J^{(-)} + \overline{V}_J^0 + \overline{V}_J^1,$$

where

$$\begin{aligned} \overline{V}_J^1 &= V_J^1, \text{ (homogeneous, weight} = -2p - 2) \\ \overline{V}_J^0 &= \sum_{j \in J} \frac{B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{pmatrix}}{B(0 \star \partial_j J)} V_{\partial_j J}^1 + V_J^0, \text{ (homogeneous, weight} = -2p) \\ \overline{V}_J^{(-)} &= \sum_{j \in J} \frac{B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{pmatrix}}{B(0 \star \partial_j J)} (\overline{V}_{\partial_j J}^{(-)} + \overline{V}_{\partial_j J}^0) = \sum_{\substack{K \subsetneq J \\ |K| \geq 1}} \beta_{K,J} \overline{V}_K^0. \text{ (weight} \geq -2p + 2) \end{aligned}$$

$\overline{V}_J^0$  can be more explicitly expressed as follows:

$$\overline{V}_J^0 = \left( (\lambda_\infty + \lambda_J + n - p - 1) B(0J) - \sum_{j \in J} \lambda_j \frac{B(0 \star J) B(0 \partial_j J)}{B(0 \star \partial_j J)} \right) F_J$$

$$+ \sum_{k \in J^c} \lambda_k \sum_{j \in J} \frac{B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{pmatrix} B \begin{pmatrix} 0 & k & \partial_j J \\ 0 & \star & \partial_j J \end{pmatrix}}{B(0 \star \partial_j J)} F_{k \partial_j J} \quad (|J| \geq 2).$$

In the case  $J = \{j\}$  ( $p = 1$ ) then

$$B(0 \star j) \widehat{F}_j = \bar{V}_j^1 + \bar{V}_j^0,$$

where

$$\begin{aligned} \bar{V}_j^1 &= - \sum_{\substack{k \in N_m \\ k \neq j}} \lambda_k B \begin{pmatrix} 0 & \star & k \\ 0 & \star & j \end{pmatrix} F_{kj}, \\ \bar{V}_j^0 &= -(\lambda_\infty + \lambda_j + n - 2) F_j - \sum_{\substack{k \in N_m \\ k \neq j}} \lambda_k F_k, \\ \bar{V}_j^{(-)} &= 0. \end{aligned}$$

In the case  $J = \{j, k\}$  ( $p = 2$ )

$$B(0 \star j k) \widehat{F}_{jk} = \bar{V}_{jk}^1 + \bar{V}_{jk}^0 + \bar{V}_{jk}^{(-)},$$

where

$$\begin{aligned} \bar{V}_{jk}^1 &= - \sum_{l \neq j, k} \lambda_l \left( B(0 \star j k) - B \begin{pmatrix} 0 & l & j & k \\ 0 & \star & j & k \end{pmatrix} \right) F_{ljk}, \\ \bar{V}_{jk}^0 &= \left( (\lambda_\infty + n - 3) B(0 j k) + \lambda_j \frac{\left\{ B \begin{pmatrix} 0 & \star & k \\ 0 & j & k \end{pmatrix} \right\}^2}{B(0 \star k)} + \lambda_k \frac{\left\{ B \begin{pmatrix} 0 & \star & j \\ 0 & k & j \end{pmatrix} \right\}^2}{B(0 \star j)} \right) F_{jk} \\ &\quad + \sum_{\substack{l \in N_m \\ l \neq j, k}} \lambda_l \left( \frac{B \begin{pmatrix} 0 & \star & j \\ 0 & k & j \end{pmatrix} B \begin{pmatrix} 0 & \star & j \\ 0 & l & j \end{pmatrix}}{B(0 \star j)} F_{lj} + \frac{B \begin{pmatrix} 0 & \star & k \\ 0 & j & k \end{pmatrix} B \begin{pmatrix} 0 & \star & k \\ 0 & l & k \end{pmatrix}}{B(0 \star k)} F_{lk} \right), \\ \bar{V}_{jk}^{(-)} &= - \frac{B \begin{pmatrix} 0 & \star & j \\ 0 & k & j \end{pmatrix}}{B(0 \star j)} \left( (\lambda_\infty + \lambda_j + n - 2) F_j + \sum_{\substack{l \in N_m \\ l \neq j}} \lambda_l F_l \right) \\ &\quad - \frac{B \begin{pmatrix} 0 & \star & k \\ 0 & j & k \end{pmatrix}}{B(0 \star k)} \left( (\lambda_\infty + \lambda_k + n - 2) F_k + \sum_{\substack{l \in N_m \\ l \neq k}} \lambda_l F_l \right). \end{aligned}$$

Fix a subset  $J \subset N_m$  with  $|J| = p$  ( $1 \leq p \leq n + 1$ ). When  $B(0 \star J)$  tends to 0 the  $n$ -cohomology  $H_\nabla^n(X, \Omega)$  degenerates to the quotient by its annihilator  $Ann_J$

$$\sigma : H_\nabla^n(X, \Omega) \longrightarrow H_\nabla^n(X, \Omega) / Ann_J.$$

where  $Ann_J$  consists of all elements in  $H_\nabla^n(X, \Omega)$  vanishing for  $B(0 \star J) = 0$ .

Due to Proposition 4.3 the following Theorem holds.

**Theorem 4.4.** Fix  $J \in N_m$  for  $1 \leq |J| \leq n+1$ . Assume that  $B(0 \star J)$  vanishes. Then  $\text{Ann}_J$  is a one dimensional subspace generated by the following element:

$$\varphi_J := \sigma \bar{V}_J^1 + \sigma \bar{V}_J^0 + \sigma \bar{V}_J^{(-)} \quad (4.6)$$

in the case  $n+1 \geq |J| \geq 2$ , where

$$\begin{aligned} \sigma \bar{V}_J^1 &= \sum_{k \in J^c} \lambda_k B \begin{pmatrix} 0 & k & J \\ 0 & \star & J \end{pmatrix} F_{kJ}, \\ \sigma \bar{V}_J^0 &= (\lambda_\infty + \lambda_J + n - p - 1) B(0 \ J) F_J \\ &\quad + \sum_{k \in J^c} \lambda_k \sum_{j \in J} \frac{B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & j & \partial_j J \end{pmatrix} B \begin{pmatrix} 0 & \star & \partial_j J \\ 0 & k & \partial_j J \end{pmatrix}}{B(0 \star \partial_j J)} F_{k \partial_j J}, \\ \sigma \bar{V}_J^{(-)} &= \sum_{\substack{K \subsetneq J \\ |K| \geq 1}} \beta_{K,J} \sigma \bar{V}_K^{(0)}. \end{aligned}$$

In the case  $J = \{j\}$  ( $|J| = 1$ ). Since  $B(0 \star j) = 0$ ,  $\varphi_j$  is expressed as

$$\varphi_j = \sigma \bar{V}_j^0 + \sigma \bar{V}_j^1, \quad (4.7)$$

where

$$\begin{aligned} \sigma \bar{V}_j^0 &= -(\lambda_\infty + \lambda_j + n - 2) F_j - \sum_{\substack{k \in N_m \\ k \neq j}} \lambda_k F_k, \\ \sigma \bar{V}_j^1 &= - \sum_{\substack{k \in N_m \\ k \neq j}} \lambda_k B \begin{pmatrix} 0 & \star & k \\ 0 & \star & j \end{pmatrix} F_{kj} = \sum_{\substack{k \in N_m \\ k \neq j}} \lambda_k B \begin{pmatrix} 0 & \star & j \\ 0 & k & j \end{pmatrix} F_{kj}, \\ \sigma \bar{V}_j^{(-)} &= 0. \end{aligned}$$

As a consequence, if  $\lambda_\infty + \lambda_J + n - p - 1 \neq 0$  ( $|J| \geq 1$ ) then  $F_J$  is cohomologous to a linear combination of the other elements of NBC basis expressed by the identity in  $H_{\nabla}^n(X, \Omega)/\text{Ann}_J$ , i.e.,

$$\varphi_J \sim 0.$$

As a special case the following contiguity relation holds:

**Corollary 4.5.** Suppose that

$$r_j = 0 \quad \text{for all } j \in N_m,$$

i.e., every hypersphere  $S_j$  ( $j \in N_m$ ) reduces to a cone such that  $\Re S_j$  is a point in  $\mathbb{R}^n$ . Then we have

$$\begin{aligned} &(\lambda_\infty + n - 2)(2\lambda_\infty + n - 2) \langle F_j, \mathfrak{z} \rangle \\ &= - \sum_{\substack{k, l \in N_m \\ k \neq l}} \lambda_k \lambda_l \rho_{kl}^2 \langle F_{kl}, \mathfrak{z} \rangle + (2\lambda_\infty + n - 2) \sum_{\substack{k \in N_m \\ k \neq j}} \lambda_k \rho_{jk}^2 \langle F_{jk}, \mathfrak{z} \rangle, \end{aligned} \quad (4.8)$$

i.e.,  $H_{\nabla}^n(X, \Omega)$  has an admissible basis

$$\{F_J \mid J \subset N_m, 2 \leq |J| \leq n+1\} \cup \{F_K \mid \{1\} \subset K \subset N_m, |K| = n+1\}.$$

**Proof.** Since  $B(0 \star j) = 2r_j^2$  vanishes for all  $j \in N_m$  by hypothesis, we have  $B\begin{pmatrix} 0 & \star & k \\ 0 & \star & j \end{pmatrix} = r_k^2 + r_j^2 - \rho_{kj}^2 = -\rho_{kj}^2$ . Then, Eq. (4.7) shows

$$(\lambda_\infty + n - 2)F_j \sim - \sum_{k \in N_m} \lambda_k F_k + \sum_{\substack{k \in N_m \\ k \neq j}} \lambda_k \rho_{jk}^2 F_{jk}, \quad (4.9)$$

so that we have

$$(\lambda_\infty + n - 2) \sum_{j \in N_m} \lambda_j F_j \sim -\lambda_\infty \sum_{k \in N_m} \lambda_k F_k + \sum_{\substack{j, k \in N_m \\ k \neq j}} \lambda_j \lambda_k \rho_{jk}^2 F_{jk}.$$

Hence we obtain

$$(2\lambda_\infty + n - 2) \sum_{j \in N_m} \lambda_j F_j \sim \sum_{1 \leq j < k \leq m} 2\rho_{jk}^2 \lambda_j \lambda_k F_{jk}. \quad (4.10)$$

Eliminating  $\sum_{j \in N_m} \lambda_j F_j$  from (4.9) and (4.10), we therefore obtain (4.8) in Corollary 4.5.  $\square$

#### 5. ASYMPTOTIC BEHAVIOR OF INTEGRAL FOR VANISHING OF $B(0 \star J)$ UNDER $|J| = n + 1$

In addition to the genericity conditions (1.2) and (1.3) we impose the following:

$$B(0 \star J)(-1)^{|J|-1} > 0 \quad (J \subset N_m, 1 \leq |J| \leq n). \quad (5.1)$$

For each set of indices  $J = \{j_1, \dots, j_{n+1}\} \subset N_m$  satisfying  $|J| = n + 1$  and a system of  $n + 1$  signs  $\varepsilon_J := \{\varepsilon_{j_1}, \dots, \varepsilon_{j_{n+1}}\}$ , where  $\varepsilon_j \in \{-1, 1\}$  for  $j \in J$ , let  $D_J\langle\varepsilon_J\rangle$  be closed domain specified by

$$D_J\langle\varepsilon_J\rangle = D_J\langle\varepsilon_{j_1}, \dots, \varepsilon_{j_{n+1}}\rangle := \bigcap_{\nu=1}^{n+1} \{x \in \mathbb{R}^n \mid \varepsilon_{j_\nu} f_{j_\nu}(x) \geq 0\}.$$

If  $B(0 \star J)$  tends to 0, the domain  $D_J\langle\varepsilon_J\rangle$  degenerates to the one point. Accordingly, our aim in this section is to discuss the explicit asymptotic behavior of the integral  $\int_{D_J\langle\varepsilon_J\rangle} \Phi(x) \varpi_n$  as  $B(0 \star J) \rightarrow 0$ .

Hereafter, for simplicity we consider the case  $J = N_{n+1}$  only. However the remaining general cases of  $|J| = n + 1$  can be straightforwardly obtained from it. By the assumption (1.3) we have

$$B(0 \star N_{n+1})(-1)^n \neq 0.$$

Further we assume that  $B(0 \star N_{n+1})$  is approximately equal to 0. We use the following symbol for this.

$$\tau := B(0 \star N_{n+1}) \asymp 0.$$

In this section we assume that the points  $O_1, \dots, O_{n+1} \in \mathbb{R}^n$  make an  $n$ -simplex, so that without loss of generality we may assume the following:

$$\alpha_{j\nu} = 0 \quad (1 \leq j \leq n + 1, n - j + 1 < \nu \leq n) \quad \text{and} \quad \alpha_{j, n-j+1} > 0 \quad (1 \leq j \leq n), \quad (5.2)$$

i.e.,

$$\begin{aligned}
O_1 &= -\alpha_1 = -(\alpha_{11}, \alpha_{12}, \dots, \alpha_{1,n-2}, \alpha_{1,n-1}, \alpha_{1n}), & \alpha_{1n} &> 0, \\
O_2 &= -\alpha_2 = -(\alpha_{21}, \alpha_{22}, \dots, \alpha_{2,n-2}, \alpha_{2,n-1}, 0), & \alpha_{2,n-1} &> 0, \\
O_3 &= -\alpha_3 = -(\alpha_{31}, \alpha_{32}, \dots, \alpha_{3,n-2}, 0, 0), & \alpha_{3,n-2} &> 0, \\
&\vdots & & \vdots \\
O_n &= -\alpha_n = -(\alpha_{n1}, 0, \dots, 0, 0, 0), & \alpha_{n1} &> 0, \\
O_{n+1} &= -\alpha_{n+1} = (0, 0, \dots, 0, 0, 0).
\end{aligned}$$

Using the Cayley–Menger determinants, the latter assumption of (5.2) is rewritten as

$$\prod_{k=1}^{n-j+1} \alpha_{n+1-k,k} = \sqrt{\frac{(-1)^{n-j} B(0 \ j \ j+1 \ \dots \ n+1)}{2^{n-j+1}}} > 0 \quad (5.3)$$

for  $j = 1, 2, \dots, n$ . In particular, we have  $\prod_{j=1}^n \alpha_{j,n+1-j} = \sqrt{(-1)^{n+1} \frac{B(0 \ N_{n+1})}{2^n}}$ . We also have

$$\begin{aligned}
\alpha_{j,0} - \alpha_{n+1,0} &= B \begin{pmatrix} 0 & j & n+1 \\ 0 & \star & n+1 \end{pmatrix} \quad (1 \leq j \leq n), \\
\sum_{\nu=1}^{n+1-k} \alpha_{j\nu} \alpha_{k\nu} &= \frac{1}{2} B \begin{pmatrix} 0 & j & n+1 \\ 0 & k & n+1 \end{pmatrix} \quad (1 \leq k \leq n).
\end{aligned}$$

If  $\tau = B(0 \star N_{n+1})$  tends to 0, then the closed domain

$$D_{N_{n+1}} \langle \varepsilon_{N_{n+1}} \rangle = \bigcap_{j=1}^{n+1} \{x \in \mathbb{R}^n \mid \varepsilon_j f_j(x) \geq 0\}$$

shrinks to the one point set  $\{P\}$ , which satisfies

$$f_1(P) = f_2(P) = \dots = f_{n+1}(P) = 0, \quad (5.4)$$

i.e.,  $P \in \bigcap_{j=1}^{n+1} \mathfrak{R}S_j$  when  $\tau = B(0 \star N_{n+1}) = 0$ . The point  $P$  satisfying (5.4) is written in terms of the coordinates  $\xi_\nu = x_\nu(P)$  as

$$\xi_1 = -\frac{B \begin{pmatrix} 0 & \star & n+1 \\ 0 & n & n+1 \end{pmatrix}}{2\rho_{nn+1}},$$

and

$$\xi_\nu = (-1)^\nu \frac{B \begin{pmatrix} 0 & \star & n+2-\nu & \dots & n+1 \\ 0 & n+1-\nu & n+2-\nu & \dots & n+1 \end{pmatrix}}{\sqrt{-2B(0 \ n+1-\nu \ n+2-\nu \ \dots \ n+1) B(0 \ n+2-\nu \ \dots \ n+1)}}$$

for  $2 \leq \nu \leq n$ . In particular,  $\xi_n$  is also written as

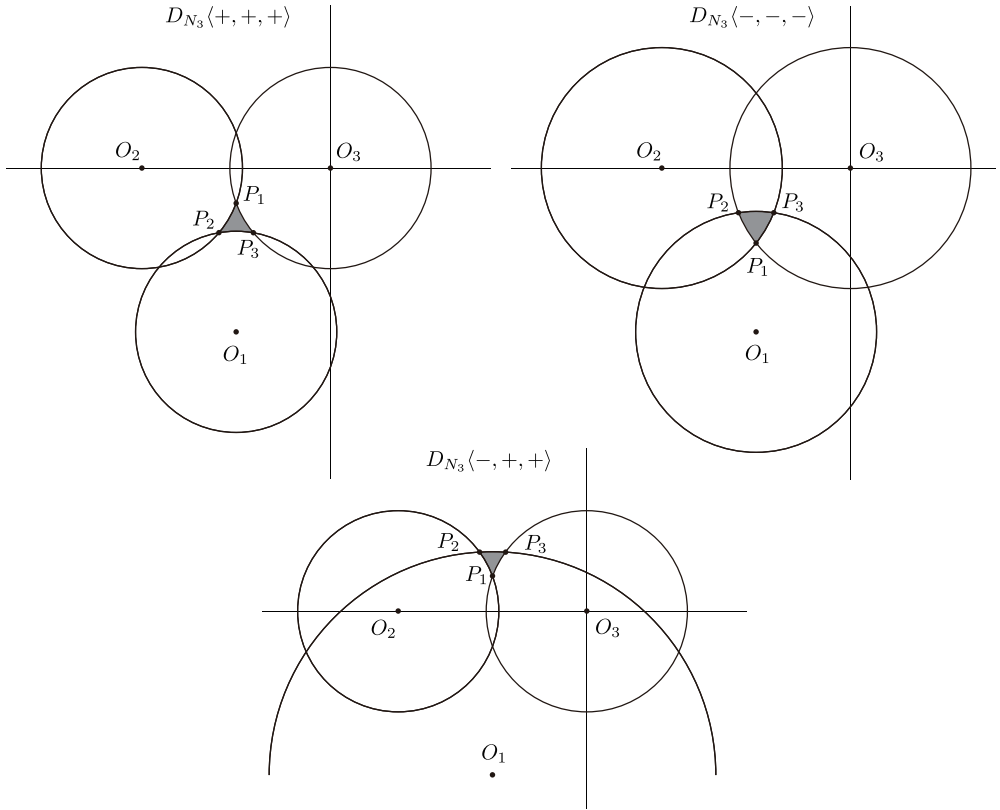
$$\xi_n = \pm \frac{\sqrt{(-1)^{n+1} B(0 \star \partial_1 N_{n+1})}}{\sqrt{(-1)^{n+1} 2 B(0 \ \partial_1 N_{n+1})}} \quad (5.5)$$

from the Jacobi identity in view of  $\tau = B(0 \star N_{n+1}) = 0$ . The point  $P$  is uniquely determined by the sequence of signs  $\varepsilon_{N_{n+1}} = \{\varepsilon_1, \dots, \varepsilon_{n+1}\}$ . In the sequel this  $P$  will be denoted by  $P \langle \varepsilon_{N_{n+1}} \rangle = P \langle \varepsilon_1, \dots, \varepsilon_{n+1} \rangle$ .

When  $\tau = B(0 \star N_{n+1})$  is sufficiently small, the domain  $D_{N_{n+1}}\langle \varepsilon_{N_{n+1}} \rangle$  is approximated by  $n$ -simplex  $\triangle P_1 P_2 \dots P_{n+1}$ , where each vertex  $P_j$  satisfies the simultaneous equations

$$f_k(x) = 0 \quad (k \in N_{n+1}, k \neq j). \quad (5.6)$$

(We remark that for each  $j \in N_{n+1}$  two distinct points are determined as solutions to the simultaneous equations (5.6). The point  $P_j$  is automatically determined as one of two solutions that converges to  $P\langle \varepsilon_{N_{n+1}} \rangle$  as  $\tau = B(0 \star N_{n+1}) \rightarrow 0$ .) Thus the domain  $D_{N_{n+1}}\langle \varepsilon_{N_{n+1}} \rangle$  is regarded as a spherically faced  $n$ -simplex with  $n+1$  vertices  $P_1, P_2, \dots, P_{n+1}$ . In this sense we also use the same symbol  $\triangle P_1 P_2 \dots P_{n+1}$  instead of  $D_{N_{n+1}}\langle \varepsilon_{N_{n+1}} \rangle$  and call it the *pseudo  $n$ -simplex with vertices  $P_1, P_2, \dots, P_{n+1}$* .



The regularization of the pseudo  $n$ -simplex  $\triangle P_1 P_2 \dots P_{n+1}$  defines in a natural way an  $n$ -dimensional twisted cycle (denoted by  $\mathfrak{z}(N_{n+1})$ ) in the space  $X = \mathbb{C}^n - \bigcup_{j=1}^m S_j$ . The integral of  $\Phi$  over the twisted  $n$ -cycle  $\mathfrak{z}(N_{n+1})$  can be defined by the pairing

$$I_{n,m}(N_{n+1}) := I\langle \varpi_n, \mathfrak{z}(N_{n+1}) \rangle = \int_{\triangle P_1 P_2 \dots P_{n+1}} \Phi(x) \varpi_n. \quad (5.7)$$

**Lemma 5.1.** For  $\tau = B(0 \star N_{n+1}) \asymp 0$  the asymptotic behavior of  $f_k(P_j)$  is given as follows:

$$\begin{aligned} & \varepsilon_j f_j(P_j) \\ &= \varepsilon_j \frac{B(0 \star N_{n+1})}{2B \begin{pmatrix} 0 & \star & \partial_j N_{n+1} \\ 0 & j & \partial_j N_{n+1} \end{pmatrix}} (1 + O(|\tau|)) = \frac{(-1)^n B(0 \star N_{n+1})}{2\sqrt{B(0 \ N_{n+1})B(0 \star \partial_j N_{n+1})}} (1 + O(|\tau|)) \\ & \hspace{20em} (j \in N_{n+1}), \quad (5.8) \end{aligned}$$

$$\begin{aligned} f_k(P_j) &= \frac{B(0 \star k N_{n+1})}{B \begin{pmatrix} 0 & \star & N_{n+1} \\ 0 & k & N_{n+1} \end{pmatrix}} + O(|\tau|) = -\frac{B \begin{pmatrix} 0 & k & N_{n+1} \\ 0 & \star & N_{n+1} \end{pmatrix}}{B(0 \ N_{n+1})} + O(|\tau|) \\ & \hspace{20em} (k \notin N_{n+1}). \quad (5.9) \end{aligned}$$

There exist the unique  $\varepsilon_k$  ( $k \in N_m$ ) such that

$$\varepsilon_j f_j(P_j) > 0 \quad (j \in N_{n+1}) \quad \text{and} \quad \varepsilon_k f_k(P_j) > 0 \quad (k \notin N_{n+1}),$$

and that  $\varepsilon_k f_k(x)$  are all positive inside  $\triangle P_1 P_2 \dots P_{n+1}$ .

**Remark.**  $\varepsilon_j$  ( $j \in N_{n+1}$ ) coincides with the sign of  $\frac{B(0 \star N_{n+1})}{2B \begin{pmatrix} 0 & \star & \partial_j N_{n+1} \\ 0 & j & \partial_j N_{n+1} \end{pmatrix}}$ .

**Proof.** We may assume that  $j = 1$  without loss of generality. Owing to (2.1) in Lemma 2.1, we have the relation

$$B(0 \ \partial_1 N_{n+1}) f_1^2 - 2u f_1 + v = 0, \quad (5.10)$$

where  $u$  and  $v$  are functions of  $f_2, \dots, f_{n+1}$  given by

$$\begin{aligned} u &:= \sum_{j=2}^{n+1} B \begin{pmatrix} 0 & 1 & \partial_1 \partial_j N_{n+1} \\ 0 & j & \partial_1 \partial_j N_{n+1} \end{pmatrix} f_j - B \begin{pmatrix} 0 & \star & \partial_1 N_{n+1} \\ 0 & 1 & \partial_1 N_{n+1} \end{pmatrix}, \\ v &:= -2 \sum_{2 \leq j < k \leq n+1} B \begin{pmatrix} 0 & j & \partial_j \partial_k N_{n+1} \\ 0 & k & \partial_j \partial_k N_{n+1} \end{pmatrix} f_j f_k \\ & \quad + 2 \sum_{j=2}^{n+1} B \begin{pmatrix} 0 & 1 & j & \partial_1 \partial_j N_{n+1} \\ 0 & 1 & \star & \partial_1 \partial_j N_{n+1} \end{pmatrix} f_j - B(0 \star N_{n+1}). \end{aligned}$$

Solving equation (5.10) with respect to  $f_1$  we see that  $f_1$  can be expressed as the function of  $f_2, \dots, f_{n+1}$  given by

$$B(0 \ \partial_1 N_{n+1}) f_1 = u - u \sqrt{1 - \frac{B(0 \ \partial_1 N_{n+1}) v}{u^2}}$$

such that  $f_1$  vanishes if  $f_2 = \dots = f_{n+1} = B(0 \star N_{n+1}) = 0$ . Since we have

$$u|_{f_2=\dots=f_{n+1}=0} = -B \begin{pmatrix} 0 & \star & \partial_1 N_{n+1} \\ 0 & 1 & \partial_1 N_{n+1} \end{pmatrix}, \quad v|_{f_2=\dots=f_{n+1}=0} = -B(0 \star N_{n+1}) = -\tau,$$

we therefore obtain

$$f_1(P_1) = f_1 \Big|_{f_2=\dots=f_{n+1}=0} = \frac{u}{B(0 \ \partial_1 N_{n+1})} \left( 1 - \sqrt{1 - \frac{B(0 \ \partial_1 N_{n+1}) v}{u^2}} \right) \Big|_{f_2=\dots=f_{n+1}=0}$$

$$= \frac{1}{2} \frac{v}{u} (1 + O(|v|)) \Big|_{f_2=\dots=f_{n+1}=0} = \frac{1}{2} \frac{B(0 \star N_{n+1})}{B \begin{pmatrix} 0 & \star & \partial_1 N_{n+1} \\ 0 & 1 & \partial_1 N_{n+1} \end{pmatrix}} (1 + O(|\tau|)). \quad (5.11)$$

Since we have

$$B(0 \star N_{n+1})B(0 \star \partial_1 N_{n+1}) - \left\{ B \begin{pmatrix} 0 & \star & \partial_1 N_{n+1} \\ 0 & 1 & \partial_1 N_{n+1} \end{pmatrix} \right\}^2 = B(0 \star \partial_1 N_{n+1})B(0 \star N_{n+1})$$

from (2.8) in Lemma 2.5 for  $J = N_{n+1}$ , (5.11) coincides with (5.8) for  $j = 1$ .

On the other hand, since  $f_j(P_1) = 0$  ( $2 \leq j \leq n+1$ ), (2.7) in Lemma 2.4 shows

$$B(0 \star k \star N_{n+1}) - B \begin{pmatrix} 0 & \star & k & \partial_1 N_{n+1} \\ 0 & 1 & k & \partial_1 N_{n+1} \end{pmatrix} f_1(P_1) - B \begin{pmatrix} 0 & \star & N_{n+1} \\ 0 & k & N_{n+1} \end{pmatrix} f_k(P_1) = 0 \quad (5.12)$$

for  $k \notin N_{n+1}$ . From (5.11) we have  $f_1(P_1) = O(|\tau|)$  when  $\tau \rightarrow 0$ . Hence, (5.12) indicates

$$f_k(P_1) = \frac{B(0 \star k \star N_{n+1})}{B \begin{pmatrix} 0 & \star & N_{n+1} \\ 0 & k & N_{n+1} \end{pmatrix}} + O(|\tau|) = - \frac{B \begin{pmatrix} 0 & \star & N_{n+1} \\ 0 & k & N_{n+1} \end{pmatrix}}{B(0 \star N_{n+1})} + O(|\tau|), \quad (5.13)$$

for  $k \notin N_{n+1}$ . The second equal sign of (5.13) is due to (2.9) in Lemma 2.5 for  $J = N_{n+1}$ . Therefore (5.13) means (5.9) for  $j = 1$ .  $\square$

We now state one of our main theorems.

**Theorem 5.2.** *Under the conditions (1.2), (1.3) and (5.1) for  $n+1 \leq m$ , let  $I'_{n,m}(N_{n+1})$  be the integral specified by*

$$I'_{n,m}(N_{n+1}) := \int_{\Delta_{P_1 P_2 \dots P_{n+1}}} \prod_{j=1}^m \{\varepsilon_j f_j(x)\}^{\lambda_j} \varpi_n. \quad (5.14)$$

Then the integral  $I_{n,m}(N_{n+1})$  given by (5.7) is written as

$$I_{n,m}(N_{n+1}) = \left( \prod_{j=1}^m \varepsilon_j^{-\lambda_j} \right) I'_{n,m}(N_{n+1}).$$

If  $\tau = B(0 \star N_{n+1})$  tends to 0, the pseudo  $n$ -simplex  $\Delta_{P_1 P_2 \dots P_{n+1}}$  shrinks to the point  $P = P\langle \varepsilon_1, \dots, \varepsilon_{n+1} \rangle$ , and the following asymptotic formula holds true:

$$\begin{aligned} I'_{n,m}(N_{n+1}) &= \frac{\prod_{j=1}^{n+1} \Gamma(\lambda_j + 1)}{\Gamma(\sum_{j=1}^{n+1} \lambda_j + n + 1)} \prod_{j=1}^{n+1} \{\varepsilon_j f_j(P_j)\}^{\lambda_j} \left( \prod_{k=n+2}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \right) \\ &\quad \times \frac{(-1)^n \{B(0 \star N_{n+1})\}^n}{\sqrt{(-1)^{n+1} 2^{3n} \{B(0 \star N_{n+1})\}^n \prod_{j=1}^{n+1} B(0 \star \partial_j N_{n+1})}} (1 + O(|\tau|)) \end{aligned} \quad (5.15)$$

as  $\tau \rightarrow 0$ , where  $\varepsilon_j f_j(P_j)$  and  $f_k(P)$  ( $P = P\langle \varepsilon_1, \dots, \varepsilon_{n+1} \rangle$ ) are given by

$$\begin{aligned} \varepsilon_j f_j(P_j) &= \varepsilon_j \frac{B(0 \star N_{n+1})}{2B \begin{pmatrix} 0 & \star & \partial_j N_{n+1} \\ 0 & j & \partial_j N_{n+1} \end{pmatrix}} (1 + O(|\tau|)) \\ &= \frac{(-1)^n B(0 \star N_{n+1})}{2\sqrt{B(0 \star N_{n+1})B(0 \star \partial_j N_{n+1})}} (1 + O(|\tau|)) > 0 \quad (1 \leq j \leq n+1) \end{aligned}$$

and

$$f_k(P) = -\frac{B\begin{pmatrix} 0 & k & N_{n+1} \\ 0 & \star & N_{n+1} \end{pmatrix}}{B(0 \ N_{n+1})} = \frac{B(0 \star k \ N_{n+1})}{B\begin{pmatrix} 0 & k & N_{n+1} \\ 0 & \star & N_{n+1} \end{pmatrix}} \neq 0 \quad \text{such that } \varepsilon_k f_k(P) > 0 \quad (n+2 \leq k \leq m).$$

Before proving Theorem 5.2, we will show two lemmas. From now on we assume that  $|\tau|$  is sufficiently small. i.e.,  $\tau \asymp 0$ . Then  $f_1$  can be expanded as a power series in  $f_2, f_3, \dots, f_{n+1}$  for small  $|f_j|$  ( $2 \leq j \leq n+1$ ), i.e.,

$$\varepsilon_1 f_1 = c_0 - \sum_{j=2}^{n+1} c_j \varepsilon_j f_j + \text{higher order} \quad (5.16)$$

such that  $c_j$  are all positive. For the coefficients  $c_j$  we have the following.

**Lemma 5.3.** *When  $\tau = B(0 \star N_{n+1}) \rightarrow 0$ ,  $c_0$  converges to 0 and every  $c_j$  ( $2 \leq j \leq n+1$ ) converges to a non-zero definite value. More precisely*

$$\begin{aligned} c_0 &= \varepsilon_1 f_1(P_1) = \varepsilon_1 \frac{B(0 \star N_{n+1})}{2B\begin{pmatrix} 0 & 1 & \partial_1 N_{n+1} \\ 0 & \star & \partial_1 N_{n+1} \end{pmatrix}} (1 + O(|\tau|)) \\ &= \frac{(-1)^n B(0 \star N_{n+1})}{2\sqrt{B(0 \ N_{n+1}) B(0 \star \partial_1 N_{n+1})}} (1 + O(|\tau|)), \end{aligned} \quad (5.17)$$

$$\begin{aligned} c_j &= \frac{\varepsilon_1 f_1(P_1)}{\varepsilon_j f_j(P_j)} = (-1)^{j-1} \frac{B\begin{pmatrix} 0 & \star & \partial_j N_{n+1} \\ 0 & \star & \partial_1 N_{n+1} \end{pmatrix}}{B(0 \star \partial_1 \partial_j N_{n+1})} (1 + O(|\tau|)) \\ &= \sqrt{\frac{B(0 \star \partial_j N_{n+1})}{B(0 \star \partial_1 N_{n+1})}} (1 + O(|\tau|)) \quad (2 \leq j \leq n+1). \end{aligned} \quad (5.18)$$

**Proof.** (5.17) and (5.18) are immediately obtained from (5.16) applying the explicit form (5.8) of  $\varepsilon_j f_j(P_j)$  ( $j \in N_{n+1}$ ) in Lemma 5.1.  $\square$

Take the new coordinates

$$t_j = \varepsilon_{j+1} \frac{c_{j+1}}{c_0} f_{j+1} > 0 \quad (1 \leq j \leq n) \quad (5.19)$$

in a neighborhood of  $P_1$ . Then  $\varepsilon_1 f_1$  can be rewritten as

$$\varepsilon_1 f_1 = c_0 \left( 1 - t_1 - \dots - t_n + c_0 g(t_1, \dots, t_n) \right), \quad (5.20)$$

where  $g$  is a power series in  $t_1, t_2, \dots, t_n$  of higher degree and their coefficients are all bounded for  $\tau \rightarrow 0$ , i.e.,

$$\varepsilon_1 f_1 = c_0 \left( 1 - t_1 - \dots - t_n + O(|\tau|) \right).$$

**Lemma 5.4.** *Assume that  $\tau = B(0 \star N_{n+1}) = 0$ . Then*

$$\left. \frac{\partial(f_2, \dots, f_{n+1})}{\partial(x_1, \dots, x_n)} \right|_P = \pm \sqrt{(-1)^{n+1} 2^n B(0 \star \partial_1 N_{n+1})}, \quad (5.21)$$

where  $P = P(\varepsilon_1, \dots, \varepsilon_{n+1})$ .

**Proof.** By definition we have

$$\begin{aligned}
\frac{\partial(f_2, \dots, f_{n+1})}{\partial(x_1, \dots, x_n)} \Big|_P &= \det \left( \frac{\partial f_{j+1}}{\partial x_k} \right)_{j,k=1}^n \Big|_P = \det \left( 2(x_k + \alpha_{j+1,k}) \right)_{j,k=1}^n \Big|_P \\
&= 2^n \det(x + \alpha_2, \dots, x + \alpha_{n+1}) \Big|_P = 2^n \det(\alpha_2, \dots, \alpha_n, x) \Big|_P \quad (\alpha_{n+1} = 0 \text{ by (5.2)}) \\
&= 2^n (-1)^{\frac{(n-1)(n-2)}{2}} x_n(P) \prod_{j=2}^n \alpha_{j,n+1-j}. \quad (\text{by assumption (5.2)})
\end{aligned}$$

Since we have  $\prod_{j=2}^n \alpha_{j,n+1-j} = \sqrt{(-1)^n B(0 \star \partial_1 N_{n+1}) / 2^{n-1}}$  by (5.3) and the explicit form of  $x_n(P) = \xi_n$  is given by (5.5), we obtain

$$\frac{\partial(f_2, \dots, f_{n+1})}{\partial(x_1, \dots, x_n)} \Big|_P = \pm 2^n \frac{\sqrt{(-1)^{n+1} B(0 \star \partial_1 N_{n+1})}}{\sqrt{(-1)^n 2 B(0 \star \partial_1 N_{n+1})}} \sqrt{\frac{(-1)^n B(0 \star \partial_1 N_{n+1})}{2^{n-1}}},$$

which coincides with (5.21).  $\square$

**Proof of Theorem 5.2.** According to Lemmas 5.3 and 5.4, we have

$$\begin{aligned}
\frac{\partial(x_1, x_2, \dots, x_n)}{\partial(t_1, t_2, \dots, t_n)} \Big|_P &= \frac{\partial(\varepsilon_2 f_2, \dots, \varepsilon_{n+1} f_{n+1})}{\partial(t_1, t_2, \dots, t_n)} \Big|_P / \frac{\partial(\varepsilon_2 f_2, \dots, \varepsilon_{n+1} f_{n+1})}{\partial(x_1, x_2, \dots, x_n)} \Big|_P \\
&= \frac{c_0}{c_2} \frac{c_0}{c_3} \dots \frac{c_0}{c_{n+1}} \Big|_P / \frac{\partial(\varepsilon_2 f_2, \dots, \varepsilon_{n+1} f_{n+1})}{\partial(x_1, x_2, \dots, x_n)} \Big|_P \\
&= \left( \prod_{j=2}^{n+1} \frac{(-1)^n B(0 \star N_{n+1})}{2 \sqrt{B(0 \star N_{n+1}) B(0 \star \partial_j N_{n+1})}} \right) \frac{1}{\sqrt{(-1)^{n+1} 2^n B(0 \star \partial_1 N_{n+1})}} \\
&= \frac{(-1)^n \{B(0 \star N_{n+1})\}^n}{\sqrt{(-1)^{n+1} 2^{3n} \{B(0 \star N_{n+1})\}^n \prod_{j=1}^{n+1} B(0 \star \partial_j N_{n+1})}}. \tag{5.22}
\end{aligned}$$

Hence by change of integration variables from  $x = (x_1, x_2, \dots, x_n)$  to  $t = (t_1, t_2, \dots, t_n)$  satisfying (5.19) and (5.20), the integral (5.14) is written as

$$\begin{aligned}
I'_{n,m}(N_{n+1}) &= \int_{\substack{t_j \geq 0 \ (1 \leq j \leq n), \\ 1 - \sum_{k=1}^n t_k + c_0 g(t) \geq 0}} \left( 1 - \sum_{k=1}^n t_k + c_0 g(t) \right)^{\lambda_1} \prod_{j=1}^n t_j^{\lambda_{j+1}} \\
&\quad \times \prod_{j=1}^n \left( \frac{c_0}{c_{j+1}} \right)^{\lambda_{j+1}} \prod_{k=n+2}^m \{\varepsilon_k f_k(x)\}^{\lambda_k} \left| \frac{\partial(x_1, \dots, x_n)}{\partial(t_1, \dots, t_n)} \right| dt_1 \wedge \dots \wedge dt_n.
\end{aligned}$$

If  $\tau = B(0 \star N_{n+1}) \rightarrow 0$ , then  $c_0$  tends to 0. In a neighborhood of the  $n$ -simplex  $\Delta_0$  defined as

$$\Delta_0 : \quad t_j \geq 0 \ (1 \leq j \leq n) \quad \text{and} \quad 1 - \sum_{j=1}^n t_j \geq 0,$$

the locus  $S_1$  defined by  $1 - \sum_{j=1}^n t_j + c_0 g(t) = 0$  tends to the hyperplane  $H_1 : 1 - \sum_{j=1}^n t_j = 0$ . Hence there exists a twisted  $n$ -cycle  $\mathfrak{z}_{N_{n+1}}$  detoured around  $S_1$  and  $H_1$  invariant under the isotopic variation  $\tau = B(0 \star N_{n+1}) \rightarrow 0$ . It regularizes the  $n$ -simplex  $\Delta P_1 \dots P_n$ , including “finite part of divergent integral” for  $\Re \lambda_j \leq -1$  (see [2, Section 3.2] for *regularization of integrals*):

$$\begin{aligned}
I'_{n,m}(N_{n+1}) &= \text{f.p.} I'_{n,m}(N_{n+1}) \\
&= \int_{\mathfrak{z}_{N_{n+1}}} \left(1 - \sum_{j=1}^n t_j\right)^{\lambda_1} \left(\frac{1 - \sum_{k=1}^n t_k + c_0 g(t)}{1 - \sum_{j=1}^n t_j}\right)^{\lambda_1} \prod_{j=1}^n t_j^{\lambda_{j+1}} \\
&\quad \times \prod_{j=1}^n \left(\frac{c_0}{c_{j+1}}\right)^{\lambda_{j+1}} \prod_{k=n+2}^m \{\varepsilon_k f_k(x)\}^{\lambda_k} \left| \frac{\partial(x_1, \dots, x_n)}{\partial(t_1, \dots, t_n)} \right| dt_1 \wedge \dots \wedge dt_n. \quad (5.23)
\end{aligned}$$

Since on  $\mathfrak{z}_{N_{n+1}}$  there exists a positive constant  $C$  independent of  $\tau$  such that  $|1 - \sum_{j=1}^n t_j| \geq C$ , one can estimate (5.23) when  $\tau = B(0 \star N_{n+1}) \asymp 0$  as follows:

$$\begin{aligned}
&\text{f.p.} I'_{n,m}(N_{n+1}) \\
&\equiv \int_{\mathfrak{z}_{N_{n+1}}} \left(1 - \sum_{j=1}^n t_j\right)^{\lambda_1} \prod_{j=1}^n t_j^{\lambda_{j+1}} dt_1 \wedge \dots \wedge dt_n \\
&\quad \times c_0^{\lambda_1} \prod_{j=1}^n \left(\frac{c_0}{c_{j+1}}\right)^{\lambda_{j+1}} \left( \prod_{k=n+2}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \right) \left| \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(t_1, t_2, \dots, t_n)} \right|_{t_1=\dots=t_n=0} \\
&\quad \times (1 + O(|\tau|)) \\
&\equiv \int_{\Delta_0} \left(1 - \sum_{j=1}^n t_j\right)^{\lambda_1} \prod_{j=1}^n t_j^{\lambda_{j+1}} dt_1 \wedge \dots \wedge dt_n \\
&\quad \times c_0^{\lambda_1} \prod_{j=1}^n \left(\frac{c_0}{c_{j+1}}\right)^{\lambda_{j+1}} \left( \prod_{k=n+2}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \right) \left| \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(t_1, t_2, \dots, t_n)} \right|_P \\
&\quad \times (1 + O(|\tau|)) \\
&= \frac{\prod_{j=1}^{n+1} \Gamma(\lambda_j + 1)}{\Gamma(\sum_{j=1}^{n+1} \lambda_j + n + 1)} c_0^{\lambda_1} \prod_{j=1}^n \left(\frac{c_0}{c_{j+1}}\right)^{\lambda_{j+1}} \left( \prod_{k=n+2}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \right) \\
&\quad \times \left| \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(t_1, t_2, \dots, t_n)} \right|_P (1 + O(|\tau|)),
\end{aligned}$$

which coincides with the right-hand side of (5.15) using (5.17), (5.18) in Lemmas 5.3 and (5.22). Theorem 5.2 has thus been proved.  $\square$

**Remark on Theorem 5.2.** Theorem 5.2 is intimately related to the monodromy formula concerning the integral over  $D_J(\varepsilon_J)$  ( $J = \{j_1, \dots, j_{n+1}\}$ ) around the point where  $\tau = B(0 \star J)$  vanishes in the space of parameters. This formula has been presented in [1]. Here we explain it roughly.

Without loss of generality we may assume  $J = N_{n+1}$  as well. Consider a special isotopic deformation  $V(\theta)$  of  $\bigcup_{1 \leq k \leq m} S_k$  depending on the parameter  $\theta$  ( $0 \leq \theta \leq 2\pi$ ) in the following way. The center  $\tilde{O}_1 = -\alpha_1$  of  $S_1$  only varies, while it leaves fixed the radius  $r_1$  of  $S_1$  and all the other hyperspheres  $S_k$  ( $2 \leq k \leq m$ ) such that  $\tau = B(0 \star N_{n+1})$  turns counter-clockwise around the origin in a neighborhood of the origin in the complex plane  $\mathbb{C}$ :

$$V(\theta) : \begin{cases} B(0 \star N_{n+1}) & \rightarrow \tilde{B}(0 \star N_{n+1}) = B(0 \star N_{n+1}) e^{\sqrt{-1}\theta} & (0 \leq \theta \leq \pi) \\ -B(0 \star N_{n+1}) & \rightarrow \tilde{B}(0 \star N_{n+1}) = B(0 \star N_{n+1}) e^{\sqrt{-1}\theta} & (\pi \leq \theta \leq 2\pi) \end{cases},$$

where the symbol  $\tilde{f}_1$  means that  $f_1$  is replaced with the transformed  $\tilde{f}_1$  by  $V(\theta)$ . We put  $X_\nu = \varepsilon_{\nu+1} f_{\nu+1}$  ( $0 \leq \nu \leq n$ ) and denote the domains  $D_{N_{n+1}}(\varepsilon'_0, \varepsilon'_1, \dots, \varepsilon'_n)$  by  $\Delta X_0^{\varepsilon'_0} X_1^{\varepsilon'_1} \dots X_n^{\varepsilon'_n}$ . According to F. Pham's generalized Picard–Lefschetz formula (see [1, Propositions 6 and 7]), we have

$$V(\pi) : \begin{cases} \Delta X_0^- X_1^- \dots X_n^- & \rightarrow (-1)^n e^{2\pi\sqrt{-1}\lambda_1} \Delta \tilde{X}_0^+ X_1^+ \dots X_n^+ \\ \Delta X_0^{\varepsilon'_0} X_1^{\varepsilon'_1} \dots X_n^{\varepsilon'_n} & \rightarrow \Delta X_0^{\varepsilon'_0} X_1^{\varepsilon'_1} \dots X_n^{\varepsilon'_n} \\ & - \varepsilon'_0 \varepsilon'_1 \dots \varepsilon'_n e^{\pi\sqrt{-1}(1-\varepsilon'_0)\lambda_1} \Delta \tilde{X}_0^+ X_1^+ \dots X_n^+ \end{cases},$$

where the case  $\varepsilon'_\nu$  ( $0 \leq \nu \leq n$ ) being all negative is excluded. In the same way, we have

$$V(2\pi) : \begin{cases} \Delta \tilde{X}_0^+ X_1^+ \dots X_n^+ & \rightarrow (-1)^n e^{2\pi\sqrt{-1}(\lambda_2 + \dots + \lambda_{n+1})} \Delta X_0^- X_1^- \dots X_n^- \\ \Delta \tilde{X}_0^{\varepsilon'_0} X_1^{\varepsilon'_1} \dots X_n^{\varepsilon'_n} & \rightarrow \Delta X_0^{\varepsilon'_0} X_1^{\varepsilon'_1} \dots X_n^{\varepsilon'_n} \\ & + (-1)^n \varepsilon'_0 \varepsilon'_1 \dots \varepsilon'_n e^{\pi\sqrt{-1} \sum_{\nu=2}^n (\varepsilon'_\nu + 1)\lambda_{\nu+1}} \Delta X_0^- X_1^- \dots X_n^- \end{cases},$$

where the case  $\varepsilon'_\nu$  ( $0 \leq \nu \leq n$ ) being all positive is excluded.

## 6. ASYMPTOTIC BEHAVIOR OF INTEGRAL FOR VANISHING OF $B(0 \star J)$ UNDER $J \subset N_{n+1}$ AND $1 \leq |J| \leq n$

In this section we consider the case when  $J$  is a proper subset of  $N_{n+1}$ . Without loss of generality we may assume that  $J := N_p = \{1, 2, \dots, p\}$ , where  $p$  is fixed as  $1 \leq p \leq n$ .

In addition to the genericity conditions (1.2) and (1.3) we impose the following:

$$B(0 \star K)(-1)^{|K|-1} > 0 \quad (K \subset N_m, 1 \leq |K| \leq n+1)$$

unless  $N_p \subsetneq K$ . Further we assume that  $B(0 \star N_p)$  is approximately equal to 0:

$$\tau := B(0 \star N_p) \simeq 0.$$

In this section we may assume the following for  $O_1, \dots, O_p \in \mathbb{R}^n$ :

$$\alpha_{j\nu} = 0 \quad (1 \leq j \leq p, p-j+1 \leq \nu \leq n) \quad \text{and} \quad \alpha_{j,p-j} > 0 \quad (1 \leq j \leq p-1), \quad (6.1)$$

i.e.,

$$\begin{aligned} O_1 &= -\alpha_1 = -(\alpha_{11}, \alpha_{12}, \dots, \alpha_{1,p-3}, \alpha_{1,p-2}, \alpha_{1,p-1}, 0, \dots, 0), & \alpha_{1,p-1} &> 0, \\ O_2 &= -\alpha_2 = -(\alpha_{21}, \alpha_{22}, \dots, \alpha_{2,p-3}, \alpha_{2,p-2}, 0, 0, \dots, 0), & \alpha_{2,p-2} &> 0, \\ O_3 &= -\alpha_3 = -(\alpha_{31}, \alpha_{32}, \dots, \alpha_{3,p-3}, 0, 0, 0, \dots, 0), & \alpha_{3,p-3} &> 0, \\ & \vdots & & \vdots \\ O_{p-1} &= -\alpha_{p-1} = -(\alpha_{p-1,1}, 0, \dots, 0, 0, 0, \dots, 0), & \alpha_{p-1,1} &> 0, \\ O_p &= -\alpha_p = (0, 0, \dots, 0, 0, 0, 0, \dots, 0), \end{aligned}$$

so that

$$f_j(x) = |x|^2 + \sum_{\nu=1}^{p-j} 2\alpha_{j\nu} x_\nu + \alpha_{j0} \quad (1 \leq j \leq p). \quad (6.2)$$

Using the Cayley–Menger determinants, the latter assumption of (6.1) is rewritten as

$$\prod_{k=1}^{p-j} \alpha_{p-k,k} = \sqrt{\frac{(-1)^{p-j-1} B(0 \ j \ j+1 \ \cdots \ p)}{2^{p-j}}} > 0 \quad (6.3)$$

for  $j = 1, 2, \dots, p-1$ . In particular, we have  $\prod_{k=1}^{p-1} \alpha_{k,p-k} = \sqrt{(-1)^p \frac{B(0 \ N_p)}{2^{p-1}}}$ . We also

have

$$\alpha_{j,0} - \alpha_{p,0} = B \begin{pmatrix} 0 & j & p \\ 0 & \star & p \end{pmatrix} \quad (1 \leq j \leq p-1),$$

$$\sum_{\nu=1}^{p-k} \alpha_{j\nu} \alpha_{k\nu} = \frac{1}{2} B \begin{pmatrix} 0 & j & p \\ 0 & k & p \end{pmatrix} \quad (1 \leq k \leq p-1).$$

For each set of indices  $K = \{k_1, \dots, k_p\} \subset N_m$  satisfying  $|K| = p$  and a system of  $p$  signs  $\varepsilon_K := \{\varepsilon_{k_1}, \dots, \varepsilon_{k_p}\}$ , where  $\varepsilon_k \in \{-1, 1\}$  for  $k \in K$ , let  $D_K \langle \varepsilon_K \rangle$  be closed domain specified by

$$D_K \langle \varepsilon_K \rangle = D_K \langle \varepsilon_{k_1}, \dots, \varepsilon_{k_p} \rangle := \bigcap_{\nu=1}^p \{x \in \mathbb{R}^n \mid \varepsilon_{k_\nu} f_{k_\nu}(x) \geq 0\}. \quad (6.4)$$

There exist uniquely the signs  $\varepsilon_{N_p} = \{\varepsilon_1, \dots, \varepsilon_p\}$  such that  $D_{N_p} \langle \varepsilon_{N_p} \rangle$  is non-empty. The intersection  $\bigcap_{1 \leq j \leq p} \mathfrak{R}S_j$  is an  $n-p$  dimensional sphere with the radius

$$r_{N_p} = \sqrt{\frac{-B(0 \star N_p)}{2B(0 \ N_p)}} \quad (6.5)$$

and with the center  $M(\xi_1, \dots, \xi_{p-1}, 0, \dots, 0)$ , which is given by

$$\xi_1 = -\frac{B \begin{pmatrix} 0 & \star & p \\ 0 & p-1 & p \end{pmatrix}}{2\rho_{p-1p}} \quad (6.6)$$

$$\xi_\nu = \frac{(-1)^\nu B \begin{pmatrix} 0 & \star & p-\nu+1 & \cdots & p \\ 0 & p-\nu & p-\nu+1 & \cdots & p \end{pmatrix}}{\sqrt{-2B(0 \ p-\nu \ p-\nu+1 \ \cdots \ p) B(0 \ p-\nu+1 \ \cdots \ p)}} \quad (2 \leq \nu \leq p-1). \quad (6.7)$$

Let  $L_p$  be the  $p-1$  dimensional subspace specified by

$$L_p : x_p = x_{p+1} = \cdots = x_n = 0.$$

Then we have the following:

**Lemma 6.1.** *If  $\tau = B(0 \star N_p)$  tends to 0, then  $r_{N_p}$  tends to 0 and  $D_{N_p} \langle \varepsilon_{N_p} \rangle$  shrinks to the unique point  $\{P\}$ , where  $P = (\xi_1, \dots, \xi_{p-1}, 0, \dots, 0)$  in  $L_p$  satisfies*

$$f_j(P) = 0 \quad (1 \leq j \leq p) \quad (6.8)$$

under  $\tau = B(0 \star N_p) = 0$ , and  $\xi_1, \dots, \xi_{p-1}$  are explicitly given by (6.6) and (6.7).

When  $\tau = B(0 \star N_p)$  is sufficiently small, the cross section of  $D_{N_p} \langle \varepsilon_{N_p} \rangle$  by  $L_p$  is approximated by  $(p-1)$ -simplex  $\triangle P_1 P_2 \dots P_p$ , where each vertex  $P_j \in L_p$  satisfies the simultaneous equations

$$f_k(x) = 0 \quad (k \in N_p, k \neq j) \quad (6.9)$$

restricted on  $L_p$ . For each  $j \in N_p$  two distinct points in  $L_p$  are determined as solutions to the simultaneous equations (6.9). The point  $P_j \in L_p$  is automatically selected as one of two solutions that converges to  $P \in L_p$  given by (6.8) as  $\tau = B(0 \star N_p) \rightarrow 0$ . Thus the cross section  $D_{N_p} \langle \varepsilon_{N_p} \rangle \cap L_p$  is regarded as a nonempty spherical pseudo  $(p-1)$ -simplex with the  $p$  vertices  $P_1, P_2, \dots, P_p$ . In this sense we also use the same symbol  $\Delta P_1 P_2 \dots P_p$  instead of  $D_{N_p} \langle \varepsilon_{N_p} \rangle \cap L_p$ , so that

$$D_{N_p} \langle \varepsilon_{N_p} \rangle \cap L_p = \Delta P_1 P_2 \dots P_p,$$

which we call the pseudo  $(p-1)$ -simplex with vertices  $P_1, P_2, \dots, P_p$ .

The following lemma is similar to Lemma 5.1.

**Lemma 6.2.** *Suppose that  $\tau = B(0 \star N_p) \asymp 0$ . Then*

$$\begin{aligned} \varepsilon_j f_j(P_j) &= \varepsilon_j \frac{B(0 \star N_p)}{2B \begin{pmatrix} 0 & \star & \partial_j N_p \\ 0 & j & \partial_j N_p \end{pmatrix}} (1 + O(|\tau|)) \\ &= \frac{(-1)^{p-1} B(0 \star N_p)}{2\sqrt{B(0 N_p) B(0 \star \partial_j N_p)}} (1 + O(|\tau|)) \quad (j \in N_p), \\ f_k(P_j) &= -\frac{B \begin{pmatrix} 0 & \star & \partial_j N_p \\ 0 & k & \partial_j N_p \end{pmatrix}}{B(0 \partial_j N_p)} + O(|\tau|) \\ &= \frac{B(0 \star k N_p)}{B \begin{pmatrix} 0 & \star & N_p \\ 0 & k & N_p \end{pmatrix}} + O(|\tau|) \quad (j \in N_p \text{ and } k \notin N_p). \end{aligned} \quad (6.10)$$

**Remark.** The expression in the right-hand side of (6.10) does not depend on  $P_j$ .

**Lemma 6.3.** *Suppose that  $\tau = B(0 \star N_p) = 0$ . Then*

$$f_k(P) = \frac{B(0 \star k N_p)}{B \begin{pmatrix} 0 & k & N_p \\ 0 & \star & N_p \end{pmatrix}} = -\frac{B \begin{pmatrix} 0 & k & N_p \\ 0 & \star & N_p \end{pmatrix}}{B(0 N_p)} \neq 0 \quad (p+1 \leq k \leq m),$$

due to the Jacobi identity (2.9) for  $J = N_p$ . There exist the unique signs  $\varepsilon_k$  ( $k \in N_m$ ) such that

$$\varepsilon_j f_j(P_j) > 0 \quad (j \in N_p)$$

and

$$\varepsilon_k f_k(P) > 0, \quad \varepsilon_k f_k(P_j) > 0 \quad (j \in N_p \text{ and } k \notin N_p).$$

Then  $\varepsilon_k f_k(x)$  are all positive inside  $\Delta P_1 P_2 \dots P_p$ .

For an arbitrary  $n$  dimensional vector  $x^{(p)} := (0, \dots, 0, x_p, \dots, x_n)$  satisfying  $0 \leq |x^{(p)}| \leq r_{N_p}$ , where  $|x^{(p)}| = \sqrt{x_p^2 + \dots + x_n^2}$ , we denote by  $L_p + x^{(p)}$  the affine subspace parallel to  $L_p$  with displacement  $x^{(p)}$ . When we set

$$\mathcal{B}_{N_p} := \{x^{(p)} \mid 0 \leq |x^{(p)}| \leq r_{N_p}\},$$

for the vector  $x^{(p)} \in \mathcal{B}_{N_p}$  the cross section of  $D_{N_p}\langle \varepsilon_{N_p} \rangle$  by  $L_p + x^{(p)}$  defines the family of nonempty pseudo  $(p-1)$ -simplices  $\Delta P_1 \dots P_p(x^{(p)})$ , which are given by

$$\Delta P_1 \dots P_p(x^{(p)}) := D_{N_p}\langle \varepsilon_{N_p} \rangle \cap (L_p + x^{(p)}) = D_{N_p}\langle \varepsilon_{N_p} \rangle \cap \iota^{-1}(x^{(p)}),$$

where  $\iota$  is the natural projection defined by

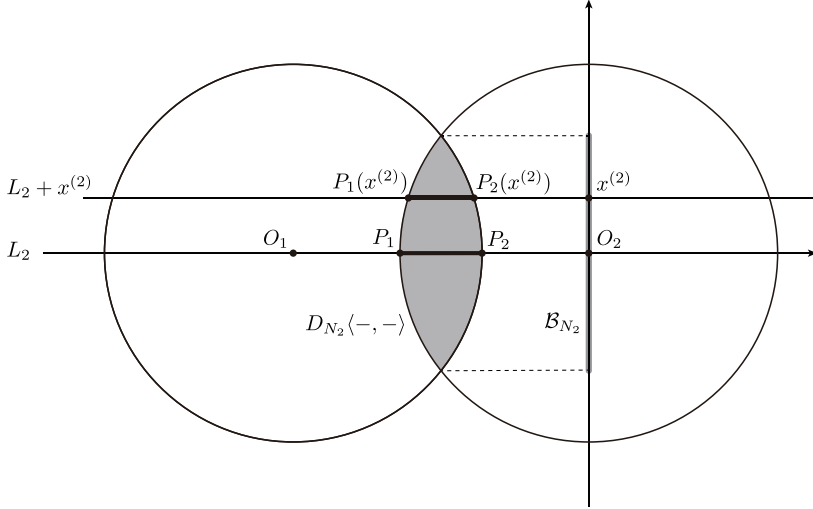
$$\iota : \mathbb{R}^n \rightarrow \mathbb{R}^{n-p+1}; \quad x = (x_1, \dots, x_n) \mapsto x^{(p)} = (x_p, \dots, x_n). \quad (6.11)$$

For  $x^{(p)} \in \mathcal{B}_{N_p}$  one sees that  $\Delta P_1 \dots P_p(x^{(p)})$  is a pseudo  $(p-1)$ -simplex with the  $p$  vertices  $P_1(x^{(p)}), \dots, P_p(x^{(p)})$ , where each vertex  $P_j(x^{(p)}) \in L_p + x^{(p)}$  satisfies the simultaneous equations (6.9) restricted on  $L_p + x^{(p)}$ . For each  $j \in N_p$  the point  $P_j(x^{(p)}) \in L_p + x^{(p)}$  is automatically determined as one of two solutions of (6.9) on  $L_p + x^{(p)}$  that continuously converges to  $P_j \in L_p$  as  $|x^{(p)}| \rightarrow 0$ .

The  $n$ -dimensional domain  $D_{N_p}\langle \varepsilon_{N_p} \rangle$  defined by (6.4) coincides with the disjoint union of the pseudo  $(p-1)$ -simplices  $\Delta P_1 \dots P_p(x^{(p)})$ , i.e.,

$$D_{N_p}\langle \varepsilon_{N_p} \rangle = \bigsqcup_{x^{(p)} \in \mathcal{B}_{N_p}} \Delta P_1 \dots P_p(x^{(p)}). \quad (6.12)$$

Since the natural projection  $\iota$  defined by (6.11) induces the fibration  $\iota : D_{N_p}\langle \varepsilon_{N_p} \rangle \rightarrow \mathcal{B}_{N_p}$  with the fibre  $\iota^{-1}(x^{(p)}) = \Delta P_1 \dots P_p(x^{(p)})$  above  $x^{(p)} \in \mathcal{B}_{N_p}$ , (6.12) indicates that the domain  $D_{N_p}\langle \varepsilon_{N_p} \rangle$  is regarded as a fibre space.



For an arbitrary  $h \geq 0$  let  $f_j(x_1, \dots, x_{p-1}; h)$  ( $1 \leq j \leq m$ ) be functions on  $L_p \simeq \mathbb{R}^{p-1}$  specified by

$$f_j(x_1, \dots, x_{p-1}; h) := f_j(x_1, \dots, x_{p-1}, 0, \dots, 0) + h^2.$$

In particular, from (6.2), for  $x^{(p)} \in \mathcal{B}_{N_p}$  the functions  $f_j(x_1, \dots, x_{p-1}; |x^{(p)}|)$  coincide with  $f_j(x)$  if  $1 \leq j \leq p$ , i.e., we have

$$f_j(x_1, \dots, x_{p-1}; |x^{(p)}|) = f_j(x_1, \dots, x_{p-1}, 0, \dots, 0) + |x^{(p)}|^2 = f_j(x) \quad (6.13)$$

for  $1 \leq j \leq p$ . We remark that the  $n$ -dimensional domain  $D_{N_p}\langle \varepsilon_{N_p} \rangle$  is also described

by the equations

$$\varepsilon_j f_j(x_1, \dots, x_{p-1}; |x^{(p)}|) \geq 0 \quad (1 \leq j \leq p).$$

$B(0 \star N_p)$  corresponding to the defining equations

$$f_j(x_1, \dots, x_{p-1}; h) = 0 \quad (1 \leq j \leq p)$$

on  $L_p$  is denoted by  $B(0 \star N_p; h)$ . This is obtained from the original  $B(0 \star N_p)$  by replacing  $r_j^2$  with  $r_j^2 - h^2$  ( $1 \leq j \leq p$ ), i.e.,

$$B(0 \star N_p; h) := B(0 \star N_p) \Big|_{\substack{r_j^2 \rightarrow r_j^2 - h^2 \\ (1 \leq j \leq p)}}.$$

For  $j \in N_p$  we also define  $B(0 \star \partial_j N_p; h) := B(0 \star \partial_j N_p) \Big|_{\substack{r_j^2 \rightarrow r_j^2 - h^2 \\ (1 \leq j \leq p)}}.$

Then by direct calculation we have the following.

**Lemma 6.4.** *One has the equalities*

$$\begin{aligned} B(0 \star N_p; h) &= B(0 \star N_p) + 2h^2 B(0 \star N_p), \\ B(0 \star \partial_j N_p; h) &= B(0 \star \partial_j N_p) + 2h^2 B(0 \star \partial_j N_p) \quad (j \in N_p). \end{aligned} \tag{6.14}$$

Here we define the pseudo  $(p-1)$ -simplex with the  $p$  vertices  $P_1(h), \dots, P_p(h)$  in  $L_p$  as

$$\Delta P_1 \dots P_p(h) := \{(x_1, \dots, x_{p-1}) \in L_p \mid \varepsilon_j f_j(x_1, \dots, x_{p-1}; h) \geq 0 \ (1 \leq j \leq p)\},$$

which is nonempty if  $0 \leq h \leq r_{N_p}$ . For  $x^{(p)} \in \mathcal{B}_{N_p}$  if  $|x^{(p)}| = h$ , then we have

$$\Delta P_1 \dots P_p(x^{(p)}) \simeq \Delta P_1 \dots P_p(h) \subset L_p.$$

$\Delta P_1 \dots P_p(h)$  and  $B(0 \star N_p; h)$  will be used later in the proof of Theorem 6.5.

We now state our main theorem of this section.

**Theorem 6.5.** *Suppose  $1 \leq p \leq n+1$ . Let*

$$I_{n,m}(N_p) := \int_{D_{N_p} \langle \varepsilon_{N_p} \rangle} \prod_{j=1}^m f_j(x)^{\lambda_j} \varpi_n, \quad I'_{n,m}(N_p) := \int_{D_{N_p} \langle \varepsilon_{N_p} \rangle} \prod_{j=1}^m \{\varepsilon_j f_j(x)\}^{\lambda_j} \varpi_n. \tag{6.15}$$

Then  $I_{n,m}(N_p) = (\prod_{j=1}^m \varepsilon_j^{-\lambda_j}) I'_{n,m}(N_p)$ . If  $\tau = B(0 \star N_p)$  tends to 0, the  $n$ -dimensional domain  $D_{N_p} \langle \varepsilon_{N_p} \rangle$  shrinks to the point  $P$  given as (6.8), and the following asymptotic formula holds true:

$$\begin{aligned} I'_{n,m}(N_p) &= \frac{\pi^{\frac{n-p+1}{2}} \prod_{j=1}^p \Gamma(\lambda_j + 1)}{\Gamma(\sum_{j=1}^p \lambda_j + \frac{n+p+1}{2})} \prod_{j=1}^p \{\varepsilon_j f_j(P_j)\}^{\lambda_j} \left( \prod_{k=p+1}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \right) \\ &\quad \times \sqrt{\frac{\{(-1)^{p-1} B(0 \star N_p)\}^{n+p-1}}{2^{n+2(p-1)} \{(-1)^p B(0 \star N_p)\}^n \prod_{j=1}^p \{(-1)^p B(0 \star \partial_j N_p)\}}} (1 + O(|\tau|)), \end{aligned}$$

as  $\tau \rightarrow 0$ , where  $\varepsilon_j f_j(P_j)$  and  $f_k(P)$  are given by

$$\begin{aligned} & \varepsilon_j f_j(P_j) \\ &= \varepsilon_j \frac{B(0 \star N_p)}{2B \begin{pmatrix} 0 & \star & \partial_j N_p \\ 0 & j & \partial_j N_p \end{pmatrix}} (1 + O(|\tau|)) = \frac{(-1)^{p-1} B(0 \star N_p)}{2\sqrt{B(0 \ N_p)B(0 \star \partial_j N_p)}} (1 + O(|\tau|)) > 0 \end{aligned} \quad (1 \leq j \leq p),$$

$$f_k(P) = \frac{B(0 \star k N_p)}{B \begin{pmatrix} 0 & k & N_p \\ 0 & \star & N_p \end{pmatrix}} = -\frac{B \begin{pmatrix} 0 & k & N_p \\ 0 & \star & N_p \end{pmatrix}}{B(0 \ N_p)} \neq 0 \quad \text{such that} \quad \varepsilon_k f_k(P) > 0 \quad (p+1 \leq k \leq m).$$

**Proof.** Since Theorem 6.5 for  $p = n + 1$  is equivalent to Theorem 5.2, which was already proven, we assume  $1 \leq p \leq n$  below. Since the coefficients of the functions  $f_j(x)$  ( $p + 1 \leq j \leq m$ ) are all independent of  $\tau = B(0 \star N_p)$ , under the genericity conditions the functions  $f_j(x)$  ( $p + 1 \leq j \leq m$ ) are continuous on the neighborhood of  $x = P$ . In particular we have

$$\prod_{k=p+1}^m \{\varepsilon_k f_k(x)\}^{\lambda_k} = \prod_{k=p+1}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \quad (\tau \rightarrow 0), \quad (6.16)$$

where  $f_k(P) = f_k(\xi_1, \dots, \xi_{p-1}, 0, \dots, 0)$  and  $\xi_1, \dots, \xi_p$  are explicitly given as (6.6)–(6.7). On the other hand, by definition (6.13) shows

$$f_j(x) = f_j(x_1, \dots, x_{p-1}; |x^{(p)}|) \quad \text{for} \quad 1 \leq j \leq p. \quad (6.17)$$

From (6.16) and (6.17) the integral (6.15) is then expressed as

$$I'_{n,m}(N_p) = \int_{D_{N_p}(\varepsilon_{N_p})} \prod_{j=1}^p \{\varepsilon_j f_j(x_1, \dots, x_{p-1}; |x^{(p)}|)\}^{\lambda_j} \left( \prod_{k=p+1}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \right) \varpi_n.$$

According to Fubini's theorem, (6.12) reads

$$I'_{n,m}(N_p) = \int_{\mathcal{B}_{N_p}} dx^{(p)} \int_{\Delta_{P_1 \dots P_p}(x^{(p)})} \varpi_{p-1} \prod_{j=1}^p \{\varepsilon_j f_j(x_1, \dots, x_{p-1}; |x^{(p)}|)\}^{\lambda_j} \left( \prod_{k=p+1}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \right),$$

where  $dx^{(p)} := dx_p \wedge \dots \wedge dx_n$  and  $\varpi_{p-1} := dx_1 \wedge \dots \wedge dx_{p-1}$ . For this expression of  $I'_{n,m}(N_p)$ , we apply the change of integration variables by  $x^{(p)} = h\eta = h(0, \dots, 0, \eta_1, \dots, \eta_{n-p+1})$ , where  $h \geq 0$  and  $|\eta| = 1$ , i.e.,

$$x_{p-1+j} = h\eta_j \quad (1 \leq j \leq n-p+1), \quad \text{where} \quad h \geq 0 \quad \text{and} \quad |\eta| = \sqrt{\eta_1^2 + \dots + \eta_{n-p+1}^2} = 1.$$

Then we have

$$dx^{(p)} = dx_p \wedge \dots \wedge dx_n = h^{n-p} dh \wedge d\omega_{n-p},$$

where  $d\omega_{n-p}$  denotes the volume form of the  $n - p$  dimensional unit sphere  $U_{n-p} := \{\eta = (\eta_1, \dots, \eta_{n-p+1}) \mid |\eta| = 1\}$ , which is explicitly given as

$$d\omega_{n-p} = \sum_{j=1}^{n-p+1} (-1)^{j-1} \eta_j d\eta_1 \wedge \dots \wedge d\eta_{j-1} \wedge d\eta_{j+1} \wedge \dots \wedge d\eta_{n-p+1}.$$

In particular, the volume of the unit sphere  $U_{n-p}$  is known to be

$$\int_{U_{n-p}} d\omega_{n-p} = \frac{2\pi^{\frac{n-p+1}{2}}}{\Gamma(\frac{n-p+1}{2})}.$$

We also have  $|x^{(p)}| = |h\eta| = h$  and  $\mathcal{B}_{N_p} = [0, r_{N_p}] \times U_{n-p}$ . Since the functions

$$f_j(x_1, \dots, x_{p-1}; |x^{(p)}|) = f_j(x_1, \dots, x_{p-1}; h) \quad (1 \leq j \leq m)$$

are all independent of  $\eta \in U_{n-p}$ , we therefore obtain

$$\begin{aligned} I'_{n,m}(N_p) &= \int_{[0, r_{N_p}]} dh h^{n-p} \int_{U_{n-p}} d\omega_{n-p} \int_{\Delta_{P_1 \dots P_p}(h)} \varpi_{p-1} \prod_{j=1}^p \{\varepsilon_j f_j(x_1, \dots, x_{p-1}; h)\}^{\lambda_j} \\ &\quad \times \left( \prod_{k=p+1}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \right) \\ &= \int_{U_{n-p}} d\omega_{n-p} \times \int_0^{r_{N_p}} dh h^{n-p} \int_{\Delta_{P_1 \dots P_p}(h)} \varpi_{p-1} \prod_{j=1}^p \{\varepsilon_j f_j(x_1, \dots, x_{p-1}; h)\}^{\lambda_j} \\ &\quad \times \left( \prod_{k=p+1}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \right) \\ &= \frac{2\pi^{\frac{n-p+1}{2}}}{\Gamma(\frac{n-p+1}{2})} \int_0^{r_{N_p}} dh h^{n-p} \int_{\Delta_{P_1 \dots P_p}(h)} \varpi_{p-1} \prod_{j=1}^p \{\varepsilon_j f_j(x_1, \dots, x_{p-1}; h)\}^{\lambda_j} \\ &\quad \times \left( \prod_{k=p+1}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \right). \quad (6.18) \end{aligned}$$

Since we have  $h^2 = \frac{B(0 \star N_p; h) - B(0 \star N_p)}{2B(0 N_p)} = \frac{B(0 \star \partial_j N_p; h) - B(0 \star \partial_j N_p)}{2B(0 \partial_j N_p)}$  due to Lemma 6.4, using  $0 \leq h^2 \leq r_{N_p}^2 = -B(0 \star N_p)/2B(0 N_p)$  we obtain

$$0 \leq -\frac{B(0 \star N_p; h)}{B(0 N_p)} \leq -\frac{B(0 \star N_p)}{B(0 N_p)}$$

and

$$0 \leq \frac{B(0 \star \partial_j N_p; h) - B(0 \star \partial_j N_p)}{B(0 \partial_j N_p)} \leq -\frac{B(0 \star N_p)}{B(0 N_p)},$$

so that

$$|B(0 \star N_p; h)| \leq |\tau| \quad \text{and} \quad |B(0 \star \partial_j N_p; h) - B(0 \star \partial_j N_p)| \leq \left| \frac{B(0 \partial_j N_p)}{B(0 N_p)} \right| |\tau|.$$

In particular,  $\tau \rightarrow 0$  implies  $B(0 \star N_p; h) \rightarrow 0$  and  $B(0 \star \partial_j N_p; h) \rightarrow B(0 \star \partial_j N_p)$ . Then, we can apply Theorem 5.2 to the integral over the domain  $\Delta_{P_1 \dots P_p}(h)$  in (6.18) as follows:

$$\begin{aligned} &\int_{\Delta_{P_1 \dots P_p}(h)} \prod_{j=1}^p \{\varepsilon_j f_j(x_1, \dots, x_{p-1}; h)\}^{\lambda_j} \left( \prod_{k=p+1}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \right) \varpi_{p-1} \\ &= \frac{\prod_{j=1}^p \Gamma(\lambda_j + 1)}{\Gamma(\sum_{j=1}^p \lambda_j + p)} \prod_{j=1}^p \{\varepsilon_j f_j(P_j(h))\}^{\lambda_j} \left( \prod_{k=p+1}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \right) \\ &\quad \times \frac{(-1)^{p-1} \{B(0 \star N_p; h)\}^{p-1}}{\sqrt{(-1)^p 2^{3(p-1)} \{B(0 N_p)\}^{p-1} \prod_{j=1}^p B(0 \star \partial_j N_p; h)}} \left( 1 + O(|B(0 \star N_p; h)|) \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{\prod_{j=1}^p \Gamma(\lambda_j + 1)}{\Gamma(\sum_{j=1}^p \lambda_j + p)} \prod_{j=1}^p \{\varepsilon_j f_j(P_j(h))\}^{\lambda_j} \left( \prod_{k=p+1}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \right) \\
&\quad \times \frac{(-1)^{p-1} \{B(0 \star N_p; h)\}^{p-1}}{\sqrt{(-1)^p 2^{3(p-1)} \{B(0 N_p)\}^{p-1} \prod_{j=1}^p B(0 \star \partial_j N_p)}} (1 + O(|\tau|)), \tag{6.19}
\end{aligned}$$

where  $\varepsilon_j f_j(P_j(h))$  is written as

$$\begin{aligned}
\varepsilon_j f_j(P_j(h)) &= \frac{(-1)^{p-1} B(0 \star N_p; h)}{2\sqrt{B(0 N_p) B(0 \star \partial_j N_p; h)}} \left( 1 + O(|B(0 \star N_p; h)|) \right) \\
&= \frac{(-1)^{p-1} B(0 \star N_p; h)}{2\sqrt{B(0 N_p) B(0 \star \partial_j N_p)}} (1 + O(|\tau|)). \tag{6.20}
\end{aligned}$$

Applying (6.19) and (6.20) to (6.18), we have

$$\begin{aligned}
&I'_{n,m}(N_p) \\
&= \frac{2\pi^{\frac{n-p+1}{2}}}{\Gamma(\frac{n-p+1}{2})} \int_0^{r_{N_p}} \frac{\prod_{j=1}^p \Gamma(\lambda_j + 1)}{\Gamma(\sum_{j=1}^p \lambda_j + p)} \prod_{j=1}^p \left\{ \frac{(-1)^{p-1} B(0 \star N_p; h)}{2\sqrt{B(0 N_p) B(0 \star \partial_j N_p)}} (1 + O(|\tau|)) \right\}^{\lambda_j} \\
&\quad \times \left( \prod_{k=p+1}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \right) \frac{(-1)^{p-1} \{B(0 \star N_p; h)\}^{p-1}}{\sqrt{(-1)^p 2^{3(p-1)} \{B(0 N_p)\}^{p-1} \prod_{j=1}^p B(0 \star \partial_j N_p)}} \\
&\quad \times (1 + O(|\tau|)) h^{n-p} dh. \tag{6.21}
\end{aligned}$$

The integral with respect to  $h$  reduces to the beta integral as follows. By change of variable  $h = r_{N_p} \sqrt{s}$  ( $0 \leq s \leq 1$ ), using (6.14) in Lemma 6.4 and (6.5), we have

$$\begin{aligned}
&\int_0^{r_{N_p}} \{(-1)^{p-1} B(0 \star N_p; h)\}^{\sum_{j=1}^p \lambda_j + p-1} h^{n-p} dh \\
&= \frac{1}{2} \{(-1)^{p-1} B(0 \star N_p)\}^{\sum_{j=1}^p \lambda_j + p-1} r_{N_p}^{n-p+1} \int_0^1 s^{\frac{n-p-1}{2}} (1-s)^{\sum_{j=1}^p \lambda_j + p-1} ds \\
&= \frac{\Gamma(\frac{n-p+1}{2}) \Gamma(\sum_{j=1}^p \lambda_j + p)}{2\Gamma(\sum_{j=1}^p \lambda_j + \frac{n+p+1}{2})} r_{N_p}^{n-p+1} \{(-1)^{p-1} B(0 \star N_p)\}^{\sum_{j=1}^p \lambda_j + p-1} \\
&= \frac{\Gamma(\frac{n-p+1}{2}) \Gamma(\sum_{j=1}^p \lambda_j + p)}{2\Gamma(\sum_{j=1}^p \lambda_j + \frac{n+p+1}{2})} \left( \frac{-B(0 \star N_p)}{2B(0 N_p)} \right)^{\frac{n-p+1}{2}} \{(-1)^{p-1} B(0 \star N_p)\}^{\sum_{j=1}^p \lambda_j + p-1}. \tag{6.22}
\end{aligned}$$

Applying (6.22) to (6.21), we therefore obtain

$$\begin{aligned}
I'_{n,m}(N_p) &= \frac{\pi^{\frac{n-p+1}{2}} \prod_{j=1}^p \Gamma(\lambda_j + 1)}{\Gamma(\sum_{j=1}^p \lambda_j + \frac{n+p+1}{2})} \prod_{j=1}^p \left\{ \frac{(-1)^{p-1} B(0 \star N_p)}{2\sqrt{B(0 N_p) B(0 \star \partial_j N_p)}} (1 + O(|\tau|)) \right\}^{\lambda_j} \\
&\quad \times \sqrt{\frac{\{(-1)^{p-1} B(0 \star N_p)\}^{n+p-1}}{2^{n+2(p-1)} \{(-1)^p B(0 N_p)\}^n \prod_{j=1}^p \{(-1)^p B(0 \star \partial_j N_p)\}}} \\
&\quad \times \left( \prod_{k=p+1}^m \{\varepsilon_k f_k(P)\}^{\lambda_k} + O(|\tau|) \right) (1 + O(|\tau|)).
\end{aligned}$$

This completes the proof.  $\square$

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